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WAVE PROPAGATION IN INHOMOGENEOUS MEDIA

by
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ABSTRACT

In this dissertation a method based on orthogonal expansions as well as the invariant imbedding technique has been developed to solve a class of problems of wave propagation in media which are inhomogeneous in more than one direction. By expanding the solution to Maxwell's equations in terms of orthogonal functions, an infinite set of coupled first order linear differential equations with two point boundary conditions results. The technique of invariant imbedding is then employed to transform the linear boundary value problem into a non-linear initial value problem, which makes the solution by digital computers more tractable.

Numerical results for the reflection and transmission coefficients as well as the field distribution are obtained for a slab whose index of refraction is a parabolic function in the direction of propagation and sinusoidal in the transverse direction. To make use of the numerical method of solution, the resulting equations for the modal reflection and transmission coefficients are truncated to include only five modes. The truncation is justified by the fact that the coupling coefficients for higher order modes are negligible. Thus, a system of one hundred simultaneous non-linear first order differential equations are solved by the Runga-Kutta integration method. A number of numerical examples are used to illustrate the dependence of the modal reflection and

transmission coefficients on various parameters. These coefficients are also used to calculate the field distributions. The results obtained for the variation of the transmitted field indicate that it is possible to devise a two-dimensional ionospheric model for diagnostic purposes.

Although a specific type of variation of the index of refraction is used, the method developed is general and can be applied to a large class of problems involving two dimensional inhomogeneities. Further possible applications and extensions of this work are also discussed.

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INTRODUCTION

The problem of wave propagation in inhomogeneous media is important in many situations in which electromagnetic waves are caused to propagate through media whose index of refraction has arbitrary spatial variations. Typical examples arise in: radar signals intercepting ionized wakes, radio waves entering the ionosphere, microwave diagnostics of ionized gas, inhomogeneous filled waveguides or waveguides with non-uniform cross-sections. In general, the available methods of solution to these problems are mostly approximate and applicable conveniently to one dimensional variations. For this so-called stratified model, the inhomogeneities in other directions, if any, are neglected. From a realistic point of view, nature contains inhomogeneous media more often than it does homogeneous media, and the inhomogeneities are usually not limited to one direction. In fact, Komesuroff and others (References 1, 2), have presented experimental results that indicate that the stratified model for the ionosphere is inadequate. Pitteway (Reference 3) discusses this problem and proposes a number of two-dimensional models for the ionosphere. Consequently, it is important to develop methods to cope with this problem. The approximate methods that are available to deal with these problems are the Born approximation or the ray methods. However, these are limited to slow variations

of the index of refraction. Thus, a rigorous approach to the problem of wave propagation in a two-dimensional inhomogeneous media is lacking.

The primary objective of this dissertation is to develop a method which makes use of digital computers to solve a class of problems of wave propagation in media which are inhomogeneous in more than one direction. The method followed here consists of using orthogonal expansions as well as the invariant imbedding technique. By expanding the solution in terms of orthogonal functions, Maxwell's equations are transformed to an infinite set of coupled first order linear ordinary differential equations. Thus, the problem is reduced to solving a system of equations with two point boundary conditions. Because of recent developments in high speed digital computers, numerical analysis has made great progress, and for many problems numerical methods are ideally suited. Since the mechanics of solving differential equations by means of digital computers is such that it is preferable to solve non-linear initial value problems as opposed to linear problems with two point conditions, the method of invariant imbedding to transform the boundary value problem to an initial value problem is employed. The formulation developed is used to solve a specific example in wave propagation in an inhomogeneous medium, but it is equally applicable to such diverse problems as scattering by nuclear potentials, electromagnetic or acoustic waves propagating in inhomogeneously filled waveguides or in waveguides with non-uniform cross-sections.

The dissertation is divided into six chapters. Chapter I provides a general discussion of wave propagation and contains a survey of the approximate methods of solution of the problem of wave propagation in stratified media.

Chapter II deals with the problem of wave propagation in a medium whose dielectric constant is a periodic function along the transverse coordinate. This case serves as a canonical problem for the method to be developed for solving the problem of wave propagation in a two-dimensional inhomogeneous medium.

In Chapter III the mathematical formulation is presented for the solution of the problem of a plane wave incident obliquely upon a dielectric slab whose index of refraction is a function of two coordinates. The method presented is based upon the use of modal expansions to eliminate one of the variables.

In Chapter IV the two point boundary value problem derived in Chapter III is transformed to an initial value problem by means of invariant imbedding.

In Chapter V as an illustration of the method, the reflected and transmitted field distribution as well as the reflection and transmission coefficients are computed for various parameters of a two-dimensional inhomogeneous slab. Finally, in Chapter VI, the conclusions, possible applications and extension of the work are discussed.

CHAPTER I

GENERAL DISCUSSION OF THE WAVE EQUATION

1.1. MAXWELL'S EQUATIONS

The general problem to be considered here is the determination of the electromagnetic fields which may exist in a given linear medium in response to a known source. The only restriction on the medium is that there be a definite time invariant relation among the electromagnetic fields. The time invariance permits considerable simplification of the problem in that harmonic steady state analysis will suffice. The electrical constitution of the medium is completely specified by its permittivity and permeability functions. By virtue of the inhomogeneity of the medium the permittivity and permeability are functions of space and may be discontinuous, as for example, at the interface between a dielectric and free space.

The behavior of the electromagnetic fields in the medium under consideration will be described mathematically by Maxwell's equations and in the subsequent discussion, MKS units will be used throughout. Before exhibiting Maxwell's equations, the following definitions will be presented:

permeability of free space:	μ_o
permittivity of free space:	ϵ_o
speed of light in vacuo:	$c = \frac{1}{\sqrt{\mu_o \epsilon_o}}$
angular wave frequency:	ω
permeability of an arbitrary medium:	μ
permittivity of an arbitrary medium:	ϵ
index of refraction:	$n = c\sqrt{\mu\epsilon}$
electric wave-field vector:	E
magnetic wave-field vector:	H
wave number in vacuo:	$k = \omega/c$

In terms of these quantities, Maxwell's equations take the form

$$\nabla \times E = -\mu \frac{\partial H}{\partial t} \tag{1.1.1}$$

$$\nabla \times H = \epsilon \frac{\partial E}{\partial t} \tag{1.1.2}$$

$$\nabla \cdot \epsilon E = 0 \tag{1.1.3}$$

$$\nabla \cdot \mu H = 0 \tag{1.1.4}$$

By eliminating from the above equations E or H, we obtain:

$$\mu \nabla \times \left(\frac{1}{\mu} \nabla \times E\right) + \frac{n^2}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \tag{1.1.5}$$

$$\epsilon \nabla \times \left(\frac{1}{\epsilon} \nabla \times H\right) + \frac{n^2}{c^2} \frac{\partial^2 H}{\partial t^2} = 0 \tag{1.1.6}$$

However, these equations can be reduced to the following forms:

$$\nabla^2 E - \frac{n^2}{c^2} \frac{\partial^2 E}{\partial t^2} = - (p \cdot E) - q \times \nabla \times E \qquad (1.1.7)$$

$$\nabla^2 H - \frac{n^2}{c^2} \frac{\partial^2 H}{\partial t^2} = - (q \cdot H) - p \times \nabla \times H \qquad (1.1.8)$$

where

$$p = \frac{1}{\epsilon} \text{ grad } \epsilon = \text{grad } (\log \epsilon)$$

$$q = \frac{1}{\mu} \text{ grad } \mu = \text{grad } (\log \mu)$$

The vectors p and q are functions of the medium and are related by the following equation:

$$\frac{1}{2} (p+q) = \text{grad } \log n.$$

In general, equations (1.1.7) and (1.1.8) cannot be decoupled into separate equations for each component.

These equations are to be satisfied at every point in space including the boundary conditions which would have to be added to the above partial differential equations to prescribe the fields at the interfaces between regions of different electrical properties.

For monochromatic waves with time dependence $e^{-i\omega t}$, equations (1.1.1) and (1.1.2) may be simplified to:

$$\nabla \times E = i\omega \mu H \qquad (1.1.9)$$

$$\nabla \times H = -i\omega \epsilon E \qquad (1.1.10)$$

and equations (1.1.7) and (1.1.8) take the form:

$$\nabla^2 E + k^2 n^2 E = -(p \cdot E) - q \times \nabla \times E \qquad (1.1.11)$$

$$\nabla^2 H + k^2 n^2 H = -(q \cdot H) - p \times \nabla \times H \qquad (1.1.12)$$

1.2. EQUATIONS FOR A HOMOGENEOUS MEDIUM

In the special case of a homogeneous medium, the vectors p and q are zero. Equations (1.1.11) and (1.1.12) become

$$\nabla^2 E + k^2 n^2 E = 0 \qquad (1.2.1)$$

$$\nabla^2 H + k^2 n^2 H = 0 \qquad (1.2.2)$$

Every component of E and H satisfies the appropriate partial differential equations. To solve these partial differential equations, one may employ the standard method of separation of variables technique.

1.3. EQUATIONS FOR A STRATIFIED MEDIUM

For a stratified medium, where the dielectric constant varies in the z-direction, or in a radial direction in a spherical system, the problem of wave propagation in this type of medium reduces to the solution of the scalar wave equation. Let one consider, for example, the case where $\mu = 1$ and $\epsilon = \epsilon(z)$ (Figure 1). Then,

$$p = 2 \operatorname{grad} (\log n) = \left(0,\ 0,\ 2\,\frac{n'(z)}{n(z)}\right)$$

$$q = 0.$$

The differential equations (1.1.11) and (1.1.12) reduce to:

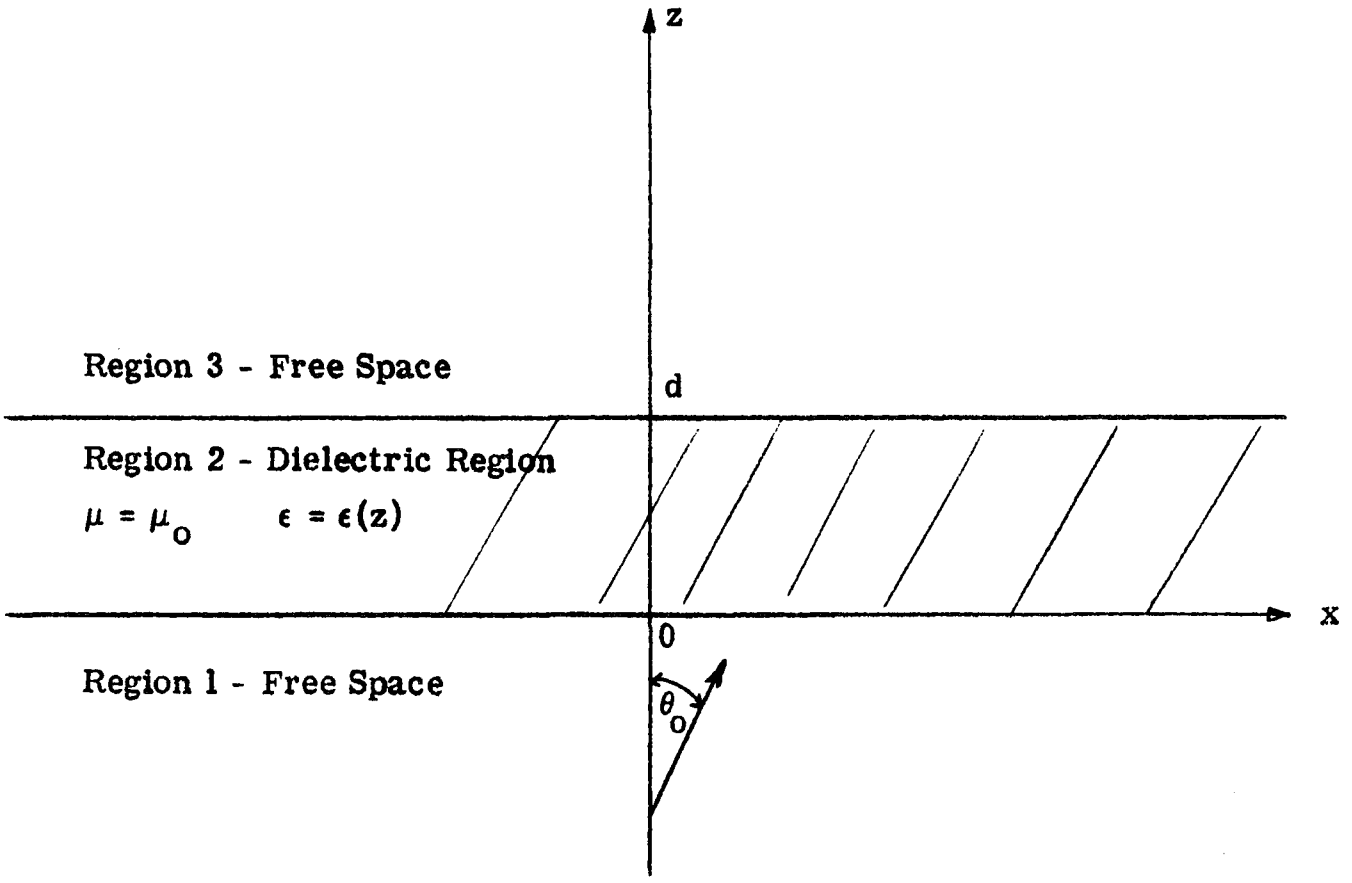


FIGURE 1. A UNIFORM PLANE WAVE INCIDENT UPON A STRATIFIED DIELECTRIC LAYER

$$\left.\begin{aligned}
\nabla^2 E_x + k^2 n^2(z) E_x &= -2 \frac{n'(z)}{n(z)} \frac{\partial E_z}{\partial x} \\
\nabla^2 E_y + k^2 n^2(z) E_y &= -2 \frac{n'(z)}{n(z)} \frac{\partial E_z}{\partial y} \\
\nabla^2 E_z + k^2 n^2(z) E_z &= -2 \frac{\partial}{\partial z} \left(\frac{n'(z)}{n(z)} E_z \right)
\end{aligned}\right\} \qquad (1.3.1)$$

$$\left.\begin{aligned}
\nabla^2 H_x + k^2 n^2(z) H_x + 2 \frac{n'}{n} \frac{\partial H_z}{\partial z} &= 2 \frac{n'}{n} \frac{\partial H_z}{\partial x} \\
\nabla^2 H_y + k^2 n^2(z) H_y - 2 \frac{n'}{n} \frac{\partial H_y}{\partial z} &= 2 \frac{n'}{n} \frac{\partial H_z}{\partial x} \\
\nabla^2 H_z + k^2 n^2(z) H_z &= 0
\end{aligned}\right\} \qquad (1.3.2)$$

Thus, two partial differential equations for E_z and H_z are obtained which enable one to find exact solutions for particular variations of $n(z)$.

The above equations simplify further for the special cases when the wave is linearly polarized with the electric vector perpendicular to the plane of incidence and when it is linearly polarized with the magnetic vector perpendicular to the plane of incidence. One designates these cases by transverse electric wave (denoted by TE) and transverse magnetic wave (denoted by TM) respectively. Any arbitrarily polarized plane wave may be resolved into two waves, one of which is the TE wave:

$$\text{TE wave } E = E_y;\ H_y = E_z = E_x = 0$$

and the other a TM wave

$$\text{TM wave} \quad H = H_y; \; E_y = H_x = H_z = 0.$$

If one takes the plane of incidence to be the xz plane, z being the direction of stratification, Maxwell's equations for the two cases reduce to the following scalar equations (assuming time dependence $e^{-i\omega t}$)

TE wave:

$$\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} = -i\omega\epsilon E_y$$

$$\frac{\partial E_y}{\partial z} = -i\omega\mu H_x$$

$$\frac{\partial E_y}{\partial x} = i\omega\mu H_z$$

TM wave:

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = -i\omega\mu H_y$$

$$\frac{\partial H_y}{\partial z} = i\omega\epsilon E_x$$

$$\frac{\partial H_y}{\partial x} = -i\omega\epsilon E_z$$

The equations show that the fields are functions of x and z only. The first group involves only E_y, H_z, H_x and the second only H_y, E_x, E_z. Thus one can obtain solutions for any polarization by separating any solution into a linear combination of the two solutions.

For the TE wave, equation (1. 3. 1) reduces to

$$\nabla^2 E_y + k^2 n^2(z) E_y = 0 \tag{1. 3. 3}$$

and for the TM wave, equation (1. 3. 2) reduces to

$$\nabla^2 H_y + k^2 n^2(z) H_y - 2 \frac{n'(z)}{n(z)} \frac{\partial H_y}{\partial z} = 0 \tag{1. 3. 4}$$

Thus, equations (1. 3. 3) and (1. 3. 4) represent the wave equations for TE and TM polarizations, respectively.

By employing the method of separation of variables and assuming a solution for the TE wave of the form

$$E_y = u(z) e^{i(k\alpha x - \omega t)}$$

and for the TM wave,

$$H_y = v(z) e^{i(k\alpha x - \omega t)}$$

where α is the separation constant, Equations (1. 3. 3) and (1. 3. 4) become, respectively,

$$\frac{d^2 u}{dz^2} + k^2(n^2(z) - \alpha^2)u = 0 \tag{1. 3. 5}$$

and

$$\frac{d^2 v}{dz^2} - 2 \frac{n'(z)}{n(z)} \frac{dv}{dz} k^2(n^2(z) - \alpha^2)v = 0 \tag{1. 3. 6}$$

Thus, the problem has been reduced to solving a second order linear differential equation.

1. 4. EQUATIONS FOR A TWO-DIMENSIONAL INHOMOGENEOUS MEDIUM

In the case of non-homogeneous media, where the index of refraction is a function of all three coordinates, equations (1. 1. 11) and (1. 1. 12) do not reduce to one equation for each component, since the operators on the right side involve all three components in each equation. However, one can consider a special case of a plane wave propagating through a medium, whose index of refraction is a function of x and z, with no variation in the y-direction. By representing the field as a superposition of TE and TM waves, equations (1. 1. 11) and (1. 1. 12) can be reduced to the following form, respectively:

$$\nabla^2 E_y + k^2 n^2(x, z) E_y = 0 \qquad (1.4.1)$$

and

$$\nabla^2 H_y + k^2 n^2(x, z) H_y + \frac{1}{n^2}\left(\frac{\partial n^2(x, z)}{\partial x}\frac{\partial H_y}{\partial x} + \frac{\partial n^2(x, z)}{\partial z}\frac{\partial H_y}{\partial z}\right) = 0 \qquad (1.4.2)$$

Thus, for this special case, the problem reduces to solving equations (1. 4. 1) and (1. 4. 2).

1. 5. EXACT SOLUTIONS OF THE WAVE EQUATION FOR A STRATIFIED MEDIUM

The mathematical results pertaining to equation (1. 3. 5) have applications in a variety of technical fields. Therefore, it is not surprising that a great amount of research has been devoted to the study of differential equations of this type. This equation has no

general solution that can be written in terms of known functions. Thus, particular cases for which one can obtain exact solutions acquire great interest. A rather small number of exact solutions to equation (1. 3. 5) have been obtained. A short review of the known solutions that are expressible in terms of tabulated functions are presented. The following cases, which differ from one another in the form of the function $n^2(z)$, have been studied by many researchers and are discussed in Reference (4). The form of the dielectric constant and the solutions to the wave equation are as follows:

(1) A linear layer without absorption $n^2(z) = 1 - az$, solutions in terms of Bessel functions of one-third order or Airy functions.

(2) A parabolic layer without absorption, $n^2(z) = a - bz^2$, solution in terms of Weber functions or parabolic cylindrical functions.

(3) A layer characterized by an index of refraction in the form $n^2(z) = a(b+z)^{-2}$, solution in terms of polynomials in z.

(4) A layer characterized by an index of refraction of the form

$$n^2(z) = a+b\,\frac{e^{mz}}{1+e^{mz}} - c\,\frac{e^{mz}}{(1+e^{mz})^2},$$

the solutions are hypergeometric functions.

(5) For the profile $n^2(z) = (1+az)^n$ solutions for the wave equations in terms of Hankel functions have been obtained.

(6) A cosine profile in the index of refraction leads to solutions of the wave equations in terms of Mathieu functions.

(7) For the profile $n^2(z) = ae^{\alpha z}$ solutions of the wave equations may be represented in terms of cylindrical Bessel functions.

(8) Closed form solutions to equation (1. 3. 5) in terms of hypergeometric series can be obtained by transforming the hypergeometric equation into the normal form (equation (1. 3. 5))and identifying the resulting function for n(z).

In the last few years, numerical results have appeared for many of the above cases mentioned.

1. 6. GEOMETRIC OPTICS APPROXIMATION

In the previous section, a number of profiles of the index of refraction for which exact solutions of the wave equations are possible were enumerated. Of great interest are the approximate methods of solution to equation (1. 3. 5), since the approximate methods can be applied to arbitrary functions of the index of refraction. The electromagnetic wave transmission and reflection characteristics from a plane medium whose index of refraction varies in the direction perpendicular to the plane interfaces, have been studied by many investigators employing various approximate methods.

The form most convenient for this discussion is given by

$$\frac{d^2u}{dz^2} + h^2q^2(z,h)u = 0 \qquad (1.6.1)$$

where u(z) and q(z) are scalars, and the parameter h is taken to be large and positive. This restriction involves very little loss in generality, since any second order differential equation, particularly the Sturm-Louville equation, may be transformed to this form. In but a few cases, can one obtain a closed form solution for the above equation. For the general case, one has to seek approximate solutions. The Wentzel-Kramers-Brillouin (WKB) method constitutes a powerful technique for dealing with these approximations. Essentially, employing such a technique is to restrict oneself to the first term of an asymptotic series.

The two solutions to equation (1. 6. 1) are given by

$$u_1 = q^{-1/2} \exp (ih \int q \, dz) \qquad (1.6.2)$$

$$u_2 = q^{-1/2} \exp (-ih \int q \, dz) \qquad (1.6.3)$$

and are known as the WKB approximations.

To extend the usefulness of the WKB method, it becomes necessary to obtain correction terms. Bremmer (Reference 5) showed that the WKB approximation is the first term of a rigorous series solution. The solution is obtained by using successive generation of waves of different orders for which the method of successive approximation is ideally suited. Thus, the obtained solution allows one to estimate the limits of applicability of the WKB method. Clearly, the WKB method may be considered as sufficiently good approximation if the secondary

waves, which were obtained as a result of reflections at the inhomogeneities of the medium, can be neglected in comparison with the primary waves.

Wait, in Reference (6), discusses these correction terms and shows that the WKB method is valid whenever the following condition is satisfied:

$$\left|\frac{d}{dz}\left(\frac{1}{q}\right)\right| \ll 1.$$

In the previous discussion, turning points have been excluded from the domain of validity of the approximate solutions. To obtain solutions valid in the whole domain, Jeffrey's (Reference 7) used a special solution through the turning point to join the two sides together. This technique patches the results together into a more or less complete representation. Langer (Reference 8) attacked the problem from a more fundamental viewpoint, namely that of obtaining one and the same representation which is valid over the whole domain of the variable. Langer was able to obtain an approximate solution valid at the turning point, such that the asymptotic expressions of the solution were merely the WKB solutions in the domain away from the turning point.

From a realistic point of view, nature contains inhomogeneous media more often than it does with homogeneous media. Consequently, it is important to develop methods to cope with the problem of propagation through media whose index of refraction has an arbitrary spatial variation. The most widely used method is the "ray method"

which is essentially the WKB approximation to the scalar wave equation. In deriving the eikonal equation, which describes the ray path, one has to assume that the properties of the medium are slowly varying within a distance of a wavelength. A complete treatment of the method is given by Born and Wolf (Reference 9). The development of high speed digital computers has permitted numerical solutions of the eikonal equation for different variations of the index of refraction as a function of space.

1. 7. THE BORN APPROXIMATION

One can find an expression for the amplitude in terms of quadratures by using the so-called Born approximation. This approximation is adaptable to numerical computation and is applicable only when the true wave function in the scattering region does not differ very much from the incident wave.

Consider the reduced wave equation given by the relation

$$\nabla^2 u + k^2[1 - n^2(\underset{\sim}{X})]\, u = 0 \qquad (1.7.1)$$

where u is a scalar function and

n(X) is the perturbation in the index of refraction

$X = (x_1, x_2, x_3)$.

Equation (1. 7. 1) can be transformed to a Fredholm integral equation with the aid of the Green's function technique. One partitions equation (1. 7. 1) as follows:

$$\nabla^2 u + k^2 u = [k^2 n^2(X)]\, u \tag{1.7.2}$$

In the scattering formulation of the problem, the solution of equation (1.7.2) can be expressed as the sum of two wave functions,

$$u = u_o + u_s \tag{1.7.3}$$

where the unperturbed wave u_o satisfies the homogeneous equation

$$\nabla^2 u_o + k^2 u_o = 0 \tag{1.7.4}$$

The scattered wave satisfies

$$\nabla^2 u_s + k^2 u_s = [k^2 n^2(X)]\, u \tag{1.7.5}$$

and can be expressed as an integral equation by employing the Green's function

$$\nabla^2 G + k^2 G = \delta(\overline{X} - \overline{X}'),$$

where

$$G = - \frac{\exp ik\, |\overline{X} - \overline{X}'|}{4\pi\, |\overline{X} - \overline{X}'|} \tag{1.7.6}$$

It follows from the theory of Green's functions that

$$u_s = - \frac{1}{4\pi} \int \frac{\exp ik\, |\overline{X} - \overline{X}'|}{|\overline{X} - \overline{X}'|} [k^2 - k^2 n^2(X')]\, u(X')\, dX' \tag{1.7.7}$$

and by using the relation $u = u_o + u_s$ one obtains the Fredholm integral equation of the second kind in terms of u

$$u = u_o - \frac{1}{4\pi} \int \frac{\exp ik\, |\overline{X} - \overline{X}'|}{|\overline{X} - \overline{X}'|} [k^2 - k^2 n^2(X')]\, u\, dX' \tag{1.7.8}$$

By substituting the unperturbed wave function under the integral sign, and denoting $|\bar{X} - \bar{X}'|$ by ρ, one obtains an expression for the Born approximation

$$u = u_o - \frac{1}{4\pi} \int \frac{\exp ik\rho}{\rho} [k^2 - k^2 n^2(X')] u_o dX' \qquad (1.7.9)$$

To obtain the asymptotic form of equation (1.7.9), one uses the boundary condition at infinity. In that limit

$$\rho \rightarrow \bar{X} - \bar{X}' \cos\theta ; \; \bar{X} \rightarrow \infty$$

where θ is the angle between the vectors $\bar{X}$ and $\bar{X}'$. Since $\bar{X}$ is the vector to the point of observation, it is convenient to define the vector $\bar{k}_s$ in the direction of $\bar{X}$ but having the magnitude of k, so that

$$\bar{k}_s \cdot \bar{X}' = kX' \cos\theta$$

Therefore, in the limit as $X \rightarrow \infty$, and using the expnession for u_o, one obtains

$$u \sim \exp ik\bar{X} - \frac{1}{4\pi} \frac{\exp ikX}{X} \cdot$$

$$\int \exp\{i[k\bar{X}' - \bar{k}_s \cdot \bar{X}']\} [k^2 - k^2 n^2(X')] dX' \qquad (1.7.10)$$

1.8. IMPROVING THE BORN APPROXIMATION

Goldman and Migdal (10) present a method given to improve the Born approximation. The method consists of the following: the wave

function of the problem is written in the form $u = u_o + u_1$, where u_o is the ordinary WKB solution, and for u_1 an inhomogeneous equation is obtained, the solution of which is found by the use of Green's function.

To improve the Born approximation, Saxon and Schiff (Reference 11), developed a method whereby the basic equation is reformulated into a single integral expression. The essential idea of the method is the following: the wave equation is rewritten as an integral equation by introducing a modified Green's function in place of the usual Green's function. This integral equation, which is exact, gives as its zero th approximation the WKB result. Higher order corrections depend on the variation of the local wave number along the ray and on the curvature of the ray. The actual construction of such an integral equation is accomplished by first choosing a Green's function with the desired character and then finding the differential equation it satisfies. The integral equation follows directly upon the application of Green's theorem. This procedure, while physically appealing, is very complicated mathematically.

In general, it can be said that the methods of solution to the problem of wave propagation in inhomogeneous media are approximate and applicable conveniently to one-dimensional variations. It is the primary objective of this dissertation to develop a technique which makes use of digital computers to solve the problem of wave propagation in media inhomogeneous in more than one dimension. Some work on such problems

appears in the investigation of waveguides filled with non-uniform dielectric material or varying cross-sections (References 12, 13). Extension of this work to exterior problems shall be considered in this thesis. One shall start by considering a special class of problems, that of an inhomogeneous slab with the index of refraction varying in the transverse direction. These problems have recently attracted attention and shall serve as the first step toward the solution to more complex problems.

CHAPTER II

REFLECTION AND TRANSMISSION BY A PERIODIC DIELECTRIC LAYER

2.1. INTRODUCTION

The study described in this chapter deals with the reflection and transmission of electromagnetic waves by a medium possessing a dielectric constant which varies periodically along the transverse coordinate. The interest in this specific problem was stimulated by several topics in which this work has direct application, and it also serves as a basis for further developments. Aside from the areas of direct application, the case to be discussed will serve as a canonical problem in the theory of wave propagation in a two-dimensional inhomogeneous medium. However, the treatment of this topic contained in this chapter should be of value in itself.

In the present study, one obtains a formal solution for the transmission and reflection coefficients for a dielectric layer whose index of refraction is a periodic function of the transverse coordinate. The special case of a sinusoidally modulated slab of finite thickness will also be discussed.

2. 2. WAVE SOLUTION FOR H-MODES

A plane wave is assumed to be incident at an angle θ_o upon a periodic dielectric layer (Figure 2). For simplicity, the plane of incidence is taken parallel to the x, z plane, thus obtaining a two-dimensional problem with $\frac{\partial}{\partial y} \equiv 0$. With this geometry, it is easy to verify from Maxwell's equations that the solution may be represented in terms of H modes only (transverse electric) with respect to a propagation direction along the z-coordinate. The field then consists of electric E_y and magnetic H_x and H_z components only. Hence the solution can be represented by the scalar component E_y with H_x and H_z obtainable from it by differentiation. The electric field vector of the incident field is given by the expression

$$E_y^i(x, z) = Ae^{i(\beta_o x + \bar{\gamma}_o z)}$$

where

$$\begin{aligned} \beta_o &= k \sin \theta \\ \bar{\gamma}_o &= k \cos \theta \end{aligned} \qquad (2.2.1)$$

and a time dependence $e^{-i\omega t}$ is assumed.

The electromagnetic field components outside the layer are satisfied by a scalar wave equation of the form

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + k^2 E_y = 0 \qquad \begin{matrix} z > d \\ z < 0 \end{matrix} \qquad (2.2.2)$$

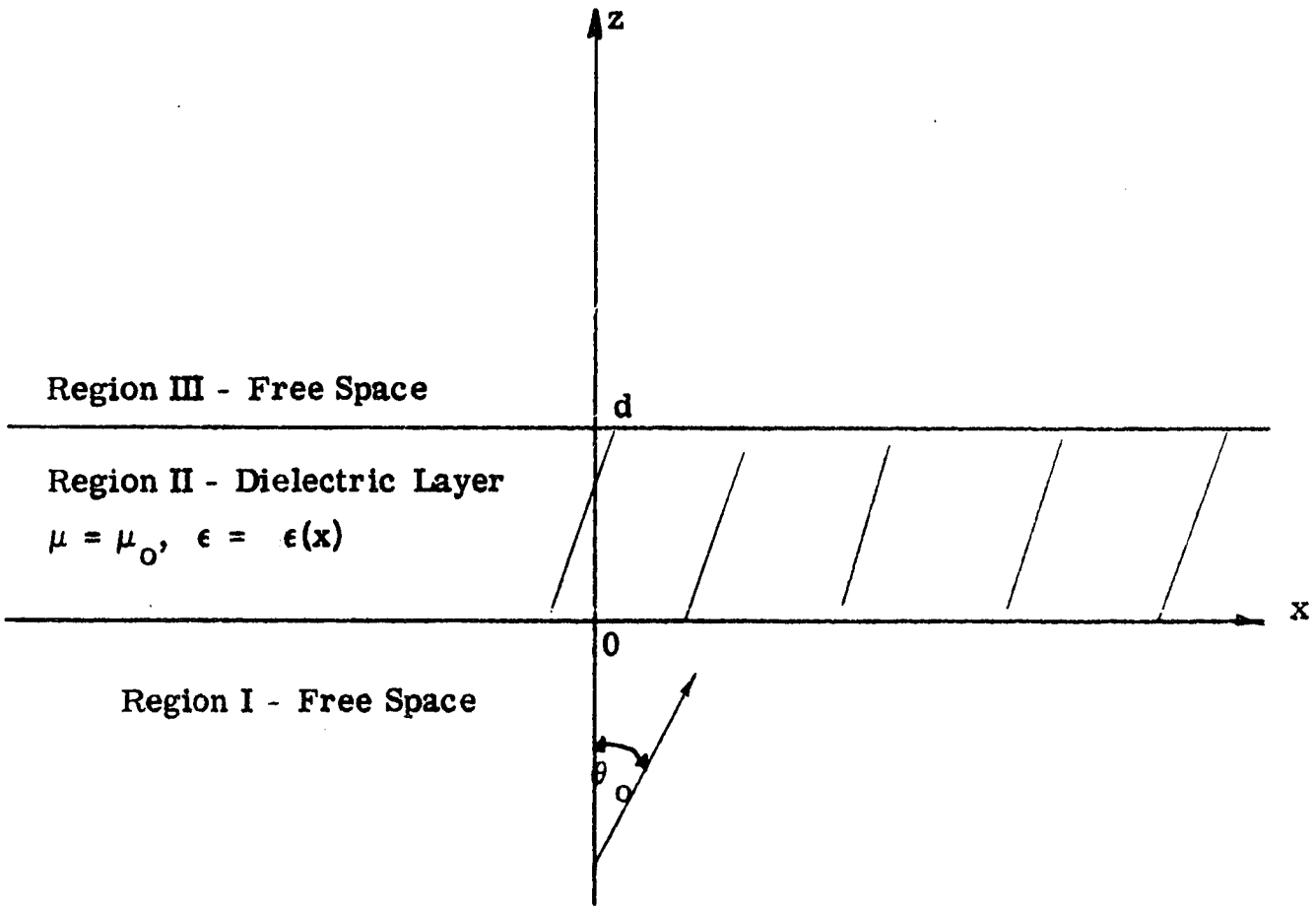


FIGURE 2. UNIFORM PLANE WAVE INCIDENT UPON A PERIODIC DIELECTRIC LAYER

The dielectric constant of the layer will be assumed to be a periodic function of x, and is given by the following expression:

$$\epsilon(x) = \epsilon_1[1 - p(x)]$$

and

$$p(x) = p(x+a) \tag{2. 2. 3}$$

where a is the period.

The solution to the field outside the dielectric layer may be represented as a summation over characteristic waves of the form

$$e^{i\beta_\nu x} \quad e^{i\bar{\gamma}_\nu z}$$

One considers modes propagating along the z-direction with a transverse distribution $e^{i\beta_\nu x}$. The wave numbers β_ν form a discrete set and because of periodicity at $z = 0$, it must satisfy the Floquet condition

$$\beta_\nu = \beta_0 + \frac{2\pi\nu}{a} \quad (\nu = 0, \pm 1, \pm 2, \ldots)$$

The excitation by a single incident plane wave prescribes a unique value to β_0 which is given by equation (2. 2. 1), so that the fundamental (n = 0) mode is identified by the incident wave.

The wave numbers $\bar{\gamma}_\nu$ must satisfy the condition of separation of variables of equation (2. 2. 2), thus

$$\bar{\gamma}_\nu = (k^2 - \beta_\nu^2)^{1/2}$$

This representation is prescribed by the modulated dielectric, which acts essentially as a grating, hence the index ν in β_ν and $\bar{\gamma}_\nu$ corresponds to the νth spectral order of diffraction.

For the particular problem under consideration, i. e., TE polarization, Maxwell's equations yield a scalar wave equation in the dielectric layer given by

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial E_y}{\partial z^2} + k^2 \epsilon(x) E_y = 0 \qquad (2.2.4)$$

Since in equation (2. 2. 4) the coefficients are functions of x only, one may assume a solution of the form

$$E_y = \xi(x)\eta(z) \qquad (2.2.5)$$

Upon separation of variables, one obtains

$$\frac{d^2\eta}{dz^2} + (k^2\epsilon_1 - \alpha^2)\eta = 0 \qquad (2.2.6)$$

or

$$\frac{d^2\xi}{dx^2} + [\alpha^2 - k^2\epsilon_1 p(x)]\xi = 0 \qquad (2.2.7)$$

where α^2 is the separation constant.

The solution to equation (2. 2. 6) can be written as

$$\eta(z) = Ae^{i(k^2\epsilon_1 - \alpha^2)^{1/2} z} + Be^{-i(k^2\epsilon_1 - \alpha^2)^{1/2} z} \qquad (2.2.8)$$

The coefficient of equation (2. 2. 7) is periodic in x of period a. The properties of the solution of this equation (eigenvalues, eigenfunctions, orthogonality, etc.) are discussed in Appendix I.

One notes from the geometry of the problem that the dielectric constant is invariant with respect to the z-direction. Thus, one expects the characteristic solutions to be of the form

$$e^{i\gamma_n z}\,\xi_n(x)$$

where the quantity γ_n relates to a set of eigenvalues identified by the index n, and is given by the relation

$$\gamma_n = (k^2\epsilon_1 - \alpha_n^2)^{1/2}$$

With the field expanded in terms of the function $\xi_n(x)$ in the dielectric constant, one again obtains the picture of individual modes propagating in the z-direction, each one of these modes possesses a wave number γ_n and a transverse distribution $\xi_n(x)$ along the x-direction. By assuming an amplitude of unity for the incident electric field and accounting for the radiation condition, one finds that for region I, the total field can be expressed as

$$E_y^{(1)}(x, z) = e^{i(\beta_o x + \bar{\gamma}_o z)} + \sum_\nu R_\nu\, e^{i(\beta_\nu x - \bar{\gamma}_\nu z)} \qquad (2.2.9)$$

In the dielectric layer, the field can be expressed in the following manner:

$$E_y^{(2)}(x, z) = \sum_n \left[A_n e^{i\gamma_n z} + B_n e^{-i\gamma_n z} \right] \xi_n(x) \qquad (2.2.10)$$

and in region III, is given by the expression

$$E_y^{(3)}(x, z) = \sum_\nu T_\nu \, e^{i(\beta_\nu x + \bar{\gamma}_\nu z)} \qquad (2.2.11)$$

The boundary conditions at $z = 0$ and $z = d$ are given, respectively, by

$$E_y^{(1)}(x, o) = E_y^{(2)}(x, o)$$

$$\left. \frac{\partial E_y^{(1)}}{\partial z} \right|_{z=o} = \left. \frac{\partial E_y^{(2)}}{\partial z} \right|_{z=o} \qquad (2.2.12)$$

and

$$E_y^{(2)}(x, d) = E_y^{(3)}(x, d)$$

$$\left. \frac{\partial E_y^{(2)}}{\partial z} \right|_{z=d} = \left. \frac{\partial E_y^{(3)}}{\partial z} \right|_{z=d} \qquad (2.2.13)$$

Employing equations (2.2.9) and (2.2.10) in conjunction with the boundary conditions (2.2.12), one obtains two infinite sets of linear equations. The system of equations is derived by performing the following operations. Both sides of the equations are multiplied by the factor $\frac{1}{a} \exp(-i\beta_\mu x)$ and integrated along the period a. Using the orthogonality properties and following the definitions of the Fourier coefficients given in Appendix II, one obtains the system

of equations

$$\delta_{\mu,o} + R_\mu = \sum_n (A_n + B_n) a_n(\beta_\mu)$$

$$\bar{\gamma}_o \delta_{\mu,o} - \bar{\gamma}_\mu R_\mu = \sum_n (\gamma_n A_n - \gamma_n B_n) a_n(\beta_n) \qquad (2.2.14)$$

$$\mu = 0, \pm 1, \pm 2, \ldots$$

where

$$a_n(\beta_\mu) = \frac{1}{a} \int_{-a/2}^{a/2} \xi_n(x)\, e^{-i\beta_\mu x}\, dx.$$

In order to eliminate the reflection coefficient, one divides the second equation by $\bar{\gamma}_\mu$ and adds this to the first equation:

$$2\delta_{\mu,o} = \sum_n \left[A_n \left(1 + \frac{\gamma_n}{\bar{\gamma}_\mu} \right) a_n(\beta_\mu) + B_n \left(1 - \frac{\gamma_n}{\bar{\gamma}_\mu} \right) a_n(\beta_\mu) \right] \qquad (2.2.15)$$

$$\mu = 0, \pm 1, \pm 2, \ldots$$

Now employing equations (2.2.10) and (2.2.11) together with the boundary conditions at $z = d$, given by equation (2.2.13), one obtains two infinite sets of linear equations. The system of equations is obtained by performing the following operations; both sides are multiplied by $\frac{1}{a} \xi^*_m(x)$, ($\xi^*_m(x)$ is the conjugate of $\xi_m(x)$), and integrated along the period "a". Employing the orthogonality properties of $\xi_m(x)$, one obtains the following equations:

$$\sum_{\nu} T_{\nu} e^{i\bar{\gamma}_{\nu} d} a_m^*(\beta_{\nu}) = A_m e^{i\gamma_m d} + B_m e^{-i\gamma_m d} \tag{2. 2. 16}$$

$$\sum_{\nu} \bar{\gamma}_{\nu} T_{\nu} e^{i\bar{\gamma}_{\nu} d} a_m^*(\beta_{\nu}) = A_m \gamma_m e^{i\gamma_m d} - B_m \gamma_m e^{-i\gamma_m d} \tag{2. 2. 17}$$

$$m = 0, 1, 2, \ldots$$

From these equations, one can solve for A_m and B_m in terms of T_{ν}, one thus obtains

$$A_m = \frac{1}{2} \sum_{\nu} T_{\nu} e^{i(\gamma_{\nu} - \gamma_m)d} a_m^*(\beta_{\nu}) \left[1 + \frac{\bar{\gamma}_{\nu}}{\gamma_m}\right] \tag{2. 2. 18}$$

$$B_m = \frac{1}{2} \sum_{\nu} T_{\nu} e^{i(\bar{\gamma}_{\nu} \pm \gamma_m)d} a_m^*(\beta_{\nu}) \left[1 - \frac{\bar{\gamma}_{\nu}}{\gamma_m}\right]. \tag{2. 2. 19}$$

Now substituting equations (2. 2. 18) and (2. 2. 19) into equation (2. 2. 15), we obtain an infinite set of inhomogeneous linear equations for the coefficients T_{ν}

$$\sum_{\nu} T_{\nu} e^{i\bar{\gamma}_{\nu} d} \sum_{n} \cos \gamma_n d \, \frac{1}{2} a_n^*(\beta_{\nu}) a_n(\beta_{\mu}) \cdot \left[1 + \frac{\bar{\gamma}_{\nu}}{\bar{\gamma}_{\mu}}\right]$$

$$- i \sin \gamma_n d \, \frac{1}{2} a_n^*(\beta_{\nu}) a_n(\beta_{\mu}) \left[\frac{\bar{\gamma}_{\mu}}{\gamma_n} + \frac{\gamma_n}{\bar{\gamma}_{\mu}}\right] = \delta_{\mu, o} \tag{2. 2. 20}$$

$$\mu = 0, \pm 1, \pm 2, \ldots$$

Thus the problem reduces to solving this system of equations for the amplitude T_{ν}, from which the reflection coefficients may be found.

By the method of truncation, the solution to such systems of equations can be obtained providing one can solve equation (2. 2. 7).

2. 3. SPECIAL CASE OF A SINUSOIDALLY VARIABLE DIELECTRIC LAYER

Consider a plane wave incident at angle θ_o on a sinusoidally variable dielectric layer. The dielectric constant of the layer will be assumed to be a periodic function of x and is given by the following expression:

$$\epsilon(x) = \epsilon_1 \left[1 - m \cos\left(\frac{2\pi x}{a}\right)\right] \tag{2. 3. 1}$$

For the case of TE polarization, Maxwell's equations yield a scalar wave equation in the dielectric layer given by

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + k^2 \epsilon_1 \left(1 - m \cos \frac{2\pi x}{a}\right) E_y = 0 \tag{2. 3. 2}$$

Using the same procedure as above, one obtains the following two equations:

$$\frac{d^2 \eta}{dz^2} + (k^2 \epsilon_1 - \alpha^2)\eta = 0 \tag{2. 3. 3}$$

$$\frac{d^2 \xi}{dx^2} + (\alpha^2 - k^2 \epsilon_1 m \cos \frac{2\pi x}{a})\xi = 0 \tag{2. 3. 4}$$

where α^2 is the separation constant.

The solution to equation (2. 3. 3.) can be written as

$$\eta(z) = Ae^{i(k^2\epsilon_1 - \alpha^2)^{1/2} z} + Be^{-i(k^2\epsilon_1 - \alpha^2)^{1/2} z}$$

In Equation (2. 3. 4), let

$$x' = \frac{\pi}{a} x$$

Thus

$$\frac{d^2\xi_n}{dx'} + \frac{a^2}{\pi^2} (\alpha^2 - k^2\epsilon_1 m \cos 2x')\xi_n = 0.$$

Now let

$$2h^2 = \frac{a^2k^2\epsilon_1 m}{\pi^2}$$

and

$$\lambda_n = \frac{a^2\alpha_n^2}{\pi^2} = (\frac{a}{\pi})^2[k_1^2\epsilon - \gamma_n^2] \qquad (2.3.5)$$

The above equation takes the canonical form of Mathieu's differential equation

$$\frac{d^2\xi_n}{dx'^2} + (\lambda_n - 2h^2 \cos 2x')\xi_n = 0 \qquad (2.3.6)$$

with λ_n and h^2 purely real.

The solution to the Mathieu differential equation may be put into the Floquet form:

$$\xi_n(x') = \exp[i\beta_n(\frac{a}{\pi})\, x'] \sum_{r=-\infty}^{\infty} a_r^n \exp\left[i\left(\frac{2\pi r}{a}\right)x'\right]$$

where a_r^n are the coefficients of the Fourier series of a periodic function of x with period a and β_n plays the role of the characteristic exponent. The index n refers to the solution pertinent to the eigenvalues λ_n in equation (2. 3. 6). In the present case, β_n plays the role of wave numbers denoting the phase progression along x. It should also be recognized that the field continuity condition at the interface ($z = 0$) requires that the set β_n possess the same values as outside the dielectric layer, hence

$$\beta_n = \beta_o + \frac{2\pi n}{a} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

The coefficients a_r^n and the wave number β_o are related to the eigenvalues λ_n by means of complicated transcendental relations involving continued fractions. For a complete discussion of the Mathieu equation, see Appendix I.

In principle, however, when h^2 and β_n are given, then λ_n and a_r^n are specified and therefore $\xi_n(x)$ is well defined. It also should be noted that the wave number γ_n in the dielectric region is simply related to λ_n by relation (2. 3. 5). By assuming the same condition as for the general periodic slab, one finds that the field is given by

$$E_y^{(1)}(x', z) = e^{i[\beta_o(\frac{a}{\pi}x') + \gamma_o z]} + \sum R_\nu e^{i(\beta_\nu(\frac{a}{\pi}x') - \bar{\gamma}_\nu z)}$$

$$(z < 0)$$

where

$$\beta_\nu = \beta_o + \frac{2\pi\nu}{a}$$

and

$$\bar{\gamma}_\nu = [k^2 - \beta_\nu^2]^{1/2}$$

The number of propagation modes may be determined from the above equation, that is, if $k^2 > \beta_\nu^2$ then the νth mode is propagating. In the dielectric layer, the field can be expressed in the following manner:

$$E_y^{(2)}(x', z) = \sum_n \left[Ae^{i\gamma_n z} + B_n e^{-i\gamma_n z} \right] \xi_n(x')$$

$$0 < z < d$$

where

$$\gamma_n = \left[k^2 \epsilon_1 - (\frac{\pi}{a})^2 \lambda_n \right]^{1/2}$$

and in region $z > d$ the field can be expressed as

$$E_y^{(3)}(x', z) = \sum_\nu T_\nu \; e^{i\left[\beta_\nu(\frac{\pi}{a}x') + \bar{\gamma}_\nu z\right]}$$

when the indices ν and n in the summations are understood to run over all values $n, \nu = 0, \pm 1, \pm 2, \ldots$

By following the same approach outlined in the previous section, one can obtain solutions to this problem by truncation. This problem was discussed by Wang (Reference 14) who obtained a solution for a number of modes by this method.

For the normal incidence case ($\theta_o = 0$), the Mathieu functions are of integral order. That is, $\beta_n(\frac{\pi}{a})$ is an integer. The field in the dielectric constant can be expressed in terms of the following functions;:

$$ce_{2n}(x', h^2) = \sum_{r=o}^{\infty} A_{2r}^{2n}(h^2) \cos 2rx'$$

$$ce_{2n+1}(x', h^2) = \sum_{r=o}^{\infty} A_{2r+1}^{2n+1}(h^2) \cos (2r+1)x'$$

$$se_{2n+1}(x', h^2) = \sum_{r=o}^{\infty} B_{2r+1}^{2n+1}(h^2) \sin (2r+1)x'$$

$$se_{2n+2}(x', h^2) = \sum_{r=o}^{\infty} B_{2r+2}^{2n+2}(h^2) \sin (2r+2)x'$$

and the eigenvalues $a_n(h^2)$ and $b_n(h^2)$ correspond to the eigenfunctions $ce_n(x', h^2)$ and $se_n(x', h^2)$ respectively. The properties of these functions as well as other relations are discussed in Appendix I.

For the dielectric constant given by equation (2.3.1), the solutions to equation (2.3.6) have the symmetry $\xi(x') = \xi(-x')$, and so there can be no antisymmetric modes excited. Thus, the field in the dielectric constant can be expressed as

$$E_y^{(2)}(x', z) = \sum_n \left[A_n e^{i\gamma_n z} + B_n e^{-i\gamma_n z} \right] ce_{2n}(x')$$

where

$$\gamma_n = \left[k^2\epsilon_1 - \left(\frac{\pi}{a}\right)^2 a_{2n}\right]^{1/2}$$

The modes will be propagating or evanescent according to whether γ_n is real or imaginary, respectively. It is observed that the number of propagating modes in a dielectric with $\epsilon_1 > 1$ is, in general, larger than that in the air. The reverse obviously holds for isotropic electron plasmas where $\epsilon_r < 1$.

One should recognize that, when a wave is incident on a modulated dielectric slab, the periodicity manifests itself already at the interface by producing fields that are expressible in terms of modes of the Floquet type. These modes are coupled only at the interface and propagate independently in the dielectric slab. At the other side of the slab, the second interface produces coupling analogous to the preceding one. The waves emerge in the air region and travel as independent modes.

The problem discussed here will serve as a canonical problem in the solution of more complex problems. If for the dielectric constant of a slab, one assumes an arbitrary spatial variation in both x and z coordinates, one should be able to divide the slab into a finite number of slabs along the direction of propagation assume a constant variation in the z-direction for each slab. Thus, the method of separation of variables can be used to reduce the two-dimensional wave equation to a problem of solving a second

order differential equation in each slab. The solution in each slab must satisfy the boundary conditions at the interfaces between each slab. This results in a set of algebraic equations for each particular slab which besides determining the arbitrary coefficients, provide a relation for the separation of constants. However, in general, the separation constants are not discrete, and thus the solutions to the wave equation in each slab must be represented in terms of integrals. This approach is very cumbersome and for the general case quite difficult.

In the following chapter, this approach will be developed for the two-dimensional inhomogeneous slab which has an arbitrary variation in the z-direction and a periodic variation in the x-direction. As one divides the slab into a finite number of slabs, and assumes constant variation in the z-direction in each slab, the resulting problem is reduced to the problem discussed in this chapter. Thus, the periodic slab will serve as the canonical problem for our future work. As the slab is subdivided into finer divisions, the solution to the problem becomes more exact. Thus, as the subdivision approaches zero, the solution to the problem becomes exact. This approach will be developed into a method of solving this class of problems.

CHAPTER III

MATHEMATICAL FORMULATION OF THE PROPAGATION PROBLEM THROUGH AN INHOMOGENEOUS SLAB

3.1. INTRODUCTION

A plane wave is assumed to be incident at an angle θ_o upon a dielectric slab. The dielectric constant of the slab is assumed to be periodic in the x-direction and arbitrary in the z-direction (Figure 3). If one assumes that the slab is divided into a large number of slabs along the direction of propagation, then the field in each slab can be approximately represented in the form of a series of waves. This approximate solution becomes exact as the divisions approach zero. This method has been employed by Katzenelenbaum and others (12, 15, 16) to analyze interior problems such as non-uniform wave guides and transmission lines. The method is based on the fact that at any point in the non-uniform section, the fields $\bar{E}$ and $\bar{H}$ can be resolved into fields of waves which may exist in a section in which ϵ is the same function of the coordinate x for a particular value of $z = z'$. Thus, formally, the representation for the solution of this problem is exact. However, one still has to solve an infinite set of first-order differential equations with prescribed boundary conditions.

This method will be extended to solve the exterior type problem by using the solution to the periodic slab discussed in the previous chapter.

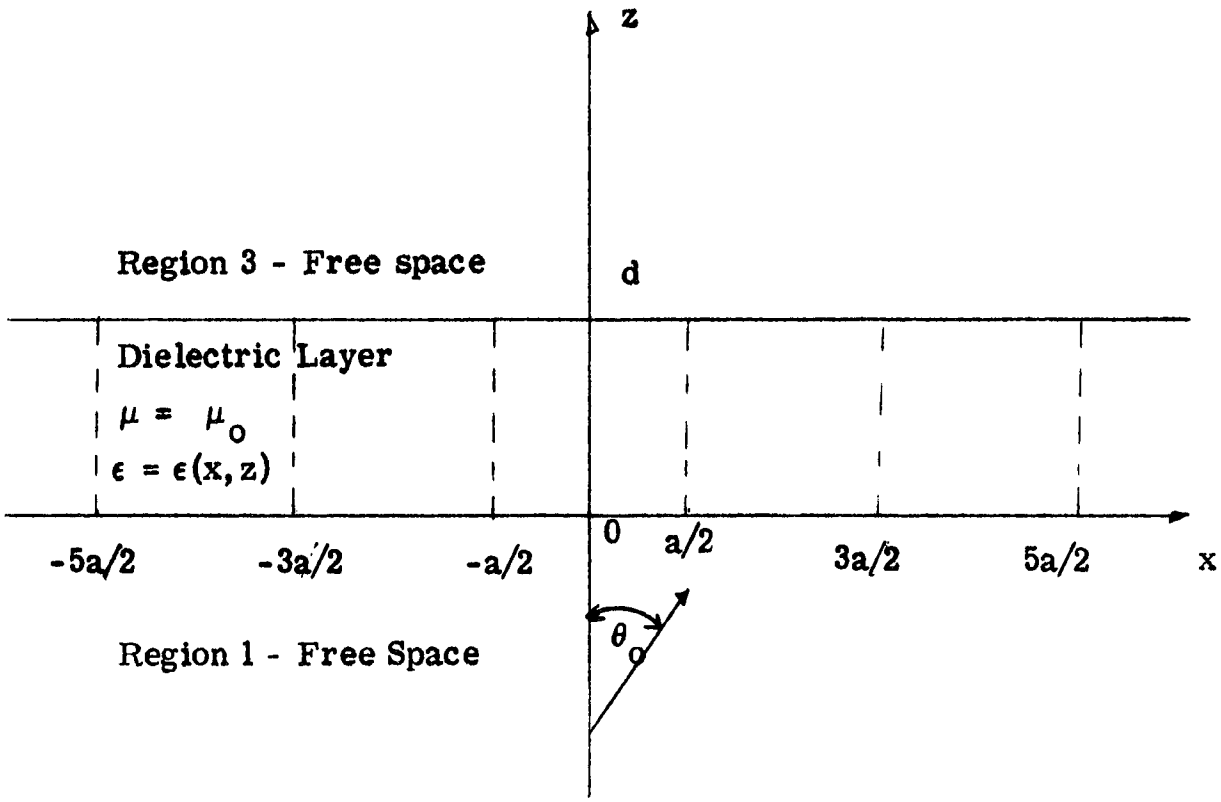


FIGURE 3. UNIFORM PLANE WAVE INCIDENT UPON AN INHOMOGENEOUS DIELECTRIC LAYER

3. 2. GENERAL SYSTEM OF EQUATIONS

Let one first consider a slab whose dielectric constant $\epsilon(x)$ is a periodic function of x of period "a" and no variation in the z-direction. For TE polarization, Maxwell's equations are given by

$$\frac{\partial E_y}{\partial z} = -i\omega\mu H_x$$

$$\frac{\partial E_y}{\partial x} = +i\omega\mu H_z \qquad (3.2.1)$$

$$\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} = +i\omega\epsilon(x)E_y$$

E_x, E_z, H_y are all equal to zero.

Since the properties of the medium do not depend on z, it follows that the solution to the above system of equations can be written in the form

$$E_y(x, z) = u(x)e^{+i\gamma z}$$

$$H_x(x, z) = \bar{v}_x(x)e^{+i\gamma z} \qquad (3.2.2)$$

$$H_z(x, z) = \bar{v}_z(x)e^{+i\gamma z}$$

By substituting equation (3. 2. 2) into (3. 2. 1), one obtains the second order differential equation

$$\frac{d^2u}{dx^2} + (\omega^2\mu\epsilon(x) - \gamma^2)u(x) = 0 \qquad (3.2.3)$$

For the boundary conditions we require that the eigenfunctions of the above equation be of the Floquet type. The solution to this problem was discussed in the previous chapter. Various properties of equation (3. 2. 3) are discussed in Appendix I.

The magnetic field components are determined from Maxwell's equations and are given by the relations

$$H_x(x, z) = \bar{v}_x(x)e^{+i\gamma z} = -\frac{1}{i\omega\mu}\frac{\partial}{\partial z}[u(x)e^{+i\gamma z}]$$

$$H_z(x, z) = \bar{v}_z(x)e^{+i\gamma z} = +\frac{1}{i\omega\mu}\frac{\partial}{\partial x}[u(x)e^{+i\gamma z}] \qquad (3.2.4)$$

In the ensuing discussion, $u(x)$, $\bar{v}_x(x)$, $\bar{v}(x)$ and γ are considered to be determined. One also assumes that the eigenfunctions are chosen such that one number, γ_n, corresponds to one function, $u^n(x)$. The functions $u^n(x)$ and $\bar{v}^n_x(x)$ corresponding to the propagation constant γ_n will be called cross-section eigenfunctions for the medium under discussion. Thus, the wave eigenfunctions are given by

$$E^n_y(x, z) = u^n(x)e^{+i\gamma_n z}$$

$$H^n_x(x, z) = \bar{v}^n_x(x)e^{+i\gamma_n z} \qquad (3.2.5)$$

$$H^n_z(x, z) = \bar{v}^n_z(x)e^{+i\gamma_n z}$$

One denotes waves propagating in the positive z-direction by the positive index n. For waves propagating in the negative z-direction, it is convenient to introduce negative subscripts. The fields and the

wave number (index n) and the opposite waves and wave numbers are connected by the relationship (Reference 16):

$$\begin{aligned} u^{-n} &= -u^{n} \\ \bar{v}_x^{-n} &= \bar{v}_x^{n} \\ \bar{v}_z^{-n} &= -\bar{v}_z^{n} \end{aligned} \qquad (3.2.6)$$

$$\gamma_{-n} = -\gamma_n$$

By substituting the fields given by equation (3.2.5) (these exist in a media in which ϵ is independent of z) into Maxwell's equations (3.2.1), one obtains

$$+i\gamma_n u^n(x) e^{+i\gamma_n z} = -i\omega\mu \bar{v}_x^n(x) e^{+i\gamma_n z} \qquad (3.2.7)$$

$$\frac{\partial u^n(x)}{\partial x} e^{+i\gamma_n z} = +i\omega\mu \bar{v}_z^n(x) e^{+i\gamma_n z}$$

$$\frac{\partial \bar{v}_z(x)}{\partial x} e^{+i\gamma_n} - i\gamma_n \bar{v}_x^n e^{+i\gamma_n z} = +i\epsilon(x) u^n(x) e^{i\gamma_n z}$$

or

$$i\gamma_n u^n(x) = -i\omega\mu \bar{v}_x^n(x) \qquad (3.2.8)$$

$$\frac{\partial u^n(x)}{\partial x} = +i\omega\mu\epsilon(x) \bar{v}_z^n(x) \qquad (3.2.9)$$

$$i\gamma_n \bar{v}_x^n - \frac{\partial \bar{v}_z^n(x)}{\partial x} = -i\omega\epsilon u^n(x) \qquad (3.2.10)$$

The cross-section eigenfunctions provide one with the apparatus necessary for the analysis of the field in an inhomogeneous medium, where the dielectric constant is a function of both x and z coordinates. Thus the periodic slab will serve as the canonical problem in the development of the method of solution to the more complex problem.

Let one consider the case of a plane wave incident upon a dielectric slab at angle θ_o, with the electric field in the y-direction (Figure 3). The dielectric constant is assumed to be periodic in the x-direction and arbitrary in the z-direction. The fields will be represented in any cross-section $z = z'$ in the form of a series in terms of the cross-section eigenfunctions for an auxiliary medium for which $\epsilon(x)$ coincides with $\epsilon(x,z)$ in a given cross-section. The dependence of the fields of natural waves on z is expressed by the factor $\exp(i\gamma_n z)$. For $\epsilon(x,z)$ real, a finite number of propagation constants, γ_n, are real and the remainder imaginary. The sign of γ_n is chosen so that for $n > 0$ the waves are propagating or evanescent waves, in the positive z-direction (direct waves) and for $n < 0$ in the negative z-direction (return waves). The cross-section eigenfunctions as well as the propagation constants are functions of z'. Thus for an inhomogeneous medium where ϵ is a function of x and z, the field can be expressed in the form

$$E_y(x,z') = \sum_{n=1}^{\infty} Q_n(z')u^n(x,z')$$

$$H_x(x,z') = \sum_{n=1}^{\infty} R_n(z')\bar{v}_x^n(x,z') \qquad (3.2.11)$$

Let one introduce new modal coefficients, $P_n(z')$ and $P_{-n}(z')$ which are connected with the coefficients $Q_n(z')$ and $R_n(z')$ in the following manner

$$P_{-n}(z') - P_n(z') = Q_n(z')$$

and

$$P_{-n}(z') + P_n(z') = R_n(z') \tag{3. 2. 12}$$

Thus, equation (3. 2. 11) can be written as

$$E_y(x, z') = \sum_{n=-\infty}^{\infty} P_n(z') u^n(x, z')$$

$$H_x(x, z') = \sum_{n=-\infty}^{\infty} P_n(z') \bar{v}_x^n(x, z') \tag{3. 2. 13}$$

Now by assuming that one is interested in a very small region at $z = z'$, then if equations (3. 2. 8) and (3. 2. 10) are multiplied by $P_n(z')$ and summed, one obtains

$$+i \sum_n P_n(z')\gamma_n(z')u^n(x, z') = -i\omega\mu \sum_n P_n(z') \bar{v}_x^n(x, z') \tag{3. 2. 14}$$

$$i \sum_n \gamma_n(z')P_n(z') \bar{v}_x^n(x, z') - \sum_n \frac{\partial \bar{v}_z^n(x, z')}{\partial x} P_n(z')$$

$$= -i\omega\epsilon \sum_n P_n(z')u^n(x, z') \tag{3. 2. 15}$$

Now for the inhomogeneous medium, where ϵ is a function of x and z, one substitutes the field expression (3. 2. 13) into equations (3. 2. 1)

$$\sum_n \frac{\partial}{\partial z'} [P_n(z')u^n(x, z')] = -i\omega\mu \sum_n P_n(z') \bar{v}_x^n(x, z') \tag{3. 2. 16}$$

$$\sum_n \left\{ \frac{\partial}{\partial x} [P_n(z')\bar{v}^n_z(x,z')] - \frac{\partial}{\partial z'} [P_n(z')\bar{v}^n_x(x,z')] \right\}$$

$$= +i\omega\epsilon(x,z') \sum_n P_n(z')u^n(x,z') \qquad (3.2.17)$$

Substituting equations (3.2.14) and (3.2.15) into equations (3.2.16) and (3.2.17), respectively, one obtains

$$\sum_n [P'_n(z') - i\gamma_n(z')P_n(z')]\, u^n(x,z') = -\sum_n P_n(z')u^{n'}(x,z') \qquad (3.2.18)$$

and

$$\sum_n [P'_n(z') - i\gamma_n(z')P_n(z')]\, \bar{v}^n_x(x,z') = -\sum_n P_n(z')\bar{v}^{n'}_x(x,z') \qquad (3.2.19)$$

Here, the prime indicates the derivative with respect to z'.

To obtain the equations for $P_n(z')$ one multiplies equations (3.2.18) and (3.2.19) by $\bar{v}^m_x$ and u^m, respectively, add them and integrate over the cross-section, the result is

$$\int_{-a/2}^{a/2} \left[\sum_n \left(u^n \bar{v}^m_x + \bar{v}^n_x u^m \right) \left(P'_n(z') - i\gamma_n(z')P_n(z') \right) \right] dx$$

$$= -\int_{-a/2}^{a/2} \sum_n \left(u^{n'}\bar{v}^m_x + \bar{v}^{n'}_x u^m \right) P_n(z')dx \qquad (3.2.20)$$

The fields u^n and $\bar{v}^n_x$ are solutions of equation (3.2.3), and satisfy the orthogonality conditions described in Appendix I. Thus,

$$\int_{-a/2}^{a/2} (u^n \bar{v}_x^m + u^m \bar{v}_x^n)dx = 2N^m \delta_{mn}$$

$$= 0 \text{ for } m \neq n \qquad (3.2.21)$$

$$= 2N^m \text{ for } m = n$$

where

$$N^m = \int_{-a/2}^{a/2} (u^m \bar{v}_x^m)dx \qquad (3.2.22)$$

By substituting equation (3.2.22), expression (3.2.20) reduces to a system of equations

$$P_m'(z') - i\gamma_m(z')P_m(z') = \sum_n S_{mn}(z')P_n(z') \qquad (3.2.23)$$

where

$$S_{mn} = -\frac{1}{2N^m} \int_{-a/2}^{a/2} (u^{n'} \bar{v}_x^m + u^m \bar{v}_x^{n'})dx \qquad (3.2.24)$$

The coefficients S_{mn} characterize the coupling between waves in the inhomogeneous media. Thus the problem reduces to solving an infinite set of ordinary differential equations represented by equation (3.2.23).

From a mathematical point of view, the determination of the existence and uniqueness of a solution to such a system of equations is extremely difficult. It is general engineering practice to truncate such a system of equations and find the solutions. The numerical solution to such a system of

equations is also carried out in this spirit. This procedure can be used in a numerical solution if the coupling coefficients and boundary condition are known. For convenience of numerical calculations, the coupling coefficients are transformed and represented by a simple integration. The derivation of this form is given below.

3. 3. COUPLING COEFFICIENTS

In certain concrete problems the coupling coefficients are not very suitable for computations, thus one would like to find other forms for the $S_{mn}(z')$ expression. For a medium in which ϵ is independent of $z(\frac{\partial \epsilon}{\partial z} = 0)$, the coupling coefficients must be zero. Thus, one should be able to find an expression for $S_{mn}(z')$ in which derivatives of ϵ with respect to z' appear explicitly. To accomplish this, the Lorentz reciprocity theorem (Reference 16) will be used.

Let one write Maxwell's equations for the fields

$$E_y^n(x, z, z') = \hat{y} u^n(x, z') e^{+i\gamma_n(z')z}$$
$$H^n(x, z, z') = \bar{v}^n(x, z') e^{i\gamma_n(z')z} \qquad (3.3.1)$$

in the x, z coordinate system, and differentiate with respect to z', one obtains

$$\nabla \times E_y^{n'} = +i\omega\mu H^{n'}$$
$$\nabla \times H^{n'} = -i\omega\epsilon E_y^{n'} - i\omega\epsilon' E_y^n \qquad (3.3.2)$$

where

$$E_y^{n'} = \hat{y}[u^{n'}(x,z') + iz\gamma_n(z')u^n(x,z')]e^{+i\gamma_n(z')z}$$
$$H^{n'} = [\bar{v}^n(x,z') + iz\gamma_n(z')\bar{v}^n(x,z')]e^{+i\gamma_n(z')z} \qquad (3.3.3)$$

Multiplying equation (3.3.2) by $+H^{-j}$ and $-E_y^{-j}$, one obtains

$$+H^{-j}\nabla x E_y^{n'} = +i\omega\mu H^{n'}H^{-j} \qquad (3.3.4)$$

$$-E_y^{-j}\nabla x H^{n'} = +i\omega\epsilon E_y^{n'}E_y^{-j} + i\omega\epsilon' E_y^n E_y^{-j} \qquad (3.3.5)$$

Now let one write Maxwell's equations for the index (-j) and multiply them by $H^{n'}$ and $E_y^{n'}$, the equations are

$$H^{n'}\nabla x E^{-j} = +i\omega\mu H^{-j}H^{n'}$$
$$E_y^{n'}\nabla x H^{-j} = -i\omega\epsilon E_y^{-j}E_y^{n'} \qquad (3.3.6)$$

By subtracting equation (3.3.5) from (3.3.4) and adding equation (3.3.6), one obtains

$$E_y^{n'}\nabla x H^{-j} - H^{-j}\nabla x E_y^{-n'} - E_y^{-j}\nabla x H^{-n'}$$
$$+ H^{n'}\nabla x E^{-j} = +i\omega\epsilon' E_y^n E_y^{-j} \qquad (3.3.7)$$

Making use of the known identity from vector analysis

$$\text{div}[AB] = B\nabla x A - A\nabla x B \qquad (3.3.8)$$

one gets the relation

$$\operatorname{div}[H^{n'}E_y^{-j} - H^{-j}E_y^{n'}] = i\omega\epsilon' E_y^n E_y^{-j} \tag{3.3.9}$$

At $x = \pm a/2$ one assumes E_y' is continuous and that

$$E_y(x + a/2, z') = E_y(x - a/2, z') \exp(-i\alpha)$$

where $\alpha = ak \sin\theta_o$. Substituting equation (3.3.3) into equation (3.3.9) and integrating over the cross-section for the case $j \neq n$, one obtains

$$\begin{aligned} \operatorname{div} \int_{-a/2}^{a/2} & \left\{ [\bar{v}^{n'}(\hat{y}u^{-j}) - \bar{v}^{-j}(\hat{y}u^{n'}) + iz\gamma_n] \right. \\ & \left. \cdot [\bar{v}^{n}(\hat{y}u^{-j}) - \bar{v}^{-j}(\hat{y}u^{n})] \right\} e^{i[\gamma_n - \gamma_j]z} dx \\ & = e^{i[\gamma_n - \gamma_j]z} i\omega \int_{-a/2}^{a/2} \epsilon' u^n u^{-j} dx \end{aligned} \tag{3.3.10}$$

The term under the integral sign, which is multiplied by $iz\gamma_n$, is zero by the orthogonality relation given by equation (3.2.21).

Upon taking the divergence of the integral, one obtains

$$\begin{aligned} & i(\gamma_n - \gamma_j) \int_{-a/2}^{a/2} (\bar{v}_x^{n'} u^{-j} - \bar{v}_x^{-j} u^{n'}) dx \\ & + \left[\bar{v}_z^{n'} u^{-j} - \bar{v}_z^{-j} u^{n'} \right]_{-a/2}^{a/2} = + i\omega \int_{-a/2}^{a/2} \epsilon' u^n u^{-j} dx \end{aligned} \tag{3.3.11}$$

Now, substituting the relations given in equations (3. 2. 6) and (3. 2. 24) into equation (3. 3. 11), one obtains

$$2i(\gamma_n - \gamma_j)S_{jn}N^j + \left[\bar{v}_z^j u^{n'} - \bar{v}_z^{n'} u^j\right]_{-a/2}^{a/2} = -i\omega \int_{-a/2}^{a/2} \epsilon' u^n u^j dx \tag{3. 3. 12}$$

Solving for S_{jn}

$$S_{jn} = \frac{\omega}{2N^j(\gamma_j - \gamma_n)} \int_{-a/2}^{a/2} \epsilon' u^n u^j dx - \frac{i}{2N^j(\gamma_j - \gamma_n)} \left[\bar{v}_z^j u^{n'} - \bar{v}_z^{n'} u^j\right]_{-a/2}^{a/2} \tag{3. 3. 13}$$

By substituting equation (3. 2. 4)

$$v_z^j = +\frac{i}{\omega\mu} \frac{\partial u^j}{\partial x} \tag{3. 3. 14}$$

into equation (3. 3. 13), one obtains the following expression for S_{jn}

$$S_{jn} = \frac{\omega}{2N^j(\gamma_j - \gamma_n)} \int_{-a/2}^{a/2} \epsilon' u^n u^j dx + \frac{1}{2N^j(\gamma_j - \gamma_n)\omega\mu} \left[\frac{\partial u^j}{\partial x} u^{n'} - \frac{\partial}{\partial x}(u^{n'})u^j\right]_{-a/2}^{a/2} \tag{3. 3. 15}$$

or

$$S_{jn} = \frac{\omega}{2N^j(\gamma_j - \gamma_n)} \int_{-a/2}^{a/2} \epsilon' u^n u^j dx$$

$$+ \frac{[1 - \exp(-i\alpha)]}{2N^j(\gamma_j - \gamma_n)\omega\mu} \left[\frac{\partial u^j}{\partial x} u^{n'} - \frac{\partial}{\partial x}(u^{n'}) u^j \right] \tag{3.3.16}$$

at $x = a/2$.

To obtain the expression for $S_{jj}(z')$, one uses equation (3.2.24) and lets $m = n = j$

$$S_{jj} = - \frac{1}{2N^j} \int_{-a/2}^{a/2} (u^{j'} v_x^j + u^j v_x^{j'}) dx \tag{3.3.17}$$

or

$$S_{jj} = - \frac{1}{2N^j} \int_{-a/2}^{a/2} \frac{d}{dz} (u^j v^j) dx$$

By substituting equation (3.2.22), one obtains

$$S_{jj} = - \frac{N^{j'}}{2N^j} . \tag{3.3.18}$$

Equations (3.3.16) and (3.3.18) are the expressions for the general coupling coefficients for arbitrary incidence. Numerically, it is found that as the difference between n and j increases, the coupling coefficient S_{jn} becomes very small. Thus, only the first few modes

are excited and the other modes are practically negligible. Hence, for engineering applications this method of truncation is justified.

3.4. BOUNDARY CONDITIONS FOR $P_m(z')$

The boundary conditions of system (3.2.23) are set by the requirements that at $z' = 0$, the field of the incident wave be equal to one of the natural waves. Specifically, if a wave number "j" with a unit amplitude is incident from below (Figure 3), the boundary conditions for $P_m(z')$ will have the form

$$P_j(0) = 1, \quad P_m(0) = 0 \quad \text{for} \quad m > 0, \quad m \neq j$$
$$P_m(d) = 0 \quad \text{for} \quad m < 0 \tag{3.4.1}$$

System (3.2.23) and (3.4.1) are complete and thus, one can find all the amplitudes $P_m(z')$. Solving this system of equations, gives the field on the plane $z' = 0$ and $z' = d$, and then also the fields in the inhomogeneous semi-infinite regions $z' > d$ and $z' < 0$. The amplitude of the reflected waves are given by $P_m(0)$ for $m < 0$. The amplitudes of the direct waves at $z' = d$ determine the waves which penetrate the region $z' > d$.

It should be noted that the boundary conditions imposed on the problem are two point conditions. From a computational point of view, the mechanics of solving differential equations by means of digital computers is such that it is preferable to solve non-linear

initial value problems as opposed to linear problems with two point conditions. Thus, for the numerical computations equation (3. 2. 23) is transformed to an initial value problem. The method of transforming the equations and the subsequent numerical work is discussed in the next chapter.

CHAPTER IV

THE NUMERICAL METHOD FOR THE DETERMINATION OF THE MODAL COEFFICIENTS FOR A TWO DIMENSIONAL INHOMOGENEOUS SLAB BY MEANS OF INVARIANT IMBEDDING

4.1. TRANSFORMATION OF A BOUNDARY VALUE PROBLEM TO AN INITIAL VALUE PROBLEM

Generally speaking, the mechanics of solving differential equations by means of digital computers is such that it is preferable to solve non-linear initial value problems as opposed to linear problems with two-point conditions. The reason for this is that the initial value problem can be resolved by means of a simple iteration procedure, which is ideally suited to digital computers, whereas the two-point boundary problem requires at some state, the solution of a large system of linear equations. Thus, one wishes to transform the boundary value problem to an initial value problem by employing the method of invariant imbedding (Reference 17). The invariant imbedding approach has another advantage since it grinds out the solution for all intermediate thickness, whereas in two-point boundary formulation the solution must be carried out for each thickness.

One can write the system of equations (3. 2. 23) in the following form:

$$\frac{dP_i}{dz} = [S_{ij}(z) + i\gamma_i(z)\delta_{ij}]P_j + S_{i,-j}(z)P_{-j} \tag{4.1.1}$$

$$-\frac{dP_{-i}}{dz} = -S_{-i,j}(z)P_j = [S_{-i,-j} + i\gamma_{-i}\delta_{-i,-j}]P_{-j} \tag{4.1.2}$$

The boundary conditions are given by

$$P_m(0) = 1, \quad P_i(0) = 0 \quad i \neq m$$

$$P_{-i}(d) = 0.$$

In order to simplify the notation, let one define the variables

$$\begin{aligned} P_i &= U \\ P_{-i} &= V \end{aligned} \tag{4.1.3}$$

where U and V are column matrices.

The boundary conditions are

$$\begin{aligned} U(0) &= I_m \\ V(d) &= 0 \end{aligned} \tag{4.1.4}$$

where I_m is a column matrix whose m^{th} element is 1 and all other elements are zero.

Equations (4.1.2) and (4.1.3) can be written as

$$\frac{dU}{dz} = AU + BV \tag{4.1.5}$$

$$-\frac{dV}{dz} = CU + DV \tag{4.1.6}$$

where A, B, C, and D are matrices and are given by the following

expressions:

$$\begin{aligned} A &= S_{ij} + i\gamma_i\delta_{ij} \\ B &= S_{i-j} \\ C &= -S_{-ij} \\ D &= -S_{ij} + i\gamma_i\delta_{ij} \end{aligned} \tag{4.1.7}$$

Since one is not really interested in the interior of $0 \leq z \leq d$ but their values at the ends, namely, $V(0)$ and $U(d)$, one can employ the method of invariant imbedding. One begins by considering the family of problems generated by the parameters x_1 and x_2 in the range of $x_1 \leq z \leq x_2$ and boundary condition given by

$$\begin{aligned} U(x_1) &= U_1 \\ V(x_2) &= V_2 \end{aligned} \tag{4.1.8}$$

Then by linearity the "scattering matrix" description of the slab $x_1 \leq z \leq x_2$ follows,

$$\begin{bmatrix} V(x_1) = V_1 \\ U(x_2) = U_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ V_2 \end{bmatrix} \tag{4.1.9}$$

By employing the principle of invariant imbedding, we can transform this boundary value problem to an initial value problem. By this transformation a set of first order non-linear differential equations with appropriate initial values is obtained.

In Appendix III the differential equations for M_{11} and M_{21} have been obtained and are given by

$$\frac{-dM_{11}(y_1, x_2)}{dy_1} = C(y_1) + D(y_1)M_{11}(y_1, x_2) + M_{11}(y_1, x_2)A(y_1) + M_{11}(y_1, x_2)B(y_1)M_{11}(y_1, x_2) \tag{4.1.10}$$

and

$$\frac{-dM_{21}(y_1, x_2)}{dy_1} = M_{21}(y_1, x_2)A(y_1) + M_{21}(y_1, x_2)B(y_1)M_{21}(y_1, x_2) \tag{4.1.11}$$

where the initial conditions are given respectively as

$$M_{11}(y_1, x_2) = 0 \qquad y_1 \to x_2 \tag{4.1.12}$$

and

$$M_{21}(y_1, x_2) = I \qquad y_1 \to x_2 \tag{4.1.13}$$

Thus, one has obtained the so-called "invariant imbedding" equations corresponding to a given linear system of differential equations with two point boundary conditions.

4.2. MODAL COEFFICIENTS

Let one consider a plane wave of unit amplitude incident normally upon a dielectric slab. For the model considered above, one would like to calculate the amplitude and phase of the modal coefficients. These coefficients can be obtained from the relation

$$\begin{bmatrix} V_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ O \end{bmatrix} \tag{4.2.1}$$

Thus one has to obtain the scattering coefficients. The equations for the scattering coefficients are given by equations (4. 1. 10) and (4. 1. 11).

By defining

$$R_{ij}(y_1, x_2) = M_{11}(y_1, x_2) \tag{4. 2. 3}$$

and

$$T_{ij}(y_1, x_2) = M_{21}(y_1, x_2) \tag{4. 2. 4}$$

and substituting equations (4. 2. 3) and (4. 2. 4) into equations (4. 1. 10) and (4. 1. 11), respectively, one obtains:

$$\begin{aligned} -\frac{dR_{ij}(y_1, x_2)}{dy_1} &= C_{ij}(y_1) + D_{ik}(y_1)R_{kj}(y_1, x_2) \\ &\quad + R_{ik}(y_1, x_2)A_{kj}(y_1, x_2) \\ &\quad + R_{ik}(y_1, x_2)B_{k\ell}(y_1)R_{\ell j}(y_1, x_2) \end{aligned} \tag{4. 2. 5}$$

with initial conditions given by

$$R_{ij}(y_1, x_2)\Big|_{y_1 = x_2} = 0 \tag{4. 2. 6}$$

and

$$-\frac{dT_{ij}(y_1, x_2)}{dy_1} = T_{ik}(y_1, x_2)A_{kj} + T_{ik}(y_1, x_2)B_{k\ell}R_{\ell j} \tag{4. 2. 7}$$

where the initial conditions are given by

$$T_{ij}(y_1, x_2)\Big|_{y_1 = x_2} = I \tag{4. 2. 8}$$

where I is the identity matrix.

The parameters are defined by:

$$
\begin{aligned}
A_{kj}(y_1) &= S_{kj}(y_1) + i\gamma^{k}(y_1)\delta_{kj} \\
B_{k\ell}(y_1) &= T'_{k\ell}(y_1) \\
C_{ij}(y_1) &= -T'_{ij}(y_1) \\
D_{ik}(y_1) &= -S_{ik}(y_1) + i\gamma^{i}(y_1)\delta_{ik}
\end{aligned}
\qquad (4.2.9)
$$

The coupling coefficients S and T' are presented in Appendix II.

To use a numerical method of solution for complex differential equations, one has to decompose the functions into real and imaginary parts. This doubles the number of equations, but it becomes readily handled by high speed computers. Thus, let one decompose the coefficients and the parameters in the following manner:

$$
\begin{aligned}
R_{ij} &= R^{r}_{ij} + iR^{i}_{ij} \\
T_{ij} &= T^{r}_{ij} + iT^{i}_{ij} \\
A_{kj} &= A^{r}_{kj} + iA^{i}_{kj} \\
B_{k\ell} &= B^{r}_{k\ell} + iB^{i}_{k\ell} \\
C_{k\ell} &= C^{r}_{k\ell} + iC^{i}_{k\ell} \\
D_{k\ell} &= D^{r}_{k\ell} + iD^{i}_{k\ell}
\end{aligned}
\qquad (4.2.10)
$$

where all the functions are real quantities.

Rewriting equations (4. 2. 5) and (4. 2. 7), one obtains:

$$
\begin{aligned}
-\frac{dR^r_{ij}}{dy_1} &= C^r_{ij} + D^r_{ik}R^r_{kj} - D^i_{ik}R^i_{kj} \\
&\quad + R^r_{ik}A^r_{kj} - R^i_{ik}A^i_{k\ell} + [R^r_{ik}B^r_{k\ell} - R^i_{ik}B^i_{k\ell}]R^r_{\ell j} \\
&\quad - [R^i_{ik}B^r_{k\ell} + R^r_{ik}B^i_{k\ell}]R^i_{\ell j}
\end{aligned}
\tag{4.2.11}
$$

$$
\begin{aligned}
-\frac{dR^i_{ij}}{dy_1} &= C^i_{ij} + D^i_{ik}R^r_{kj} + D^r_{ik}R^i_{kj} \\
&\quad + R^i_{ik}A^r_{k\ell} + R^r_{ik}A^i_{k\ell} + [R^i_{ik}B^r_{k\ell} + R^r_{ik}B^i_{k\ell}]R^r_{\ell j} \\
&\quad + [R^r_{ik}B^r_{k\ell} - R^i_{ik}B^i_{k\ell}]R^i_{\ell j}
\end{aligned}
\tag{4.2.12}
$$

where the initial conditions are given by

$$
\begin{aligned}
R^r_{ij}(y_1, x_2)\big|_{y_1=x_2} &= 0 \\
R^i_{ij}(y_1, x_2)\big|_{y_1=x_2} &= 0
\end{aligned}
\tag{4.2.13}
$$

and

$$
\begin{aligned}
-\frac{dT^r_{ij}}{dy_1} &= T^r_{ik}A^r_{kj} - T^i_{ik}A^i_{kj} \\
&\quad + [T^r_{ik}B^r_{k\ell} - T^i_{ik}B^i_{k\ell}]R^r_{\ell j} - [T^r_{ik}B^i_{k\ell} + T^i_{ik}B^r_{k\ell}]R^i_{\ell j}
\end{aligned}
\tag{4.2.14}
$$

$$
\begin{aligned}
-\frac{dT^i_{ij}}{dy_1} &= T^r_{ik}A^i_{kj} + T^i_{ik}A^r_{kj} \\
&\quad + [T^r_{ik}B^r_{k\ell} - T^i_{ik}B^i_{k\ell}]R^i_{\ell j} + [T^r_{ik}B^i_{k\ell} + T^i_{ik}B^r_{k\ell}]R^r_{\ell j}
\end{aligned}
\tag{4.2.15}
$$

where the initial conditions are given by

$$T^{r}_{ij}(y_1, x_2)\big|_{y_1=x_2} = I$$

$$T^{i}_{ij}(y_1, x_2)\big|_{y_1=x_2} = 0 \qquad (4.2.16)$$

To make use of the numerical method of solution, the resulting equations for R and T are truncated to include only five modes. This truncation was made on the basis that the coupling between the main mode and the high order modes greater than five is negligible. Thus, a system of one hundred simultaneous non-linear first order differential equations were solved by the Runga-Kutta numerical integration method. A step size of $\frac{\lambda}{20}$ to $\frac{\lambda}{50}$ for the numerical integration was used, which was found by trial. The results of these computations are presented in the next chapter.

CHAPTER V

DISCUSSION OF NUMERICAL RESULTS

5. 1. NUMERICAL RESULTS

The infinite system of equations derived in the previous chapter is truncated on the basis that the coupling between the main mode and the higher order modes, greater than five, is negligible. The resulting system consists of one hundred simultaneous, non-linear first order differential equations with prescribed initial conditions. One uses the Runga-Kutta numerical integration method to solve this system of equations. A few examples will be used to illustrate the dependence of the modal coefficients on the following parameters: slab width, period length, modulation constant and wavelength. The modal coefficients will be used to calculate the field distribution as well as the reflection and transmission coefficients for various combinations of the above parameters.

For the numerical calculations we shall consider a dielectric constant given by

$$\epsilon(x, z) = \epsilon_0[1 - m(z) \cos \frac{2\pi x}{a}] \qquad (5.1.1)$$

where a is the period length.

The variation in the z-direction is given by a parabolic function. Thus

$$m(z) = b \left[1 - \frac{(z-d/2)^2}{(d/2)^2}\right] \qquad (5.1.2)$$

where

b = the modulation constant

d = the thickness of the slab.

By using the transformation $x' = \frac{\pi x}{a}$, equation (5. 1. 1) becomes

$$\epsilon(x', z) = \epsilon_0[1 - m(z) \cos 2x']$$

By assuming a wave of unit amplitude e^{ikz} for the incident field and accounting for the radiation condition, one can write the field in the air region as

$$E_y(x, z) = e^{ikz} + \sum_{n=1}^{\infty} R_{n1} e^{-i\sqrt{k^2 - (\frac{2\pi(n-1)}{a})^2}\, z} e^{i \frac{2\pi(n-1)x}{a}} \qquad z < 0 \qquad (5. 1. 3)$$

$$E_y(x, z) = \sum_{n=1}^{\infty} T_{n1} e^{i\sqrt{k^2 - [\frac{2\pi}{a}(n-1)]^2}\,(z-d)} e^{i \frac{2\pi(n-1)x}{a}} \qquad z > d \qquad (5. 1. 4)$$

The terms R_{n1} and T_{n1} are the modal coefficients of the slab, and correspond to P_{-n} and P_n, respectively. R_{n1} designates the coefficient of mode n traveling in the negative z-direction due to an incident wave of mode m = 1. The same notation applies to the T_{n1} terms.

Typical examples of the dependence of the modal coefficients on the slab width "d", period length "a" in units of wavelength, and modulation constant b are presented in Tables 1 through 4.

$\frac{a}{\lambda}$	$\frac{d}{\lambda}$	$R_{11}^{r} \times 10^{-1}$	$R_{11}^{i} \times 10^{-1}$	$R_{21}^{r} \times 10^{-1}$	$R_{21}^{i} \times 10^{-1}$	$R_{31}^{r} \times 10^{-1}$	$R_{31}^{i} \times 10^{-1}$	$R_{41}^{r} \times 10^{-1}$	$R_{41}^{i} \times 10^{-1}$	$R_{51}^{r} \times 10^{-1}$	$R_{51}^{i} \times 10^{-1}$
0.75	1	0.022	-0.049	-0.065	-0.900	0.004	-0.007				
1.5	1	-1.238	-0.512	1.465	-0.741	0.041	0.237	-0.005	0.011		
2.5	1	1.010	-0.076	-0.581	-0.819	-1.109	0.865	0.135	-0.444	0.027	-0.036
3.5	1	1.020	-0.886	-0.580	0.201	0.064	-0.542	-0.826	-0.601	-0.168	0.744
5.0	1	0.074	-0.507	0.164	-0.256	0.044	0.377	0.004	0.064	0.482	0.495
0.75	2.5	0.002	-0.002	0.011	-0.406	0.001	-0.001				
1.5	2.5	0.404	0.027	-0.173	0.161	-0.049	0.137	-0.002	0.002		
2.5	2.5	0.034	-0.1645	0.316	0.093	0.267	-0.074	0.116	-0.100	0.009	-0.004
3.5	2.5	0.015	0.378	0.041	-0.496	0.202	0.402	-1.469	0.359	-0.177	0.352
5.0	2.5	-0.168	-0.433	0.031	0.150	-0.100	-0.137	-0.071	0.223	0.197	-0.006

Table 1. Modal Reflection Coefficients for b = 0.8 for Various Parameters of a and d in Units of Wavelength.

$\frac{a}{\lambda}$	$\frac{d}{\lambda}$	$T_{11}^{r} \times 10^{-1}$	$T_{11}^{i} \times 10^{-1}$	$T_{21}^{r} \times 10^{-1}$	$T_{21}^{i} \times 10^{-1}$	$T_{31}^{r} \times 10^{-1}$	$T_{31}^{i} \times 10^{-1}$	$T_{41}^{r} \times 10^{-1}$	$T_{41}^{i} \times 10^{-1}$	$T_{51}^{r} \times 10^{-1}$	$T_{51}^{i} \times 10^{-1}$
0.75	1.0	8.820	4.350	0.445	-0.775	-0.001	-0.008				
1.5	1.0	3.019	2.012	7.641	6.145	0.739	-0.143	0.016	-0.009		
2.5	1.0	-5.991	1.132	-1.801	-5.435	2.545	1.108	0.918	0.379	0.075	0.034
3.5	1.0	0.926	0.525	3.825	6.750	-4.250	1.551	0.664	-3.541	-0.294	-1.330
5.0	1.0	2.061	-0.155	2.430	7.970	-3.832	0.214	2.382	-1.158	-0.245	1.742
0.75	2.5	-4.300	-9.000	-0.360	0.186	-0.001	0.002				
1.5	2.5	2.158	-9.121	2.848	-1.859	0.089	-0.157	0.001	-0.003		
2.5	2.5	2.372	-0.023	5.135	-5.520	-6.020	0.984	0.082	0.552	0.015	0.019
3.5	2.5	2.319	-0.351	3.833	-1.645	3.250	-0.687	-0.303	-7.852	-1.085	-0.187
5.0	2.5	2.670	-0.937	4.035	-0.383	3.110	-0.612	4.010	1.510	1.020	6.790

Table 2. Modal Transmission Coefficients for b = 0.8 for Various Parameters of a and d in Units of Wavelength.

$\frac{a}{\lambda}$	$\frac{d}{\lambda}$	$R_{11}^{r} \times 10^{-1}$	$R_{11}^{i} \times 10^{-1}$	$R_{21}^{r} \times 10^{-1}$	$R_{21}^{i} \times 10^{-1}$	$R_{31}^{r} \times 10^{-1}$	$R_{31}^{i} \times 10^{-1}$	$R_{41}^{r} \times 10^{-1}$	$R_{41}^{i} \times 10^{-1}$	$R_{51}^{r} \times 10^{-1}$	$R_{51}^{i} \times 10^{-1}$
0.75	1	0.009	-0.011	-0.038	-0.465						
1.5	1	-0.392	-0.211	0.643	0.318	0.012	0.107				
2.5	1	0.065	-0.421	-0.056	-0.078	0.026	0.242	0.046	-0.103		
3.5	1	0.140	-0.387	-0.013	-0.222	-0.088	-0.312	-0.742	0.069	-0.050	0.172
5.0	1	0.018	-0.381	0.171	-0.182	-0.087	-0.122	0.106	0.000	0.320	0.075
0.75	2.5	-0.003	-0.001	0.008	-0.212						
1.5	2.5	0.027	0.042	0.010	0.165	-0.014	0.031				
2.5	2.5	-0.068	-0.072	0.100	0.035	0.062	-0.075	0.019	-0.012		
3.5	2.5	0.053	-0.134	0.046	0.011	0.116	0.004	-0.032	0.131		
5.0	2.5	0.022	-0.048	-0.065	-0.890	0.004	-0.007				

Table 3. Modal Reflection Coefficients for b = 0.4 for Various Parameters of a and d in Units of Wavelength.

$\frac{a}{\lambda}$	$\frac{d}{\lambda}$	$T_{11}^{r} \times 10^{-1}$	$T_{11}^{i} \times 10^{-1}$	$T_{21}^{r} \times 10^{-1}$	$T_{21}^{i} \times 10^{-1}$	$T_{31}^{r} \times 10^{-1}$	$T_{31}^{i} \times 10^{-1}$	$T_{41}^{r} \times 10^{-1}$	$T_{41}^{i} \times 10^{-1}$	$T_{51}^{r} \times 10^{-1}$	$T_{51}^{i} \times 10^{-1}$
0.75	1	7.619	6.473	0.329	-0.329						
1.5	1	7.641	8.029	4.562	4.390	0.258	0.013				
2.5	1	7.292	0.245	1.910	5.171	-2.730	0.636	-0.194	0.202		
3.5	1	7.140	-0.728	1.430	5.671	-2.342	-0.735	1.162	-1.245	0.049	-0.277
5.0	1	7.369	-1.054	1.129	6.150	-1.606	-0.829	1.282	0.781	-0.348	0.427
0.75	2.5	1.752	-9.839	-0.210	-0.293						
1.5	2.5	-4.560	-5.761	-6.029	3.032	0.037	0.108				
2.5	2.5	-1.892	-2.621	-4.020	-6.622	-4.550	-3.062	-0.153	1.289		
3.5	2.5	-0.749	-1.245	-2.595	-6.870	2.790	-3.965	3.587	-2.726	-0.065	-0.271
5.0	2.5	0.321	-0.800	-2.570	-6.920	4.450	-2.062	0.653	3.030	-2.015	1.200

Table 4. Modal Transmission Coefficients for b = 0.4 for Various Parameters of a and d in Units of Wavelength.

From these tables, one notes that for $b = 0.4$ and $a \leq 1.5$, which is a case of weak coupling, the amplitudes of the higher order modes are very small compared to the main mode. For such cases a perturbation solution would be sufficient. The solution would be based on the assumption that none of the higher order modes nor the reflection in the main mode have any effect on the forward propagation of the main mode. This method has been used in the propagation in non-uniform waveguides with impedance walls by R. L. Gallawa (Reference 13). However, this assumption would give very poor results when $a \geq 1.5$ and $b = 0.4$ or 0.8.

5. 2. ELECTRIC FIELD DISTRIBUTION

The field in the air region appears in the form that is common to all reflection gratings and consists of the incident wave plus an infinite number of diffracted waves, as given by equations (5. 1, 3) and (5. 1. 4). These waves are viewed as modes traveling in the z-direction, but only those with γ_n real are propagating. These γ_n are usually few in number and the remaining γ_n are imaginary and account for evanescent modes that decay in the z-direction as one moves away from the slab. Hence the field in the air region is given in terms of a finite number of propagating modes, except near the boundaries ($z = 0$ and $z = d$) where the evanescent waves may have an appreciable magnitude. In the region $z < 0$, only the incident wave and its specular reflection appear in the far field. If several modes propagate, then each of them appears as a plane wave traveling at an angle

$$\theta_n = \tan^{-1}\left(\frac{\gamma_n}{\frac{2\pi n}{a}}\right)$$

The number of propagation modes can be determined from the relation

$$\gamma_n = \left[k^2 - \left(\frac{2\pi n}{a}\right)^2\right]^{1/2}$$

for all n for which γ_n is real.

In the region $z > d$, only the modes with a real γ_n appear in the far field, and each of them appears as a plane wave traveling at angle

$$\theta_n = \tan^{-1}\left(\frac{\gamma_n}{\frac{2\pi n}{a}}\right) \qquad (5.2.1)$$

The field in the modulated dielectric region is more complicated since each mode consists of an infinite number of space harmonics, rather than a single harmonic as is seen from equations (5.1.3) and (5.1.4). The modes will be propagating or evanescent according to whether $\gamma_n(z)$ is real or imaginary.

In Table (5) the various θ_n are presented for five values of periodic length a.

The reflected field at the boundaries $z = 0$ is given by

$$E_y(x,0) = \frac{R_{11}}{\sqrt{2}} + R_{21}\cos\frac{2\pi x}{a} + R_{31}\cos\frac{4\pi x}{a} + R_{41}\cos\frac{6\pi x}{a} + R_{51}\cos\frac{8\pi x}{a} \qquad (5.2.2)$$

and the transmitted field at $z = d$ is given by

$$E_y(x,d) = \frac{T_{11}}{\sqrt{2}} + T_{21}\cos\frac{2\pi x}{a} + T_{31}\cos\frac{4\pi x}{a} + T_{41}\cos\frac{6\pi x}{a} + T_{51}\cos\frac{8\pi x}{a} \qquad (5.2.3)$$

These fields change as one moves away from the boundary since some of these modes for a particular value of "a" may be evanescent.

a	θ_1	θ_2	θ_3	θ_4	θ_5
0.75	90^o	—	—	—	—
1.5	90^o	48.2^o	—	—	—
2.5	90^o	66.4^o	36.9^o	—	—
3.5	90^o	73.5^o	55.1^o	30^o	—
5.0	90^o	78.4^o	66.4^o	53.1^o	36.9^o

Table 5. Angle of Propagation of the Various Modes for Different Values of Periodicity a.

For the case where only two modes can propagate, the transmitted field is given by

$$E_y(x,z) = \frac{T_{11}}{\sqrt{2}} e^{ik(z-d)} + T_{21} e^{i\gamma_2(z-d)} \cos \frac{2\pi x}{a} \qquad (5.2.4)$$

In Figures 4 through 7 the transmitted field distribution is given for two cases of modulation constant (b = 0.4 and 0.8) and period length (a = 1.5) for values of $z = d$ and 10d. One observes from Figures 4 and 5 that as the slab width increases, the variation of the transmitted field decreases. Also, the field distribution given in figures 6 and 7, which is 10d away from the slab, has a smoother variation than at the boundary. The reason for this is that at the boundary, the evanescent waves also add to the field distribution. In Figures 8 and 9 we present the field distribution at the boundary for two cases of modulation constant (b = 0.4 and 0.8) and a period length of (a = 2.5). By comparing these field distributions to the ones given in Figures 4 and 5, one notes that as the period length increases, more modes are able to propagate and thus, the transmitted field variation is quite abrupt at boundary of the slab.

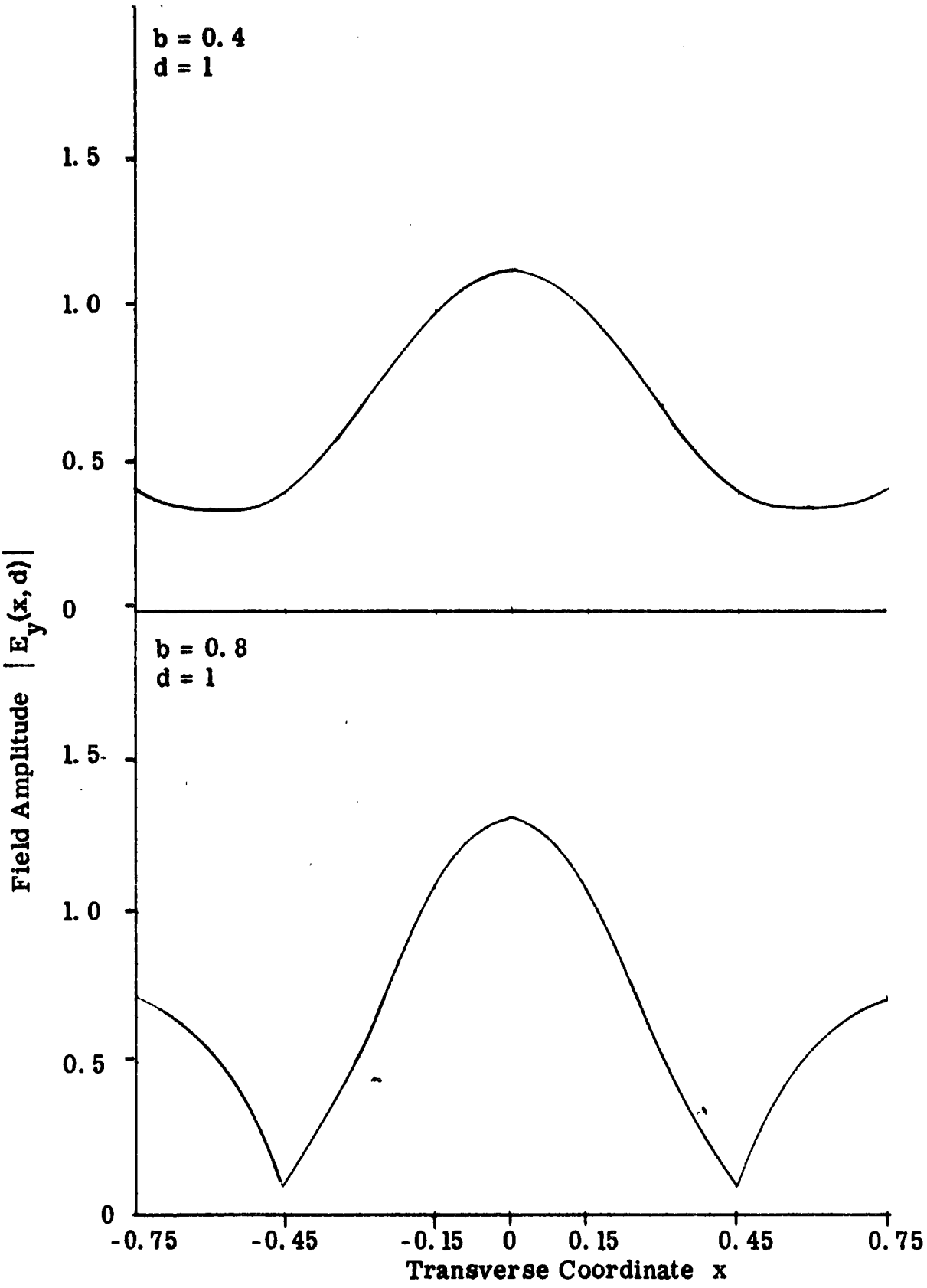


Figure 4. Variation of the Field Amplitude at the Boundary (z=d) for a Slab Width of d = 1.0, Periodic Length a = 1.5 and Modulation Constants b = 0.4 and 0.8.

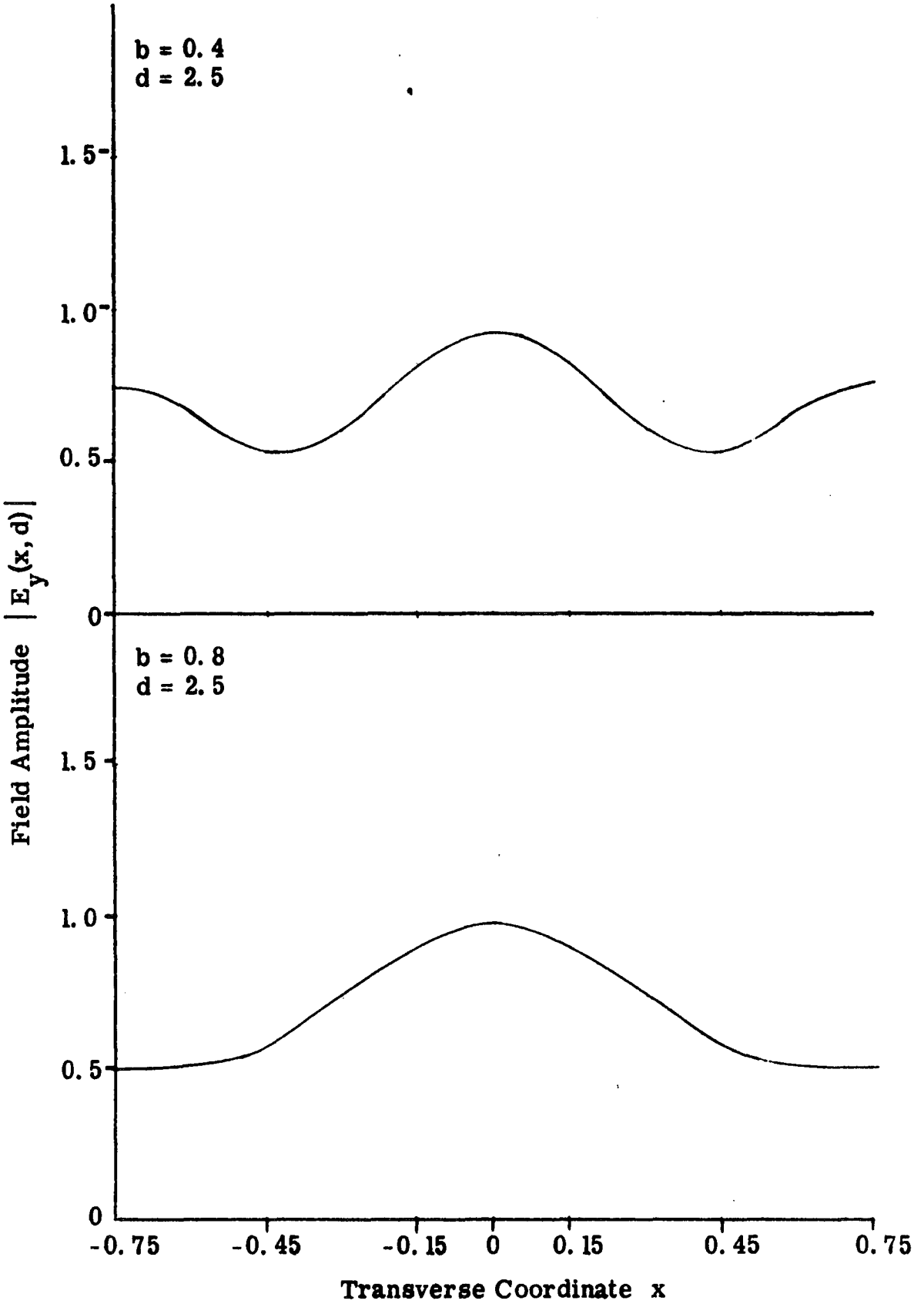


Figure 5. Variation of the Field Amplitude at the Boundary (z=d) for a Slab Width of d = 2.5, Periodic Length a = 1.5 and Modulation Constants b = 0.4 and 0.8.

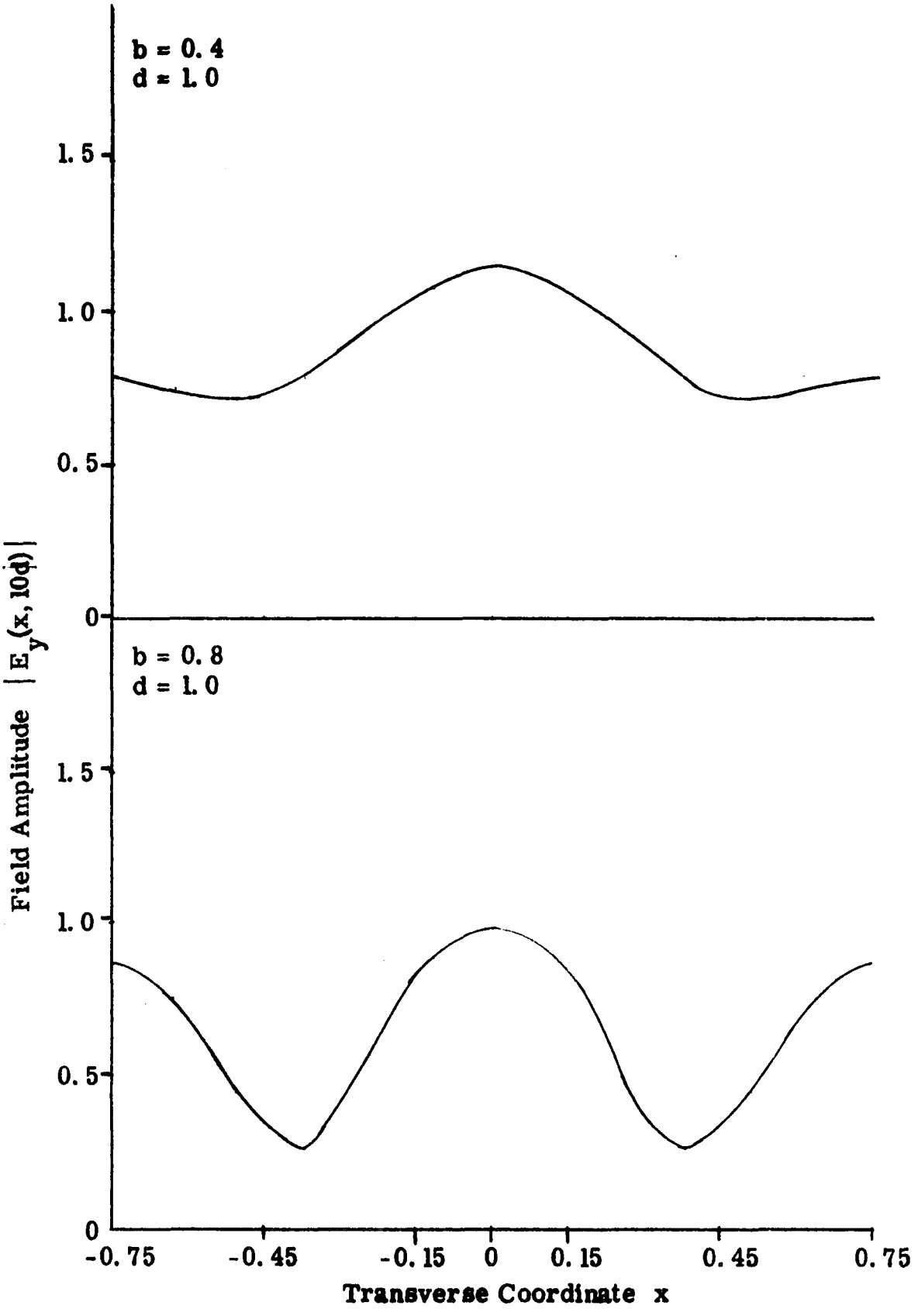


Figure 6. Variation of the Field Amplitude at a Distance 10d From the Boundary (z=d) for a Slab Width of d=1, Periodic Length a = 1.5 and Modulation Constants b = 0.4 and 0.8.

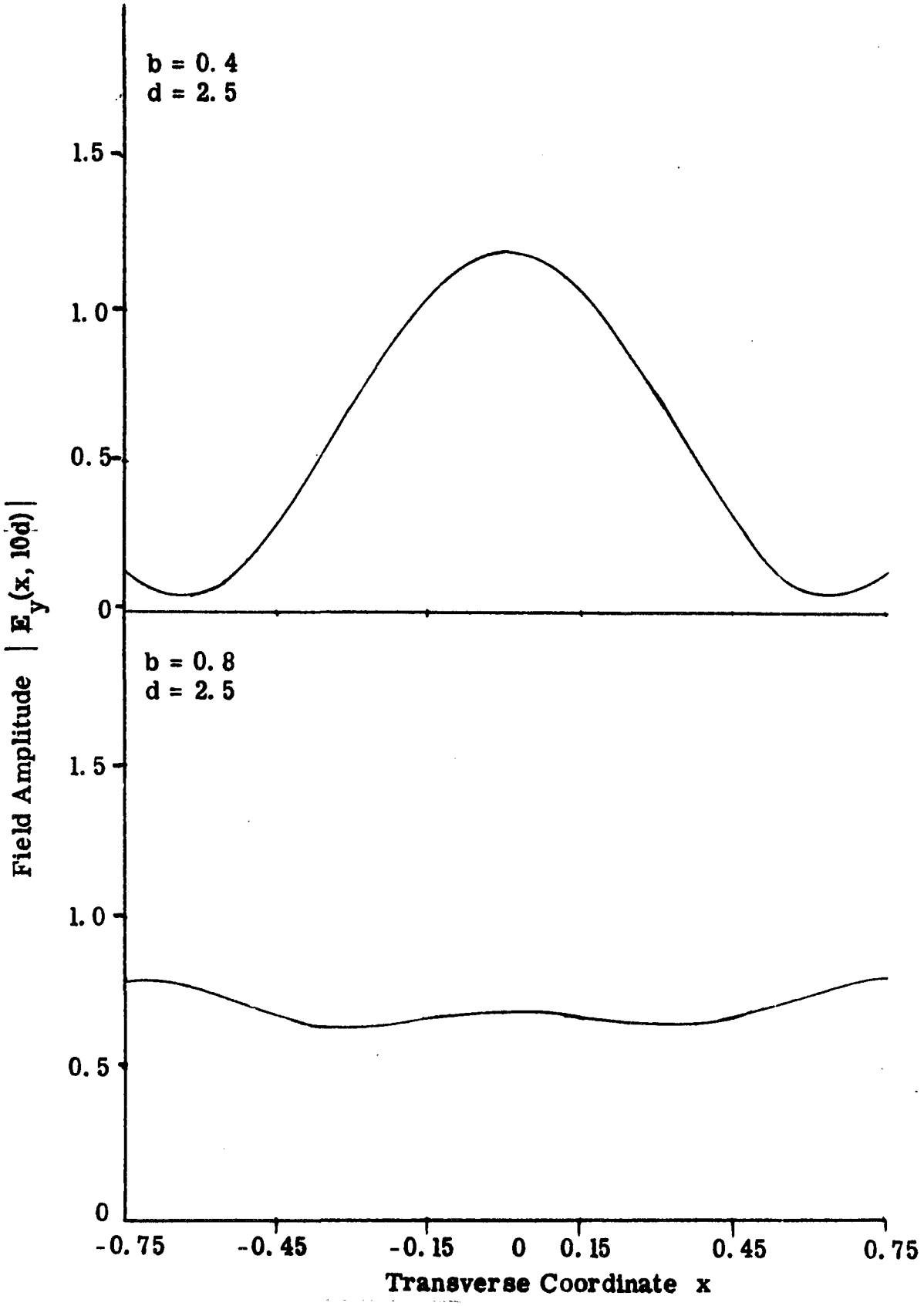


Figure 7. Variation of the Field Amplitude at a Distance 10d from the Boundary (z=d) for a Slab Width of d=2.5, Periodic Length a = 1.5 and Modulation Constants b = 0.4 and 0.8.

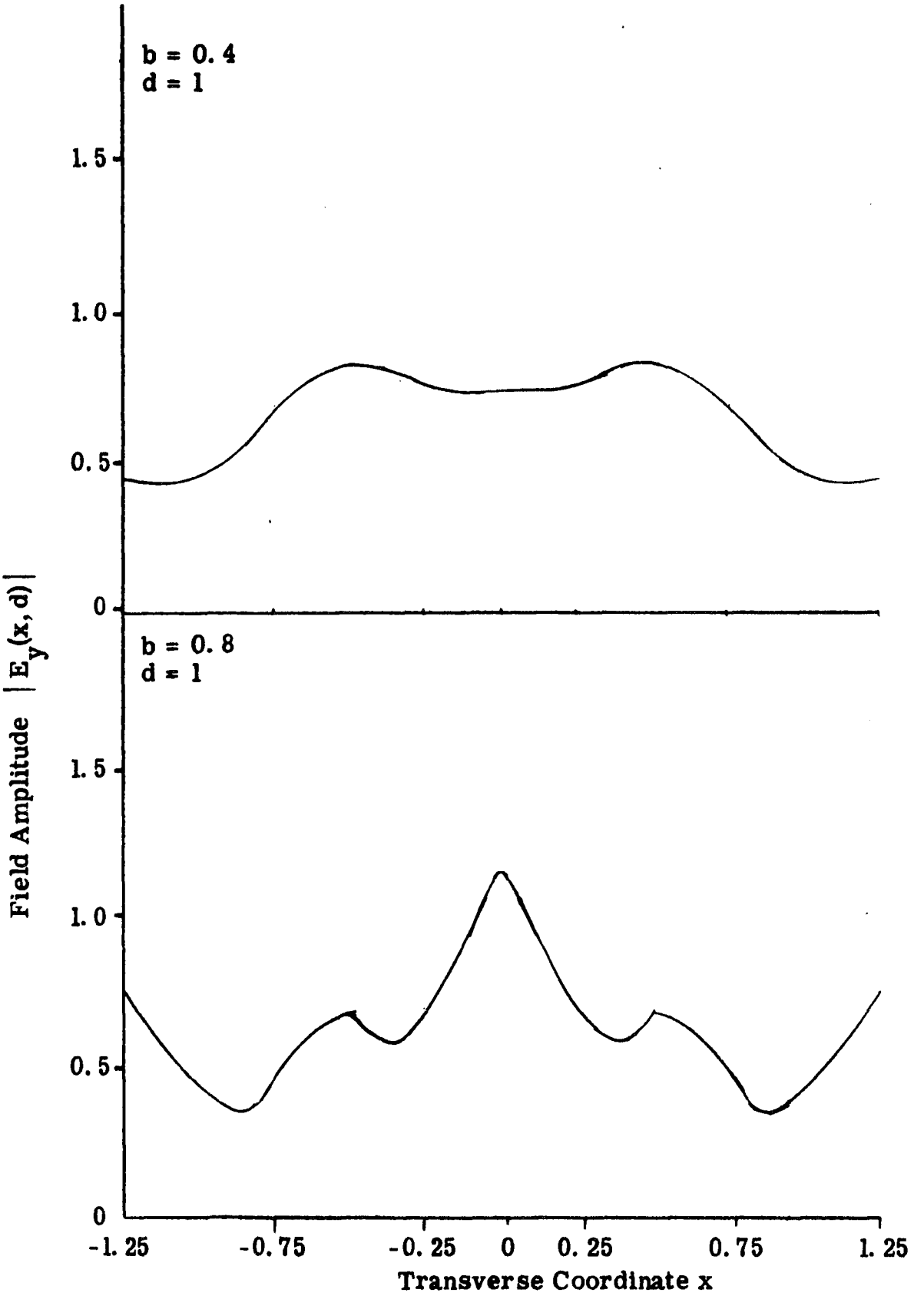


Figure 8. Variation of the Field Amplitude at the Boundary (z=d) for a Slab Width of d = 1, Periodic Length a = 2.5 and Modulation Constants b = 0.4 and 0.8.

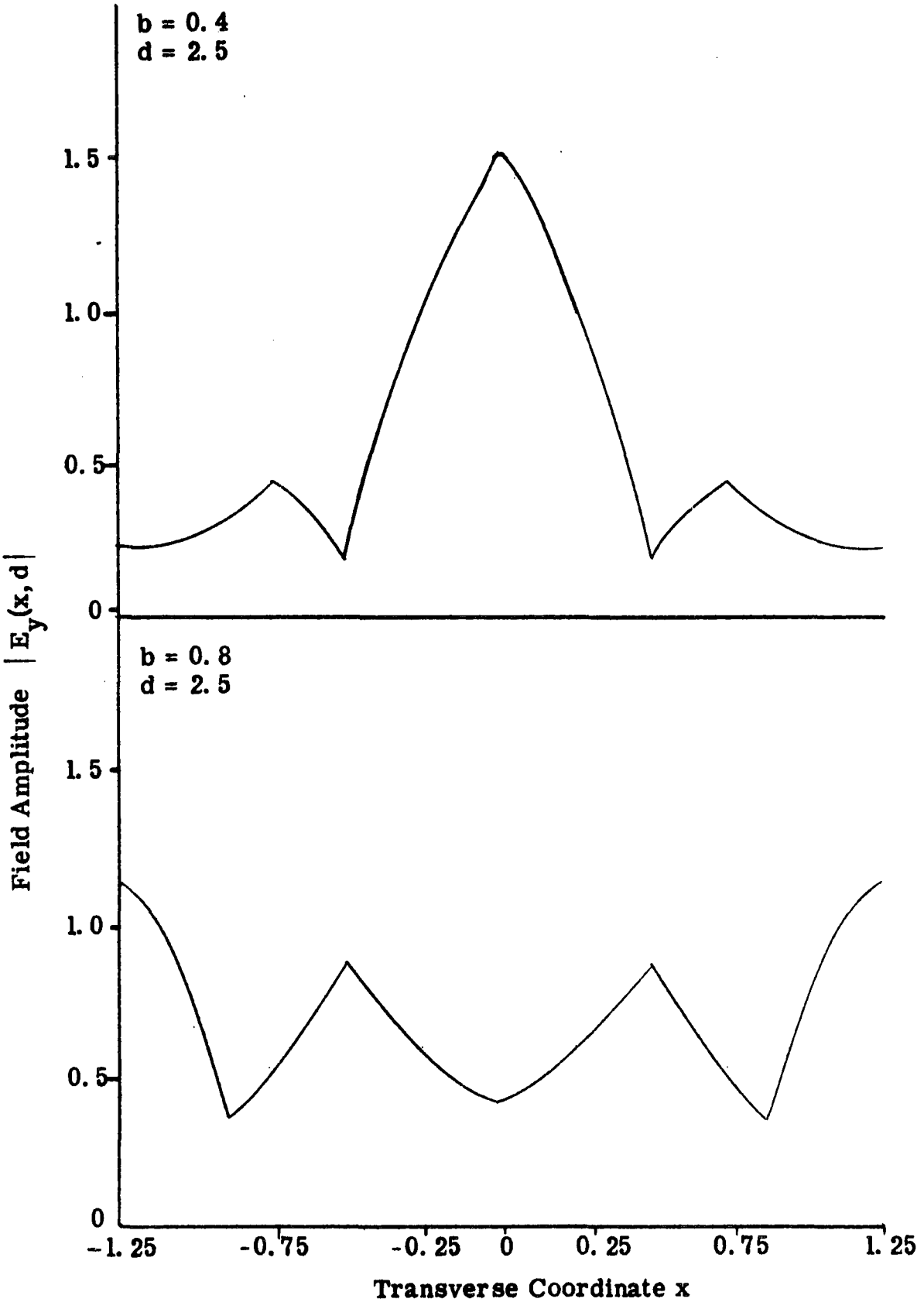


Figure 9. Variation of the Field Amplitude at the Boundary (z=d) for a Slab Width of d = 2.5, Periodic Length a = 2.5 and Modulation Constants b = 0.4 and 0.8.

5. 3. THE TRANSMISSION AND REFLECTION COEFFICIENTS

As already pointed out, the field in the air region far away from the slab consists only of those modes with γ_n real. All of the remaining γ_n are imaginary and account for the evanescent modes that decay as $|z|$ gets large. Thus it becomes necessary to talk about modal reflection and transmission coefficients. That is, to each propagating mode in the air region $z < 0$, one can call the amplitudes of the propagating modes the modal reflection coefficients. The same procedure can be applied to the region $z > d$ where one can call the amplitude of each propagating mode the modal transmission coefficient.

In Figures 10 and 11, the first two modal reflection coefficients are presented for a modulation constant $b = 0.8$ and for slab width $d = 1$ and 2. 5. The coefficients are plotted as a function of normalized period length (a). One notes from Figure 10 that the reflection coefficients follow a periodic variation and converge to a certain value for increasing a. From Figure 11, one notes that as the period increase, the reflection coefficients for the second mode approach a very small value.

In Figures 12 and 13, the first two modal transmission coefficients are presented for the same parameters as in Figures 10 and 11. As can be seen from these graphs, at the point at which the first harmonic comes into existence the transmission coefficient of the main mode decreases and the energy distributes itself into the other modes.

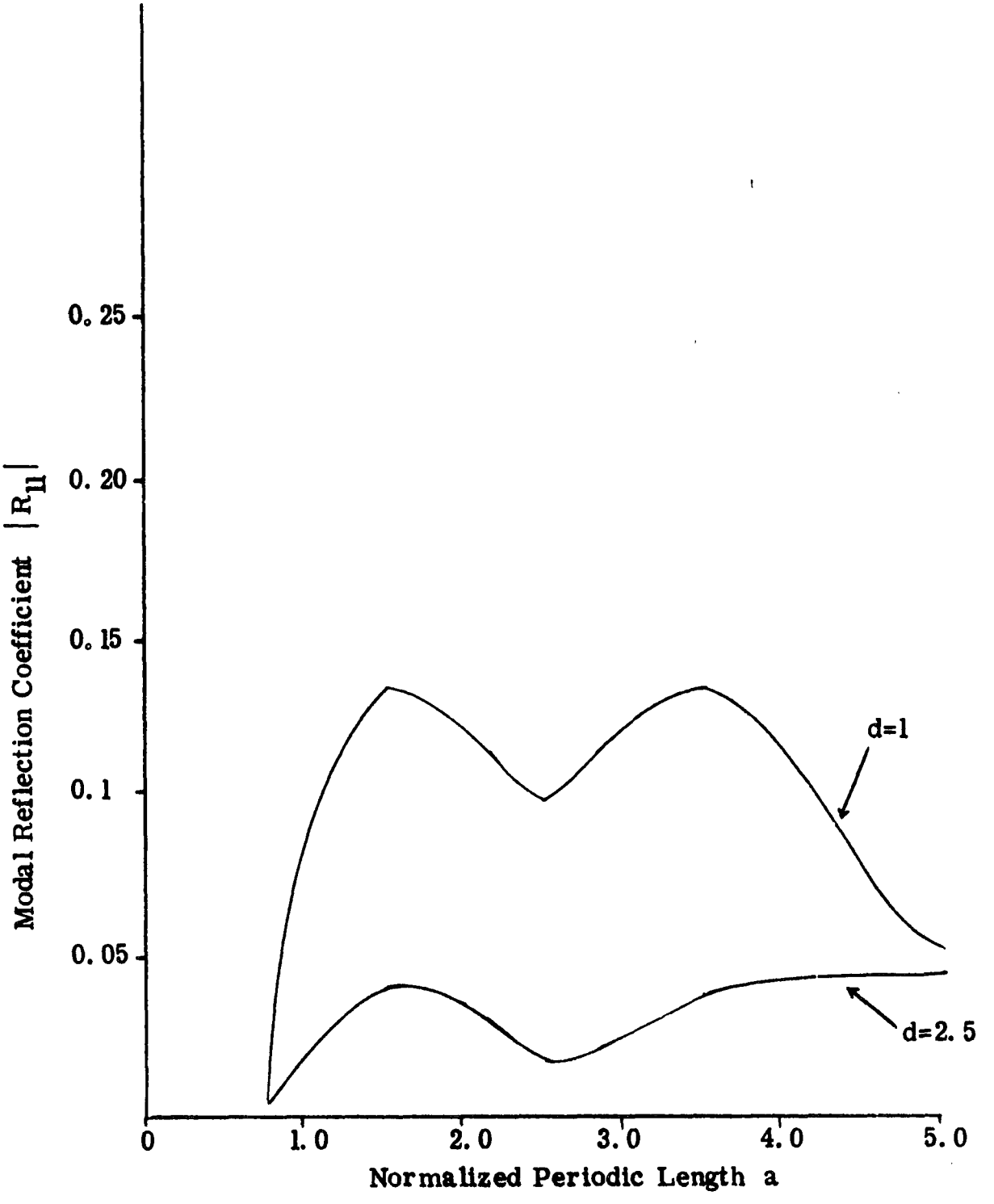


Figure 10. Modal Reflection Coefficients as a function of Periodic Length a for b = 0.8 and d = 1 and 2.5.

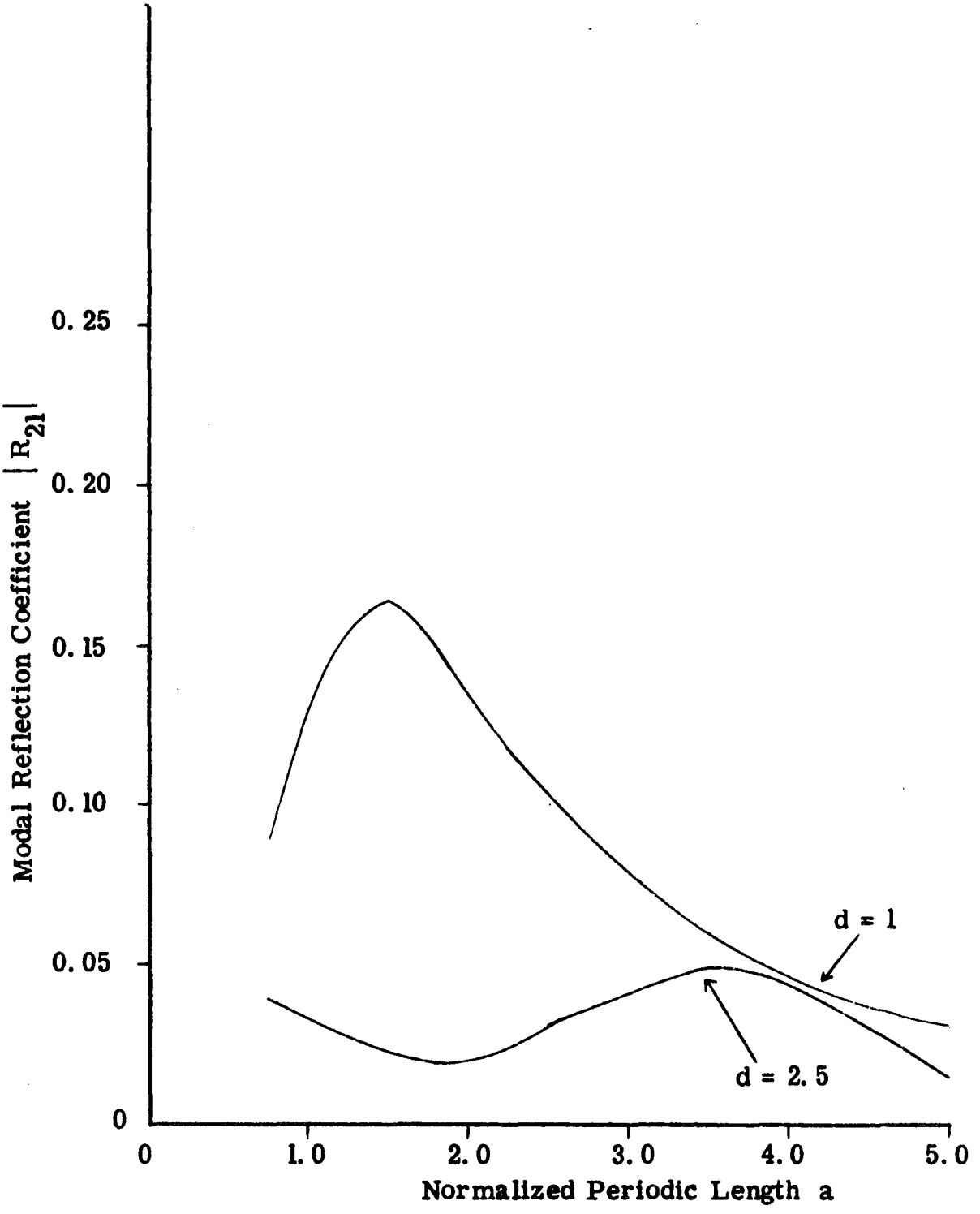


Figure 11. Modal Reflection Coefficients as a function of Periodic Length a for b = 0.8 and d = 1 and 2.5.

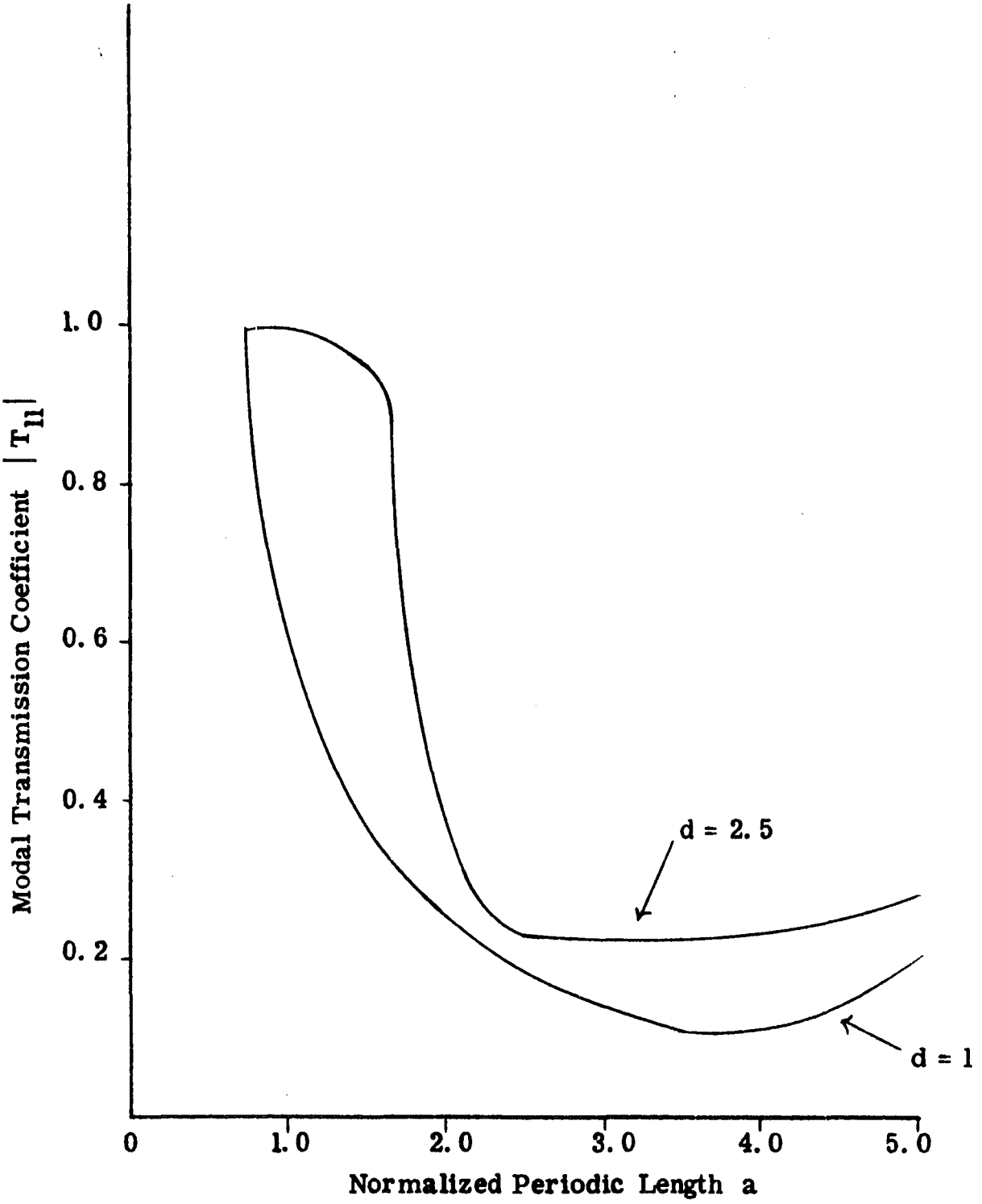


Figure 12. Modal Transmission Coefficients as a Function of Periodic Length for b = 0.8 and d = 1 and 2.5.

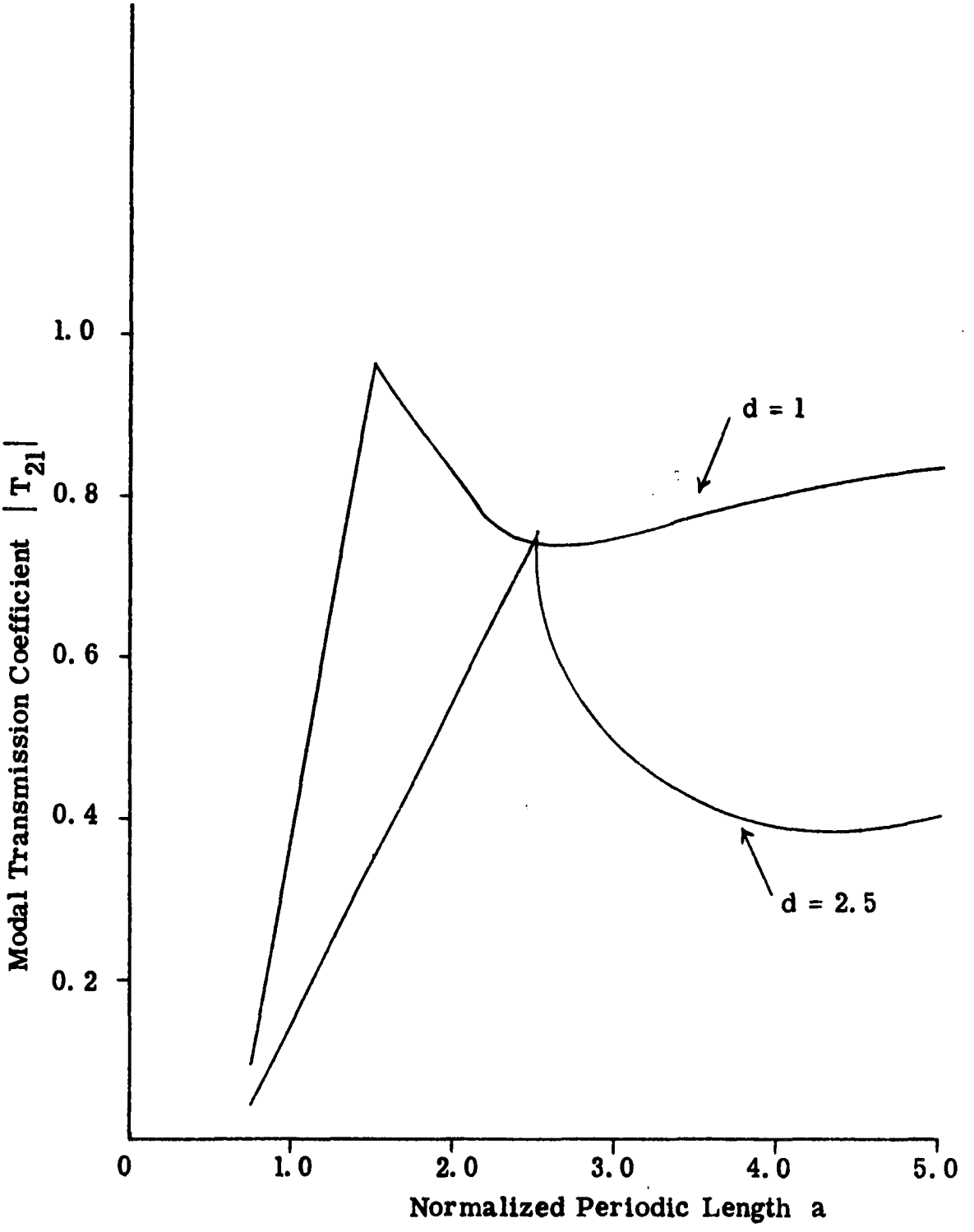


Figure 13. Modal Transmission Coefficients as a Function of Periodic Length for b = 0.8 and d = 1 and 2.5.

In Figures 14 through 17, one presents the first and second modal reflection and transmission coefficients for a modulation constant b = 0. 4, respectively. The coefficients are plotted as a function of period length for slab width d = 1 and 2. 5. These results follow quite closely to the ones presented for a modulation constant b = 0. 8, however, the amplitudes for the reflection coefficients are much lower than for b = 0. 8.

The transmitted and reflected energy for this slab (b = 0. 4), was compared to a one-dimensional slab, whose dielectric constant has the same parabolic variation in the z-direction. The calculations for the one -dimensional slab are presented in Reference (18) and they compare very closely to the results presented here for small modulation constants.

5. 4. IONOSPHERIC MODEL

One of the models proposed by Pitteway (Reference 3) for the ionosphere is of the form

$$\epsilon(x, z) = \epsilon_o[1 - m(z) \cos^2 \frac{\pi x}{a}], \qquad (5.4.1)$$

however, a solution was not attempted. This equation can be rewritten as

$$\epsilon(x, z) = \epsilon_o \left[1 - \frac{1}{2} m(z) - \frac{1}{2} m(z) \cos 2 \left(\frac{\pi x}{a}\right)\right] \qquad (5.4.2)$$

where

$$m(z) = b \left[1 - \frac{(z - \frac{d}{2})^2}{(\frac{d}{2})^2}\right] \qquad (5.4.3)$$

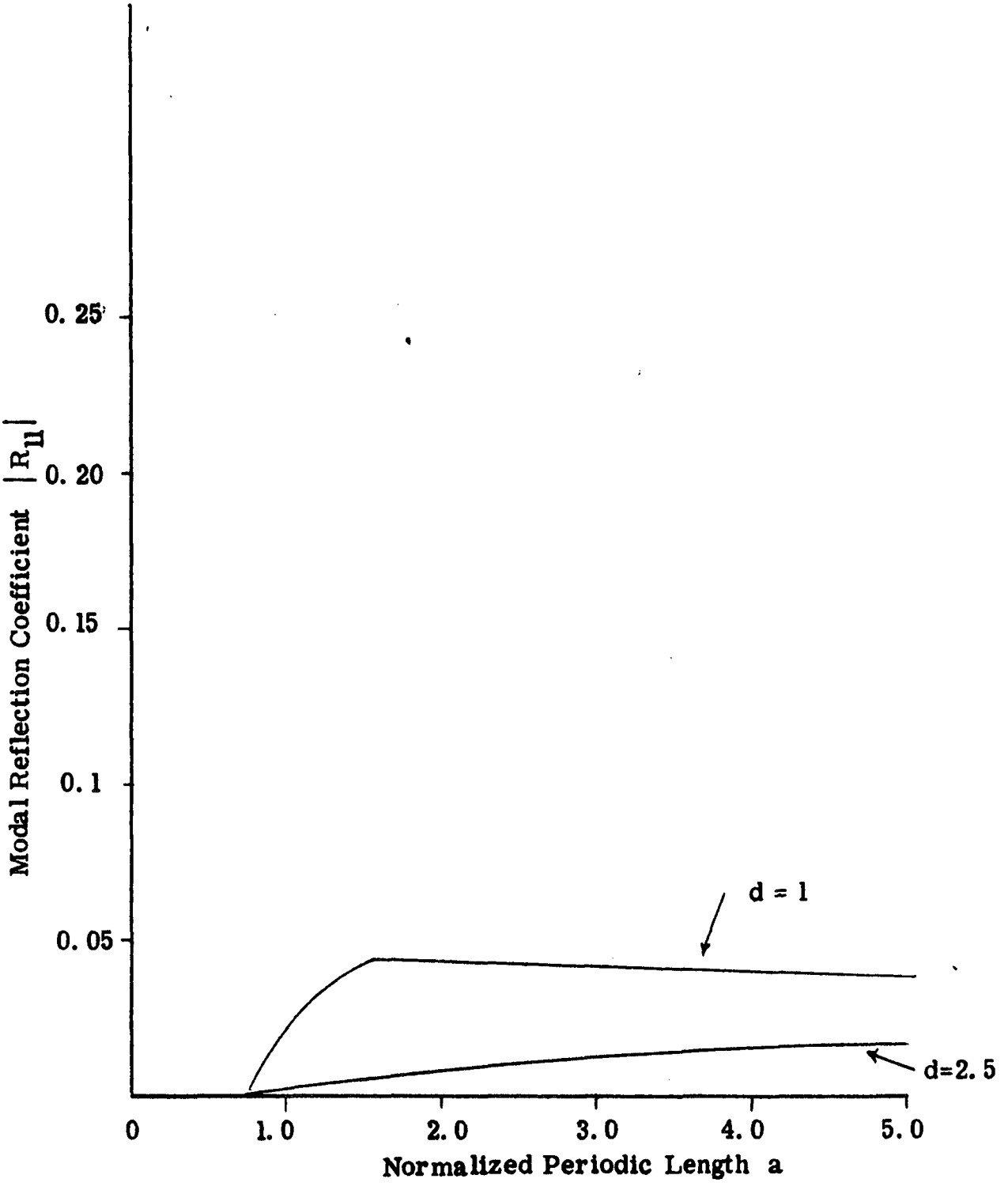


Figure 14. Modal Reflection Coefficients as a Function of Periodic Length for b = 0. 4 and d = 1 and 2. 5.

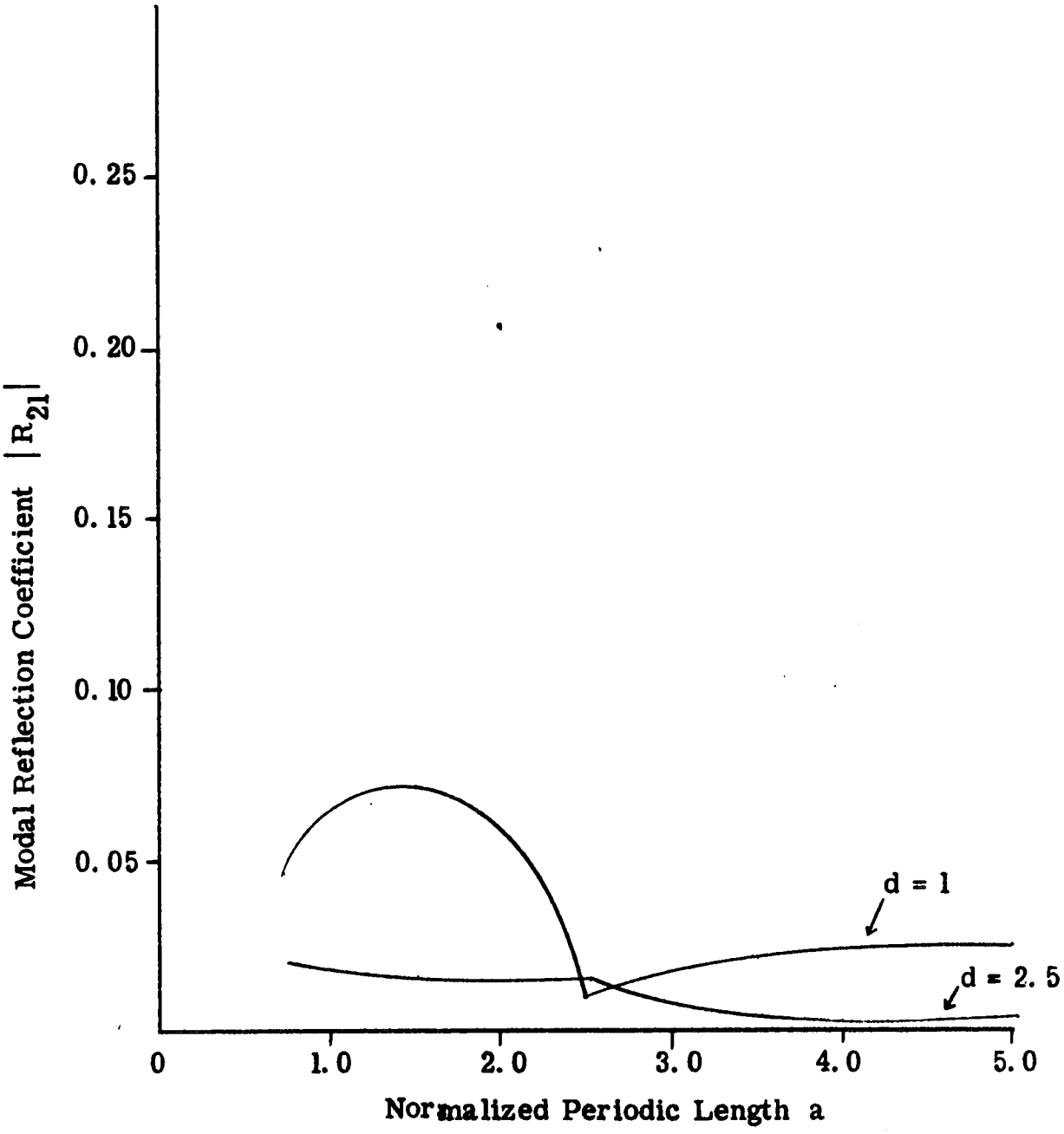


Figure 15. Modal Reflection Coefficients as a Function of Periodic Length for b = 0.4 and d = 1 and 2.5.

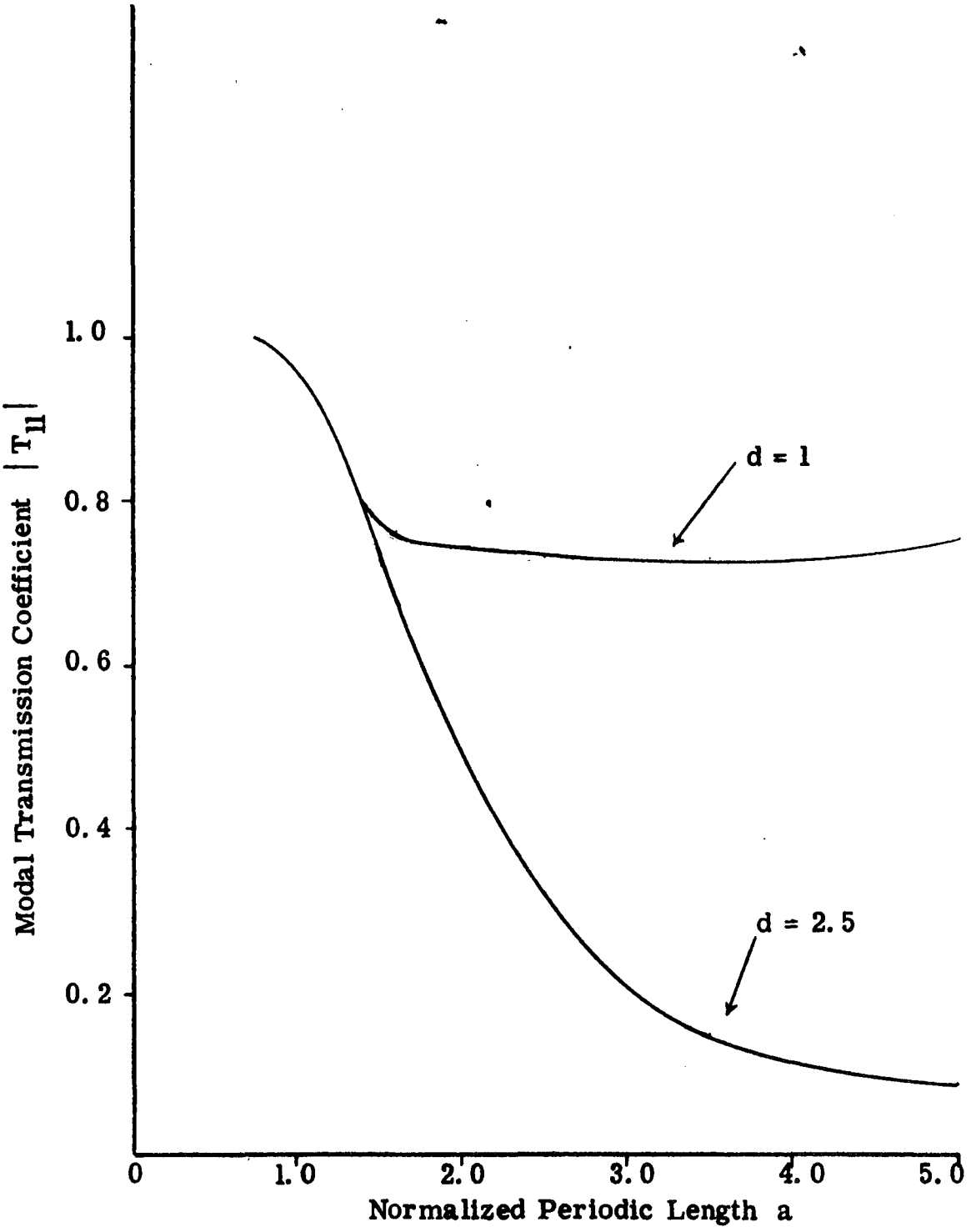


Figure 16. Modal Transmission Coefficients as a Function of Periodic Length for b = 0.4 and d = 1 and 2.5.

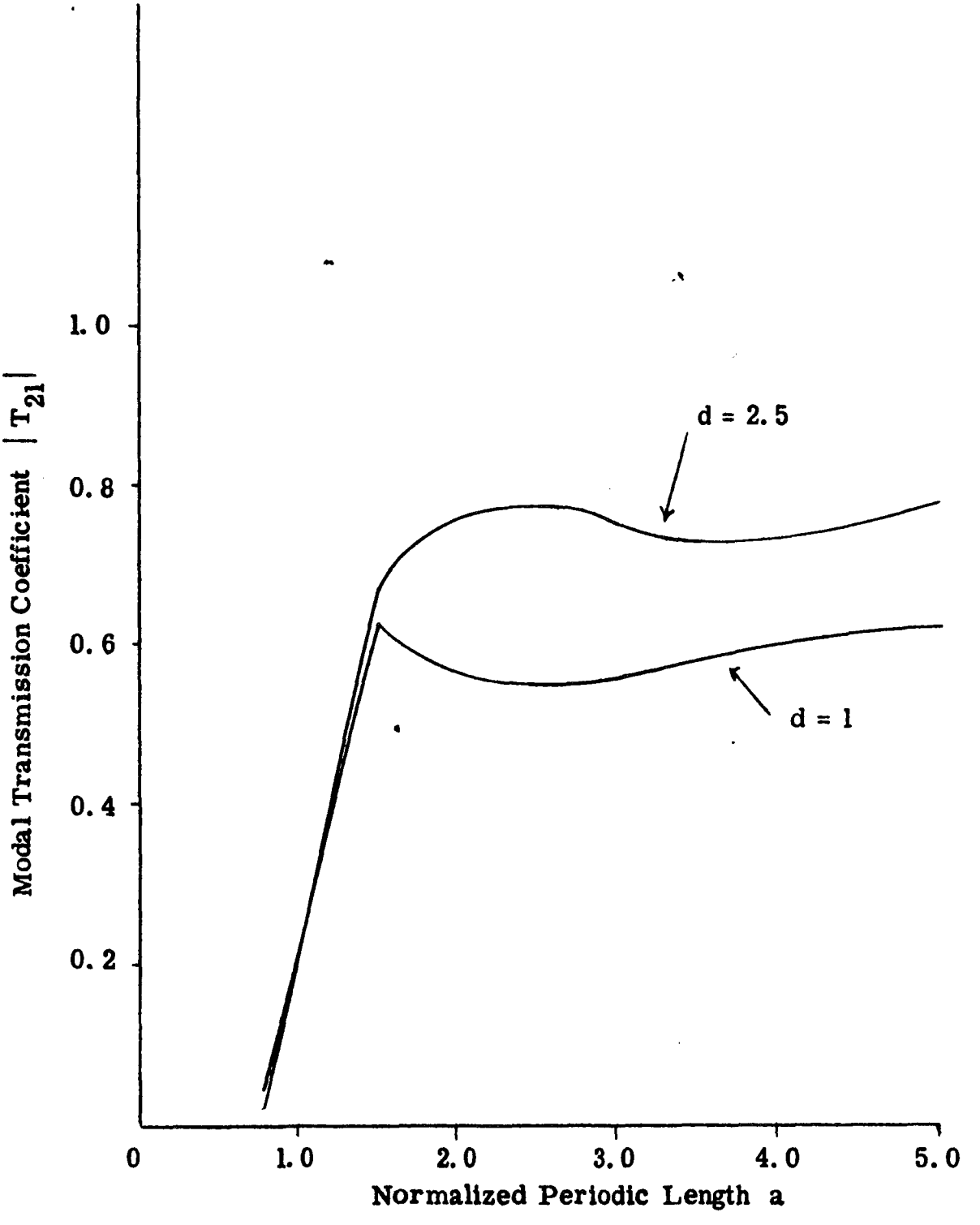


Figure 17. Modal Transmission Coefficients as a Function of Periodic Length for b = 0.4 and d = 1 and 2.5.

the constant b is equal to

$$b = \frac{e^2}{\epsilon_o m \omega^2} \dot{n}_o \qquad (5.4.4)$$

where

e = charge of the electron

m = mass of the electron

n_o = number of electrons per cm^3

If one defines

$$f_c = \left[\frac{e^2 n_o}{4\pi^2 \epsilon_o m} \right]^{1/2} \qquad (5.4.5)$$

then from equation (5.4.3)

$$b = \frac{f_c^2}{f^2} = \frac{8.061 \times 10^7 \, n_o}{f^2} \qquad (5.4.6)$$

where

$$f = \frac{c}{\lambda} \text{ cycles per second}$$

and

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$$

The term b is always positive, but one must distinguish whether it is less than or greater than unity.

If $b < 1$ because $f > f_c$, then ϵ is real positive.

If $b > 1$ because $f < f_c$, then ϵ is real negative.

Now, using the transformation

$$x' = \frac{\pi}{a} x$$

equation (5. 4. 2) becomes

$$\epsilon(x', z) = \epsilon_o \left[1 - \frac{1}{2} m(z) - \frac{1}{2} m(z) \cos 2x'\right] \qquad (5.4.7)$$

The solution to this model follows the same procedure as above.

The particular point which one wishes to illustrate is that propagation of electromagnetic waves will continue even when $b > 1$. This can be seen by observing the relation for the modal propagation constants

$$\gamma_n(z) = \left\{k^2\left[1 - \frac{1}{2} m(z)\right] - \left(\frac{\pi}{a}\right)^2 \lambda_n\right\}^{1/2} \qquad (5.4.8)$$

that for a certain combination of parameters, γ_n can be real. From Figure 22, Appendix I, one observes that λ_o is always negative, thus for a particular choice of period length, $\gamma_o(z)$ will be real and transmission will take place.

To illustrate this phenomena, one shall choose the modulation constant $b = 3.36$, $d = 2$, and for a number of values of a. The modal coefficients are presented in Table 6. In Figures 18 and 19 the transmitted field distribution at the boundary and at a distance of 10d from the slab, for two values of period length $a = 1.5$ and 2. 5 are presented. First one notes that energy is being transmitted, and for the two cases of field distribution shown, the variation of the field does not change appreciably as one moves away from the boundary.

$\frac{a}{\lambda}$	$\frac{d}{\lambda}$	R_{11}^{r} x 10^{-1}	R_{11}^{i} x 10^{-1}	R_{21}^{r} x 10^{-1}	R_{21}^{i} x 10^{-1}	R_{31}^{r} x 10^{-1}	R_{31}^{i} x 10^{-1}	R_{41}^{r} x 10^{-1}	R_{41}^{i} x 10^{-1}	R_{51}^{r} x 10^{-1}	R_{51}^{i} x 10^{-1}
0.75	2	5.794	-4.261	-0.505	-1.551	-0.002	-0.011				
1.5	2	-1.215	-1.140	2.023	0.080	0.381	0.200				
2.5	2	-1.947	-1.601	2.811	-0.345	0.562	0.592	-0.227	-0.088	-0.021	-0.010
5.0	2	-0.647	-0.576	-0.734	-0.329	0.199	0.078	1.223	-1.064	-0.702	-0.112

$\frac{a}{\lambda}$	$\frac{d}{\lambda}$	T_{11}^{r} x 10^{-1}	T_{11}^{i} x 10^{-1}	T_{21}^{r} x 10^{-1}	T_{21}^{i} x 10^{-1}	T_{31}^{r} x 10^{-1}	T_{31}^{i} x 10^{-1}	T_{41}^{r} x 10^{-1}	T_{41}^{i} x 10^{-1}	T_{51}^{r} x 10^{-1}	T_{51}^{i} x 10^{-1}
0.75	2	1.239	-4.089	-0.335	-0.228						
1.5	2	-6.743	0.127	-4.302	2.471	-0.029	0.151				
2.5	2	-0.346	-4.018	-2.344	-5.025	-1.319	-0.613	-0.065	0.041		
5.0	2	3.479	-2.580	3.722	-3.382	-2.239	-3.174	-2.251	-1.178	-0.673	-0.010

Table 6. Modal Reflection and Transmission Coefficients for b = 3.36 for Various Parameters of a and d = 2 in Units of Wavelength.

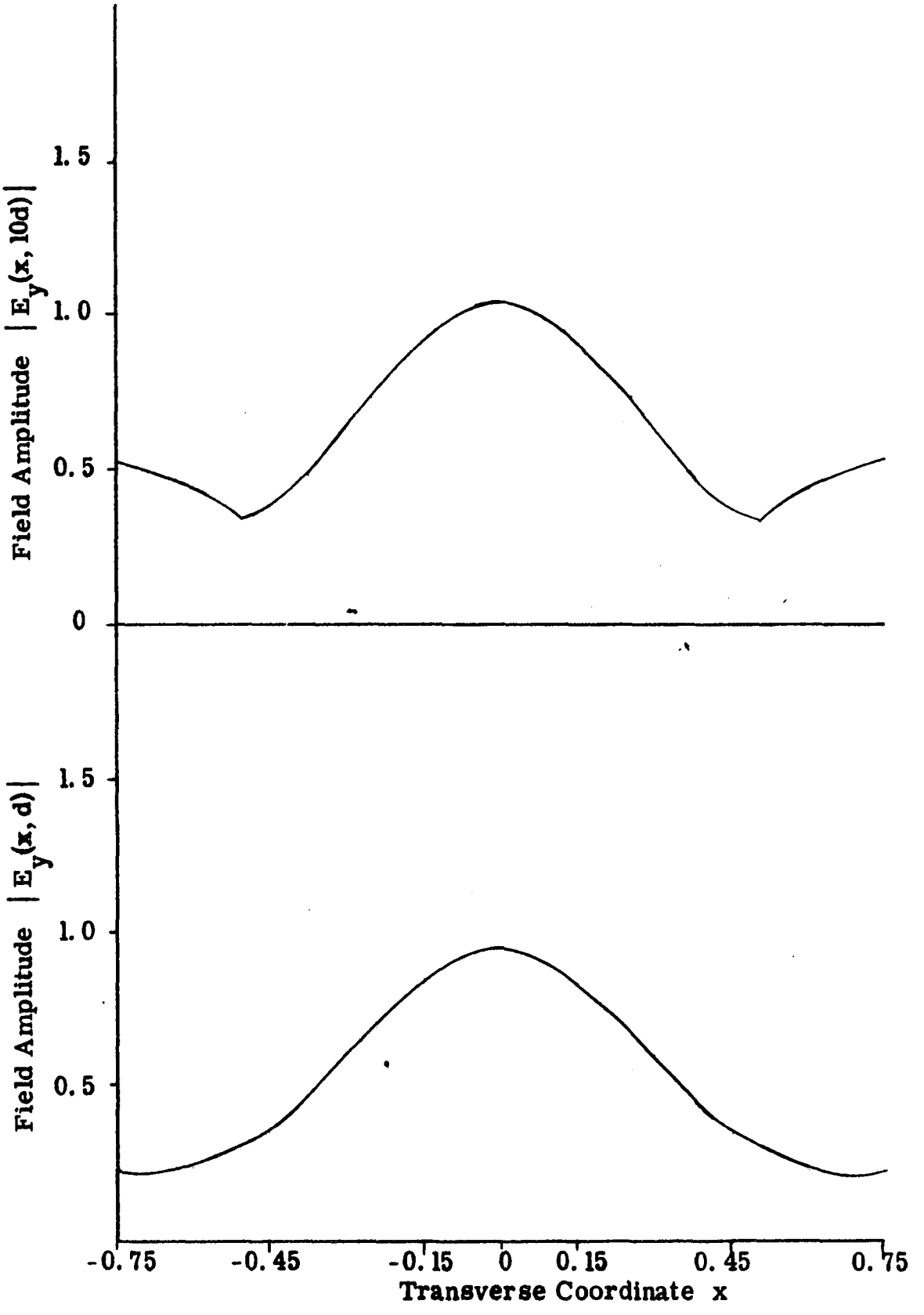


Figure 18. Variation of Field Amplitude at the Boundary (z=d) and 10d for a Slab Width of d = 2, Periodic Length a = 1.5 and Modulation Constant b = 3.36

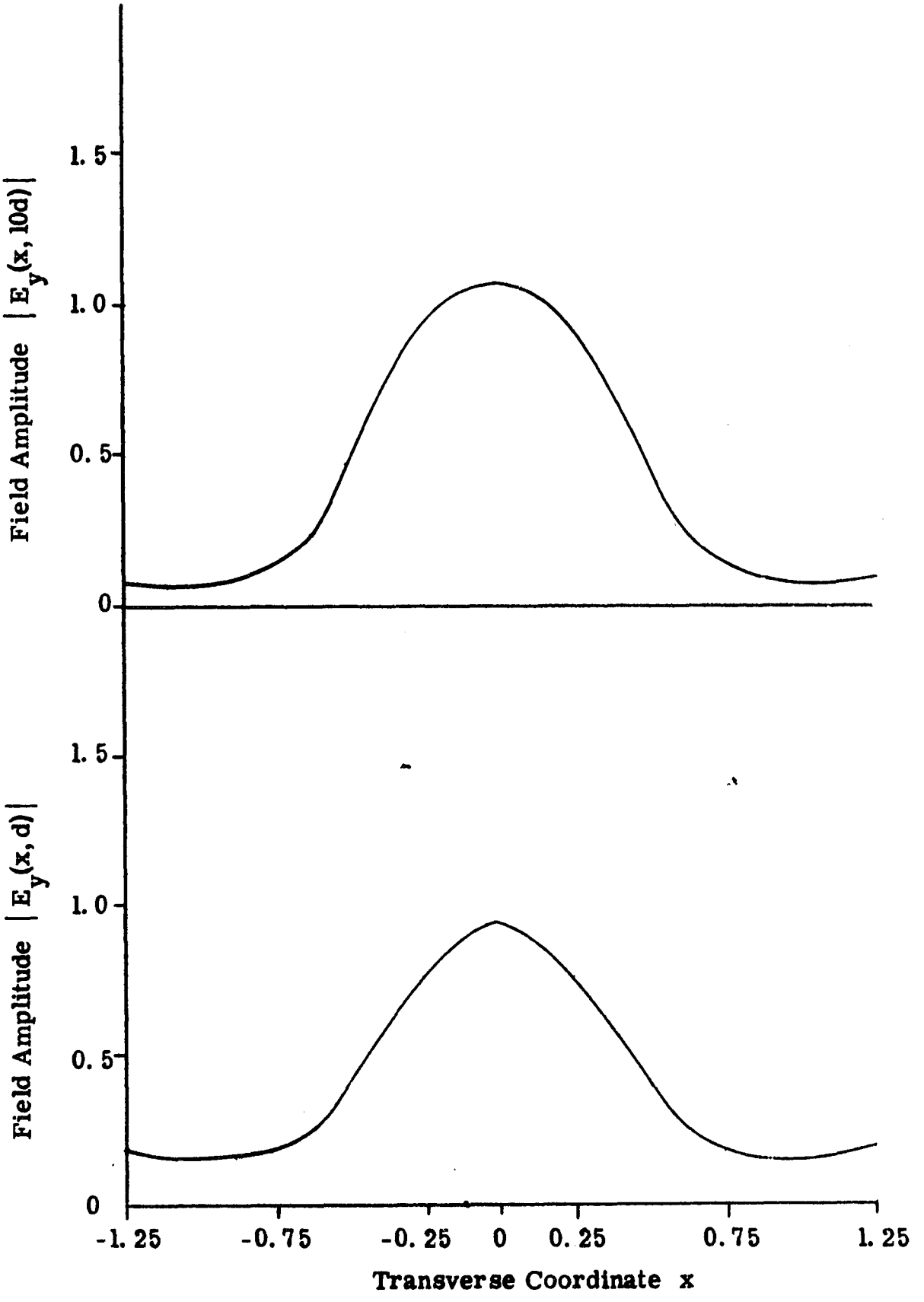


Figure 19. Variation of Field Amplitude at the Boundary (z=d) and 10d for a Slab Width of d = 2, Periodic Length a = 2.5 and Modulation Constant b = 3.36.

In Figures 20 and 21, the first two modal transmission coefficients are presented. One notes that for a small periodic length "a" the transmission coefficients go to zero. However, as the period increases, the transmission increases and seems to approach a certain value. Extensive calculations for this model were not performed since the only interest was to illustrate that transmission is possible. This may explain the occasional transparency of the ionosphere for frequencies below the maximum usable frequency.

In this chapter numerical results for the transmission and reflection coefficients as well as the field distribution for a specific index of refraction were obtained. Although a special variation of the index of refraction was used to demonstrate the method, it can be extended to a much larger class of problems.

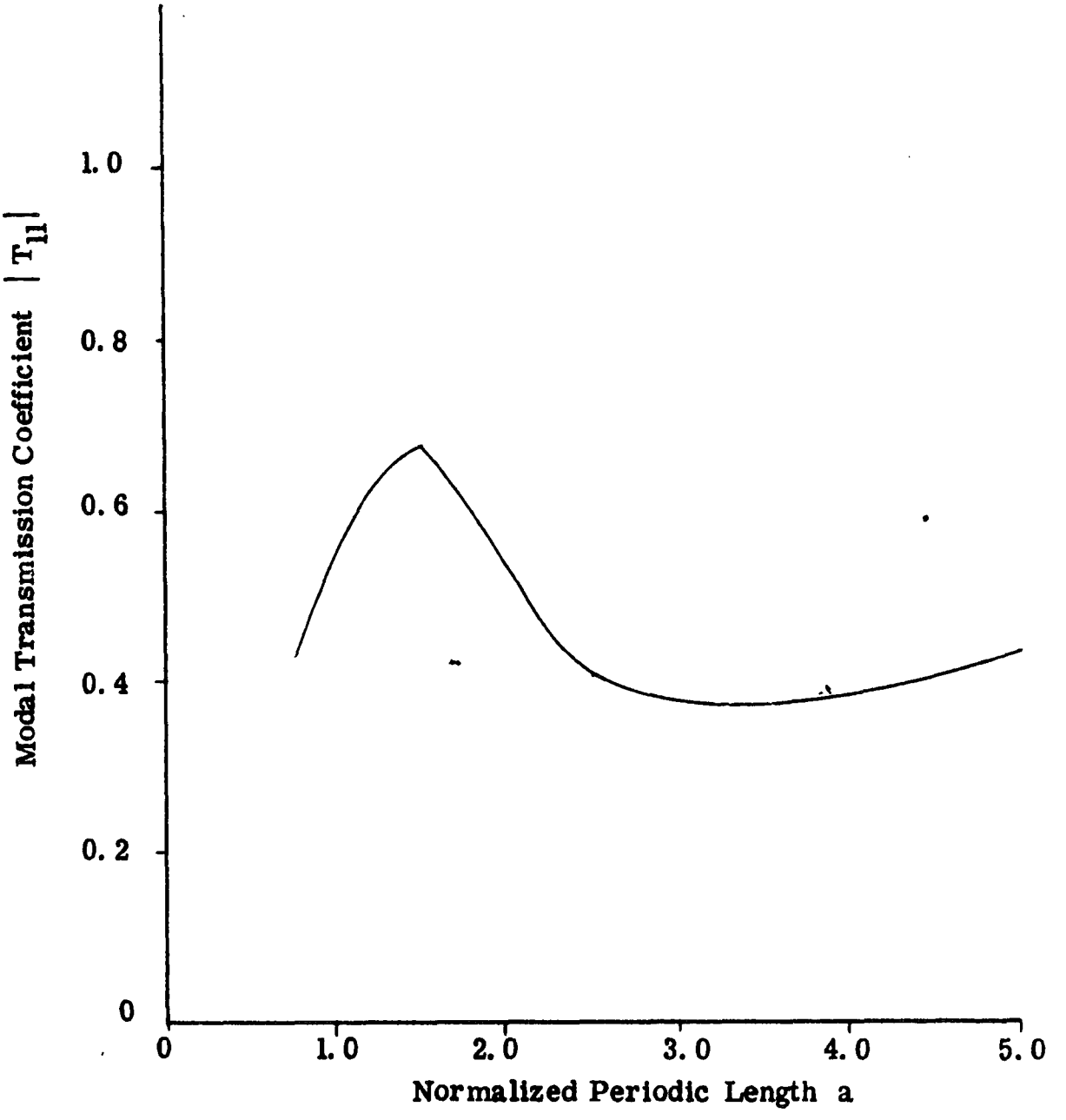


Figure 20. Modal Transmission Coefficients as a Function of Periodic Length for b = 3.36 and d = 2.

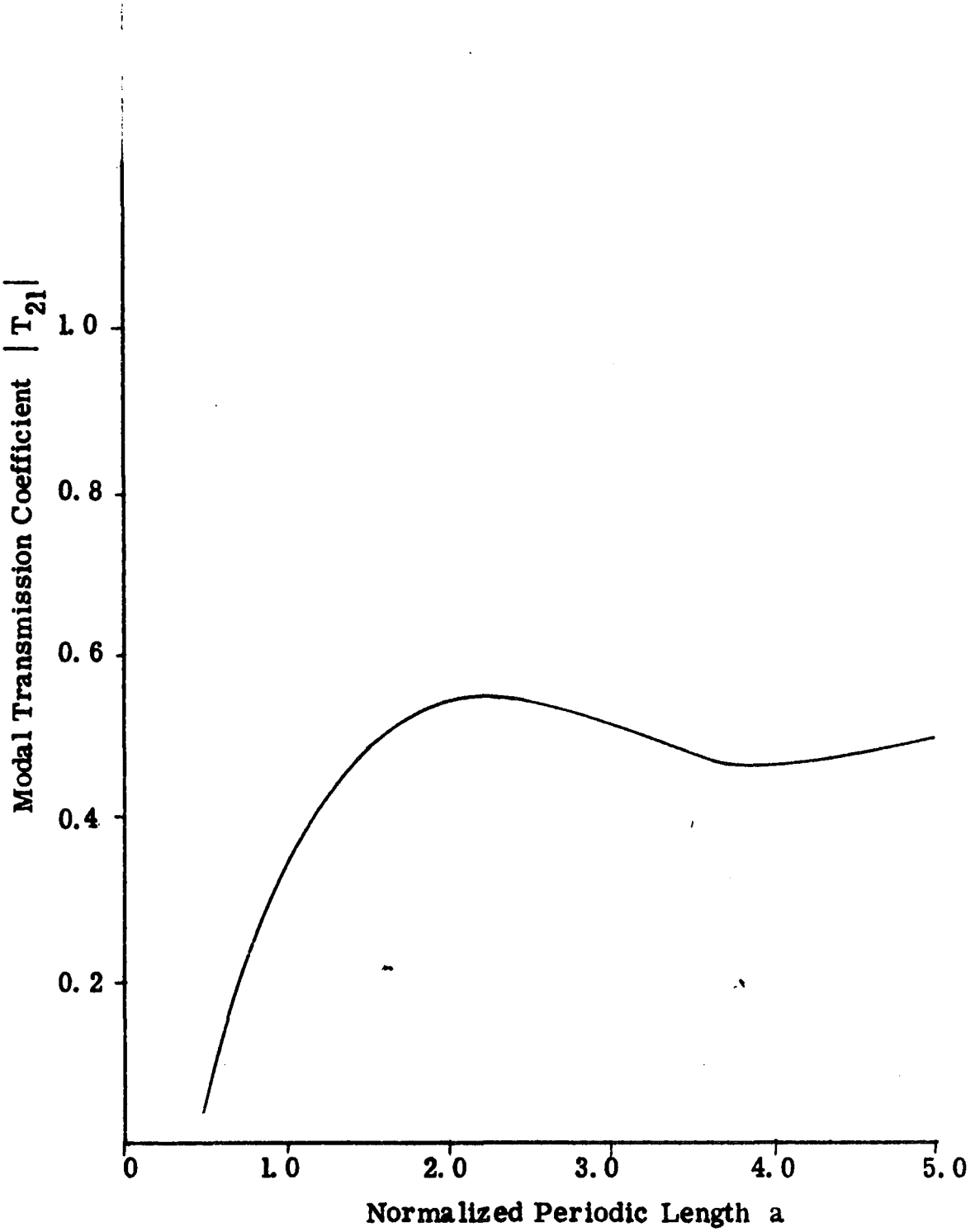


Figure 21. Modal Transmission Coefficients as a Function of Periodic Length for b = 3.36 and d = 2.

CHAPTER VI

SUMMARY AND CONCLUSIONS

In this dissertation a method has been developed which makes use of digital computers to solve a large class of problems in two dimensional inhomogeneous media. The method consists of orthogonal expansions as well as the invariant imbedding technique. This method was used to obtain numerical results for the reflection and transmission coefficients as well as the field distribution, for an index of refraction which is a parabolic function in the direction of propagation and sinusoidal in the transverse direction. Although a specific type of variation of the index of refraction was used, the method developed is general and can be used to solve a large class of problems involving two dimensional inhomogeneities. As far as physical problems are concerned, the choice of a sinusoidal variation for mathematical convenience is not restrictive.

The results obtained in this thesis, even though not as extensive as we would like, can be used as a canonical problem to check the approximate methods such as geometric optics or the Born approximation. The numerical results obtained for the variation of the transmitted field indicate that it is possible to devise a two dimensional ionospheric

model for diagnostic purposes. However, more numerical work should be done to obtain conclusive results. For the ionospheric model one has illustrated that electromagnetic energy will be transmitted for frequencies much lower than the maximum usable frequency, for certain combinations of parameters. This result might have interesting applications in the blackout problem.

In conclusion, the method presented in this dissertation has direct applications to the problems of focussing or scanning due to periodicity, and can be extended to solve two dimensional canonical problems for cylindrical and spherical geometries.

APPENDIX I

SUMMARY OF PROPERTIES OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

In this appendix the properties of the solution of a second order linear differential equation with periodic coefficients will be considered briefly. One shall consider equation (2. 2. 7) which was obtained in Chapter II and is of the form

$$\frac{d^2\xi}{dx^2} + [\lambda + p(x)]\xi = 0 \qquad \text{(I. 0. 1)}$$

where

$$p(x) = p(x+a)$$

and "a" is the period.

For the boundary conditions one requires that the eigenfunctions of equation (I. 0. 1) be of the Floquet type. Thus, they take the form

$$\xi_n(x, \gamma) = e^{i\beta_o x} v_n(x, \beta_o) \qquad \text{(I. 0. 2)}$$

for β_o ($\beta_o = \frac{2\pi}{\lambda} \sin \theta_o$) and $v_n(x, \beta_o)$ is periodic of period a.

The following properties follow directly on the basis of the Floquet theory:

(1) $\xi_n(x, \beta_o)$ is bounded for all x.

(2) $\xi_n(x+a, \beta_o) = e^{i\beta_o a} \xi_n(x, \beta_o)$ for real β_o.

(3) $\xi_n(x,\beta_o)$ is a periodic function of β_o with a period $2(\frac{\pi}{a})$, thus

$$\xi_n(x,\gamma) = \xi_n(x,\beta_o + 2(\tfrac{\pi}{a}))$$

(4) The eigenfunctions ξ_n and ξ_m for different eigenvalues are orthogonal in the interval "a",

$$\frac{1}{a}\int_{(a)} \xi_n(x,\beta_o)\xi_m^*(x,\beta_o) = \delta_{nm} \qquad \text{(I. 0. 3)}$$

(5) The function $u_n(x,\beta_o)$ can be represented in the following manner

$$\xi_n(x,\gamma) = e^{i[\beta_o+2n(\frac{\pi}{a})]x} \sum_r a_r^n e^{2r(\frac{\pi}{a})x} \qquad \text{(I. 0. 4)}$$

Let $\beta_n = \beta_o + 2n\frac{\pi}{a}$. Thus

$$a_r^n = \frac{1}{a}\int_{(a)} \xi_n(x,\beta_o)e^{-i[\beta_n+2r(\frac{\pi}{a})]x}dx \qquad \text{(I. 0. 5)}$$

(6) The orthogonality relations for the coefficients are given by

$$\sum_r a_n^{*r} a_m^r = \delta_{nm} \qquad \text{(I. 0. 6)}$$

$$\sum_n a_n^{*r} a_n^s = \delta_{rs} \qquad \text{(I. 0. 7)}$$

δ_{nm} denotes the Kroneckers delta function and the asterisk denotes the complex conjugate.

I. 1. SUMMARY OF PROPERTIES OF THE SOLUTION OF THE MATHIEU EQUATION

By substituting for the function p(x) a sinusoidal variation, which corresponds to an index of refraction of the form

$$\epsilon(x) = \epsilon_1 \left[1 - \cos \frac{2\pi x}{a} \right] \tag{I. 1. 1}$$

One obtains the Mathieu equation (2. 3. 6) which is of the form

$$\frac{d^2\xi}{dx'^2} + [\lambda - 2h^2 \cos 2x'] \xi = 0 \tag{I. 1. 2}$$

with λ and h^2 real, where

$$\lambda = (k^2\epsilon_1 - \gamma_n)\left(\frac{a}{\pi}\right)^2 \tag{I. 1. 3}$$

$$2h^2 = \frac{a^2k^2\epsilon_1 m}{\pi^2} = \left(\frac{ak}{\pi}\right)^2 \epsilon_1 m \tag{I. 1. 4}$$

and

$$x' = \frac{\pi}{a} x.$$

Equation (I. 1. 2) is the canonical form of the Mathieu differential equation. All of the properties listed above for the general, differential equation are equally valid for the Mathieu differential equation. Let one express the solution to equation (I. 1. 2) in the Floquet form by

$$\xi(x', \nu) = e^{i\nu x'} \sum_{r=-\infty}^{\infty} a_r e^{i2rx'} \tag{I. 1. 5}$$

where $\nu = \frac{a}{\pi} \beta_o$. By introducing equation (I. 1. 5) into (I. 1. 2), one obtains

$$\sum_{n=-\infty}^{\infty} \lambda - (\nu+2r)^2 - 2h^2(e^{2ix'} + e^{-2ix'})a_r e^{i2rx'} = 0 \qquad \text{(I. 1. 6)}$$

Since the above must hold for any value of x' , equation (I. 1. 6) can be written as an infinite set of homogeneous equations.

$$a_{r-1} + D_r a_r + a_{r+1} = 0 \qquad \text{(I. 1. 7)}$$

where

$$D_r = \frac{(\nu+2r)^2 - \lambda}{2h^2}$$

Employing an iterative process on equation (I. 1. 7), one obtains the two continued fractions:

$$\frac{a_r}{a_{r-1}} = -\cfrac{1}{D_r - \cfrac{1}{D_{r+1} - \cfrac{1}{D_{r+2} - \cdots}}} \qquad \text{(I. 1. 8)}$$

$$\frac{a_r}{a_{r+1}} = -\cfrac{1}{D_r - \cfrac{1}{D_{r-1} - \cfrac{1}{D_{r-2} - \cdots}}} \qquad \text{(I. 1. 9)}$$

which when combined yield

$$D_r = \cfrac{1}{D_{r+1} - \cfrac{1}{D_{r+2} - \cdots}} + \cfrac{1}{D_{r-1} - \cfrac{1}{D_{r-2} - \cfrac{1}{D_{r-3}}}} \qquad \text{(I. 1. 10)}$$

The continued fractions involved in equations (I. 1. 8), (I. 1. 9), and (I. 1. 10) can be shown to converge (Reference 19) if $D_r \geq 2$ for $r > R$ where R is the finite integer. An inspection of the expressions for D_r shows that this condition is always satisfied for the case considered here.

Now, the problem reduces to solving equation (I. 1. 10). To obtain a relation between λ, h^2 and ν, a so-called "stability chart" is often used. In some cases, simple approximate relations are desirable, these relations for λ will be given subsequently.

The eigenfunctions of the Mathieu differential equation are given by

$$\xi_n(x) = \exp\left(i \nu_n x\right) \sum_{r=-\infty}^{\infty} a_r^n \exp\left[i\left(\frac{2r\pi}{a}\right)x\right]$$

where

$$\nu_n = \frac{a}{\pi} \beta_o + 2n$$

These functions form a complete set of orthogonal functions, and for convenience, a normalization is adopted of the form

$$\frac{1}{a} \int_{-a/2}^{a/2} \xi_m(x) \xi_n^*(x) dx = \delta_{mn} \qquad \text{(I. 1. 11)}$$

where δ_{mn} is the usual Kronecker's delta function and the asterisk denotes the complex conjugate.

Equation (I. 1.11) yields the normalization condition

$$\sum_r (a_r^n)^2 = 1$$

This result essentially defines the values of a_o^n since all other a_r^n are related to a_o^n (Reference 19).

I. 2. GRAPHICAL METHOD

The stability chart is a set of constant ν curves plotted in a λ vs h^2 plane. Three independent variables are interrelated by the chart. When any two parameters are known, the third can be found uniquely on the chart.

Consider a case of interest where the medium is excited by a fixed frequency as shown in Figure (4). Since ν and h^2 are known, the corresponding values of λ_n can be uniquely determined by noting that the various modes corresponding to values of $\nu_n = \nu + 2n$, thus locating the pertinent point for each mode on the stability chart.

I. 3. APPROXIMATE METHODS

For small values of h^2, the continued fractions in equation (I. 1. 10) converge very rapidly. Hence they may be truncated to derive a simple analytic formula relating λ_n to h^2 and ν_n. In Reference (19), the authors present a power series for the eigenvalues λ_n, which is given by the expression

$$\lambda_n = \nu_n^2 + \frac{1}{2(\nu_n^2 - 1)} h^4 + \frac{5\nu_n^2 + 7}{32(\nu_n^2 - 1)^3(\nu_n^2 - 4)} h^8 + \frac{9\nu_n^4 + 58\nu_n^2 + 29}{64(\nu_n^2 - 1)^5(\nu_n^2 - 4)(\nu_n^2 - 9)} h^{12} + O(h^{16}) \qquad \text{(I. 3. 1)}$$

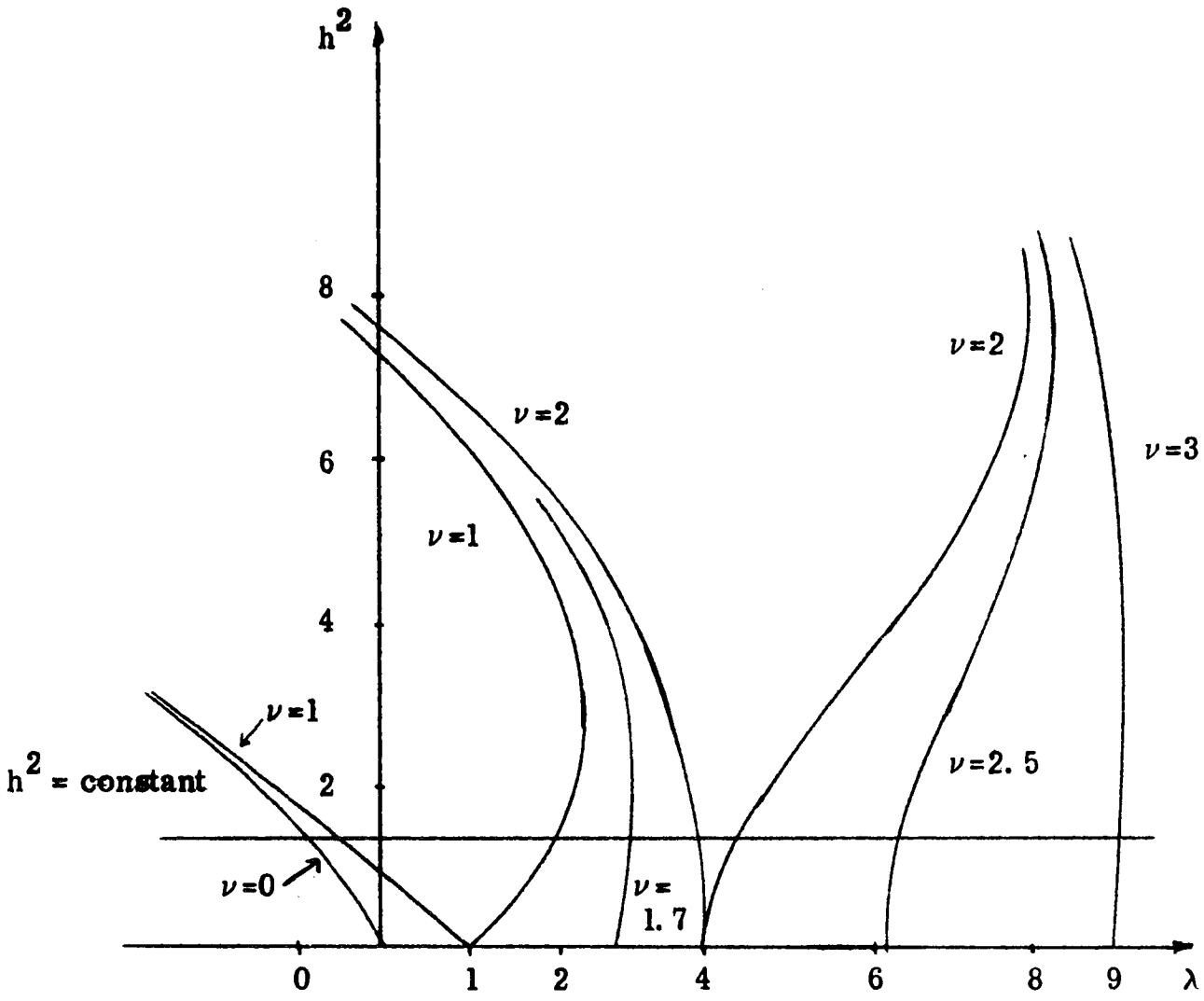


Figure 22. Mathieu Stability Chart

Obviously, the above formula is not valid for ν_n close to an integer. One notes that small h^2 due to relation

$$h^2 = \frac{1}{2} m\epsilon_r \left(\frac{ka}{\pi}\right)^2$$

means either small m or small ka.

(b) For Large Values of ν.

For large values of ν and h^2, the following expression for the eigenvalues are given in Reference (18), respectively:

$$\lambda_n = (\nu_n)^2 + \frac{h^4}{2(\nu_n^2 - 1)} + O(n^{-6}) \qquad (n \to \pm \infty) \qquad \text{(I. 3. 2)}$$

and

$$\lambda_n = -2h^2 + 2(\nu_n)h - \frac{1}{8}[\nu_n^2 + 1] - \frac{1}{128h}(\nu_n^3 + 3\nu_n) - \frac{1}{4096h^2}[5\nu_n^4 + 34\nu_n^2 + 9] + O\left(\frac{1}{h^3}\right) \qquad \text{(I. 3. 3)}$$

I. 4. MATHIEU FUNCTIONS OF INTEGRAL ORDER

It is natural, in view of the periodic properties, to express the Mathieu functions of integral order (ν = integer) as a trigonometric series in x'. The Mathieu functions of integral order are the solutions to Mathieu differential equation

$$\frac{du}{dx'} + (\lambda - 2h^2 \cos 2x')u = 0$$

and are given by the following Fourier series:

$$ce_{2n}(x', h^2) = \sum_{r=0}^{\infty} A_{2r}^{2n}(h^2) \cos 2rx'$$

$$ce_{2n+1}(x', h^2) = \sum_{r=0}^{\infty} A_{2r+1}^{2n+1}(h^2) \cos (2r+1)x'$$

$$se_{2n+1}(x', h^2) = \sum_{r=0}^{\infty} B_{2r+1}^{2n+1}(h^2) \sin (2r+1)x'$$

$$se_{2n+2}(x', h^2) = \sum_{r=0}^{\infty} B_{2r+2}^{2n+2}(h^2) \sin (2r+2)x'$$

where the coefficients A and B are subject to the following conditions:

$$2(A_0^{2n})^2 + \sum_{r=1}^{\infty} (A_{2r}^{2n})^2 = 1$$

$$\sum_{r=0}^{\infty} (A_{2r+1}^{2n+1}) = 1$$

$$\sum_{r=0}^{\infty} (B_{2r+1}^{2n+1})^2 = 1$$

$$\sum_{r=0}^{\infty} (B_{2r+2}^{2n+2})^2 = 1$$

$$A_k^n(o) = B_k^n(o) = \begin{cases} 2^{-1/2} & (k = n = 0) \\ 1 & (k = n \neq 0) \\ 0 & (k \neq n) \end{cases}$$

and

$$ce_0(x', o) = 2^{-1/2}$$

$$ce_k(x', o) = \cos kx', \quad se_k(x', o) = \sin kx' \qquad \text{for } k = 1, 2, 3, \ldots$$

$$\int_0^{2\pi} ce_n^2(x', h^2) = \int_0^{2\pi} se_n^2(x', h^2) = \pi$$

We designate the eigenvalues by $a_n(h^2)$ and $b_n(h^2)$ for real h^2 for the even and odd eigenfunction respectively.

$$\lambda = a_n(h^2)$$

$$\lambda = b_n(h^2)$$

For $h^2 > 0$ these eigenvalues are ordered

$$a_o < b_1 < a_1 < b_2 < a_2 < \ldots < b_n < a_n$$

for $h^2 < 0$

$$a_o < a_1 < b_1 < b_2 < a_2 < a_3 < b_3 < b_4 \ldots$$

For the particular problem considered in Chapter (V), one is only interested in the Fourier coefficients $A_{2r}^{2n}(h^2)$ and the eigenvalues $a_{2n}(h^2)$.

In reference (19), power series expansions are given in terms of h^2, for the Fourier coefficients and the eigenvalues. These expansions are valid for $h < 0.707$ and $h > 5$. For the region of $0.707 \leq h \leq 5$ we shall use the tables given in reference (20). In this presentation the eigenvalues and eigenfunctions will be given up to $n = 5$. Thus one shall subdivide the representation of the Fourier coefficients and eigenvalues into the following three regions:

Region I $\quad 0 \leq h(z) < 0.707$

Region II $\quad 0.707 \leq h(z) \leq 5$

Region III $\quad h(z) > 5$

Region I

For the region $h < 0.707$, the Fourier coefficients and the eigenvalues are given by the following expressions, respectively (Reference 19):

$$A_0^0(h^2) = \frac{1}{\sqrt{2}}\left[1 - \frac{h^4}{16}\right] \qquad A_0^2(h^2) = \frac{h^2}{4} - \frac{49}{1152}\,h^6$$

$$A_2^0(h^2) = \frac{1}{\sqrt{2}}\left(\frac{h^2}{2} - \frac{11}{128}\,h^6\right) \qquad A_2^2(h^2) = 1 - \frac{19}{288}\,h^4$$

$$A_4^0(h^2) = \frac{1}{\sqrt{2}}\left(\frac{h^4}{32}\right) \qquad A_4^2(h^2) = -\frac{h^2}{12} + \frac{19}{3456}\,h^6$$

$$A_6^0(h^2) = -\frac{1}{\sqrt{2}}\left(\frac{h^6}{9 \times 128}\right) \qquad A_6^2(h^2) = \frac{h^4}{384}$$

$$A_8^0(h^2) = \frac{1}{\sqrt{2}}\left(\frac{h^8}{286 \times 256}\right) \qquad A_8^2(h^2) = \frac{h^8}{286 \times 256}$$

For $m \geq 4 \quad r > 0$

$$A_{n+2r}^n(h^2) = \left[(-1)^r \frac{n!}{r!\,(n+r)!}\left(\frac{h^2}{4}\right)^r\right] A_n^n(h^2)$$

$$A_{n-2r}^n(h^2) = \left[\frac{(n-r-1)!}{r!(n-1)!}\left(\frac{h^2}{4}\right)^r\right] A_n^n(h^2)$$

$$A_n^n(h^2) = 1 - \frac{1}{32}\left[\frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right] h^4$$

The eigenvalues are:

$$a_0(h^2) = -\frac{1}{2}h^2 + \frac{7}{128}h^8$$

$$a_2(h^2) = 4 + \frac{5}{12}h^4 + \frac{5}{13824}h^8$$

$$a_4(h^2) = 16 + \frac{1}{30}h^4 + \frac{433}{864000}h^8$$

$$a_6(h^2) = 36 + \frac{1}{20}h^4 + \frac{187}{43904000}h^8$$

$$a_8(h^2) = 64 + \frac{h^4}{126} + \frac{h^8}{1437585}$$

Region II

For the region $h > 5$, the Fourier coefficients and the eigenvalues are given by the following expressions, respectively (Reference 19):

$$A_o^{2n}(h^2) = 2^{-h-1/4}\,\frac{\sqrt{(2n)!}}{n!}\,(\pi h)^{-1/4}$$

for $s > 0$

$$A_{2r}^{2n}(h^2) = 2^{-h+3/4}\,\frac{\sqrt{(2n)!}}{n!}\,(-1)^r(\pi h)^{-1/4}\,\frac{D_{2n}(\frac{2r}{h^{1/2}})}{D_{2n}(o)}$$

where

$$D_n(x) = (-1)^n\, e^{x^2/4}\,\frac{d^n}{dx^n}\,(e^{-x^2/2})$$

and

$$D_0(x) = e^{-x^2/4}, \quad D_0(o) = 1$$

$$D_2(x) = e^{-x^2/4}(x^2-1)$$

$$D_4(x) = e^{-x^2/4}(x^4-6x^2+3)$$

$$D_6(x) = e^{-x^2/4}(x^6 - 15x^4 + 45x^2 - 15)$$

$$D_8(x) = e^{-x^2/4}(x^8 - 28x^6 + 210x^4 - 420x^2 + 105)$$

The eigenvalues are given by:

$$a_n(h^2) = -2h^2 + 2(2n+1)h - \frac{1}{8}[(2n+1)^2+1] - \frac{1}{128h}[(2n+1)^3 + 3(2n+1)]$$

$$- \frac{1}{4096h^2}[5(2n+1)^4 + 34(2n+1)^2 + 9]$$

$$- \frac{1}{2^{17}h^3}[33(2n+1)^5 + 410(2n+1)^3 + 405(2n+1)]$$

$$- \frac{1}{2^{20}h^4}[63(2n+1)^6 + 1260(2n+1)^4 + 2943(2n+1)^2 + 486]$$

$$- \frac{1}{2^{25}h^5}[527(2n+1)^7 + 15617(2n+1)^5 + 69001(2n+1)^3 + 41607(2n+1)]$$

$n = 2, 4, 6, 8$

Region III

For this region $0.707 \leq h \leq 5$ the tables (Reference 20) were used to obtain the Fourier coefficients and the eigenvalues.

APPENDIX II

COUPLING COEFFICIENTS

The coupling coefficients for the normal incidence case are given by the expressions

$$S_{ij}(z) = \frac{\omega}{2N^i(\gamma_i - \gamma_j)} \int_{-a/2}^{a/2} \frac{d\epsilon}{dz} u^i(x)u^j(x)dx \qquad (I.0.1)$$

for $i \neq j$, and

$$S_{ii} = -\frac{dN^i(z)}{dz} \Big/ 2N^i(z) \qquad (I.0.2)$$

where

$$N^i = \int_{-a/2}^{a/2} u^i \bar{v}_x^i dx \qquad (I.0.3)$$

The natural waves $u^i(x)$ and the propagation constants $\gamma_i(z)$ are obtained by solving the following second order differential equation:

$$\frac{d^2u}{dx^2} + [\omega\mu\epsilon(x,z) - \gamma^2]u = 0 \qquad (I.0.4)$$

where

$$\epsilon(x,z) = \epsilon_0[1 - m(z)\cos\frac{2\pi x}{a}] \qquad (I.0.5)$$

Let us substitute equation (I. 0. 5) into equation (I. 0. 4) which yields:

$$\frac{d^2u}{dx^2} + [k^2 - \gamma^2 - k^2 m(z) \cos \frac{2\pi x}{a}]u = 0 \qquad \text{(I. 0. 6)}$$

where

$$k^2 = \omega^2 \mu \epsilon_0 \qquad \text{(I. 0. 7)}$$

By making the following transformations:

$$x' = \frac{\pi}{a} x \qquad \text{(I. 0. 8)}$$

$$\lambda = (\frac{a}{\pi})^2 (k^2 - \gamma^2) \qquad \text{(I. 0. 9)}$$

$$2h^2 = (\frac{ak}{\pi})^2 m(z) \qquad \text{(I. 0. 10)}$$

Thus, equation (I. 0. 6) takes the form

$$\frac{d^2u^i}{dx'^2} + [\lambda_i - 2h^2(z) \cos 2x']u^i = 0 \qquad \text{(I. 0. 11)}$$

and

$$\gamma_i(z) = [k^2 - (\frac{\pi}{a})^2 \lambda_i(z)]^{1/2}$$

Equation (I. 0. 11) is the Mathieu equation which is fully described in Appendix I.

In the normal incidence case, the problem has the symmetry $u(x) = u(-x)$, and so there can be no antisymmetric modes excited. Thus it is sufficient to represent the field by even Mathieu functions of the form

$$u^i(x') = \sum_{n=o}^{\infty} A_{2n}^{2(i-1)}(h^2) \cos 2nx' \qquad \text{(I. 0. 12)}$$

where the coefficients $A_{2n}^{2(i-1)}(h^2)$ and the eigenvalues $a_{2(i-1)}$ are given in Appendix I for all values of $h^2 \geq 0$, $i = 1, 2, \ldots$

The normalization factor has been defined by

$$N^i = \int_{-a/2}^{a/2} u^i \bar{v}_x^i dx \qquad \text{(I. 0. 13)}$$

where u^i and $\bar{v}_x^i$ are the cross-section eigenfunctions. By substituting the expressions for $\bar{v}_x^i$ given by equation (3. 2. 8)

$$\bar{v}_x^i = -\frac{\gamma_i}{\omega\mu} u^i \qquad \text{(I. 0. 14)}$$

into equation (I. 0. 13), one obtains

$$N^i = -\frac{\gamma_i}{\omega\mu} \int_{-a/2}^{a/2} u^i u^i dx \qquad \text{(I. 0. 15)}$$

Using the transformation

$$x' = \frac{\pi}{a} x$$

and the orthogonality property given in Appendix I

$$\int_{-\pi/2}^{\pi/2} u^n u^n dx' = \frac{\pi}{2} , \qquad \text{(I. 0. 16)}$$

the normalization factor is equal to

$$N^i = -\frac{\gamma_i a}{2\omega\mu} \qquad \text{(I. 0. 17)}$$

By substituting the expression for the normalization factor (I. 0. 3) and the dielectric constant (I. 0. 5) into equation (I. 0. 1), and using the transformation (I. 0. 8), one obtains

$$S_{i,j}(z) = \frac{k^2 \frac{dm(z)}{dz}}{\pi\gamma_i(\gamma_i - \gamma_j)} \int_{-\pi/2}^{\pi/2} \cos 2x' u^i u^j dx' \qquad \text{(I. 0. 18)}$$

By examining the integral

$$\int_{-\pi/2}^{\pi/2} \cos 2x' \, u^i u^j dx'$$

and substitute the appropriate eigenfunctions given by equation (I. 0. 12) one obtains

$$\int_{-\pi/2}^{\pi/2} \left[\cos 2x' \sum_{o}^{\infty} A_{2n}^{2(i-1)} \cos 2nx' \sum_{o}^{\infty} A_{2m}^{2(j-1)} \cos mx' \right] dx'$$

Let one write the above expression as

$$\sum_{n=o}^{\infty} \sum_{m=o}^{\infty} A_{2n}^{2(i-1)} A_{2m}^{2(j-1)} \int_{-\pi/2}^{\pi/2} \left[\cos 2x' \cos 2nx' \cos 2mx' \right] dx'$$

or

$$\sum_{n=o}^{\infty} \sum_{m=o}^{\infty} A_{2n}^{2(i-1)} A_{2m}^{2(j-1)} \int_{-\pi/2}^{\pi/2} \cos 2x' \left\{ \frac{1}{2} \left[\cos 2(n-m)x' + \cos 2(n+m)x' \right] \right\} dx'$$

or

$$\frac{\pi}{2} \left[A_0^{2(i-1)} A_2^{2(j-1)} + A_2^{2(i-1)} A_0^{2(j-1)} \right] + \frac{\pi}{4} \sum_{n=1}^{\infty} \left[A_{2n+2}^{2(i-1)} A_{2n}^{2(j-1)} + A_{2n}^{2(i-1)} A_{2n+2}^{2(j-1)} \right]$$

Now one can write the equation for the coupling coefficients as

$$S_{2i,\,2j}(z) = + \frac{k^2 \frac{dm(z)}{dz}}{2\gamma_i(\gamma_i - \gamma_j)} \left[A_0^{2(i-1)} A_2^{2(j-1)} + A_2^{2(i-1)} A_0^{2(j-1)} + \frac{1}{2} \sum_{n=1}^{\infty} \left(A_{2n}^{2(i-1)} A_{2n+2}^{2(j-1)} + A_{2n+2}^{2(i-1)} A_{2n}^{2(j-1)} \right) \right] \tag{I. 0. 19}$$

This equation is valid for

$$i \neq j, \quad i, j > 0, \quad \text{and} \quad i, j < 0.$$

For

$$i > 0 \quad \text{and} \quad j < 0$$

or

$$i < 0 \quad \text{and} \quad j > 0$$

$$S_{i,-j}(z) = S_{-i,j}(z)$$

Thus, from equation (I. 0. 19)

$$S_{-i,j} = S_{i,-j} = T'_{i,j} = - \frac{k^2 \frac{d}{dz} m(z)}{2\gamma_i(\gamma_i + \gamma_j)} \left[A_0^{2(i-1)} A_2^{2(j-1)} + A_2^{2(i-1)} A_0^{2(j-1)} + \frac{1}{2} \left(\sum_{n=1}^{\infty} A_{2n+2}^{2(i-1)} A_{2n}^{2(j-1)} \right) \right] \tag{I. 0. 20}$$

Now by substituting the normalization factor (I. 0. 3) into equation (I. 0. 2) one obtains

$$S_{i,i}(z) = -d\gamma_i / dz \Big/ 2\gamma_i \tag{I. 0. 21}$$

Thus the coupling coefficients for normal incidence are given by the equations (I. 0. 19), (I. 0. 20) and (I. 0. 21).

APPENDIX III

TRANSFORMATION OF A BOUNDARY VALUE PROBLEM TO AN INITIAL VALUE PROBLEM BY MEANS OF INVARIANT IMBEDDING

Let one employ the method of invariant imbedding to transform a boundary value problem to an initial value problem (Reference 17). To illustrate the invariant imbedding technique, one can begin with a simple problem described by a pair of coupled first order linear differential equations, and then extend it to higher-order systems.

Let one start by considering a coupled first order differential equation

$$\frac{du(z)}{dz} = a(z)u(z) + b(z)v(z) \qquad \text{(III. 0. 1)}$$

$$-\frac{dv(z)}{dz} = c(z)u(z) + d(z)v(z) \qquad \text{(III. 0. 2)}$$

in the range $x_1 \leq z \leq x_2$.

Let one imagine $u(z)$ as some forward transport quantity, and $v(z)$ as some backward transport quantity. Then the quantities incident are

$$u_1 = u(x_1) \qquad \text{(III. 0. 3)}$$

$$v_2 = v(x_2)$$

while the reflected quantities are

$$v_1 = v(x_1)$$

$$u_2 = u(x_2) \qquad \text{(III. 0. 4)}$$

Since this is a linear problem, one can describe the slab by a "scattering matrix", that is

$$\begin{bmatrix} v_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ v_2 \end{bmatrix} \qquad \text{(III. 0. 5)}$$

where M_{11}, M_{21} are the solutions of a boundary value problem for $v(x_1)$ and $u(x_2)$ with $v(x_2) = 0$, and $u(x_1) = 1$, while M_{12} and M_{22} are the solutions for $v(x_1)$ and $u(x_2)$ with $u(x_1) = 0$ and $v(x_2) = 1$. By using the principle of invariant imbedding, the scattering matrices can be expressed as the solution of a first order non-linear differential equation, with appropriate initial values.

To illustrate the principle, let one imbed the original problem given by equations (III. 0. 1) and (III. 0. 2), into a wider class of problems with the x_1 variabl denoted by y_1 and keep x_2 fixed. The differential equations (III. 0.1) and (III. 0. 2) can be rewritten as:

$$u_z(z, y_1, x_2) = a(z)u(z, y_1, x_2) + b(z)v(z, y_1, x_2) \qquad \text{(III. 0. 6)}$$

$$-v_z(z, y_1, x_2) = c(z)u(z, y_1, x_2) + d(z)v(z, y_1, x_2) \qquad \text{(III. 0. 7)}$$

where the subscript z denotes differentiation with respect to z.

The above equations are subject to the boundary conditions

$$\begin{aligned} u(z, y_1, x_2)\big|_{z=y_1} &= 1 \\ v(z, y_1, x_2)\big|_{z=x_2} &= 0 \end{aligned} \qquad \text{(III. 0. 8)}$$

Let one denote the solution to equations (III. 0. 6) and (III. 0. 7) by

$$u(z, y_1, x_2)$$

and

$$v(z, y_1, x_2)$$

and define

$$v(z, y_1, x_2)\big|_{z=y_1} = M_{11}(y_1, x_2) \tag{III. 0. 9}$$

$$u(z, y_1, x_2)\big|_{z=x_2} = M_{21}(y_1, x_2) \tag{III. 0. 10}$$

Thus if $y_1 \to x_2$, then

$$M_{11}(y_1, x_2)\big|_{y_1=x_2} = 0$$

$$M_{21}(y_1, x_2)\big|_{y_1=x_2} = 1 \tag{III. 0. 11}$$

Now, let one obtain the differential equations for M_{11} and M_{21}. By differentiating the definitions given in equations (III. 0. 9) and (III. 0. 10), it follows that

$$\frac{dM_{11}}{dy_1} = v_z(z, y_1, x_2)\big|_{z=y_1} + v_{y_1}(z, y_1, y_2)\big|_{z=y_1} \tag{III. 0. 12}$$

and

$$\frac{dM_{21}(y_1, x_2)}{dy_1} = u_{y_1}(z, y_1, x_2)\big|_{z=x_2} \tag{III. 0. 13}$$

Now the goal is to express the right sides of equation (III. 0. 12) and (III. 0. 13) in terms of known functions. Thus, one is interested in investigating the three right hand terms. Evidently this can be done for the first term of equation (III. 0. 12) by evaluating equation (III. 0. 7) at $z = y_1$ and using the boundary condition $u(z, y_1, x_2)\big|_{z=y_1} = 1$. Thus, one obtains

$$v_z(z, y_1, x_2)\big|_{z=y_1} = -[c(y_1)u(z, y_1, x_2)\big|_{z=y_1} + d(y_1)v(z, y_1, x_2)\big|_{z=y_1}]$$

$$= -[c(y_1) + d(y_1)M_{11}(y_1, x_2)] \qquad \text{(III. 0. 14)}$$

For the second term on the right side of equation (III. 0. 12), one obtains it by differentiating equations (III. 0. 6) and (III. 0. 7) with respect to y_1,

$$(u_z)_{y_1}(z, y_1, x_2) = a(z)u_{y_1}(z, y_1, x_2) + b(z)v_{y_1}(z, y_1, x_2) \qquad \text{(III. 0. 15)}$$

$$(-v_z)_{y_1}(z, y_1, x_2) = c(z)u_{y_1}(z, y_1, x_2) + d(z)v_{y_1}(z, y_1, x_2) \qquad \text{(III. 0. 16)}$$

Thus u_{y_1} and v_{y_1} satisfy the same differential equations as u and v, but with different boundary conditions. By differentiating the original boundary conditions (III. 0. 8), one obtains

$$u_{y_1}(z, y_1, x_2)\big|_{z=y_1} = -u_z(z, y_1, x_2)\big|_{z=y_1}$$

$$= -[a(y_1) + b(y_1)M_{11}(y_1, x_2)] \qquad \text{(III. 0. 17)}$$

$$v_{y_1}(z, y_1, x_2)\big|_{z=x_2} = 0 \qquad \text{(III. 0. 18)}$$

By the linearity of equations (III. 0. 6) and (III. 0. 7) it follows that $u_{y_1}(z, y_1, x_2)\big|_{z=x_2}$ and $v_{y_1}(z, y_1, x_2)\big|_{z=y_1}$ are just multiples of u and v. Thus,

$$v_{y_1}(z, y_1, x_2)\big|_{z=y_1} = -[a(y_1)+b(y_1)M_{11}(y_1, x_2)]M_{11}(y_1, x_2) \qquad \text{(III. 0. 19)}$$

$$u_{y_1}(z, y_1, x_2)\big|_{z=x_2} = -[a(y_1)+b(y_1)M_{11}(y_1, x_2)]M_{21}(y_1, x_2) \qquad \text{(III. 0. 20)}$$

Substituting equation (III. 0. 14) and (III. 0. 19) into equation (III. 0. 12), one obtains

$$\frac{dM_{11}(y_1, x_2)}{dy_1} = -[c(y_1)+d(y_1)M_{11}(y_1, x_2)] - [a(y_1)+b(y_1)M_{11}(y_1, x_2)]M_{11}(y_1, x_2) \qquad \text{(III. 0. 21)}$$

or

$$-\frac{dM_{11}(y_1, x_2)}{dy_1} = c(y_1)+[a(y_1)+d(y_1)]M_{11}(y_1, x_2)+b(y_1)M_{11}^2(y_1, x_2) \qquad \text{(III. 0. 22)}$$

with an initial condition

$$M_{11}(y_1, x_2)\big|_{y_1=x_2} = 0 \qquad \text{(III. 0. 23)}$$

The differential equation for $M_{21}(y_1, x_2)$ is obtained by substituting equation (III. 0. 20) into equation (III. 0. 13), one obtains

$$-\frac{dM_{21}(y_1, x_2)}{dy_1} = [a(y_1)+b(y_1)M_{11}(y_1, x_2)]M_{21}(y_1, x_2) \qquad \text{(III. 0. 24)}$$

with an initial condition

$$M_{21}(y_1, x_2)\big|_{y_1=x_2} = 1. \qquad \text{(III. 0. 25)}$$

Similarly, one can imbed the original problem given by equations (III. 0. 1) and (III. 0. 2) into a wider class of problems, with x_1 fixed and let $x_2 \equiv y_2$ be variable. The differential equations can be rewritten as

$$u_z(z, x_1, y_2) = a(z)u(z, x_1, y_2) + b(z)v(z, x_1, y_2) \qquad \text{(III. 0. 26)}$$

$$-v_z(z, x_1, y_2) = c(z)u(z, x_1, y_2) + d(z)v(z, x_1, y_2) \qquad \text{(III. 0. 27)}$$

subject to the boundary conditions

$$v(z, x_1, y_2)\big|_{z=y_2} = 1$$

$$u(z, x_1, y_2)\big|_{z=x_1} = 0 \qquad \text{(III. 0. 28)}$$

and define

$$M_{22}(x_1, y_2) = u(z, x_1, y_2)\big|_{z=y_2} \qquad \text{(III. 0. 29)}$$

$$M_{12}(x_1, y_2) = v(z, x_1, y_2)\big|_{z=x_1} \qquad \text{(III. 0. 30)}$$

To derive the differential equations for M_{22} and M_{12} one follows the same procedure as above. Thus by differentiating equations (III. 0. 29) and (III. 0. 30), we obtain

$$\frac{dM_{22}(x_1, y_2)}{dy_2} = u_z(z, x_1, y_2)\big|_{z=y_2} + u_{y_2}(z, x_1, y_2)\big|_{z=y_2} \qquad \text{(III. 0. 31)}$$

$$\frac{dM_{12}(x_1, y_2)}{dy_2} = v_{y_2}(z, x_1, y_2)\big|_{z=x_1} \qquad \text{(III. 0. 32)}$$

Similarly, the goal is to express the terms on the right sides of equations (III. 0. 31) and (III. 0. 32) in terms of known functions.

Following the procedure as before, one can obtain the following expressions:

$$u_z(z, x_1, y_2)\Big|_{z=y_2} = a(y_2)M_{22}(x_1, y_2) + b(y_2) \qquad \text{(III. 0. 33)}$$

$$u_{y_2}(z, x_1, y_2)\Big|_{z=y_2} = [c(y_2)M_{22}(x_1, y_2)+d(y_2)]M_{22}(x_1, y_2) \qquad \text{(III. 0. 34)}$$

and

$$v_{y_2}(z, x_1, y_2)\Big|_{z=x_1} = [c(y_2)M_{22}(x_1, y_2)+d(y_2)]M_{12}(x_1, y_2) \qquad \text{(III. 0. 35)}$$

Thus, by substituting expressions (III. 0. 33) and (III. 0. 34) into equation (III. 0. 31), one obtains

$$\frac{dM_{22}(x_1, y_2)}{dy_2} = b(y_2) + [a(y_2)+d(y_2)]M_{22}(x_1, y_2) + c(y_2)M_{11}^2(x_1, y_2) \qquad \text{(III. 0. 36)}$$

With the initial condition

$$M_{22}(x_1, y_2) = 0 \qquad \text{(III. 0. 37)}$$

To obtain the differential equation for $M_{12}(x_1, y_2)$, one substitutes equation (III. 0. 35) into equation (III. 0. 32) and obtain

$$\frac{dM_{12}(x_1, y_2)}{dy_2} = [c(y_2)M_{22}(x_1, y_2) + d(y_2)]M_{12}(x_1, y_2) \qquad \text{(III. 0. 38)}$$

with the initial condition given by

$$M_{12}(x_1, x_1) = 1 \qquad \text{(III. 0. 39)}$$

III. 1. LARGE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

The extension of the invariant imbedding technique to a large system of linear differential equations is straightforward. The method will be illustrated for a large system of equations of the form

$$\frac{dU(z)}{dz} = A(z)U(z) + B(z)V(z) \tag{III. 1. 1}$$

$$-\frac{dV(z)}{dz} = C(z)U(z) + D(z)V(z) \tag{III. 1. 2}$$

in the range $x_1 \leq z \leq x_2$ where A, B, C, D are n x n matrices and U, V are column matrices (vectors) of order n.

Subject to the boundary conditions

$$\begin{aligned} U(x_1) &= U_1 \\ V(x_2) &= V_2 \end{aligned} \tag{III. 1. 3}$$

Since this is a linear problem, one can describe the slab by a "scattering matrix" of the form

$$\begin{bmatrix} V(x_1) = V_1 \\ U(x_2) = U_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ V_2 \end{bmatrix} \tag{III. 1. 4}$$

where M_{11}, M_{12}, M_{21} and M_{22} are matrices.

To illustrate the method, let one imbed the original problem given by equations (III. 1. 1) and (III. 1. 2) into a wider class of problems with x_1 variable denoted by y_1 and x_2 fixed. The differential equations (III. 1. 1) and (III. 1. 2) can be rewritten as

$$U_z(z, y_1, x_2) = A(z)U(z, y_1, x_2) + B(z)V(z, y_1, x_2) \qquad \text{(III. 1. 5)}$$

$$-V_z(z, y_1, x_2) = C(z)U(z, y_1, x_2) + D(z)V(z, y_1, x_2) \qquad \text{(III. 1. 6)}$$

One assumes the boundary conditions to be

$$\begin{aligned} U(z, y_1, x_2)\big|_{z=y_1} &= I \\ V(z, y_1, x_2)\big|_{z=x_2} &= 0 \end{aligned} \qquad \text{(III. 1. 7)}$$

where I is a column matrix which possess one element that is equal to unity and all other elements are zero.

By defining

$$V(z, y_1, x_2)\big|_{z=y_1} = M_{11}(y_1, x_2) \qquad \text{(III. 1. 8)}$$

$$U(z, y_1, x_2)\big|_{z=x_2} = M_{21}(y_1, x_2) \qquad \text{(III. 1. 9)}$$

and differentiating the above equations one can obtain the differential equations for $M_{11}(y_1, x_2)$ and $M_{21}(y_1, x_2)$. The equations are given by the following expressions,

$$\frac{dM_{11}(y_1, x_2)}{dy_1} = V_z(z, y_1, x_2)\big|_{z=y_1} + V_{y_1}(z, y_1, x_2)\big|_{z=y_1} \qquad \text{(III. 1. 10)}$$

and

$$\frac{dM_{21}(y_1, x_2)}{dy_1} = U_{y_1}(z, y_1, x_2)\big|_{z=x_2} \qquad \text{(III. 1. 11)}$$

Now, it remains to express the right sides of equations (III. 1. 10) and (III. 1. 11) in terms of known functions. Thus, by following the same

procedure as was done for the previous case one can obtain the following expressions:

$$V_z(z, y_1, x_2)\big|_{z=y_1} = -[C(y_1)+D(y_1)M_{11}(y_1, x_2)] \qquad \text{(III. 1. 12)}$$

$$V_{y_1}(z, y_1, x_2)\big|_{z=y_1} = -M_{11}(y_1, x_2)[A(y_1)+B(y_1)M_{11}(y_1, x_2)] \qquad \text{(III. 1. 13)}$$

and

$$U_{y_1}(z, y_1, x_2)\big|_{z=x_2} = -M_{21}(y_1, x_2)[A(y_1)+B(y_1)M_{11}(y_1, x_2)] \qquad \text{(III. 1. 14)}$$

Therefore, by substituting equations (III. 1. 12) and (III. 1. 13) into equation (III. 1. 10), one obtains

$$\begin{aligned} -\frac{dM_{11}(y_1, x_2)}{dy_1} &= C(y_1)+D(y_1)M_{11}(y_1, x_2) \\ &\quad +M_{11}(y_1, x_2)A(y_1) \\ &\quad +M_{11}(y_1, x_2)B(y_1)M_{11}(y_1, x_2) \end{aligned} \qquad \text{(III. 1. 15)}$$

with the initial condition

$$M_{11}(y_1, x_2)\big|_{y_1=x_2} = 0 \qquad \text{(III. 1. 16)}$$

To obtain the differential equation for $M_{12}(y_1, x_2)$, one substitutes equation (III. 1. 14) into equation (III. 1. 11) and obtains

$$\begin{aligned} -\frac{d}{dy_1} M_{21}(y_1, x_2) &= M_{21}(y_1, x_2)A(y_1) \\ &\quad + M_{21}(y_1, x_2)B(y_1)M_{11}(y_1, x_2) \end{aligned} \qquad \text{(III. 1. 17)}$$

with the initial condition given by

$$M_{21}(y_1, x_2)\big|_{y_1=x_2} = I \tag{III. 1. 18}$$

With the parallel derivation as in the previous case, the following can easily be deduced:

$$\begin{aligned}\frac{d}{dy_2} M_{22}(x_1, y_2) = {} & B(y_2)+A(y_2)M_{22}(x_1, y_2) \\ & + M_{22}(x_1, y_2)D(y_2) \\ & + M_{22}(x_1, y_2)C(y_2)M_{22}(x_1, y_2)\end{aligned} \tag{III. 1. 19}$$

with the initial condition given by

$$M_{22}(x_1, y_2)\big|_{y_2=x_1} = 0 \tag{III. 1. 20}$$

and

$$\begin{aligned}\frac{d}{dy_2} M_{12}(x_1, y_2) = {} & M_{12}(x_1, y_2)D(y_2) \\ & + M_{12}(x_1, y_2)C(x_2)M_{22}(x_1, y_2)\end{aligned} \tag{III. 1. 21}$$

with the initial condition given by

$$M_{12}(x_1, y_2)\big|_{y_2=x_1} = I \ . \tag{III. 1. 22}$$

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