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DUALITIES IN GENERALIZED MANIFOLDS AND HIGHER DIMENSIONAL CYCLIC ELEMENT THEORY

by

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DUALITIES IN GENERALIZED MANIFOLDS AND HIGHER DIMENSIONAL CYCLIC ELEMENT THEORY

The purpose of this dissertation is to analyze the structure of the homology and cohomology groups of a space by means of (1) a fundamental system of cycles and (2) higher order cyclic element theory. The fundamental system of cycles is used to extend the Poincare and Alexander dualities to non-perfectly normal generalized manifolds. Our homology and cohomology is always taken over a field.

First, a fundamental system of cycles is defined for general Hausdorff spaces. This definition preserves the essential properties of the fundamental system of cycles, as defined by R. L. Wilder, for compact G_δ subsets of an lc^n space. Let M be a Hausdorff space and let $\{\gamma^r_\alpha\}$ be a basis for $H^r(M)$, where $H^r(M)$ is the cohomology space of M taken over a field A. Then, by reference to duality theorems of S. Lefschetz, there exists, for each γ^r_α, a cycle $Z^\alpha_r \in H_r(M)$ such that

$$Z^\alpha_r \cdot \gamma^r_\beta = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases} .$$

The set $\{Z^\alpha_r\}$ is called the fundamental system of cycles for M. A complete uniform structure is placed on $H_r(M)$ such that the space of linear combinations of the set $\{Z^\alpha_r\}$ is dense in $H_r(M)$ relative to the uniform

topology. Suppose we have a mapping which assigns to each γ_α^r an element $a_\alpha \in A$, where A is the coefficient field for the homology. By the duality of S. Lefschetz, there exists an element $Z_r \in H_r(M)$ such that $Z_r . \gamma_\alpha^r = a_\alpha$. The infinite sum $\sum_\alpha a_\alpha Z_r^\alpha$ is defined to be this element of $H_r(M)$.

Our main application of the fundamental system of cycles is in the proof of the Alexander and Poincare dualities for non-perfectly normal generalized manifolds. A second application shows the representation of $H^r(M)$ as a weak direct sum of spaces $H^r(M_\rho)$ induces a representation of $H_r(M)$ as the strong direct product of the spaces $H_r(M_\rho)$ (each M_ρ is a closed subset of M).

The second part of this dissertation extends the higher order cyclic element theory of Whyburn to compact Hausdorff spaces. Whyburn's theory is restricted to compact subsets of Euclidean n-space. The main result shows that if E is any closed subset of a compact space M, where E contains all the (r-1)-dimensional cyclic elements of M, then the mapping of $H_r(E)$ into $H_r(M)$, induced by inclusion, is an isomorphism onto. Also, if $E_{r-1}^1, \ldots, E_{r-1}^n$ are (r-1)-dimensional cyclic elements of M, then the inclusion mappings of $H_r(E_{r-1}^i)$ into $H_r(M)$ induce an isomorphism of $\prod_1^n H_r(E_{r-1}^i)$ into $H_r(M)$. Some other relations between homology and cyclic elements are discussed.

Whyburn has shown that 0-dimensional cyclic elements have special properties in locally connected spaces. We will show that some of these properties do not generalize

to higher dimensional cyclic elements. For example, if M is compact and locally connected, and f is a monotone mapping of M onto N, then for every 0-dimensional cyclic element E_0^N in N there exists a 0-dimensional cyclic element E_0^M in M such that $f(E_0^M) \supset E_0^N$. Also, the 0-dimensional cyclic elements of M form a null sequence. These results are not true in higher dimensions.

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TABLE OF CONTENTS

Page

INTRODUCTION 1

Chapter

I. COHOMOLOGY AND HOMOLOGY IN GENERAL SPACES . . . 8

II. APPLICATIONS TO THE TOPOLOGY OF MANIFOLDS . . . 29

III. HIGHER DIMENSIONAL CYCLIC ELEMENT THEORY IN COMPACT SPACES 39

IV. HIGHER DIMENSIONAL CYCLIC ELEMENTS IN LOCALLY CONNECTED SPACES. 58

BIBLIOGRAPHY 68

INTRODUCTION

It is the purpose of this dissertation to employ the recent advances that have been made in topology (1) to remove the restriction of perfect normality in the basic dualities for orientable generalized manifolds and (2) to place on an intrinsic basis and extend to general compact spaces the higher dimensional cyclic element theory that Whyburn created over twenty years ago for compact subsets of Euclidean space.[1]

The dualities for generalized manifolds have been given by Wilder, but under the assumption that the spaces considered were perfectly normal (that is, the spaces are normal and every closed subset is a G_δ subset).[2] In chapter II this requirement is shown to be unnecessary. Furthermore, the proofs are placed on an intrinsic basis. This is in contrast to the methods employed by Roberts in his dissertation which depended on imbedding in lc^n spaces.[3]

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[1] G. T. Whyburn, Higher Order Cyclic Element Theory, American Journal of Mathematics, Vol. 56 (1935), pp. 133-146.

[2] R. L. Wilder, Topology of Manifolds (hereafter denoted by T.M.) American Mathematical Society Colloquium Publications, 1949, p. 185.

[3] R. A. Roberts, Dualities in Generalized Manifolds, Doctoral Dissertation, University of Michigan, (1953).

Chapter I is devoted to the preliminary material on homology and cohomology needed in the later chapters, particularly to the development of the notion of a fundamental system of cycles. The fundamental system of cycles provides a method for analyzing the structure of the homology space $H_r(M)$ and for analyzing the relation between the homology space $H_r(M)$ and the cohomology space $H^r(M)$. In the sequel the homology and cohomology will be taken over the same field A as coefficient group.

The construction of a fundamental system of n-dimensional cycles as given by Wilder requires, again, that M be a compact Hausdorff space and also that M be imbedded as a G_δ set in a locally compact lc^n space. Under these conditions the construction of a fundamental system of cycles produces the following structure.

1. A metric is assigned to the space $H_r(M)$. This metric space is complete.

2. A countable set $\{Z_r^i\}$ of cycles, called the fundamental cycles of M, is defined. If $H_r^f(M)$ is the subspace of $H_r(M)$ which is generated by all linear combinations of the set $\{Z_r^i\}$, then $H_r^f(M)$ is dense in $H_r(M)$. Let $\{a_i\} \in \prod_i^\infty A_i$, where $A_i = A$ is the coefficient field of $H_r(M)$ and $H^r(M)$. Let $S_n = \sum_i^n a_i Z_r^i$. Then the sequence $\{S_n\}$ is a Cauchy sequence in $H_r(M)$, hence converges to an element $Z_r \in H_r(M)$. Therefore, a mapping $\psi: \prod_i^\infty A_i \longrightarrow H_r(M)$ can be defined. This mapping is an isomorphism onto. Thus, any $Z_r \in H_r(M)$ is uniquely represented as an "infinite sum"

of the form $\sum_{i}^{\infty} a_i Z_r^i$.

3. A set $\{\gamma_i^r\}$ of cocycles of M is defined such that $\gamma_i^r \cdot Z^j = \delta_{ij}$. The set γ_i^r is a basis for $H^r(M)$. Then, if $\rho(\gamma_i^r) = Z_r^i$, by linear extension to all of $H^r(M)$ a mapping $\rho: H^r(M) \longrightarrow H_r(M)$ can be defined. This mapping is an isomorphism of $H^r(M)$ into $H_r(M)$, such that $\rho(H^r(M)) = H_r^f(M)$.

In Chapter I these results are extended to the case of a general Hausdorff space M. Dual systems, as defined by S. Lefschetz, are used.[1] Roughly speaking, the vector spaces G and H over a field A form a dual system if G is the direct limit of finite dimensional vector spaces G^ν, and H is the inverse limit of vector spaces H_ν, such that G^ν and H_ν are dually paired by a scalar product. Thus, H corresponds to $H_r(M)$ and G corresponds to $H^r(M)$. By using the duality theory of S. Lefschetz[2] a structure analogous to that described above for $H_r(M)$ and $H^r(M)$ is constructed.

1. A complete uniform structure is placed on H.

2. A dot product between G and H is defined. This gives an orthogonal dual pairing of G and H to A. Let $\{g_\alpha\}$ be a basis (not necessarily countable) for G. Then a mapping $\varphi: G \longrightarrow H$ is defined such that, for every pair

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[1] S. Lefschetz, Algebraic Topology (hereafter denoted by A.T.), American Mathematical Society Colloquium Publications, (1942), p.67.

[2] Ibid., Chapter II.

(g_α, g_β),

$$[\varphi(g_\alpha)]\cdot g_\beta = \begin{cases} 1, \alpha = \beta \\ 0, \alpha \neq \beta \end{cases}.$$

3. If $H^f = \varphi(G)$, then H^f is dense in H. An "infinite sum", $\sum_\alpha a_\alpha \varphi(g_\alpha)$, is defined. This sum will "converge" to an element of H for any $\{a_\alpha\} \in \prod_\alpha A_\alpha$ ($A_\alpha = A$ for all α). Furthermore, $h = \sum_\alpha (h\cdot g_\alpha)\varphi(g_\alpha)$ for all $h \in H$.

Once this algebraic structure has been set up it is applied to the following four homology spaces.

1. A fundamental system of cycles is defined for $H_r(M)$, where M is a Hausdorff space and $H_r(M)$ denotes the Cech homology group over a field.

2. A fundamental system of infinite cycles for $\mathcal{H}_r(S)$ is defined, where S is a locally compact Hausdorff space and $\mathcal{H}_r(S)$ is defined according to Wilder.[1]

3. A fundamental system of cycles for $\tilde{H}_0(M)$ is defined. Here M is a Hausdorff space and $\tilde{H}_0(M)$ is the 0-dimensional reduced homology group on M.

4. Let M be a closed subset of a compact space S. Let $H_r(M;\tilde{S}) = i_*^{-1}(0)$, where $i_*: H_r(M) \longrightarrow H_r(S)$. A fundamental system of cycles for $H_r(M;\tilde{S})$ is defined.

Roberts, using a different method, has constructed a fundamental system of cycles for $H_r(M)$ in case M is a compact subset of a locally compact lc^n space.[2] He has

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[1] R. L. Wilder, T. M., p. 248.

[2] R. A. Roberts, Dualities in Generalized Manifolds.

also obtained a fundamental system of infinite cycles for $\mathcal{F}_r(S)$ in case S is an open, lc^n subset of a compact space.

In Chapter II the fundamental system of cycles is used to remove the restriction of perfect normality in the proofs of the Poincare and Alexander dualities for generalized manifolds. A slight modification of the proof given by Wilder in T. M. for the Poincare duality shows, without the restriction of perfect normality; that if S is an orientable n-gm, then $h^r(S) = h_{n-r}(S)$ for $o \leqq r \leqq n$. Here $h^r(S)$ denotes the space of compact r-cocycles of S and $h_{n-r}(S)$ denotes the space of compact (n-r)-cycles of S.

If S is an orientable n-gcm and M a proper closed subset of S, then the above form of the Poincare duality is used to show that $H^r(M;\tilde{S}) = h_{n-r-1}(S-M;\tilde{S})$. Here $H^r(M;\tilde{S}) = \frac{H^r(M)}{i^*(H^r(S))}$, where $i^*:H^r(S) \longrightarrow H^r(M)$ is induced by inclusion and $h_{n-r-1}(S-M;\tilde{S})$ is the space of compact n-r-1 cycles in S-M which bound in S.

The notion of a fundamental system allows these dualities to be stated in terms of homology alone. For instance, if S is a locally compact space, then a subspace $\mathcal{F}_r^f(S)$ of $\mathcal{F}_r(S)$ is defined such that $\mathcal{F}_r^f(S) = h^r(S)$. Then the Poincare duality becomes $\mathcal{F}_r^f(S) = h_{n-r}(S)$.

Chapters III and IV represent an attempt to generalize and expand the higher dimensional cyclic element theory. This theory, as originally given by Whyburn in 1935, was restricted to compact subsets of Euclidean n-space.[1]

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[1] G. T. Whyburn, Higher Order Cyclic Element Theory, pp. 133-146.

In Chapter III the cyclic element theory is developed for compact Hausdorff spaces. An r-dimensional cyclic element is a maximal set which can not be disconnected by any set which does not carry an essential r-cycle (augmented homology).[1] If r = o, this gives a maximal set which can not be disconnected by a single point. An r-dimensional cyclic element is called an E^{r+1} set. The main theorem states that if E is a closed subset of a compact Hausdorff space M, such that E contains all the E^{r+1} sets in M, then $i_*: H_r(E) \rightarrow H_r(M)$ is an isomorphism onto (i_* is the mapping induced by inclusion). Puckett has shown that this mapping is onto in case M is a compact metric space.[2]

Chapter IV is concerned with the properties of cyclic elements in lc^r spaces. Whyburn gives several useful properties of the 0-dimensional cyclic elements in lc^0 spaces.[3] Unfortunately, some of these properties do not generalize to the higher dimensional cyclic elements. For example, if M is compact and locally connected and if f is a monotone mapping of M onto N, then, for every 0-dimensional cyclic element E_0^N in N, there exists a 0-dimensional cyclic element E_0^M in M such that $f(E_0^M) \supset E_0^N$.

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[1] A cycle Z_r on a space M is essential if Z_r has a carrier K in M such that Z_r does not bound on K.

[2] W. J. Puckett, Jr., A Theorem Concerning Homologies in a Compact Space, Fundamenta Mathematicae, Vol. 31 (1938), p. 22.

[3] G. T. Whyburn, Analytic Topology, American Mathematical Society Colloquium Publications, 1942.

Also, the 0-dimensional cyclic elements of M are locally connected and form a null sequence. It is shown that these results are not true in higher dimensions. There does not seem to be any connection between the r-cut points of M and the Er sets of M. The connection between E_n sets and 0-cut points is useful in the analysis of the structure of cactoids.[1]

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[1] G. T. Whyburn, Analytic Topology, Chapter IX.

CHAPTER I

COHOMOLOGY AND HOMOLOGY IN GENERAL SPACES

1. The definition of a dual system of vector spaces.

DEFINITION 1.1. After S. Lefschetz we call a structure satisfying the three conditions given below a dual system of vector spaces.[1]

1. Let $\{G^\nu, \pi^{\mu\nu}\}$ be a directed system of vector spaces over the directed set V such that each G^ν is a finite dimensional vector space over a field A. Let $G = \varinjlim \{G^\nu, \pi^{\mu\nu}\}$.

2. Let $\{H_\nu, \pi_{\mu\nu}\}$ be an inverse system over V such that each H_ν is a finite dimensional vector space over A. Let $H = \varprojlim \{H_\nu, \pi_{\mu\nu}\}$.

3. For each $\nu \in V$, G^ν and H_ν are orthogonally dually paired to A.[2] If (μ,ν) is a pair with $\nu > \mu$ and if $g^\mu \in G^\mu$ and $h_\nu \in H_\nu$, then $[\pi_{\mu\nu}(h_\nu)] \cdot g^\mu = h_\nu \cdot [\pi^{\nu\mu}(g^\mu)]$.

We define the projections $\pi^\nu: G^\nu \longrightarrow G$ and $\pi_\nu: H \longrightarrow H_\nu$ according to Eilenberg and Steenrod.[3] Thus, $\pi_\nu(h)$ is the

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[1] S. Lefschetz, A. T., p. 67.

[2] R. L. Wilder, T. M., p. 162.

[3] S. Eilenberg and N. Steenrod, Foundations of Algebraic Topology, Princeton (1952), Chapter VIII.

ν^{th} component of h and $\pi^{\nu}(g^{\nu})$ is the "equivalence class determined by g^{ν}". We call G and H the base spaces of the dual system.

Lefschetz has shown that the following scalar product induces a dual pairing between G and H.[1]

DEFINITION 1.2. Given $g \in G$ and $h \in H$ define g.h by selection of a G^{μ} and a $g^{\mu} \in G^{\mu}$ such that $\pi^{\mu}(g^{\mu}) = g$ and then setting $g.h = g^{\mu}.[\pi_{\mu}(h)]$.

2. The construction of a uniform structure on the vector space H of the dual system.

DEFINITION 2.1. For each $\nu \in V$ let $\mathcal{V}_{\nu} = \left\{(h_1, h_2) \middle| \pi_{\nu}(h_1) = \pi_{\nu}(h_2)\right\}$. Thus, $\mathcal{V}_{\nu}$ is a subset of H x H.[2,3]

THEOREM 2.2. The set $\{\mathcal{V}_{\nu}\}$ taken over all $\nu \in V$ forms a base for a uniformity on H.[4]

PROOF. The following statements are easily verified:

(1) each $\mathcal{V}_{\nu}$ contains the diagonal;

(2) for any $\mathcal{V}_{\nu}$, $\mathcal{V}_{\nu} = \mathcal{V}_{\nu}^{-1}$;

(3) for any $\mathcal{V}_{\nu}$, $\mathcal{V}_{\nu} \circ \mathcal{V}_{\nu} = \mathcal{V}_{\nu}$;

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[1] S. Lefschetz, A. T., Chapter II, (20.8).

[2] Roberts, in Dualities in a Gen. Manifold, has defined a similiar topology.

[3] The mapping $\pi_{\nu}: H \longrightarrow H_{\nu}$ is defined in section 1 and V is defined in definition 1.1.

[4] J. L. Kelley, General Topology, New York (1955), p.176.

(4) given any pair $\mathcal{V}_{\nu_1}$, $\mathcal{V}_{\nu_2}$ there exists a $\mathcal{V}_{\nu_3}$ such that $\mathcal{V}_{\nu_3} \subset \mathcal{V}_{\nu_1} \cap \mathcal{V}_{\nu_2}$.

Hence, H is a uniform space with $\{\mathcal{V}_\nu\}$ as neighborhoods of the diagonal. This uniform-space topology is equivalent to the linear topology defined by Lefschetz.[1]

THEOREM 2.3. The above uniform space is Hausdorff.

We shall omit the proof since it is quite straightforward.

THEOREM 2.4. The uniform structure on H is complete.

PROOF. Assume that $\{S_\alpha, \alpha \in D\}$ is a Cauchy net on H (that is, $\{S_\alpha, \alpha \in D\}$ is a net and for any $\mathcal{V}_\nu$ there exists an α_ν such that $\alpha, \beta > \alpha_\nu$ implies $(S_\alpha \times S_\beta) \subset \mathcal{V}_\nu$). We shall define an element $h^o = \{h_\nu^o\} \in \varprojlim \{H_\nu, \pi_{\mu\nu}\}$ and show that the net converges to h^o. Given $\nu \in V$ choose an α_ν such that $S_\alpha \times S_\beta \subset \mathcal{V}_\nu$ for $\alpha, \beta > \alpha_\nu$. Now, choose an $h(\alpha_\nu) \in S_{\alpha_\nu}$. Let $h_\nu^o = \pi_\nu(h(\alpha_\nu))$. We must show that $\nu > \mu$ implies $\pi_{\mu\nu}(h_\nu^o) = h_\mu^o$. Since $\{S_\alpha, \alpha \in D\}$ is a net, we may choose an S_β such that $S_\beta \times S_{\alpha_\nu} \subset \mathcal{V}_\nu$ and $S_\beta \times S_{\alpha_\mu} \subset \mathcal{V}_\mu$. Select an element $h(\beta) \in S_\beta$. Then since $S_\beta \times S_{\alpha_\nu} \subset \mathcal{V}_\nu$, and $h(\alpha_\nu) \in S_{\alpha_\nu}$, we have $\pi_\nu(h(\beta)) = \pi_\nu(h(\alpha_\nu)) = h_\nu^o$. By the same reasoning $\pi_\mu(h(\beta)) = \pi_\mu(h(\alpha_\mu)) = h_\mu^o$. Then $\pi_{\mu\nu}(h_\nu^o) = \pi_{\mu\nu}(\pi_\nu(h(\beta)) = \pi_\mu(h(\beta)) = h_\mu^o$. Thus, $h^o = \{h_\nu^o\} \in H$. Now, we will show that the net converges to h^o. That is, given any neighborhood U of h^o, (here the

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[1] S. Lefschetz, A. T., p. 75.

neighborhood is of course taken in the uniform-space topology) we must show the existence of an α such that $S_\beta \subset U$ for $\beta > \alpha$. Any neighborhood of h^0 can be taken in the form $U_\nu = \{h \mid h \in H, \quad \pi_\nu(h) = \pi_\nu(h^0)\}$, where ν is some element of V. Given U_ν, take α_ν, $h(\alpha_\nu)$ and h_ν as defined above in the construction of h^0. Then $S_\beta \times S_{\alpha_\nu} \subset \mathcal{V}_\nu$ for $\beta > \alpha_\nu$ (that is $\pi_\nu(h) = \pi_\nu(h(\alpha_\nu)) = h^0_\nu = \pi_\nu(h^0)$ for all $h \in S_\beta$). Thus, $S_\beta \subset U_\nu$ for $\beta > \alpha_\nu$. This completes the proof of the theorem. We have shown that H is a Hausdorff, complete uniform space.

3. Construction of an isomorphism $\varphi: G \longrightarrow H$.

LEMMA 3.1. *Let $B = \{g_\alpha\}$ be a basis for G. For any $g_\alpha \in B$, there exists an $h_\alpha \in H$ such that,*

$$g_\beta \cdot h_\alpha = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases},$$

for all $g_\beta \in B$.

PROOF. For any $g_\alpha \in B$, we may define a linear mapping $f_\alpha: G \longrightarrow A$ (A is the coefficient field) by taking for each $g_\beta \in B$

$$f_\alpha(g_\beta) = \begin{cases} 1 \text{ if } \alpha = \beta \\ 0 \text{ if } \alpha \neq \beta \end{cases}$$

and then defining $f_\alpha: G \longrightarrow A$ by linear extension. Since G and H are dually paired, there exists an $h_\alpha \in H$ such that $h_\alpha \cdot g = f_\alpha(g)$ for all $g \in G$.[1] In particular, $h_\alpha \cdot g_\beta =$

$$f_\alpha(g_\beta) = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}.$$

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[1] S. Lefschetz, A. T., Chapt. II.

DEFINITION 3.2. Let $\{g_\alpha\}$ be a basis for G. Then $g_\alpha \longrightarrow h_\alpha$ (h_α is the element of H orthogonal to g_α) defines a mapping of $\{g_\alpha\}$ into H. Let $\varphi: G \longrightarrow H$ be the linear extension of this mapping. Let $\varphi(G)$ be denoted by H^f. Thus, H^f is defined relative to the given basis for G.

THEOREM 3.3. The mapping $\varphi: G \longrightarrow H$ is an isomorphism into; furthermore, φ is unique relative to a fixed basis $\{g_\alpha\}$ under the requirement that

$$\left[\varphi(g_\alpha)\right] \cdot g_\beta = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

PROOF. First, φ is an isomorphism. Suppose $\varphi(g) = 0$ for some $g \in G$. Now, $g = \sum_i^n a_i g_{\alpha_1}$, where $g_{\alpha_1} \in \{g_\alpha\}$ for $1 \leq i \leq n$. But then

$$\left[\varphi(g)\right] \cdot g_{\alpha_i} = \varphi\left[\sum_i^n a_i g_{\alpha_i}\right] \cdot g_{\alpha_1} = a_1 = 0 \text{ for } 1 \leq i \leq n.$$

Hence, $g = 0$.

Suppose there were two mappings φ_1 and φ_2 such that

$$\left[\varphi_1(g_\alpha)\right] \cdot g_\beta = \left[\varphi_2(g_\alpha)\right] \cdot g_\beta = \begin{cases} 0, & \alpha \neq \beta \\ 1, & \alpha = \beta \end{cases}.$$

Then $\left[\varphi_1(g_\alpha) - \varphi_2(g_\alpha)\right] \cdot g_\beta = 0$ for all g_β, and thus $\varphi_1(g_\alpha) = \varphi_2(g_\alpha)$.

LEMMA 3.4. Let $\{h_p\}$ be a subset of H such that, for any $\nu \in V$, $\pi_\nu(h_p) = 0$ for all h_p, except those in a finite set $(h_{p_1}, \ldots, h_{p_n})$. Then there exists an element $h^0 \in H$ such that $\pi_\nu(h^0) = \pi_\nu(h_{p_1}) + \ldots + \pi_\nu(h_{p_n})$.

PROOF. Define $h^0 = \pi_\nu(h_{p_1}) + \ldots + \pi_\nu(h_{p_n})$. Then

$\{h_\nu^0\}$ determines an element h^0 of H, since $\pi_{\mu\nu}(h_\nu^0) = h_\mu^0$ if $\nu > \mu$. This follows from the fact that $\pi_\nu(h_p) = 0$ implies $\pi_\mu(h_p) = 0$.

DEFINITION 3.5. If the set $\{h_p\}$ of elements of H satisfies the condition stated in lemma 3.4, then define $\sum_p(h_p)$ to be the element $h^0 \in H$ whose existence is demonstrated in lemma 3.4. Then we say that $\sum_p h_p$ converges to h^0.

LEMMA 3.6. Let $\{g_\alpha\} = B$ be a basis for G. Let φ be the mapping defined in definition 3.2. Then for each $\nu \in V$, $\pi_\nu(\varphi(g_\alpha)) = 0$, except for a finite set $(g_{\alpha_1}, \cdots, g_{\alpha_n})$.

PROOF. Since G^ν has finite dimension, there is a finite subset $(g_{\alpha_1}, \cdots, g_{\alpha_n})$ of $B = \{g_\alpha\}$ whose linear combinations generate $\pi^\nu(G^\nu)$. Thus, for any $g^\nu \in G^\nu$, we have $\pi^\nu(g^\nu) = \sum_i^n a_i g_{\alpha_i}$. From lemma 3.1 we have $[\pi_\nu(\varphi(g_\beta))] \cdot g^\nu = [\varphi(g_\beta)] \cdot [\pi^\nu(g^\nu)] = [\varphi(g_\beta)] \cdot \left[\sum_i^n a_i g_{\alpha_i}\right] = \sum_i^n a_i [\varphi(g_\beta)] \cdot g_{\alpha_i} = 0$ if $g_\beta \notin (g_{\alpha_1}, \cdots, g_{\alpha_n})$. But G^ν is orthogonal to H_ν, and therefore $\pi_\nu(\varphi(g_\beta)) = 0$ if $g_\beta \notin (g_{\alpha_1}, \cdots, g_{\alpha_n})$.

LEMMA 3.7. Let $\{g_\alpha\}$ be a basis for G. Then $h = \sum_\alpha (h \cdot g_\alpha)\varphi(g_\alpha)$ for all $h \in H$. Furthermore, all sums of the form $\sum_\alpha a_\alpha \varphi(g_\alpha)$ (where $a_\alpha \in A$) converge to an element of H.[1]

PROOF. The convergence of any sum of the form $\sum_\alpha a_\alpha \varphi(g_\alpha)$ is assured by lemma 3.6. If h is an element of

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[1] S. Lefschetz, A. T., p. 80. Lefschetz has proven this lemma using a different definition of the infinite sum $\sum_\alpha a_\alpha \varphi(g_\alpha)$.

H, let $h^0 = \sum_\alpha (h \cdot g_\alpha)\rho(g_\alpha)$. Then by definition 3.5 and lemma 3.6 there is, for each ν, a set $(g_{\alpha_1}, \cdots, g_{\alpha_n})$ such that $\pi_\nu(h^0) = \sum_i^n (h \cdot g_\alpha)\pi_\nu(\rho(g_{\alpha_i}))$. For any $g_\beta \in B$, there is a $\nu_0 \in V$ and a $g_\beta^{\nu_0} \in G^{\nu_0}$ with $\pi^{\nu_0}(g_\beta^{\nu_0}) = g_\beta$. Then $[\pi_{\nu_0}(\rho(g_\beta))] \cdot g_\beta^{\nu_0} = [\rho(g_\beta)] \cdot [\pi^{\nu_0} g_\beta^{\nu_0}] = [\rho(g_\beta)] \cdot g_\beta = 1$; hence, $\pi_{\nu_0}\rho(g_\beta) \neq 0$. Thus, we may assume $g_{\alpha_1} = g_\beta$ in the representation $\pi_{\nu_0}(h^0) = \sum_i^n (h \cdot g_{\alpha_i})\pi_{\nu_0}(\rho(g_{\alpha_i}))$. Therefore, $g_\beta \cdot h^0 = g_\beta^{\nu_0} \cdot [\pi_{\nu_0}(h^0)] = \sum_i^n (h \cdot g_{\alpha_i})[\pi_{\nu_0}(\rho(g_{\alpha_i}))] \cdot g_\beta^{\nu_0} = h \cdot g_\beta$. Thus, $h^0 \cdot g_\beta = h \cdot g_\beta$ for all g_β, which shows that $h = \sum_\alpha (h \cdot g_\alpha)\rho(g_\alpha)$.

THEOREM 3.8. The space H^f defined in definition 3.2 is dense in H (relative to the uniform-space topology).

PROOF. We must show that any neighborhood of any point $h^0 \in H$ contains a point of H^f. If $h^0 \in H$, then any neighborhood of h^0 is of the form $U_\nu(h^0) = \{h \mid \pi_\nu(h) = \pi_\nu(h^0)\}$, where ν is some element of V. By lemma 3.7 $\pi_\nu(h^0) = \sum_i^n (h^0 \cdot g_{\alpha_i})\pi_\nu(\rho(g_{\alpha_i}))$. But $\sum_i^n (h^0 \cdot g_{\alpha_i})\rho(g_{\alpha_i})$ is an element of H^f, and therefore H^f is dense in H.

THEOREM 3.9. If $(h^1, \cdots h^k)$ are linearly independent elements of H, then there exists a basis $(g_{\alpha_1}, \cdots g_{\alpha_k}, \cdots g_\alpha \cdots)$ for G such that $\rho(g_{\alpha_i}) = h^i$ for $1 \leq i \leq k$.

PROOF. By theorem 18.25, p. 164, in T. M. and the dual pairing of G and H there exists a subset $(g_{\alpha_1}, \ldots, g_{\alpha_k})$ of G such that $h^i \cdot g_{\alpha_j} = \delta_{ij}$. Consider all sets $C_\gamma \subset G$ such that:

(1) $(g_{\alpha_1}, \ldots, g_{\alpha_k}) \subset C_\gamma$;

(2) the elements of C_γ are linearly independent:

(3) if $g \in C_\gamma$ and $g \notin (g_{\alpha_1}, \ldots, g_{\alpha_k})$, then $g.h_i = 0$ for $1 \leq i \leq k$.

Order the sets C_γ by inclusion, that is $C_\beta > C_\gamma$ if $C_\gamma \supset C_{\gamma'}$. Then, given any simply ordered collection of the sets C_γ, the union of the elements of the collection produces an upper bound for the collection. The union is easily seen to be a C_γ set. There exists at least one C_γ set, namely $(g_{\alpha_1}, \ldots, g_{\alpha_k})$. Hence, the hypothesis of Zorn's Lemma is satisfied, and therefore the collection of sets $\{C_\beta\}$ has a maximal element. Denote one such maximal element by C_m. Then C_m is a basis for G. We know by definition that the elements of C_m are linearly independent. We must show C_m generates G. If not, then we could augment C_m by an element $\tilde{g}_o \in G$ such that $\tilde{g}_o$ was independent of the elements of C_m. Let $g_o = \tilde{g}_o - \sum_i (h^i . \tilde{g}_o) g_{\alpha_i}$. Then g_o is independent of the elements of C_m. Also, $g_o . h^i = 0$ for $1 \leq i \leq k$. Hence, C_m is a basis.

By lemma 3.4 there exists a mapping $\varphi: G \longrightarrow H$ such that

$$\varphi(g_\alpha) . g_\beta = \begin{cases} 0, & \alpha \neq \beta \\ 1, & \alpha = \beta \end{cases},$$

where $\{g_\alpha\}$ denotes the elements of the basis C_m. Since

$$[\varphi(g_{\alpha_i})] . g_\alpha = \begin{cases} 0, & \alpha \neq \alpha_i \\ 1, & \alpha = \alpha_i \end{cases}$$

for $1 \leq i \leq k$, the uniqueness of φ implies that $\varphi(g_{\alpha_i}) = h_i$.

REMARK. Theorem 3.9 does not hold for a countably infinite set of linearly independent elements of H. We

will construct an example to demonstrate this fact. First, we must construct a dual system of vector spaces. Take the directed set V to be the set of positive integers with the usual ordering. For each $n \in V$ let $G_n = \prod_{k=1}^{n} Z_2^k = H_n$, where $Z_2^k = Z_2$ is the field of integers modulo two. Let H_n and G_n be paired to Z_2 by the usual scalar product. If $m>n$ then $H_m = H_n \times \prod_{n+1}^{m} Z_2^k$. The projection π_{nm} of H_m into H_n is defined by mapping $\prod_{n+1}^{m} Z_2^k$ into zero and mapping H_n onto H_n by the identity. Since V is well ordered, π_{mn} can be defined such that the required permanence relations hold. The mapping of G_n into G_m is defined to be the dual of π_{nm}.

Choose a basis $\{g_i\}$ for $G = \varinjlim \{G_n, \pi_{nm}^*\}$. This basis will be countably infinite. Let $\{h_i\}$ be the fundamental system dual to $\{g_i\}$; that is, $h_i = \varphi(g_i)$, where φ is the mapping defined in definition 3.6. We have shown in lemma 3.7 that any $h \in H = \varprojlim \{H_n, \pi_{nm}\}$ can be represented uniquely by the infinite sum $\sum_i a_i h_i$, where $a_i = h \cdot g_i$. Conversely, any infinite sum $\sum_i a_i h_i$ is equal to an element $h \in H$ with $h \cdot g_i = a_i$.

For each positive integer n define $\bar{h}_n = \sum_i^{\infty} a_{nj} h_j$, where

$$a_{nj} = 1 \quad n \neq j$$
$$a_{nn} = 0 \quad n = n$$

We will show that the elements $\bar{h}_n$ are linearly independent. Suppose $\bar{h} = \sum_i^{k} \bar{h}_{n_i} = 0$. Now, $g_{n_1} \cdot \bar{h} = g_{n_1} \cdot \left[\sum_i^{k} h_{n_i}\right] = \sum_i^{k} g_{n_1} \cdot h_{n_i} = \sum_i^{k} a_{n_i n_1} = k-1$, since $a_{n_i n_1} = 1$ for $n_i \neq n_1$ and $a_{n_1 n_1} = 0$.

But $\bar{h} = 0$, and therefore $g_{n_1} \cdot \bar{h} = k-1 = 0$. This shows that k must be odd. We may assume $n_k + 1 \neq n_i$ for $i = 1, 2, \ldots, k$. Then we have $(g_{n_k+1}) \cdot \bar{h} = \sum_{i=1}^{k} a_{n_i n_k+1} = k = 0$. But this is not possible, and therefore the elements $\{\bar{h}_n\}$ are independent.

Suppose there exist elements $\bar{g}_i \in G$ such that $\bar{g}_i \cdot \bar{h}_j = \delta_{ij}$. Since $\{g_i\}$ is a basis, we would have $\bar{g}_1 = \sum_{i=1}^{k} a_i g_i$ for some set $(a_1, \ldots . a_k)$. We have $\bar{g}_1 \cdot \bar{h}_n = \sum_{i=1}^{k} a_i g_i \bar{h}_n = \sum_{i=1}^{k} a_i a_{n_i} = a_1 + a_2 + \ldots + a_{n-1} + 0 + a_{n+1} + \ldots + a_k$, since

$$a_{n_i} = \begin{cases} 1, i \neq n \\ 0, i = n \end{cases}.$$

Therefore, we have

$$\bar{g}_1 \cdot \bar{h}_1 = 0 + a_2 + \ldots + a_n = 1$$

$$\bar{g}_1 \cdot \bar{h}_2 = a_1 + 0 + a_3 + \ldots + a_n = 0$$

$$\bar{g}_1 \cdot \bar{h}_n = a_1 + a_2 + \ldots + 0 = 0$$

This implies $(n-1)(a_1 + \ldots + a_n) = 1$. But $\bar{g}_1 \cdot \bar{h}_{n+1} = a_1 + a_2 + \ldots + a_n = 0$. Thus, we can not have elements $\bar{g}_i$ such that $\bar{h}_i \cdot \bar{g}_j = \delta_{ij}$.

REMARK. One attempt to set up a fundamental system of cycles led us to the consideration of a problem concerning direct limits. Suppose $\{F_\nu, \pi_{\nu\mu}\}$ and $\{G_\nu, \tau_{\nu\mu}\}$ are direct systems over a directed set V such that:

(1) $F_\nu = G_\nu$ for all $\nu \in V$;

(2) F_ν is a finite dimensional vector space over a fixed field A for all $\nu \in V$;

(3) $\pi_{\nu\mu}$ and $\tau_{\nu\mu}$ are isomorphisms into for any $\nu, \mu \in V$. Does it follow that $\varinjlim \{G_\nu, \tau_{\nu\mu}\} = \varinjlim \{H_\nu, \pi_{\mu\nu}\}$?

We will construct an example in which $\varinjlim\{G_\nu, \tau_{\nu\mu}\} \neq \varinjlim\{H_\nu, \pi_{\nu\mu}\}$.

Let $\mathfrak{H} = \{\theta\}$ be the set of all ordinals of the first and second kind. Then $\mathfrak{H}$ is uncountable. For each positive integer n take

$$V_n = \left\{(\theta_1, \theta_2, \ldots \theta_n) \middle| \theta_i \in \mathfrak{H}, \theta_i \neq \theta_j \text{ for } i \neq j\right\}.$$

Then define $V = \bigcup_{n=1}^{\infty} V_n$. Define an order relation in V by setting $(\theta_1, \ldots, \theta_n) < (\theta_1^1, \ldots \theta_m^1)$ if and only if for each $i \leq n$ there exists a θ_j^1 such that $\theta_i = \theta_j^1$ (that is $\bigcup_1^n \theta_i \subset \bigcup_1^m {}^1_j$). Then V becomes a directed set under "$<$", since, for ν, μ, and $\rho \in V$:

(1) $\nu < \nu$;

(2) $\nu < \mu$, $\nu < \rho$ implies $\nu < \rho$;

(3) given ν, μ there exists a ρ such that $\rho > \nu$ and $\rho > \nu$.

If $\nu = (\theta_1, \ldots, \theta_n)$, then define $F_\nu = G_\nu = \prod_1^n Z_2^i$, where $Z_2^i = Z_2$ represents the integers modulo two. Then a basis for $F_\nu = G_\nu$ is:

$$f_1^\nu = g_1^\nu = (1, 0, \ldots, 0)$$
$$f_2^\nu = g_2^\nu = (0, 1, 0 \ldots, 0)$$
$$f_n^\nu = g_n^\nu = (0, \ldots 0, 1).$$

For $\mu > \nu$ define $\tau_{\mu\nu}: G_\nu \longrightarrow G_\mu$ by $\tau_{\mu\nu}(g_i^\nu) = g_i^\mu$ (extend linearly to all of G_ν). Then we can show:

(1) $\tau_{\nu\mu}$ is an isomorphism into;

(2) if $\mu > \nu > \rho$, then $\tau_{\mu\rho} = \tau_{\mu\nu}\tau_{\nu\rho}$

If $\nu = (\theta_1, \ldots, \theta_n)$, $\mu = (\theta_1^1, \theta_2^1, \ldots \theta_m^1)$ and $\mu > \nu$, then define $\pi_{\mu\nu}: F_\nu \longrightarrow F_\mu$ by $\pi_{\mu\nu}(f_i^\nu) = f_j^\mu$, where j is such that $\theta_i = \theta_j^1$ (extend linearly to all of H_ν). From

the definition of the partial order in V we can show:

(1) $\pi_{\mu\nu}$ is well defined;

(2) $\pi_{\mu\nu}$ is an isomorphism;

(3) $\mu > \nu > \rho$ implies $\pi_{\mu\rho} = \pi_{\mu\nu}\pi_{\nu\rho}$

Now, we will show that $\varinjlim\{G_\nu, \tau_{\nu\mu}\}$ is countable. For each $n \geq 1$ choose a $\nu_n = (\theta_1, \ldots \theta_n) \in V$. Then the set

$$\bigcup_{n=1}^{\infty}\left[\bigcup_{i=1}^{n} \bar{g}_i^{\nu_n}\right]$$

generates $\varinjlim\{G_\nu, \tau_{\nu\mu}\}$ ($\bar{g}_i^{\nu_n}$ is the element of $\varinjlim\{G_\nu, \tau_{\nu\mu}\}$ determined by $g_i^{\nu_n}$). For, given any $\mu = (\theta_1^1, \ldots, \theta_n^1)$, there exists a $\rho \in V$ such that $\rho > \mu$ and $\rho > \nu_n$. Consequently, $\tau_{\rho\mu}(g_1^\mu) = g_1^\rho = \tau_{\rho\nu_n}(g_1^{\nu_n})$, and therefore $\bar{g}_1^\mu = g_1^{\nu_n}$. But, the set $\{\bar{g}_1^\mu\}$ taken over all μ, and all allowable i, generates $\varinjlim\{G_\nu, \tau_{\nu\mu}\}$. Therefore,

$$\bigcup_{n=1}^{\infty}\left[\bigcup_{i=1}^{n} \bar{g}_i^{\nu_n}\right]$$

generates $\varinjlim\{G_\nu, \tau_{\nu\mu}\}$. Since we are working over Z_2, $\varinjlim\{G_\nu, \tau_{\mu\nu}\}$ is countable.

Next, we will show that $\varinjlim\{F_\nu, \pi_{\nu\mu}\}$ is uncountable. If $\nu = (\theta_1)$ and $\mu = (\theta_2)$, then $\bar{f}_1^\nu \neq \bar{f}_1^\mu$, for otherwise we would have an element $\rho = (\theta_1^1, \ldots, \theta_n^1)$ such that $\pi_{\rho\nu}(f_1^\nu) = \pi_{\rho\mu}(f_1^\mu)$. But this would imply the existence of an integer j such that $\theta_1 = \theta_j^1 = \theta_2$, which is impossible. Since V is uncountable, $\varinjlim\{F_\nu, \pi_{\mu\nu}\}$ must also be uncountable.

4. Applications to homology theory.

THEOREM 4.1. Let G be the weak direct sum $\sum_{p \in P} G_p$

of subspaces G_ρ *(that is, for any* $g \in G$ *there is a unique representation* $g = g_{\rho_1} + \ldots + g_{\rho_n}$, *where* $g_{\rho_1} \in G_{\rho_1}$*). For each* $\rho \in P$, *let* $H_\rho \subset H$ *be the annihilator of* $\sum_{\rho' \in P-\rho} G_{\rho'}$. *Then H is isomorphic to the strong direct product of the spaces* H_ρ. *If* $\{h_\rho\} \in \prod_{\rho \in P} H_\rho$, *then the image of* $\{h_\rho\}$ *is* $\sum_{\rho \in P} h_\rho$, *where the infinite sum is defined in definition 3.5.*

PROOF. Choose a basis B_ρ for each space G_ρ. Then $B = \bigcup_{\rho \in P} B_\rho$ is a basis for G since $G = \sum_P G_\rho$. Let $\varphi: G \longrightarrow H$ be the isomorphism defined relative to the basis B according to lemma 3.1. We will show that $h \in H_\rho$ if and only if

$$h = \sum_{g_\alpha \in B_\rho} a_\alpha \varphi(g_\alpha)$$

for some set $\{a_\alpha\}$, where $a_\alpha \in A$. If $g_\beta \in B_{\rho'}$ $(\rho \neq \rho')$, then

$$g_\beta \cdot h = g_\beta \cdot \left[\sum_{g_\alpha \in B_\rho} a_\alpha \varphi(g_\alpha)\right] = \sum_{g_\alpha \in B_\rho} a_\alpha g_\beta \cdot \varphi(g_\alpha) = 0,$$

since $B_{\rho'}$ and B_ρ are disjoint. Thus, h annihilates $\sum_{\rho' \in P-\rho} G_{\rho'}$. Any $h \in H$ can be represented as

$$h = \sum_{g_\alpha \in B_\rho} (h \cdot g_\alpha) \varphi(g_\alpha).$$

If h annihilates $\sum_{\rho' \in P-\rho} G_{\rho'}$, then $h \cdot g_\alpha = 0$ for all $g_\alpha \in B - B_\rho$, and therefore

$$h = \sum_{g_\alpha \in B_\rho} (h \cdot g_\alpha) \varphi(g_\alpha).$$

Let $h \in H$ be represented by

$$h = \sum_{g_\alpha \in B} (h \cdot g_\alpha) \varphi(g_\alpha).$$

For this given h, let

$$h_\rho = \sum_{g_\alpha \in B_\rho} (h \cdot g_\alpha)\rho(g_\alpha).$$

It is easily seen from definition 3.5 and lemma 3.6 that

$$h = \sum_{g_\alpha \in B} (h \cdot g_\alpha)\rho(g_\alpha) = \sum_{\rho} \sum_{g_\alpha \in B_\rho} (h \cdot g_\alpha)\rho(g_\alpha) = \sum_{\rho} h_\rho$$

Note that $h_\rho \in H_\rho$. For any $h_\rho \in H_\rho$, we have

$$h_\rho = \sum_{g_\alpha \in B_\rho} (h_\rho \cdot g_\alpha)\rho(g_\alpha).$$

Therefore, the mapping of H into $\prod_\rho H_\rho$, induced by the representation of $h \in H$ as $\sum_\rho h_\rho$, is an isomorphism onto.

COROLLARY 4.2. Let M be a compact Hausdorff space and let $\{M_\rho\}$, for $\rho \in P$, be a collection of closed subsets of M. Let $i_\rho^*: H^r(M) \longrightarrow H^r(M_\rho)$ be the mapping induced by inclusion. For each $\gamma^r \in H^r(M)$, suppose $i_\rho^*(\gamma^r) = 0$ except for a finite set $i_{\rho_1}^*(\gamma^r), \ldots, i_{\rho_n}^*(\gamma^r)$. Also, suppose there exists for any $\gamma_\rho^r \in H^r(M_\rho)$ a $\gamma^r \in H^r(M)$ such that $i_\rho^*(\gamma^r) = \gamma_\rho^r$ and $i_{\rho'}^*(\gamma^r) = 0$ if $\rho \neq \rho'$. In other words assume the mappings $\{i_\rho^*\}$ define an isomorphism of $H^r(M)$ onto the weak direct sum $\sum_{\rho \in P} H^r(M_\rho)$ of the spaces $H^r(M_\rho)$. Under this assumption $H^r(M)$ is isomorphic to the strong direct product $\prod_{\rho \in P} H_r(M_\rho)$. Furthermore, this isomorphism is natural in the sense that the image of an element $\{Z_r^\rho\} \in \prod_{\rho \in P} H_r(M_\rho)$ is $\sum_{\rho \in P} i_*^\rho(Z_r^\rho)$, where the infinite sum is defined in definition 3.5.

PROOF. By assumption, we have an isomorphism $\chi: H^r(M) \longrightarrow \sum_{\rho \in P} H^r(M)$) such that $\chi(\gamma^r) = \sum_{\rho \in P} i_\rho^*(\gamma^r)$. Let $G = H^r(M)$, $H = H_r(M)$, and $G_\rho = \chi^{-1}(H^r(M)$). Then G is the weak direct sum of the subspaces G_ρ and we can apply theorem 4.1.

If $Z_r = i_{\rho *}^{\rho}(Z_r^{\rho})$, for some $Z_r^{\rho} \in H_r(M_\rho)$, then $Z_r . \gamma^r = i_{\rho *}^{\rho}(Z_r^{\rho}) . \gamma^r = Z_r . i_{\rho *}^{\rho}(\gamma^r)$, for any $\gamma^r \in H^r(M) = G$. For any $\gamma^r \in \sum_{\rho' \in P-\rho} G_{\rho'}$, we have $i_\rho^*(\gamma^r) = 0$. Thus, $i_{\rho *}^{\rho}(H_r(M_\rho))$ is contained in the annihilator of $\sum_{\rho' \in P-\rho} G_{\rho'}$. Assume a basis B_ρ for G_ρ and a basis B of G have been chosen according to theorem 4.1. Let H_ρ be the annihilator of $\sum_{\rho' \in P-\rho} G_{\rho'}$. Then if $h_\rho \in H_\rho$, we have

$$h_\rho = \sum_{g_\alpha \in B} (h . g_\alpha) \varphi(g_\alpha), \text{ where } \varphi \text{ is}$$

defined in theorem 4.1. If B_ρ is the basis for G_ρ, then the set $\{i_\rho^*(g_\alpha)\}$, taken over all $g_\alpha \in B_\rho$, is a basis for $H^r(M_\rho)$. Therefore, we can define a mapping $\varphi' : H^r(M_\rho) \longrightarrow H_r(M)$, relative to $\{i_\rho^*(g_\alpha)\}$ for $g_\alpha \in B_\rho$, according to lemma 3.1. Then

$$\bar{h} = \sum_{g_\alpha \in B_\rho} (h . g_\alpha) \varphi'(i_\rho^*(g_\alpha))$$

is an element of $H_r(M_\rho)$. Now, $g_\alpha . i_{\rho *}^{\rho}(h_\rho) = 0$ if $g_\alpha \notin B$, since $i_\rho^*(H^r(M_\rho))$ is contained in the annihilator of $\sum_{\rho' \in P-\rho} G_{\rho'}$. Also, $g_\alpha . [i_\rho^*(\bar{h}_\rho)] = i_\rho^*(g_\alpha) . \bar{h}_\rho = h_\rho . g_\alpha$ for $g_\alpha \in B$. Therefore $i_{\rho *}^{\rho}(\bar{h}_\rho) = h_\rho$. Hence, $i_{\rho *}^{\rho}(H_r(M_\rho)) = H_\rho$ is the annihilator of $\sum_{\rho' \in P-\rho} G$. By theorem 4.1 $H = \prod_\rho H_\rho = \prod_\rho i_{\rho *}^{\rho}(H_r(M_\rho))$. Furthermore, by theorem 4.1, the image of any $\{h_\rho\} \in \prod_\rho i_{\rho *}^{\rho}(H_r(M_\rho))$ is $\sum_\rho h_\rho$, where this sum is defined in definition 3.5.

THEOREM 4.3. *Let* $H = \varprojlim \{ H_\nu, \pi_{\mu\nu} \}$ *and* $G = \varinjlim \{ G^\nu, \pi^{\mu\nu} \}$ *be base spaces of a dual system over the directed set* V. *Assume* $\bar{H} = \varprojlim \{ \bar{H}_\nu, \bar{\pi}_{\mu\nu} \}$ *and* $\bar{G} = \varinjlim \{ \bar{G}^\nu, \bar{\pi}^{\mu\nu} \}$ *are also base spaces of another dual system over the same directed set* V. *Let* $f_*: H \longrightarrow \bar{H}$ *and* $f^*: \bar{G} \longrightarrow G$ *be defined by mappings* $f_\nu: H_\nu \longrightarrow \bar{H}_\nu$ *and* $f^\nu: \bar{G}^\nu \longrightarrow G^\nu$ *such that* $\bar{\pi}_{\mu\nu}(f_\nu(h_\nu)) = f_\mu(\pi_{\mu\nu}(h_\nu))$ *and* $\pi^{\nu\mu}(f^\mu(\bar{g}^\mu)) = f^\nu(\bar{\pi}^{\nu\mu}(\bar{g}^\mu))$ *for* $\nu > \mu$. *Also, assume that* f_ν *and* f^ν *are dual mappings. Define* $\tilde{H} = f_*^{-1}(0)$ *and* $\tilde{G} = \frac{G}{f^*(\bar{G})}$. *Then* $\tilde{H}$ *and* $\tilde{G}$ *are base spaces for a dual system of vector spaces.*

PROOF. We must construct the required dual system. Let V be the directed set for the dual system. Define $\tilde{H}_\nu = (f^\nu)^{-1}(\theta)$. Then $\tilde{H}_\nu \subset H_\nu$ is a finite dimensional vector space. Let $\tilde{\pi}_{\mu\nu}$ be the restriction of $\pi_{\mu\nu}$ to $\tilde{H}_\nu$. Then $\tilde{\pi}_{\mu\nu}$ maps $\tilde{H}_\nu$ into $\tilde{H}_\mu$ for $\nu > \mu$, since $\bar{\pi}_{\mu\nu}(f_\nu(h_\nu)) = f_\mu(\pi_{\mu\nu}(h_\nu))$. Also, $\tilde{\pi}_{\nu_1\nu_2}\tilde{\pi}_{\nu_2\nu_3} = \tilde{\pi}_{\nu_1\nu_3}$ for $\nu_3 > \nu_2 > \nu_1$. Thus, $\{\tilde{H}_\nu, \tilde{\pi}_{\mu\nu}\}$ is an inverse system. If $h \in \tilde{H}$, then $h = \{h_\nu\}$ and $f_*(h) = \{f_\nu(h_\nu)\} = 0$. Then $f_\nu(h_\nu) = 0$ for all $\nu \in V$, and hence, $h_\nu \in \varprojlim\{\tilde{H}_\nu, \tilde{\pi}_{\mu\nu}\}$. If $\{h_\nu\} \in \varprojlim\{\tilde{H}_\nu, \tilde{\pi}_{\mu\nu}\}$, then $\{h_\nu\} \in \tilde{H}$, since $f^*(\{h_\nu\}) = \{f_\nu(h_\nu)\} = 0$. Thus $\tilde{H} = \varprojlim\{\tilde{H}_\nu, \tilde{\pi}_{\mu\nu}\}$.

Let $\tilde{G}^\nu = \frac{G^\nu}{f^\nu(\bar{G}^\nu)}$. If $\nu > \mu$, then $\pi^{\nu\mu}(f^\mu(\bar{G}^\mu)) = f^\nu\bar{\pi}^{\nu\mu}(\bar{G}^\mu) \subset f^\nu(\bar{G}^\nu)$. Therefore, $\pi^{\nu\mu}$ induces a map $\tilde{\pi}^{\nu\mu}: \tilde{G}^\mu \longrightarrow \tilde{G}^\nu$. Consequently, $\tilde{\pi}^{\nu_1\nu_2}\tilde{\pi}^{\nu_2\nu_3} = \tilde{\pi}^{\nu_1\nu_3}$ for $\nu_1 > \nu_2 > \nu_3$. Also, $\tilde{G}^\nu$ is a finite dimensional vector space. Thus $\{\tilde{G}^\nu, \tilde{\pi}^{\mu\nu}\}$ is a direct system. We will show $\tilde{G}$ is isomorphic to $\varinjlim\{\tilde{G}^\nu, \tilde{\pi}^{\mu\nu}\}$. If $\tilde{g} \in \tilde{G}$, then choose a representative $g \in G$

of $\tilde{g}$. Then choose a representative g^ν of g. Consider g^ν as an element $\tilde{g}^\nu$ of $\tilde{G}^\nu$. Let $\varphi(\tilde{g}) = \tilde{\pi}^\nu(\tilde{g}^\nu)$. If $g_1 = g_2 + f^*(\bar{g})$ for some $\bar{g} \in \bar{G}$, and if $g_1^{\nu_1}$ and $g_2^{\nu_2}$ are representatives of g_1 and g_2, then there exists a μ, ν_1, ν_2 such that $\pi^{\mu\nu_1}(g_1^{\nu_1}) = \pi^{\mu\nu_2}(g_2^{\nu_2}) + f^\mu \bar{g}^\mu$. Thus, φ is well defined since $\pi^{\nu_1}(g_1^{\nu_1}) = \pi^\mu \pi^{\mu\nu_1}(g_1^{\nu_1}) = \pi^\mu \pi^{\mu\nu_2}(g_2^{\nu_2}) + \pi^\mu f^\mu(\bar{g}^\mu) = \pi^{\nu_2}(g_2^{\nu_2}) + f^*(\bar{g})$. It is not difficult to show that φ is an isomorphism onto. Thus, $\tilde{G} = \varinjlim\{\tilde{G}^\nu, \tilde{\pi}^{\nu\mu}\}$.

Now, we will show that $\tilde{H}_\nu$ and $\tilde{G}_\nu$ are orthogonally dually paired. First, define a dot product between $\tilde{H}_\nu$ and $\tilde{G}^\nu$. Given $\tilde{H}_\nu \in \tilde{H}_\nu$ and $\tilde{g}^\nu \in \tilde{G}^\nu$ choose a representative $g^\nu \in G^\nu$ of $\tilde{g}^\nu$, and then define $\tilde{h}_\nu \cdot \tilde{g}^\nu = \tilde{h}_\nu \cdot g^\nu$ (note that $\tilde{h}_\nu \in H_\nu$ and H_ν and G^ν are orthogonally dually paired). If $g_1^\nu = g_2^\nu + f^\nu(\bar{g}^\nu)$, then $\tilde{h}_\nu \cdot g_1^\nu = \tilde{h}_\nu \cdot g_2^\nu + \tilde{h}_\nu \cdot f^\nu(\bar{g}^\nu) = \tilde{h}_\nu \cdot g_2^\nu + f_\nu(\tilde{h}_\nu) \cdot \bar{g}^\nu = \tilde{h}_\nu \cdot g_2^\nu$. Therefore, this dot product is well defined. If $\tilde{h}_\nu \neq 0$, then there exists a $g^\nu \in G^\nu$ such that $\tilde{h}_\nu \cdot g^\nu = 1$. Let $\tilde{g}^\nu$ be the element of $\tilde{G}^\nu$ determined by g^ν, then $\tilde{h}_\nu \cdot \tilde{g}^\nu = \tilde{h}_\nu \cdot g^\nu = 1$. If $\tilde{g}^\nu \neq 0$, we will show the existence of an $\tilde{h}_\nu$ such that $\tilde{h}_\nu \cdot \tilde{g}^\nu \neq 0$. Choose a representative $g^\nu \in G^\nu$. Then $g^\nu = \sum_{i=1}^{n} a_i g_i^\nu$, where $\{g_i^\nu\}$ is a basis for G^ν. Assume we have chosen this basis such that $g_1^\nu, \ldots, g_s^\nu$ is a basis for $f^\nu(\bar{G}^\nu)$. Since $\tilde{g}^\nu \neq 0$, $g^\nu \notin f^\nu(\bar{G}^\nu)$. Therefore, $a_{i_0} \neq 0$ for some i_0's. Let $H_\nu^{i_0} \in H_\nu$ such that $h_\nu^{i_0} \cdot g_j^\nu = \delta_{i_0 j}$ (note that G^ν and H_ν are orthogonally dually paired). Then $h_\nu^{i_0} \cdot g^\nu = a^{i_0} \neq 0$. Also, $f_\nu(h_\nu^{i_0}) = 0$, for if not, then there exists a $\bar{g}^\nu$ with $f_\nu(h_\nu^{i_0}) \cdot \bar{g}^\nu \neq 0$ (note that $\bar{G}^\nu$ and $\bar{H}_\nu$ are orthogonally dually paired). But, $f_\nu(h_\nu^{i_0}) \cdot \bar{g}^\nu =$

$h^i_{\nu_0}.f^\nu(\bar{g}^\nu) \neq 0$ and $f^\nu(\bar{g}^\nu) = \sum_1^s a_i g_i^\nu$ implies $h^i_\nu.f^\nu(\bar{g}^\nu) = 0$. Thus, $h^i_{\nu_0} \in \tilde{H}_\nu$, and therefore $\tilde{G}^\nu$ and $\tilde{H}_\nu$ are orthogonally dually paired. Also, $\tilde{\pi}_{\mu\nu}(\tilde{h}_\nu).\tilde{g}^\mu = h_\nu . \tilde{\pi}^{\nu\mu}(\tilde{g}^\mu)$ for any $\tilde{h}_\nu \in \tilde{H}_\nu$ and any $\tilde{g}^\mu \in \tilde{G}^\mu$ since $\pi_{\mu\nu}(h_\nu).g^\mu = h_\nu . \pi^{\nu\mu}(g^\mu)$ for any $h_\nu \in H_\nu$ and any $g^\mu \in G^\mu$. Thus, we have constructed a system with base spaces $\tilde{G}$ and $\tilde{H}$.

Let M be a compact Hausdorff space. Let $H_r(M)$ and $H^r(M)$ be the r-dimensional Cech homology and cohomology spaces taken over a field A. Then $H_r(M) = \varprojlim \{H_r(\mathcal{U}), \pi_{\mathcal{U}\mathcal{V}}\}$ and $H^r(M) = \varinjlim \{H^r(\mathcal{U}), \pi^*_{\mathcal{U}\mathcal{V}}\}$, where $H_r(\mathcal{U})$ and $H^r(\mathcal{U})$ are finite dimensional vector spaces over A and $\pi_{\mathcal{U}\mathcal{V}}$ and $\pi^*_{\mathcal{U}\mathcal{V}}$ are linear mappings. It is well known that $H^r(\mathcal{U})$ and $H_r(\mathcal{U})$ are orthogonally dually paired to A. If $\mathcal{U}, \mathcal{V}$, $Z_r(\mathcal{U}) \in H_r(M)$, and $\gamma^r(\mathcal{V}) \in H^r(\mathcal{V})$, then $Z_r(\mathcal{U}).\pi^*_{\mathcal{U}\mathcal{V}}(\gamma^r(\mathcal{V})) = \pi_{\mathcal{U}\mathcal{V}}(Z_r(\mathcal{U}).\gamma^r(\mathcal{V}))$.[1] If we let $H_r(M) = H$ and $G = H^r(M)$, we obtain a dual system of vector spaces. It must be noted that the definition of the dot product given in definition 1.2 is exactly the same as the usual dot product between $H_r(M)$ and $H^r(M)$. Therefore, the theorems proven above all hold when H and G are replaced by $H_r(M)$ and $H^r(M)$, that is, $H_r(M)$ and $H^r(M)$ are the base spaces for a dual system.

DEFINITION 4.4. Let M be a compact Hausdorff space. Given a basis $\{\gamma^r_\alpha\}$ for $H^r(M)$, a mapping $\varphi: H^r(M) \longrightarrow H_r(M)$ may be defined by definition 3.2. The set $\{\varphi(\gamma^r_\alpha)\}$ is called a fundamental system of cycles for $H_r(M)$. Define $H^f_r(M)$ to be $\varphi(H^r(M))$.

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[1] R. L. Wilder, T. M., pp. 160-161.

Let S be a locally compact Hausdorff space. Let $\hat{S}$ be the one point compactification of S by the ideal point $\hat{p}$. Define the "infinite homology" space of S to be $H_r(\hat{S},\hat{p})$ and denote it by $\mathcal{H}_r(S)$. Define compact cohomology as $H^r(\hat{S},\hat{p})$ and denote it by $h^r(S)$. These definitions agree with those given in Chapter VIII of T. M. Let $\mathcal{U}$ be a finite covering of $\hat{S}$. Let $\hat{S}_{\mathcal{U}}$ be the nerve of $\mathcal{U}$. Let $\hat{p}_{\mathcal{U}}$ be the subcomplex of $\hat{S}_{\mathcal{U}}$ composed of simplexes which are on $\hat{p}$. Let $H_r(\hat{S}_{\mathcal{U}}, \hat{p}_{\mathcal{U}})$ be the r-dimensional homology space of $\hat{S}_{\mathcal{U}}$, modulo $\hat{p}_{\mathcal{U}}$, taken over the field A. If $\mathcal{M} > \mathcal{U}$ let $\pi_{\mathcal{U}\mathcal{M}}: H_r(\hat{S}_{\mathcal{M}}, \hat{p}_{\mathcal{M}}) \longrightarrow H_r(\hat{S}_{\mathcal{U}}, \hat{p}_{\mathcal{U}})$ be the map induced by the projection of $\mathcal{M}$ into $\mathcal{U}$. Then we have an inverse system $\{H_r(\hat{S}_{\mathcal{U}}, \hat{p}_{\mathcal{U}}), \pi_{\mathcal{U}\mathcal{M}}\}$. By the definition of $H_r(\hat{S}, \hat{p})$, $\mathcal{H}_r(S) = H_r(\hat{S},\hat{p}) = \varprojlim\{H_r(\hat{S}_{\mathcal{U}},\hat{p}_{\mathcal{U}}), \pi_{\mathcal{U}\mathcal{M}}\}$. We may define $H^r(\hat{S}_{\mathcal{U}}, \hat{p}_{\mathcal{U}})$ and $\pi^*_{\mathcal{U}\mathcal{M}}$ in a manner analogous to that given above for homology. Then $\{H^r(\hat{S}_{\mathcal{U}}, \hat{p}_{\mathcal{U}}), \pi^*_{\mathcal{U}\mathcal{M}}\}$ is a direct system. By the definition of $H^r(\hat{S}, \hat{p})$, $h^r(S) = H^r(\hat{S}, \hat{p}) = \varinjlim\{H^r(\hat{S}_{\mathcal{U}}, \hat{p}_{\mathcal{M}}), \pi^*_{\mathcal{U}\mathcal{M}}\}$. By exactly the same reasoning as used above for the case of $H_r(M)$ and $H^r(M)$, we can show that $\mathcal{H}_r(S)$ and $h^r(S)$ may be taken as the base spaces, H and G, of a dual system of vector spaces.

DEFINITION 4.5. Let $\{\gamma^r_\alpha\}$ be a basis for $h^r(S)$ and let $\varphi: h^r(S) \longrightarrow \mathcal{H}_r(S)$ be the associated isomorphism. Define $\mathcal{H}^f_r(S) = \varphi(h^r(S))$ and call $\{\varphi(\gamma^r_\alpha)\}$ a fundamental system of cycles for $\mathcal{H}_r(S)$.

We will now show that the 0-dimensional reduced

homology and cohomology, $\tilde{H}_0(M)$ and $\tilde{H}^0(M)$, over a Hausdorff space M are the base spaces for a dual system. We will use theorem 4.3 to prove this.

Let P_0 be an ideal space consisting of a single point. Define a mapping $F:M \longrightarrow P_0$ by $F(x) = P_0$ for all $x \in M$. Then F induces mappings $F_*:H_0(M) \longrightarrow H_0(P_0)$ and $F^*:H^0(P_0) \longrightarrow H^0(M)$. The reduced 0-dimensional homology and cohomology spaces are defined to be $\tilde{H}_0(M) = F_x^{-1}(o)$ and

$$\tilde{H}^0(M) = \frac{H^0(M)}{F^*(H^0(P^0))} .$$

Let $\{\mathcal{U}\}$ be the set of finite open coverings of M. Let $\{\mathcal{U}\}$ be the directed set V of theorem 4.3. Then let $H = H_0(M) = \varprojlim\{H_0(\mathcal{U}), \pi_{\mathcal{V}\mathcal{U}}\}$ and $G = H^0(M) = \varinjlim\{H^0(\mathcal{U}), \pi^*_{\mathcal{V}\mathcal{U}}\}$. Then H and G are base spaces of a fundamental system over $\{\mathcal{U}\} = V$. For each $\mathcal{U}$, let $\bar{H}_0(\mathcal{U}) = H_0(P_0)$ and $\bar{H}^0(\mathcal{U}) = H^0(P_0)$. For $\mathcal{W} > \mathcal{U}$, let $\bar{\pi}_{\mathcal{U}\mathcal{W}}$ and $\bar{\pi}^*_{\mathcal{W}\mathcal{U}}$ be the identity mappings. Let $\bar{H} = \varprojlim\{\bar{H}_0(\mathcal{U}), \bar{\pi}_{\mathcal{U}\mathcal{W}}\}$ and $\bar{G} = \varinjlim\{\bar{H}^0(\mathcal{U}), \bar{\pi}^*_{\mathcal{U}\mathcal{W}}\}$. Then $\bar{H}$ and $\bar{G}$ are base spaces of a dual system over $\{\mathcal{U}\} = V$. It is obvious that $\bar{H} = H_0(P_0)$ and $\bar{G} = H^0(P_0)$. For each $\mathcal{U}$, let $f_{\mathcal{U}}:H_0(\mathcal{U}) \longrightarrow \bar{H}_0(\mathcal{U})$ and $f^{\mathcal{U}}:\bar{H}^0(\mathcal{U}) \longrightarrow H^0(\mathcal{U})$ be the mappings induced by the simplicial map which carries each element of $\mathcal{U}$ into P_0. Then $f_{\mathcal{U}}$ and $f^{\mathcal{U}}$ are dual mappings such that $\bar{\pi}_{\mathcal{U}\mathcal{W}} f_{\mathcal{W}} = f_{\mathcal{U}} \pi_{\mathcal{U}\mathcal{W}}$ and $\bar{\pi}^*_{\mathcal{W}\mathcal{U}} f^{\mathcal{U}} = f^{\mathcal{W}} \pi^*_{\mathcal{W}\mathcal{U}}$. Therefore, $\{f_{\mathcal{U}}\}$ and $\{f^{\mathcal{U}}\}$ induce mappings $f_*:H = H_0(M) \longrightarrow \bar{H} = H_0(P_0)$ and $f^*:\bar{G} = H^0(P_0) \longrightarrow H^0(M) = G$. We will show that $f_* = F_*$ and $f^* = F^*$. Let $\mathcal{U}_M$ be the covering of M consisting of

the single open set M. Let $f_{\mathcal{U}_M}$ be the isomorphism of $H_0(\mathcal{U}_M)$ onto $H_0(P_0)$ defined by the simplicial map which takes the element M of $\mathcal{U}_M$ into P_0. If $Z_0 \varepsilon H_0(M)$ then $F_*(Z_0) = f_{\mathcal{U}_M}(Z_0(\mathcal{U}_M))$. The mapping $\overline{\pi}_{\mathcal{U}_M \mathcal{U}}$ is the identity. Therefore, for any $Z_0(\mathcal{U}) \varepsilon H_0(\mathcal{U})$, $f_{\mathcal{U}_M}(\overline{\pi}_{\mathcal{U}_M \mathcal{U}}(Z_0(\mathcal{U}))) = f_{\mathcal{U}_M}(Z_0(\mathcal{U}_M)) = F_*(Z_0) = \overline{\pi}_{\mathcal{U}_M \mathcal{U}}(f_{\mathcal{U}}(Z_0(\mathcal{U}))) = f_{\mathcal{U}}(Z_0(\mathcal{U}))$. But, $f_*(Z_0) = \{f_{\mathcal{U}}(Z_0(\mathcal{U}))\} = \{F_*(Z_0)\} = F_*(Z_0)$. Similarly we can show that $f^* = F^*$. Hence, $\tilde{H} = f_*^{-1}(0) = F_*^{-1}(0) = H_0(M)$ and

$$\tilde{G} = \frac{G}{f^*(G)} = \frac{H^0(M)}{f^*(H^0(P_0))} = \frac{H^0(M)}{F^*(H^0(P_0))} = \tilde{H}^0(M).$$

By theorem 4.3 $\tilde{G}$ and $\tilde{H}$ are base spaces for a dual system of vector spaces. Therefore, the same is true of $\tilde{H}_0(M)$ and $\tilde{G}^0(M)$.

DEFINITION 4.6. Let M be a Hausdorff space. Let $\{\tilde{\gamma}^0_\alpha\}$ be a basis for $\tilde{H}^0(M)$ and let $\varphi: \tilde{H}^0(M) \longrightarrow \tilde{H}_0(M)$ be the associated isomorphism. The elements, $\{\varphi(\tilde{\gamma}^0_\alpha)\}$, of $\tilde{H}_0(M)$ are the fundamental reduced 0-cycles on M. The space $\tilde{H}^f_0(M)$ is defined to be $\varphi(\tilde{H}^0(M))$.

CHAPTER II

APPLICATIONS TO THE TOPOLOGY OF MANIFOLDS

Next we shall apply the fundamental system in the proof of the Alexander and Poincare dualities. Assume Q and P are open sets in a locally compact space S such that $\overline{Q} < P$ and $\overline{P}$ is compact. Let $H^r(S,Q,P)$ denote the space of compact r-cocycles in Q modulo compact cohomology in P. Let $h_r(S,Q,P)$ denote the space of compact r-cycles in Q modulo compact homology in P. Assume S is an orientable n-gm and take Γ^1_n to be the fundamental n-cycle on S.[1] It is not difficult to see that the cap product induces a linear mapping $\varphi: h^r(S,Q,P) \longrightarrow h_{n-r}(S,Q,P)$.[2] This mapping is defined by $\varphi(\gamma^r) = \gamma^r \cap \Gamma_n$.

LEMMA 1.1. *Let S be an orientable n-gm. Let Q and P be open sets in S such that $\overline{Q} < \overline{P}$ and $\overline{P}$ is compact. Then the mapping $\varphi: h^r(S,Q,P) \longrightarrow h_{n-r}(S,Q,P)$ defined above is an isomorphism onto for $0 \leq r \leq n$.*

PROOF. The components of S are open since S is

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[1] R. L. Wilder, T. M., p. 250.

[2] Ibid., p. 248.

locally connected. Therefore, only a finite number of the components of S meet $\overline{P}$. Thus, $P = \bigcup_1^S P_i$, where P_i is open and closed in P. Also, $Q = \bigcup_1^S Q_i$, where $Q_i \subset P_i$ and Q_i is open and closed in Q. Let C_i be the component of S which contains P_i. Then the closure of P_i is compact and $\overline{Q}_i \subset P_i$. Any component of an orientable n-gm is a connected orientable n-gm. We will take the fundamental cycle, Γ_n^i, of C_i to be the restriction of Γ_n to C_i. By lemma 5.6, p.256, in T. M. there exists a mapping of $h^r(S,Q_i,P_i)$ onto $h_{n-r}(S,Q_i,P_i)$. By lemmas 5.7 and 5.8, p. 247, lemma 18.31, p. 166, and theorem 1.1, p. 244, in T. M., there exists a mapping of $h_{n-r}(S,Q_i,P_i)$ onto $h^r(S,Q_i,P_i)$. But $h^r(S,Q_i, P_i)$ is a finite dimensional vector space by theorem 1.1, p. 244, in T. M. Therefore, the vector spaces $h_{n-r}(S,Q_i,P_i)$ and $h^r(S,Q_i,P_i)$ have the same dimension. The map $\varphi_i: h^r(S,Q_i,P_i) \longrightarrow h_{n-r}(S,Q_i,P_i)$ of lemma 5.6 in T.M., defined by $\varphi_i(\gamma_i^r) = \gamma_i^r \cap \Gamma_n^i$, is onto. Therefore, the equality of the dimensions of the two spaces implies that φ_i is an isomorphism. The lemmas of T.M. used above do not require that S be perfectly normal.

It is not difficult to show that $h^r(S,Q,P) = \prod_1^S h^r(S,Q_i,P_i)$ and $h_{n-r}(S,Q,P) = \prod_1^S h_{n-r}(S,Q_i,P_i)$. Then any $\gamma^r \in h^r(S,Q,P)$ is represented uniquely as $\gamma^r = \gamma_1^r + \cdots + \gamma_s^r$, where $\gamma_i^r \in h^r(S,Q_i,P_i)$. Now, $\varphi(\gamma^r) = \sum_1^S \varphi_i(\gamma_i^r)$ and, since each $\varphi_i: h^r(S,Q_i,P_i) \longrightarrow h_{n-r}(S,Q_i,P_i)$ is an isomorphism onto, φ is an isomorphism onto.

THEOREM 1.2. Let S be an orientable n-gm (S is

not assumed to be connected). Then $h^r(S) = h_{n-r}(S)$, for $0 \leq r \leq n$.

PROOF. We will define a mapping $\chi: h^r(S) \longrightarrow h_{n-r}(S)$. For any $\gamma^r \in h^r(S)$, let $\chi(\gamma^r) = \gamma^r \cap \Gamma_n$, where Γ_n is the fundamental n-cycle on S.[1] Then χ is onto, since any $Z_{n-r} \in h_{n-r}(S)$ has a representative $Z^1_{n-r} \in h_{n-r}(S,Q,P)$, for some Q and P satisfying the conditions of lemma 1.1. By lemma 1.1 there exists a $\gamma^r_1 \in h^r(S,Q,P)$ such that $\gamma^r_1 \cap \Gamma_n = Z^1_{n-r}$. But γ^r_1 determines an element γ^r of $h^r(S)$, and $\chi(\gamma^r) = \gamma^r \cap \Gamma_n = Z_{n-r}$. Thus χ is onto.

If $\chi(\gamma^r) = Z_{n-r} = 0$, then Z_{n-r} has a representative $Z^1_{n-r} \in h_{n-r}(S,Q,P)$, for some P and Q satisfying the conditions of lemma 1.1, such that $Z^1_{n-r} = 0$. We may enlarge Q and P so that γ^r has a representative γ^r_1 in $h^r(S,Q,P)$. Then $\gamma^r_1 \cap \Gamma_n = Z^1_{n-r} = 0$, and thus by lemma 1.1 $\gamma^r_1 = 0$, which implies $\gamma^r = 0$.

COROLLARY 1.3. Under the conditions of theorem 1.2 $\mathcal{H}^f_r(S) = h_{n-r}(S)$, for $0 \leq r \leq n$.

PROOF. This follows immediately from lemma 4.5 in Chapter I.

LEMMA 1.4. If M is a proper closed subset of an orientable n-gcm S then, S-M is an orientable n-gm.

PROOF. Since S-M is an open subset of S, conditions

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[1] R. L. Wilder, T. M., p. 248.

A, B and C on p. 244 in T. M. are obviously satisfied.

We will demonstrate the existence of an infinite cycle, $\hat{\Gamma}_n \in \mathcal{F}_n(S-M)$, such that $\hat{\Gamma}_n \neq 0$, modulo $(S-M)-U$, for any open set U (non-empty) in S-M. It is easy to see that the fundamental n-cycle Γ_n on S determines an element $\hat{\Gamma}_n$ of $\mathcal{F}_n(S-M)$. If $\hat{\Gamma}_n = 0$, modulo $(S-M)-U$, then it is easy to prove $\Gamma_n = 0$, modulo S-U. But this is impossible by the definition of Γ_n.

We will show that condition D' is satisfied for each component C of S-M.[1] Let $\hat{C}$ be the one point compactification of C by $\hat{p}$. Suppose $Z_r \in \mathcal{F}_n(C) = H_n(\hat{C},\hat{p})$, where Z_n is on a proper closed subset $\hat{F}$ of $\hat{C}$. Define $F = \hat{F}-\hat{p}$. Let $f:S \to \hat{C}$ be defined by $f(x) = \hat{p}$ for $x \in S-C$ and $f(x) = x$ for $x \in C$. Then f is continuous, since S is lc^0 and C is a component of a open set in S. In addition, f is a homeomorphism of $S-(S-C) = C$ onto $\hat{C}-\hat{p} = C$. Also, f induces the mappings f_*^1 and f_* shown below. These mappings are isomorphisms onto, since f is a relative homeomorphism. The diagram below is commutative.

$$\begin{array}{ccc} H_n(C,F) & \xleftarrow{i_*} & H_n(C,\hat{p}) \\ \uparrow{\scriptstyle f_*^1} & & \uparrow{\scriptstyle f_*} \\ H_n(S-(S-C),F) & \xleftarrow{i_*^1} & H_n(S,S-C) \end{array}$$

There exists a $Z_n^s \in H_n(S,S-C)$ such that $f_*(Z_n^s) = Z_n$. Since

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[1] R. L. Wilder, T. M., p. 254.

Z_n is on F, $i_*f_*(Z_n^s) = 0 = f_*^1 i_*^1(Z_n^s)$. Since f_*^1 is an isomorphism this implies $i_*^1(Z_n^s) = 0$. By lemma 2.3, p. 205, in T. M. there exists a minimal closed set K such that $i_*^\pi(Z_n^s) = 0$ in $H_n(S,K)$, where $i_*^\pi: H_n(S,S-C) \longrightarrow H_n(S,K)$, and $(S-C) \cup F \supset K \supset S-C$. Obviously, $K \neq S-C$ since $Z_n^s \neq 0$. Also, $K \cap C \neq C$ since F is a proper subset of C. Thus $K \cap C \neq \phi$ and $K \cap C \neq C$. Since C is connected $F(K) \cap C \neq \phi$. For any $\chi \in F(K) \cap C$ choose U and V to be neighborhoods of χ such that $\bar{V} < U$ and $P_n(S,V,U) = 1$.[1] Let Γ_n be the fundamental n-cycle of S. Since C is open, we may choose U such that $S-U \supset S-C$. Then Z_n^s may be regarded as a cycle modulo (S-U). Thus $aZ_n^s = b\Gamma_n = 0$, modulo (S-V), regarding Γ_n as a cycle modulo (S-U) (a and b not both zero). If $b = 0$, then $aZ_n^s = 0$ in V and thus K is not minimal, since $\chi \in F(K)$ (note that $b = 0$ implies $a \neq 0$). Therefore $b \neq 0$, and $\Gamma_n = \frac{a}{b} Z_n^s$. But this is impossible since $Z_n^s = 0$, modulo (S-Q), where Q is open and $Q \subset (S-K) \cap V$.[2]

Next, we will prove a form of the Alexander duality. First, we will define a dual system of vector spaces.

DEFINITION 1.5. Let M be a closed subset of a compact space S. Let $i^*: H^r(S) \longrightarrow H^r(M)$ and $i_*: H_r(M) \longrightarrow H_r(S)$ be the maps induced by inclusion. Define $H_r(M;\tilde{s}) = i_*^{-1}(0)$ and

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[1] R. L. Wilder, T. M., p. 191.

[2] Ibid., p. 214.

$$H^r(M;\hat{s}) = \frac{H^r(M)}{i^*(H^r(S))}$$

We will show $H^r(M;\hat{s})$ and $H_r(M;\hat{s})$ are base spaces of a dual system. Let $G = H^r(M) = \varinjlim \{H^r(\mathcal{U}_M), \pi^*_{\mathcal{U}M}\}$, $H = H_r(M) = \varprojlim \{H_r(\mathcal{U}_M), \pi_{\mathcal{U}M}\}$, $\bar{G} = H^r(S) = \varinjlim \{H^r(\mathcal{U}_S), \pi^*_{\mathcal{U}M}\}$ and $\bar{H} = H_r(S) = \varprojlim \{H_r(\mathcal{U}_S), \pi_{\mathcal{U}M}\}$. Then G and H are base spaces for a dual system over the directed set $\{\mathcal{U}\} = V$. The same is true of $\bar{G}$ and $\bar{H}$. Also, the mappings $i^*:\bar{G} = H^r(S) \longrightarrow H^r(M) = G$ and $i_*:H = H_r(M) \longrightarrow H_r(S) = \bar{H}$ satisfy the hypothesis of theorem 4.3 in Chapter I. That is, i^* and i_* are determined by mappings $i^{\mathcal{U}}:H^r(\mathcal{U}_S) \longrightarrow H^r(\mathcal{U}_M)$ and $i_{\mathcal{U}}:H_r(\mathcal{U}_M) \longrightarrow H_r(\mathcal{U}_S)$, where $i^{\mathcal{U}}$ is dual to $i_{\mathcal{U}}$. Then, by theorem 4.3, $\tilde{H} = i_*^{-1}(0) = H_r(M;\tilde{s})$ and

$$\tilde{G} = \frac{G}{i^*(\bar{G})} = \frac{H^r(M)}{i^*(H^r(S))} = H^r(M;\hat{s})$$

are base spaces for a dual system.

DEFINITION 1.6. Let $\{\gamma^r_\alpha\}$ be a basis for $H^r(M;\tilde{s})$ and let $\varphi:H^r(M;\hat{s}) \longrightarrow H_r(M;\tilde{s})$ be the associated isomorphism. Then the elements of the set $\{\varphi(\gamma^r_\alpha)\}$ are called the fundamental cycles of $H_r(M;\tilde{s})$ and $\varphi(H^r(M;\tilde{s}) = H^f_r(M;\tilde{s})$.

DEFINITION 1.7. Let M be a closed subset of a compact space S. Let $j^*:h^r(S-M) \longrightarrow H^r(S)$ and $K_*:h_r(S-M) \longrightarrow H_r(S)$ be injections. Then define $h^r(S-M;\hat{s})$ to be $(j^*)^{-1}(0)$ and $h_r(S-M;\tilde{s})$ to be $K_*^{-1}(0)$.

LEMMA 1.8. Under the conditions in definition 1.7 $h^{r+1}(S-M;\tilde{s}) = H^r(M;\tilde{s})$.

PROOF. Consider the exact sequence:
$H^{r+1}(S) \xleftarrow{j^*} h^{r+1}(S-M) \xleftarrow{\delta} H^r(M) \xleftarrow{i^*} H^r(S)$. Since $\delta i^*(H^r(s)) = 0$ and $j^* \delta (H^r(M)) = 0$, δ induces a mapping $\tilde{\delta}$ of

$$H^r(M;\hat{s}) = \frac{H^r(M)}{i^*(H^r(S))}$$

into $h^{r+1}(S-M;\tilde{s})$. Since $h^{r+1}(S-M;\hat{s}) = (j^*)^{-1}(0)$ and since the above sequence is exact, $\tilde{\delta}$ is onto. That is, given any $\gamma^{r+1} \in h^{r+1}(S-M)$ such that $j^*(\gamma^{r+1}) = 0$ then there exists a $\gamma^r \in H^r(M)$ such that $\delta(\gamma^r) = \gamma^{r+1}$. Also, $\tilde{\delta}$ is an isomorphism, since $\tilde{\delta}(\gamma^r) = 0$ for $\gamma^r \in H^r(M;\hat{s})$ implies γ^r has a representative $\gamma^r_1 \in H^r(M)$ such that $\delta(\gamma^r_1) = 0$. But, by exactness, $\gamma^r_1 \in i^*(H^r(S))$. Therefore, $\gamma^r = 0$ in $H^r(M;\hat{s})$.

LEMMA 1.9. Let M be a compact proper closed subset of an orientable n-gcm S. Then $h^r(S-M;\hat{s}) = h_{n-r}(S-M;\tilde{s})$.

PROOF. By lemma 1.4 S-M is an orientable n-gm. Thus $\varphi: h^r(S-M) \longrightarrow h_{n-r}(S-M)$, defined by $\varphi(\gamma^r) = \gamma^r \cap \hat{\Gamma}_n$, is an isomorphism onto ($\hat{\Gamma}_n$ is the fundamental infinite n-cycle on S-M). We will show that the restriction of φ to $h^r(S-M;\hat{s})$ defines an isomorphism of $h^r(S-M;\hat{s})$ onto $h_{n-r}(S-M;\tilde{s})$. Let $\tilde{\varphi}$ be the restriction of φ to $h^r(S-M;\hat{s})$. Note that $\varphi(\gamma^r) = \gamma^r \cap \hat{\Gamma}_n = \gamma^r \cap \Gamma_H$, where Γ_n is a fundamental n-cycle on S. If we let $\psi: H^r(S) \longrightarrow H_{n-r}(S)$ be defined by $\psi(\gamma^r) = \gamma^r \cap \Gamma_n$, for $\gamma^r \in H^r(S)$, then the diagram below is commutative (note that j^* and k_* are injections).

$$\begin{array}{ccc} h^r(S-M) & \xrightarrow{\varphi} & h_{n-r}(S-M) \\ \uparrow{\scriptstyle j^*} & & \uparrow{\scriptstyle k^*} \\ H^r(S) & \xrightarrow{\gamma} & H_{n-r}(S) \end{array}$$

Wilder has shown that the mapping γ is an isomorphism of $H^r(S)$ onto $H_{n-r}(S)$.[1] If $\gamma^r \in h^r(S-M;\tilde{s})$, then $j^*(\gamma^r) = 0$. Therefore, by the commutativity of the diagram, $k_*(\varphi(\gamma^r)) = \gamma(j^*(\gamma^r)) = 0$. Thus $\varphi(\gamma^r) \in h_{n-r}(S-M;\tilde{s})$ and $\tilde{\varphi}$ maps $h^r(S-M;\tilde{s})$ into $h_{n-r}(S-M;\tilde{s})$. If $Z_{n-r} \in h_{n-r}(S-M;\tilde{s}) \subset h_{n-r}(S-M)$, then $K_*(Z_{n-r}) = 0$. Since φ is an isomorphism onto, there exists a $\gamma^r \in h^r(S-M)$ such that $\varphi(\gamma^r) = Z_{n-r}$. Now, by commutativity in the diagram above, $\gamma(j^*(\gamma^r)) = K_*(\varphi(\gamma^r)) = K_*(Z_{n-r}) = 0$. Therefore, $j^*(\gamma^r) = 0$, since γ is an isomorphism. This implies $\gamma^r \in h^r(S-M;\tilde{s})$; therefore $\tilde{\varphi}$ maps $h^r(S-M;\tilde{s})$ onto $h_{n-r}(S-M;\tilde{s})$. Also, $\tilde{\varphi}$ is an isomorphism since φ is an isomorphism.

THEOREM 1.10. Let S be an orientable n-gcm. Then $H^r(M;\tilde{s}) = h_{n-r-1}(S-M;\tilde{s})$ and $H^f_r(M;\tilde{s}) = h_{n-r-1}(S-M;\tilde{s})$, for $0 \leq r \leq n-1$.

PROOF. By lemma 1.8 $H^r(M;\tilde{s}) = h^{r+1}(S-M;\tilde{s})$. By lemma 1.9 $h^{r+1}(S-M;\tilde{s}) = h_{n-r-1}(S-M;\tilde{s})$. By definition 1.6 $H^f_r(M;\tilde{s}) = H^r(M;\tilde{s})$.

COROLLARY 1.11. Let S be an orientable n-gcm,

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[1] R. L. Wilder, T. M., p. 253.

M a proper closed subset of S. Assume $\tilde{H}^r(S) = H^{r+1}(S) = 0$ for some r such that $0 \leq r < n-1$. ($\tilde{H}^r(S)$ denotes the reduced or augmented homology on S). If $r = n-1$, we require that $H^{n-1}(S) = 0$ and $\tilde{H}^0(S) = 0$. Then $\tilde{H}^r(M) = \tilde{h}_{n-r-1}(S-M)$.

PROOF. If $0 < r < n-1$, then

$$H^r(M;\tilde{s}) = \frac{H^r(M)}{i^*(H^r(S))} = H^r(M),$$

since $H^r(S) = 0$. Also, $h_{n-r-1}(S-M;\tilde{s}) = (j^*)^{-1}(0) = h_{n-r-1}(S-M)$, since $H^{r+1}(S) = 0$ implies $H_{n-r-1}(S) = 0$ by theorem 4.2, p. 253, in T. M. Thus, $H^r(M) = h_{n-r-1}(S-M)$ by theorem 1.10.

If $r = 0$, then $H^0(M) = \tilde{H}^0(M) \oplus A$, where A is the coefficient field for the cohomology. Also, $\tilde{H}^0(S) = 0$ which implies $H^0(S) = A$. It is easy to see that $i^*(H^0(S)) \neq 0$; therefore, $i^*(H^0(S)) = A$. Therefore, the definition of $H^0(M;\tilde{s})$ implies that $H^0(M) = H^0(M;\tilde{s}) \oplus A$, or $\tilde{H}^0(M) = H^0(M;\tilde{s})$. Since $H^1(S) = 0$, by duality $H_{n-1}(S) = 0$; therefore, $h_{n-1}(S-M;\tilde{s}) = h_{n-1}(S-M)$. Therefore, $\tilde{H}^0(M) = h_{n-1}(S-M)$ by theorem 1.10.

If $r = n-1$, then

$$H^{n-1}(M;\tilde{s}) = \frac{H^{n-1}(M)}{i^*(H^{n-1}(S))} = H^{n-1}(M).$$

The mapping $k_*: h_0(S-M) \longrightarrow H_0(S)$ is onto, because $\tilde{H}_0(S) = 0$ implies $H_0(S) = A$. Thus, $h_0(S-M) = (k_*)^{-1}(0) \oplus A = h_0(S-M;\tilde{s}) \oplus A$, and $h_0(S-M;\tilde{s}) = \tilde{h}_0(S-M)$. By theorem 1.10 $H^{n-1}(M) = \tilde{h}_0(S-M)$.

COROLLARY 1.12. Under the conditions of corollary 1.11 (1) $H_r^f(M) = \tilde{h}_{n-r-1}(S-M)$ if $o < r \leq n-1$, and (2) $\tilde{H}_o^f(M) = h_{n-1}(S-M)$. (Note that $H_r^f(M)$ and $\tilde{H}_o^f(M)$ are defined in definition 4.4 and definition 4.6, in Chapter II).

PROOF. The proof is an immediate consequence of definition 4.4 and definition 4.6, in Chapter II.

CHAPTER III

HIGHER DIMENSIONAL CYCLIC ELEMENT THEORY IN COMPACT SPACES

1. Preliminary lemmas. Hereafter M will always denote a compact Hausdorff space. We will use only the Cech homology and cohomology over a field.

DEFINITION 1.1. A T_r set in M is a closed subset T of M such that $H_r(K) = 0$ for all closed sets $K \subset T$. In other words, a T_r set is a closed set which carries no essential r-cycle.[1]

DEFINITION 1.2. An E_r set in M is a non-degenerate subset of M which is maximal with respect to the property that it can not be separated by a T_r set of M.

With the exception of lemmas 1.9, 1.11, and 1.13, the following lemmas have been proven by Whyburn in case M is a compact subset of a Euclidean space.[2]

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[1] A cycle $Z_r \in H_r(M)$ is essential if there is a carrier K for Z_r such that Z_r does not bound on K.

[2] G. T. Whyburn, Higher Order Cyclic Element Theory, pp. 133-146.

LEMMA 1.3. Let M be a compact set. Then M can be disconnected by a T_r set if and only if there exist closed sets M_1 and M_2 with (1) $M = M_1 \cup M_2$, (2) $M_1 \cap M_2$ a T_r set, and (3) M_1 and M_2 proper closed subsets of M.

PROOF. Suppose there exists a T_r set T such that $M-T = \tilde{M}_1 \cup \tilde{M}_2$, where $\tilde{M}_1$ and $\tilde{M}_2$ are nonempty and $\overline{\tilde{M}}_1 \cap \tilde{M}_2 = \tilde{M}_1 \cap \overline{\tilde{M}}_2 = \phi$. Let $M_1 = \tilde{M}_1 \cup T$ and $M_2 = \tilde{M}_2 \cup T$. Then $M = M_1 \cup M_2$, and $M_1 \cap M_2 = T$ is a T_r set. Note that M_1 and M_2 are proper subsets of K. Since $\overline{\tilde{M}}_1 \cap \tilde{M}_2 = \tilde{M}_1 \cap \overline{\tilde{M}}_2 = \phi$, we have $\overline{M}_1 = \overline{\tilde{M}}_1 \cup T = \tilde{M}_1 \cup T = M_1$, for i = 1,2.

Suppose $M = M_1 \cup M_2$, where M_1 and M_2 satisfy the conditions above. Let $\tilde{M}_1 = M_1 - (M_1 \cap M_2) = M-M_2$ and $\tilde{M}_2 = M_2 - (M_1 \cap M_2) = M-M_1$. Then $M-(M_1 \cap M_2) = \tilde{M}_1 \cup \tilde{M}_2$ where $\tilde{M}_1$ and $\tilde{M}_2$ are non-empty, disjoint, open subsets of $M-(M_1 \cap M_2)$. Then M is separated by a T_r set, namely $M_1 \cap M_2$.

LEMMA 1.4. Let K be a subset of M which can not be disconnected by a T_r set. If $M = M_1 \cup M_2$, T_r separated, (by this we mean that M_1 and M_2 satisfy the conditions of lemma 1.3) then $K \subset M_1$ (or $K \subset M_2$).

PROOF. Suppose the contrary. By lemma 1.3 $M-(M_1 \cap M_2) = (M-M_2) \cup (M-M_1)$ and $(M-M_1) \cap (\overline{M-M_2}) = (\overline{M-M_1}) \cap (M-M_2) = \phi$ We must have $K \cap (M-M_1) \neq \phi$ and $K \cap (M-M_2) \neq \phi$, since we have assumed K is not contained in either M_1 or M_2. But then $K-(M_1 \cap M_2) = \left[K \cap (M-M_1)\right] \cup \left[K \cap (M-M_2)\right]$ and K is disconnected by a T_r set. Note that $M_1 \cap M_2$ is a T_r set by lemma 1.3.

LEMMA 1.5. *If K is an E_r set, then K is closed and connected.*

PROOF. Since the empty set is a T_r set, K can not be disconnected by the empty set, thus K is connected. We will show $\overline{K}$ can not be disconnected by a T_r set. Since K is maximal with this property, we will then have $K = \overline{K}$. Suppose $\overline{K}$ can be disconnected by a T_r set. Then $\overline{K} = K_1 \cup K_2$, T_r - separated, by lemma 1.3. By lemma 1.4 $K \subset K_1$, and since K_1 is closed, this implies $\overline{K} \subset K_1$. But K_1 is a proper subset of $\overline{K}$ by lemma 1.3, and thus we have a contradiction.

LEMMA 1.6. *If K_1 and K_2 are both E_r sets and $K_1 \neq K_2$, then $K_1 \cap K_2$ is a T_r set.*

PROOF. Let $M = K_1 \cup K_2$. If M can not be disconnected by a T_r, set then K_1 is not maximal. Thus we have $M = M_1 \cup M_2$, T_r - separated. We may assume $K_1 \subset M_1$, by lemma 1.4. If $K_2 \subset M_1$ then $M_1 = M$, which is impossible by lemma 1.3. By lemma 1.4 we must have $K_2 \subset M_2$. Then $K_1 \cap K_2$ is a closed subset of $M_1 \cap M_2$. But $M_1 \cap M_2$ is a T_r set. Hence, $K_1 \cap K_2$ is a T_r set.

LEMMA 1.7. *If K is a non-degenerate subset of M, which can not be disconnected by a T_r set, then K is contained in a unique E_r set in M.*

PROOF. Let $\{K_\alpha\}$ be the set of all subsets of M such that no K_α can be disconnected by a T_r set, and each

K_α contains K. Order the set $\{K_\alpha\}$ by inclusion, that is, $K_\alpha > K_\beta$ if $K_\alpha \supset K_\beta$. Let $\{K_{\alpha'}\}$ be any simply ordered subset of $\{K_\alpha\}$. Then $\bigcup_{\alpha'} K_{\alpha'}$ is an element of $\{K_\alpha\}$. Suppose $(\bigcup_{\alpha'} K_{\alpha'})-T = \tilde{K}_1 \cup \tilde{K}_2$, separated, where T is a T_r set. Since $\{K_{\alpha'}\}$ is simply ordered, there exists a $K_\beta \in \{K_{\alpha'}\}$ such that $K_\beta \cap \tilde{K}_1 \neq \phi$ and $K_\beta \cap \tilde{K}_2 \neq \phi$. But then $K_\beta - T = (K_\beta \cap \tilde{K}_1) \cup (K_\beta \cap \tilde{K}_2)$, which is impossible since $K_\beta \in \{K_\alpha\}$. Thus, by Zorn's lemma, the set $\{K_\alpha\}$ has a maximal element. By definition 1.2 this maximal element is an E_r set.

If $E^1 \neq E^2$, $E^1 \supset K$, $E^2 \supset K$ and both E^1 and E^2 are E_r sets, then $E^1 \cap E^2 \supset K$. Since K is non-degenerate, we can find an open set U of M such that $\bar{U}-U$ separates K. Then $(E^1 \cap E^2) \cap (\bar{U}-U)$ still separates K since $K \subset (E^1 \cap E^2)$. But $(E^1 \cap E^2) \cap (\bar{U}-U)$ is a closed subset of $E^1 \cap E^2$ and $E^1 \cap E^2$ is a T_r set. Hence, K is separated by a T_r set, which is impossible. Thus, the E_r set containing K is unique.

DEFINITION 1.8. Suppose we are given an element $\gamma^r \in H^r(M)$. A closed subset D of M is called a D_{r-1} set relative to γ^r if $i^*(\gamma^r) \neq 0$ (where $i^*: H^r(M) \longrightarrow H^r(D)$ is the mapping induced by the inclusion mapping of D into M) and if D is minimal with respect to this property.

LEMMA 1.9. *If $\gamma^r \neq 0$ is an element of $H^r(M)$, then there exists at least one D_{r-1} set relative to γ^r.*

PROOF. Consider the set of all $\{D_\alpha\}$, where D_α is a closed subset of M such that $i_\alpha^*(\gamma^r) \neq 0$ (here

$i^*_\alpha: H^r(M) \longrightarrow H^r(D_\alpha)$ is the mapping induced by the inclusion of D_α into M). Order this set by taking $D_\alpha > D_\beta$ if and only if $D_\alpha \subset D_\beta$. Consider a simply ordered subset $\{D_{\alpha'}\}$ of $\{D_\alpha\}$; let $D_o = \bigcap_{\alpha'} D_{\alpha'}$. If $D_{\alpha'} > D_{\beta'}$ then define $i^*_{\alpha'\beta'}: H^r(D_{\beta'}) \longrightarrow H^r(D_{\alpha'})$ to be the mapping induced by inclusion. Then $\{H^r(D_{\alpha'}), i^*_{\alpha'\beta'}\}$ is a direct system. By the continuity of the Cech theory we have an isomorphism $\varphi: H^r(D_o) \longrightarrow \varinjlim\{H^r(D_{\alpha'}), i^*_{\alpha'\beta'}\}$. The set $\{i^*_{\alpha'}(\gamma^r)\}$, taken over all α', is an element of $\varinjlim\{H^r(D_{\alpha'}), i^*_{\alpha'\beta'}\}$. Let $i^*_\delta: H^r(M) \longrightarrow H^r(D_o)$ be the mapping induced by inclusion. Since φ is an isomorphism, $i^*_\delta(\gamma^r) = 0$ implies $i^*_{\alpha'}(\gamma^r) = 0$, for some α'. But this is impossible. Hence, D_o is an element of $\{D_\alpha\}$. Now, by Zorn's lemma, there exists a D_{r-1} set relative to γ^r.

COROLLARY 1.10. *A D_{r-1} set can not be disconnected by a T_{r-1} set. Hence, if $H_r(M) \neq 0$ $(r \geq 1)$, then there exists an E_{r-1} set in M.*

PROOF. Let D be a D_{r-1} set relative to $\gamma^r \in H^r(M)$. If $D = D_1 \cup D_2$, T_{r-1} - separated, we will consider the Mayer-Vietoris sequence,

$$\longrightarrow H^{r-1}(D_1 \cap D_2) \longrightarrow H^r(D) \xrightarrow{\phi} H^r(D_1) \oplus H^r(D_2) \longrightarrow .$$

Since $D_1 \cap D_2$ is a T_{r-1} set, the mapping ϕ is an isomorphism. Thus, if we let $i^*_D: H^r(M) \longrightarrow H^r(D)$ be the map induced by inclusion, we must have $\phi(i^*_\delta(\gamma^r)) = (i^*_{D_1 D}(i^*_D(\gamma^r)), i^*_{D_2 D}(i^*_D(\gamma^r))) \neq 0$. (Here $i^*_{D_1 D}: H^r(D) \longrightarrow H^r(D_1)$, $i = 1,2$, are the maps induced by inclusion.) But $i^*_{D_1 D} i^*_D = i^*_{D_1}$

where $i^*_{D_1}: H^r(M) \longrightarrow H^r(D_1)$. Thus we can assume $i^*_{D_1}(\gamma^r) \neq 0$, which would imply that D was not a D_{r-1} set, since D_1 is a proper subset of D and D is minimal with respect to the property that $i^*_D(\gamma^r) \neq 0$. Thus, a D_{r-1} set can not be disconnected by a T_{r-1} set.

If $H_r(M) \neq 0$ then $H^r(M) \neq 0$ and by lemma 1.9 we have a D_{r-1} set in M. But, since a D_{r-1} set can not be disconnected by a T_{r-1} set, lemma 1.7 implies the existence of an E_{r-1} set contained in M.

LEMMA 1.11. *If D is a D_{r-1} set relative to $\gamma^r \in H^r(M)$, then D is minimal relative to the property of being a closed carrier of a cycle Z_r such that $Z_r . \gamma^r \neq 0$.*

PROOF. First, there exists a $Z_r \in H_r(D)$ such that $Z_r . i^*_D(\gamma^r) = i^D_*(Z_r) . \gamma^r \neq 0$, where $i^*_D: H^r(M) \longrightarrow H^r(D)$ and $i^D_*: H_r(D) \longrightarrow H_r(M)$. This follows from the orthogonal dual pairing of $H_r(D)$ and $H^r(D)$. If D_1 is a proper closed subset of D and $Z^1_r \in H_r(D^1)$, then $i^{D_1}_*(Z^1_r) . \gamma^r = Z^1_r . i^*_{D_1}(\gamma^r) = 0$, since $i^*_{D_1}(\gamma^r) = 0$.

LEMMA 1.12. *If $A = \bigcup_{i=1}^{\infty} T_i$, where each T_i is a T_{r-1} set in M and A is closed in M, then A is a T_r set.*

PROOF. Suppose K is a closed subset of A such that $H^r(K) \neq 0$. Then K contains a closed subset D such that D is a D_{r-1} set. We have $D = \bigcup_1^{\infty} D_i$, where $D_i = T_i \cap D$. We will show each D_i is nowhere dense in D. Suppose U is open in D and $U \subset D_i$. Then there exists a V open in

D such that $\overline{V}-V$ separates D and $\overline{V}-V \subset U < D_i$. Since $\overline{V}-V$ is a closed subset of D_i, $\overline{V}-V$ is a T_{r-1} set. But D is a D_{r-1} set, and can not be separated by a T_{r-1} set, according to corollary 1.10. Thus each D_i is nowhere dense in D; D is a closed subset of M, hence D is compact. Then $D \neq \bigcup_{i=1}^{\infty} D_i$, by the generalized Baire density theorem. Hence, our original assumption is wrong and A is a T_r set.

LEMMA 1.13. Given $E^1, \ldots, E^n$, where each E^i $(1 \leq i \leq n)$ is an E_{r-1} set and $M \neq \bigcup_{i=1}^{n} E^i$, then there exist closed subsets M_1 and M_2 of M such that $M = M_1 \cup M_2$, T_r - separated, (in fact, $M_1 \cap M_2$ is the union of a finite number of T_{r-1} sets) and $M_1 \supset \bigcup_{i}^{n} E^i$.

PROOF. The proof will be by induction on n. Consider the case n = 1. Since $M \neq E^1$, M is not an E_{r-1} set. Thus, $M = M_1 \cup M_2$, T_{r-1} - separated. By lemma 1.4 we may take $E^1 \subset M_1$. By lemma 1.12 $M_1 \cap M_2$ is a T_r set, hence $M = M_1 \cup M_2$, T_r - separated.

Assume the lemma is true up to n-1. Since M is not an E_{r-1} set, we must have $M = M_1 \cup M_2$, T_{r-1} - separated. Let $E = \bigcup_{i}^{\infty} E^i_{r-1}$. If $M-E \subset M_1 \cap M_2$, then there exists a non-empty open set V such that $\overline{V} \subset M-E$. Then $\overline{V}-V$ separates M and $\overline{V}-V \subset M-E \subset M_1 \cap M_2$. Thus, $\overline{V}-V$ is a T_{r-1} set; therefore $M = ((M-V) \cup V)$, T_{r-1} - separated, where $E \subset M-V$. Thus, we may assume $(M-E) \cap (M-M_1) = (M-E) \cap (M_2-(M_1 \cap M_2)) \neq 0$. Also, we may assume $\bigcup_{s+1}^{n} E^i_{r-1} \subset M_1$ and $\bigcup_{i}^{s} E^i_{r-1} \subset M_2$, where $1 \leq s \leq n-1$, since $E \subset M_1$ or $E \subset M_2$ implies that the conclusion of the lemma is satisfied.

If T is a T_{r-1} set in M, we will show $\overline{(E_{r-1}-T)} = E_{r-1}$, where E_{r-1} is any E_{r-1} set in M. Suppose $x \in E_{r-1} \cap T$ and suppose there exists an open set U in M such that $U \cap (E_{r-1}-T) = 0$. Then there exists a $\overline{V} \subset U$ such that $(\overline{V}-V) \cap E_{r-1} \subset T$ and $\overline{V} \cap E_{r-1}$ is non-empty. But then $((M-V) \cap E_{r-1}) \cup (\overline{V} \cap E_{r-1}) = E_{r-1}$, T_{r-1} - separated, and this is impossible. Hence, $x \in E_{r-1} \cap T$ implies $x \in \overline{(E_{r-1}-T)}$.

Thus, since $M_1 \cap M_2$ is a T_{r-1} set, we have $\overline{(E^i_{r-1}-(M_1 \cap M_2))} = E^i_{r-1}$, for $1 \leq i \leq n$. Therefore, $E^i_{r-1} = \overline{E^i_{r-1}-(M_1 \cap M_2)} \subset \overline{M_2-(M_1 \cap M_2)} = M-M_1$, for $1 \leq i \leq s$. Let $M_3 = \overline{M-M_1}$. Note that $M_3 \supset E^i_{r-1}$ for $1 \leq i \leq s$, or $M_3 \supset \bigcup_1^s E^i_{r-1}$. Then $(M-E) \cap (M-M_1) \neq \phi$ and $M_3 \supset \bigcup_1^s E^i_{r-1}$, $M_3 \neq \bigcup_1^s E^i_{r-1}$. Thus, by the induction assumption, we have $M_3 = M_4 \cup M_5$, T_r - separated, where $M_4 \cap M_5$ is the union of a finite number of T_{r-1} sets and $\bigcup_1^s E^i_{r-1} \subset M_4$. Now, $M_3-M_4 \neq \phi$ and thus there exists an $x \in M_3-M_4$ and an open set U in M, such that $x \in U \subset M-M_4$. Since $x \in M_3 = \overline{M-M_1}$, there exists a $y \in U \cap (M-M_1)$. Therefore, $y \in (M-M_4) \cap (M-M_1) = M-(M_1 \cup M_4)$. Thus, $M = M_1 \cup M_3 = (M_1 \cup M_4) \cup M_5$ where $(M_1 \cup M_4)$ and M_5 are proper closed subsets of M. Also, $\bigcup_{s+1}^n E^i_{r-1} \subset M_1$ and $\bigcup_1^s E^i_{r-1} \subset M_4$; therefore $E = \bigcup_1^n E^i_{r-1} \subset M_1 \cup M_4$. We have $(M_1 \cup M_4) \cap M_5 = (M_1 \cap M_5) \cup (M_4 \cap M_5)$, where $(M_1 \cap M_5) \subset (M_1 \cap M_2)$. Therefore, $(M_1 \cup M_4) \cap M_5$ is the union of a finite number of T_{r-1} sets. Thus, the conclusion of the lemma is satisfied.

LEMMA 1.14. *Let E_r be an E_r set in M. Then there exists a sequence $E_0, E_1, \ldots, E_{r-1}$ such that E_1 is an*

E_i *set for* $o \leq i \leq r-1$, *and* $E_o \supset E_1 \supset \ldots \supset E_{r-1} \supset E_r$.

PROOF. Since any T_{r-1} set is a T_r set, no T_{r-1} set can disconnect E_r. Thus, by lemma 1.7, $E_r \subset E_{r-1}$ where E_{r-1} is an E_{r-1} set. This shows that we may construct a proof by induction.

LEMMA 1.15. *Let* E_r *be an* E_r *set in* M. *Let* $i^*:H_s(E_r) \longrightarrow H_s(M)$ *be the mapping induced by inclusion. If* $s \geq r$, *then this mapping is an isomorphism.*

PROOF. Suppose $i_*(Z_s) = 0$ for some $Z_s \in H_s(E_r)$. Then there exists a minimal set K such that $E_r \subset K \subset M$ and $i_*(Z_s) = 0$ ($i^K_*:H_s(E_r) \longrightarrow H_s(K)$ is induced by inclusion).[1] If $K \neq E_r$ then, since K is not an E_r set, we must have $K = K_1 \cup K_2$, T_r - separated. We may assume $E_r \subset K_1$ by lemma 1.4. Since K is compact we may apply the following Mayer-Vietoris sequence,

$$\longleftarrow H^s(K) \xleftarrow{\phi} H^s(K_1) \oplus H^s(K_2) \longleftarrow H^s(K_1 \cap K_2) \longleftarrow .$$

Consider the element $(i^{K_1}_*(Z_s), 0) \in H^s(K_1) \oplus H^s(K_1)$, where $i^{K_1}_*:H_s(E_r) \longrightarrow H_s(K_1)$ is induced by inclusion. Then $\phi((i^{K_1}_*(Z_s), 0)) = i^K_*(Z_s) = 0$. But $H^s(K_1 \cap K_2) = 0$, since $K_1 \cap K_2$ is a T_r set, hence a T_s set by lemma 1.12. Thus, by exactness, $i^{K_1}_*(Z_s) = 0$, which contradicts the minimal property of K. Thus, $K = E_r$, and $Z_s = 0$. Hence, $i_*:H_s(E_r) \longrightarrow H_s(M)$ is an isomorphism into.

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[1] R. L. Wilder, T. M., p. 206, lemma 2.8.

2. The relation between homology and cyclic elements.

THEOREM 2.1. *Let* $\{E^{\alpha}_{r-1}\}$ *be the set of all* E_{r-1} *sets in* M *such that* $H_r(E^{\alpha}_{r-1}) \neq 0$, *and let* $E = \overline{(\bigcup_{\alpha} E^{\alpha}_{r-1})}$. *Then* $i_*: H_r(E) \longrightarrow H_r(M)$ *is onto and* $i^*: H^r(M) \longrightarrow H^r(E)$ *is one-to-one.* (*These mappings are induced by inclusion*).

PROOF. We will show that $i^*: H^r(M) \longrightarrow H^r(E)$ is one-to-one. Consider the exact sequence,

$$\longrightarrow H^r(M,E) \xrightarrow{j^*} H^r(M) \xrightarrow{i^*} H^r(E) \longrightarrow .$$

Suppose $j^*(\gamma^r) \neq 0$ where $\gamma^r \varepsilon H^r(M,E)$. Then there exists a D_{r-1} set relative to $J^*(\gamma^r)$ by lemma 1.9. By lemma 1.7, there exists an E_{r-1} set containing the above D_{r-1} set; let E^1_{r-1} be this set. Then, if $i^*_1: H^r(M) \longrightarrow H^r(E^1_{r-1})$, we have $i^*_1(J^*(\gamma^r)) \neq 0$ by the definition of the D_{r-1} set relative to $j^*(\gamma^r)$ and the fact that E^1_{r-1} contains this D_{r-1} set. Then $E^1_{r-1} \subset E$ by definition of E, and $i^*_1(j^*(\gamma r)) = i^*_2(i^*(j^*(\gamma^r))$, where $i^*_2: H^r(E) \longrightarrow H^r(E^1_{r-1})$. Note that $i^*j^*(\gamma^r) = 0$, by exactness. Thus, $j^*(\gamma^r) = 0$ for all $\gamma^r \varepsilon H^r(M,E)$, and $i^*: H^r(M) \longrightarrow H^r(E)$ is an isomorphism into by the exact sequence.

Now, we will show $i_*: H_r(E) \longrightarrow H_r(M)$ is onto. We must show that $j_*(Z_r) = 0$ for any $Z_r \varepsilon H_r(M)$, where $j_*: H_r(M) \longrightarrow H_r(M,E)$. Then the exactness of the sequence,

$$\longrightarrow H_r(E) \longrightarrow H_r(M) \longrightarrow H_r(M,E) \longrightarrow$$

will imply that i_* is onto. Suppose $j_*(Z_r) \neq 0$. Then there exists a $\gamma^r \varepsilon H^r(M,E)$ such that $\gamma^r . j_*(Z_r) = 1$, by

the orthogonal dual pairing of $H^r(M,E)$ and $H_r(M,E)$. But then we would have $j^*(\gamma^r).Z_r = 1$, and we have just shown that $j^*(\gamma^r) = 0$, for all $\gamma^r \in H^r(M,E)$; hence, $j_*(Z_r) = 0$.

THEOREM 2.2. *Let $\{E^\nu_{r-1}\}$ be the set of all E_{r-1} sets in M. Let $\tilde{E} = (\overline{\bigcup_\nu E^\nu_{r-1}})$. Then $i^*:H^r(M) \longrightarrow H^r(E)$ and $i_*:H_r(E) \longrightarrow H_r(M)$ are both isomorphisms onto.*

PROOF. Since $E \subset \tilde{E}$, theorem 2.1 implies $i_*:H_r(\tilde{E}) \longrightarrow H_r(M)$ is onto and $i^*:H^r(M) \longrightarrow H^r(\tilde{E})$ is one-to-one. We will show that $i_*:H_r(\tilde{E}) \longrightarrow H_r(M)$ is one-to-one. Suppose $i_*(Z_r) = 0$ for some $Z_r \in H_r(\tilde{E})$. Then there exists a minimal closed set K such that $M \supset K \supset \tilde{E}$ and $i^\kappa_*(Z_r) = 0$, where $i^\kappa_*:H_r(\tilde{E}) \longrightarrow H_r(K)$ is the mapping induced by inclusion.[1] If $K \neq E$, then there exists an open set U of M such that $U \cap K \neq \phi$ and $\overline{U} \subset M-\tilde{E}$. We will show that $\overline{U}-U$ is a T_r set. Suppose not, then there exists a closed subset $F \subset \overline{U}-U$ such that $H^r(F) \neq 0$. Then by corollary 1.10 there exists an E_{r-1} set of F. This set can not be disconnected by a T_{r-1} set. Hence, by lemma 1.7, there exists an E_{r-1} set in M which contains the above E_{r-1} set of F. Hence, there is a set E^o_{r-1} which is an E_{r-1} set of M such that $E^o_{r-1} \cap F \neq \phi$. Therefore, $E^o_{r-1} \cap \overline{U} \neq \phi$ and $E^o_{r-1} \cap (M-\tilde{E}) \neq \phi$. But this is impossible, since $E^o_{r-1} \subset \tilde{E}$ by definition of $\tilde{E}$. Therefore, $\overline{U}-U$ is a T_r set. Let $K_1 = \overline{U} \cap K$ and $K_2 = (M-U) \cap K$. Then $K = K_1 \cup K_2$, where K_1 and K_2 are proper closed subsets of K and $\tilde{E} \subset K_2$.

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[1] R. L. Wilder, T. M., p. 206, lemma 2.8.

Also, $K_1 \cap K_2 \subset \bar{U}-U$, and is therefore a T_r set. Consider the Mayer-Vietoris sequence,

$$\longleftarrow H_r(K) \xleftarrow{\phi} H_r(K_1) \oplus H_r(K_2) \longleftarrow H_r(K_1 \cap K_2).$$

By exactly the same reasoning as in lemma 1.15, we have $i_*^{K2}(Z_r) = 0$, where $i_*^{K2}: H_r(\tilde{E}) \longrightarrow H_r(K_2)$ is induced by inclusion. But this contradicts the minimal property of K, hence we must have $K = \tilde{E}$ and $Z_r = 0$. Hence, i_* is an isomorphism onto.

We will show that $i^*: H^r(M) \longrightarrow H^r(\tilde{E})$ is onto. Consider the exact sequence,

$$\longleftarrow H^{r+1}(M, \tilde{E}) \xleftarrow{\delta^*} H^r(\tilde{E}) \longleftarrow H^r(M) \longleftarrow .$$

Suppose $\delta^*(\gamma^r) \neq 0$ for some $\gamma^r \in H^r(\tilde{E})$. Then we must have a $Z_{r+1} \in H_r(M, \tilde{E})$ such that $Z_{r+1} \cdot \delta^*(\gamma^r) = 1$, by the orthogonal dual pairing of $H^{r+1}(M, \tilde{E})$ to $H_{r+1}(M, \tilde{E})$. But δ^* is the dual map of $\partial^*: H^{r+1}(M, \tilde{E}) \longrightarrow H^r(\tilde{E})$, and we have shown above that $\partial^*(Z_{r+1}) = 0$ for all $Z_{r+1} \in H^{r+1}(M, \tilde{E})$. Thus, $Z_{r+1} \cdot \delta^*(\gamma^r) = \partial Z_{r+1} \cdot \gamma^r = 0$ which is impossible. Hence, $\delta^*(\gamma^r) = 0$ for all $\gamma^r \in H^r(\tilde{E})$. Therefore, by exactness, $i^*: H^r(M) \longrightarrow H^r(\tilde{E})$ is onto.

LEMMA 2.3. *Let* $E_{r-1}^1, \ldots, E_{r-1}^n$ *be* E_{r-1} *sets in* M. *Let* $E = \bigcup_i^n E_{r-1}^i$. *Define* $\pi^*: H^r(E) \longrightarrow \prod_{i=1}^n H^r(E_{r-1}^i)$ *as follows: for* $\gamma^r \in H^r(E)$, $\pi^*(\gamma^r) = (i_{E1}^*(\gamma^r), \ldots, i_{En}^*(\gamma^r))$, *where* $i_{Ei}^*: H^r(E) \longrightarrow H^r(E_{r-1}^i)$ *is induced by inclusion. Define* $\pi_*: \prod_i^n H_r(E_{r-1}^i) \longrightarrow H_r(E)$ *as follows: for* $(Z_r^1, \ldots, Z_r^n) \in \prod_i^n H_r(E_{r-1}^i)$ *take* $\pi_*(Z_r^1, \ldots, Z_r^n) = i_*^{\varepsilon'}(Z_r^1) + \ldots + i_*^{\varepsilon^n}(Z_r^n)$, *where* $i_*^{\varepsilon^i}: H_r(E_{r-1}^i) \longrightarrow H_r(E)$.

Then both π^* and π_* are isomorphisms onto.

PROOF. The proof will be inductive. The case where n = 1 is obvious. Suppose the lemma true for n-1. Since E is not an E_{r-1} set, we have $E = N_1 \cup N_2$, T_{r-1} - separated. By reindexing the E^i_{r-1} we may assume $N_1 = \bigcup_1^s E^i_{r-1}$, $N_2 = \bigcup_{s+1}^n E^i_{r-1}$. Consider the Mayer-Vietoris sequence,

$$\longleftarrow H_{r-1}(N_1 \cap N_2) \longleftarrow H_r(E) \longleftarrow H_r(N_1) \oplus H_r(N_2) \longleftarrow H_r(N_1 \cap N_2) \longleftarrow .$$

Now, $H_{r-1}(N_1 \cap N_2) = H_r(N_1 \cap N_2) = 0$ because $N_1 \cap N_2$ is a T_{r-1} set; therefore, a T_r set. By the induction assumption $H_r(N_1) = \prod_1^s H_r(E^i_{r-1})$ and $H_r(N_2) = \prod_{s+1}^n H_r(E^i_{r-1})$. The sequence is exact; therefore, $H_r(E) = H_r(N_1) \oplus H_r(N_2)$. It is easy to see that the combination of these isomorphisms is exactly π_*. The proof for π^* is obtained by using the cohomology version of the Mayer-Vietoris sequence.

THEOREM 2.4. Let $E^1_{r-1}, \ldots, E^n_{r-1}$ be E_{r-1} sets in M. Let $\tilde{i}_*: \prod_1^n H_r(E^i_{r-1}) \longrightarrow H_r(M)$ be defined as follows: for $(Z^1_r), \ldots, Z^n_r) \in \prod_1^n H_r(E^i_{r-1})$ take $\tilde{i}_*(Z^1_r, \ldots, Z^n_r) = \sum_{k=1}^n i^k_*(Z^k_r)$, where $i^k_*: H_r(E^k_{r-1}) \longrightarrow H_r(M)$ is induced by inclusion. In a similar manner define $\tilde{i}^*: H^r(M) \longrightarrow \prod_1^n H^r(E^i_{r-1})$. That is, let $\tilde{i}^*(\gamma^r) = (i^*_1(\gamma^r), \ldots, i^*_n(\gamma^r))$, where $i^*_k: H^r(M) \longrightarrow H^r(E^k_{r-1})$ is induced by inclusion. Then $\tilde{i}_*: \prod_1^n H_r(E^i_{r-1}) \longrightarrow H_r(M)$ is one-to-one and $\tilde{i}^*: H^r(M) \longrightarrow \prod_1^n H^r(E^i_{r-1})$ is onto.

PROOF. Assume $\tilde{i}_*(Z^1_r, \ldots, Z^n_r) = 0$. Let $E = \bigcup_1^n E^i_{r-1}$ and define $i^E_*: H_r(E) \longrightarrow H_r(M)$ to be the mapping induced

by inclusion. Let $\pi_*: \prod_1^n H_r(E_{r-1}^i) \longrightarrow H_r(E)$ be the mapping defined in lemma 2.3. Then $\tilde{i}_* = i_*^\varepsilon \pi_*$. Define Z_r^ε to be $Z_r^\varepsilon = \pi_*(Z_r^1, \ldots, Z_r^n)$, where $Z_r^i \in H_r(E_{r-1}^i)$. If $\tilde{i}_*(Z_r^1, \ldots, Z_r^n) = 0$, then $i_*^\varepsilon(Z_r^\varepsilon) = 0$. Then there is a minimal set K such that $M \supset K \supset E$ and $i_*^K(Z_r^\varepsilon) = 0$, where $i_*^K: H_r(E) \longrightarrow H_r(K)$ is induced by inclusion. We may assume $K \neq E$; hence, by lemma 1.13, we have $K = K_1 \cup K_2$, T_r-separated, where $E \subset K_1$. Now, by using the same method as in lemma 1.15, we can show $K = E$ and $Z_r^\varepsilon = 0$. Hence, since π_* is an isomorphism, $(Z_r^1, \ldots, Z_r^n) = 0$.

The proof that $\tilde{i}^*$ is onto follows from the fact that $\tilde{i}_*$ is one-to-one. Let $\tilde{i}_\varepsilon^*: H^r(M) \longrightarrow H^r(E)$ be induced by inclusion. Then $\tilde{i}^* = \pi^* \tilde{i}_\varepsilon^*$, where $\pi^*: H^r(E) \longrightarrow \prod_1^n H^r(E_{r-1}^i)$ is defined in lemma 2.3. If $\tilde{i}_\varepsilon^*$ is onto, then $\tilde{i}^*$ is onto since π^* is onto. But $\tilde{i}_*^\varepsilon$ is the dual of i_ε^* and $\tilde{i}_*^\varepsilon$ is one-to-one, as we have shown above. Thus, by the same method as used in theorem 2.1, we can prove that i_ε^* is onto.

THEOREM 2.5. *If $H_r(E_{r-1}^\alpha) \neq 0$ for only a finite number of the E_{r-1} sets in M, say $E_{r-1}^1, \ldots, E_{r-1}^n$, then the mappings $\tilde{i}_*: \prod_1^n H_r(E_{r-1}^i) \longrightarrow H_r(M)$ and $\tilde{i}^*: H^r(M) \longrightarrow \prod_1^n H^r(E_{r-1}^i)$ are both isomorphisms onto.*

PROOF. By theorem 2.1 the mapping $i_*^\varepsilon: H_r(E) \longrightarrow H_r(M)$ is onto. By lemma 2.3 the mapping $\pi_*: \prod_1^n H_r(E_{r-1}^i) \longrightarrow H_r(E)$ is onto. Note that $\tilde{i}_* = i_*^\varepsilon \pi_*$ and therefore $\tilde{i}_*$ is onto. By theorem 2.4 $\tilde{i}_*$ is one-to-one. A similar argument will show that $\tilde{i}^*$ is an isomorphism onto.

COROLLARY 2.6. *If $H_r(M)$ has finite dimension, then $H_r(E^\alpha_{r-1}) \neq 0$ for only a finite number of the E_{r-1} sets in M. Thus, the hypothesis of theorem 2.5 is satisfied.*

PROOF. Suppose $H_r(E^\alpha_{r-1}) \neq 0$ for an infinite number of the E_{r-1} sets of M. Thus, we have a countable set $\{E^i_{r-1}\}$ of distinct E_{r-1} sets such that $H_r(E^i_{r-1}) \neq 0$. By theorem 2.4, $\tilde{i}_*: \prod^n H_r(E^i_{r-1}) \longrightarrow H_r(M)$ is an isomorphism into for all n. Thus, the number of linearly independent elements in $H_r(M)$ will not be bounded. But this is impossible, since $H_r(M)$ has finite dimension.

THEOREM 2.7. *Given any $\gamma^r \neq 0 \in H^r(M)$, let $\{D^\alpha_{r-1}\}$ be the set of all D_{r-1} sets relative to γ^r (that is, $i^*_\alpha(\gamma^r) \neq 0$, where $i^*_\alpha: H^r(M) \longrightarrow H^r(D^\alpha_{r-1})$ is induced by inclusion and D^α_{r-1} is a minimal closed set with this property). Given $U^{\text{open}} \supset \bigcup_\alpha D^\alpha_{r-1}$, then there exists a $\gamma^r_U \in H^r(M, M-U)$ such that $j^*(\gamma^r_U) = \gamma^r$, where $j^*: H^r(M, M-U) \longrightarrow H^r(M)$ (that is, γ^r may be considered a cocycle of M in U).*

PROOF. Consider the exact sequence,

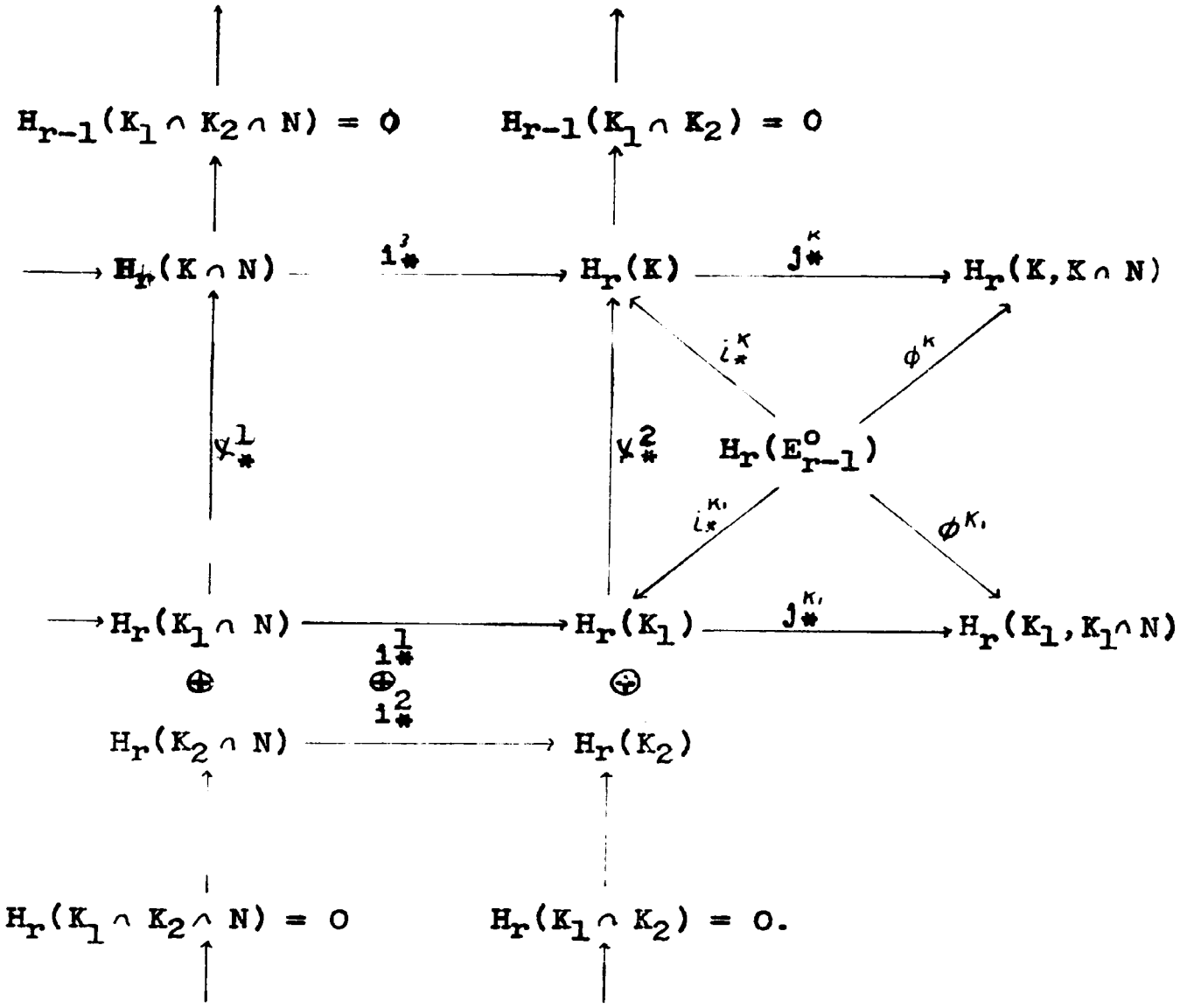
$$\longrightarrow H^r(M, M-U) \xrightarrow{j^*} H^r(M) \xrightarrow{i^*_U} H^r(M-U) \longrightarrow .$$

We will show $i^*_U(\gamma^r) = 0$, where γ^r is the element of $H^r(M)$ given above. Suppose $i^*_U(\gamma^r) \neq 0$; then, by lemma 1.9, there exists a closed subset K of M-U such that K is a D_{r-1} set relative to $i^*_U(\gamma^r)$. But K is also a D_{r-1} set relative to γ^r, since $i^*_K = i^*_{KU}\, i^*_U$. Here, $i^*_K: H^r(M) \longrightarrow H^r(K)$, $i^*_{KU}: H^r(M-U) \longrightarrow H^r(K)$, and $i^*_U: H^r(M) \longrightarrow H^r(M-U)$ are the

mappings induced by inclusion. But $K \cap (M-U) \neq \phi$, which is impossible since $U \supset \bigcup_\alpha D^\alpha_{r-1}$. Therefore, $i^*_U(\gamma^r) = 0$, and by the exactness of the sequence above there exists a $\gamma^r_U \in H^r(M, M-U)$ such that $j^*(\gamma^r_U) = \gamma^r$.

THEOREM 2.8. *Let* E^o_{r-1} *be an* E_{r-1} *set in* M, *and let* N *be a closed set in* M *such that* $E^o_{r-1} \cap N = \phi$. *Let* $i^M_*: H_r(E^o_{r-1}) \longrightarrow H_r(M)$ *be induced by inclusion, and let* $j^M_*: H_r(M) \longrightarrow H_r(M,N)$ *be the natural injection. Then* $\phi^M = j^M_* i^M_*: H_r(E^o_{r-1}) \longrightarrow H_r(M,N)$ *is an isomorphism into.*

PROOF. Suppose $\phi^M(\tilde{Z}_r) = 0$, for some $\tilde{Z}_r \in H_r(E^o_{r-1})$. Then there exists a minimal closed set K in M such that $K \supset E^o_{r-1}$ and $\phi^K(\tilde{Z}_r) = 0$, where $\phi^K: H_r(E^o_{r-1}) \longrightarrow H_r(K, K \cap N)$ is analogous to ϕ^M defined above. This follows from Zorn's lemma and continuity. We will assume that $K \neq E^o_{r-1}$. If $K = E^o_{r-1}$, we have $\tilde{Z}_r = 0$ and thus ϕ^M is an isomorphism. We may assume $K \neq E^o_{r-1}$ and thus K can be separated by a T_{r-1} set, that is, $K = K_1 \cup K_2$, T_{r-1}- separated. By lemma 1.4 we may assume $E^o_{r-1} \subset K_1$. We can show that the following diagram is commutative (the mappings are the usual inclusion or injection mappings unless otherwise defined),



The two vertical sequences are Mayer-Vietoris sequences. Also, the two horizontal sequences are exact. We have $H_{r-1}(K_1 \cap K_2) = H_{r-1}(K_1 \cap K_2 \cap N) = H_r(K_1 \cap K_2) = H_r(K_1 \cap K_2 \cap N) = 0$, since $K_1 \cap K_2$ is a T_{r-1} set. Now, $\phi^K(\tilde{Z}_r) = j^K_* i^K_*(\tilde{Z}_r) = 0$, therefore there exists a $Z^3_r \in H_r(K \cap N)$ such that $i^3_*(Z^3_r) = i^K_*(\tilde{Z}_r)$. This follows from the exact sequence. There exists $(Z^1_r, Z^2_r) \in H_r(K_1 \cap N) \oplus H_r(K_2 \cap N)$ such that

$\forall^1_*(Z^1_r, Z^2_r) = Z^3_r$. By commutativity, $\forall^2_*(i^1_*(Z^1_r), i^2_*Z^2_r) = i^3_*\forall^1_*(Z^1_r, Z^2_r) = i^3_*(Z^3_r) = i_*(Z_r) = \forall^2_*(i^{K1}_*(Z_r), 0)$. By exactness $\forall^2_*$ is an isomorphism, hence $i^1_*(Z^1_r) = i^{K1}_*(Z_r)$. Thus, $j^{K1}_* i^{K1}_*(Z_r) = j^{K1}_* i^1_*(Z^1_r) = 0$ by exactness. But this is impossible, since K was minimal. Thus, $K = E^o_{r-1}$ and ϕ^M is an isomorphism into.

COROLLARY 2.9. Let E^o_{r-1} be an E_{r-1} set in M and U an open set containing E^o_{r-1}. Let $j^*:H^r(M,M-U) \longrightarrow H^r(M)$ and $i^*:H^r(M) \longrightarrow H^r(E^o_{r-1})$ be the usual injection and inclusion maps. Then the mapping $\pi = i^*j^*:H^r(M,M-U) \longrightarrow H^r(E^o_{r-1})$ is onto.

PROOF. For any $\gamma^r_u \in H^r(M,M-U)$ and $Z^o_r \in H_r(E^o_{r-1})$, we have $\pi(\gamma^r_u).Z^o_r = i^*j^*(\gamma^r_u).Z^o_r = j^*(\gamma^r_u).i_*(Z^o_r) = \gamma^r_u.j^*i_*(Z^o_r) = \gamma^r_u.\phi^M(Z^o_r)$, where ϕ^M is defined in theorem 2.8. Thus, π and ϕ^M are dual mappings.

Choose a basis B_u for the subspace $\pi(H^r(M,M-U)$. Augment B_u to form a basis B of $H^r(E^o_{r-1})$. If $\pi(H^r(M,M-U)) \neq H^r(E^o_{r-1})$ (that is, if π is not onto), then $B-B_u$ is not empty. Now, $H^r(M,M-U)$ and $H_r(M,M-U)$ are base spaces for a dual system (see Chapter I). Thus, if $\varphi:H^r(M,M-U) \longrightarrow H_r(M,M-U)$ is the mapping determined by the basis B, and if $\gamma^r_o \in B-B_u$, then $\varphi(\gamma^r_o).\gamma^r_a = 0$, for all $\gamma^r_a \in B_u$. Since B_u is a basis for $\pi(H^r(M,M-U))$, we have $\varphi(\gamma^r_o).\pi(\gamma^r_u) = 0$ for all $\gamma^r_u \in H^r(M,M-U)$. Now, $\varphi(\gamma^r_o) \neq 0$ since φ is an isomorphism; therefore, $\phi^M(\varphi(\gamma^r_o)) \neq 0$ since ϕ^M is an isomorphism. Since the vector spaces

$H^r(M,M-U)$ and $H_r(M,M-U)$ are orthogonally dually paired, there exists a $\gamma_u^r \in H^r(M,M-U)$ such that $\phi^\wedge(\varphi(\gamma_o^r)).\gamma_u^r = 1$. But $\phi^\wedge(\varphi(\gamma_o^r)).\gamma_u^r = \varphi(\gamma_o^r).\pi(\gamma_o^r) = 0$. Thus π is onto.

CHAPTER IV

HIGHER DIMENSIONAL CYCLIC ELEMENTS IN LOCALLY CONNECTED SPACES

A. Simon has proven the following theorem.[1]

THEOREM 1.1. *Let M be a compact Hausdorff space. There exists a minimal closed set A in M such that; if U is an open set containing A, then U contains all except a finite number of the E_{r-1} sets of M. If A is a T_r set, then $\tilde{i}^*:H^r(M) \longrightarrow \prod_\alpha H^r(E^\alpha_{r-1})$ becomes an isomorphism of $H^r(M)$ onto the weak direct sum $\sum_\alpha H^r(E^\alpha_{r-1})$. Here, $\{E^\alpha_{r-1}\}$ is the set of all E_{r-1} sets in M; and $\tilde{i}^*$ is the mapping induced by projection onto each component of the direct sum (that is, $\tilde{i}^*(\gamma^r) = \{i^*_\alpha(\gamma^r)\}$, where $i^*_\alpha:H^r(M) \longrightarrow H^r(E^\alpha_{r-1})$ is induced by inclusion).*

COROLLARY 1.2. *Under the conditions of theorem 1.2, there exists an isomorphism Θ mapping $H_r(M)$ onto $\prod_\alpha H_r(E^\alpha_{r-1})$. Furthermore, $Z_r \in H_r(M)$ can be uniquely represented as $\sum_\alpha i^\alpha_*(Z^\alpha_r)$, where $Z^\alpha_r \in H_r(E^\alpha_{r-1})$ ($i^\alpha_*:H_r(E^\alpha_{r-1}) \longrightarrow H_r(M)$ is induced by inclusion).*

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[1] A. Simon, n-Cyclic Elements, Bulletin of the American Mathematical Society, Vol. 62, no.1 (Jan. 1956), p.36.

PROOF. The proof is an immediate consequence of corollary 4.2, Chapter I.

For any M, $\tilde{i}^*$ is an isomorphism of $H^r(M)$ into $\prod_\alpha H^r(E^\alpha_{r-1})$ (note that $\gamma^r \neq 0$ implies $i^*_\alpha(\gamma^r) \neq 0$ for some $i^*_\alpha: H^r(M) \longrightarrow H^r(E^\alpha_{r-1})$). If M is an lc^{r-1} space, it is probably true that $\tilde{i}^*$ becomes an isomorphism of $H^r(M)$ onto the weak direct sum $\sum_\alpha H^r(E^\alpha_{r-1})$. Also, if the set $\{E^\alpha_{r-1}\}$ of all E_{r-1} sets in M is a null sequence, the same conclusion probably holds. We have been unable to verify these conjectures.

THEOREM 1.3. *If* M *is* s-lc *and* E^o_r *is an* E_r *set of* M, *where* $s \geq r$, *then* E^o_r *is* s-lc.

PROOF. Given any $x \in E^o_r$, and an open set U^o of E^o_r containing x, then there exists an open set U of M such that $U \cap E^o_r = U^o$. Since M is s-lc, there exists an open set V containing x, such that $\overline{V} \subset U$ and any compact s-cycle in V bounds on a compact subset of U. Let Z_s be a compact cycle on $V \cap E^o_r = V^o$. Then Z_s bounds on a compact subset of U. Then there exists a minimal closed set K in M such that $\overline{V}^o \subset K \subset U$ and Z_s bounds on K. We must show $K \subset U^o = U \cap E^o_r$. Suppose K is not contained in $U \cap E^o_r$, then K is not contained in E^o_r, since $K \subset U$. If we let $N = K \cup E^o_r$, then $N = N_1 \cup N_2$, T_r - separated. Thus, $K = (N_1 \cap K) \cup (N_2 \cap K) = K_1 \cup K_2$. We may assume $N_1 \supset E^o_r$ by lemma 1.4. in Chapter III. Therefore, there exists an $x \in K \cap (N - N_1)$, since N_1 is a proper subset of N. Thus,

$K_1 = (N_1 \cap K)$ is a proper subset of K. Also, $K_1 \cap E_r^o = K \cap E_r^o$, since $K_1 = N_1 \cap K$ and $N_1 \supset E_r^o$. Thus, $K_1 \supset \bar{V}_o$ since $\bar{V}_o \subset E_r^o \cap K$. Consider the Mayer Vietoris sequence below,

$$\longleftarrow H_s(K) \xleftarrow{\phi} H_s(K_1) \oplus H_s(K_2) \longleftarrow H_s(K_1 \cap K_2) \longleftarrow .$$

But, $K_1 \cap K_2 \subset N_1 \cap N_2$ and $N_1 \cap N_2$ is a T_r set, hence $K_1 \cap K_2$ is a T_r set. Also, $s \leq r$ and therefore $K_1 \cap K_2$ is a T_s set by lemma 1.12 in Chapter III. We may consider Z_s as an element of $H_s(K_1)$ since $K_1 \supset \bar{V}_o$. Now, Z_s bounds on K, hence $\phi(Z_s, 0) = 0$; therefore, $Z_s = 0$ in $H_s(K_1)$. But K is minimal, and K_1 is a proper subset of K. Thus, by contradiction, $K \subset U^o$.

Next, we shall prove three lemmas which will be useful in the construction of certain examples.

LEMMA 1.4. *Let N_1 and N_2 be subsets of M which can not be disconnected by a T_r set. Suppose that $\bar{N}_1 \cap \bar{N}_2$ is not a T_r set. Then $\bar{N}_1 \cup \bar{N}_2$ can not be disconnected by a T_r set.*

PROOF. Let $K = \bar{N}_1 \cup \bar{N}_2$ and suppose $K = K_1 \cup K_2$, T_r - separated. Then we may assume, by lemma 1.4 in Chapter III, $N_1 \subset K_1$ and $N_2 \subset K_2$. Thus, $\bar{N}_1 \subset K_1$ and $\bar{N}_2 \subset K_2$, since K_1 and K_2 are closed. Therefore, $\bar{N}_1 \cap \bar{N}_2 \subset K_1 \cap K_2$. But this is impossible, since $\bar{N}_1 \cap \bar{N}_2$ is not a T_r set and $K = K_1 \cup K_2$, T_r - separated. Thus, K can not be separated by a T_r set.

LEMMA.1.5. *Let $f: M \longrightarrow N$ and suppose $T_r^o \subset N$ is a T_r set such that $f^{-1}(T_r^o)$ is also a T_r set. If f is a*

homeomorphism of $M-f^{-1}(T_r^o)$ onto $N-T_r^o$, then $f^{-1}(T_r^{\sim})$ is a T_r set in M, where $T_r^{\sim}$ is any T_r set in N.

PROOF. Let K be an closed subset of $f^{-1}(T_r^{\sim})$, where $T_r^{\sim}$ is any T_r set in N. Denote $f^{-1}(T_r^o)$ by $\tilde{T}_r^o$. Consider the following exact sequence,

$$\longrightarrow H^r(K\cap\tilde{T}_r^o)\longrightarrow H^r(K)\longrightarrow H^r(K,(\tilde{T}_r^o\cap K))\longrightarrow H^{r+1}(K\cap\tilde{T}_r^o)\longrightarrow .$$

Since $\tilde{T}_r^o$ is a T_r set, we have $H^r(K,\cap\tilde{T}_r^o) \neq H^{r+1}(K\cap\tilde{T}_r^o) = 0$, and $H^r(K) = H^r(K,\tilde{T}_r^o\cap K)$. We also have the following exact sequence (M is compact, hence f maps closed sets into closed sets),

$$\longrightarrow H^r(f(K\cap\tilde{T}_r^o))\longrightarrow H^r(f(K))\longrightarrow H^r(f(K),f(\tilde{T}_r^o\cap K))\longrightarrow H^{r+1}(f(K\cap\tilde{T}_r^o))\longrightarrow .$$

Since $f(K)\subset T_r^{\sim}$, we have $H^r(f(K)) = H^r(f(K\cap\tilde{T}_r^o)) = H^{r+1}(f(K\cap\tilde{T}_r^o)) = 0$; therefore, $H^r(f(K),f(\tilde{T}_r^o\cap K)) = H^r(f(K)) = 0$. Now, $f(K-\tilde{T}_r^o) = f(K)-f(\tilde{T}_r^o) = f(K)-f(K\cap\tilde{T}_r^o)$ and f is a homeomorphism on $M-\tilde{T}_r^o$, hence on $K-\tilde{T}_r^o$. Therefore, $H^r(K) = H^r(K,K\cap\tilde{T}_r^o) = H^r(f(K),f(K\cap\tilde{T}_r^o)) = 0$ and $f^{-1}(T_r^{\sim})$ is a T_r set in M.

LEMMA.1.6. Assume f is a mapping of M onto N such that $f^{-1}(T_r^{\sim})$ is a T_r set, where $T_r^{\sim}$ is any T_r set in N. If $K\subset M$ can not be disconnected by a T_r set in M, then f(K) can not be disconnected by a T_r set in N.

PROOF. Suppose $f(K) = \tilde{K}_1\cup\tilde{K}_2$, T_r - separated. Then $K = K_1\cup K_2$ with $K_i = K\cap f^{-1}(\tilde{K}_i)$, $i = 1,2$, proper closed subsets. Also, $f^{-1}(\tilde{K}_1)\cap f^{-1}(\tilde{K}_2) = f^{-1}(\tilde{K}_1\cap\tilde{K}_2)$ is a T_r set, since $\tilde{K}_1\cap\tilde{K}_2$ is a T_r set. But this is impos-

sible, since K can not be separated by a T_r set.

DEFINITION 1.7. A non-degenerate subset K of M is called a simple r-link of M, if K is maximal with respect to the following property: $M = M_1 \cup M_2$, T_r - separated, implies $K \subset M_1$ (or $K \subset M_2$). In other words, K is a maximal set which can not be separated by a T_r set which also separates M.

LEMMA 1.8. Any simple r-link in M is closed.

PROOF. Let K be a simple r-link in M. Consider the collection of all pairs $\{(M_1^\alpha, M_2^\alpha)\}$, such that for every α, $M = M_1^\alpha \cup M_2^\alpha$, T_r - separated. We may index these pairs such that $K \subset M_1^\alpha$ for all α. Then $K = \bigcap_\alpha M_1^\alpha$, since $\bigcap_\alpha M_1^\alpha$ can not be disconnected by a T_r set which also separates M. Thus, K is closed since the M_1^α are closed.

LEMMA 1.9. If K_1 and K_2 are two distinct simple r-links in M, then $K_1 \cap K_2$ is a T_r set. If L is a non-degenerate subset of M that is not disconnected by any T_r set in M which disconnects M, then L is contained in a simple r-link of M. We may have L contained in more than one simple r-link of M.

PROOF. First, $K_1 \cup K_2$ is not a simple r-link. Thus, there exist sets M_1, M_2 such that $M = M_1 \cup M_2$, T_r - separated, and $K_1 \subset M_1$, $K_2 \subset M_2$. Hence, $K_1 \cap K_2 \subset M_1 \cap M_2$; therefore, $K_1 \cap K_2$ is a T_r set. The existence of a simple

r-link containing L is proved by the same methods as in lemma 1.7 in Chapter III. The uniqueness proof will not apply here since L may be a T_r set.

EXAMPLE 1.10. We will construct an lc^r space M in which the collection $\{E_r^\alpha\}$ of E_r sets does not form a null sequence. This example will also show that in an lc^r space the simple r-links need not be the same as the E_r sets. We must have $r \geq 1$ in order to produce such an example.[1]

For each positive integer n, let R_n be a solid, three-dimensional rod of height one and diameter $1/2^n$. In Euclidean three-space, let $I = \{(x, y, z) \mid x = 0, y = 0, 0 \leq z \leq 1.\}$ Then I is a unit interval on the Z-axis. Imbed R_n in Euclidean three-space so that $R_n = \{(x, y, z) \mid (y-3/2^{n+1})^2 + x^2 \leq 1/2^n, 0 \leq z \leq 1\}$. Then R_n is tangent to R_{n+1} and the sequence of sets $\{R_n\}$ converges to I. Let M be the set $\left[\overset{\infty}{\underset{1}{\cup}} R_n\right] \cup I$. Then M is a compact lc^1 space, each R_n is an E_1 set in M, but the collection $\{R_n\}$ is not a null sequence. Note that, for any pair (R_i, R_j), $i \neq j$, R_i can be separated from R_j by a unit interval and the latter is a T_1 set. Also, I is a simple 1-link. If $M = M_1 \cup M_2$, T_1 - separated, then $R_n \subset M_1$ or $R_n \subset M_2$, for any R_n. Hence, we can assume M_1 contains infinitely many of the spaces R_n. Thus, $I \subset M_1$ since M_1 is closed.

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[1] G. T. Whyburn, Analytic Topology, p. 66 and p. 71.

In this example we have $H_2(M) = 0$. We can modify the example so that $H_2(M)$ is infinitely generated.

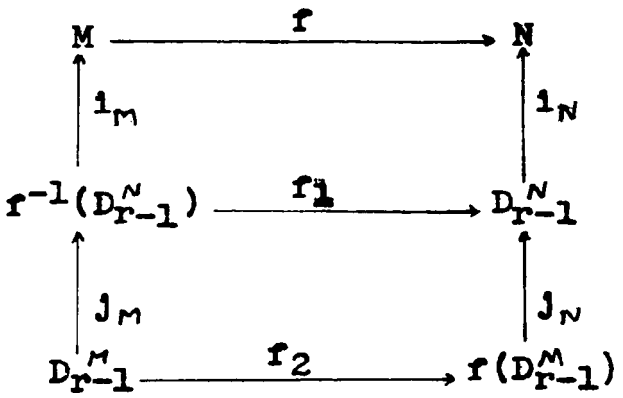
EXAMPLE 1.11. We will construct a compact lc^r space which contains an E_r set which is not lc^r. Consider the following curve in three-space:

$$x = 0,\ y = t,\ z = \sin(\pi/t),$$

for $0 < t \leq 1$. Expand this curve slightly so that it becomes a solid, three-dimensional rod, which oscillates as it approaches the origin. Let N be this space, along with its limiting line segment on the z-axis. Then N is compact. Let $P = \{(x, y, z) \mid x = 0,\ 0 \leq y \leq 1, -1 \leq z \leq 1\}$; then define $M = P \cup N$. Now, N is an E_1 set in M and M is lc^1, but N is not 0-lc.

THEOREM 1.12. *Let f be an (r-1)-monotone mapping of M onto N, where M and N are compact Hausdorff spaces. If D^N_{r-1} is a D_{r-1} set in N relative to $\gamma^r_N \in H^r(N)$, then there exists a set D^M_{r-1}, which is a D_{r-1} set relative to $f^*(\gamma^r_N)$, such that $f(D^M_{r-1}) = D^N_{r-1}$.*

PROOF. Since f is (r-1)-monotone (note that our homology and cohomology is taken on compact spaces over a field) $f^*: H^r(N) \longrightarrow H^r(M)$ is a one-to-one mapping; therefore, $f^*(\gamma^r_N) \neq 0$. Consider the commutative diagram below. The vertical mappings are inclusion mappings.



The mapping f_1 is the restriction of f to $f^{-1}(D^N_{r-1})$. Therefore, f_1 is (r-1)-monotone. Now, $i^*_N(\gamma^r_N) \neq 0$, since D^N_{r-1} is a D_{r-1} set relative to γ^r_N. Therefore, $i^*_M f^*(\gamma^r_N) = f_1^* i^*_N(\gamma^r_N) \neq 0$ since $f_1^*: H^r(D^N_{r-1}) \longrightarrow H^r(f^{-1}(D^N_{r-1}))$ is one-to-one. Since $i^*_M(f^*(\gamma^r_N)) \neq 0$, $f^{-1}(D^N_{r-1})$ contains a D_{r-1} set relative to $f^*(\gamma^r_N)$. Denote this D_{r-1} set by D^M_{r-1} and let $f_2: D^M_{r-1} \longrightarrow f(D^M_{r-1})$ be the restriction of f to D^M_{r-1}. Now, $j^*_M i^*_M(f^*(\gamma^r_N)) \neq 0$ by the definition of a D_{r-1} set. Since $j^*_M i^*_M f^*(\gamma^r_N) = f_2^* j^*_N i^*_N(\gamma^r_N)$, we must have $j^*_N i^*_N(\gamma^r_N) \neq 0$. This implies $f(D^M_{r-1}) = D^N_{r-1}$, since D^N_{r-1} is a D_{r-1} set relative to γ^r_N.

EXAMPLE 1.13. It is not possible to substitute E_{r-1} sets for D_{r-1} sets in theorem 1.12. In other words, if f is an (r-1)-monotone mapping of M onto N, where M and N are lc^r spaces and E^N_{r-1} is an E_{r-1} set in N, we may not have an E_{r-1} set in M such that $f(E^M_{r-1}) \supset E^N_{r-1}$. However, we must take $r > 1$ in order to find an example. The fact that a 0-monotone mapping will cover any E_0 set in N by the image of an E_0 set in M is basic to the 0-dimen-

sional cyclic element theory.[1]

We will construct the example in three-space. Consider the following solid cylinders:

$$M_1 = \{(x, y, z) | x^2 + y^2 \leq 1, 0 \leq z \leq 1\};$$

$$M_2 = \{(x, y, z) | x^2 + (y-2)^2 \leq 1, 0 \leq z \leq 1\}.$$

These cylinders have their axes parallel to the z-axis and each has its base in the (x, y)-plane. Let I be the line segment defined by, $I = \{(x, y, z) | x = 0, y = 1, 0 \leq z \leq 1\}$. Then M_1 and M_2 are tangent along I. Let P_1 and P_2 be the endpoints, (0,1,0) and (0,1,1), of I. Let M_3 be a closed arc joining P_1 and P_2, which does not meet $M_1 \cup M_2$, except at its endpoints, P_1 and P_2. Now, define the space M as, $M = M_1 \cup M_2 \cup M_3$. We shall define an upper semi-continuous decomposition of M. Then the decomposition space, which we will denote by N, will be a compact Hausdorff space and the decomposition will define a continuous mapping $f: M \longrightarrow N$.

To form N identify all the points in M_3 into a single point. Do not identify any other points of M. Thus, the mapping $f: M \longrightarrow N$ restricted to $M-M_3$ is a homeomorphism. If $x \in N-f(M_3)$, then $f^{-1}(x)$ is a single point. If $x = f(M_3)$, then $f^{-1}(x) = M_3$ and M_3 is an arc. Therefore, f is r-monotone for all r.

We will show that N is an E_1 set. First, neither M_1 nor M_2 can be disconnected by a T_1 set. The Alexander

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[1] G. T. Whyburn, Analytic Topology.

duality can be used to prove this. Now, f restricted to M_1 is a homeomorphism on $M_1-(p_1 \cup p_2)$. Since $p_1 \cup p_2$ and $f(p_1 \cup p_2) = f(M_3)$ are both T_1 sets, lemmas 1.5 and 1.6 imply that $f(M_1)$ can not be disconnected by a T_1 set. Similarly, $f(M_2)$ can not be disconnected by a T_1 set. Now, $f(M_1) \cap f(M_2) = f(M_1 \cap M_2)$, and $f(M_1 \cap M_2)$ is obtained by identifying the end points, p_1 and p_2, of a unit interval. Therefore, $f(M_1 \cap M_2)$ carries an essential 1-cycle, hence, can not be a T_1 set. By lemma 1.4, $f(M_1) \cup f(M_2) = f(M) = N$, can not be disconnected by a T_1 set. Therefore, N is an E_1 set. If K is a closed subset of M such that $f(K) \supset N$, then $K \supset (M_1 \cup M_2)$. But, then K can be disconnected by a T_1 set, namely $M_1 \cap M_2$. Therefore, there is no E_1 set in M, say E_1^M, such that $f(E_1^M) \supset N$.

Note that M is obviously lc^r for all r. Therefore, N is also lc^r for all r, since f is r-monotone for all r.

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