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THE TAIL FIELD OF A MARKOV CHAIN

by
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THE TAIL FIELD OF A MARKOV CHAIN

By Allan Francis Abrahamse

Let E be a subset of the integers, containing zero. Let Ω be the space of sequences $\omega = (x_0, x_1, \dots)$, $x_j \in E$. Let $X_n: \Omega \rightarrow E$ denote the coordinate function, $X_n(\omega) = x_n$ for ω as shown. Let $\mathcal{F}_n$ denote the σ -field of subsets of Ω induced by the set $X_n, X_{n+1}, \dots$ and let $\mathcal{F} = \mathcal{F}_0$ and $\mathcal{I}_\omega = \bigcap \mathcal{F}_n$. $\mathcal{I}_\omega$ is called the tail field. Let $\theta: \Omega \rightarrow \Omega$ denote the shift function, $\theta[(x_0, x_1, \dots)] = (x_1, x_2, \dots)$. Let $\mathcal{F}_I$ denote the class of sets $F \in \mathcal{F}$ such that $\theta^{-1}F = F$. $\mathcal{F}_I$ is called the invariant field. Let p be a stochastic matrix on $E \times E$, and let P_0 denote the measure on $\mathcal{F}$ such that $P_0(X_0 = 0) = 1$ and $P_0(X_{n+1} = y | X_n = x) = p(x, y)$. We study the fields $\mathcal{I}_\omega$ and $\mathcal{F}_I$ as they relate to the Markov chain $X = \{X_n; n \geq 0\}$ over the probability space $(\Omega, \mathcal{F}, P_0)$.

Blackwell defined a class $\mathcal{A}$ of subsets of E , closed under finite unions and differences, containing E and the empty set. He showed that this class is isomorphic to $\mathcal{F}_I$, modulo certain equivalence relations.

We show that θ maps $\mathcal{I}_\omega$ one-to-one onto itself, and $\mathcal{F}_I = \{F \in \mathcal{I}_\omega | \theta F = F\}$. We define a class of sequences

$A = (A_n; n \geq 0)$ of subsets of E . Denoting this class of sequences by $\mathcal{A}$, we can define operations of finite union and difference in a natural way, and also define a shift function θ mapping $\mathcal{A}$ into itself.

We show that $\mathcal{A}$ is isomorphic to $\mathcal{I}_\omega$, modulo certain equivalence relations, and that θ commutes with this isomorphism. We show that $\mathcal{A}$ is essentially the same as the subclass of sequences A in $\mathcal{A}$ such that $\theta A = A$. From this observation a proof of Blackwell's result can be obtained.

Finally, a zero-one law is obtained for the tail field in terms of solutions to the system of equations

$$f_n(x) = \sum_{y \in E} p(x, y) f_{n+1}(y).$$

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CHAPTER I

INTRODUCTION

1.1 Outline.

This thesis is concerned with the structure of certain δ -fields which occur in the theory of Markov chains. The main results will be announced in this chapter. The proofs of the theorems will be given in Chapter 2, and Chapter 3 contains a few applications and elaborations.

We adopt the practice of placing the symbol $\parallel$ at the conclusion of a proof.

1.2 Boolean δ -algebras.

Objects known as δ -fields are familiar members of the probabilist's entourage. In this thesis, however, it is necessary to use some of the more remote aspects of the theory of such objects. Therefore, we begin with a brief discussion of Boolean algebras.

The following definition of a Boolean algebra is more general than the one that is usually given. Let $\mathcal{A}$ be a non-empty set, and let $\leq$ be a partial ordering of the elements of $\mathcal{A}$. The pair $(\mathcal{A}, \leq)$ is a Boolean algebra if the following three conditions hold:

1) For each pair of elements A and B in $\mathcal{A}$, the least upper bound with respect to the ordering $\leq$ of the pair A and B exists, as well as the greatest lower bound. The least upper bound of the pair A and B is denoted by the symbol $A \vee B$, and the greatest lower bound by the symbol $A \wedge B$.

2) There is an element U and an element $\emptyset$ in $\mathcal{A}$ such that $\emptyset \leq A \leq U$ for each element $A \in \mathcal{A}$. U is called the maximal element of $\mathcal{A}$, and $\emptyset$ is called the minimal element.

3) For each $A \in \mathcal{A}$ there is an element denoted by $A^c \in \mathcal{A}$, such that $A \vee A^c = U$ and $A \wedge A^c = \emptyset$.

A Boolean algebra is a Boolean σ -algebra if in addition to these conditions, for each countable sequence $A_1, A_2, \dots$ of elements of $\mathcal{A}$, the least upper bound of the sequence, denoted by the symbol $\bigvee A_j$, exists.

A familiar example of a Boolean algebra (Boolean

σ -algebra) is that for which $\mathcal{A}$ is a class of subsets of some space Ω such that $\mathcal{A}$ is closed under the operations of taking **complements** and (countable) unions and intersections. The partial ordering is the usual containment relation $\subseteq$. We note, for example, that the least upper bound of two elements of $\mathcal{A}$ with respect to the containment relation is the union of the two elements. It is usual to call a Boolean algebra (Boolean σ -algebra) of this sort a field (σ -field) of subsets of the space Ω .

For another example of a Boolean algebra (Boolean σ -algebra) let Ω be some set of points, and suppose $\mathcal{Q}$ is a field (σ -field) of subsets of Ω . Let P be a probability measure defined on the elements of $\mathcal{Q}$. We say that two sets A and B in $\mathcal{Q}$ are P-a.s. equal if $P(A \Delta B) = 0$ ($A \Delta B$ denotes the symmetric difference, $(A-B) \cup (B-A)$). The relation of being P-a.s. equal is an equivalence relation. Let $\mathcal{Q}[P]$ denote the family of equivalence classes. If A' and B' are elements of $\mathcal{Q}[P]$, we write $A' \leq B'$ if there is a set A in A' and a set B in B' such that $P(A-B) = 0$. It is not hard to see that $\leq$ is a well-defined partial ordering of the elements of $\mathcal{Q}[P]$, and that the pair $(\mathcal{Q}[P], \leq)$ is a Boolean algebra (Boolean σ -algebra). We note, for example, that the least upper bound of two elements A' and B' of $\mathcal{Q}[P]$ is the element of $\mathcal{Q}[P]$ which contains the set $A \cup B$, where $A \in A'$ and $B \in B'$.

This thesis is concerned with isomorphisms between certain Boolean σ -algebras. We now give a precise definition of this concept.

1.21 Definition.

Let $(\mathcal{A}, \leq)$ and $(\mathcal{B}, \leq)$ be two Boolean algebras or two Boolean σ -algebras. If there exists a one-to-one map T from $\mathcal{A}$ onto $\mathcal{B}$ such that $A_1 \leq A_2$ if and only if $T(A_1) \leq T(A_2)$, then $(\mathcal{A}, \leq)$ and $(\mathcal{B}, \leq)$ are said to be isomorphic, and T is called an isomorphism.

It is easy to show that if T is an isomorphism between $(\mathcal{A}, \leq)$ and $(\mathcal{B}, \leq)$, then it preserves least upper bounds and greatest lower bounds, maps the maximal and minimal element of $\mathcal{A}$ to the maximal and minimal element of $\mathcal{B}$ respectively, and commutes with the operation $A \rightarrow A^c$. For example, suppose $(\mathcal{A}, \leq)$ and $(\mathcal{B}, \leq)$ are σ -algebras, and T is an isomorphism between them. Let $A_1, A_2, \dots$ be a sequence of elements of $\mathcal{A}$. We have $T(A_j) \leq T(\bigcup A_i)$ for each j , hence $\bigcup T(A_j) \leq T(\bigcup A_j)$. On the other hand, suppose B is an element of $\mathcal{B}$ such that $T(A_j) \leq B$ for each j . Then $A_j \leq T^{-1}(B)$ for each

j , so $\cup A_j \leq T^{-1}(B)$, hence $T(\cup A_j) \leq B$. Thus, $T(\cup A_j)$ is the least upper bound of the set $T(A_1), T(A_2), \dots$ that is to say, $\cup T(A_j) = T(\cup A_j)$.

Finally, we point out that an isomorphism can be used to prove that a Boolean algebra is in fact a Boolean σ -algebra, under certain conditions.

1.22 Lemma.

Let $(\alpha, \leq)$ and $(\beta, \leq)$ be isomorphic Boolean algebras. If in addition, $(\beta, \leq)$ is a Boolean σ -algebra, then so is $(\alpha, \leq)$.

Proof. Let $A_1, A_2, \dots$ be a countable sequence in α . Let B be the least upper bound of the sequence $T(A_1), T(A_2), \dots$, where T is the isomorphism. It is trivial to show that $T^{-1}(B)$ is the least upper bound of the sequence $A_1, A_2, \dots$, and this proves the result. ||

1.3 Probabilistic notations.

Let E be a subset of the integers which contains zero, and let Ω denote the space of sequences $\omega = (x_0, x_1, \dots)$, for $x_j \in E$, $j \geq 0$. For each non-negative integer n , let $X_n: \Omega \rightarrow E$ be the n -th coordinate function, where $X_n(\omega) = x_n$ for ω as shown. Let $\mathcal{F}$ denote the smallest σ -field with respect to which all of the functions $X_0, X_1, \dots$ are measurable.

Let $\theta: \Omega \rightarrow \Omega$ denote the "shift" function, defined by setting

$$\theta[(x_0, x_1, \dots)] = (x_1, x_2, \dots)$$

Let $\mathcal{F}_I$ denote the class of sets $A \in \mathcal{F}$ such that $\theta^{-1}A = A$. $\mathcal{F}_I$ is a σ -field, and is called the invariant field.

Let $p(x, y)$ be a stochastic matrix on $E \times E$. For each $b \in E$, there is a unique probability measure P_b on $\mathcal{F}$ such that

$$P_b(X_0 = b) = 1$$

$$P_b(X_{n+1} = y | X_n = x) = p(x, y)$$

With respect to the probability space $(\Omega, \mathcal{F}, P_b)$, the sequence $X = (X_n; n \geq 0)$ is a Markov chain, with stationary transition function p . We will be chiefly interested in this chain over the probability space $(\Omega, \mathcal{F}, P_0)$. We assume that E is the minimal state-space for this chain, i.e., for each $x \in E$, there is a non-negative integer n such that $P_0(X_n = x) > 0$.

1.4 Blackwell's characterization of $\mathcal{I}_I$.

In [1] Blackwell derived a characterization of the invariant field $\mathcal{I}_I$, and it is a generalization of this result that is the main contribution of this thesis. In this section, we will outline Blackwell's characterization. We will express it in a manner that is somewhat different from the original, and will follow more along the lines of Chung's exposition of Blackwell's results, found in [3]. In addition, our version is slightly stronger.

A subset A of E is recurrent if $P_0(X_n \in A \text{ i.o.}) > 0$, otherwise, A is transient. A recurrent set A is almost closed if

$$P_0(X_n \in A \text{ all large } n | X_n \in A \text{ i.o.}) = 1$$

We point out that for any set $A \subseteq E$, $X_n \in A$ for all large n implies that $X_n \in A$ i.o.. Therefore, the condition that the set A be almost closed amounts to requiring that the converse implication hold, P_0 -a.s.. Let $\mathcal{A}$ denote the class of subsets of E which are either almost closed or transient. It is easy to verify that $\mathcal{A}$ is a field under the usual containment relation. It is also easy to

construct examples in which $\mathcal{A}$ is not a σ -field.

On $\mathcal{A}$ we define a measure μ by setting

$$\mu(A) = P_0(X_n \in A \text{ i.o.})$$

That μ is in fact a (finitely additive) measure requires some proving, but it is not difficult, and we omit it. We consider the Boolean algebra of equivalence classes, $\mathcal{A}[\mu]$, the construction of which was explained in Section 1.2. Two sets are contained in the same element of $\mathcal{A}[\mu]$ if they differ by a transient set.

We also consider the Boolean σ -algebra $\mathcal{G}_I[P_0]$. Two sets of $\mathcal{G}_I$ are contained in the same element of $\mathcal{G}_I[P_0]$ if they are P_0 -a.s. equal.

Using these definitions, Blackwell's characterization can be stated in the following way:

1.41 Theorem.

$\mathcal{A}[\mu]$ is a Boolean σ -algebra, and is isomorphic to $\mathcal{G}_I[P_0]$.

Blackwell proved that $\mathcal{A}[\mu]$ is a Boolean algebra, isomorphic to $\mathcal{G}_I[P_0]$. Our version is stronger to the extent that we apply 1.22 to obtain that $\mathcal{A}[\mu]$ is in fact a σ -algebra.

We will derive 1.41 as a corollary to our main result, in Section 2.4.

1.5 The tail field.

In this section we will outline the main result of the thesis.

Let n be a non-negative integer, and let $\mathcal{I}_n$ denote the smallest σ -field of subsets of Ω with respect to which the functions $X_n, X_{n+1}, \dots$ are measurable. The tail field $\mathcal{I}_\infty$ is the σ -field defined by

$$\mathcal{I}_\infty = \bigcap_{n \geq 0} \mathcal{I}_n$$

If F is any subset of Ω , then the set θF is defined by the relation

$$\theta F = \{ \omega \mid \text{There is a } \nu \in F, \text{ and } \theta \nu = \omega \}.$$

The action of θ on subsets of Ω is of special interest when it is restricted to the tail field. We state our first theorem.

1.51 Theorem.

θ maps $\mathcal{I}_\infty$ one-to-one and onto itself.

Furthermore, $\mathcal{I}_I = \{F \in \mathcal{I}_\infty \mid \theta F = F\}$.

In other words, if $\mathcal{I}_\infty$ is regarded as a set of "points", then θ acts as a permutation operator on this set, and $\mathcal{I}_I$ is the set of "fixed points" under this permutation. The proof of this theorem will be given in Section 2.1.

Since $\mathcal{I}_I$ is a σ -field contained in $\mathcal{I}_\infty$ which is characterized by the set of almost closed sets in the sense of Theorem 1.41, it seems reasonable to expect that some set of objects characterizes $\mathcal{I}_\infty$ in a similar way. Just such a characterization will be discussed next.

In what follows, the symbols A and B will denote sequences of subsets of E : $A = \{A_n; n \geq 0\}$, $B = \{B_n; n \geq 0\}$, with $A_n \subseteq E$ and $B_n \subseteq E$ for each non-negative integer n . Such a sequence A will be called a transient sequence if for each integer k ,

$$P_0(X_n \in A_{n+k} \text{ i.o.}) = 0$$

The sequence A is a tail sequence if it is not a transient sequence, and for each integer k such that $P_0(X_n \in A_{n+k} \text{ i.o.}) > 0$, we have the relation

$$P_0(X_n \in A_{n+k} \text{ all large } n | X_n \in A_{n+k} \text{ i.o.}) = 1$$

For any sequence A , we have

$$\{X_n \in A_{n+k} \text{ i.o.}\} = \theta^k \{X_n \in A_n \text{ i.o.}\}$$

Therefore, A is transient if and only if for each integer k , $P_0(\theta^k \{X_n \in A_n \text{ i.o.}\}) = 0$. We also have for any sequence A ,

$$\{X_n \in A_n \text{ all large } n\} \subseteq \{X_n \in A_n \text{ i.o.}\}$$

Hence, A is a tail sequence if it is not a transient sequence, and for each integer k

$$\theta^k(X_n \in A_n \text{ i.o.}) \subseteq \theta^k(X_n \in A_n \text{ all large } n)$$

P_0 -a.s.

If A is the sequence $(D, D, D, \dots)$ where D is some subset of E , then A is a transient sequence or a tail sequence if and only if D is a transient set or an almost closed set.

Let $\mathcal{X}$ denote the class of sequences of subsets of E which are either transient sequences or tail sequences. If $A = \{A_n; n \geq 0\}$ and $B = \{B_n; n \geq 0\}$ are elements of $\mathcal{X}$ we write:

$$A^c = \{A_n^c; n \geq 0\}$$

$$A \circ B = \{A_n \circ B_n; n \geq 0\}$$

$$\theta A = \{A_{n+1}; n \geq 0\}$$

In the second expression, "o" is meant to represent any set-theoretic connective, e.g. $\cup$ or $\cap$. We will say that $A \subseteq B$ if $A_n \subseteq B_n$ for each $n \geq 0$.

1.52 Lemma.

If A and B are elements of $\mathcal{X}$, then so are A^c , θA , $A \cup B$, and $A \cap B$. The pair $(\mathcal{X}, \subseteq)$ is a Boolean algebra.

The proof of this assertion will be given in Section 2.3. We point out that $\mathcal{X}$ is not necessarily a Boolean σ -algebra. For example, let $A = \{A_n; n \geq 0\}$ be a sequence of subsets of E which is neither a transient sequence nor a tail sequence. (It is easy to construct a Markov chain in which such a sequence exists.) For each pair of non-negative integers n and k , set

$$A_n^k = \begin{cases} A_n & n \leq k \\ \emptyset & n > k \end{cases}$$

and let $A^k = \{A_n^k; n \geq 0\}$. Each A^k is a transient

sequence, so each is an element of $\mathcal{A}$. But $\bigcup A^k = A$, which is not in $\mathcal{A}$.

Two sequences A and B in $\mathcal{A}$ will be said to be equivalent if the sequence $A \Delta B$ is a transient sequence. Let $\mathcal{A}^*$ denote the family of equivalence classes. The elements of $\mathcal{A}^*$ will be denoted by symbols A^* , B^* , etc. We write $A^* \leq B^*$ if for some element $A \in A^*$, and $B \in B^*$, $A - B$ is a transient sequence. This defines a partial ordering of the elements of $\mathcal{A}^*$, and we will see that the pair $(\mathcal{A}^*, \leq)$ is a Boolean algebra.

A set F in $\mathcal{I}_\infty$ will be called a null set if $P_0(\theta^k F) = 0$ for every integer k . This is stronger than the usual definition of a null set. We see from the above definitions that the sequence $A = (A_n; n \geq 0)$ is a transient sequence if and only if the set $\{X_n \in A_n \text{ i.o.}\}$ is a null set in $\mathcal{I}_\infty$.

Two sets F and G are equivalent if $F \Delta G$ is a null set. This is indeed an equivalence relation, so we let $\mathcal{I}_\infty^*$ denote the family of equivalence classes. The elements of $\mathcal{I}_\infty^*$ will be denoted by F^* , G^* , etc. We say that $F^* \leq G^*$ if there is an element $F \in F^*$ and an element $G \in G^*$ such that $F - G$ is a null set. The pair $(\mathcal{I}_\infty^*, \leq)$ is a Boolean σ -algebra.

A shift function Θ can be defined over both $\mathcal{X}^*$ and $\mathcal{Q}_\omega^*$. We will show for A and B elements of $\mathcal{X}$, that A is equivalent to B if and only if ΘA is equivalent to ΘB . Given this, if A^* is an element of $\mathcal{X}^*$, choose $A \in A^*$, and define ΘA^* to be the element of $\mathcal{X}^*$ which contains ΘA . Similarly, for F^* an element of $\mathcal{Q}_\omega^*$, let $F \in F^*$, and let ΘF^* be the element of $\mathcal{Q}_\omega^*$ which contains ΘF .

We are now in a position to state our main result.

1.55 Theorem.

$\mathcal{X}^*$ is a Boolean σ -algebra and is isomorphic to $\mathcal{Q}_\omega^*$. Θ commutes with the isomorphism.

The proof of this assertion will be found in Section 2.3.

In Section 2.4, we will show that Blackwell's result, Theorem 1.41, can be obtained from 1.55. We will also investigate further relations between the objects defined in this section and those in Section 1.4.

CHAPTER II

THE MAIN RESULTS

2.1 Proof of Theorem 1.51.

For convenience, we restate this theorem.

2.11 Theorem.

- a) θ maps $\mathcal{I}_\infty$ one-to-one onto itself.
- b) $\mathcal{I}_I = \{F \in \mathcal{I}_\infty \mid \theta F = F\}$.

Proof. For $m \geq 0$, $k > 0$, let C be the cylinder set

$$C = \{\omega \mid X_m(\omega) \in C_0, \dots, X_{m+k}(\omega) \in C_k\}$$

where $C_0, \dots, C_k$ are subsets of E . Then

$$\theta^{-1}C = \{\omega \mid X_{m+1}(\omega) \in C_0, \dots, X_{m+k+1}(\omega) \in C_{m+k}\}$$

Therefore, if $\mathcal{C}_m$ denotes the ring of subsets of Ω induced by the cylinders in $\mathcal{I}_m$, then $\theta^{-1}\mathcal{C}_m = \mathcal{C}_{m+1}$. Since θ^{-1} is a countably additive map from $\mathcal{I}$ into $\mathcal{I}$, we see that $\theta^{-1}\mathcal{I}_m = \mathcal{I}_{m+1}$. For any set A , $\theta(\theta^{-1}A) = A$, hence $\mathcal{I}_m = \theta\mathcal{I}_{m+1}$.

For each $m \geq 0$, $\mathcal{I}_\infty \subseteq \mathcal{I}_{m+1}$, and so $\theta\mathcal{I}_\infty \subseteq \mathcal{I}_m$, hence $\theta\mathcal{I}_\infty \subseteq \mathcal{I}_\infty$. This shows that θ maps $\mathcal{I}_\infty$ into itself. A similar argument shows that $\theta^{-1}\mathcal{I}_\infty \subseteq \mathcal{I}_\infty$. Therefore, if $A \in \mathcal{I}_\infty$, it follows that $\theta^{-1}A$ is in $\mathcal{I}_\infty$ too, and since $\theta(\theta^{-1}A) = A$, this shows that θ maps $\mathcal{I}_\infty$ onto itself.

We now show that θ is one-to-one. Suppose A and B are sets in $\mathcal{I}_\infty$, and $\theta A = \theta B$. Let $\omega \in A$. Then $\theta\omega \in \theta B$. Hence there is a $\nu \in B$, such that $\theta\omega = \theta\nu$. Let $\omega = (x_0, x_1, \dots)$ and $\nu = (y_0, y_1, \dots)$. The fact that $\theta\omega = \theta\nu$ implies that $x_j = y_j$ for $j \geq 1$. Since B is in $\mathcal{I}_\infty$, it is in $\mathcal{I}_m$ for each $m \geq 1$, hence, the point $\nu' = (x_0, y_1, \dots)$ is in B . But now $\nu' = \omega$, hence we have shown that $A \subseteq B$. The symmetry of the argument proves that $A = B$, and this completes the proof of part a)

of the theorem.

To prove part b), if $A \in \mathcal{D}_I$, then in particular $A \in \mathcal{D}$, and for each integer n , $A = \theta^{-n}A \in \theta^{-n}\mathcal{D} = \mathcal{D}_n$, hence $A \in \mathcal{D}_\infty$. Furthermore, since $\theta^{-1}A = A$, $\theta(\theta^{-1}A) = \theta A$. Since $\theta(\theta^{-1}A) = A$ for any set, we see that

$$(2.12) \quad \mathcal{D}_I \subseteq \{A \in \mathcal{D}_\infty \mid \theta A = A\}.$$

On the other hand, if A is an element of the right-hand side of (2.12), then $A = \theta A$, and so $\theta^{-1}A = \theta^{-1}(\theta A)$. Since $A \in \mathcal{D}_\infty$, part a) of the theorem implies that $\theta^{-1}(\theta A) = A$, hence $A \in \mathcal{D}_I$, and this proves part b). ||

2.2 Quasi-regular functions.

A regular function is a non-negative function f on E which satisfies the relation

$$f(x) = \sum_{y \in E} p(x,y)f(y)$$

for each $x \in E$. Such functions play an important role in Blackwell's proof of Theorem 1.41. The corresponding concept in the present work is that of a quasi-regular function, which we will define shortly.

2.21 Lemma.

For any set $F \in \mathcal{F}_\infty$, and for each pair of points b, c in E , and for any integer $n \geq 0$ such that $P_c(X_n = b) > 0$, we have the relation

$$P_c(F | X_n = b) = P_b(\theta^n F)$$

Proof. Let n be any non-negative integer, and let

integer m be chosen larger than n . Let C be a cylinder set in $\mathcal{I}_m$,

$$C = \{\omega \mid X_m(\omega) \in C_0, \dots, X_{m+k}(\omega) \in C_k\}$$

where $C_0, \dots, C_k$ are subsets of E . Then

$$\theta^n C = \{\omega \mid X_{m-n}(\omega) \in C_0, \dots, X_{m+k-n}(\omega) \in C_k\}$$

and so we obtain the following relations:

$$\begin{aligned} P_c(C \mid X_n = b) &= P_c(X_m \in C_0, \dots, X_{m+k} \in C_k \mid X_n = b) \\ &= P_b(X_{m-n} \in C_0, \dots, X_{m+k-n} \in C_k) \\ &= P_b(\theta^n C) \end{aligned}$$

Hence, for fixed integer n , and for any $m \geq n$, the assertion of the lemma holds for every cylinder in $\mathcal{I}_m$,

and therefore the assertion holds for every set in $\mathcal{I}_m$.

Since $\mathcal{I}_\alpha \subseteq \mathcal{I}_m$, the lemma follows. ||

2.22 Lemma.

For any set $F \in \mathcal{I}_\infty$, for any integer n ,

$$P_x(\theta^n F) = \sum_{y \in E} p(x, y) P_y(\theta^{n+1} F)$$

Proof. Select any integer n , and let m be a positive integer greater than n . Let C be a cylinder set in $\mathcal{I}_m$, say the one defined above in the proof of 2.21. Then

$$\theta^{n+1} C = \{ \omega \mid X_{m-(n+1)}(\omega) \in C_0, \dots, X_{m+k-(n+1)}(\omega) \in C_k \}$$

and so we have the relation

$$P_y(\theta^{n+1} C) = \sum' p^{m-(n+1)}(y, x_m) p(x_m, x_{m+1}) \dots p(x_{m+k-1}, x_{m+k})$$

where the summation $\sum'$ is taken over $x_m \in C_0, \dots, x_{m+k} \in C_k$.

From this, it follows that

$$\begin{aligned}
 & \sum_{y \in E} p(x, y) P_y(\theta^{n+1} C) \\
 &= \sum_{m=0}^{\infty} p^{m-n}(x, x_m) p(x_m, x_{m+1}) \dots p(x_{m+k-1}, x_{m+k}) \\
 &= P_x(\theta^n C)
 \end{aligned}$$

Hence, the assertion of the lemma holds for all sets in

$\mathcal{I}_m$, and since $\mathcal{I}_\omega \subseteq \mathcal{I}_m$ for each m , the lemma follows. ||

We call a non-negative function f of two variables n and x , notated $f_n(x)$, where n is any integer, and $x \in E$, a quasi-regular function if

$$f_n(x) = \sum_{y \in E} p(x, y) f_{n+1}(y)$$

for each integer n and each $x \in E$. Lemma 2.22 asserts that for each set $F \in \mathcal{I}_\omega$, the function $f_n(x) = P_x(\theta^n F)$ is a quasi-regular function. This is the only kind of

quasi-regular function that will be encountered in this chapter. We now demonstrate a limit theorem for such functions.

2.23 Lemma.

Let $F \in \mathcal{H}_\infty$, and let $f_n(x) = P_x(\theta^n F)$. Then for each integer k ,

$$\lim_{n \rightarrow \infty} f_{n+k}(X_n) = \chi_{\theta^k F} \quad P_0\text{-a.s.}$$

Proof. From 2.21, it follows that

$$f_{n+k}(x) = P_0(\theta^k F | X_n = x)$$

and therefore,

$$f_{n+k}(X_n) = P_0(\theta^k F | X_n) \quad P_0\text{-a.s.}$$

Since $\theta^k F \in \mathcal{H}_\infty$, by the Markov property the left-hand side of this equation is equal to

$$P_0(\theta^k_F | X_n, X_{n-1}, \dots, X_0) \quad P_0\text{-a.s.}$$

and by a martingale theorem ([4], Cor. 1, p. 332) it converges P_0 -a.s. to $P_0(\theta^k_F | \mathcal{F}) = \chi_{\theta^k_F} \quad P_0\text{-a.s.} \parallel$

A relation between quasi-regular functions and tail sequences can now be given.

2.24 Lemma.

Let $F \in \mathcal{F}_0$, and let $f_n(x) = P_x(\theta^n_F)$. For each $n \geq 0$, let $A_n = \{x | f_n(x) > 1/2\}$. Then $A = \{A_n; n \geq 0\}$ is a tail sequence or a transient sequence.

Proof. For any integer k ,

$$\{X_n \in A_{n+k} \text{ i.o.}\} = \{f_{n+k}(X_n) > 1/2 \text{ i.o.}\}$$

$$\subseteq \{\limsup_{n \rightarrow \infty} f_{n+k}(X_n) \geq 1/2\}$$

From 2.23 we know that $\limsup_{n \rightarrow \infty} f_{n+k}(X_n) = \liminf_{n \rightarrow \infty} f_{n+k}(X_n)$
 $= \chi_{\theta_F^k} \quad P_0\text{-a.s.}$ Hence, the above set is $P_0\text{-a.s.}$ equal to

$$\theta_F^k = \{ \liminf_{n \rightarrow \infty} f_{n+k}(X_n) = 1 \}$$

$$\subseteq \{X_n \in A_{n+k} \text{ all large } n\}.$$

Thus for each integer k , $\{X_n \in A_{n+k} \text{ i.o.}\}$ is $P_0\text{-a.s.}$
a subset of $\{X_n \in A_{n+k} \text{ all large } n\}$. Hence, if
 $P_0(X_n \in A_{n+k} \text{ i.o.}) > 0$, we have

$$\begin{aligned} & P_0(X_n \in A_{n+k} \text{ all large } n | X_n \in A_{n+k} \text{ i.o.}) \\ &= P_0(X_n \in A_{n+k} \text{ large } n \cap X_n \in A_{n+k} \text{ i.o.}) / P_0(X_n \in A_{n+k} \text{ i.o.}) \\ &= 1 \end{aligned}$$

and so $A = \{A_n; n \geq 0\}$ is a tail sequence. ||

2.3 Tail sequences and Theorem 1.55.

In this section we will give a proof of Theorem 1.55. Some of the steps are of separate interest, so we will present the proof in a series of lemmas. We will also review most of the definitions as we go along. We begin by restating Lemma 1.52.

2.31 Lemma.

Let $A = \{A_n; n \geq 0\}$ and $B = \{B_n; n \geq 0\}$ be elements of $\mathcal{X}$. Then $A \cup B$, $A \cap B$, and A^c are also elements of $\mathcal{X}$. Hence, $(\mathcal{X}, \subseteq)$ is a Boolean algebra.

Proof. Let $D = \{D_n; n \geq 0\}$ be any sequence of subsets of E . An alternative to the definition given in Section 1.5 that D be a tail sequence or a transient sequence is that for each integer k , the limit of the sequence of functions

$$\chi_{\{X_n \in D_{n+k}\}}; n \geq -k, 0)$$

exist, P_0 -a.s. We have the relations:

$$\chi_{\{X_n \in (A_{n+k} \cap B_{n+k})\}} = \chi_{\{X_n \in A_{n+k}\}} \cdot \chi_{\{X_n \in B_{n+k}\}}$$

$$\chi_{\{X_n \in A_{n+k}^c\}} = 1 - \chi_{\{X_n \in A_{n+k}\}}.$$

It therefore follows from the standard theory of limits that if A and B are elements of $\mathcal{L}$, then so are $A \cap B$ and A^c . Therefore, $A \cup B$ is an element of $\mathcal{L}$ as well, and this proves the lemma. ||

We review the definition of $\mathcal{L}^*$. For A, B in $\mathcal{L}$, $A \Delta B = (A-B) \cup (B-A)$ is in $\mathcal{L}$ too. A is equivalent to B if $A \Delta B$ is a transient sequence. Then $\mathcal{L}^*$ is the family of equivalence classes. For A^*, B^* in $\mathcal{L}^*$, we write $A^* \leq B^*$ if there is an $A \in A^*$ and a $B \in B^*$ such that $A-B$ is a transient sequence. This is clearly a well-defined partial ordering of $\mathcal{L}^*$, and therefore, Lemma 2.31 implies that $(\mathcal{L}^*, \leq)$ is a Boolean algebra.

A set $F \in \mathcal{F}_\infty$ is a null set if $P_0(\theta^n F) = 0$ for every integer n . Two sets F and G in $\mathcal{F}_\infty$ are said to be equivalent if $F \Delta G$ is a null set. Let $\mathcal{F}_\infty^*$ denote

the family of equivalence classes. If F^* and G^* are in $\mathcal{I}_\omega^*$, we write $F^* \leq G^*$ if there is an element $F \in F^*$ and an element $G \in G^*$ such that $F-G$ is a null set. Again, this is a well-defined partial ordering of $\mathcal{I}_\omega^*$, and $(\mathcal{I}_\omega^*, \leq)$ is a Boolean σ -algebra.

When $A = \{A_n; n \geq 0\}$, let $S(A)$ be the subset of $\mathcal{I}_\omega$ defined by the relation

$$S(A) = \{\omega \mid X_n(\omega) \in A_n \text{ i.o.}\}.$$

It is clear from the definitions that $S(A)$ is a null set of $\mathcal{I}_\omega$ if and only if A is a transient sequence. We will presently show that S induces the isomorphism between $\mathcal{I}^*$ and $\mathcal{I}_\omega^*$.

The main properties of the correspondence S are given in the following lemma.

2.32 Lemma.

The **correspondence** $A \rightarrow S(A)$ between elements of $\mathcal{X}$ and elements of $\mathcal{Q}_\omega$ preserves set-theoretic relations and is 1:1, modulo transient sets in $\mathcal{X}$ and null sets in $\mathcal{Q}_\omega$. More precisely, if A and B are elements of $\mathcal{X}$, then

$S(A \cup B)$ is equivalent to $S(A) \cup S(B)$

$S(A \cap B)$ is equivalent to $S(A) \cap S(B)$

$S(A^c)$ is equivalent to $S(A)^c$

and $S(A)$ is equivalent to $S(B)$ if and only if A is equivalent to B .

Proof. For any two sequences $A = \{A_n; n \geq 0\}$ and $B = \{B_n; n \geq 0\}$, the following two relations always hold:

$$\{X_n \in (A_n \cup B_n) \text{ i.o.}\} = \{X_n \in A_n \text{ i.o.}\} \cup \{X_n \in B_n \text{ i.o.}\}$$

$$\begin{aligned}
& \{X_n \in (A_n - B_n) \text{ all large } n\} \\
&= \{X_n \in A_n \text{ all large } n\} - \{X_n \in B_n \text{ i.o.}\}
\end{aligned}$$

The first relation implies that $S(A \cup B) = S(A) \cup S(B)$. By definition, a sequence $D = \{D_n; n \geq 0\}$ is in $\mathcal{T}$ if the two sets $\{X_n \in D_n \text{ i.o.}\}$ and $\{X_n \in D_n \text{ all large } n\}$ are equivalent in $\mathcal{I}_\omega$. Therefore, the second relation implies that if A and B are in $\mathcal{T}$, then $S(A-B) = S(A) - S(B)$. This proves the assertions concerning set-theoretic relations.

To show that S is essentially 1:1, we note that for A and B in $\mathcal{T}$, $S(A \Delta B)$ is a null set if and only if $A \Delta B$ is a transient sequence. By the first part of the lemma, $S(A \Delta B)$ is equivalent to $S(A) \Delta S(B)$. This completes the proof. ||

Let A^* be an element of $\mathcal{T}^*$. Lemma 2.32 implies that if A and B are both in A^* , then $S(A)$ and $S(B)$ belong to the same equivalence class in $\mathcal{I}_\omega^*$. We let $S^*(A^*)$ denote the equivalence class of $\mathcal{I}_\omega^*$ which contains $S(A)$. S^* is a well-defined function from $\mathcal{T}^*$ into $\mathcal{I}_\omega^*$. In the next three lemmas, we will show that S^* is an isomorphism.

2.33 Lemma.

S^* is one-to-one.

Proof. Suppose for $A^*, B^* \in \mathcal{X}^*$, that $S^*(A^*) = S^*(B^*)$. Then for $A \in A^*$ and $B \in B^*$, $S(A)$ is equivalent to $S(B)$. From 2.32, A is equivalent to B , hence, $A^* = B^*$. ||

2.34 Lemma.

For $A^*, B^* \in \mathcal{X}^*$, $A^* \leq B^*$ if and only if $S^*(A^*) \leq S^*(B^*)$.

Proof. For A and B two elements of $\mathcal{X}$, $A-B$ is transient if and only if $S(A-B)$ is a null set. By 2.32, $S(A-B)$ is equivalent to $S(A)-S(B)$, hence, we can assert that $A-B$ is transient if and only if $S(A)-S(B)$ is a null set.

Now let A^* and B^* be elements of $\mathcal{X}^*$, and choose $A \in A^*$ and $B \in B^*$. Then $A^* \leq B^*$ if and only if $A-B$ is a transient set, which is equivalent to the assertion

that $S(A) - S(B)$ is a null set, which is equivalent to saying that $S^*(A^*) \leq S^*(B^*)$. ||

The proofs of the last two lemmas follow directly from 2.32. That S^* is also an onto function is a little deeper, and follows from the following construction.

2.35 Lemma.

Let $F \in \mathcal{I}_\omega$, and let $f_n(x) = P_x(\theta^n F)$. For each $n \geq 0$, let $A_n = \{x | f_n(x) > 1/2\}$, and let $A = \{A_n; n \geq 0\}$. Then A is an element of $\mathcal{I}$, and $S(A)$ is equivalent to F .

Proof. Lemma 2.24 implies that A is in $\mathcal{I}$. In the proof of that assertion, we showed that

$$\theta^k S(A) = \{X_n \in A_{n+k} \text{ i.o.}\}$$

$$\subseteq \theta^k F \subseteq \{X_n \in A_{n+k} \text{ all large } n\} \subseteq \theta^k S(A)$$

P_0 -a.s., hence, $\theta^k F$ is P_0 -a.s. equal to $\theta^k S(A)$, for every integer k . This means that for each k ,

$$P_0(\theta^k F \Delta \theta^k S(A)) = P_0(\theta^k (F \Delta S(A))) = 0$$

and the lemma follows. ||

Lemma 2.35 shows that S^* maps $\mathcal{X}^*$ onto $\mathcal{Q}_\omega^*$. For let $F^* \in \mathcal{Q}_\omega^*$, and choose $F \in F^*$. Then if A is the element of $\mathcal{X}$ constructed as in the statement of 2.35, and A^* is the equivalence class of $\mathcal{X}^*$ containing A , then $S(A)$ is equivalent to F , so $S^*(A^*) = F^*$.

We have thus exhibited a function S^* which maps $\mathcal{X}^*$ one-to-one onto $\mathcal{Q}_\omega^*$, such that $A^* \leq B^*$ if and only if $S^*(A^*) \leq S^*(B^*)$, so by definition 1.21, $\mathcal{X}^*$ and $\mathcal{Q}_\omega^*$ are isomorphic Boolean algebras. But $\mathcal{Q}_\omega^*$ is a Boolean σ -algebra, hence, $\mathcal{X}^*$ is also a Boolean σ -algebra, by Lemma 1.22.

To complete the proof of Theorem 1.55, we need to show that θ commutes with S^* . We will review how θ is defined over $\mathcal{X}^*$ and $\mathcal{Q}_\omega^*$ after the next result.

2.36 Lemma.

If A is an element of $\mathcal{X}$, then so is ΘA . If A and B are two elements of $\mathcal{X}$, then A is equivalent to B if and only if ΘA is equivalent to ΘB .

Proof. If $A = \{A_n; n \geq 0\}$, then we recall that $\Theta A = \{A_{n+1}; n \geq 0\}$. Now A is a transient sequence if for each integer k , $P_0(X_n \in A_{n+k} \text{ i.o.}) = 0$. Hence, if A is transient, using $k+1$ instead of k in this definition, we see that $P_0(X_n \in A_{n+k+1} \text{ i.o.}) = 0$ for every k . This implies that ΘA is transient. Conversely, if ΘA is transient, we have $P_0(X_n \in A_{(n+1)+(k-1)} \text{ i.o.}) = 0$ for each k , hence A is transient. This shows that A is transient if and only if ΘA is. The proof that A is a tail sequence if and only if ΘA is follows the same line of reasoning. This establishes the first assertion of the lemma.

Clearly $\Theta(A \triangle B) = \Theta A \triangle \Theta B$. Now A is equivalent to B if and only if $A \triangle B$ is transient, and this is equivalent to the assertion that $\Theta(A \triangle B) = \Theta A \triangle \Theta B$ is transient, hence, the second assertion of the lemma is obtained. ||

For $A^* \in \mathcal{X}^*$, if A and B are elements of A^* , Lemma 2.36 shows that ΘA and ΘB are in the same equivalence class of $\mathcal{X}^*$. Let ΘA^* denote the equivalence class of $\mathcal{X}^*$ which contains ΘA . Θ is a well-defined operation on $\mathcal{X}^*$.

From the definition of the equivalence relation on $\mathcal{Q}_\omega^*$, it is clear that for F and G in $\mathcal{Q}_\omega$, F is equivalent to G if and only if ΘF is equivalent to ΘG . Θ can be defined over $\mathcal{Q}_\omega^*$ similarly to the way it was defined over $\mathcal{X}^*$ --if $F^* \in \mathcal{Q}_\omega^*$, select $F \in F^*$, and let ΘF^* be the equivalence class of $\mathcal{Q}_\omega^*$ which contains ΘF .

We can complete the proof of Theorem 1.55 after the next lemma.

2.37 Lemma.

Let A and B be elements of $\mathcal{X}$. Then A is equivalent to ΘB if and only if $S(A)$ is equivalent to $\Theta S(B)$.

Proof. Let $A = \{A_n; n \geq 0\}$ and $B = \{B_n; n \geq 0\}$.

We note that $\theta S(B) = \theta\{X_n \in B_n \text{ i.o.}\} = \{X_n \in B_{n+1} \text{ i.o.}\}$.
 From 2.32, $S(A) \Delta \theta S(B)$ is equivalent to the set
 $\{X_n \in (A_n \Delta B_{n+1}) \text{ i.o.}\} = S(A \Delta \theta B)$. Hence, $S(A) \Delta \theta S(B)$
 is a null set if and only if $A \Delta \theta B$ is a transient
 sequence. ||

The proof of Theorem 1.55 can now be completed.
 We assert that for $A^* \in \mathcal{I}^*$, $S^*(\theta A^*) = \theta S^*(A^*)$.
 F is an element of $S^*(\theta A^*)$ if and only if there is
 an element $A \in A^*$ such that F is equivalent to
 $S(\theta A)$. By 2.37, $S(\theta A)$ is equivalent to $\theta S(A)$, hence,
 F is an element of $S^*(\theta A^*)$ if and only if it is an
 element of $\theta S^*(A^*)$. This finishes the proof of the
 theorem.

2.4 Almost closed sets and Theorem 1.41.

In this section we will return to the objects defined by Blackwell, and see how they compare with the objects we have defined. We will obtain a proof of Theorem 1.41.

We begin by examining the relation between the class $\mathcal{A}$ of almost closed sets and transient sets, and the class $\mathcal{T}$ of tail sequences and transient sequences. Let D be a subset of E . We will write $\{D\}$ as a short-hand way of writing the sequence $\{D, D, D, \dots\}$. We have already pointed out the fact that D is an element of $\mathcal{A}$ if and only if $\{D\}$ is an element of $\mathcal{T}$. It is also clear from the definitions that for D' some other subset of E , $\{D \cup D'\} = \{D\} \cup \{D'\}$, $\{D - D'\} = \{D\} - \{D'\}$ and $D \subseteq D'$ if and only if $\{D\} \subseteq \{D'\}$. We also note that $\Theta\{D\} = \{D\}$.

It follows that if D and D' are equivalent elements of $\mathcal{A}$, then $\{D\}$ and $\{D'\}$ are equivalent elements of $\mathcal{T}$. Thus we can define a function from $\mathcal{A}[\mu]$ into $\mathcal{T}^*$, whose value at the class D^* in $\mathcal{A}[\mu]$ we denote by the symbol $\{D^*\}^*$, by setting $\{D^*\}^*$ equal to the element of $\mathcal{T}^*$ which contains $\{D\}$, where D is some element of D^* . This is clearly a 1:1 map which preserves the Boolean algebraic structure of the Boolean algebras involved. It is not necessarily onto, since $\Theta\{D^*\}^* = \{D^*\}^*$ for each

element D^* of $\mathcal{A}[\mu]$, and in general it is not the case that $\Theta A^* = A^*$ for each $A^* \in \mathcal{T}^*$. We turn next to studying the elements in $\mathcal{T}^*$ which do satisfy this relation.

2.41 Lemma.

Let $A \in \mathcal{T}$, and suppose ΘA is equivalent to A . Then there is an element $D \in \mathcal{A}$ such that A is equivalent to the sequence $(D, D, \dots)$.

Proof. Let A be given in $\mathcal{T}$, such that ΘA is equivalent to A . Let $F = S(A) = \{X_n \in A_n \text{ i.o.}\}$, where $A = \{A_n; n \geq 0\}$. From Lemma 2.37, F is equivalent to ΘF . Set $f_n(x) = P_x(\Theta^n F)$, and for each non-negative integer n let $B_n = \{x | f_n(x) > 1/2\}$. From Lemma 2.35, if B is the sequence $\{B_n; n \geq 0\}$, then $S(B)$ is equivalent to $F = S(A)$, hence from Lemma 2.32, B is equivalent to A . Now since F is equivalent to ΘF , $f_n(x) = f_0(x)$ for every n , and so $B_n = B_0$ for every n . So if we set $D = B_0$, then $B = (D, D, \dots)$, and since B is in $\mathcal{T}$, D is in $\mathcal{A}$. ||

We claim that $\mathcal{T}^*$ "contains" $\mathcal{A}[\mu]$, in the sense that $\mathcal{T}^*$ contains a sub-Boolean algebra which is an isomorphic image of $\mathcal{A}[\mu]$. This image is the set $\mathcal{T}_I^*$,

defined by the relation

$$\mathcal{X}_I^* = \{A^* \in \mathcal{X}^* \mid \theta A^* = A^*\}.$$

2.42 Lemma.

$\mathcal{A}[\mu]$ and $\mathcal{X}_I^*$ are isomorphic.

Proof. The isomorphism is the correspondance $D^* \rightarrow \{D^*\}^*$. We have already remarked that this correspondance maps $\mathcal{A}[\mu]$ one-to-one into $\mathcal{X}_I^*$, and preserves the Boolean algebraic structure, so we have only to show that it is onto. Let A^* be an element of $\mathcal{X}_I^*$, so that if A is an element of A^* , then θA is equivalent to A . From Lemma 2.41, there is an element $D \in \mathcal{A}$ such that $\{D\}$ is equivalent to A . So if we let D^* denote the class in $\mathcal{A}[\mu]$ containing D , then $\{D^*\}^* = A^*$. ||

We now consider the Boolean σ -algebra $\mathcal{F}_\omega^*$, and begin by defining

$$\mathcal{F}_I^* = \{F^* \in \mathcal{F}_\omega^* \mid \theta F^* = F^*\}.$$

$\mathcal{I}_I^*$ is a Boolean σ -algebra which is contained in $\mathcal{I}_\infty^*$. The elements of $\mathcal{I}_I^*$ can be characterized in the following way: $F^* \in \mathcal{I}_I^*$ if $F^* \in \mathcal{I}_\infty^*$, and for each $F \in F^*$, F and ΘF differ by a null set.

$\mathcal{I}_I^*$ is a family of classes of sets in $\mathcal{I}_\infty$. $\mathcal{I}_I[P_0]$ is a family of classes of sets in $\mathcal{I}_I$, and so if F' is an element of $\mathcal{I}_I[P_0]$, if $F \in F'$, then F is equal to ΘF .

2.43 Lemma.

For each element F' of $\mathcal{I}_I[P_0]$ there exists a unique element of $\mathcal{I}_I^*$ such that $F' \subseteq F^*$. Conversely, every element F^* of $\mathcal{I}_I^*$ contains one and only one element F' of $\mathcal{I}_I[P_0]$.

Proof. Choose F' a member of $\mathcal{I}_I[P_0]$, and select F from F' . Since $F \in \mathcal{I}_I$, by 1.51, $F \in \mathcal{I}_\infty$. Let F^* be the element of $\mathcal{I}_\infty^*$ containing F . There is only one such set F^* . This establishes the existence and uniqueness of a set F^* , and we now show that $F' \subseteq F^*$. Suppose $G \in F'$. By definition, $P_0(G \Delta F) = 0$. Now F and G are elements of $\mathcal{I}_I$, hence for every integer n , $\Theta^n(F \Delta G) = F \Delta G$. From this

we obtain that $P_0(\theta^n(F \Delta G)) = P_0(F \Delta G) = 0$ for every integer n , hence F and G differ by a null set. F and G are both elements of $\mathcal{Q}_0$, and so they are equivalent in $\mathcal{Q}_0$, hence $G \in F^*$. Hence, we see that $F' \subseteq F^*$.

To prove the converse, suppose $F^* \in \mathcal{Q}_I^*$. Let $F \in F^*$, and let $f_n(x) = P_x(\theta^n F)$. For each integer n , let $A_n = \{x | f_n(x) > 1/2\}$. Then letting $A = \{A_n; n \geq 0\}$, we know that $S(A)$ is equivalent to F . Since $F^* \in \mathcal{Q}_I^*$, we have by definition that $\theta F^* = F^*$, hence $f_n(x) = f_0(x)$ for every integer n , and therefore $A_n = A_0$ for every integer n . Hence, $S(A) = \{X_n \in A_0 \text{ i.o.}\}$ is an element of $\mathcal{Q}_I$. Let F' denote the element of $\mathcal{Q}_I[P_0]$ containing $S(A)$, then $F' \subseteq F^*$. Thus we have established the existence of F' . To prove uniqueness, suppose G' is an element of $\mathcal{Q}_I[P_0]$ such that $G' \subseteq F^*$. Then if $G \in G'$, G is in F^* , hence G is equivalent to $S(A)$, and so $G' = F'$. ||

The correspondence of Lemma 2.43 is clearly an isomorphism. Thus, we have obtained:

2.44 Corollary.

$\mathcal{Q}_I[P_0]$ and $\mathcal{Q}_I^*$ are isomorphic.

A proof of Theorem 1.41 now follows. We notice that

$$\mathcal{Q}_I^* = \{F^* \in \mathcal{Q}_\infty^* \mid \theta F^* = F^*\}$$

$$\mathcal{X}_I^* = \{A^* \in \mathcal{X}^* \mid \theta A^* = A^*\}.$$

Since θ commutes with the isomorphism function S^* , it follows that $\mathcal{Q}_I^*$ and $\mathcal{X}_I^*$ are also isomorphic. Theorem 1.41 follows from this fact together with Lemma 2.42 and Corollary 2.44.

CHAPTER III

ELABORATIONS

3.1 Cyclic sequences.

A tail sequence $A = \{A_n; n \geq 0\}$ is said to be minimal if for any sequence $B = \{B_n; n \geq 0\}$ in $\mathcal{X}$, if $B-A$ is a transient sequence, then B is either a transient sequence or is equivalent to A . It turns out that a minimal sequence, if one exists, is cyclic, in a sense to be explained below.

A set $F \in \mathcal{Q}_\infty$ is an atom of $\mathcal{Q}_\infty$ if it is not a null set, and every subset $G \in \mathcal{Q}_\infty$ of F is either a null set or is equivalent to F . It is easy to see that under the correspondance of Theorem 1.51, the atoms of $\mathcal{Q}_\infty$ correspond to the minimal tail sequences, if any of either exist.

Let B be a minimal tail sequence. For each integer j , either $\Theta^j B$ is equivalent to B , or $\Theta^j B \cap B$ is a transient tail sequence. Let N_B be defined by the following relation:

$$N_B = \inf \{j | j > 0, \theta^j B \text{ is equivalent to } B\}.$$

According to the usual convention, N_B is infinite if the set on the right is empty. N_B will be known as the period of the sequence B .

3.11 Theorem.

Let B be a minimal tail sequence with period N_B .

i) $N_B = \infty$. Then there exists an infinite sequence $C = \{C_n; n \geq 0\}$ of mutually disjoint subsets C_n of E , which is a tail sequence equivalent to B .

ii) $N_B < \infty$. Then there exists a set of N_B mutually disjoint sets $C_0, \dots, C_{N_B-1}$, such that $C = \{C_n \pmod{N_B}; n \geq 0\}$ is a tail sequence equivalent to B .

Proof. We will prove only assertion i). The proof of ii) follows the same lines, with appropriate

changes of indices, modulo N_B .

Let $F = (X_n \in B_n \text{ i.o.})$. For $i \neq j$, the sets $\theta^i F$ and $\theta^j F$ have a null intersection. Let $G = \cup \theta^i F$. Then G is an element of $\mathfrak{I}_I$.

Define $f_n(x) = P_x(\theta^n F)$, and $g(x) = P_x(G)$. Since for $i \neq j$, $P_x(\theta^n(\theta^i F \Delta \theta^j F)) = 0$, we have

$$g(x) = \sum_{n=-\infty}^{\infty} f_n(x)$$

We define the sets C_n by

$$C_n = \{x \in B_n \mid g(x) > 0, f_n(x)/g(x) > 1/2\}$$

From the way in which f and g are related, it follows that the sets $\{C_n; n \geq 0\}$ are mutually disjoint.

It follows from Lemma 2.23 that for any integer k ,

$$f_{n+k}(X_n) \rightarrow \chi_{\theta^k F} \quad P_0\text{-a.s.}$$

Lemma 2.23 also applies to the function g , and we have

$$g(X_n) \rightarrow \chi_G \quad P_0\text{-a.s.}$$

and these two limits imply that over the set G ,

$$\frac{f_{n+k}(X_n)}{g(X_n)} \rightarrow \chi_{\theta^{k_F}} \quad P_0\text{-a.s.}$$

F is not a null set, so neither is G . Hence, for some integer k , $f_{n+k}(X_n)/g(X_n) \rightarrow 1$ with positive probability, which means that $X_n \in C_{n+k}$ for all large n with positive probability, hence, the sets $\{C_n; n \geq 0\}$ are not all empty. We obtain the following string of P_0 -a.s. equal sets:

$$\begin{aligned} \theta^{k_F} &= (X_n \in B_{n+k} \text{ i.o.}) \\ &= (f_{n+k}(X_n)/g(X_n) \rightarrow 1) \cap G = \{X_n \in C_{n+k} \text{ i.o.}\} \cap G. \end{aligned}$$

Hence, $P_0((X_n \in B_{n+k} \text{ i.o.}) - (X_n \in C_{n+k} \text{ i.o.}) | G) = 0$.

Since for each k , $(X_n \in B_{n+k} \text{ i.o.}) \in G$, we also have

$P_0((X_n \in B_{n+k}) | G^c) = 0$. These two equalities imply:

$$P_0((X_n \in B_{n+k} \text{ i.o.}) - (X_n \in C_{n+k} \text{ i.o.}))$$

$$= P_0((X_n \in B_{n+k} - C_{n+k})) = 0$$

Now $C_n \subseteq B_n$ for each n , so this last expression implies that C is equivalent to B , which completes the proof. ||

3.2 Trivial tail fields.

The tail field $\mathcal{I}_\infty$ is said to be trivial if for each $F \in \mathcal{I}_\infty$, $P_0(F)$ is either zero or one. One of Blackwell's results in [1] was that the invariant field $\mathcal{I}_1$ is trivial if and only if the only bounded regular functions are the constant functions. Analogous to this, we obtain:

3.21 Theorem.

$\mathcal{I}_\infty$ is trivial if and only if the only bounded solutions to

$$f_n(x) = \sum_{y \in E} p(x, y) f_{n+1}(y)$$

on the set $E^0 = \{(n, x) | P_0(X_n = x) > 0\}$ are constant over this set.

Proof. Suppose $\mathcal{I}_\infty$ is trivial. Let f be a bounded solution to the given equation on E^0 . Then $f_n(X_n)$ is a martingale, and since $\mathcal{I}_\infty$ is trivial, the P_0 -a.s. limit, which is measurable with respect to $\mathcal{I}_\infty$, is some constant, say λ . Now for each $(n, x) \in E^0$,

$$f_n(x) = E_0(f_m(X_m) | X_n = x)$$

where E_0 is the expectation with respect to the measure P_0 . By bounded convergence, this proves that $f_n(x) = \lambda$ for each $(n, x) \in E^0$.

Conversely, suppose the only bounded solutions are constants. For $F \in \mathcal{G}_\infty$, let $f_n(x) = P_x(\theta^n F)$. By Lemma 2.22 f is a bounded solution, and so by hypothesis is equal to some constant λ on E^0 . By Lemma 2.23, the P_0 -a.s. limit of the sequence $f_n(X_n)$ is χ_F , and clearly $f_n(X_n) = \lambda$ for each n , hence λ is either zero or one. In particular, $f_0(0) = P_0(F)$ is either zero or one. ||

3.3 A theorem of Blackwell and Freedman.

A Markov chain X is recurrent if for each $x \in E$, $P_0(X_n = x \text{ i.o.}) > 0$. We conclude this thesis with a new proof of a result of Blackwell and Freedman concerning such chains [2].

3.31 Theorem.

The tail field of a recurrent chain is trivial.

Proof. Let f be a bounded solution to

$$f_n(x) = \sum_{y \in E} p(x, y) f_{n+1}(y)$$

over the set E^0 . We will show that f is constant over E^0 , and the theorem follows from Theorem 3.21.

Let T_n denote the index of the n -th return to 0, with $T_0 = 0$. The sequence $T = \{T_n; n \geq 0\}$ is a random walk.

The sequence $f_n(X_n)$ is a P_0 -martingale, and therefore $\lim f_n(X_n)$ exists P_0 -a.s. Hence,

$$\lim_n f_{T_n}(X_{T_n}) = \lim_n f_{T_n}(0)$$

exists, P_0 -a.s., and is equal to the same limiting value. But this limit is measurable with respect to the invariant field of the random walk T . The invariant field of a random walk is trivial, therefore, this limit is some constant λ . By bounded convergence, $f_n(x)$ assumes this constant value on E^0 . ||

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