Negative microfilm 65–5311
received from university

GROBE, Elizabeth Mendell, 1934–
SOME p–ADIC AND q–ADIC VERSIONS
OF ROTH'S THEOREM.

The University of Michigan, Ph.D., 1964
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

SOME p-ADIC AND q-ADIC VERSIONS OF ROTH'S THEOREM

by

Elizabeth Mendell Grobe

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1964

Doctoral Committee:

Professor William J. LeVeque, Chairman
Professor Leslie Kish
Professor Donald J. Lewis
Professor Jack E. McLaughlin
Professor Allen L. Shields
Doctor John H. Smith

ACKNOWLEDGMENTS

The writer is especially indebted to Professor W. J. LeVeque for all the time and energy which he spent in reading the manuscript and in discussing the many problems which arose therefrom. It is thanks to him that so many of the difficulties did indeed prove to be nonessential. Thanks are also due to Professor D. J. Lewis for a critical reading of the manuscript.

Work on this thesis was supported in part by N.S.F. Grant GP-88.

TABLE OF CONTENTS

		Page
ACKNOWLEDMENTS		ii
INTRODUCTION		1

Notation and Valuation Theory

Chapter

I.	STATEMENT OF THE THEOREMS	8
II.	PROOF OF THEOREM I	11
III.	A BOUND ON THE NUMBER OF SOLUTIONS IN THEOREM I	34
IV.	EQUIVALENCE OF THEOREMS I AND II	46
V.	PROOF OF THEOREM III	53
VI.	THEOREM III IMPLIES THEOREM I	63
VII.	A BOUND ON THE NUMBER OF SOLUTIONS IN THEOREM III	66
VIII.	EQUIVALENCE OF THEOREM III AND THEOREM IV	71
IX.	RESTRICTION TO INTEGERS IN K	76
X.	SOME RESTATEMENTS OF THE THEOREMS	88
XI.	SOME APPLICATIONS INVOLVING BINARY FORMS	105
APPENDIX A		122
BIBLIOGRAPHY		126

INTRODUCTION

NOTATION AND VALUATION THEORY

Let α be a real algebraic number of degree $n \geq 2$ over the rationals. As early as 1844, Liouville had proved that there exists a constant $c > 0$ such that

$$|\alpha - \frac{p}{q}| > \frac{c}{q^n}$$

for all rational integers p and q with $q > 0$. Since then, the inequality has been sharpened and generalized in a number of ways. Thue, Siegel, and Dyson replaced the exponent n by successively smaller values, and in 1955 Roth obtained the best possible exponent independent of q, namely $2 + \epsilon$, where $\epsilon > 0$ is arbitrarily small. Siegel, Mahler, Parry, LeVeque, and Ridout all obtained generalizations. These involved modifying the class of numbers with which α is to be approximated, broadening the class of valuations used in the inequality, or a combination of the two. One can, for example, replace $\frac{p}{q}$ by an element β of a fixed algebraic number field of finite degree over the rationals, and q on the right-hand side by the height of β (the maximum of the absolute values of the coefficients in the defining equation of β); or one can replace the archimedean absolute value by other valuations, or one can do both simultaneously.

Mahler [8] proved several theorems involving rational approximations with p-adic and g-adic valuations. In Appendix C of his book, he stated six conjectured generalizations of these theorems in which

the approximation is by elements of a fixed algebraic number field of finite degree, pointing out that "detailed proofs would be rather long and involved; but there is no reason to foresee any essential difficulties." His conjectures are Theorems I-VI of this thesis. Theorems I-IV are mutually equivalent, and Theorems V and VI are specializations, so that the main result is the proof of Theorem I. A similar proof is given in Mahler [9]. A sketch of such a proof is also to be found in Lang [4]. The basic technique of the proof, which allows us to obtain an exponent independent of n, is Roth's, while the necessary extension of these ideas to cover algebraic approximations is due to Siegel and LeVeque. The treatment of different valuations is the logical extension of Mahler's treatment of p-adic and g-adic valuations, and owes a great deal to his extremely clear development.

In Chapters I-VIII, the proofs of Theorems I-IV are given and some bounds are obtained on the number of solutions in case this number is finite. In obtaining the first bound, we follow Parry [10] and Davenport and Roth [1]. Chapter IX consists of the proofs of Theorems V and VI and several other specializations, and follows Mahler [9]. Chapter X consists of some restatements of the theorems, and Chapter XI consists of some results involving binary forms, several of which are generalizations of theorems proved by Mahler in [8].

Before stating Theorem I, it is necessary to establish some notation.

Let K be an algebraic number field of finite degree n over the rationals with R_1 real and $2R_2$ imaginary conjugate fields $K^{(1)},\ldots,K^{(n)}$, ordered so that the first R_1 are real and $K^{(R_1+i)}$ and $K^{(R_1+R_2+i)}$ are complex conjugate fields for $i = 1,\ldots,R_2$.

Put

$$g_i = \begin{cases} 1 & \text{if } 1 \le i \le R_1 \\ 2 & \text{if } R_1 + 1 \le i \le R_1 + R_2. \end{cases}$$

Let $\mathbb{Z}$ denote the rational integers and $\mathbb{Q}$ the rational numbers.

For $\alpha \in K$, put $N(\alpha) = |N_{K/\mathbb{Q}}(\alpha)|$, and for $\mathfrak{a}$ an ideal of K, put $N(\mathfrak{a}) = N_{K/\mathbb{Q}}(\mathfrak{a})$. $H(\alpha)$, the height of α, is the maximum of the absolute values of the coefficients of the defining polynomial of α, the latter being the polynomial of lowest degree over $\mathbb{Z}$ having α as a zero and having relatively prime coefficients.

For $\alpha \in K$ and $i = 1,\ldots,R_1 + R_2$, let $|\alpha|_i = |\alpha^{(i)}|$, where $\alpha^{(i)}$ is the conjugate of α in $K^{(i)}$, and let K_i be the completion of K with respect to this valuation, so that K_i is the real field for $i = 1,\ldots,R_1$ and the complex field for $i = R_1 + 1,\ldots,R_1 + R_2$. For $\xi \in K_i$, let $|\xi|_i$ denote the continuation of $|..|_i$ onto K_i.

If p is a rational prime, let the principal ideal (p) in K have the factorization

$$(p) = \prod_{j=1}^{\pi(p)} \mathfrak{P}_j^{e(\mathfrak{P}_j)}$$

where the $\mathfrak{P}_j$ are distinct prime ideals in K. Let $N(\mathfrak{P}_j) = p^{f(\mathfrak{P}_j)}$, and set $g(\mathfrak{P}_j) = e(\mathfrak{P}_j)\, f(\mathfrak{P}_j)$ for $j = 1,\ldots,\pi(p)$. Clearly

$$(1) \qquad \sum_{j=1}^{\pi(p)} g(\mathfrak{P}_j) = n.$$

If α is a nonzero element of K, and $\mathfrak{P}$ is a prime ideal in K which divides the rational prime p, and if the fractional ideal (α) has the representation

$$(\alpha) = \frac{\mathfrak{m}\mathfrak{P}^h}{\mathfrak{n}}$$

where $\mathfrak{m}$ and $\mathfrak{n}$ are integral ideals relatively prime to $\mathfrak{P}$, then we put

$$|\alpha|_{\mathfrak{P}} = p^{\frac{-h}{e(\mathfrak{P})}}.$$

For completeness, define

$$|0|_{\mathfrak{P}} = 0.$$

Let $K_{\mathfrak{P}}$ be the completion of K with respect to the valuation $|\ldots|_{\mathfrak{P}}$ on K. It is a field and its elements are $\mathfrak{P}$-adic numbers.

On the other hand, if $\mathfrak{q}$ is a nontrivial integral ideal of K with factorization $\mathfrak{q} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$ into distinct prime ideals of K, where $e_1,\ldots,e_r$ are positive rational integers, and if $\mathfrak{P}_j$ divides the rational prime p_j for $j = 1,\ldots,r$, then we put

$$|\alpha|_{\mathfrak{q}} = \max\left(|\alpha|_{\mathfrak{P}_1}^{\frac{e(\mathfrak{P}_1)\log N(\mathfrak{q})}{e_1 \log p_1}}, \ldots, |\alpha|_{\mathfrak{P}_r}^{\frac{e(\mathfrak{P}_r)\log N(\mathfrak{q})}{e_r \log p_r}}\right).$$

$|\ldots|_{\mathfrak{q}}$ is a pseudo-valuation; that is, a real valued function which satisfies the properties of a valuation except for the product equality, which is replaced by $|\alpha\beta|_{\mathfrak{q}} \leq |\alpha|_{\mathfrak{q}}\ |\beta|_{\mathfrak{q}}$. (See Mahler [8, p. 4].) The completion of K with respect to this $\mathfrak{q}$-adic pseudo-valuation is the ring of $\mathfrak{q}$-adic numbers, and is denoted by $K_{\mathfrak{q}}$.

We collect a few properties of these valuations and pseudo-valuations. Clearly, for all $\alpha \in K$,

$$N(\alpha)\prod_{\mathfrak{P}}|\alpha|_{\mathfrak{P}}^{g(\mathfrak{P})} = \begin{cases} 1 & \alpha \neq 0 \\ 0 & \alpha = 0 \end{cases} \tag{2}$$

where $\mathfrak{P}$ runs over all prime ideals of K. Consequently,

$$N(\alpha)\prod_{j=1}^{r}|\alpha|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \geq 1 \tag{3}$$

if $\alpha \neq 0$ is an integer in K and $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ are distinct prime ideals of K.

It should be noted that if $\mathfrak{q} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$ and $(\alpha) = \mathfrak{P}_1^{h_1} \cdots \mathfrak{P}_r^{h_r}\frac{\mathfrak{m}}{\mathfrak{n}}$, where $\alpha \in K$ and $\mathfrak{m}$, $\mathfrak{n}$, and $\mathfrak{q}$ are pairwise relatively prime ideals of K, then

$$|\alpha|_{\mathfrak{q}} = \max_i \left(\frac{1}{N(\mathfrak{q})}\right)^{\frac{h_i}{e_i}}. \tag{4}$$

From this it easily follows that if c is a positive constant and if

α is an element of $K_{\mathfrak{q}}$ such that $|\alpha|_{\mathfrak{q}} \leq c$, then $\mathfrak{q}^{w} || (\alpha)$ for some nonnegative integer w, and

$$N(\mathfrak{q})^{w} > \frac{1}{cN(\mathfrak{q})} . \tag{5}$$

For let $(\alpha) = \mathfrak{q}^{w} \mathfrak{p}_1^{w_1} \cdots \mathfrak{p}_r^{w_r} \frac{\mathfrak{m}}{\mathfrak{n}}$ with $\mathfrak{m}$ and $\mathfrak{n}$ as above and $w \geq 0$. Then $w_i < e_i$ for some i, $1 \leq i \leq r$. Now by equation (4),

$$|\alpha|_{\mathfrak{q}} = \max_i \left(N(\mathfrak{q})^{-\frac{e_i w + w_i}{e_i}}\right)$$

$$= \left\{\frac{1}{N(\mathfrak{q})}\right\}^{w + \min_i \left(\frac{w_i}{e_i}\right)},$$

so that

$$N(\mathfrak{q})^{w} = \frac{\left\{\frac{1}{N(\mathfrak{q})}\right\}^{\min_i \left(\frac{w_i}{e_i}\right)}}{|\alpha|_{\mathfrak{q}}} > \frac{1}{cN(\mathfrak{q})} .$$

Finally, we note the correspondence between $\mathfrak{q}$-adic numbers and $\mathfrak{p}_j$-adic numbers where $\mathfrak{q} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ as above. It follows directly from Mahler's decomposition theorem [8, pp. 22-23], for instance, that there exists a one to one correspondence

$$\Xi \longleftrightarrow (\eta_1, \ldots, \eta_r)$$

between the elements Ξ of $K_{\mathfrak{q}}$ and the ordered sets $(\eta_1, \ldots, \eta_r)$ with one element from each of $K_{\mathfrak{p}_1}, \ldots, K_{\mathfrak{p}_r}$. This correspondence

preserves arithmetic operations, the operations being performed componentwise on the r-tuples. Moreover, if $\Xi \longleftrightarrow (\eta_1,\ldots,\eta_r)$ as above, then

$$(6) \qquad |\varkappa - \Xi|_{\mathfrak{q}} = \max_i \left(|\varkappa - \eta_i|_{\mathfrak{P}_i}^{\frac{e(\mathfrak{P}_i)\log N(\mathfrak{q})}{e_i \log p_i}} \right).$$

$\eta_1,\ldots,\eta_r$ are called the components of Ξ.

We are now in a position to state Mahler's six assertions.

CHAPTER I

STATEMENT OF THE THEOREMS

The following notation is used in the statement of the theorems and throughout this thesis.

Let $\mathfrak{q}, \mathfrak{q}', \mathfrak{q}''$ be integral ideals of K different from (0) and (1) and pairwise relatively prime.

Let r, s, t be nonnegative integers and let $\mathfrak{P}_1,\ldots,\mathfrak{P}_r$; $\mathfrak{P}_{r+1},\ldots,\mathfrak{P}_{r+s}$; $\mathfrak{P}_{r+s+1},\ldots,\mathfrak{P}_{r+s+t}$ be distinct prime ideals of K.

We adopt the convention that

(1) $$\xi_i \in K_i, \quad i = 1,\ldots,R_1 + R_2$$

$$\eta_i \in K_{\mathfrak{P}_i}, \quad i = 1,\ldots,r.$$

Let u be an integer, $0 \le u \le R_1 + R_2$, and let $i_1,\ldots,i_u$ be integers such that $1 \le i_1 < i_2 < \ldots < i_u \le R_1 + R_2$, where in case $u = 0$, the set $i_1,\ldots,i_u$ is empty.

Let $\rho_1,\ldots,\rho_u$, σ, λ, and μ be nonnegative real numbers, where $0 \le \lambda \le 1$ and $0 \le \mu \le 1$.

Let $c_1,\ldots,c_u$; c, c', c"; C, C', C", $C_1,\ldots,C_u$ be positive constants.

If x is any nonnegative number, let $x^* = \min(x, 1)$.

THEOREM I: Let $\xi_{i_1},\ldots,\xi_{i_u}$ and $\eta_1,\ldots,\eta_r$ be algebraic numbers satisfying (1). Let Ξ be the $\mathfrak{q}$-adic number defined by $\Xi \longleftrightarrow (\eta_1,\ldots,\eta_r)$. Assume that there are infinitely many distinct nonzero elements $\varkappa$ of K such that

(A) $|\xi_{i_k} - \varkappa^{(i_k)}|^* \leq c_k H(\varkappa)^{-\rho_k}, \quad k = 1,\ldots,u$

(B) $|\Xi - \varkappa|^*_{\mathfrak{q}} \leq cH(\varkappa)^{-\sigma}$

(C) $|\varkappa|^*_{\mathfrak{q}'} \leq c'H(\varkappa)^{\lambda-1}$

(D) $|\frac{1}{\varkappa}|^*_{\mathfrak{q}''} \leq c''H(\varkappa)^{\mu-1}.$

Then $\sum_{k=1}^{u} g_{i_k} \rho_k + \sigma \leq \lambda + \mu.$

THEOREM II: Let $F(x)$ be a nonconstant polynomial over K which has no multiple factors. Assume that there exists an infinite sequence of distinct nonzero elements $\varkappa$ of K satisfying

(A') $|F(\varkappa)|^*_{i_k} \leq C_k H(\varkappa)^{-\rho_k}, \quad k = 1,\ldots,u$

(B') $|F(\varkappa)|^*_{\mathfrak{q}} \leq CH(\varkappa)^{-\sigma}$

(C') $|\varkappa|^*_{\mathfrak{q}'} \leq C'H(\varkappa)^{\lambda-1}$

(D') $|\frac{1}{\varkappa}|^*_{\mathfrak{q}''} \leq C''H(\varkappa)^{\mu-1}.$

Then $\sum_{k=1}^{u} g_{i_k} \rho_k + \sigma \leq \lambda + \mu.$

THEOREM III: Let $\xi_{i_1},\ldots,\xi_{i_u}$ and $\eta_1,\ldots,\eta_r$ be algebraic numbers satisfying (1). Assume that there exists an infinite sequence of distinct nonzero elements $\varkappa$ of K such that

(E) $$\prod_{k=1}^{u} |\xi_{i_k} - \varkappa^{(i_k)}|^{g_{i_k}} \prod_{j=1}^{r} |\eta_j - \varkappa|^{*\,g(\mathfrak{P}_j)}_{\mathfrak{P}_j} \prod_{j=r+1}^{r+s} |\varkappa|^{*\,g(\mathfrak{P}_j)}_{\mathfrak{P}_j} \prod_{j=r+s+1}^{r+s+t} |\tfrac{1}{\varkappa}|^{*\,g(\mathfrak{P}_j)}_{\mathfrak{P}_j}$$

$$\leq cH(\varkappa)^{-\tau}.$$

Then $\tau \leq 2$.

THEOREM IV: Let $F_1(x),\ldots,F_u(x)$, $G_1(x),\ldots,G_r(x)$ be equal or distinct nonconstant polynomials over K which have no multiple factors. Assume that there exists an infinite sequence of distinct nonzero elements $\varkappa$ of K satisfying

$$(F)\quad \prod_{k=1}^{u} |F_k(\varkappa)|_{\mathfrak{p}_j}^{*e_{ik}} \prod_{j=1}^{r} |G_j(\varkappa)|_{\mathfrak{p}_j}^{*g(\mathfrak{p}_j)} \prod_{j=r+1}^{r+s} |\varkappa|_{\mathfrak{p}_j}^{*g(\mathfrak{p}_j)} \prod_{j=r+s+1}^{r+s+t} \left|\frac{1}{\varkappa}\right|_{\mathfrak{p}_j}^{*g(\mathfrak{p}_j)}$$

$$\leq C\, H(\varkappa)^{-\tau}.$$

Then $\tau \leq 2$.

THEOREM V: If, in Theorem III, $t = 0$ and the numbers $\varkappa$ are distinct nonzero integers in K, then the conclusion becomes $\tau \leq 1$.

THEOREM VI: If, in Theorem IV, $t = 0$ and the numbers $\varkappa$ are distinct nonzero integers in K, then the conclusion becomes $\tau \leq 1$.

CHAPTER II

PROOF OF THEOREM I

The proof of Theorem I depends mainly on two previous results, one due to Mahler [8, p. 105][1] and one to LeVeque [5, p. 140][2]. These are Lemmas 1 and 2 respectively.

LEMMA 1: Let $F(x) = F_0x^f + F_1x^{f-1} + \ldots + F_f$ be a nonconstant polynomial of exact degree f over $\mathbb{Z}$ which has no multiple factors. Put $d_1 = 2 \max(|F_0|, \ldots, |F_f|)$. Let $r_1, \ldots, r_m$ be positive integers, and let d be a real number not less than $4f\sqrt{2m}$.

Then there exists a polynomial

$$A(x_1, \ldots, x_m) = \sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} a_{k_1 \ldots k_m} x_1^{k_1} \ldots x_m^{k_m} \not\equiv 0$$

with the following properties.

(a) Its coefficients are integers satisfying

$$|a_{k_1 \ldots k_m}| \leq 5(4d_1)^{r_1 + \ldots + r_m},$$

and they vanish unless

---

1 The hypothesis that $F_f \neq 0$ is unnecessary.

2 This is Theorem 4-12 with $N = 1$. In the proof of Theorem 4-11 [5, p. 138], it is stated that $|V| < M$. This should read $|V| < M^N$, necessitating subsequent minor changes in Theorems 4-11 and 4-12. However for $N = 1$, clearly no changes are necessary.

$$\frac{m-d}{2} < \sum_{h=1}^{m} \frac{k_h}{r_h} < \frac{m+d}{2} .$$

(b) Its partial derivative

$$A_{j_1 \dots j_m}(x,\dots,x) = \frac{1}{j_1! \dots j_m!} \left(\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{j_m} A(x,\dots,x)$$

is divisible by $F(x)$ whenever $0 \le j_1 \le r_1, \dots, 0 \le j_m \le r_m$ and $\sum_{h=1}^{m} \frac{j_h}{r_h} \le \frac{m-d}{2}$.

(c) The absolute values of the coefficients in $A_{j_1 \dots j_m}(x_1,\dots,x_m)$ are less than or equal to those of the corresponding coefficients in

$$5(8d_1)^{r_1 + \dots + r_m} (1 + x_1)^{r_1} \dots (1 + x_m)^{r_m} ;$$

that is, the former polynomial is majorized by the latter.

LEMMA 2: Let m be a positive integer, $m \ge 2$, and suppose that $0 < \delta < 1/(m2^{m+2})$. Let $a \ge 1$ be a real number, and let $r_1,\dots,r_m$ and $q_1,\dots,q_m$ be positive integers such that

$$r_m > 10/\delta$$

$$r_h/r_{h-1} < \delta, \quad h = 2,\dots,m$$

$$\log q_1 > \max\left(2m(2m+1)/\delta,\ 6/\delta\right)$$

$$r_h \log q_h \ge r_1 \log q_1, \quad h = 2,\dots,m$$

$$a \le q_1^{\delta r_1} .$$

Let $\alpha_1,\ldots,\alpha_m$ be algebraic numbers of heights $q_1,\ldots,q_m$ respectively. Let $A(x_1,\ldots,x_m)$ be a polynomial of the form

$$A(x_1,\ldots,x_m) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} a_{i_1\cdots i_m} x_1^{i_1} \cdots x_m^{i_m} \neq 0$$

with integral coefficients, each coefficient having absolute value at most a.

Then there exist nonnegative integers $j_1,\ldots,j_m$ satisfying

$$\sum_{h=1}^{m} \frac{j_h}{r_h} \leq 10^m a^{(1/2)^m}$$

such that

$$A_{j_1\cdots j_m}(\alpha_1,\ldots,\alpha_m) \neq 0.$$

THEOREM I: Let $\xi_{i_1},\ldots,\xi_{i_u}$ and $\eta_1,\ldots,\eta_r$ be algebraic numbers as specified in Chapter I. Let Ξ be the q-adic number defined by $\Xi \longleftrightarrow (\eta_1,\ldots,\eta_r)$. Assume that there are infinitely many distinct nonzero elements α of K such that

(A) $|\xi_{i_k} - \alpha^{(i_k)}|^* \leq c_k H(\alpha)^{-\rho_k}$, $k = 1,\ldots,u$

(B) $|\Xi - \alpha|_q^* \leq cH(\alpha)^{-\sigma}$

(C) $|\alpha|_{q'}^* \leq c'H(\alpha)^{\lambda-1}$

(D) $|\frac{1}{\alpha}|_{q''}^* \leq c''H(\alpha)^{\mu-1}$.

Then $\sum_{k=1}^{u} g_{i_k} \rho_k + \sigma \leq \lambda + \mu$.

PROOF OF THEOREM I:

We shall prove the theorem by contradiction, so we start by supposing that

(1) $$\sum_{k=1}^{u} g_{i_k} \rho_k + \sigma = \lambda + \mu + 4\epsilon \quad \text{where} \quad \epsilon > 0.$$

Since decreasing the numbers ρ_k and σ, and increasing c_k, c, c', and c'', only weaken the hypotheses of the theorem, we can suppose without loss of generality that

(2) $0 < \epsilon \leq 1/12$, $c \geq 1$, $c' \geq 1$, $c'' \geq 1$, and $c_k \geq 1$, $k = 1,\ldots,u$.

The contradiction arises from the study of a large number m of solutions $\alpha_1,\ldots,\alpha_m$ of the inequalities (A)-(D). These numbers can be supposed to be distinct primitive elements of K, since every element of K is a primitive element of some subfield of K and any algebraic number field K has only finitely many subfields. Because we are assuming that there are infinitely many candidates for the numbers $\alpha_1,\ldots,\alpha_m$, we can pick them so that their heights are arbitrarily large. These heights will be the $q_1,\ldots,q_m$ of Lemma 2, where $q_i = H(\alpha_i)$ for $i = 1,\ldots,m$. Let $F(x) = F_0x^f+\ldots+F_f$ be a polynomial of lowest degree $f \geq 1$ over $\mathbb{Z}$ having $\zeta_{i_1},\ldots,\zeta_{i_u}$ and $\boxed{\ }$ as zeros. This will be the polynomial of Lemma 1. Since $F(x)$ is a minimal polynomial for a set of algebraic numbers, it has no multiple factors. Let $d_1 = 2\max(|F_0|,\ldots,|F_f|)$,

and let

$$m = [2\,(\tfrac{24f}{\epsilon})^2] + 1 \tag{3}$$

and

$$d = \epsilon\, m/6. \tag{4}$$

Then $m \geq 2(24f/\epsilon)^2$ implies that

$$d^2 = \frac{\epsilon^2 m^2}{36} \geq \frac{\epsilon^2 m}{36}\,(2(\tfrac{24f}{\epsilon})^2) = 2m(\tfrac{24f}{6})^2 = 2m(4f)^2,$$

or

$$d \geq 4f\sqrt{2m}, \tag{5}$$

as required in Lemma 1.

Suppose that δ is such that

$$0 < \delta < \frac{1}{m2^{m+2}}\left(\frac{m\epsilon}{6\cdot 10^m}\right)^{2^m}. \tag{6}$$

By (3) and (6), we also have

$$0 < \delta < \frac{1}{m2^{m+2}}. \tag{7}$$

Choose the numbers $\alpha_1, \ldots, \alpha_m$ so that their heights $q_1, \ldots, q_m$ satisfy

$$\log q_1 > \max\left(\frac{2m(2m+1)}{\delta}\,,\ \frac{6}{\delta}\right), \tag{8}$$

$$(9) \qquad \frac{\log q_h}{\log q_{h-1}} > \frac{2}{\delta}, \quad h = 2,\ldots,m,$$

and

$$(10) \quad q_1 \geq \max\Big((20d_1)^{\frac{m}{\delta}}, \max_k(c_k^{\frac{1}{\rho_k}}), c^{\frac{1}{\delta}}, c'^{\frac{1}{1-\lambda}}, c''^{\frac{1}{1-\mu}}, R\Big),$$

where here and hereafter we adopt the convention that $a^{1/0} = 0$, and where R is to be determined later. The inequalities

$$q_1 \geq \max\big(\max_k(c_k^{1/\rho_k}), c^{1/\sigma}, c'^{1/(1-\lambda)}, c''^{1/(1-\mu)}\big)$$

of (10) allow us to remove the asterisks from hypotheses (A)-(D) of the theorem in case ρ_k, σ, $1-\lambda$, or $1-\mu$, respectively, are nonzero.

Finally, choose r_1 to be any integer such that

$$(11) \qquad r_1 > \max\Big(\frac{10 \log q_m}{\delta \log q_1}, \frac{2 \log q_m}{\epsilon \log q_1}\Big),$$

and define $r_2,\ldots,r_m$ by

$$(12) \qquad \frac{r_1 \log q_1}{\log q_h} < r_h \leq \frac{r_1 \log q_1}{\log q_h} + 1, \quad h = 2,\ldots,m.$$

Then (12) and (11) yield

$$(13) \qquad r_m > \frac{r_1 \log q_1}{\log q_m} > \frac{10}{\delta},$$

and applications of (12), (9), (12), (9), (11), and (7) successively, yield

$$\frac{r_h}{r_{h-1}} < \frac{r_h \log q_{h-1}}{r_1 \log q_1} = \frac{\log q_{h-1}}{\log q_h}\left(\frac{r_h \log q_h}{r_1 \log q_1}\right) < \frac{\delta}{2}\left(\frac{r_h \log q_h}{r_1 \log q_1}\right)$$

$$\leq \frac{\delta}{2}\left(1 + \frac{\log q_h}{r_1 \log q_1}\right) \leq \frac{\delta}{2}\left(1 + \frac{\log q_m}{r_1 \log q_1}\right) < \frac{\delta}{2}\left(1 + \frac{\delta}{10}\right) < \delta,$$

or

$$r_h/r_{h-1} < \delta, \quad h = 2,\ldots,m. \tag{14}$$

Thus by (7), (13), (14), (8), and (12), the hypotheses of Lemma 2 will be satisfied as soon as we choose a suitable polynomial $A(x_1,\ldots,x_m)$.

Before applying Lemmas 1 and 2, we collect a few useful properties of the parameters just defined. Obviously

$$2 < q_1 < \ldots < q_m \tag{15}$$

and

$$r_m < \ldots < r_1, \quad \text{so that} \quad r_1 + \ldots + r_m < mr_1. \tag{16}$$

Also, if $k_1,\ldots,k_m$ are arbitrary nonnegative integers, then

$$q_1^{r_1 \sum_{h=1}^{m} \frac{k_h}{r_h}} \leq q_1^{k_1}\cdots q_m^{k_m} \leq q_1^{(1+\epsilon) r_1 \sum_{h=1}^{m} \frac{k_h}{r_h}}. \tag{17}$$

This property follows from (12), (11), and (15), which imply that

$$r_h > \frac{r_1 \log q_1}{\log q_h} > \frac{2 \log q_m}{\epsilon \log q_h} \geq \frac{2}{\epsilon},$$

so that

$$r_h - 1 = r_h\left(1 - \frac{1}{r_h}\right) > \left(1 - \frac{\epsilon}{2}\right)r_h > \frac{r_h}{1+\epsilon},$$

since $(1+\epsilon)\left(1 - \frac{\epsilon}{2}\right) = 1 + \frac{\epsilon}{2}(1-\epsilon) > 1$ by inequality (2). Hence

$$r_h < (1+\epsilon)(r_h - 1). \tag{18}$$

By (12), (18), and (12),

$$r_1 \log q_1 < r_h \log q_h < (1+\epsilon)(r_h - 1)\log q_h \leq (1+\epsilon) r_1 \log q_1,$$

or

$$r_1 \log q_1 \leq r_h \log q_h \leq (1+\epsilon) r_1 \log q_1, \quad h = 1, \ldots, m,$$

and this is easily seen to yield the inequality (17).

We are now in a position to apply Lemma 1, which guarantees the existence of a polynomial

$$A(x_1, \ldots, x_m) = \sum_{k_1=0}^{r_1} \cdots \sum_{k_m=0}^{r_m} a_{k_1 \ldots k_m} x_1^{k_1} \cdots x_m^{k_m} \not\equiv 0 \tag{19}$$

with coefficients in Z such that

$$|a_{k_1 \ldots k_m}| \begin{cases} \leq 5(4d_1)^{r_1 + \ldots + r_m} & \text{always} \\ = 0 & \text{unless} \quad \frac{m-d}{2} < \sum_{h=1}^{m} \frac{k_h}{r_h} < \frac{m+d}{2}. \end{cases} \tag{20}$$

Moreover,

$A_{j_1 \dots j_m}(x, \dots, x)$ is divisible by $F(x)$ whenever

(21) $$0 \le j_1 \le r_1, \dots, 0 \le j_m \le r_m \quad \text{and} \quad \sum_{h=1}^{m} \frac{j_h}{r_h} \le \frac{m-d}{2},$$

and

$A_{j_1 \dots j_m}(x_1, \dots, x_m)$ is majorized by

(22) $$5(8d_1)^{r_1+\dots+r_m}(1+x_1)^{r_1}\dots(1+x_m)^{r_m}.$$

To apply Lemma 2 to this polynomial, we need only verify that $|a_{k_1 \dots k_m}| \le q_1^{8r_1}$ for all $k_1, \dots, k_m$, a condition guaranteed by (20) and (10), since (16) implies that $5(4d_1)^{r_1+\dots+r_m} < (20d_1)^{mr_1}$. We conclude that there exist integers $t_1, \dots, t_m$, fixed throughout the remainder of the proof, such that $0 \le t_1 \le r_1, \dots, 0 \le t_m \le r_m$,

(23) $$A_{t_1 \dots t_m}(\varkappa_1, \dots, \varkappa_m) \neq 0,$$

and

(24) $$\sum_{h=1}^{m} \frac{t_h}{r_h} \le 10^m \delta (\tfrac{1}{2})^m .$$

We shall abbreviate the quantity occurring on the left-hand side in (23) by Ψ. Then clearly

(25) $$\Psi^{(i)} = A_{t_1 \dots t_m}(\varkappa_1^{(i)}, \dots, \varkappa_m^{(i)}) \neq 0, \quad i = 1, \dots, n.$$

To simplify the notation somewhat, let

(26) $$L = \sum_{h=1}^{m} \frac{t_h}{r_h}, \quad S_1 = \frac{m-d}{2} - L, \quad S_2 = \frac{m+d}{2} - L, \quad S_3 = m - L.$$

Note that by inequalities (24) and (6),

$$L \leq 10^m \, 8^{(1/2)^m} < 10^m \left\{ \frac{1}{m2^{m+2}} \left(\frac{m\varepsilon}{6 \cdot 10^m} \right)^{2^m} \right\}^{(1/2)^m} = \left\{ \frac{1}{m2^{m+2}} \right\}^{(1/2)^m} \frac{m\varepsilon}{6} ,$$

so that

$$L < \frac{m\varepsilon}{6} . \tag{27}$$

Finally, let $b_i \geq 1$ be the leading coefficient in the defining polynomial for α_i, where $i = 1,\ldots,m$, and let

$$\varphi = b_1^{r_1 - t_1} \ldots b_m^{r_m - t_m} N(\Psi).$$

The remainder of the proof consists in finding upper and lower bounds for φ. A comparison of these bounds will yield the desired contradiction.

AN UPPER BOUND FOR φ :

Taylor's Theorem gives the following representations for the various conjugates $\Psi^{(i_k)}$, $k = 1,\ldots,n$:

$$\Psi^{(i_k)} = \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} A_{j_1 \cdots j_m}(\xi_{i_k}, \ldots, \xi_{i_k}) \binom{j_1}{t_1} \cdots \binom{j_m}{t_m} \times (\alpha_1^{(i_k)} - \xi_{i_k})^{j_1 - t_1} \ldots (\alpha_m^{(i_k)} - \xi_{i_k})^{j_m - t_m} \tag{28}$$

where the binomial coefficient $\binom{j_h}{t_h} = 0$ if $j_h < t_h$. By (21),

$A_{j_1 \cdots j_m}(\xi_{i_k}, \ldots, \xi_{i_k}) = 0$ in case $0 \le j_h \le r_h$ for $h = 1, \ldots, m$ and $\sum_{h=1}^{m} \frac{j_h}{r_h} \le \frac{m-d}{2}$, so in expression (28) for $\Psi^{(i_k)}$ we sum over the set of all m-tuples $(j) = (j_1, \ldots, j_m)$ belonging to the set J defined by

$$J: \quad 0 \le j_h - t_h \le r_h - t_h, \ h = 1, \ldots, m, \ \sum_{h=1}^{m} \frac{j_h - t_h}{r_h} > \frac{m-d}{2} - \sum_{h=1}^{m} \frac{t_h}{r_h} = s_1.$$

By (22) and (16),

$$|A_{j_1 \cdots j_m}(\xi_{i_k}, \ldots, \xi_{i_k})| \le 5(8d_1)^{r_1 + \ldots + r_m}(1 + |\xi_{i_k}|)^{r_1} \ldots (1 + |\xi_{i_k}|)^{r_m}$$

$$\le (40d_1)^{mr_1}(1 + |\xi_{i_k}|)^{mr_1}.$$

Also $\binom{j_1}{t_1} \cdots \binom{j_m}{t_m} \le 2^{j_1 + \ldots + j_m}$, so that equality (28) yields

$$(29) \quad |\Psi^{(i_k)}| \le \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} (40d_1)^{mr_1}(1 + |\xi_{i_k}|)^{mr_1} 2^{j_1 + \ldots + j_m}$$

$$\times \max_{(j) \in J} \left(|\kappa_1^{(i_k)} - \xi_{i_k}|^{j_1 - t_1} \ldots |\kappa_m^{(i_k)} - \xi_{i_k}|^{j_m - t_m} \right).$$

In this expression,

$$\sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} 2^{j_1 + \ldots + j_m} \le (2^{r_1+1} - 1) \ldots (2^{r_m+1} - 1)$$

$$\le 2^{2r_1} \ldots 2^{2r_m}$$

$$\le 4^{mr_1},$$

so that (29) becomes

$$(30)\quad |\Psi^{(i_k)}| \le (160d_1(1 + |\xi_{i_k}|))^{mr_1} \max_{(j)\in J}\left\{\prod_{h=1}^{m} |\kappa_h^{(i_k)} - \xi_{i_k}|^{j_h - t_h}\right\}.$$

To find an upper bound for the second factor in this expression, we call on property (A) of the theorem, which states that

$$\prod_{h=1}^{m} |\kappa_h^{(i_k)} - \xi_{i_k}|^{*\,j_h - t_h} \le \prod_{h=1}^{m} (c_k q_h^{-\rho_k})^{j_h - t_h}.$$

Using this, inequalities (17) and (16), and the definition of J, we see that for $(j) \in J$,

$$\begin{aligned}\prod_{h=1}^{m} |\kappa_h^{(i_k)} - \xi_{i_k}|^{*\,j_h - t_h} &\le \prod_{h=1}^{m} c_k^{\,j_h - t_h} \prod_{h=1}^{m} (q_h^{\,j_h - t_h})^{-\rho_k} \\ &\le c_k^{\sum_{h=1}^{m} (j_h - t_h)} \max_{(j)\in J}\left\{q_1^{-\rho_k r_1 \sum_{h=1}^{m} \frac{j_h - t_h}{r_h}}\right\} \\ &\le c_k^{r_1 m}\, q_1^{-\rho_k r_1 s_1}.\end{aligned}$$

Suppose that $\rho_k \neq 0$. Then inequalities (10) and (15) guarantee that $c_k q_h^{-\rho_k} \le 1$, so that by property (A),

$$|\kappa_h^{(i_k)} - \xi_{i_k}|^{*} = |\kappa_h^{(i_k)} - \xi_{i_k}| \quad \text{for} \quad h = 1, \ldots, m.$$

Therefore whenever $\rho_k \neq 0$, inequality (30) yields

$$(31)\quad |\Psi^{(i_k)}| \le \left(160d_1c_k(1 + |\xi_{i_k}|)\right)^{mr_1} q_1^{-\rho_k r_1 s_1} = (c_k')^{mr_1} q_1^{-\rho_k r_1 s_1},$$

where $c_k' = 160d_1c_k(1 + |\xi_{i_k}|) > 1$.

Suppose that $i_{u'} \le R_1 < i_{u'+1}$, and let H be the set of all integers i_k for which $\rho_k \neq 0$ and $1 \le k \le u$, together with all integers of the form $R_2 + i_k$ for which $\rho_k \neq 0$ and $u' < k \le u$. Then by inequality (31),

$$(32)\quad N(\Psi) = \prod_{h\in H} |\Psi^{(h)}| \prod_{h\notin H} |\Psi^{(h)}| \le \Big(\prod_{k=1}^{u} (c_k')^{g_{i_k}}\Big)^{mr_1} q_1^{-r_1 S_1 \sum_{k=1}^{u} \rho_k g_{i_k}} \Big(\prod_{h\notin H} |\Psi^{(h)}|\Big).$$

It remains to find an upper bound for $b_1^{r_1-t_1}\cdots b_m^{r_m-t_m} \prod_{h\notin H} |\Psi^{(h)}|$. Observe that

$$(33)\quad \Psi^{(h)} = \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} a_{j_1\cdots j_m} \binom{j_1}{t_1}\cdots\binom{j_m}{t_m} (\alpha_1^{(h)})^{j_1-t_1}\cdots(\alpha_m^{(h)})^{j_m-t_m},$$

where, by virtue of condition (20) and definition (26), $j_1,\ldots,j_m$ may be restricted to range over the set

$$I:\quad 0 \le j_1 - t_1 \le r_1 - t_1, \ldots, 0 \le j_m - t_m \le r_m - t_m,\ S_1 < \sum_{i=1}^{m} \frac{j_i - t_i}{r_i} < S_2.$$

Conditions (16) and (20) yield

$$|a_{j_1\cdots j_m}| \le (20d_1)^{mr_1},$$

and, as previously noted,

$$\binom{j_1}{t_1}\cdots\binom{j_m}{t_m} \le 2^{j_1+\cdots+j_m} \le 2^{mr_1}.$$

Hence (33) yields

$$(34)\quad |\Psi^{(h)}| \le (40d_1)^{mr_1} \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} \max(1,|\varkappa_1^{(h)}|)^{j_1-t_1} \cdots \max(1,|\varkappa_m^{(h)}|)^{j_m-t_m}$$

$$\le (40d_1)^{mr_1}(r_1+1)\cdots(r_m+1) \max_{(j)\in I}\left\{\prod_{i=1}^{m} \max(1,|\varkappa_i^{(h)}|)^{j_i-t_i}\right\}$$

$$\le (80d_1)^{mr_1} \max_{(j)\in I}\left\{\prod_{i=1}^{m} \max(1,|\varkappa_i^{(h)}|)^{j_i-t_i}\right\}.$$

In Chapter IX we shall use expression (34) to obtain a stronger upper bound. Now, however, we use (34) only to obtain the rough estimate

$$(35)\quad \Big|\prod_{h\notin H} \Psi^{(h)}\Big| < (80d_1)^{nmr_1} \prod_{h=1}^{n} \prod_{i=1}^{m} \max(1,|\varkappa_i^{(h)}|)^{r_i-t_i}.$$

This, in turn, can be estimated with the help of a result of Mahler [7].

LEMMA 3: *If* $\varkappa_i$ *is a primitive element of* K, *then*

$$\prod_{h=1}^{n} \max(1,|\varkappa_i^{(h)}|) \le (n+1)\frac{q_i}{b_i}.$$

Hence (35) and (17) yield

$$\Big|\prod_{h\notin H}\Psi^{(h)}\Big| < (80d_1)^{nmr_1} \prod_{i=1}^{m} \left\{(n+1)\frac{q_i}{b_i}\right\}^{r_i-t_i}$$

$$\le \frac{((80d_1)^n(n+1))^{mr_1}\, q_1^{(1+\epsilon)mr_1}}{b_1^{r_1-t_1}\cdots b_m^{r_m-t_m}}.$$

From this, inequality (32), and the definition of φ, (page 20), we deduce the inequality

$$\varphi < \left\{(80d_1)^n(n+1)\prod_{k=1}^{u}(c_k')^{g_{i_k}}\right\}^{mr_1} q_1^{(1+\epsilon)mr_1 - r_1 s_1 \sum_{k=1}^{u} \rho_k g_{i_k}}.$$

Letting $d_2 = (80d_1)^n(n+1)\prod_{k=1}^{u}(c_k')^{g_{i_k}}$, so that $d_2 > 1$, we obtain the desired upper bound for φ :

$$\varphi < d_2^{mr_1} q_1^{(1+\epsilon)mr_1 - r_1 s_1 \sum_{k=1}^{u} \rho_k g_{i_k}}. \tag{36}$$

A LOWER BOUND FOR φ :

We shall use hypotheses (B)-(D) of the theorem to establish a lower bound for φ.

As a principal ideal, (Ψ) can be written as the quotient of two relatively prime integral ideals,

$$(\Psi) = \frac{\mathfrak{m}}{\mathfrak{n}},$$

and so we need a lower bound for $N(\mathfrak{m})$ and an upper bound for $N(\mathfrak{n})$. If

$$\mathfrak{m} = \mathfrak{q}^{v}\mathfrak{q}'^{v'}\mathfrak{m}'$$

where v and v' are nonnegative integers and $\mathfrak{m}'$ is an integral ideal divisible by neither $\mathfrak{q}$ nor $\mathfrak{q}'$, then

$$N(\mathfrak{m}) \geq N(\mathfrak{q}^{v})N(\mathfrak{q}'^{v'}),$$

so we begin by finding lower bounds for $N(\mathfrak{q}^{v})$ and $N(\mathfrak{q}'^{v'})$.

APPLICATION OF HYPOTHESIS (B):

In case $\sigma \neq 0$, inequality (10) and the inequalities $|\Xi - \varkappa_i|_q^* \leq cq_i^{-\sigma}$ of Theorem I imply

$$|\Xi - \varkappa_i|_q^* = |\Xi - \varkappa_i|_q \leq cq_i^{-\sigma}, \quad i = 1, \ldots, m. \tag{37}$$

Let

$$M = \max\,(|\Xi|_q\,,\ 1). \tag{38}$$

Now

$$\Psi = \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} A_{j_1 \cdots j_m}(\Xi, \ldots, \Xi)\binom{j_1}{t_1}\cdots\binom{j_m}{t_m}(\varkappa_1 - \Xi)^{j_1-t_1}\ldots(\varkappa_m - \Xi)^{j_m-t_m}$$

where $\binom{j_h}{t_h} = 0$ if $j_h < t_h$. Because $|\ldots|_q$ is a non-archimedean pseudo-valuation,

$$|\Psi|_q \leq B_1 B_2 \tag{39}$$

where

$$B_1 = \max_{(j)\in J}\left(|A_{j_1 \cdots j_m}(\Xi, \ldots, \Xi)\binom{j_1}{t_1}\cdots\binom{j_m}{t_m}|_q\right)$$

and

$$B_2 = \max_{(j)\in J}(|\varkappa_1 - \Xi|_q^{j_1-t_1}\ldots|\varkappa_m - \Xi|_q^{j_m-t_m}).$$

Here $B_1 \leq \max\limits_{(j)\in J}\left(\max(1, |\Xi|_q^{r_1-j_1+\ldots+r_m-j_m})\right)$, by the non-archimedean property of $|\ldots|_q$ and because the binomial coefficients and the

integral coefficients of $A_{j_1 \dots j_m}(\Xi, \dots, \Xi)$ all have $\mathfrak{q}$-adic values at most 1. Hence by (38),

$$B_1 \leq M^{mr_1} \quad \text{in case} \quad \sigma \neq 0. \tag{40}$$

By inequality (37), still assuming that $\sigma \neq 0$,

$$B_2 \leq \max_{(j)\in J} \left(c^{(j_1-t_1)+\dots+(j_m-t_m)} \right) \max_{(j)\in J} \left((q_1^{j_1-t_1}, \dots q_m^{j_m-t_m})^{-\sigma} \right),$$

or, by (17),

$$B_2 \leq c^{mr_1} q_1^{-\sigma r_1 s_1}. \tag{41}$$

Finally, (39), (40), and (41) imply that $|\Psi|_{\mathfrak{q}} \leq (cM)^{mr_1} q_1^{-\sigma r_1 s_1}$ for $\sigma \neq 0$, so that by inequality (5) of the introduction,

$$\boxed{N(\mathfrak{q})^{\nu} \geq \frac{1}{(cM)^{mr_1} N(\mathfrak{Q})} q_1^{\sigma r_1 s_1}}, \tag{42}$$

and this bound holds trivially in case $\sigma = 0$.

APPLICATION OF HYPOTHESIS (C):

We suppose that $\lambda \neq 1$ and apply the above argument to the expression

$$\Psi = \sum_{j_1=0}^{r_1} \dots \sum_{j_m=0}^{r_m} a_{j_1 \dots j_m} \binom{j_1}{t_1} \dots \binom{j_m}{t_m} \varkappa_1^{j_1-t_1} \dots \varkappa_m^{j_m-t_m}$$

obtained from equation (19). Property (C) guarantees that

$|\kappa_i|_{\mathfrak{q}'} \leq c' q_1^{\lambda-1}$ in case $c' q_1^{\lambda-1} \leq 1$, which is true by inequality (10), so that

$$|\Psi|_{\mathfrak{q}'} \leq \max_{(j)\in J} \left(|\kappa_1|_{\mathfrak{q}'}^{j_1-t_1} \cdots |\kappa_m|_{\mathfrak{q}'}^{j_m-t_m} \right)$$

$$\leq c'^{mr_1} \max_{(j)\in J} \left((q_1^{j_1-t_1} \cdots q_m^{j_m-t_m})^{\lambda-1} \right)$$

$$\leq c'^{mr_1} q_1^{r_1(\lambda-1)s_1}.$$

Hence

$$N(\mathfrak{q}')^{\nu'} \geq \frac{1}{c'^{mr_1} N(\mathfrak{q}')} q_1^{r_1(1-\lambda)s_1}, \tag{43}$$

where again the bound holds trivially in case $\lambda = 1$.

APPLICATION OF HYPOTHESIS (D):

It remains to find an upper bound for $N(\mathfrak{n})$.

By virtue of the result stated in Appendix A, we can put $\kappa_i = \frac{\alpha_i}{\beta_i}$, where α_i and β_i are algebraic integers in K such that

$$N(\beta_i) \leq c(K) b_i \quad \text{for} \quad i = 1, \ldots, m, \tag{44}$$

and $c(K)$ is a constant which depends only on K. Then

$$\Psi = \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} a_{j_1 \cdots j_m} \binom{j_1}{t_1} \cdots \binom{j_m}{t_m} \frac{\alpha_1^{j_1-t_1} \cdots \alpha_m^{j_m-t_m}}{\beta_1^{j_1-t_1} \cdots \beta_m^{j_m-t_m}}, \tag{45}$$

where, as before, by virtue of condition (20) and definition (26), $j_1,\dots,j_m$ may be restricted to range over the set

$$\text{I:}\quad 0 \le j_1 - t_1 \le r_1 - t_1,\dots,0 \le j_m - t_m \le r_m - t_m,\ S_1 < \sum_{h=1}^{m} \frac{j_h - t_h}{r_h} < S_2.$$

It is easily seen that if γ_i and δ_i are integers in K for $i = 1,\dots,s$, then $\left(\sum_{i=1}^{s} \frac{\gamma_i}{\delta_i}\right)$ l.c.m. $\left((\delta_1),\dots,(\delta_s)\right)$ is an integral ideal of K. Thus (45) yields

$$N(\mathfrak{n}) \mid N\left(\text{l.c.m.}_{(j)\in I} \left((\beta_1)^{j_1 - t_1}\cdots(\beta_m)^{j_m - t_m}\right)\right). \tag{46}$$

First assume that $\mu \neq 1$, so that inequality (10) guarantees that $c''\, q_1^{\mu-1} \le 1$. Then by property (D),

$$|\beta_i|_{\mathfrak{q}''} \le \left|\frac{1}{\kappa_i}\right|_{\mathfrak{q}''} = \left|\frac{1}{\kappa_i}\right|^{*}_{\mathfrak{q}''} \le c''\, q_i^{\mu-1}, \tag{47}$$

so that, by (47) and inequality (5) of the introduction, there are integers h_i such that

$$(\beta_i) = \mathfrak{q}''^{h_i}\mathfrak{n}_i \quad \text{for} \quad i = 1,\dots,m, \tag{48}$$

where $\mathfrak{n}_i$ is an integral ideal, and further,

$$\frac{1}{c''N(\mathfrak{q}'')}\, q_1^{1-\mu} < N(\mathfrak{q}'')^{h_i}.$$

Clearly the numbers h_i need not be such that $\mathfrak{q}''^{h_i} \| (\beta_i)$, but may be chosen to be smaller, if necessary, so that

$$(49)\qquad \frac{1}{c''N(\mathfrak{q}'')}\, q_i^{1-\mu} < N(\mathfrak{q}'')^{h_i} \le \frac{q_i^{1-\mu}}{c''} \le q_i^{1-\mu} \quad \text{for} \quad i = 1,\dots,m.$$

Thus by (48), (49), and (44),

$$(50)\qquad N(\eta_i) = \frac{N(\beta_i)}{N(\mathfrak{q}'')^{h_i}} < \frac{N(\beta_i)}{q_i^{1-\mu}}\, c''N(\mathfrak{q}'') \le c''c(K)b_iN(\mathfrak{q}'')q_i^{\mu-1}$$

for $i = 1,\dots,m$, so that by (48), (49) and (50),

$$(51)\qquad N\Big(\underset{(j)\in I}{\text{l.c.m.}}\big((\beta_1)^{j_1-t_1}\cdots(\beta_m)^{j_m-t_m}\big)\Big) \le \prod_{i=1}^{m}\big(c''c(K)b_iN(\mathfrak{q}'')q_i^{\mu-1}\big)^{r_i-t_i}$$

$$\times \max_{(j)\in I} \prod_{i=1}^{m} (q_i^{1-\mu})^{j_i-t_i}.$$

Using (46), (51), and (17), we obtain

$$N(\eta) \le \big(c''c(K)\, N(\mathfrak{q}'')\big)^{mr_1}\Big\{\prod_{i=1}^{m} b_i^{\,r_i-t_i}\Big\}\, q_1^{\,r_1(\mu-1)S_3}\, q_1^{\,r_1(1+\epsilon)(1-\mu)S_2},$$

so that if we set $d_3 = c''c(K)\, N(\mathfrak{q}'')$, still provided that $\mu \neq 1$, we have

$$(52)\qquad \boxed{N(\eta) \le d_3^{\,mr_1} b_1^{r_1-t_1}\cdots b_m^{r_m-t_m}\, q_1^{\,r_1(\mu-1)S_3+r_1(1+\epsilon)(1-\mu)S_2}}\ .$$

In case $\mu = 1$, we use the trivial estimate

$$N(\eta) \le \prod_{i=1}^{m} (c(K)b_i)^{r_i-t_i} \le c(K)^{mr_1} b_1^{r_1-t_1}\cdots b_m^{r_m-t_m}$$

obtained from (46) and (44), and see that the above inequality remains

valid.

THE LOWER BOUND:

Now since $\mathfrak{q}$ and $\mathfrak{q}'$ are relatively prime, we can combine the inequalities (42), (43), and (52) to obtain the desired lower bound,

$$b_1^{r_1-t_1}\cdots b_m^{r_m-t_m} N(\Psi) \geq \frac{1}{(cc'd_3M)^{mr_1} N(\mathfrak{q})\, N(\mathfrak{q}')}\, q_1^{\sigma r_1 S_1} \times q_1^{r_1(1-\lambda)S_1 - r_1(\mu-1)S_3 - r_1(1+\epsilon)(1-\mu)S_2},$$

or

$$\varphi \geq \frac{1}{d_4^{mr_1}}\, q_1^{r_1\left((1+\sigma-\lambda)S_1-(1+\epsilon)(1-\mu)S_2+(1-\mu)S_3\right)} \tag{53}$$

where $d_4 = cc'd_3M\, N(\mathfrak{q})\, N(\mathfrak{q}') \geq 1$.

CONTRADICTION:

Combining the two bounds (36) and (53) for φ, we obtain the inequality

$$d_4^{-mr_1} q_1^{r_1\left((1+\sigma-\lambda)S_1-(1+\epsilon)(1-\mu)S_2+(1-\mu)S_3\right)} < d_2^{mr_1} q_1^{r_1\left((1+\epsilon)m-S_1\sum_{k=1}^{m}\rho_k g_{i_k}\right)},$$

or

$$q_1^{r_1\left\{\left(\sum_{k=1}^{m}\rho_k g_{i_k}+\sigma-\lambda+1\right)S_1-(1+\epsilon)(1-\mu)S_2+(1-\mu)S_3-(1+\epsilon)m\right\}} < (d_2d_4)^{mr_1}. \tag{54}$$

If we take the number R of inequality (10) to be $(d_2 d_4)^{\frac{12}{\epsilon}}$, then $d_2 d_4 \leq q_1^{\frac{\epsilon}{12}}$, so that (54) implies that

$$\Big(\sum_{k=1}^{m} \rho_k g_{i_k} + \sigma - \lambda + 1\Big) S_1 - (1+\epsilon)(1-\mu) S_2 + (1-\mu) S_3 - (1+\epsilon) m < m\epsilon/12,$$

or, by (1),

$$(55) \qquad (1+\mu+4\epsilon) S_1 - (1+\epsilon)(1-\mu) S_2 + (1-\mu) S_3 - (1+\epsilon) m < m\epsilon/12.$$

Applying definitions (26) to inequality (55) yields

$$m\left\{\frac{1+\mu+4\epsilon-(1+\epsilon)(1-\mu)}{2} + (1-\mu)-(1+\epsilon) - \frac{\epsilon}{12}\right\}$$
$$- d\left\{\frac{1+\mu+4\epsilon+(1+\epsilon)(1-\mu)}{2}\right\}$$
$$- L\left\{1+\mu+4\epsilon-(1+\epsilon)(1-\mu)+(1-\mu)\right\} < 0,$$

or

$$(56) \qquad \frac{5/2+3\mu}{6}\,\epsilon m - \frac{2+5\epsilon-\epsilon\mu}{2}\, d - (\mu+1+3\epsilon+\epsilon\mu) L < 0.$$

But $d = \epsilon m/6$, and by (27), $L < \epsilon m/6$, so that (56) implies that

$$\frac{\epsilon m}{6}\left(5/2+3\mu-1-\frac{5-\mu}{2}\,\epsilon-\mu-1-3\epsilon-\epsilon\mu\right) < 0,$$

or

$$(57) \qquad 1/2 < \epsilon\left(3+\frac{5-\mu}{2}+\mu\right) - 2\mu.$$

But $0 < \epsilon \leq 1/12$ and $0 \leq \mu \leq 1$, so that

$$(58) \qquad \epsilon\left(3+\frac{5-\mu}{2}+\mu\right) - 2\mu \leq \frac{1}{12}\left(3+5/2+1/2\right) = 1/2,$$

and a comparison of inequalities (57) and (58) yields the desired contradiction.

CHAPTER III

A BOUND ON THE NUMBER OF SOLUTIONS IN THEOREM I

Retain the notation of Theorem I. In case $\mu = 1$ and

$$\sum_{k=1}^{u} \rho_k g_{i_k} + \sigma - \lambda - 1 = 4\epsilon > 0,$$

we know that the inequalities of that theorem have only finitely many solutions. Following Parry [10, pp. 44 ff.] and Davenport and Roth [1], a bound on the number of these solutions can be extracted from the proof of Theorem I.

THEOREM Ia: Let $\xi_{i_1}, \ldots, \xi_{i_u}$ and Ξ be as in Theorem I. Let $\sum_{k=1}^{u} \rho_k g_{i_k} + \sigma - \lambda = 1 + 4\epsilon$ where $0 < \epsilon \leq 1/12$. Then the number of distinct nonzero solutions in primitive elements of K of the inequalities

(A) $\quad |\xi_{i_k} - x^{(i_k)}|^* \leq c_k H(x)^{-\rho_k}, \quad k = 1, \ldots, u$

(B) $\quad |\Xi - x|_q^* \leq cH(x)^{-\sigma}$

(C) $\quad |x|_{q'}^* \leq c'H(x)^{\lambda - 1}$

is at most

$$F^{1/\epsilon^2 - \frac{6}{\epsilon} \log D}$$

where $F > 1$ is a constant depending only on $c_1, \ldots, c_u$, c, c', $\xi_{i_1}, \ldots, \xi_{i_u}$, Ξ, q, q', and K, where $D = 1$ in case the asterisks are omitted from inequalities (A)-(C), and where D is the smallest of those of the numbers ρ_k, σ, $1 - \lambda$ which are not zero

otherwise.

PROOF: The proof consists in finding an upper bound for the number of solutions needed to obtain the contradiction reached in Chapter II. In that chapter, we used the fact that ϵ could be chosen to be at most $1/12$, and so chose it in order to obtain a contradiction. Therefore we must also so choose it in the present discussion. However if $\epsilon > 1/12$, we can, as before, decrease any nonzero $\rho_1,\ldots,\rho_u$ or σ, and increase λ subject to the condition that $\lambda \leq 1$, so that a new and sufficiently small ϵ is obtained. This will not decrease the number of solutions to be counted. Thus the restriction that $\epsilon \leq 1/12$, while necessary in the final bound on the number of solutions, does not limit the applicability of the theorem. As in the proof of Theorem I, we suppose, without loss of generality, that the numbers $c_1,\ldots,c_u$, c, and c' are all at least 1.

The contradiction obtained in the proof of Theorem I stemmed from the existence of at least m solutions in primitive elements of the set of inequalities (A)-(C). The heights of these solutions were $q_1,\ldots,q_m$ where $q_1 > B$ for some large bound B, and where $q_j < q_{j+1}^{\delta/2}$ for $j = 1,\ldots,m-1$ and for some small positive number δ. We first investigate the number of solutions with heights greater than B, and then those with smaller heights.

Recall the previous notation:

$F(x)$ is a polynomial of lowest degree over the integers having

ζ_{i_k} and Ξ as zeros,

f is the degree of $F(x)$;

d_1 is twice the maximum of the absolute values of the coefficients of $F(x)$,

(1) $$m = \left[2\left(\frac{24f}{\epsilon}\right)^2\right] + 1,$$

and

(2) $$0 < \delta < \frac{1}{m2^{m+1}} \left(\frac{m}{6 \cdot 10^m}\right)^{2^m}.$$

By inequalities (8) and (10) of Chapter II,

(3) $$q_1 > \max\left(e^{\frac{2m(2m+1)}{\delta}}, e^{\frac{6}{\delta}}, (20d_1)^{\frac{m}{\delta}}, \max_k(c_k^{\frac{1}{p_k}}), c^{\frac{1}{\sigma}}, c'^{\frac{1}{1-\lambda}}, R\right).$$

Here $a^{1/0} = 0$ as before, and

$$R = (d_2 d_4)^{12/\epsilon},$$

$$d_4 = cc'd_3 MN(\mathfrak{q})\, N(\mathfrak{q}'),$$

$d_3 = c''c(K)\, N(\mathfrak{q}'') = c(K)$, since $c'' = N(\mathfrak{q}'') = 1$ in the present case,

$c(K)$ is the constant occurring in Appendix A,

$$d_2 = (240d_1)^n \prod_{k=1}^{u} (c_k')^{g_{i_k}},$$

and

$$c_k' = 160d_1\, c_k(1 + |\zeta_{i_k}|).$$

The constant M is defined to be 1 in case $\sigma = 0$, and is otherwise not smaller than 1 and dependent only on Ξ and $\mathfrak{q}$ (see condition (II-38)). If $K = \mathbb{Q}(\theta)$ as in Corollary 2 of Appendix A, then, by the result there stated, $c(K)$ can be determined explicitly provided we know the defining equation for θ. Hence

$$(4)\quad R = \left\{ (240d_1)^n \prod_{k=1}^{u} \left(160d_1\, c_k (1 + |\zeta_{i_k}|)^{g_{i_k}}\right) c(K)\, cc' N(\mathfrak{q})\, N(\mathfrak{q}')M \right\}^{\frac{12}{\epsilon}}.$$

An examination of conditions (3) and (4) shows that

$$q_1 > \max\left(A_1^{m(2m+1)/\delta},\ \max_k(c_k^{1/\rho_k}),\ c^{1/\sigma},\ c'^{1/1-\lambda}\right)$$

for some constant $A_1 > 1$. The only purpose served by the conditions $q_1 > \max\left(\max_k(c_k^{1/\rho_k}),\ c^{1/\sigma},\ c'^{1/1-\lambda}\right)$ was that they allowed the asterisks to be omitted from the hypotheses (A)-(C) for the m solutions in question. Thus we can replace inequality (3) by

$$(5)\qquad q_1 > B \quad \text{where} \quad B = A^{m(2m+1)/\delta D}.$$

Here D is the number defined in the theorem, and $A > 1$ is a constant depending on d_1, c, c', Ξ, $\mathfrak{q}$, $\mathfrak{q}'$, K, and the c_k and ζ_{i_k}.

We next show that if the heights associated with two solutions are sufficiently large, then one height must be substantially larger than the other. This will yield an upper bound on the number of solutions necessary to yield the m heights used in the proof of

Theorem I.

In case there are two distinct solutions α_1 and α_2 with heights q_1 and q_2 respectively, where $q_1 \leq q_2$, and in case we let empty products be 1, we have

$$(6)\quad \prod_{k=1}^{u} \max\left(|\alpha_1^{(i_k)} - \xi_{i_k}|^{g_{i_k}}, |\alpha_2^{(i_k)} - \xi_{i_k}|^{g_{i_k}}\right) \cdot \max\left(|\alpha_1 - \Xi|_{\mathfrak{q}}, |\alpha_2 - \Xi|_{\mathfrak{q}}\right)$$

$$\times \max\left(|\alpha_1|_{\mathfrak{q}'}, |\alpha_2|_{\mathfrak{q}'}\right) \leq cc' \prod_{k=1}^{u} c_k q_1^{-(2+4\epsilon)}.$$

Now let

$$\mathfrak{q} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r} \quad \text{and} \quad \mathfrak{q}' = \mathfrak{P}_{r+1}^{e_{r+1}} \cdots \mathfrak{P}_{r+s}^{e_{r+s}}.$$

By definition,

$$|\alpha|_{\mathfrak{q}} = \max_i \left\{ |\alpha|_{\mathfrak{P}_i}^{\frac{e(\mathfrak{P}_i) \log N(\mathfrak{q})}{e_i \log p_i}} \right\},$$

so that

$$|\alpha|_{\mathfrak{P}_i} \leq |\alpha|_{\mathfrak{q}}^{\frac{e_i \log p_i}{e(\mathfrak{P}_i) \log N(\mathfrak{q})}}, \quad i = 1, \ldots, r.$$

Thus

$$\prod_{i=1}^{r} |\alpha|_{\mathfrak{P}_i}^{g(\mathfrak{P}_i)} \leq |\alpha|_{\mathfrak{q}}^{\sum_{i=1}^{r} \frac{g(\mathfrak{P}_i) e_i \log p_i}{e(\mathfrak{P}_i) \log N(\mathfrak{q})}} = |\alpha|_{\mathfrak{q}},$$

or

$$(7)\quad \prod_{j=1}^{r} |\alpha|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \leq |\alpha|_{\mathfrak{q}} \quad \text{and} \quad \prod_{j=r+1}^{r+s} |\alpha|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \leq |\alpha|_{\mathfrak{q}'}$$

for any $\alpha \in K$. Also

$$(8)\quad 2^{g_{i_k}} \max\left\{|x_1^{(i_k)} - \xi_{i_k}|^{g_{i_k}},\ |x_2^{(i_k)} - \xi_{i_k}|^{g_{i_k}}\right\} \geq |x_1^{(i_k)} - x_2^{(i_k)}|^{g_{i_k}},$$

$$(9)\quad \max\left\{|x_1 - \Xi|_{\mathfrak{q}},\ |x_2 - \Xi|_{\mathfrak{q}}\right\} \geq |x_1 - x_2|_{\mathfrak{q}},$$

and

$$(10)\quad \max\left\{|x_1|_{\mathfrak{q}'}, |x_2|_{\mathfrak{q}'}\right\} \geq |x_1 - x_2|_{\mathfrak{q}'}.$$

Inequalities (6), (7), (8), (9), and (10) yield

$$(11)\quad \prod_{k=1}^{u}\left\{\frac{|x_1^{(i_k)} - x_2^{(i_k)}|}{2}\right\}^{g_{i_k}} \prod_{j=1}^{r+s}|x_1 - x_2|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \leq cc' \prod_{k=1}^{u} c_k q_1^{-(2+4\epsilon)}.$$

Now

$$(12)\quad N(x_1 - x_2) = \prod_{k=1}^{u}|x_1^{(i_k)} - x_2^{(i_k)}|^{g_{i_k}} \prod_{\substack{i=1\\ i\neq i_1,\ldots,i_u}}^{R_1+R_2}|x_1^{(i)} - x_2^{(i)}|^{g_i},$$

where

$$(13)\quad \prod_{\substack{i=1\\ i\neq i_1,\ldots,i_u}}^{R_1+R_2}|x_1^{(i)} - x_2^{(i)}|^{g_i} \leq \prod_{\substack{i=1\\ i\neq i_1,\ldots,i_u}}^{R_1+R_2} 2^{g_i}\max\left\{|x_1^{(i)}|^{g_i}, |x_2^{(i)}|^{g_i}\right\}$$

$$\leq 2^n \prod_{i=1}^{n}\max(1, |x_1^{(i)}|) \prod_{i=1}^{n}\max(1, |x_2^{(i)}|)$$

$$\leq 2^n \cdot \frac{(n+1)q_1}{b_1} \cdot \frac{(n+1)q_2}{b_2}.$$

Here b_1 and b_2 are the leading coefficients in the defining polynomials of α_1 and α_2 respectively, and the final inequality above follows from Lemma 3 of Chapter II, (page 24).

Thus (11), (12), and (13) yield

$$2^{-\sum_{k=1}^{u} g_{i_k}} N(\alpha_1 - \alpha_2)\, 2^{-n} \left\{ \frac{b_1 b_2}{(n+1)^2 q_1 q_2} \right\} \prod_{j=1}^{r+s} |\alpha_1 - \alpha_2|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \leq cc' \prod_{k=1}^{u} c_k q_1^{-(2+4\epsilon)},$$

or

$$(14)\quad b_1 b_2\, N(\alpha_1 - \alpha_2) \prod_{j=1}^{r+s} |\alpha_1 - \alpha_2|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} < 2^{5n} cc' \left\{ \prod_{k=1}^{u} c_k \right\} q_1^{-(1+4\epsilon)} q_2.$$

By the first corollary of the theorem proved in Appendix A, the left-hand side of (14) is at least 1, so that

$$(15)\qquad q_1^{1+4\epsilon} \leq (2^{5n} cc' \prod_{k=1}^{u} c_k) q_2.$$

Let $E = 2^{5n} cc' \prod_{k=1}^{u} c_k$. In case $E^{\frac{2}{7\epsilon}} \leq q_1$, relation (15) implies that $q_1^{1+4\epsilon} \leq q_1^{7\epsilon/2} q_2$ or $q_1^{1+\epsilon/2} \leq q_2$. Thus we have proved the following lemma.

LEMMA 1: *If α_1 and α_2 are any two distinct solutions of inequalities (A)-(C) of Theorem I with heights q_1 and q_2 respectively, and if*

$$(2^{5n} cc' \prod_{k=1}^{u} c_k)^{2/7\epsilon} \leq q_1 \leq q_2,$$

then

$$q_1^{1+\epsilon/2} \leq q_2 .$$

Suppose now that there are at least s_2 solutions of inequalities (A)-(C) with heights greater than B. By (3), (4), and (5),

$$B \geq (2^{5n_{cc'}} \prod_{k=1}^{u} c_k)^{2/7\epsilon} = E^{2/7\epsilon},$$

so that if we denote the heights of these solutions by $q_1, \cdots, q_{s_2}$ with $q_1 \leq q_2 \leq \cdots \leq q_{s_2}$, then by Lemma 1,

$$q_1 \leq q_2^{\frac{2}{2+\epsilon}} \leq q_3^{\left(\frac{2}{2+\epsilon}\right)^2} \leq \cdots \leq q_{s_2}^{\left(\frac{2}{2+\epsilon}\right)^{s_2-1}} .$$

In case $(\frac{2}{2+\epsilon})^{s_2-1} < \frac{\delta}{2}$, this yields $q_1 < q_{s_2}^{\delta/2}$, and therefore we can use q_{s_2} as the q_2 in the proof of Theorem I. Hence we suppose that

$$s_2 = 2 + \left[\frac{\log \delta/2}{\log \frac{2}{2+\epsilon}}\right], \tag{16}$$

thus ensuring that $q_1 < q_{s_2}^{\delta/2}$.

Continuing, let us suppose that there are s_3 additional solutions with heights $q_{s_2} \leq q_{s_2+1} \leq q_{s_2+2} \leq \cdots \leq q_{s_2+s_3}$. Then, again by Lemma 1,

$$q_{s_2} \leq q_{s_2+1}^{\frac{2}{2+\epsilon}} \leq \cdots \leq q_{s_2+s_3}^{\left(\frac{2}{2+\epsilon}\right)^{s_3}} ,$$

and in case $(\frac{2}{2+\epsilon})^{s_3} < \frac{\delta}{2}$, we have the q_3 of Theorem I. This last inequality is true if

$$s_3 = 1 + \left[\frac{\log \delta/2}{\log \frac{2}{2+\epsilon}}\right] .$$

In general, let

$$s_i = 1 + \left[\frac{\log \delta/2}{\log \frac{2}{2+\epsilon}}\right] , \quad i = 3,\ldots,v, \tag{17}$$

and suppose that there are $s_2 + s_3 + \ldots + s_v$ solutions with heights at least B. By definitions (16) and (17),

$$s_2 + s_3 + \ldots + s_v = v + (v-1) \left[\frac{\log \delta/2}{\log \frac{2}{2+\epsilon}}\right] . \tag{18}$$

Thus we obtain heights $q_1,\ldots,q_v$ as in the proof of Theorem I. Since there cannot be m such heights, v can be at most $m - 1$. Hence if there are s_1 solutions with heights at most B, equality (18) ensures that there are at most

$$s_1 + m + (m-1) \left[\frac{\log \delta/2}{\log \frac{2}{2+\epsilon}}\right] - 1 \tag{19}$$

solutions in primitive elements of K of the inequalities (A)-(C).

It is clear from equation (1) that for $\epsilon \leq 1/12$, m is sufficiently large so that

$$(20) \qquad \left(\frac{1}{10^m}\right)^{2^m} < \frac{1}{m2^{m+2}} \left(\frac{m}{6\cdot 10^m}\right)^{2^m} .$$

Let

$$(21) \qquad \delta = \left(\frac{1}{10^m}\right)^{2^m} .$$

This is permissible by (2) and (20). Then definitions (1) and (21) yield

$$\begin{aligned} m + (m-1)\left[\frac{\log \delta/2}{\log \frac{2}{2+\epsilon}}\right] - 1 &\le (m-1)\left(1 + \frac{\log \delta/2}{\log \frac{2}{2+\epsilon}}\right) \\ &\le \frac{2(24)^2 f^2}{\epsilon^2}\left(1 + \frac{\log(2\cdot 10^{m2^m})}{\log(1+\epsilon/2)}\right) \\ &< \frac{2(24)^2 f^2}{\epsilon^2}\left(\frac{3(24)^2 f^2}{\epsilon^2}\, 2^{2(24)^2 f^2/\epsilon^2}}{\log(1+\epsilon/2)}\right) \\ &< \frac{2^{4(24)^2 f^2/\epsilon^2}}{\frac{2}{3}\left(\frac{\epsilon}{2}\right)} , \end{aligned}$$

since $6\alpha^2 < 2^{2\alpha}$ for $\alpha \ge 2(24)^2(12)^2$ and $\log(1 + \frac{\epsilon}{2}) > \frac{2}{3}(\frac{\epsilon}{2})$ for $0 < \epsilon \le \frac{1}{12}$. Therefore

$$(22) \qquad m + (m-1)\left[\frac{\log \delta/2}{\log \frac{2}{2+\epsilon}}\right] - 1 < 2^{5(24)^2 f^2/\epsilon^2} .$$

It remains to estimate s_1, the number of solutions with heights at most B. We have shown in Lemma 1 that if q_1 and q_2

are the heights of two distinct solutions, where $E^{\frac{2}{7\epsilon}} \leq q_1 \leq q_2$, then $q_1^{1+\epsilon/2} \leq q_2$. Let there be t_1 solutions with heights less than $E^{2/7\epsilon}$ and t_2 solutions with heights between $E^{2/7\epsilon}$ and B. Then

$$s_1 = t_1 + t_2. \tag{23}$$

Now clearly

$$t_1 \leq n(2E^{2/7\epsilon})^{n+1}, \tag{24}$$

there being at most $(2E^{2/7\epsilon})^{n+1}$ irreducible polynomials of degree n with integral coefficients and heights $< E^{2/7\epsilon}$, each having at most n roots in K.

Suppose that the heights between $E^{2/7\epsilon}$ and B are numbered so that $E^{2/7\epsilon} \leq q_1 \leq q_2 \leq \cdots \leq q_{t_2} \leq B$. Then by Lemma 1,

$$E^{2/7\epsilon} \leq q_1 \leq q_2^{\frac{2}{2+\epsilon}} \leq \cdots \leq q_{t_2}^{\left(\frac{2}{2+\epsilon}\right)^{t_2-1}} \leq B^{\left(\frac{2}{2+\epsilon}\right)^{t_2-1}}.$$

Hence

$$E^{\frac{2}{7\epsilon}(1+\epsilon/2)^{t_2-1}} \leq B,$$

or

$$(1 + \epsilon/2)^{t_2-1} < \log B \,/\, \log(E^{2/7\epsilon}).$$

This implies that

$$t_2 < 1 + (1/\log(1+\epsilon/2))\log\left(\frac{\log B}{\log(E^{2/7\epsilon})}\right)$$

$$< \frac{\log\log B}{\epsilon/6}$$

$$= \frac{6}{\epsilon}\log\log(A^{m(2m+1)/\delta D})$$

by definition (5) of B. Applying definitions (1) and (21), we obtain

$$t_2 < \frac{6}{\epsilon}\left\{\log(m(2m+1)10^{m2^m}/D) + \log\log A\right\}$$

$$< \frac{6}{\epsilon}\left\{\frac{3(24)^2f^2}{\epsilon^2}\left(2^{2(24)^2f^2/\epsilon^2}\right) + 3\log(2(24)^2f^2/\epsilon^2) - \log D + \log\log A\right\},$$

or

$$(25)\qquad t_2 < 2^{4(24)^2f^2/\epsilon^2} - (6/\epsilon)\log D + (6/\epsilon)\log\log A.$$

Hence by (19), (22), (23), (24), and (25), the total number of solutions is less than

$$n(2E^{2/7})^{\frac{n+1}{\epsilon}} + 2^{4(24)^2f^2/\epsilon^2} + (6/\epsilon)\log\log A - (6/\epsilon)\log D + 2^{5(24)^2f^2/\epsilon^2},$$

or less than $F^{1/\epsilon^2} - (6/\epsilon)\log D$ where F is a constant depending on K, $c_1,\ldots,c_u$, c, c', $\xi_{i_1},\ldots,\xi_{i_u}$, Ξ, q, and q'. This is just the bound given in Theorem Ia.

It should be noted that F can be given explicitly provided we know the defining equations for $\xi_{i_1},\ldots,\xi_{i_u}$, Ξ, and θ, where $K = \mathbb{Q}(\theta)$.

CHAPTER IV

EQUIVALENCE OF THEOREMS I AND II

THEOREM II: Let $F(x)$ be a nonconstant polynomial over K which has no multiple factors. Assume that there exists an infinite sequence of distinct nonzero elements $\varkappa$ of K satisfying

(A') $|F(\varkappa)|_{i_k}^{*} \leq C_k H(\varkappa)^{-\rho_k}, \quad k = 1, \ldots, u$

(B') $|F(\varkappa)|_{q}^{*} = CH(\varkappa)^{-\sigma}$

(C') $|\varkappa|_{q'}^{*} \leq C'H(\varkappa)^{\lambda-1}$

(D') $|\frac{1}{\varkappa}|_{q''}^{*} \leq C''H(\varkappa)^{\mu-1}.$

Then

$$\sum_{k=1}^{u} g_{i_k} \rho_k + \sigma \leq \lambda + \mu.$$

Eventually, we shall show that the hypotheses of any one of Theorems I-IV imply those of any other. The conclusions of these theorems, while not always identical, will be shown to be restatements of one another. In this sense, we shall call the four theorems equivalent. We start by demonstrating the equivalence of Theorems I and II.

Properties (C') and (D') are the same as properties (C) and (D) for suitable choice of constants. We shall show that properties (A') and (B') of Theorem II imply properties (A) and (B) of Theorem I, at least for an infinite subsequence of $\varkappa$'s and

for some algebraic numbers $\xi_{i_k} \in K_{i_k}$ with $k = 1,\ldots,u$, and for some algebraic $\mathfrak{q}$-adic number Ξ. The following chain of lemmas gives this result and its converse, that properties (A') and (B') imply properties (A) and (B) for some polynomial $F(x)$ and appropriate constants $C_1,\ldots,C_u$, and C.

LEMMA 1: Suppose that every infinite sequence of elements of K which is bounded with respect to the valuation w contains an infinite subsequence which has a limit in the completion K_w of K with respect to w. Suppose also that $F(x)$ is a nonconstant polynomial in $K[x]$ with no multiple zeros in K_w and that $\Sigma = \{\alpha_1, \alpha_2, \cdots\}$ is an infinite sequence in K such that $\lim_{j\to\infty} w(F(\alpha_j)) = 0$. Then there exist an infinite subsequence $\Sigma' = \{\alpha_{i_1}, \alpha_{i_2}, \cdots\}$ of Σ, a zero ξ of $F(x)$ in K_w, and a constant $\gamma_w > 0$, such that $w(\alpha_{i_j} - \xi) \leq \gamma_w w(F(\alpha_{i_j}))$ for $j = 1,2,\ldots$.

See [8, p. 127] for the lemma and its proof.

LEMMA 2: If $w(\alpha) = |\alpha^{(i)}|$ for $\alpha \in K$ and some i, $1 \leq i \leq n$, then K_w has the compactness property listed as the first hypothesis of Lemma 1.

This is, of course, a classical result.

LEMMA 3: If $w(\alpha) = |\alpha|_{\mathfrak{P}}$ for $\alpha \in K$ and $\mathfrak{P}$ a prime ideal in K, then $K_{\mathfrak{P}}$ has the compactness property listed as the first hypothesis of Lemma 1.

PROOF: Let $\{\alpha_1, \alpha_2, \ldots\}$ be a sequence Σ of elements of K

bounded with respect to the $\mathfrak{P}$-adic valuation. There is no loss of generality in supposing that the α_i are all $\mathfrak{P}$-adic integers. Now each α_i corresponds to a unique formal sum $a_{oi} + a_{1i}\, q + a_{2i}\, q^2 + \cdots$ where q us an integer in K such that $\mathfrak{P} \| (q)$ and the a_{ji} are elements of K chosen from some complete residue system modulo $\mathfrak{P}$. (See, for instance, [18, pp. 99-100].) There are only finitely many choices for each a_{ji}, so that for any positive integer j, there are infinitely many $\alpha_k \in \Sigma$ such that the $j+1$-tuples $(a_{ok}, \ldots, a_{jk})$ are identical for all k. Let Σ_1 be a subsequence of Σ for which every element has the same first two "digits" in its formal sum. In general, let Σ_{j+1} be an infinite subsequence of Σ_j for which every element has the same first $j + 2$ "digits" in its formal sum. If $\Sigma_j = \{\alpha_{j1}, \alpha_{j2}, \cdots\}$, let $\Sigma' = \{\alpha_{11}, \alpha_{22}, \cdots\}$. Σ' is an infinite subsequence of Σ, and if $\mathfrak{P}$ divides the rational prime p, then

$$|\alpha_{ii} - \alpha_{jj}|_{\mathfrak{P}} \leq \left(\frac{1}{p}\right)^{\frac{\min(i,j)+1}{e(\mathfrak{P})}},$$

so that Σ' is a fundamental sequence, and thus converges to some element of $K_{\mathfrak{P}}$.

LEMMA 4: If $F(x)$ and Σ are as in Lemma 1 and $\mathfrak{q}$ is an integral ideal in K for which $\lim_{j\to\infty} |F(\alpha_j)|_{\mathfrak{q}} = 0$, then there exist an infinite subsequence $\Sigma' = \{\alpha_{i_1}, \alpha_{i_2}, \cdots\}$ of Σ, a $\mathfrak{q}$-adic zero Ξ of $F(x)$, and a constant $\gamma_{\mathfrak{q}} > 0$, such that

$$|\alpha_{i_j} - \Xi|_{\mathfrak{q}} \leq \gamma_{\mathfrak{q}} |F(\alpha_{i_j})|_{\mathfrak{q}} \quad \text{for} \quad j = 1,2,\ldots .$$

PROOF: Clearly $\lim_{j\to\infty} |F(\alpha_j)|_{\mathfrak{p}_i} = 0$ for $i = 1,\ldots,r$ where $\mathfrak{q} = \mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_r^{e_r}$. By lemmas 1 and 3, there exist an infinite subsequence $\Sigma_1 = \{\alpha_{i1}\}$ of Σ, a $\mathfrak{p}_1$-adic zero η_1 of $F(x)$, and a constant $\gamma_1 > 0$ such that $|\alpha_{i1} - \eta_1|_{\mathfrak{p}_1} \leq \gamma_1 |F(\alpha_{i1})|_{\mathfrak{p}_1}$ for $i = 1,2,\ldots$. Continuing, we can obtain sequences $\Sigma_r \subseteq \Sigma_{r-1} \subseteq \cdots \subseteq \Sigma_1 \subseteq \Sigma$, $\mathfrak{p}_j$-adic zeros η_j of $F(x)$, and constants $\gamma_j > 0$ such that if $\Sigma_j = \{\alpha_{ij}\}$ for $j = 1,\ldots,r$, then

$$|\alpha_{ij} - \eta_j|_{\mathfrak{p}_j} \leq \gamma_j |F(\alpha_{ij})|_{\mathfrak{p}_j} \quad \text{for} \quad i = 1,2,\ldots .$$

Let $\Sigma' = \Sigma_r$, let Ξ be the $\mathfrak{q}$-adic number such that $\Xi \longleftrightarrow (\eta_1,\ldots,\eta_r)$, and put

$$\gamma_{\mathfrak{q}} = \max_i \left\{ \gamma_i^{e(\mathfrak{p}_i)\log N(\mathfrak{q})/(e_i \log p_i)} \right\}.$$

Note that Ξ is algebraic and is a zero of $F(x)$. Thus we have

$$|\alpha_{ir} - \Xi|_{\mathfrak{q}} = \max_j \left\{ |\alpha_{ir} - \eta_j|_{\mathfrak{p}_j}^{e(\mathfrak{p}_j)\log N(\mathfrak{q})/(e_j \log p_j)} \right\}$$

$$\leq \max_j \left\{ (\gamma_j |F(\alpha_{ir})|_{\mathfrak{p}_j})^{e(\mathfrak{p}_j)\log N(\mathfrak{q})/(e_j \log p_j)} \right\}$$

$$\leq \gamma_{\mathfrak{q}} |F(\alpha_{ir})|_{\mathfrak{q}} \quad \text{for} \quad i = 1,2,\ldots .$$

LEMMA 5: If $F(x)$ is as in Theorem II, and if Σ is an infinite sequence of distinct nonzero elements α of K which satisfy the

conditions

$$(A') \quad |F(\alpha)|_{i_k}^{*} \leq C_k H(\alpha)^{-\rho_k}, \quad k = 1,\ldots,u$$

and

$$(B') \quad |F(\alpha)|_{q}^{*} \leq CH(\alpha)^{-\sigma},$$

then there exist a subsequence Σ' of Σ, algebraic numbers $\xi_{i_k} \in K_{i_k}$, and a q-adic algebraic number Ξ, such that each α in Σ' satisfies the conditions

$$(A) \quad |\xi_{i_k} - \alpha^{(i_k)}|^{*} \leq c_k H(\alpha)^{-\rho_k}, \quad k = 1,\ldots,u$$

and

$$(B) \quad |\Xi - \alpha|_{q}^{*} \leq cH(\alpha)^{-\sigma}$$

for some positive constants c and c_k where $k = 1,\ldots,u$.

Conversely, given a sequence Σ satisfying properties (A) and (B), we can find a polynomial $F(x)$ and appropriate constants $C_1,\ldots,C_u$, and C such that Σ satisfies properties (A') and (B').

PROOF: First let (A) and (B) hold for each term of the sequence $\Sigma = \{\alpha_1, \alpha_2, \ldots\}$. Let $F(x)$ be a polynomial of lowest degree over $\mathbb{Z}$ having $\xi_{i_1},\ldots,\xi_{i_u}$, and Ξ as zeros. As the minimum polynomial for a set of algebraic numbers, $F(x)$ has no multiple zeros. Then the polynomial

$$\Phi(x,y) = \frac{F(x) - F(y)}{x - y}$$

has coefficients in Z, and

$$|F(\varkappa_j)|_{i_k} = |\varkappa_j^{(i_k)} - \xi_{i_k}| \, |\Phi(\varkappa_j^{(i_k)}, \xi_{i_k})| \quad \text{for} \quad k = 1,\ldots,u$$

and

$$|F(\varkappa_j)|_q \leq |\varkappa_j - \Xi|_q \, |\Phi(\varkappa_j, \Xi)|_q .$$

Now if ρ_k and σ are not zero, then Σ has limits in the i_k- and q-adic valuations respectively, so that Σ is a bounded sequence with respect to these valuations, and hence $|\Phi(\varkappa_j^{(i_k)}, \xi_{i_k})|$ and $|\Phi(\varkappa_j, \Xi)|_q$ are bounded. Let their upper bounds be T_k and T_q respectively. It follows that (A') and (B') hold for the constants $C_k = c_k T_k$ for $k = 1,\ldots,u$, and $C = cT_q$. In case ρ_k or σ is zero, we can choose $C_k = 1$ or $C = 1$ respectively.

Now let (A') and (B') hold for the sequence $\Sigma = \{\varkappa_1, \varkappa_2, \ldots\}$. Again we can suppose that ρ_k and σ are not zero, so that

$$\lim_{j\to\infty} |F(\varkappa_j)|_{i_k} = 0, \quad k = 1,\ldots,u \tag{1}$$

and

$$\lim_{j\to\infty} |F(\varkappa_j)|_q = 0. \tag{2}$$

By virtue of Lemma 2 and equation (1) above, we can apply Lemma 1, which guarantees the existence of an infinite subsequence $\Sigma^{(1)} = \{\varkappa_{j1}\}$ of Σ, a zero ξ'_{i_1} of $F(x)$ in K_{i_1}, and a constant $\gamma_1 > 0$, such that

$$|\alpha_{j1} - \xi'_{i_1}|_{i_1} \leq \gamma_1 |F(\alpha_{j1})|_{i_1}, \quad j = 1,2,\ldots .$$

Successive applications of Lemma 2 give u infinite subsequences $\Sigma^{(u)} \subseteq \Sigma^{(u-1)} \subseteq \ldots \subseteq \Sigma^{(1)} \subseteq \Sigma$, zeros $\xi'_{i_1}, \ldots, \xi'_{i_u}$ of $F(x)$ in $K_{i_1}, \ldots, K_{i_u}$ respectively, and constants $\gamma_k > 0$ for $k = 1,\ldots,u$, such that

$$|\alpha_{jk} - \xi'_{i_k}|_{i_k} \leq \gamma_k |F(\alpha_{jk})|_{i_k}$$

for $k = 1,\ldots,u$ and for all $\alpha_{jk} \in \Sigma^{(j)}$. Note that there exist algebraic numbers $\xi_{i_k} \in K_{i_k}$ for $k = 1,\ldots,u$ such that, again for all $\alpha_{jk} \in \Sigma^{(j)}$,

$$|\alpha_{jk}^{(i_k)} - \xi_{i_k}| = |\alpha_{jk} - \xi'_{i_k}|_{i_k}.$$

In fact, if ξ'_{i_k} corresponds to the fundamental sequence $\{\alpha_1, \alpha_2, \ldots\}$ of elements of K, then $\xi_{i_k} = \lim_{j \to \infty} \alpha_j^{(i_k)}$.

Finally, by Lemma 4 and equation (2) we can find an infinite subsequence $\Sigma' = \{\alpha_{i_1}, \alpha_{i_2}, \ldots\}$ of $\Sigma^{(u)}$, a q-adic zero Ξ of $F(x)$, and a constant $\gamma_q > 0$, such that

$$|\alpha_{i_j} - \Xi|_q \leq \gamma_q |F(\alpha_{i_j})|_q \quad \text{for} \quad j = 1,2,\ldots .$$

This sequence Σ', the numbers ξ_{i_k} and Ξ, and the constants $c_k = \gamma_k C_k$ and $c = \gamma_q C$ satisfy the requirements (A) and (B).

CHAPTER V

PROOF OF THEOREM III

THEOREM III: Let $\xi_{i_1},\dots,\xi_{i_u}$ and $\eta_1,\dots,\eta_r$ be algebraic numbers such that $\xi_{i_j} \in K_{i_j}$ for $j = 1,\dots,u$ and $\eta_j \in K_{\mathfrak{P}_j}$ for $j = 1,\dots,r$. Assume that there exists an infinite sequence $\Sigma = \{\alpha_1, \alpha_2, \dots\}$ of distinct nonzero elements of K such that

$$(E)\quad \prod_{j=1}^{u} |\xi_{i_j} - \alpha_k^{(i_j)}|_{\mathfrak{P}_j}^{*g_{i_j}} \prod_{j=1}^{r} |\eta_j - \alpha_k|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+1}^{r+s} |\alpha_k|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+s+1}^{r+s+t} \left|\frac{1}{\alpha_k}\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \le cH(\alpha_k)^{-\tau}.$$

Then $\tau \le 2$.

PROOF: Suppose that $\tau = 2 + 4\epsilon$, where we may assume that $0 < \epsilon \le 1$. We may also assume that $H(\alpha_k) \ge 2$ for all $\alpha_k \in \Sigma$. An application of Theorem I will yield a contradiction, and hence show that $\tau \le 2$.

For every k there exists a system of nonnegative real numbers $a_1(k),\dots,a_{r+s+t}(k)$, $b_1(k),\dots,b_u(k)$ such that

$$(1)\qquad H(\alpha_k)^{-b_j(k)} = |\alpha_k^{(i_j)} - \xi_{i_j}|_{\mathfrak{P}_j}^{*g_{i_j}},\quad j = 1,\dots,u$$

and

$$(2)\quad H(\alpha_k)^{-a_j(k)\log N(\mathfrak{P}_j)} = \begin{cases} |\alpha_k - \eta_j|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}, & j=1,\dots,r \\ |\alpha_k|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}, & j=r+1,\dots,r+s \\ \left|\frac{1}{\alpha_k}\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}, & j=r+s+1,\dots,r+s+t, \end{cases}$$

where $\mathfrak{P}_j$ divides the rational prime p_j and $N(\mathfrak{P}_j) = p_j^{f(\mathfrak{P}_j)}$ for $j=1,\ldots,r+s+t$. Let

$$(3) \qquad s_k = \sum_{j=1}^{r+s+t} a_j(k)\log N(\mathfrak{P}_j) + \sum_{j=1}^{u} b_j(k),$$

so that if we denote the left-hand side of (E) by $\Phi(\varkappa)$, then

$$(4) \qquad \Phi(\varkappa_k) = H(\varkappa_k)^{-s_k} \leq cH(\varkappa_k)^{-\tau} = cH(\varkappa_k)^{-2-4\epsilon}.$$

Since $\lim_{k\to\infty} H(\varkappa_k) = \infty$, this implies that for k sufficiently large,

$$(5) \qquad s_k \geq 2 + 3\epsilon.$$

We assume that this holds for all k.

If $\varkappa_k = \alpha_k/\beta_k$ where α_k and β_k are algebraic integers in K, then, since all the prime ideals $\mathfrak{P}_1,\ldots,\mathfrak{P}_{r+s+t}$ are distinct, inequality (3) of the introduction yields

$$(6) \qquad \prod_{j=r+1}^{r+s} |\alpha_k|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \geq \frac{1}{N(\alpha_k)} \quad \text{and} \quad \prod_{j=r+s+1}^{r+s+t} |\beta_k|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \geq \frac{1}{N(\beta_k)}$$

for $k = 1,2,\ldots$.

As before, there is no loss of generality in assuming that $\varkappa_k$ is a primitive element of K for all k. By the result given in Appendix A, we can choose α_k and β_k so that $\varkappa_k = \alpha_k/\beta_k$ where

$$(7) \qquad N(\alpha_k) \leq c(K)\, H(\varkappa_k) \quad \text{and} \quad N(\beta_k) \leq c(K)\, H(\varkappa_k).$$

Because $|\varkappa_k|_{\mathfrak{P}_j} \geq |\alpha_k|_{\mathfrak{P}_j}$ and $|1/\varkappa_k|_{\mathfrak{P}_j} \geq |\beta_k|_{\mathfrak{P}_j}$ for all j and k, the inequalities (6) and (7) yield

$$(8)\qquad \prod_{j=r+1}^{r+s} |\varkappa_k|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \geq \frac{1}{c(K)\,H(\varkappa_k)} \quad \text{and} \quad \prod_{j=r+s+1}^{r+s+t} \left|\frac{1}{\varkappa_k}\right|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \geq \frac{1}{c(K)\,H(\varkappa_k)}$$

for all k, or, by the definition (2) of $a_j(k)$,

$$(9)\qquad H(\varkappa_k)^{-\sum_{j=r+1}^{r+s} a_j(k)\log N(\mathfrak{P}_j)} \geq \frac{1}{c(K)\,H(\varkappa_k)}$$

and

$$H(\varkappa_k)^{-\sum_{j=r+s+1}^{r+s+t} a_j(k)\log N(\mathfrak{P}_j)} \geq \frac{1}{c(K)\,H(\varkappa_k)},$$

also for all k. Hence by (9), at least for all sufficiently large k,

$$(10)\qquad \sum_{j=r+1}^{r+s} a_j(k)\log N(\mathfrak{P}_j) \leq 1+\epsilon \quad \text{and} \quad \sum_{j=r+s+1}^{r+s+t} a_j(k)\log N(\mathfrak{P}_j) \leq 1+\epsilon.$$

Again we assume that this is true for all k.

Put

$$(11)\qquad A_j(k) = \frac{a_j(k)}{s_k},\ j = 1,\ldots,r+s+t, \quad \text{and} \quad B_j(k) = \frac{b_j(k)}{s_k},\ j = 1,\ldots,u,$$

so that by equation (3),

$$(12)\qquad \sum_{j=1}^{u} B_j(k) + \sum_{j=1}^{r+s+t} A_j(k)\log N(\mathfrak{P}_j) = 1.$$

The numbers $A_j(k)$ and $B_j(k)$ are nonnegative and bounded; suppose that

$$0 \le A_j(k) \le A \quad \text{and} \quad 0 \le B_j(k) \le B, \tag{13}$$

where A and B are positive and independent of both j and k. Let

$$M = \left[\frac{5}{\epsilon}\left\{u + \sum_{j=1}^{r+s+t} \log N(\mathfrak{P}_j)\right\}\right] + 1, \tag{14}$$

so that

$$u + \sum_{j=1}^{r+s+t} \log N(\mathfrak{P}_j) < \frac{\epsilon M}{5}. \tag{15}$$

Let

$$\begin{aligned} e_j(k) &= [A_j(k)M], \quad j = 1,\ldots,r+s+t \\ f_j(k) &= [B_j(k)M], \quad j = 1,\ldots,u, \end{aligned} \tag{16}$$

so that $0 \le e_j(k) \le [AM]$ and $0 \le f_j(k) \le [BM]$.

There are only finitely many choices of integers between 0 and $[AM]$ and between 0 and $[BM]$ for the $e_j(k)$ and $f_j(k)$ respectively, so there is an infinite subsequence Σ' of Σ, $\Sigma' = \{\chi_{i_k}\}$, and a system of nonnegative integers $f_1,\ldots,f_u$, $e_1,\ldots,e_{r+s+t}$ such that $e_j(i_k) = e_j$ and $f_j(i_k) = f_j$ for all k. Replace Σ by Σ', so that for all k,

$$e_j(k) = [A_j(k)M] = e_j \quad \text{and} \quad f_j(k) = [B_j(k)M] = f_j, \tag{17}$$

and consequently

$$(18)\qquad e_j \le A_j(k)M < e_j + 1 \quad \text{and} \quad f_j \le B_j(k)M < f_j + 1.$$

Hence by (12) and (18),

$$M = \left\{ \sum_{j=1}^{u} B_j(k) + \sum_{j=1}^{r+s+t} A_j(k)\log N(\mathfrak{P}_j) \right\} M$$

$$< \left\{ \sum_{j=1}^{u} f_j + \sum_{j=1}^{r+s+t} e_j \log N(\mathfrak{P}_j) \right\} + \left\{ u + \sum_{j=1}^{r+s+t} \log N(\mathfrak{P}_j) \right\},$$

so that by (15),

$$(19)\qquad (1 - \epsilon/5)M < \sum_{j=1}^{u} f_j + \sum_{j=1}^{r+s+t} e_j \log N(\mathfrak{P}_j).$$

We are now in a position to show that Σ satisfies the hypotheses of Theorem 1. Put

$$\mathfrak{q} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}, \quad \mathfrak{q}' = \mathfrak{P}_{r+1}^{e_{r+1}} \cdots \mathfrak{P}_{r+s}^{e_{r+s}}, \quad \text{and} \quad \mathfrak{q}'' = \mathfrak{P}_{r+s+1}^{e_{r+s+1}} \cdots \mathfrak{P}_{r+s+t}^{e_{r+s+t}},$$

where an empty product is 1. By the definitions (1) and (11) of $b_j(k)$ and $B_j(k)$ and by inequalities (18) and (5),

$$|\alpha_k^{(i_j)} - \xi_{i_j}|^{*g_{i_j}} = H(\alpha_k)^{-b_j(k)}$$

$$= H(\alpha_k)^{-B_j(k)\, s_k}$$

$$\le H(\alpha_k)^{-s_k f_j/M}$$

$$\leq H(\alpha_k)^{-(2+3\epsilon)f_j/M},$$

or

$$|\alpha_k^{(i_j)} - \zeta_{i_j}|^* \leq H(\alpha_k)^{-\rho_j}, \tag{20}$$

where

$$\rho_j = \frac{(2+3\epsilon)f_j}{g_{i_j}M} \geq 0. \tag{21}$$

Similarly, using definition (2), we obtain

$$|\alpha_k - \eta_j|_{\mathfrak{p}_j}^{*g(\mathfrak{p}_j)} = H(\alpha_k)^{-a_j(k)\log N(\mathfrak{p}_j)}$$

$$= H(\alpha_k)^{-A_j(k)s_k \log N(\mathfrak{p}_j)}$$

$$\leq H(\alpha_k)^{-s_k e_j \log N(\mathfrak{p}_j)/M},$$

or

$$|\alpha_k - \eta_j|_{\mathfrak{p}_j}^{*g(\mathfrak{p}_j)} \leq H(\alpha_k)^{-(2+3\epsilon)e_j \log N(\mathfrak{p}_j)/M} \tag{22}$$

Let

$$\sigma = \frac{(2+3\epsilon)\log N(\mathfrak{a})}{M}, \tag{23}$$

and observe that $\sigma \geq 0$. If $\sigma = 0$, then $N(\mathfrak{a}) = 1$, or $e_i = 0$ for $i = 1,\ldots,r$. Otherwise we may renumber the e's so

that $e_1 \cdots e_v \neq 0$ and $e_{v+1} = \cdots = e_r = 0$; then $\mathfrak{q} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_v^{e_v}$ and $N(\mathfrak{q}) > 1$. Let Ξ be the $\mathfrak{q}$-adic number such that $\Xi \longleftrightarrow (\eta_1, \ldots, \eta_v)$. Then inequality (22) together with the assumption that $H(\alpha_k) \geq 2$ yield

$$(24) \qquad |\alpha_k - \eta_j|^*_{\mathfrak{P}_j} = |\alpha_k - \eta_j|_{\mathfrak{P}_j} \leq H(\alpha_k)^{-\frac{(2+3\epsilon)e_j \log N(\mathfrak{P}_j)}{g(\mathfrak{P}_j)M}}$$

for $j = 1, \ldots, v$. Now

$$(25) \qquad |\alpha_k - \Xi|_{\mathfrak{q}} = \max \left\{ |\alpha_k - \eta_1|_{\mathfrak{P}_1}^{\frac{e(\mathfrak{P}_1)\log N(\mathfrak{q})}{e_1 \log p_1}}, \ldots, |\alpha_k - \eta_v|_{\mathfrak{P}_v}^{\frac{e(\mathfrak{P}_v)\log N(\mathfrak{q})}{e_v \log p_v}} \right\},$$

so, since $N(\mathfrak{P}_j) = p_j^{f(\mathfrak{P}_j)}$ and $e(\mathfrak{P}_j)f(\mathfrak{P}_j)/g(\mathfrak{P}_j) = 1$, we have by (23), (24), and (25) that $|\alpha_k - \Xi|_{\mathfrak{q}} \leq H(\alpha_k)^{-\sigma}$ if $\sigma > 0$. If $\sigma = 0$, we can replace $\mathfrak{q}$ by any nontrivial integral ideal, and in either case,

$$(26) \qquad |\alpha_k - \Xi|^*_{\mathfrak{q}} \leq H(\alpha_k)^{-\sigma}.$$

Finally, put

$$(27) \qquad \lambda = 1 - \frac{(2+3\epsilon)\log N(\mathfrak{q}')}{(1+\epsilon)M}$$

and

$$(28) \qquad \mu = 1 - \frac{(2+3\epsilon)\log N(\mathfrak{q}'')}{(1+\epsilon)M}.$$

By (18), (11), (10), and (5),

$$\log N(\mathfrak{q}') = \sum_{j=r+1}^{r+s} e_j \log N(\mathfrak{P}_j)$$

$$\leq M \sum_{j=r+1}^{r+s} A_j(k) \log N(\mathfrak{P}_j)$$

$$= \frac{M}{s_k} \sum_{j=r+1}^{r+s} a_j(k) \log N(\mathfrak{P}_j)$$

$$\leq \frac{(1+\epsilon)M}{2+3\epsilon}.$$

Similarly, $\log N(\mathfrak{q}'') \leq \frac{(1+\epsilon)M}{2+3\epsilon}$, so that $0 \leq \lambda \leq 1$ and $0 \leq \mu \leq 1$, where $\lambda = 1$ if and only if $N(\mathfrak{q}') = 1$, and $\mu = 1$ if and only if $N(\mathfrak{q}'') = 1$. By (2), (11), (5), and (18),

$$|\chi_k|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} = H(\chi_k)^{-a_j(k)\log N(\mathfrak{P}_j)} = H(\chi_k)^{-A_j(k)s_k \log N(\mathfrak{P}_j)}$$

$$\leq H(\chi_k)^{-\frac{(2+3\epsilon)e_j \log N(\mathfrak{P}_j)}{M}}$$

for all j and k. It follows that unless $N(\mathfrak{q}') = 1$,

$$|\chi_k|_{\mathfrak{q}'} = \max\left\{ |\chi_k|_{\mathfrak{P}_{r+1}}^{\frac{e(\mathfrak{P}_{r+1})\log N(\mathfrak{q}')}{e_{r+1}\log p_{r+1}}}, \ldots, |\chi_k|_{\mathfrak{P}_{r+s}}^{\frac{e(\mathfrak{P}_{r+s})\log N(\mathfrak{q}')}{e_{r+s}\log p_{r+s}}} \right\}$$

$$\leq H(\chi_k)^{-\frac{(2+3\epsilon)\log N(\mathfrak{q}')}{M}}$$

$$= H(\chi_k)^{(1+\epsilon)(\lambda-1)}$$

$$\leq H(\chi_k)^{\lambda-1}.$$

In case $N(\mathfrak{q}') = 1$, $\lambda = 1$, and the inequality $|\alpha_k|^*_{\mathfrak{q}'} \leq H(\alpha_k)^{\lambda-1}$ is valid for any nontrivial ideal $\mathfrak{q}'$. Thus for suitable $\mathfrak{q}'$,

$$|\alpha_k|^*_{\mathfrak{q}'} \leq H(\alpha_k)^{\lambda-1}. \tag{29}$$

Similarly,

$$\left|\frac{1}{\alpha_k}\right|^*_{\mathfrak{q}''} \leq H(\alpha_k)^{\mu-1}, \tag{30}$$

where in case $\mu = 1$, we replace $\mathfrak{q}''$ by any integral ideal with norm at least 2. We can easily ensure that $\mathfrak{q}$, $\mathfrak{q}'$, and $\mathfrak{q}''$ are pairwise relatively prime.

Hence by (20), (26), (29), and (30), the hypotheses of Theorem I are all satisfied, and we may conclude that

$$\sum_{j=1}^{u} g_{i_j} \rho_j + \sigma \leq \lambda + \mu. \tag{31}$$

Now by definitions (21), (23), (27), and (28) and by inequality (19),

$$\begin{aligned}
\sum_{j=1}^{u} g_{i_j} \rho_j + \sigma - \lambda - \mu &= \sum_{j=1}^{u} \frac{(2+3\epsilon)f_j}{M} + \frac{(2+3\epsilon)\log N(\mathfrak{q})}{M} - 2 \\
&\quad + \frac{(2+3\epsilon)\log N(\mathfrak{q}')}{(1+\epsilon)M} + \frac{(2+3\epsilon)\log N(\mathfrak{q}'')}{(1+\epsilon)M} \\
&= \frac{2+3\epsilon}{M}\left\{\sum_{j=1}^{u} f_j + \log N(\mathfrak{q}) + \frac{\log(N(\mathfrak{q}')N(\mathfrak{q}''))}{1+\epsilon}\right\} - 2 \\
&\geq \frac{2+3\epsilon}{(1+\epsilon)M}\left\{\sum_{j=1}^{u} f_j + \log(N(\mathfrak{q})N(\mathfrak{q}')N(\mathfrak{q}''))\right\} - 2 \\
&> \frac{2+3\epsilon}{(1+\epsilon)M}(1-\epsilon/5)M - 2.
\end{aligned}$$

Therefore (31) implies that

$$2(1+\epsilon) > (2+3\epsilon)(1-\epsilon/5),$$

or $\epsilon > 1$, which is a contradiction since we supposed that $\epsilon \leq 1$.

CHAPTER VI

THEOREM III IMPLIES THEOREM I

We have shown that Theorem III is a consequence of Theorem I. The two theorems are in fact equivalent in the aforementioned sense that the hypotheses of one yield those of the other. We shall now verify that Theorem III implies Theorem I.

Let ξ_{i_k}, η_j, Ξ, ρ_k, σ, λ, μ, $\mathfrak{q}$, $\mathfrak{q}'$, $\mathfrak{q}''$, c_k, c, c', and c'' be defined as in Theorem I where $k = 1,\ldots,u$ and $j = 1,\ldots,r$. We suppose that there is an infinite sequence $\Sigma = \{\chi_1, \chi_2, \cdots\}$ of distinct nonzero elements of K, all satisfying the hypotheses of Theorem I. Let

$$\mathfrak{q} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}, \quad \mathfrak{q}' = \mathfrak{P}_{r+1}^{e_{r+1}} \cdots \mathfrak{P}_{r+s}^{e_{r+s}}, \quad \text{and} \quad \mathfrak{q}'' = \mathfrak{P}_{r+s+1}^{e_{r+s+1}} \cdots \mathfrak{P}_{r+s+t}^{e_{r+s+t}}$$

where, by hypothesis, $\mathfrak{P}_1, \cdots, \mathfrak{P}_{r+s+t}$ are distinct prime ideals of K and $rst \neq 0$.

By definition,

$$|\chi_i - \Xi|_{\mathfrak{q}} = \max\left\{ |\chi_i - \eta_1|_{\mathfrak{P}_1}^{\frac{e(\mathfrak{P}_1)\log N(\mathfrak{q})}{e_1 \log p_1}}, \ldots, |\chi_i - \eta_r|_{\mathfrak{P}_r}^{\frac{e(\mathfrak{P}_r)\log N(\mathfrak{q})}{e_r \log p_r}} \right\},$$

so that

$$(1) \quad |\chi_i - \eta_j|_{\mathfrak{P}_j}^{*} \leq |\chi_i - \Xi|_{\mathfrak{q}}^{* \frac{e_j \log p_j}{e(\mathfrak{P}_j)\log N(\mathfrak{q})}}, \quad j = 1,\ldots,r.$$

Thus (1) and condition (B) of Theorem I imply that

$$\prod_{j=1}^{r} |\chi_i - \eta_j|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \leq |\chi_i - \Xi|_{\mathfrak{q}}^{*} \sum_{j=1}^{r} \frac{g(\mathfrak{P}_j)e_j \log p_j}{e(\mathfrak{P}_j)\log N(\mathfrak{q})}$$

$$= |\chi_i - \Xi|_{\mathfrak{q}}^{*}$$

$$\leq cH(\chi_i)^{-\sigma}.$$

Similarly,

$$\prod_{j=r+1}^{r+s} |\chi_i|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \leq c'H(\chi_i)^{\lambda-1}$$

and

$$\prod_{j=r+s+1}^{r+s+t} \left|\frac{1}{\chi_i}\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \leq c''H(\chi_i)^{\mu-1}$$

for all χ_i.

Hence for all $\chi_i \in \Sigma$,

$$\prod_{k=1}^{u} |\varphi_{i_k}^{(i_k)} - \chi_i|^{*g_{i_k}} \prod_{j=1}^{r} |\eta_j - \chi_i|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+1}^{r+s} |\chi_i|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+s+1}^{r+s+t} \left|\frac{1}{\chi_i}\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}$$

$$\leq \left(\prod_{k=1}^{u} c_k H(\chi_i)^{-\rho_k g_{i_k}}\right)\left(cH(\chi_i)^{-\sigma}\right)\left(c'H(\chi_i)^{\lambda-1}\right)\left(c''H(\chi_i)^{\mu-1}\right)$$

$$\leq CH(\chi_i)^{-\left(\sum_{k=1}^{u} \rho_k g_{i_k} + \sigma - \lambda - \mu + 2\right)}$$

where $C = cc'c'' \prod_{k=1}^{u} c_k$. By Theorem III, $\sum_{k=1}^{u} \rho_k g_{i_k} + \sigma - \lambda - \mu + 2 \leq 2$, or

$$\sum_{k=1}^{u} \rho_k g_{i_k} + \sigma \leq \lambda + \mu,$$

as asserted in Theorem I.

CHAPTER VII

A BOUND ON THE NUMBER OF SOLUTIONS

IN THEOREM III

THEOREM IIIa: Let $\xi_{i_1}, \ldots, \xi_{i_u}$ be nonzero algebraic numbers with $\xi_{i_j} \in K_{i_j}$ for $j = 1, \ldots, u$, and let η_j be a nonzero $\mathfrak{P}_j$-adic number for $j = 1, \ldots, r$. Then for $\epsilon > 0$, the inequality

$$(E') \quad \prod_{j=1}^{u} |\xi_{i_j} - \chi^{(i_j)}|^{g_{i_j}} \prod_{j=1}^{r} |\eta_j - \chi|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+1}^{r+s} |\chi|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \leq cH(\chi)^{-(2+4\epsilon)}$$

has at most

$$G^{1/\epsilon'^2}$$

solutions in distinct nonzero primitive elements χ of K. Here $\epsilon' = \min(\epsilon, 1/12)$ and G is a constant independent of ϵ and depending only on c, K, the various numbers ξ_{i_j} and η_j, and the ideals $\mathfrak{P}_j$.

PROOF: Following the notation used in the proof of Theorem III, we seek a bound on the number of solutions of (E) in case $\tau = 2 + 4\epsilon$ and $t = 0$. We can suppose that $c > 1$ and that $0 < \epsilon \leq 1/12$, thus letting $\epsilon = \epsilon'$. Now in case $H(\chi_k) > 1$, inequality (4) of Chapter V becomes

$$H(\chi_k)^{-s_k} \leq cH(\chi_k)^{-(2+4\epsilon)} = H(\chi_k)^{-(2+4\epsilon-c_k)}$$

where

(1) $$c_k = \frac{\log \epsilon}{\log H(\varkappa_k)},$$

so that $2 + 4\epsilon \leq s_k + c_k$. In the proof of Theorem III, we assumed (V-5) that $s_k \geq 2 + 3\epsilon$, which will be true in case $c_k \leq \epsilon$. By definition (1), $0 < c_k \leq \epsilon$ if $0 < \log c \leq \epsilon \log H(\varkappa_k)$, or, since $H(\varkappa_k)$ is a positive integer, if

(2) $$H(\varkappa_k) \geq \max(c^{1/\epsilon}, 2).$$

We also assumed (V-10) that

(3) $$\sum_{j=r+1}^{r+s} a_j(k) \log N(\mathfrak{P}_j) \leq 1 + \epsilon.$$

This was true for k sufficiently large because (V-9)

$$H(\varkappa_k)^{-\sum_{j=r+1}^{r+s} a_j(k) \log N(\mathfrak{P}_j)} \geq \frac{1}{c(K)\, H(\varkappa_k)},$$

or

$$\sum_{j=r+1}^{r+s} a_j(k) \log N(\mathfrak{P}_j) \leq 1 + \frac{\log c(K)}{\log H(\varkappa_k)}$$

for all k, where $c(K)$ depends only on K. Thus inequality (3) will hold in case

$$1 + \frac{\log c(K)}{\log H(\varkappa_k)} \leq 1 + \epsilon,$$

or in case

$$H(\alpha_k) \geq c(K)^{1/\epsilon}. \tag{4}$$

Let N_1 be the number of primitive elements of K with heights less than $\max(c(K)^{1/\epsilon}, c^{1/\epsilon}, 2)$. By (2) and (4) we disregarded such values of α_k throughout Chapter V. Clearly

$$N_1 \leq N^{1/\epsilon} \tag{5}$$

for some positive constant N depending only on c and K.

Recall that by (V-13), $0 \leq A_j(k) \leq A$ and $0 \leq B_j(k) \leq B$ where A and B are nonzero numbers which are independent of k, and where by (V-12),

$$\sum_{j=1}^{u} B_j(k) + \log 2 \sum_{j=1}^{r+s} A_j(k) \leq \sum_{j=1}^{u} B_j(k) + \sum_{j=1}^{r+s} A_j(k) \log N(\mathfrak{P}_j) = 1.$$

It follows that

$$\sum_{j=1}^{u} A_j(k) \leq \frac{1}{\log 2} \quad \text{and} \quad \sum_{j=1}^{r+s} B_j(k) \leq 1,$$

or

$$A \leq \frac{1}{\log 2} \quad \text{and} \quad B \leq 1. \tag{6}$$

By (V-17) there are $[AM] + 1$ choices for each e_j and $[BM] + 1$ choices for each f_j, where M is defined in (V-14) and depends only on K, ϵ, and the ideals $\mathfrak{P}_j$. Hence if N_2 is the total number of solutions in primitive elements of (E'), and if there are N_3 solutions remaining after the e_j and f_j have been fixed, then

$$(7) \qquad N_2 \leq N_1 + ([AM] + 1)^{r+s} ([BM] + 1)^{u} N_3 .$$

Recall that by definitions (V-21), (V-23), and (V-27) for non-zero ρ_j, σ, and $1 - \lambda$,

$$(8) \qquad \rho_j = \frac{(2+3\epsilon) f_j}{g_{i_j} M} \geq \frac{2+3\epsilon}{2M} \quad \text{for} \quad j = 1, \ldots, u,$$

$$\sigma = \frac{(2+3\epsilon) \log N(\mathfrak{q})}{M} \geq \frac{2+3\epsilon}{2M},$$

and

$$1 - \lambda = \frac{(2+3\epsilon) \log N(\mathfrak{q}')}{(1+\epsilon)M} \geq \frac{2+3\epsilon}{2M}.$$

Thus

$$\sum_{j=1}^{u} \rho_j g_{i_j} + \sigma - \lambda = \frac{2+3\epsilon}{M} \left\{ \sum_{j=1}^{u} f_j + \log N(\mathfrak{q}) + \frac{\log N(\mathfrak{q}')}{1+\epsilon} \right\} - 1.$$

By (V-19), (V-18), and (V-12),

$$\left(\frac{1-\epsilon/5}{1+\epsilon}\right) M < \sum_{j=1}^{u} f_j + \log N(\mathfrak{q}) + \frac{\log N(\mathfrak{q}')}{1+\epsilon} \leq M,$$

so that

$$1 < \sum_{j=1}^{u} \rho_j g_{i_j} + \sigma - \lambda \leq 1 + 3\epsilon \leq 1 + 4\epsilon.$$

We can therefore apply Theorem Ia of Chapter III, page 34, obtaining the inequality

$$(9) \qquad N_3 \leq F^{1/\epsilon^2} - \frac{6}{\epsilon} \log(\min(\rho_j, \sigma, 1 - \lambda))$$

where F is as in Theorem Ia and the minimum is taken over nonzero ρ_j, σ, and $1 - \lambda$. Therefore by (5), (6), the definition of M, (9), and (8), inequality (7) becomes

$$N_2 \leq N^{1/\epsilon} + (3M)^{r+s+u} \left(F^{1/\epsilon^2} + \frac{6}{\epsilon} \log \frac{2M}{2+3\epsilon}\right) \tag{10}$$

where N is a constant depending only on c and K, while F depends on c, K, and the various numbers ξ_{ij} and η_j and ideals $\mathfrak{F}_j$. Now by the definition (V-14) of M, $M \leq L/\epsilon$, where L is a constant depending on K and the $\mathfrak{F}_j$'s, so that finally inequality (10) yields

$$N_2 \leq G_1^{1/\epsilon^2} + \frac{H \log(1/\epsilon)}{\epsilon^{r+s+u+1}}$$

where G_1 and H are constants depending on c, K, the various numbers ξ_{ij} and η_j, and the ideals $\mathfrak{F}_j$. Thus

$$N_2 \leq G^{1/\epsilon^2},$$

where $G > G_1$ is a suitable constant depending only on G_1 and H, and hence is as specified in the theorem.

CHAPTER VIII

EQUIVALENCE OF THEOREM III AND THEOREM IV

THEOREM IV: Let $F_1(x),\ldots,F_u(x)$, $G_1(x),\ldots,G_r(x)$ be equal or distinct nonconstant polynomials over K which have no multiple factors. Assume that there exists an infinite sequence of distinct nonzero elements $\varkappa$ of K satisfying

$$(F)\quad \prod_{k=1}^{u} |F_k(\varkappa)|_{i_k}^{*g_{i_k}} \prod_{j=1}^{r} |G_j(\varkappa)|_{\mathfrak{p}_j}^{*g(\mathfrak{p}_j)} \prod_{j=r+1}^{r+s} |\varkappa|_{\mathfrak{p}_j}^{*g(\mathfrak{p}_j)} \prod_{j=r+s+1}^{r+s+t} \left|\frac{1}{\varkappa}\right|_{\mathfrak{p}_j}^{*g(\mathfrak{p}_j)}$$

$$\leq CH(\varkappa)^{-\tau}.$$

Then $\tau \leq 2$.

Theorems III and IV are equivalent. We first show that the hypotheses of Theorem III imply those of Theorem IV.

Let Σ be a sequence satisfying the hypotheses of Theorem IV. In case $L_1 = \liminf_{\varkappa \in \Sigma} |F_1(\varkappa)|_{i_1}$ is not zero, let $\Sigma^{(1)}$ be an infinite subsequence of Σ on which $F_1(x)$ is not zero and denote by ξ_{i_1} an arbitrary algebraic element of K_{i_1}, not zero. If, however, $L_1 = 0$, then Σ contains an infinite subsequence Σ' for which

$$\lim_{\substack{k\to\infty \\ \varkappa_k \in \Sigma'}} |F_1(\varkappa_k)|_1 = 0.$$

In the latter case, by Lemmas (IV-1) and (IV-2), page 47, there exist an infinite subsequence $\Sigma^{(1)}$ of Σ' (and hence of Σ), a zero ξ'_{i_1} of $F_1(x)$ with $\xi'_{i_1} \in K_{i_1}$, and a constant

$\gamma_1' \geq 1$ such that for all $\alpha_k \in \Sigma^{(1)}$,

$$|\alpha_k - \xi'_{i_1}|_{i_1} \leq \gamma_1' |F_1(\alpha_k)|_{i_1}.$$

Whether or not $L_1 \neq 0$, it is clear that there is a positive constant γ_1 such that

$$|\alpha_k - \xi'_{i_1}|^*_{i_1} \leq \gamma_1 |F_1(\alpha_k)|^*_{i_1}$$

for all $\alpha_k \in \Sigma^{(1)}$. We continue this process, examining

$$\liminf_{\alpha \in \Sigma^{(j-1)}} \left(|F_j(\alpha)|_{i_j} \right)$$

for $j = 2,\ldots,u$, to obtain infinite sequences

$$\Sigma^{(u)} \subseteq \ldots \subseteq \Sigma^{(1)} \subseteq \Sigma,$$

algebraic numbers $\xi'_{i_2} \in K_{i_2}, \ldots, \xi'_{i_u} \in K_{i_u}$, and constants $\gamma_1,\ldots,\gamma_u$, all at least 1, such that

$$|\alpha_k - \xi'_{i_j}|^*_{i_j} \leq \gamma_j |F_j(\alpha_k)|^*_{i_j}$$

for all $\alpha_k \in \Sigma^{(j)}$ and for $j = 2,\ldots,u$. Hence there is an algebraic number $\xi_{i_j} \in K_{i_j}$ such that

(1) $|\alpha_k^{(i_j)} - \xi_{i_j}|^* \leq \gamma_j |F_j(\alpha_k)|^*_{i_j}$ for all $\alpha_k \in \Sigma^{(u)}$, $j = 1,\ldots,u$.

Similarly, using Lemmas (IV-1) and (IV-3), page 47, we find

infinite sequences

$$\Sigma_r \subseteq \ldots \subseteq \Sigma_1 \subseteq \Sigma^{(u)} ,$$

positive constants $\delta_1,\ldots,\delta_r$, and $\mathfrak{P}_j$-adic zeros η_j of $G_j(x)$ for $j = 1,\ldots,r$ such that

$$(2)\quad |\alpha_k - \eta_j|_{\mathfrak{P}_j}^{*} \leq \delta_j |G_j(\alpha_k)|_{\mathfrak{P}_j}^{*} \quad \text{for all } \alpha_k \in \Sigma_j,\ j = 1,\ldots,r.$$

Thus if we denote the left-hand side of (E) by $\Phi(\alpha)$ and that of (F) by $\Psi(\alpha)$, we have, by inequalities (1) and (2), that

$$\Phi(\alpha_k) \leq \gamma_1^{g_{i_1}} \cdots \gamma_u^{g_{i_u}} \delta_1^{g(\mathfrak{P}_1)} \cdots \delta_r^{g(\mathfrak{P}_r)} \Psi(\alpha_k)$$

for all $\alpha_k \in \Sigma_r$. By hypothesis, $\Psi(\alpha_k) \leq CH(\alpha_k)^{-\tau}$ for all $\alpha_k \in \Sigma$, and hence for all $\alpha_k \in \Sigma_r$, so that $\Phi(\alpha_k) \leq cH(\alpha_k)^{-\tau}$ for all $\alpha_k \in \Sigma_r$ where

$$c = \gamma_1^{g_{i_1}} \cdots \gamma_u^{g_{i_u}} \delta_1^{g(\mathfrak{P}_1)} \cdots \delta_r^{g(\mathfrak{P}_r)} C.$$

By Theorem III, $\tau \leq 2$, which is the desired result.

The proof that Theorem IV implies Theorem III is similar. Let Σ satisfy the hypotheses of Theorem III, and let $F_1(x),\ldots,F_u(x)$, $G_1(x),\ldots,G_r(x)$ be irreducible monic polynomials which have $\xi_{i_1},\ldots,\xi_{i_u}, \eta_1,\ldots,\eta_r$ respectively, as roots. If we can show that there exist positive constants $\gamma_1,\ldots,\gamma_u$, $\delta_1,\ldots,\delta_r$ such that for all α_k in some subsequence Σ' of Σ,

$$(3) \qquad |F_j(\alpha_k)|^*_{i_j} \le \gamma_j |\alpha_k^{(i_j)} - \xi_{i_j}|^* , \quad j = 1,\dots,u$$

and

$$(4) \qquad |G_j(\alpha_k)|^*_{\mathfrak{r}_j} \le \delta_j |\alpha_k - \eta_j|^*_{\mathfrak{r}_j} , \quad j = 1,\dots,\mathfrak{r} ,$$

we shall have obtained the inequality

$$\Psi(\alpha_k) \le \gamma_1^{g_{i_1}} \cdots \gamma_u^{g_{i_u}} \delta_1^{g(\mathfrak{r}_1)} \cdots \delta_{\mathfrak{r}}^{g(\mathfrak{r}_{\mathfrak{r}})} \Phi(\alpha_k)$$

for all $\alpha_k \in \Sigma'$, and hence, as above, the desired result.

To show (3), we recall that ξ_{i_j} is a zero of $F_j(x)$, and hence $F_j(x) = (x - \xi_{i_j}) H_j(x)$ where $H_j(x)$ is a monic polynomial with real coefficients. Let γ_j be any number greater than or equal to 1 such that $|H_j(x)| \le \gamma_j$ for all real or complex x with $|x - \xi_{i_j}| \le 1$. Then for every k, either $|\alpha_k^{(i_j)} - \xi_{i_j}| > 1$ and hence trivially

$$|F_j(\alpha_k)|^*_{i_j} \le \gamma_j |\alpha_k^{(i_j)} - \xi_{i_j}|^* = \gamma_j,$$

or $|\alpha_k^{(i_j)} - \xi_{i_j}| \le 1$ and hence

$$|F_j(\alpha_k)|_{i_j} \le \gamma_j |\alpha_k^{(i_j)} - \xi_{i_j}|,$$

so that

$$|F_j(\alpha_k)|^*_{i_j} \le \min (1, \gamma_j |\alpha_k^{(i_j)} - \xi_{i_j}|) \le \gamma_j |\alpha_k^{(i_j)} - \xi_{i_j}|^* .$$

This gives the inequalities in (3), and the inequalities in (4) can be proved in the same manner.

CHAPTER IX

RESTRICTION TO INTEGERS IN K

THEOREMS I'-IV': In case the numbers α of Theorems I-IV satisfy the additional condition

$$(G) \qquad b(\alpha) \leq B H(\alpha)^{\nu}$$

for some constant $B > 0$ and for some ν, $0 \leq \nu \leq 1$, where $b(\alpha)$ is the leading coefficient in the defining equations of α, then the conclusion of Theorems I and II becomes

$$(1) \qquad \sum_{k=1}^{u} \rho_k g_{i_k} + \sigma \leq \lambda + \mu - (1 - \nu),$$

and the conclusion of Theorems III and IV becomes

$$\tau \leq 1 + \nu.$$

We first prove Theorem I'. The proof is due to Mahler [9] and makes use of the following lemma.

LEMMA: Let $\epsilon > 0$, and let $\Sigma = \{\alpha_1, \alpha_2, \ldots\}$ be an infinite sequence of distinct primitive elements α_i of K. For each i, let b_i be the leading coefficient in the defining polynomial of α_i, and let $q_i = H(\alpha_i)$. Then there exists a subsequence Σ' of Σ, and there exist nonnegative numbers $\beta_1, \ldots, \beta_n$, and ν' with $\nu' \leq 1$, such that

$$(a) \qquad \sum_{j=1}^{n} \beta_j < 1 + \epsilon$$

(b) $\max(1, |\alpha_i^{(j)}|) \leq \left\{(n+1)\frac{q_i}{b_i}\right\}^{\beta_j}$ for $1 \leq j \leq n$ and all $\alpha_i \in \Sigma'$

(c) $q_i^{\nu'-\epsilon/4} \leq b_i \leq q_i^{\nu'+\epsilon/4}$ for all i such that $\alpha_i \in \Sigma'$.

PROOF OF THE LEMMA: We first select a number ν', $0 \leq \nu' \leq 1$, and an infinite subsequence $\Sigma_{\nu'}$, of Σ all of whose terms satisfy (c). This is clearly possible since $0 < b_i \leq q_i$.

To show that conditions (a) and (b) can be satisfied, we make use of Lemma (II-3), page 24, which states that if α_i is a primitive element of K, then

$$\prod_{j=1}^{n} \max(1, |\alpha_i^{(j)}|) \leq (n+1)q_i/b_i. \tag{2}$$

Let

$$\max(1, |\alpha_i^{(j)}|) = ((n+1)q_i/b_i)^{\beta_j(i)}, \tag{3}$$

so that by (2),

$$\sum_{j=1}^{n} \beta_j(i) \leq 1. \tag{4}$$

Further, let

$$\beta_j^*(i) = \left[\frac{n+1}{\epsilon}\beta_j(i)\right].$$

For any j, there are clearly at most $[\frac{n+1}{\epsilon}] + 1$ choices for each integer $\beta_j^*(i)$, so that one of them, say β_j^*, must occur infinitely many times among the numbers $\beta_j^*(i)$, $i = 1,2,\ldots$. Similarly, one

n-tuple $(\beta_1^*(i),\ldots,\beta_n^*(i))$ must occur infinitely many times among the n-tuples $(\beta_1^*(i),\ldots,\beta_n^*(i))$, $i = 1,2,\ldots$. By taking a subsequence of Σ_ν, we may suppose that $\beta_j^*(i) = \beta_j^*$ for $1 \le j \le n$ and $i = 1,2,\ldots$. Let

$$\beta_j = \frac{\epsilon}{n+1}(\beta_j^* + 1),$$

so that

$$0 < \beta_j \le \beta_j(i) + \frac{\epsilon}{n+1}$$

for $1 \le j \le n$ and $i = 1,2,\ldots$. Hence, by (4),

$$\sum_{j=1}^{n} \beta_j \le 1 + \frac{n\epsilon}{n+1} < 1 + \epsilon,$$

which is just condition (a), and condition (b) follows from (3) and the fact that $\beta_j(i) \le \beta_j$ for all i and j.

It should be noted that there are no more than $([\frac{n+1}{\epsilon}] + 1)^n$ choices for the n-tuple $(\beta_1,\cdots,\beta_n)$.

PROOF OF THE THEOREMS: First we prove Theorem I'. As in the proof of Theorem I, we assume that there is an infinite sequence Σ of distinct nonzero primitive elements α_i of K which satisfy conditions (A)-(D) of that theorem. We also suppose that

$$\sum_{k=1}^{u} \rho_k g_{i_k} + \sigma = \lambda + \mu - (1 - \nu) + 8\epsilon \tag{5}$$

where $0 < \epsilon \le 1/12$ and proceed as before to obtain a contradiction.

Let b_i and q_i be as in the proof of Theorem I. By the above lemma, there is an infinite subsequence Σ' of Σ and there are nonnegative numbers $\beta_1,\ldots,\beta_n, \nu'$ with $\nu' \leq 1$ such that

$$\sum_{j=1}^{n} \beta_j < 1 + \epsilon, \tag{6}$$

$$\max(1, |\alpha_i^{(j)}|) \leq \left\{(n+1)\frac{q_i}{b_i}\right\}^{\beta_j} \quad \text{for } 1 \leq j \leq n \text{ and } \alpha_i \in \Sigma', \tag{7}$$

and

$$q_i^{\nu'-\epsilon/4} \leq b_i \leq q_i^{\nu'+\epsilon/4} \quad \text{for all } i \text{ such that } \alpha_i \in \Sigma'. \tag{8}$$

Condition (G) and (8) above allow us to suppose that

$$\nu' \leq \nu. \tag{9}$$

Following the proof of Theorem I with Σ replaced by Σ', we obtain inequality (II-34), or

$$|\Psi^{(h)}| \leq (80d_1)^{mr_1} \max_{(j)\in I}\left\{\prod_{i=1}^{m} \max(1, |\alpha_i^{(h)}|)^{j_i - t_i}\right\}. \tag{10}$$

Applying (7) and (6) to (10), we obtain

$$\left|\prod_{h\notin H} \Psi^{(h)}\right| \leq (80d_1)^{nmr_1} \prod_{h=1}^{n}\left\{\max_{(j)\in I}\left(\prod_{i=1}^{m}\left\{(n+1)\frac{q_i}{b_i}\right\}^{\beta_h(j_i-t_i)}\right)\right\}$$

$$< (80d_1)^{nmr_1} \max_{(j)\in I}\left\{\prod_{i=1}^{m}\left\{(n+1)\frac{q_i}{b_i}\right\}^{(1+\epsilon)(j_i-t_i)}\right\}.$$

Applications of (8), (II-17), and the definition of the set I as given on page 23 yield

$$\left|\prod_{h\in H}\Psi^{(h)}\right| < (80d_1)^{nmr_1}(n+1)^{(1+\epsilon)mr_1}\max_{(j)\in I}\left\{\prod_{i=1}^{m} q_i^{(1-\nu'+\epsilon/4)(1+\epsilon)(j_i-t_i)}\right\}$$

$$\leq \left\{(80d_1)^n(n+1)^{1+\epsilon}\right\}^{mr_1} q_1^{(1-\nu'+\epsilon/4)(1+\epsilon)^2 r_1 S_2}.$$

Hence by inequality (II-32) and the definition of φ (page 20), we have

$$\varphi < b_1^{r_1-t_1}\cdots b_m^{r_m-t_m}\left\{(80d_1)^n(n+1)^{1+\epsilon}\prod_{k=1}^{u}(c_k')^{g_{i_k}}\right\}^{mr_1} \times q_1^{(1-\nu'+\epsilon/4)(1+\epsilon)^2 r_1 S_2 - r_1 S_1 \sum_{k=1}^{u}\rho_k g_{i_k}}. \tag{11}$$

Finally, if we let $d_2' = (80d_1)^n(n+1)^{1+\epsilon}\prod_{k=1}^{u}(c_k')^{g_{i_k}}$, so that $d_2' > 1$, and if we apply (8) and (II-17) to (11), we obtain a new upper bound for φ:

$$\varphi < d_2'^{\,mr_1} q_1^{(\nu'+\epsilon/4)(1+\epsilon)r_1 S_3 + (1-\nu'+\epsilon/4)(1+\epsilon)^2 r_1 S_2 - r_1 S_1 \sum_{k=1}^{u}\rho_k g_{i_k}}. \tag{12}$$

Combining (12) and the previous lower bound (II-53) for φ, we obtain the inequality

$$d_4^{-mr_1} q_1^{r_1\{(1+\sigma-\lambda)S_1-(1+\epsilon)(1-\mu)S_2+(1-\mu)S_3\}} < d_2'^{\,mr_1} q_1^{r_1\{(\nu'+\epsilon/4)(1+\epsilon)S_3+(1-\nu'+\epsilon/4)(1+\epsilon)^2 S_2 - S_1\sum_{k=1}^{u}\rho_k g_{i_k}\}},$$

or

$$(13)\quad q_1^{\left(\sum_{k=1}^{u}\rho_k g_{i_k}+\sigma-\lambda+1\right)}S_1-\left((1+\epsilon)(1-\mu)+(1-\nu'+\epsilon/4)(1+\epsilon)^2\right)S_2+\left(1-\mu-(\nu'+\epsilon/4)(1+\epsilon)\right)S_3$$

$$< (d_2'd_4)^m.$$

If we take the number R of inequality (II-10) to be $(d_2'd_4)^{12/\epsilon}$, then $d_2'd_4 \le q_1^{\epsilon/12}$, so that (13) yields

$$(14)\quad \left(\sum_{k=1}^{u}\rho_k g_{i_k}+\sigma-\lambda+1\right)S_1-\left((1+\epsilon)(1-\mu)+(1-\nu'+\epsilon/4)(1+\epsilon)^2\right)S_2$$

$$+\left(1-\mu-(\nu'+\epsilon/4)(1+\epsilon)\right)S_3 < m\epsilon/12.$$

Since, by (9), $0 \le \nu' \le \nu \le 1$, we replace the assumption (5) by the less stringent assumption that

$$(15)\quad \sum_{k=1}^{u}\rho_k g_{i_k}+\sigma = \lambda+\mu-(1-\nu')+8\epsilon.$$

If we obtain a contradiction using (15), it will follow that

$$\sum_{k=1}^{u}\rho_k g_{i_k}+\sigma \le \lambda+\mu-(1-\nu'),$$

and this implies (1).

By (14) and (15),

$$\left(\mu+\nu'+8\epsilon\right)S_1-\left(2-\mu-\nu'-\epsilon\mu-2\epsilon\nu'+\frac{13}{4}\epsilon+\epsilon^2(3/2-\nu'+\epsilon/4)\right)S_2$$

$$+\left(1-\mu-\nu'-\epsilon\nu'-\epsilon/4-\epsilon^2/4\right)S_3 < m\epsilon/12,$$

or, since $0 < \epsilon \leq 1/12$, $0 \leq \mu \leq 1$, and $0 \leq \nu' \leq 1$,

$$(16)\quad \left(\mu+\nu'+8\epsilon\right)s_1 - \left(2+\mu-\nu'+\frac{13}{4}\epsilon+\frac{\epsilon}{12}\left(\frac{3}{2}+\frac{1}{48}\right)\right)s_2 + \left(1-\mu-\nu'-\frac{5}{4}\epsilon-\frac{\epsilon}{48}\right)s_3 < m\epsilon/12.$$

Applying definitions (II-26) to inequality (16), we obtain

$$(17)\qquad m\epsilon - \frac{d}{2}\left(2+\frac{23}{2}\epsilon\right) - L\left(\mu+\nu'-1+\epsilon\frac{27}{8}\right) < m\epsilon/12.$$

But $d = \epsilon m/6$, $0 \leq \mu \leq 1$, $0 \leq \nu' \leq 1$, and by (II-27), $L < \epsilon m/6$, so that (17) yields

$$m\epsilon - \frac{m\epsilon}{12}\left(2+\frac{23}{2}\epsilon\right) - \frac{m\epsilon}{6}\left(1+\frac{27}{8}\epsilon\right) < m\epsilon/12,$$

or

$$28/73 < \epsilon,$$

which contradicts the assumption that $\epsilon \leq 1/12$. Hence we have obtained inequality (1), and so proved Theorem I'.

Theorem I is, of course, just Theorem I' with $\nu = 1$.

Theorems I' and II' are equivalent by the argument used to show the equivalence of Theorems I and II.

To obtain Theorem III', suppose that $\tau = 1 + \nu + 4\epsilon$ and proceed as in the proof of Theorem III. Inequality (V-5) becomes $s_k \geq 1 + \nu + 3\epsilon$, and the numbers 2 occuring in the definitions of $\rho_j, \sigma, \lambda,$ and μ are replaced by $1 + \nu$. An application of Theorem I' instead of Theorem I yields

$$\frac{1+\nu+3\epsilon}{(1+\epsilon)M}M(1-\epsilon/5) - 2 < \sum_{j=1}^{u} \rho_j g_{i_j} + \sigma - \lambda - \mu \leq -1 + \nu,$$

or

$$3 < \epsilon + 2\nu,$$

contrary to the suppositions that $\nu \leq 1$ and $\epsilon \leq 1$. Thus $\tau \leq 1 + \nu$ as desired.

The equivalence of Theorems I' and III', and that of Theorems III' and IV', follow exactly as before.

THEOREMS I"-IV": If we restrict the numbers $\varkappa$ of Theorems I-IV to be integers in K, setting $\mu = 1$ and $t = 0$, then the conclusion of Theorems I and II becomes

$$\sum_{k=1}^{u} \rho_k g_{i_k} + \sigma \leq \lambda,$$

and the conclusion of Theorems III and IV becomes

$$\tau \leq 1.$$

PROOF: Set $\nu = 0$ in condition (G) of Theorems I'-IV'.

Theorems I"-IV" are equivalent by the arguments used to show the equivalence of Theorems I-IV. Theorems III" and IV" are just Theorems V and VI of Chapter I.

THEOREM I'a: Retain the notation of Theorems I and I'. Let

$$\sum_{k=1}^{u} \rho_k g_{i_k} + \sigma - \lambda = \nu + 8\epsilon$$

where $0 < \epsilon \leq 1/12$. Then the number of distinct nonzero solutions in

primitive elements of K of the inequalities

(A) $|\xi_{i_k} - \alpha^{(i_k)}|^* \leq c_k H(\alpha)^{-\rho_k}$, $k = 1,\dots,u$

(B) $|\Xi - \alpha|^*_{\mathfrak{q}} \leq cH(\alpha)^{-\sigma}$

(C) $|\alpha|^*_{\mathfrak{q}'} \leq c'H(\alpha)^{\lambda-1}$

(G) $b(\alpha) \leq BH(\alpha)^{\nu}$

is at most $F'^{1/\epsilon^2} - \frac{13(n+2)^n}{\epsilon^{n+2}} \log D$

where $F' > 1$ is a constant depending on the numbers c_i, ξ_{i_j}, c, c', B, and Ξ, and also on $\mathfrak{q}$, $\mathfrak{q}'$, and K, but not on ϵ or ν, and where D is as in Theorem Ia.

PROOF: By the argument used to obtain Theorem Ia, for each choice of the numbers ν', $\beta_1,\dots,\beta_n$ used in the proof of Theorem I' there are no more than $F^{1/\epsilon^2} - \frac{6}{\epsilon}\log D$ solutions α of inequalities (A)-(C). Here $F > 1$ is a constant depending on the numbers c_i, ξ_{i_j}, c, c', and Ξ, and also on $\mathfrak{q}$, $\mathfrak{q}'$, and K, but not on ϵ, ν, or B. Also D is 1 in case the asterisks are omitted from inequalities (A)-(C), and otherwise D is the smallest of those of the numbers ρ_k, σ, $1 - \lambda$ which are nonzero. In the proof of the lemma on page 76, it was remarked that it would suffice to consider no more than $([\frac{n+1}{\epsilon}] + 1)^n$ choices for the n-tuple $\beta_1,\dots,\beta_n$. Consequently, we need only find a bound on the number of choices which might be made for ν' in order to obtain a bound on the number of solutions

of inequalities (A)-(C) and (G).

Retain the notation of the lemma. Let

$$b_i = q_i^{\nu(i)}.$$

By (G),

$$q_i^{\nu(i)} \leq B\, q_i^{\nu}.$$

Assume that

$$q_i \geq B^{4/\epsilon}, \tag{18}$$

or

$$\nu(i) \leq \nu + \epsilon/4 \tag{19}$$

for $i = 1,2,\ldots$. There are clearly at most

$$n(2B^{4/\epsilon})^{n+1} \tag{20}$$

numbers α_i which do not have heights q_i satisfying (18). Let

$$\nu^*(i) = [\tfrac{2}{\epsilon}\nu(i) + 1/2],$$

so that, by (19), there are at most $\frac{2}{\epsilon}\nu + 2$ choices for $\nu^*(i)$. For each such choice, let

$$\nu'(i) = \tfrac{\epsilon}{2}\nu^*(i),$$

so that

$$\nu(i) - \epsilon/4 \le \nu'(i) \le \nu(i) + \epsilon/4$$

for all values of i yielding a given $\nu'(i)$. Let such a fixed $\nu'(i) = \nu'$ and note that for the values of i which yield this value ν',

$$q_i^{\nu'-\epsilon/4} \le b_i = q_i^{\nu(i)} \le q_i^{\nu'+\epsilon/4}.$$

Thus, under condition (18), condition (c) of the lemma can be satisfied even if we confine our choice of the number ν' to at most $\frac{2}{\epsilon}\nu + 2$ values.

Consequently, by (20) and the previous discussion, the total number of solutions sought is at most

$$n(2B^{4/\epsilon})^{n+1} + \left(\frac{2}{\epsilon}\nu + 2\right)\left(\left[\frac{n+1}{\epsilon}\right] + 1\right)^n\left(F^{1/\epsilon^2} - \frac{6}{\epsilon}\log D\right),$$

or

$$F'^{1/\epsilon^2} - \frac{13(n+2)^n}{\epsilon^{n+2}}\log D,$$

where F' is as described in the theorem.

THEOREM III'a: Let ν, ξ_{i_j}, η_j, and $\mathfrak{P}_j$ be as in Theorems III and III'. Then for $\epsilon > 0$, the inequalities

$$(E'') \quad \prod_{j=1}^{u} |\xi_{i_j} - \lambda^{(i_j)}|^{*g_{i_j}} \prod_{j=1}^{r} |\eta_j - \lambda|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+1}^{r+s} |\lambda|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \le cH(\lambda)^{-(1+\nu+4\epsilon)}$$

and

(G) $$b(\alpha) = BH(\alpha)^{\nu}$$

have at most $$G'^{1/\epsilon'^2}$$

solutions in distinct nonzero primitive elements α of K. Here $\epsilon' = \min(\epsilon, 1/12)$ and G' is a constant independent of ϵ and ν, but dependent on n, on the various numbers ξ_{i_j} and η_j, and on the ideals $\mathfrak{P}_{\nu j}$.

PROOF: Following the proof of Theorem IIIa, we need only replace the assumptions (V-5) that $s_k \geq 2 + 3\epsilon$ by the assumption that $s_k \geq 1 + \nu + 3\epsilon$ and apply Theorem I'a instead of Theorem Ia. Because 2 is replaced by $1 + \nu$ throughout the argument, the change in the lower bound for s_k does not effect the bound on the number of solutions. After we apply Theorem I'a, inequality (VII-10) becomes

$$(21) \quad N_2 \leq N^{1/\epsilon} + (3M)^{r+s+u}\left(F'^{1/\epsilon^2} + \frac{13(n+2)^n}{\epsilon^{n+2}} \log\left(\frac{2M}{1+\nu+3\epsilon}\right)\right)$$

where N_2 is the total number of solutions. Here N and M are as in Chapter VII and F' is as in Theorem I'a. As in the proof of Theorem IIIa, inequality (21) is of the form

$$N_2 \leq G'^{1/\epsilon^2}$$

where G' is as specified in Theorem III'a.

CHAPTER X

SOME RESTATEMENTS OF THE THEOREMS

We first look at a specialization of Theorem I. Setting $\sigma = 0$ and $\lambda = \mu = 1$ in that theorem, we obtain a generalization of Roth's Theorem.

THEOREM 1: Let $\xi_{i_1}, \ldots, \xi_{i_u}$ be algebraic numbers with $\xi_{i_k} \in K_{i_k}$ for $k = 1, \ldots, u$ where $1 \leq i_1 < i_2 \ldots < i_u \leq R_1 + R_2$, and let $\rho_1, \ldots, \rho_u$, $c_1, \ldots, c_u$ be positive constants. If there are infinitely many distinct nonzero elements α of K satisfying

$$|\alpha^{(i_k)} - \xi_{i_k}| \leq c_k H(\alpha)^{-\rho_k} \text{ for } k = 1, \ldots, u,$$

then

$$\sum_{k=1}^{u} \rho_k g_{i_k} \leq 2.$$

In case $u = 1$, this is a result of Uchiyama [17], which contains LeVeque's Theorem [5, p. 158].

It should be observed that hypotheses (B)-(D) of Theorem I can be replaced by conditions more nearly resembling those of hypothesis (A).

THEOREM 2: Let u; $\xi_{i_1}, \ldots, \xi_{i_u}$; $\eta_1, \ldots, \eta_r$; and $\xi_1, \ldots, \xi_{r+s+t}$ be as in Theorem I. Let $\rho_1, \ldots, \rho_u$; $\sigma_1, \ldots, \sigma_r$; $\lambda_1, \ldots, \lambda_s$; $\mu_1, \ldots, \mu_t$; $c_1, \ldots, c_u$; and $d_1, \ldots, d_{r+s+t}$ be positive constants, where it is possible that $urst = 0$, so that some of the sets may be empty. Assume that there are infinitely many distinct nonzero elements α of K such that

(A) $|\alpha^{(i_k)} - \xi_{i_k}|^* \leq c_k H(\alpha)^{-\rho_k}, \quad k = 1,\ldots,u$

(B_1) $|\alpha - \eta_k|^*_{\mathfrak{P}_k} \leq d_k H(\alpha)^{-\sigma_k}, \quad k = 1,\ldots,r$

(C_1) $|\alpha|^*_{\mathfrak{P}_{r+k}} \leq d_{r+k} H(\alpha)^{-\lambda_k}, \quad k = 1,\ldots,s$

(D_1) $|\frac{1}{\alpha}|^*_{\mathfrak{P}_{r+s+k}} \leq d_{r+s+k} H(\alpha)^{-\mu_k}, \quad k = 1,\ldots,t.$

Then

(1) $\sum_{k=1}^{u} \rho_k g_{i_k} + \sum_{k=1}^{r} \sigma_k g(\mathfrak{P}_k) + \sum_{k=1}^{s} \lambda_k g(\mathfrak{P}_{r+k}) + \sum_{k=1}^{t} \mu_k g(\mathfrak{P}_{r+s+k}) \leq 2.$

PROOF: Assuming that the left-hand side of (1) is $2 + 6\epsilon$ where $0 < \epsilon \leq 1/12$, we proceed as in the proof of Theorem I to obtain a number $\varphi = b_1^{r_1 - t_1} \ldots b_m^{r_m - t_m} N(\Psi)$ which we wish to estimate. Condition (A) yields, of course, the upper bound (II-36) for φ. We use conditions (B_1)-(D_1) to obtain a slightly different lower bound than that obtained in Chapter II. As before, we can suppose that the asterisks in (B_1)-(D_1) are superfluous and that the numbers d_k are at least 1.

Proceeding as in Chapter II (page 25 ff.), let $(\Psi) = \frac{\mathfrak{m}}{\mathfrak{n}}$ where $\mathfrak{m}$ and $\mathfrak{n}$ are relatively prime integral ideals. Then

(2) $$\mathfrak{m} = \mathfrak{P}_1^{v_1} \ldots \mathfrak{P}_{r+s}^{v_{r+s}} \mathfrak{m}'$$

where $v_1,\ldots,v_{r+s}$ are nonnegative integers and $\mathfrak{m}'$ is an ideal divisible by none of the ideals $\mathfrak{P}_1,\ldots,\mathfrak{P}_{r+s}$. Let M be any

constant such that $M \geq 1$ and $|\eta_k|_{\mathfrak{P}_k} \leq M$ for $k = 1,\ldots,r$. As in the application of hypothesis (B) in Chapter II, we obtain the inequalities

$$|\Psi|_{\mathfrak{P}_k} \leq (d_k M)^{mr_1} q_1^{-\sigma_k r_1 S_1} \quad \text{for} \quad k = 1,\ldots,r.$$

By an argument similar to that used to obtain inequality (5) of the introduction, this yields

$$(3) \qquad N(\mathfrak{P}_k)^{v_k} \geq \left\{ \frac{1}{(d_k M)^{mr_1}} q_1^{\sigma_k r_1 S_1} \right\}^{g(\mathfrak{P}_k)} \quad \text{for} \quad k = 1,\ldots,r.$$

Similarly, condition (C_1) yields the inequalities

$$|\Psi|_{\mathfrak{P}_{r+k}} \leq d_{r+k}^{mr_1} q_1^{-\sigma_k r_1 S_1}$$

or

$$(4) \quad N(\mathfrak{P}_{r+k})^{v_{r+k}} \geq \left\{ \frac{1}{d_{r+k}^{mr_1}} q_1^{\lambda_k r_1 S_1} \right\}^{g(\mathfrak{P}_{r+k})} \quad \text{for} \quad k = 1,\ldots,s.$$

Thus (2), (3), and (4) yield

$$(5) \quad N(\mathfrak{m}) \geq C_1^{-mr_1} q_1^{r_1 S_1 \left(\sum_{k=1}^{r} \sigma_k g(\mathfrak{P}_k) + \sum_{k=1}^{s} \lambda_k g(\mathfrak{P}_{r+k}) \right)}$$

where $$C_1 = \prod_{k=1}^{r+s} (d_k M)^{g(\mathfrak{P}_k)} \geq 1$$

and where empty sums are assumed to be zero, so that if $r = s = 0$, inequality (5) becomes the trivial inequality $N(\mathfrak{m}) \geq 1$.

Finally we use condition (D_1) to obtain an upper bound for $N(\mathfrak{n})$. As before, we let $\chi_i = \alpha_i/\beta_i$ where α_i and β_i are algebraic integers in K such that $N(\beta_i) \leq c(K)b_i$ for $i = 1,\ldots,m$. Then

$$(6) \qquad N(\mathfrak{n}) \leq N\Big(\underset{(j)\in I}{\text{l.c.m.}} \left((\beta_1)^{j_1-t_1} \cdots (\beta_m)^{j_m-t_m}\right)\Big)$$

where the set I is defined on page 23. Now let

$$(7) \qquad (\beta_i) = \mathfrak{n}_i \prod_{k=1}^{t} \mathfrak{P}_{r+s+k}^{w_{ki}} \quad \text{for} \quad i = 1,\ldots,m$$

where the w_{ki} are nonnegative integers and where the $\mathfrak{n}_i$ are integral ideals of K. Hypothesis (D_1) implies that we may choose the integers w_{ki} so that

$$(8) \quad \left\{\frac{1}{d_{r+s+k}} q_i^{\mu_k}\right\}^{g(\mathfrak{P}_{r+s+k})} \leq N(\mathfrak{P}_{r+s+k})^{w_{ki}} \leq \left\{\frac{1}{d_{r+s+k}} q_i^{\mu_k}\right\}^{g(\mathfrak{P}_{r+s+k})} N(\mathfrak{P}_{r+s+k})$$

$$\leq N(\mathfrak{P}_{r+s+k})\, q_i^{\mu_k\, g(\mathfrak{P}_{r+s+k})}.$$

Now by (6), (7), and (8),

$$(9) \quad N(\mathfrak{n}) \leq \prod_{i=1}^{m} N(\mathfrak{n}_i)^{r_i-t_i} \times \max_{(j)\in I} \left\{ \prod_{i=1}^{m} \prod_{k=1}^{t} \left(N(\mathfrak{P}_{r+s+k})\, q_i^{\mu_k\, g(\mathfrak{P}_{r+s+k})}\right)^{j_i-t_i} \right\}$$

where, by (7) and (8),

$$(10)\quad N(\eta_i) \le \frac{N(\beta_i)}{\prod_{k=1}^{t} \left\{ \frac{1}{d_{r+s+k}} q_i \right\}^{\mu_k} g(\mathfrak{P}_{r+s+k})} \le \frac{C_2\, c(K) b_i}{q_i^{\sum_{k=1}^{t} \mu_k\, g(\mathfrak{P}_{r+s+k})}},$$

and where

$$C_2 = \prod_{k=1}^{t} d_{r+s+k}^{g(\mathfrak{P}_{r+s+k})}.$$

Here all empty products are assumed to be 1. Thus by (9) and (10),

$$N(\eta) \le \left(C_2\, c(K)\, N(\mathfrak{P}_{r+s+1} \cdots \mathfrak{P}_{r+s+t})\right)^{mr_1} \prod_{i=1}^{m} b_i^{r_i - t_i}$$

$$\times \prod_{i=1}^{m} \left\{ q_i^{-\sum_{k=1}^{t} \mu_k\, g(\mathfrak{P}_{r+s+k})} \right\}^{r_i - t_i} \times \max_{(j)\in I} \left\{ \prod_{i=1}^{m} \left(q_i^{\sum_{k=1}^{t} \mu_k\, g(\mathfrak{P}_{r+s+k})} \right)^{j_i - t_i} \right\};$$

or, letting $C_3 = C_2\, c(K)\, N(\mathfrak{P}_{r+s+1} \cdots \mathfrak{P}_{r+s+t})$ and applying (II-17) and the definition of the set I, page 23,

$$(11)\quad N(\eta) \le C_3^{mr_1}\, b_1^{r_1 - t_1} \cdots b_m^{r_m - t_m}\, q_1^{(-r_1 S_3 + r_1(1+\epsilon) S_2) \sum_{k=1}^{t} \mu_k\, g(\mathfrak{P}_{r+s+k})}.$$

Now the bounds (5) and (11) yield a new lower bound for φ which is, in form, exactly the old lower bound with σ replaced by $\sum_{k=1}^{r} \sigma_k\, g(\mathfrak{P}_k)$, $1 - \lambda$ replaced by $\sum_{k=1}^{s} \lambda_k\, g(\mathfrak{P}_{r+k})$, and $1 - \mu$ replaced by $\sum_{k=1}^{t} \mu_k\, g(\mathfrak{P}_{r+s+k})$. The desired contradiction is obtained by the method used in Chapter II, where we must replace the inequality $0 \le 1 - \mu \le 1$ by

$$(12) \qquad 0 < \sum_{k=1}^{t} \mu_k \, g(\mathfrak{P}_{r+s+k}) \leq 2 + 6\epsilon.$$

Inequality (II-55) becomes

$$(2+6\epsilon)S_1 + \Big(\sum_{k=1}^{t} \mu_k \, g(\mathfrak{P}_{r+s+k})\Big)\big(-S_1-(1+\epsilon)S_2+S_3\big) - (1+\epsilon)m < m\epsilon/12,$$

which leads by direct computation to

$$\frac{7}{4} - \frac{\epsilon}{2} - \Big(\frac{1}{2} + \frac{\epsilon}{12}\Big) \sum_{k=1}^{t} \mu_k \, g(\mathfrak{P}_{r+s+k}) + \frac{L}{\epsilon m}\Big((1+\epsilon) \sum_{k=1}^{t} \mu_k \, g(\mathfrak{P}_{r+s+k}) - 2 - 6\epsilon\Big) < 0.$$

Because $0 < \epsilon \leq 1/12$ and $0 < L < \epsilon m/6$, this yields

$$\sum_{k=1}^{t} \mu_k \, g(\mathfrak{P}_{r+s+k}) > 186/49,$$

which contradicts (12) and thus completes the proof.

Following Gelfond [3, pp. 26-27], one can reinterpret several of the theorems in terms of $\mathfrak{P}$-adic congruences. Let m be a positive integer and let $\mathfrak{P}$ be a prime ideal of K. If $\alpha \in K$, then $\alpha \equiv 0 \ (\mathfrak{P}^m)$ means that

$$(\alpha) = \frac{\mathfrak{P}^m \mathfrak{m}}{\mathfrak{n}}$$

where $\mathfrak{m}$ and $\mathfrak{n}$ are integral ideals of K and $\mathfrak{P}$ and $\mathfrak{n}$ are relatively prime. It is clear that

$$(13) \qquad \alpha \equiv 0 \ (\mathfrak{P}^m) \quad \text{if and only if} \quad |\alpha|_{\mathfrak{P}} \leq N(\mathfrak{P})^{-m/g(\mathfrak{P})}.$$

In case $\alpha \in K_{\mathfrak{P}}$, we can use (13) to define $\alpha \equiv 0 \ (\mathfrak{P}^m)$.

LEMMA 1: *Let* $\mathfrak{P}_1,\dots,\mathfrak{P}_R$ *be distinct prime ideals of* K, *let* $\alpha \in K_{\mathfrak{P}_i}$ *for* $i = 1,\dots,R$, *let* $\tau_1,\dots,\tau_R$ *be positive constants, and let*

$$m_i = \left[\frac{\tau_i\, g(\mathfrak{P}_i)\log q}{\log N(\mathfrak{P}_i)}\right] \quad \text{for} \quad i = 1,\dots,R,$$

where q *is a positive integer which is sufficiently large to guarantee that* m_i *is positive for* $i = 1,\dots,R$. *Suppose that*

$$\alpha \equiv 0 \quad \left(\prod_{i=1}^{R} \mathfrak{P}_i^{m_i}\right). \tag{14}$$

Then

$$|\alpha|_{\mathfrak{P}_i} \le N(\mathfrak{P}_i)^{1/g(\mathfrak{P}_i)}\, q^{-\tau_i} \quad \text{for} \quad i = 1,\dots,R. \tag{15}$$

PROOF: By (13) the congruence (14) implies that

$$|\alpha|_{\mathfrak{P}_i} \le N(\mathfrak{P}_i)^{m_i/g(\mathfrak{P}_i)} \quad \text{for} \quad i = 1,\dots,R,$$

or

$$\begin{aligned} |\alpha|_{\mathfrak{P}_i} &\le q^{-\left\{\frac{\log N(\mathfrak{P}_i)}{g(\mathfrak{P}_i)\log q}\right\}\left[\frac{\tau_i\, g(\mathfrak{P}_i)\log q}{\log N(\mathfrak{P}_i)}\right]} \\ &\le q^{-\tau_i + \frac{\log N(\mathfrak{P}_i)}{g(\mathfrak{P}_i)\log q}} \\ &\le N(\mathfrak{P}_i)^{1/g(\mathfrak{P}_i)}\, q^{-\tau_i} \quad \text{for} \quad i = 1,\dots,R, \end{aligned}$$

which is just inequality (15).

In Lemma 2, we obtain a partial converse to Lemma 1.

LEMMA 2: Let $\alpha; \mathfrak{P}_1,\ldots,\mathfrak{P}_R; \tau_1,\ldots,\tau_R; m_1,\ldots,m_R;$ and q be as in Lemma 1. Suppose that

$$|\alpha|_{\mathfrak{P}_i} \leq q^{-\tau_i} \quad \text{for} \quad i = 1,\ldots,R. \tag{16}$$

Then

$$\alpha \equiv 0 \quad (\prod_{i=1}^{R} \mathfrak{P}_i^{m_i}). \tag{17}$$

PROOF: By (16)

$$|\alpha|_{\mathfrak{P}_i} \leq N(\mathfrak{P}_i)^{-\frac{\tau_i \log q}{\log N(\mathfrak{P}_i)}}$$

$$\leq N(\mathfrak{P}_i)^{-m_i/g(\mathfrak{P}_i)} \quad \text{for} \quad i = 1,\ldots,R.$$

By (13) this yields the congruence (17) as desired.

Lemmas 1 and 2 together with Theorem 2 yield

THEOREM 3: If, in the notation of Theorem 2, we let

$$m_i(\varkappa) = \left[\frac{\sigma_i\, g(\mathfrak{P}_i)\log H(\varkappa)}{\log N(\mathfrak{P}_i)} \right] \quad \text{for} \quad i = 1,\ldots,r,$$

$$m_{r+i}(\varkappa) = \left[\frac{\lambda_i\, g(\mathfrak{P}_{r+i})\log H(\varkappa)}{\log N(\mathfrak{P}_{r+i})} \right] \quad \text{for} \quad i = 1,\ldots,s,$$

and

$$m_{r+s+i}(\varkappa) = \left[\frac{\mu_i\, g(\mathfrak{P}_{r+s+i})\log H(\varkappa)}{\log N(\mathfrak{P}_{r+s+i})} \right] \quad \text{for} \quad i = 1,\ldots,t,$$

and if we suppose that there are infinitely many distinct nonzero elements α of K such that

$$(A) \quad |\alpha^{(i_k)} - \xi_{i_k}|^* \leq c_k H(\alpha)^{-\rho_k}, \qquad k = 1,\ldots,u$$

$$(B_2) \quad \alpha - \eta_k \equiv 0 \quad (\mathfrak{P}_k^{m_k(\alpha)}), \qquad k = 1,\ldots,r$$

$$(C_2) \quad \alpha \equiv 0 \quad (\mathfrak{P}_{r+k}^{m_{r+k}(\alpha)}), \qquad k = 1,\ldots,s$$

$$(D_2) \quad \frac{1}{\alpha} \equiv 0 \quad (\mathfrak{P}_{r+s+k}^{m_{r+s+k}(\alpha)}), \qquad k = 1,\ldots,t,$$

then conditions (A), (B_1), (C_1), and (D_1) of Theorem 2 are satisfied for infinitely many α where $d_i = N(\mathfrak{P}_i)^{1/g(\mathfrak{P}_i)}$ for $i = 1,\ldots,r+s+t$, and so the conclusion (1) of Theorem 2 follows.

Conversely, if conditions (A), (B_1), (C_1), and (D_1) of Theorem 2 are satisfied for infinitely many α with $d_i = 1$ for $i = 1,\ldots,r+s+t$, then conditions (A), (B_2), (C_2), and (D_2) are also satisfied for infinitely many α.

Finally, following Mahler [8, pp. 147-150] and Lang [4, p. 95], we use Theorem III to obtain generalizations of two theorems of Ridout.

THEOREM 4: Let $\xi_{i_1},\ldots,\xi_{i_u}$ be algebraic elements of $K_{i_1},\ldots,K_{i_u}$ respectively, and let $\eta_1,\ldots,\eta_r$ be $\mathfrak{P}_1$-,$\ldots$,$\mathfrak{P}_r$-adic algebraic numbers respectively. Let τ and c be positive constants. Assume that there exist infinitely many distinct nonzero elements α of K satisfying

$$\prod_{k=1}^{u} |\varkappa^{(i_k)} - \xi_{i_k}|_{\mathfrak{P}}^{*g_{i_k}} \prod_{j=1}^{r} |1 - \frac{\eta_j}{\varkappa}|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \leq cH(\varkappa)^{-\tau}.$$

Then $\tau \leq 2$.

PROOF: Suppose that the inequality of the theorem holds for an infinite sequence $\Sigma = \{\varkappa_i\}$. We need only show that there exists a constant $L > 0$ such that

$$|\varkappa_i - \eta_j|_{\mathfrak{P}_j}^{*} \leq L |1 - \frac{\eta_j}{\varkappa_i}|_{\mathfrak{P}_j}^{*}$$

for all $\varkappa_i \in \Sigma$, and for $1 \leq j \leq r$, in order to apply Theorem III and obtain the desired result. Since

$$|\varkappa_i - \eta_j|_{\mathfrak{P}_j}^{*} = \begin{cases} 1 = \dfrac{|1 - \frac{\eta_j}{\varkappa_i}|_{\mathfrak{P}_j}^{*}}{|1 - \frac{\eta_j}{\varkappa_i}|_{\mathfrak{P}_j}^{*}} & \text{if } |\varkappa_i - \eta_j|_{\mathfrak{P}_j} \geq 1 \\ |\varkappa_i - \eta_j|_{\mathfrak{P}_j} = |1 - \frac{\eta_j}{\varkappa_i}|_{\mathfrak{P}_j} |\varkappa_i|_{\mathfrak{P}_j} & \text{otherwise,} \end{cases}$$

if suffices to show that for each j with $1 \leq j \leq r$, Σ contains no infinite subsequence Σ', henceforth identified with Σ, for which either

(a) $|\varkappa_i - \eta_j|_{\mathfrak{P}_j} \geq 1$ for all $\varkappa_i \in \Sigma$ and $\lim_{i \to \infty} |1 - \frac{\eta_j}{\varkappa_i}|_{\mathfrak{P}_j} = 0$

or

(b) $|x_i - \eta_j|_{\mathfrak{P}_j} < 1$ for all $x_i \in \Sigma$ and $\lim_{i \to \infty} |x_i|_{\mathfrak{P}_j} = \infty$.

Either alternative gives rise to an immediate contradiction. Alternative (a) implies that $\lim_{i \to \infty} |x_i|_{\mathfrak{P}_j} = \infty$ so that $\lim_{i \to \infty} |\eta_j / x_i|_{\mathfrak{P}_j} = 0$, and hence $\lim_{i \to \infty} |1 - \eta_j / x_i|_{\mathfrak{P}_j} = 1$, contrary to hypothesis; alternative (b) implies that $\lim_{i \to \infty} |x_i - \eta_j|_{\mathfrak{P}_j} = \infty$, again contrary to hypothesis.

THEOREM 5: Let $\xi_{i_1}, \ldots, \xi_{i_u}$ be nonzero algebraic numbers in $K_{i_1}, \ldots, K_{i_u}$ respectively, and let $\eta_1, \ldots, \eta_r$ be $\mathfrak{P}_1$-, $\ldots$, $\mathfrak{P}_r$-adic algebraic numbers respectively. Let τ, λ, μ be nonnegative constants, and let c_1, c_2, c_3 be positive constants. Assume that there exist infinitely many distinct nonzero elements x of K satisfying

$$(18) \qquad \prod_{k=1}^{u} |x^{(i_k)} - \xi_{i_k}|^{*g_{i_k}} \prod_{j=1}^{r} |x - \eta_j|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \leq c_1 H(x)^{-\tau}$$

where each principal ideal (x) can be written in the form

$$(19) \qquad (x) = \alpha \mathfrak{P}_{r+1}^{e_{r+1}} \cdots \mathfrak{P}_{r+s}^{e_{r+s}} / (\beta \mathfrak{P}_{r+s+1}^{e_{r+s+1}} \cdots \mathfrak{P}_{r+s+t}^{e_{r+s+t}}),$$

where $\mathfrak{P}_{r+1}, \ldots, \mathfrak{P}_{r+s+t}$ are distinct fixed prime ideals of K independent of x and where $e_{r+1}, \ldots, e_{r+s+t} \in \mathbb{Z}$ are nonnegative and depend only on x. Further, suppose that α and β are integral ideals of K which depend only on x and which are relatively prime to one another and to $\mathfrak{P}_{r+1}, \ldots, \mathfrak{P}_{r+s+t}$. Finally,

suppose that

$$0 < N(\alpha) \le c_2 H(\varkappa)^{\lambda}$$
$$0 < N(\beta) \le c_3 H(\varkappa)^{\mu} . \tag{20}$$

Then $\tau \le \lambda + \mu$.

The proof makes use of the following variation of Theorem III.

THEOREM 6: If, in the notation of Theorem III, $\zeta_{i_1} \cdots \zeta_{i_u} \neq 0$ and the inequality

$$\prod_{k=1}^{u} |\varkappa^{(i_k)} - \zeta_{i_k}|^{*g_{i_k}} \prod_{k=1}^{R_1+R_2} |\varkappa^{(k)}|^{*g_k} \prod_{k=1}^{R_1+R_2} \left|\frac{1}{\varkappa^{(k)}}\right|^{*g_k} \prod_{j=1}^{r} |\varkappa - \eta_j|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}$$
$$\times \prod_{j=r+1}^{r+s} |\varkappa|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+s+1}^{r+s+t} \left|\frac{1}{\varkappa}\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \le CH(\varkappa)^{-\tau} \tag{21}$$

has infinitely many solutions $\varkappa$ in distinct nonzero elements of K, then $\tau \le 2$.

PROOF OF THEOREM 6: We can suppose, without loss of generality, that an infinite subsequence of the numbers $\varkappa$ has been chosen which converges in each metric $|\ldots|_k$ for $k = 1,\ldots,R_1 + R_2$ and $|\ldots|_{\mathfrak{P}_j}$ for $j = 1,\ldots,r+s+t$. Also suppose that $\varkappa^{(i_k)} \neq \zeta_{i_k}$ for $k = 1,\ldots,u$. If $\{\varkappa^{(i_k)}\}$ converges to ζ_{i_k}, let $\zeta'_{i_k} = \zeta_{i_k}$. Otherwise, let $\zeta'_k = 0$. Then there is a constant $c_4 > 0$ such that, for all remaining $\varkappa$,

$$\prod_{k=1}^{R_1+R_2} |\varkappa^{(k)}|^{*g_k} \prod_{k=1}^{u} |\varkappa^{(i_k)} - \zeta_{i_k}|^{*g_{i_k}} \ge c_4 \prod_{k=1}^{R_1+R_2} |\varkappa^{(k)} - \zeta'_k|^{*g_k} . \tag{22}$$

By virtue of (22), inequality (21) yields

$$(23)\quad \prod_{k=1}^{R_1+R_2} |\alpha^{(k)} - \xi_k'|^{*g_k} \prod_{k=1}^{R_1+R_2} \left|\frac{1}{\alpha^{(k)}}\right|^{*g_k} \prod_{j=1}^{r} |\alpha - \eta_j|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+1}^{r+s} |\alpha|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}$$

$$\times \prod_{j=r+s+1}^{r+s+t} \left|\frac{1}{\alpha}\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \le \frac{c}{c_4} H(\alpha)^{-\tau}.$$

In case no conjugate sequence of α's converges to infinity, the factors $|1/\alpha^{(k)}|^{*g_k}$ are bounded away from zero for $k = 1,\ldots,R_1+R_2$, and an application of Theorem III to inequality (23) yields the desired result.

If, however, some conjugate sequence is unbounded, let

$$\alpha' = \frac{a\alpha+b}{c\alpha+d} \qquad \text{or} \qquad \alpha = \frac{b-d\alpha'}{c\alpha'-a}$$

where $a, b, c, d \in \mathbb{Z}$ are such that $ad \neq bc$, $acd \neq 0$, $c\xi_k' + d \neq 0$ for $k = 1,\ldots,R_1+R_2$, $c\eta_j + d \neq 0$ for $j = 1,\ldots,r$, and such that no conjugate sequence converges to $-d/c$. Further suppose that the sequence of α's does not converge to $|d/c|_{\mathfrak{P}_j}$ in the $\mathfrak{P}_j$-adic metric for $j = 1,\ldots,r+s+t$. Thus the numbers α' and all their conjugates are bounded in absolute value, and the α' are bounded in $\mathfrak{P}_j$-adic value. Moreover inequality (23) can be written in the form

$$(24)\quad \prod_{k=1}^{R_1+R_2} \left\{ \left|\frac{c\xi_k'+d}{c\alpha'^{(k)}-a}\left(\alpha'^{(k)} - \frac{a\xi_k'+b}{c\xi_k'+d}\right)\right|^{*} \left|\frac{c}{d\alpha'^{(k)}-b}\left(\alpha'^{(k)} - \frac{a}{c}\right)\right|^{*} \right\}^{g_k}$$

$$\times \prod_{j=1}^{r} \left|\frac{c\eta_j+d}{c\alpha'-a}\left(\alpha' - \frac{a\eta_j+b}{c\eta_j+d}\right)\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+1}^{r+s} \left|\frac{d}{c\alpha'-a}\left(\alpha' - \frac{b}{d}\right)\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}$$

$$\times \prod_{j=r+s+1}^{r+s+t} \left|\frac{c}{d\alpha'-b}\left(\alpha' - \frac{a}{c}\right)\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \le \frac{c}{c_4} H(\alpha)^{-\tau}.$$

Note that for all positive real numbers A and B, $(AB)^* \geq A^* B^*$, and also note that there exists a constant $c_5 > 0$ depending only on K and the integers a, b, c, d, such that $H(\varkappa') \leq c_5 H(\varkappa)$ for all $\varkappa \in K$. Therefore (24) yields an inequality of the type required in Theorem III, with two kinds of modification. First, $\varkappa$ has been replaced by $\varkappa'$, r has been replaced by r+s+t, and s and t by 0, and the product $\prod_{k=1}^{u} |\varkappa^{(i_k)} - \xi_{i_k}|^{*g_{i_k}}$ has been replaced by

$$\prod_{k=1}^{R_1+R_2} \left\{ \left| \varkappa'^{(k)} - \frac{a\xi_k'+b}{c\xi_k'+d} \right|^* \left| \varkappa'^{(k)} - \frac{a}{c} \right|^* \right\}^{g_k}.$$

The last replacement may be shown to yield a product of the form necessary for an application of Theorem III by the technique used to obtain inequality (22), because the sequences $\{\varkappa'^{(k)}\}$ converge for $k = 1,\ldots,R_1+R_2$, and because $\frac{a\xi_k'+b}{c\xi_k'+d} \neq \frac{a}{c}$ also for $k = 1,\ldots,R_1+R_2$. There is no loss of generality in supposing that $\varkappa'^{(k)} \neq \frac{a\xi_k'+b}{c\xi_k'+d}$ and $\varkappa'^{(k)} \neq \frac{a}{c}$ for all k. Second, the left hand side of (24) has the additional factors

$$\prod_{k=1}^{R_1+R_2} \left\{ \left| \frac{c\xi_k'+d}{c\varkappa'^{(k)}-a} \right|^* \left| \frac{c}{d\varkappa'^{(k)}-b} \right|^* \right\}^{g_k} \prod_{j=1}^{r} \left| \frac{c\eta_j+d}{c\varkappa'-a} \right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}$$

$$\times \prod_{j=r+1}^{r+s} \left| \frac{d}{c\varkappa'-a} \right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+s+1}^{r+s+t} \left| \frac{c}{d\varkappa'-b} \right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}.$$

But because $|\chi'^{(k)}|$ is bounded for $k = 1,\dots,R_1+R_2$ and $|\chi'|_{\mathfrak{P}_j}$ is bounded for $j = 1,\dots,r+s+t$, these factors are all bounded away from zero. They are also, of course, all at most one. Thus Theorem III can be applied, yielding $\tau \le 2$, which is the desired result.

PROOF OF THEOREM 5: By (18), the inequality

$$(25)\quad \prod_{k=1}^{R_1+R_2} |\chi^{(k)}|^{*g_k} \prod_{k=1}^{R_1+R_2} \left|\frac{1}{\chi^{(k)}}\right|^{*g_k} \prod_{k=1}^{u} |\chi^{(i_k)} - \xi_{i_k}|^{*g_{i_k}}$$
$$\times \prod_{j=1}^{r} |\chi-\eta_j|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+1}^{r+s} |\chi|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+s+1}^{r+s+t} \left|\frac{1}{\chi}\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}$$
$$\le c_1 H(\chi)^{-\tau} \prod_{k=1}^{R_1+R_2} \left\{ |\chi^{(k)}|^{*g_k} \left|\frac{1}{\chi^{(k)}}\right|^{*g_k} \right\} \prod_{j=r+1}^{r+s} |\chi|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \prod_{j=r+s+1}^{r+s+t} \left|\frac{1}{\chi}\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}$$

has infinitely many distinct nonzero solutions. Denote the left-hand side of (25) by $\Phi(\chi)$. Applying (19) and (20) to the right-hand side of (25), we obtain

$$(26)\quad \Phi(\chi) \le c_1 H(\chi)^{-\tau+\lambda+\mu} \prod_{k=1}^{R_1+R_2} \left\{ |\chi^{(k)}|^{*} \left|\frac{1}{\chi^{(k)}}\right|^{*} \right\}^{g_k/N(\alpha\beta\mathfrak{P}_{r+1}^{e_{r+1}}\cdots\mathfrak{P}_{r+s+t}^{e_{r+s+t}})}.$$

We shall show that there is a constant $c_5 > 0$ such that

$$(27)\quad \prod_{k=1}^{R_1+R_2} \left\{ |\chi^{(k)}|^{*} \left|\frac{1}{\chi^{(k)}}\right|^{*} \right\}^{g_k/N(\alpha\beta\mathfrak{P}_{r+1}^{e_{r+1}}\cdots\mathfrak{P}_{r+s+t}^{e_{r+s+t}})} \le c_5 H(\chi)^{-2}$$

for all χ's which are primitive elements of K. Thus (26) and (27) yield the inequality of Theorem 6 with the exponent on $H(\chi)$ replaced

by $\tau - \lambda - \mu + 2$, and this inequality must be satisfied by infinitely many $\varkappa$. Therefore Theorem 6 implies that $\tau - \lambda - \mu + 2 \leq 2$, or that $\tau \leq \lambda + \mu$ as desired.

To obtain (27), we let $\varkappa = \alpha/\beta$ as in Appendix A. Thus

$$N(\alpha\beta) \leq c(K)^2 N(\alpha\beta \mathfrak{P}_{r+1}^{e_{r+1}} \cdots \mathfrak{P}_{r+s+t}^{e_{r+s+t}}) \tag{28}$$

where $c(K)$ is the constant occuring in Appendix A. Since the defining polynomial of $\varkappa$ has degree n and divides the polynomial

$$f(x) = \prod_{i=1}^{n} (\beta^{(i)} x - \alpha^{(i)}),$$

$H(\varkappa)$ must divide some coefficient in $f(x)$. Each such coefficient is the sum of at most 2^n terms, each of the form $\delta_1^{(1)} \cdots \delta_n^{(n)}$ where δ_i is either α or β for $i = 1,\ldots,n$. Hence

$$\begin{aligned} H(\varkappa) &\leq 2^n \prod_{k=1}^{R_1+R_2} \max(|\alpha^{(k)}|, |\beta^{(k)}|)^{g_k} \\ &= 2^n N(\alpha) \prod_{k=1}^{R_1+R_2} \max(1, |1/\varkappa^{(k)}|)^{g_k} \\ &= 2^n N(\beta) \prod_{k=1}^{R_1+R_2} \max(1, |\varkappa^{(k)}|)^{g_k} . \end{aligned} \tag{29}$$

Therefore

$$H(\varkappa)^2 \le 4^n N(\alpha\beta) \prod_{k=1}^{R_1+R_2} \left\{ \max(1, |1/\varkappa^{(k)}|) \max(1, |\varkappa^{(k)}|) \right\}^{g_k}$$

$$= 4^n N(\alpha\beta) \Big/ \prod_{k=1}^{R_1+R_2} \left\{ |\varkappa^{(k)}|^* |1/\varkappa^{(k)}|^* \right\}^{g_k},$$

or, using (28),

$$H(\varkappa)^2 \le \frac{4^n c(K)^2 N(\alpha\beta \mathfrak{p}_{r+1}^{e_{r+1}} \cdots \mathfrak{p}_{r+s+t}^{e_{r+s+t}})}{\prod_{k=1}^{R_1+R_2} \left\{ |\varkappa^{(k)}|^* |1/\varkappa^{(k)}|^* \right\}^{g_k}},$$

which is just the desired inequality (27).

CHAPTER XI

SOME APPLICATIONS INVOLVING BINARY FORMS

If α and β are any two elements of K, define (α,β) to be the greatest common ideal divisor, $((\alpha),(\beta))$, of α and β. Let $\mathbb{C}$ denote the field of complex numbers.

THEOREM 7: Let $F_1(x,y),\ldots,F_{R_1+R_2}(x,y)$, $G_1(x,y),\ldots,G_r(x,y)$ be arbitrary equal or distinct binary forms over K, each of total degree at least 1. Suppose further that they have no multiple factors over $\mathbb{C}$ and that $F_1(x,y),\ldots,F_{R_1+R_2}(x,y)$ all have the same degree, say f. Let B and c be positive constants. Let $\{\alpha_k,\beta_k\}$ be an infinite sequence of pairs of nonzero integers from K with no two of the pairs associated (i.e., no one a unit multiple of another), and with $N(\alpha_k,\beta_k) \le B$ for $k = 1,2,\ldots$. If there is a constant, ω, such that

$$(1)\qquad \prod_{i=1}^{R_1+R_2} |F_i(\alpha_k,\beta_k)|_i^{g_i} \prod_{j=1}^{r} |G_j(\alpha_k,\beta_k)|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \le cH(\alpha_k/\beta_k)^{\omega}$$

for $k = 1,2,\ldots$, then

$$\omega \ge f - 2.$$

The proof of the theorem follows Parry [10, pp. 59-67] and depends on several lemmas.

LEMMA 1: In the hypotheses of Theorem 7, there is no loss of generality in assuming that the binary forms do not vanish identically for $x = 0$ or $y = 0$.

PROOF: Choose integers a and b such that none of the forms vanishes when $x = 1$, $y = a$ or when $x = b$, $y = 1 + ab$ and such that $ab(1 + ab) \neq 0$. Clearly a and b depend only on the hypothesized forms. Then the transformation

$$x = x' + by' \qquad \text{or} \qquad x' = (1 + ab)x - by$$
$$y = ax' + (1 + ab)y' \qquad\qquad y' = y - ax$$

yields new forms in x' and y' which do not vanish identically for $x' = 0$ or $y' = 0$. Let

$$\alpha_k' = (1 + ab)\alpha_k - b\beta_k$$

$$\beta_k' = \beta_k - a\alpha_k.$$

We must verify that there is an infinite subsequence of $\{ \alpha_k', \beta_k' \}$ which satisfies the hypotheses of Theorem 7.

First note that $\alpha_k' = 0$ only in case $\alpha_k/\beta_k = b/(1 + ab)$, and that $\beta_k' = 0$ only in case $\alpha_k/\beta_k = 1/a$. But the pairs α_k, β_k are nonassociated, by hypothesis, so that $\alpha_k/\beta_k = \alpha_j/\beta_j$ for $k \neq j$ implies that either $\alpha_k = \gamma\alpha_j$ and $\beta_k = \gamma\beta_j$, or $\alpha_j = \gamma\alpha_k$ and $\beta_j = \gamma\beta_k$, for some nonzero integer γ of K. Now $N(\alpha_k,\beta_k) \leq B$ and $N(\alpha_j,\beta_j) \leq B$, so that $N(\gamma) \leq B$, and there are only finitely many such nonassociated integers γ of K. Hence $\alpha_k'\beta_k' \neq 0$ for infinitely many values of k, and we assume, without loss of generality, that $\alpha_k'\beta_k' \neq 0$ for all k.

In case the pairs α_k',β_k' and α_j',β_j' are associated, we have

$\alpha_k'/\beta_k' = \alpha_j'/\beta_j'$, or

$$\frac{(1 + ab)\alpha_k - b\beta_k}{\beta_k - a\alpha_k} = \frac{(1 + ab)\alpha_j - b\beta_j}{\beta_j - a\alpha_j},$$

which implies that $\alpha_k/\beta_k = \alpha_j/\beta_j$. But, as before, for fixed k, this can be true for at most finitely many values of j. Hence we assume, again without loss of generality, that no two of the pairs α_k', β_k' are associated.

We next verify that $N(\alpha_k',\beta_k') \leq B$ for all remaining pairs α_k', β_k':

$$\begin{aligned} N(\alpha_k',\beta_k') &= N\{(1 + ab)\alpha_k - b\beta_k,\ \beta_k - a\alpha_k\} \\ &= N\{\alpha_k + b(a\alpha_k - \beta_k),\ a\alpha_k - \beta_k\} \\ &= N(\alpha_k,\ a\alpha_k - \beta_k) \\ &= N(\alpha_k,\ \beta_k) \\ &\leq B. \end{aligned}$$

Finally, observe that

$$H\left(\frac{\alpha_k}{\beta_k}\right) = H\left(\frac{\alpha_k' + b\beta_k'}{a\alpha_k' + (1 + ab)\beta_k'}\right) \leq DH\left(\frac{\alpha_k'}{\beta_k'}\right)$$

where $D > 0$ is a constant depending only on a, b, and n. Thus in case $\omega \geq 0$, inequality (1) remains true for all remaining pairs

α_k', β_k' if c is replaced by cD^{ω}. In case $\omega < 0$, there is a constant $E > 0$ depending only on a, b, and n such that

$$E\, H\left(\frac{\alpha_k'}{\beta_k'}\right) \leq H\left(\frac{\alpha_k}{\beta_k}\right),$$

so again the right-hand side of (1) changes only in the constant factor when we write it in terms of α_k' and β_k'. This completes the proof of Lemma 1.

LEMMA 2: Let $F(x,y)$ be a binary form of total degree f over K. Suppose further that $F(x,y)$ has no repeated factors over $\mathbb{C}$, that $F(x,y)$ does not vanish identically for $x = 0$ or $y = 0$, and that

$$F(x,1) = \gamma \prod_{s=1}^{f} (x - \zeta_s)$$

where $\gamma, \zeta_1, \ldots, \zeta_s$ are in $\mathbb{C}$. Then for each i, $1 \leq i \leq R_1+R_2$, there is a positive constant C_i such that

$$(2) \quad |F(\alpha,\beta)|_i \geq C_i \max(|\alpha|_i, |\beta|_i)^f \min_{1\leq s\leq f} \left(\left|\left(\frac{\alpha}{\beta}\right)^{(i)} - \zeta_s\right|^*\right)$$

for all nonzero integers α,β in K.

PROOF: Let $D = \min_{j\neq k}(|\zeta_j - \zeta_k|)$. Because $F(x,y)$ has no repeated factors, $D > 0$. Observe that if $|(\alpha/\beta)^{(i)} - \zeta_k| < D/2$ for some k, then $|(\alpha/\beta)^{(i)} - \zeta_j| \geq D/2$ for $j \neq k$. Hence

$$|F(\alpha,\beta)|_i = |\beta|_i^f \, |\gamma| \prod_{s=1}^{f} |(\tfrac{\alpha}{\beta})^{(i)} - \zeta_s|$$

$$\geq |\beta|_i^f \, |\gamma| (D/2)^{f-1} \min_{1\leq s\leq f} (|(\tfrac{\alpha}{\beta})^{(i)} - \zeta_s|).$$

In case $|\beta|_i \geq |\alpha|_i$, this yields a result of the form (2).

If $|\alpha|_i > |\beta|_i$, observe that $\zeta_1 \cdots \zeta_f \neq 0$ and let $E = \min_{j\neq k}(|\frac{1}{\zeta_j} - \frac{1}{\zeta_k}|)$. As before, $E > 0$. Then

$$(3) \qquad |F(\alpha,\beta)|_i = |\alpha|_i^f \, |\gamma| \prod_{s=1}^{f} |\zeta_s| \prod_{s=1}^{f} |\frac{1}{\zeta_s} - (\tfrac{\beta}{\alpha})^{(i)}|$$

$$\geq |\alpha|_i^f \, |\gamma| \Big(\prod_{s=1}^{f} |\zeta_s|\Big) (E/2)^{f-1} \min_{1\leq s\leq f} \Big(|\frac{1}{\zeta_s} - (\tfrac{\beta}{\alpha})^{(i)}|\Big)$$

$$\geq |\alpha|_i^{f-1} \, |\beta|_i \, E' \min_{1\leq s\leq f} (|(\tfrac{\alpha}{\beta})^{(i)} - \zeta_s|)$$

where

$$E' = |\gamma| \Big(\prod_{s=1}^{f} |\zeta_s|\Big) (E/2)^{f-1} \min_{1\leq s\leq f} (|\frac{1}{\zeta_s}|).$$

To see that (3) yields a result of the form (2), we consider two cases. First suppose that

$$(\max_s(|\zeta_s|) + 1)|\beta|_i \geq |\alpha|_i.$$

In this case the result is clear. If, however,

$$(\max_s(|\zeta_s|) + 1)|\beta|_i < |\alpha|_i,$$

then

$$\min_s\left(\left|1 - (\beta/\alpha)^{(i)} \zeta_s\right|\right) \geq 1 - |\beta/\alpha|_i \max_s(|\zeta_s|)$$

$$\geq 1 - \frac{\max_s(|\zeta_s|)}{\max_s(|\zeta_s|) + 1}$$

$$= \frac{1}{\max_s(|\zeta_s|) + 1}$$

and

$$|(\alpha/\beta)^{(i)} - \zeta_s| \geq |\alpha/\beta|_i - |\zeta_s| \geq 1 \quad \text{for} \quad s = 1,\ldots,f.$$

Hence

$$\min_s\left(\left|(\alpha/\beta)^{(i)} - \zeta_s\right|\right) \geq \left|\frac{\alpha}{\beta}\right|_i \min_s\left(\left|1 - (\beta/\alpha)^{(i)}\zeta_s\right|\right)$$

$$\geq \left|\frac{\alpha}{\beta}\right|_i \frac{1}{\max_s(|\zeta_s|) + 1}$$

$$= \left|\frac{\alpha}{\beta}\right|_i \frac{1}{\max_s(|\zeta_s|) + 1} \min_s\left(\left|\left(\frac{\alpha}{\beta}\right)^{(i)} - \zeta_s\right|^*\right),$$

which, together with (3), also yields a result of the form (2).

LEMMA 3: Let $\mathfrak{p}$ be a prime ideal of K and let P(x) be a polynomial with $\mathfrak{p}$-adic coefficients, nonzero discriminant, and no $\mathfrak{p}$-adic zeros. Then there is a positive constant $M_{\mathfrak{p}}$ depending only on P(x), such that $|P(\varkappa)|_{\mathfrak{p}} \geq M_{\mathfrak{p}}$ for all $\varkappa \in K_{\mathfrak{p}}$.

For a statement of Lemma 3 and its proof, see [10, pp. 62-65].

LEMMA 4: Let $\mathfrak{p}$ be a prime ideal of K, let B be a positive constant, and let $G(x,y)$ be a binary form of degree $d \geq 1$ over $K_{\mathfrak{p}}$. Suppose further that $G(x,y)$ has a nonzero discriminant, that it does not vanish identically for $y = 0$, and that

$$G(x,y) = G_1(x,y) \prod_{t=1}^{h} (x - \eta_t y)$$

where $\eta_t \in K_{\mathfrak{p}}$ for $t = 1,\ldots,h$, and where $G_1(x,1)$ has no $\mathfrak{p}$-adic zeros. Then there is a positive constant $L_{\mathfrak{p}}$ such that

$$(4) \qquad |G(\alpha,\beta)|_{\mathfrak{p}} \geq L_{\mathfrak{p}} \min_{1\leq t\leq h} \left(\left|\frac{\alpha}{\beta} - \eta_t\right|_{\mathfrak{p}}^{*}\right)$$

for all nonzero integers α, β in K with $N(\alpha,\beta) \leq B$.

PROOF: By Lemma 3 applied to $G_1(x,1)$, we have

$$(5) \quad |G(\alpha,\beta)|_{\mathfrak{p}} = |\beta|_{\mathfrak{p}}^{d} |G_1(\tfrac{\alpha}{\beta},1)|_{\mathfrak{p}} \prod_{t=1}^{h} \left|\frac{\alpha}{\beta} - \eta_t\right|_{\mathfrak{p}} \geq M_{\mathfrak{p}} |\beta|_{\mathfrak{p}}^{d} \prod_{t=1}^{h} \left|\frac{\alpha}{\beta} - \eta_t\right|_{\mathfrak{p}}.$$

Let $N_{\mathfrak{p}} = \min_{j\neq k} |\eta_j - \eta_k|_{\mathfrak{p}}$, so that $N_{\mathfrak{p}} > 0$. Then (5) yields

$$(6) \qquad |G(\alpha,\beta)|_{\mathfrak{p}} \geq M_{\mathfrak{p}} (N_{\mathfrak{p}}/2)^{h-1} |\beta|_{\mathfrak{p}}^{d} \min_{1\leq t\leq h} \left(\left|\frac{\alpha}{\beta} - \eta_t\right|_{\mathfrak{p}}\right).$$

Inequality (6) yields the desired inequality (4) in case $|\beta|_{\mathfrak{p}} \geq L > 0$ for some constant L. It remains to find a suitable constant L such that $|\beta|_{\mathfrak{p}} < L$ also implies inequality (4). Because $N(\alpha,\beta) \leq B$, one of $|\alpha|_{\mathfrak{p}}$, $|\beta|_{\mathfrak{p}}$ is always at least $B^{-1/g(\mathfrak{p})}$. Let

$$G(x,y) = \sum_{i=0}^{d} a_i x^{d-i} y^i.$$

Then

$$|\beta|_{\mathfrak{P}} < \frac{|G(1,0)|_{\mathfrak{P}}^{*} B^{-1/g(\mathfrak{P})}}{\max(|a_1|_{\mathfrak{P}}, \ldots, |a_d|_{\mathfrak{P}}, 1)}$$

implies that $B^{-1/g(\mathfrak{P})} \leq |\alpha|_{\mathfrak{P}}$, or that

$$|\beta|_{\mathfrak{P}} \max(|a_1|_{\mathfrak{P}}, \ldots, |a_d|_{\mathfrak{P}}, 1) < |G(1,0)|_{\mathfrak{P}}^{*} |\alpha|_{\mathfrak{P}}.$$

Since $|x + y|_{\mathfrak{P}} = |y|_{\mathfrak{P}}$ whenever $|x|_{\mathfrak{P}} < |y|_{\mathfrak{P}}$, this implies that

$$|G(\alpha,\beta)|_{\mathfrak{P}} = |G(1,0)|_{\mathfrak{P}} |\alpha|_{\mathfrak{P}}^{d}$$

$$\geq |G(1,0)|_{\mathfrak{P}} B^{-d/g(\mathfrak{P})},$$

which yields (4) with $L_{\mathfrak{P}} = |G(1,0)|_{\mathfrak{P}} B^{-d/g(\mathfrak{P})}$.

PROOF OF THE THEOREM: By virtue of Lemma 1, we suppose that none of the binary forms vanishes identically for $x = 0$ or $y = 0$. Let ζ_{is} $(1 \leq s \leq f)$ be the zeros of $F_i(x,1)$ in the complex field for $i = 1,\ldots,R_1+R_2$, and let η_{jt} $(1 \leq t \leq h_j)$ be the $\mathfrak{P}_j$-adic zeros of $G_j(x,1)$ for $j = 1,\ldots,r$. By Lemma 2,

$$(7) \quad |F_i(\alpha_k,\beta_k)|_i^{g_i} \geq c_i^{g_i} \max(|\alpha_k|_i^{g_i}, |\beta_k|_i^{g_i})^f \min_{1\leq s\leq f} (|(\frac{\alpha_k}{\beta_k})^{(i)} - \zeta_{is}|^{*g_i})$$

for all k and for $i = 1,\ldots,R_1+R_2$, where the C_i are positive

constants depending only on the forms $F_i(x,y)$. By Lemma 4,

$$(8)\quad |G_j(\alpha_k,\beta_k)|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \geq L_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \min_{1\leq t\leq h_j} \left(\left|\frac{\alpha_k}{\beta_k} - \eta_{jt}\right|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}\right)$$

for all k and for $j = 1,\ldots,r$, where the $L_{\mathfrak{P}_j}$ are positive constants chosen to be at most 1 and depending only on the forms $G_j(x,y)$ and the constant B. If $h_j = 0$, the minimum is understood to be 1.

Let $\varkappa_k = \alpha_k/\beta_k$ and note that there is an infinite sequence of values of k for which all the $\varkappa_k$ are distinct. Thus (7), (8), and (1) guarantee that, at least for an infinite subsequence of the sequence $\{\varkappa_k\}$,

$$(9)\quad \prod_{i=1}^{R_1+R_2} \max(|\alpha_k|_i^{g_i}, |\beta_k|_i^{g_i})^f \prod_{i=1}^{R_1+R_2} \min_s(|\varkappa_k^{(i)} - \zeta_{is}|^{*g_i}) \prod_{j=1}^{r} \min_t(|\varkappa_k - \eta_{jt}|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)})$$

$$\leq CH(\varkappa_k)^{\omega}$$

for some positive constant C depending only on K, the binary forms, and the hypothesized constants B and c and ideals $\mathfrak{P}_j$.

Observe that, as in (X-29), for $\varkappa$ a primitive element of K

$$\frac{1}{2^n} H(\alpha_k/\beta_k) \leq \prod_{i=1}^{R_1+R_2} \max(|\alpha_k|_i^{g_i}, |\beta_k|_i^{g_i}).$$

Thus inequality (9) yields

$$(10)\quad \prod_{i=1}^{R_1+R_2} \min_s(|\varkappa_k^{(i)} - \zeta_{is}|^{*g_i}) \prod_{j=1}^{r} \min_t(|\varkappa_k - \eta_{jt}|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)}) \leq C(2^n)^f H(\varkappa_k)^{-(f-\omega)},$$

the inequality being satisfied for infinitely many values of k. Let there be r' values of j for which $h_j \neq 0$. Then for some R_1+R_2-tuple $\zeta_1,\dots,\zeta_{R_1+R_2}$ of the ζ_{is}, and for some r'-tuple $\eta_1,\dots,\eta_{r'}$ of the η_{jt}, there are infinitely many of the χ_k satisfying

$$(11)\quad \prod_{i=1}^{R_1+R_2} |\chi_k^{(i)} - \zeta_i|^{*g_i} \prod_{j=1}^{r'} |\chi_k - \eta_j|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \leq C(2^n)^f H(\chi_k)^{-(f-\omega)}.$$

Clearly there is no loss of generality in assuming that $\zeta_i \in K_i$ for $i = 1,\dots,R_1+R_2$. Applying Theorem III to inequality (11), we obtain $f - \omega \leq 2$, or $\omega \geq f - 2$, as desired.

COROLLARY: *If, in the notation of Theorem 7, $f \geq 3$, then the inequality*

$$\prod_{i=1}^{R_1+R_2} |F_i(\alpha_k,\beta_k)|_i^{g_i} \prod_{j=1}^{r} |G_j(\alpha_k,\beta_k)|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \leq c$$

has only finitely many solutions α_k,β_k as in Theorem 7.

PROOF: This is just the theorem with $\omega = 0$.

A number of other corollaries can be obtained from Theorem 7. Let $F(x,y)$ be a binary form over K with total degree $f \geq 3$ and no repeated factors over $\mathbb{C}$. If we choose $F_i(x,y)$ to be $F(x,y)$ for $i = 1,\dots,R_1+R_2$, we can replace $\prod_{i=1}^{R_1+R_2} |F_i(\alpha_k,\beta_k)|_i^{g_i}$ by $N(F(\alpha_k,\beta_k))$. If α_k and β_k are as in Theorem 7, then we obtain three corollaries of Parry [10, pp. 75-81], but without his

bounds on the number of solutions involved.

a) As $\max(N(\alpha_k), N(\beta_k))$ increases without bound, so does the maximum of the norms of the prime ideals which divide $F(\alpha_k,\beta_k)$.

b) The number of different integers of K with norms which are products of some finite number of fixed rational primes and which are expressible as $F(\alpha_k,\beta_k)$ is bounded.

c) The number of different representations of a nonzero integer of K in the form $F(\alpha_k,\beta_k)$ is bounded.

The proofs of these three assertions follow easily from Theorem 7, and the bound to be obtained now on the number of solutions of the inequality (1) of Theorem 7 would yield bounds here by the methods used by Parry in the section cited above.

THEOREM 7a: In case $f - \omega > 2$ in Theorem 7, the inequality (1) has at most G''^{1/ϵ'^2} solutions α_k,β_k as described there for which α_k/β_k is a primitive element of K. Here $f - \omega = 2 + 4\epsilon$, $\epsilon' = \min((f - \omega - 2)/4, 1/12)$, and G'' is a constant independent of ϵ and depending only on K and on the hypothesized constants, forms, and ideals.

PROOF: Recall the proof of Theorem 7. There we let

$$\alpha_k' = (1 + ab)\alpha_k - b\beta_k$$

$$\beta_k' = \beta_k - a\alpha_k$$

and rejected all values of k for which $\alpha_k' \beta_k' = 0$. For such k, either $\alpha_k/\beta_k = b/(1 + ab)$ or $\alpha_k/\beta_k = 1/a$. But we showed that there are at most N values of k for which α_k/β_k has a fixed value, where N is the number of nonassociated integers in K with norms at most B. Thus the number of rejected values of k is a finite number depending only on B and K, and in particular is at most $2N$. We also showed that for each k, whether or not $\alpha_k' \beta_k' = 0$, there are at most N pairs α_j', β_j' associated with α_k', β_k'. Hence if there are S solutions in nonassociated pairs of nonzero integers α_k', β_k' from K of inequality (1), then the total number of solutions is at most $NS + 2N$. We also assumed that the remaining pairs α_k', β_k' had distinct ratios. As before, any given ratio can occur at most N times, so that $S = NS'$, where S' is the number of solutions still to be found.

There are at most $f^n \prod_{j=1}^{r} \max(1, h_j)$ inequalities (11), each having at most G^{1/ϵ'^2} solutions in distinct nonzero primitive elements α/β of K, where G is a constant depending only on K and the hypothesized forms, constants, and ideals. This follows from Theorem IIIa, page 66. Thus there are at most $N^2 f^n (G^{1/\epsilon'^2}) \prod_{j=1}^{r} \max(1, h_j) + 2N$ solutions in inequality (1) of the type stipulated, or at most G''^{1/ϵ'^2} solutions, where G'' is a constant which again depends only on K and the hypothesized forms, constants, and ideals. These solutions are of the desired type, since clearly α_k'/β_k' is a primitive element of K if and only if α_k/β_k is. This completes the proof.

THEOREM 8: Let B, C, and ν be positive constants, and let $\mathfrak{I}_1,\dots,\mathfrak{I}_r;\mathfrak{I}_{r+1},\dots,\mathfrak{I}_{r+s};\mathfrak{I}_{r+s+1},\dots,\mathfrak{I}_{r+s+t}$ be finitely many integers of K, where $(\mathfrak{I}_1\cdots\mathfrak{I}_r)$, $(\mathfrak{I}_{r+1}\cdots\mathfrak{I}_{r+s})$, and $(\mathfrak{I}_{r+s+1}\cdots\mathfrak{I}_{r+s+t})$ are relatively prime as principal ideals in K. Suppose that there exists an infinite sequence $\{X_k,Y_k,Z_k\}$ for $k = 1,2,\dots$ of nonassociated triplets of nonzero integers of K such that

$$X_k + Y_k + Z_k = 0$$

and

$$X_k = A_kX_k', \quad Y_k = B_kY_k', \quad Z_k = C_kZ_k'$$

where X_k', Y_k', and Z_k' are power products of $\mathfrak{I}_1,\dots,\mathfrak{I}_r$; $\mathfrak{I}_{r+1},\dots,\mathfrak{I}_{r+s}$; and $\mathfrak{I}_{r+s+1},\dots,\mathfrak{I}_{r+s+t}$ respectively, and where

$$(A_k,X_k') = (B_k,Y_k') = (C_k,Z_k') = 1$$

and

$$N(A_k,B_k) \le B, \quad N(A_k,C_k) \le B, \quad N(B_k,C_k) \le B.$$

Let

$$H_k = \max\left(H\left(\frac{X_k}{Y_k}\right), H\left(\frac{X_k}{Z_k}\right), H\left(\frac{Y_k}{Z_k}\right)\right).$$

Then $N(A_kB_kC_k) \le CH_k^{\nu}$ for $k = 1,2,\dots$ implies that $\nu \ge 1$.

PROOF: We can assume, without loss of generality, that $H_k = H(\frac{X_k}{Y_k})$ for all k. Let $\mathfrak{P}_1, \dots, \mathfrak{P}_R; \mathfrak{P}_{R+1}, \dots, \mathfrak{P}_{R+S}; \mathfrak{P}_{R+S+1}, \dots, \mathfrak{P}_{R+S+T}$ be the prime ideal divisors of $\mathfrak{S}_1 \cdots \mathfrak{S}_r, \mathfrak{S}_{r+1} \cdots \mathfrak{S}_{r+s}$, and $\mathfrak{S}_{r+s+1} \cdots \mathfrak{S}_{r+s+t}$ respectively. They are distinct by hypothesis. Then

$$N(A_kB_kC_k) = \frac{N(X_kY_kZ_k)}{N(X_k'Y_k'Z_k')}$$

$$= N(X_kY_kZ_k) \prod_{i=1}^{R} |X_k'|_{\mathfrak{P}_i}^{g(\mathfrak{P}_i)} \prod_{i=R+1}^{R+S} |Y_k'|_{\mathfrak{P}_i}^{g(\mathfrak{P}_i)} \prod_{i=R+S+1}^{R+S+T} |Z_k'|_{\mathfrak{P}_i}^{g(\mathfrak{P}_i)}.$$

Because $Z_k = -(X_k + Y_k)$ and

$$\prod_{i=1}^{R} |A_k|_{\mathfrak{P}_i}^{g(\mathfrak{P}_i)} = \prod_{i=R+1}^{R+S} |B_k|_{\mathfrak{P}_i}^{g(\mathfrak{P}_i)} = \prod_{i=R+S+1}^{R+S+T} |C_k|_{\mathfrak{P}_i}^{g(\mathfrak{P}_i)} = 1,$$

this yields

$$N(A_kB_kC_k) = \prod_{i=1}^{R_1+R_2} |X_k^2Y_k+X_kY_k^2|_i^{g_i} \prod_{i=1}^{R} |X_k|_{\mathfrak{P}_i}^{g(\mathfrak{P}_i)} \prod_{i=R+1}^{R+S} |Y_k|_{\mathfrak{P}_i}^{g(\mathfrak{P}_i)}$$

$$\times \prod_{i=R+S+1}^{R+S+T} |X_k + Y_k|_{\mathfrak{P}_i}^{g(\mathfrak{P}_i)}.$$

No two triples are associated, from which it follows that no two pairs X_k, Y_k are associated. Hence, if $N(A_kB_kC_k) \leq CH_k^{\nu}$ for all k, we have a situation to which Theorem 7 applies with $f = 3$, yielding $\nu \geq 1$.

THEOREM 8a: In case $\nu = 1 - 4\epsilon$ for some $\epsilon > 0$ in Theorem 8, the

equation $X + Y + Z = 0$ has at most G_1^{1/ϵ'^2} solutions X, Y, Z as described there for which X/Y, and therefore also X/Z and Y/Z are primitive elements of K. Here $\epsilon' = \min\left((1-\nu)/4, 1/12\right)$, and G_1 is a constant depending only on K and the hypothesized constants and ideals.

PROOF: Apply Theorem 7a and multiply the resulting bound by 3 to cover all possible choices of H_k.

THEOREM 9: Let $F(x,y)$, $G_j(x,y)$ for $j = 1,\ldots,r$ be binary forms over K having no multiple factors over $\mathbb{C}$, where the total degree of $F(x,y)$ is f. For $k = 1,\ldots,u$, let $P_k(x,y)$ be polynomials over K with total degrees s_k respectively. Let B be a positive constant, and let $i_1,\ldots,i_u$ be distinct rational integers such that $1 \le i_k \le R_1 + R_2$ for $k = 1,\ldots,u$. Suppose that $f \ge \sum_{k=1}^{u} g_{i_k} s_k$, and that

$$|F(\alpha,\beta)| \prod_{j=1}^{r} |G_j(\alpha,\beta)|_{\mathfrak{p}_j}^{g(\mathfrak{p}_j)} = \prod_{k=1}^{u} |P_k(\alpha,\beta)|_{i_k}^{g_{i_k}} \tag{12}$$

for infinitely many nonassociated pairs of nonzero integers α, β of K with $N(\alpha,\beta) \le B$. Then

$$f \le \sum_{k=1}^{u} g_{i_k} s_k + 2n.$$

PROOF: The proof follows LeVeque [5, pp. 152-154]. We suppose that there are infinitely many such pairs α, β. As before, we can assume

that none of the forms is identically zero for $x = 0$ or $y = 0$. The transformation used to obtain such forms clearly leaves the degrees of the polynomials unchanged. As in the proof of Lemma 2, for a suitable positive constant c and for some complex number ζ,

$$|F(\alpha,\beta)| \geq c|\beta|^f\left|\frac{\alpha}{\beta} - \zeta\right| \tag{13}$$

for infinitely many distinct quotients α/β. Also, for suitable $D > 0$,

$$|P_k(\alpha,\beta)|_{i_k}^{g_{i_k}} \leq D \max(\lceil\alpha\rceil, \lceil\beta\rceil)^{g_{i_k} s_k} \tag{14}$$

and

$$H\left(\frac{\alpha}{\beta}\right) \leq (2\max(\lceil\alpha\rceil, \lceil\beta\rceil))^n, \tag{15}$$

where $\lceil\gamma\rceil$ is the maximum of the absolute values of the conjugates of γ for any γ in K. By Lemma 4,

$$|G_j(\alpha,\beta)|_{\mathfrak{p}_j}^{g(\mathfrak{p}_j)} \geq d_j\left|\eta_j - \frac{\alpha}{\beta}\right|_{\mathfrak{p}_j}^{*\,g(\mathfrak{p}_j)} \tag{16}$$

for suitable $d_j > 0$ and η_j, and for infinitely many distinct α/β. Therefore by (12), (13), (14), and (16), there are infinitely many distinct nonzero elements $\varkappa = \alpha/\beta$ of K with α and β as specified above, for which

$$(17)\quad |\varkappa - \varsigma| \prod_{j=1}^{r} |\eta_j - \varkappa|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \leq C \frac{\max(|\alpha|, |\beta|)^{\sum_{k=1}^{u} g_{i_k} s_k}}{|\beta|^f},$$

where $C = D^u/(c \prod_{j=1}^{r} d_j)$. We can assume, with no loss of generality, that $\max(|\alpha|, |\beta|) = |\beta|$ for infinitely many α and β, since the roles of α and β are symmetric. Moreover, we can assume that $|\beta| = \beta$ for infinitely many α and β, since, provided we order conjugates so that $(\gamma^{(i)})^{(j)} = (\gamma^{(j)})^{(i)}$ for all γ in K and all i, j, the hypothesized equality (12) remains valid when all elements, including the ideals $\mathfrak{P}_j$, are conjugated. Hence, by (17) and (15), the inequality

$$(18)\quad |\varkappa - \varsigma| \prod_{j=1}^{r} |\eta_j - \varkappa|_{\mathfrak{P}_j}^{*g(\mathfrak{P}_j)} \leq C' H(\varkappa)^{-(f - \sum_{k=1}^{u} g_{i_k} s_k)/n}$$

holds for infinitely many distinct nonzero numbers $\varkappa$, where $C' = C\, 2^{f - \sum_{k=1}^{u} g_{i_k} s_k}$. Theorem III applied to inequality (18) yields

$$\frac{1}{n}\left(f - \sum_{k=1}^{u} g_{i_k} s_k\right) \leq 2,$$

or

$$f \leq \sum_{k=1}^{u} g_{i_k} s_k + 2n.$$

APPENDIX A

THEOREM: If K is an algebraic number field of degree n, then there exists a rational integer $c = c(K)$, depending only on K, such that every primitive element $\varkappa \in K$ has a representation $\varkappa = \frac{\alpha}{\beta}$ as the quotient of two integers of K with $N(\alpha) \leq cH(\varkappa)$ and $N(\beta) \leq cb \leq cH(\varkappa)$ where $b > 0$ is the leading coefficient in the defining polynomial of $\varkappa$.

The proof makes use of the following lemma, whose proof can be found in [5, p. 83].

LEMMA: There is a positive constant c, which depends only on the field K, such that each ideal A divides a principal ideal AB of K for which $N(AB) \leq cN(A)$.

PROOF OF THE THEOREM:

We know that $(\varkappa)$ has a unique representation $(\varkappa) = \frac{\mathfrak{m}}{\mathfrak{n}}$ as the quotient of two relatively prime integral ideals of K. By the above lemma, there exists a constant $c = c(K)$ and also an integral ideal $\mathfrak{a}$ of K such that $\mathfrak{m}\mathfrak{a}$ is principal, say $\mathfrak{m}\mathfrak{a} = (\alpha)$, where $N(\alpha) \leq c$. Thus $(\varkappa) = \frac{(\alpha)}{\mathfrak{n}\mathfrak{a}}$, and $\mathfrak{n}\mathfrak{a}$ must also be principal. If we let $\mathfrak{n}\mathfrak{a} = (\beta)$, then $(\varkappa) = \frac{(\alpha)}{(\beta)}$ where $(\alpha,\beta) = \mathfrak{a}$, so that $N(\alpha,\beta) \leq c$. We may choose α so that $\varkappa = \frac{\alpha}{\beta}$, since unit multipliers do not effect norms, and we shall show that this is the desired representation for $\varkappa$.

Observe that the greatest common divisor of the coefficients of the polynomial $\prod_{i=1}^{n} (\beta^{(i)} x - \alpha^{(i)})$ is just $N(\alpha)$. (See, for instance, [18, p. 53] where this is given as the definition of the norm.

That this definition of the norm is equivalent to the more usual one follows easily from the facts that both are multiplicative and that for both, the norm of a prime ideal is a power of a rational prime. A complete proof is given in [18, pp. 147-148].) Therefore the defining polynomial of $\varkappa$ is

$$\frac{1}{N(\alpha)} \prod_{i=1}^{n} (\beta^{(i)} x - \alpha^{(i)}) = bx^n + \ldots \pm \frac{N(\alpha)}{N(\mathfrak{a})},$$

so that

$$N(\beta) = bN(\mathfrak{a}) \leq bc \leq cH(\varkappa)$$

and

$$N(\alpha) \leq N(\mathfrak{a})\, H(\varkappa) \leq cH(\varkappa).$$

COROLLARY 1: Let $\varkappa_1$ and $\varkappa_2$ be primitive elements of K and let b_1 and b_2 be the leading coefficients in the defining polynomials of $\varkappa_1$ and $\varkappa_2$ respectively. If $\mathfrak{P}_1, \ldots, \mathfrak{P}_R$ are distinct prime ideals of K, then

$$b_1 b_2\, N(\varkappa_1 - \varkappa_2) \prod_{j=1}^{R} |\varkappa_1 - \varkappa_2|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} \geq 1.$$

PROOF: Let $\varkappa_1 = \alpha_1/\beta_1$ and $\varkappa_2 = \alpha_2/\beta_2$ be as specified in the theorem. As in the preceding proof, $N(\beta_i) = d_i b_i$ where $d_i = N(\alpha_i, \beta_i)$ for $i = 1,2$. Therefore

$$b_1 b_2 N(\chi_1-\chi_2) \prod_{j=1}^{R} |\chi_1-\chi_2|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} = \frac{N(\alpha_1\beta_2-\alpha_2\beta_1) \prod_{j=1}^{R} |\alpha_1\beta_2-\alpha_2\beta_1|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)}}{d_1 d_2 \prod_{j=1}^{R} |\beta_1\beta_2|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)}}$$

where the numerator is the product of all factors $N(\mathfrak{P})^h$ with $\mathfrak{P}^h || (\alpha_1\beta_2 - \alpha_2\beta_1)$, $\mathfrak{P} \neq \mathfrak{P}_1, \ldots, \mathfrak{P}_R$, and the denominator is at most a product of factors $N(\mathfrak{q})^k$ where $\mathfrak{q}^k || (\alpha_1,\beta_1)(\alpha_2,\beta_2)$, and hence $\mathfrak{q}^k | (\beta_1\beta_2)$, $\mathfrak{q} \neq \mathfrak{P}_1, \ldots, \mathfrak{P}_R$. The statement concerning the denominator follows from the facts that

$$d_1 d_2 \prod_{j=1}^{R} |\beta_1\beta_2|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)} = N(\alpha_1,\beta_1)\, N(\alpha_2,\beta_2) \prod_{j=1}^{R} |\beta_1\beta_2|_{\mathfrak{P}_j}^{g(\mathfrak{P}_j)}$$

and that $\mathfrak{q}^k | (\alpha_1,\beta_1)(\alpha_2,\beta_2)$ implies that $\mathfrak{q}^k | (\beta_1\beta_2)$. But $\mathfrak{q}^k | (\alpha_1,\beta_1)(\alpha_2,\beta_2)$ implies that $\mathfrak{q}^k | (\alpha_1\beta_2 - \alpha_2\beta_1)$, since clearly $(\alpha_1\beta_2 - \alpha_2\beta_1) \subseteq (\alpha_1,\beta_1)(\alpha_2,\beta_2)$, and this gives the desired result.

COROLLARY 2: Let $K = \mathbb{Q}(\theta)$ where $\mathbb{Q}$ is the rational field and where θ has defining equation $a_0 x^n + \ldots + a_n$ with $a_0, \ldots, a_n$ relatively prime rational integers, and let $q = \max(|a_0|, \ldots, |a_n|)$. Then the constant c of the theorem can be chosen to be at most

$$\left\{ \frac{1 - (n+1)^n q^n}{1 - (n+1)q} \right\}^n .$$

PROOF: We observe that $a_0\theta$ is an algebraic integer, so that, following the proof of Lemma 1 [5, pp.83-84], we may take

$$c = \prod_{i=1}^{n} \left(1 + |a_0\theta^{(i)}| + |a_0^2(\theta^2)^{(i)}| + \ldots + |a_0^{n-1}(\theta^{n-1})^{(i)}|\right).$$

By Lemma (II-3), page 24, $|\theta^{(i)}| \leq (n + 1)q/a_o$, and the desired bound is easily obtained.

BIBLIOGRAPHY

1. Davenport, H. and Roth, K. F. Rational approximations to algebraic numbers, Mathematika, 2, 1955, pp. 160-167.
2. Dyson, F. J. The approximation to algebraic numbers by rationals, Acta Math., 79, 1947, pp. 225-240.
3. Gelfond, A. O. Transcendental and Algebraic Numbers. Translated from the First Russian Edition by Leo F. Boron. New York: Dover Publications, Inc., 1960.
4. Lang, S. Diophantine Geometry, New York: Interscience Publishers, a division of John Wiley and Sons, 1962.
5. LeVeque, W. J. Topics in Number Theory, Vol. II. Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1956.
6. Liouville, J. Sur des classes très-étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationnelles algébriques, J. des Math. Pures et Appl., 16, 1851, pp. 133-142.
7. Mahler, K. An application of Jensen's Formula to polynomials, Mathematika, 7, 1960, pp. 98-100.
8. Mahler, K. Lectures on Diophantine Approximations, Part I: g-adic numbers and Roth's theorem. Prepared from the notes by R. P. Bambah of my lectures given at the University of Notre Dame in the Fall of 1957. Notre Dame, Indiana: University of Notre Dame Press, 1961.
9. Mahler, K. On the approximation of algebraic numbers by algebraic integers, J. Australian Math. Soc., 3, 1963, pp. 408-434.
10. Parry, C. J. The p-adic generalization of the Thue-Siegel Theorem, Acta Math., 83, 1950, pp. 1-100.

11. Ridout, D. Rational approximations to algebraic numbers, Mathematika, 4, 1957, pp. 125-131.

12. Ridout, D. The p-adic generalization of the Thue-Siegel-Roth Theorem, Mathematika, 5, 1958, pp. 40-48.

13. Roth, K. F. Rational approximations to algebraic numbers, Mathematika, 2, 1955, pp. 1-20 with corrigendum p. 168.

14. Schneider, T. Über die Approximation algebraischer Zahlen, J. reine angew. Math., 175, 1936, pp. 182-192.

15. Siegel, C. L. Approximation algebraischer Zahlen, Math. Z., 10, 1921, pp. 173-213.

16. Thue, A. Über Annäherungswerte algebraischer Zahlen, J. reine angew. Math., 135, 1909, pp. 284-305.

17. Uchiyama, S. On the Thue-Siegel-Roth Theorem I and II, Proc. Japan Acad., 35, 1959, pp. 413-416 and pp. 525-529.

18. Weyl, H. Algebraic Theory of Numbers. Annals of Mathematics Studies No. 1, Princeton: Princeton University Press, 1940.