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An Investigation in Algebraic Topology and Morse Theory

by

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ABSTRACT

AN INVESTIGATION IN ALGEBRAIC TOPOLOGY AND MORSE THEORY

by John Paul Roth

This investigation is an attempt on the one hand to utilize some of the newer algebraic methods of topology in Morse critical point theory and on the other hand to set the stage for further exploration of the application of the structure of Morse theory to general topology. The principal tools are (1) the Eilenberg-Steenrod formulation of homology theory and (2) the notion of filter.

In chapter I the notion of a "group defined on a filter" is examined. More generally, a homology sequence is attached to filters, a method essential in our treatment of Morse theory.

Chapter II is devoted to a development of a new algebraic invariant, called an add, suggested by Morse theory. Chapter III gives examples where the add structure distinguishes spaces (some of them manifolds) not distinguished by their homology and homotopy groups.

In chapter IV an axiomatization of Morse theory is given. The point of view is to consider an arbitrary triple (S,f,H), consisting of a general space S, an arbitrary mapping f of S into the reals and an arbitrary homology H which is "admissible" on S and f. The generalized Morse relations are derived without any finiteness assumptions (e.g. finite number of critical points). Several new theorems are proven. This axiomatization, the author has very recently been informed, is similar to one developed by S. Eilenberg five years ago, not yet published.

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INTRODUCTION

This investigation is an attempt on the one hand to utilize some of the newer algebraic methods of topology in Morse critical point theory and on the other hand to set the stage for further exploration of the application of the structure of Morse theory to general topology. The principal tools are (1) the Eilenberg-Steenrod formulation of homology theory and (2) the notion of a filter.

In chapter I the notion of a "group defined on a filter" is examined. More generally, a homology sequence is attached to filters, a method essential in our treatment of Morse theory.

Chapter II is devoted to a development of a new algebraic invariant, called an add, suggested by Morse theory. Chapter III gives examples where the add structure distinguishes spaces (some of them manifolds) not distinguished by their homology and homotopy groups.

In chapter IV an axiomatization of Morse theory is given. The point of view is to consider an arbitrary triple (S, f, H), consisting of a general space S, an arbitrary mapping f of S into the reals and an arbitrary homology theory H which is "admissible" on S and f. The generalized Morse relations are derived without any finiteness assumptions (e.g., finite number of critical points). Several new theorems are proven. This axiomatization, the author has very recently been informed, is similar to one developed by S. Eilenberg five years ago, not yet published.

CHAPTER I

FILTERS, GROUPS ON FILTERS

1. Filters

1.1. A basic tool in our investigations is the filter. A filter on a set S [NB, 20] ≠ is a collection $\mathcal{F}$ of subsets of S having the following properties: (1) every set which contains an element of $\mathcal{F}$ itself belongs to $\mathcal{F}$, (2) each finite intersection of sets of $\mathcal{F}$ belongs to $\mathcal{F}$; and (3) the empty set does not belong to $\mathcal{F}$.

For our purposes it will be necessary to deal with filters somewhat differently than in [NB], in particular with regard to the concepts of limits of filters and filter refinements.

1.2. Henceforth S shall denote a topological space, that is, one satisfying the "0 - axioms" [NB, 1], although we shall frequently require S to be Hausdorff. By "neighborhood" of a point x we mean any set containing an open set containing x. The collection $\mathcal{U}(x)$ of such neighborhoods will be termed the "neighborhood filter" of x.

A neighborhood of a subset M of S is any set V which contains an open set containing M. The collection $\mathcal{U}(M)$ of all such neighborhoods will be termed the neighborhood filter of M.

1.3. Given a filter $\mathcal{F}$, consider points for which each neighborhood of any one of the points intersects each element of the filter: The collection of all such points we shall call the terminus of the filter, denoted $T(\mathcal{F})$ or simply T. For example on the real line E^1 consider the filter

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≠ The symbol [NB, 20] denotes page 20 of the reference [NB] in the bibliography. This method of referencing will be used throughout the paper.

generated by sets of the type $(0, a)$, $a > 0$. For this filter, the point 0 is the terminus, although it is not an element of every member of the filter. On the other hand, for the filter generated by sets of the type $[0, a)$, $a > 0$, 0 will still be the terminus and 0 will be an element of each member of the filter.

If a filter has just this property, that the terminus is contained in every element of the filter, it will be called <u>firm</u>. We shall usually prefer to deal with firm filters, and in fact any filter $\mathcal{F}$ can be made into a firm filter $\overline{\mathcal{F}}$ simply by adjoining to each element F_ν of $\mathcal{F}$ the terminus T of $\mathcal{F}$:

$$\overline{\mathcal{F}} = \{F_\nu \cup T \mid F_\nu \in \mathcal{F}\} .$$

If the space S is Hausdorff, then a terminus is closed.

1.4. The <u>product</u> $\mathcal{F} \cdot \mathcal{G}$ of filters $\mathcal{F}, \mathcal{G}$ is the collection of all sets belonging to both $\mathcal{F}$ and $\mathcal{G}$. It may also be defined as the filter generated by sets of the type $F_\nu \cup G_\mu$, $F_\nu \in \mathcal{F}$, $G_\mu \in \mathcal{G}$. For example if $\mathcal{F}, \mathcal{G}$ are neighborhood filters of points x, y respectively of a space, then $\mathcal{F} \cdot \mathcal{G}$ is the neighborhood filter of $x \cup y$. The terminus of the product of two filters is the union of their termini.

1.5. <u>Induced mappings on filters</u>, <u>function-generated filters</u>. Let f be an arbitrary mapping of a space S onto a space S*. If $\mathcal{F}^*$ is a filter of S*, then the collection of inverse images of elements of $\mathcal{F}^*$ will generate a filter $\mathcal{F}$ of S.[≠] We term this $\mathcal{F}$ the filter generated by $\mathcal{F}^*$ and f, or simply a <u>function-generated filter</u>. In fact, the mapping f of S

≠ For by théorème 1 of Bourbaki [NB, 22] it is necessary and sufficient for a collection of sets to generate a filter that every finite intersection shall be nonempty: If $F_\nu^*, F_\mu^* \in \mathcal{F}^*$, then $f^{-1}(F_\nu^*) \cap f^{-1}(F_\mu^*) = f^{-1}(F_\nu^* \cap F_\mu^*) \neq \emptyset$, q. e. d.

onto S* induces a mapping of the filters of S* into those of S, a circumstance which we shall use in Chapter II and which is fundamental in Morse theory.

2. Groups Defined on Filters

In the course of our investigations we shall find it necessary to define different types of groups on a given filter and so at this point we shall define most of those subsequently needed. We shall use as far as possible an arbitrary homology-cohomology theory, pointing out as we proceed where some special properties are needed (e.g. strong excision); cf. Eilenberg-Steenrod [E-S, 1] . We shall also deal with homotopy-type groups.

Throughout we shall assume a discrete field as coefficient group.

2.1. A homology and a cohomology group for an arbitrary filter. Let $\mathcal{F}$ be any filter of S. Let H denote a homology-cohomology theory which is admissible on the category of pairs and mappings to be considered with the filter $\mathcal{F}$; thus H may be for example the Čech theory or singular theory, valid on an arbitrary category. For each F_ν in $\mathcal{F}$ we assume that the homology group $H^k(S, \not{\partial}F_\nu)^{\neq}$ and the cohomology group $H_k(F_\nu)$ is defined (we shall introduce other groups, e.g., $H^k(\not{\partial}F_\nu)$, in a later section). Likewise for $F_\nu \subset F_\mu$, we are given by the homology-cohomology theory H the mappings

$$j_\mu^\nu : H^k(S, \not{\partial}F_\mu) \longrightarrow H^k(S, \not{\partial}F_\nu)$$

$$i_\mu^\nu : H_k(F_\mu) \longrightarrow H_k(F_\nu)$$

These groups and mappings form two directed systems of groups [E-S, 212-232] and we may thus form the direct limit groups

$$\varinjlim H^k(S, \not{\partial}F_\nu)$$

$$\varinjlim H_k(F_\nu)$$

$\neq$ The symbol $(S, \not{\partial}F_\nu)$ denotes the pair $(S, S - F_\nu)$.

which we shall denote respectively by

$$H^k(S, \phi \mathcal{F})$$

$$H_k(\mathcal{F}).$$

We shall show partiality to these groups and shall call these the principal homology-cohomology groups of the filter $\mathcal{F}$ or simply the principal groups.

2.2. Groups on a function-generated filter, the critical groups. Let f be a (continuous) mapping of S into the linear continuum E^1. For any point c in E^1 consider the filter $\mathcal{F}^*(c)$ in E^1 having as a base the sets $I_{c,e} = (c - e, c]$, $e > 0$. Throughout we consider f and c fixed. In S let $S_{c,e}$ be the set of all x for which f(x) is in $I_{c,e}$; the collection $\mathfrak{B}(f,c)$ of the sets $S_{c,e}$ form the base of a filter $\mathcal{F}(f,c)$, termed the filter of f at level c.

We have then the groups (cf. 2.1.) $H^k(S, \phi\mathcal{F}(f,c))$, $H_k(\mathcal{F}(f,c))$, but these are not the critical groups.≠ Let S_c denote the subset of S of points whose functional value does not exceed c. Consider then the trace $\mathcal{F}_c(f)$ of $\mathcal{F}(f,c)$ on S_c.≠≠ The group

$$H^k(S_c, \phi\, \mathcal{F}_c(f))$$

is then termed the critical group of f at level c. The group $H_k(\mathcal{F}_c(f))$ is also defined, but we shall not use it here. We shall return to this group in Chapter IV, where we will compare it with the conventional definitions.

We next turn to a group which we shall find useful in Chapter III, in classifying the van Kampen example.

2.3. Homotopy type of groups on an arbitrary filter. Let $\mathcal{F}$ be a firm filter and T its terminus. For each point t of T and element F_ν of $\mathcal{F}$,

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≠ Unless c is a maximum or a minimum, f continuous, cf. Chapter IV.

≠≠ The trace of a filter on a subset is the collection of intersections of filter elements with the subset; in our case it is also a filter.

there is defined the nth homotopy group [S, 72]

$$\pi_n(F_\nu, t)$$

and for $F_\nu \subset F_\mu$ there is the injection

$$i_\mu^\nu : \pi_n(F_\nu, t) \longrightarrow \pi_n(F_\mu, t).$$

Let $P_n(\mathcal{F}, t)$ denote the inverse limit group

$$\varprojlim \pi_n(F_\nu, t).$$

We shall say that two points t and t' of T are equivalent if their associated groups $P_n(\mathcal{F}, t)$ and $P_n(\mathcal{F}, t')$ are isomorphic and if they lie in the same quasi-component of T [W, 18] . Let $\hat{t}$ denote an equivalence class so defined and let $P_n(\mathcal{F}, \hat{t})$ be any one of the isomorphic groups associated with $\hat{t}$. We then define the nth homotopy-type group of the filter $\mathcal{F}$, $P_n(\mathcal{F})$, as the cartesian direct product of groups $P_n(\mathcal{F}, \hat{t})$, taken over all equivalence classes of T.

2.4. Homotopy- and homology-type of groups of a filter. Since it is often stated (cf. [W, 287]) that stronger topological characterization and classification of generalized manifolds in dimensions exceeding 2 than presently available will probably require the use of both homology (cohomology) and homotopy theory, it is perhaps worthwhile to attempt to attach to a filter both a homology (cohomology) and a homotopy group, and in some sense simultaneously. A possible mode of approach is the following.

Let $\mathcal{F}$ be any filter; let $H(\mathcal{F})$ be some type of homology group attached to this filter (e.g., any of those of section 2) and $P(\mathcal{F})$ a homotopy-type of group attached to the same filter. Then the cartesian product $H(\mathcal{F}) \otimes P(\mathcal{F})$ will be a group of the desired type.

2.5a. Lemma. Let $\mathcal{G}$ be a refinement of a filter $\mathcal{F}$, $\mathcal{F} \subset \mathcal{G}$. Let $\{\mathcal{F}, H(F_\nu), \pi_\mu^\nu\}$ and $\{\mathcal{F}, H(G_\nu), \pi_\mu^\nu\}$ be direct systems of groups, where the groups and homomorphisms of the direct system of $\mathcal{G}$ when restricted to "coordinates" in $\mathcal{F}$ coincide with those of $\mathcal{F}$. Then the inclusion mapping $i : \mathcal{F} \to \mathcal{G}$ induces a homomorphism of the direct limit group $\varinjlim H(F_\nu)$ into the direct limit group $\varinjlim H(G_\nu)$.

2.5b. Lemma. If $\mathcal{G}$ is a refinement of $\mathcal{F}$, and $\{\mathcal{F}, H(F_\nu), \pi_\mu^\nu\}$, $\{\mathcal{G}, H(G_\nu), \pi_\mu^\nu\}$ are inverse systems of groups, coinciding as in 2.5a, then the inclusion mapping $i: \mathcal{F} \to \mathcal{G}$ induces a homomorphism of the inverse limit group $\varprojlim H(G_\nu)$ into $\varprojlim H(F_\nu)$.

2.5c. For topological investigations we would not in general be concerned with maximal filters containing a given set. For example, for a point x in E^n, the local connectivity group $H^n(S, \mathcal{U}(x))$ and $H_n(\mathcal{U}(x))$ are isomorphic to the coefficient group. On the other hand for the refinement $\mathcal{M}(x)$ consisting of all subsets containing x, $H^n(S, \mathcal{M}(x)) = H_n(\mathcal{M}(x)) = 0$.

3. Homology Sequences of an Arbitrary Filter

The homology sequences to be defined here will be used in treating of the Morse theory and in constructing various add structures (Chapter II). We shall consider at this point only a homology-type of sequence; the cohomology theory corresponding is dual in the sense of [E-S, xiii] with the added advantage that the cohomology sequences are always exact. Again we deal with an arbitrary homology theory. [E-S, 10].

3.1. Definition of $H^k(\mathcal{F})$. We have already defined the group $H^k(S, \mathcal{F})$; the new group under consideration, $H^k(\mathcal{F})$, will contain the image of a boundary operator ∂ to be defined on $H^{k+1}(S, \mathcal{F})$. For each F_ν

of a given filter $\mathcal{F}$, consider the group $H^k(\mathcal{C}F_\nu)$. For $F_\nu \subset F_\mu$, we have the injection

$$i_\mu^\nu : H^k(\mathcal{C}F_\mu) \longrightarrow H^k(\mathcal{C}F_\nu).$$

These groups and mapping form a directed system over $\mathcal{F}$; let $H^k(\mathcal{C}\mathcal{F})$ denote the direct limit $\varinjlim H^k(\mathcal{C}F_\nu)$.

3.2. The boundary operator $\partial : H^k(S, \mathcal{C}\mathcal{F}) \longrightarrow H^{k-1}(\mathcal{C}\mathcal{F})$. For F_ν let ∂_ν be the boundary operator mapping $H^k(S, \mathcal{C}F_\nu)$ into $H^{k-1}(\mathcal{C}F_\nu)$. By axiom 3 [E-S, 11], for $F_\nu \subset F_\mu$, commutativity holds in the diagram

$$\begin{array}{ccccc} \longrightarrow & H^k(S, \mathcal{C}F_\mu) & \xrightarrow{i_\mu^\nu} & H^k(S, \mathcal{C}F_\nu) & \longrightarrow \\ & \downarrow \partial_\mu & & \downarrow \partial_\nu & \\ \longrightarrow & H^{k-1}(\mathcal{C}F_\mu) & \xrightarrow{i_\mu^\nu} & H^{k-1}(\mathcal{C}F_\nu) & \longrightarrow \end{array}$$

The collection $\{\partial_\nu\}$ of mappings induces a homomorphism of the direct limit groups,

$$\partial : H^k(S, \mathcal{C}\mathcal{F}) \longrightarrow H^{k-1}(\mathcal{C}\mathcal{F}).$$

For let $u = \{u_\nu\}$ be an element of $H^k(S, \mathcal{C}\mathcal{F})$. Let ∂u be the collection $\{\partial_\nu u_\nu\}$; we shall show this is an element of $H^{k-1}(\mathcal{C}\mathcal{F})$: Let $\partial_\nu u_\nu$, $\partial_\mu u_\mu \in \partial u$. Then u_ν, $u_\mu \in u$ so that there exists a $\lambda > \mu, \nu$ such that

$$i_\nu^\lambda u_\nu = i_\mu^\lambda u_\mu .$$

From the commutativity of the above diagram the argument follows

$$\begin{array}{ccc} \partial_\lambda i_\nu^\lambda u_\nu = & & \partial_\lambda i_\mu^\lambda u_\mu \\ \| & & \| \\ i_\nu^\lambda \partial_\nu u_\nu = & & i_\mu^\lambda \partial_\mu u_\mu . \end{array}$$

Clearly the map $\partial : u \longrightarrow \partial u$ is linear, i.e., it is a homomorphism. Q. E. D.

3.3. $H^k(\phi\mathcal{F}) \xrightarrow{i} H^k(S)$. In a natural way, $H^k(S)$ can be considered as a direct limit group over $\mathcal{F}$: The group for each coordinate is $H^k(S)$; the mapping is the identity. For each coordinate group $H^k(\phi F_\nu)$, let i_ν denote the injection of $H^k(\phi F_\nu)$ into $H^k(S)$. Again from the axioms [E-S, 11] commutativity holds in the diagram

$$\begin{array}{ccc} H^k(\phi F_\nu) & \xrightarrow{i_\nu} & H^k(S) \\ \uparrow i^\nu_\mu & & \uparrow i \\ H^k(\phi F_\mu) & \xrightarrow{i_\mu} & H^k(S). \end{array}$$

Let i denote the injection of the limit group $H^k(\phi\mathcal{F})$ into $H^k(S)$.

3.4. $H^k(S) \xrightarrow{j} H^k(S, \phi\mathcal{F})$. Since S is an element of any filter $\mathcal{F}$, $H^k(S) = H^k(S, \phi)$ can be considered as a coordinate group of $H^k(S, \phi\mathcal{F})$. Thus the mapping j is simply the projection [E-S, 221] of $H^k(S)$ into the subgroup of $H^k(S, \phi\mathcal{F})$ that it determines.

3.5. The homology sequence of a filter $\mathcal{F}$ over a homology theory H is then the collection of groups and homomorphisms

$$\ldots \longrightarrow H^k(\phi\mathcal{F}) \xrightarrow{i} H^k(S) \xrightarrow{j} H^k(S, \phi\mathcal{F}) \xrightarrow{\partial} H^{k-1}(\phi\mathcal{F}) \xrightarrow{i} \ldots$$

Let $HS(\mathcal{F})$ denote this sequence. The cohomology sequence of $\mathcal{F}$ is defined dually.

It should be mentioned perhaps that homology and cohomology sequences of a filter can likewise be defined in terms of inverse-limit groups on the filter.

4. Exact Sequences of a Filter

4.1. We shall now assume that the coordinate sequences

$$\cdots \xrightarrow{\partial} H^k(\not{F}_\nu) \xrightarrow{i} H^k(S) \xrightarrow{j} H^k(S, \not{F}_\nu) \xrightarrow{\partial} H^{k-1}(\not{F}_\nu) \longrightarrow \cdots$$

of $HS(\mathcal{F})$ or of $HS(\mathcal{L})$, $\mathcal{L}$ a base of $\mathcal{F}$, are exact. Such is the case, for example, if: (1) S is triangulable, and $\mathcal{F}$ has a base for which the complement of each element was triangulable, since under these circumstances all homology theories agree [E-S, 100] ; (2) H is singular homology theory, S and $\mathcal{F}$ arbitrary; (3) S is locally compact and $\mathcal{F}$ has a base of open sets, and the Čech groups of the point compactification are used [ES_1, IX, X] ; (4) S is compact, $\mathcal{F}$ has a base of open sets and Čech theory is used, (cf. span theory of Morse, replacing S by S_c); or (5) S is normal, $\mathcal{F}$ has a base of open sets, and Čech groups based on the Tychonoff compactification are used.

Note that if we are dealing with the groups of the trace (cf. 2.2) of a filter on a subset, say S_c, of the space, these conditions need apply only in the subspace topology; for applications to Morse theory this is important.

For a cohomology sequence, exactness would hold for arbitrary S and $\mathcal{F}$; this suggests the desirability of a cohomology treatment. If the coordinate sequences are exact, then so will be the limit sequence $HS(\mathcal{F})$.

CHAPTER II

ADDS OF A TOPOLOGICAL SPACE

It will be the purpose of this chapter to define various add structures of a space S.

0. Motivation for Introduction of the Add

Besides the important role that Morse theory has played in modern analysis, it has given rather striking information in pure topology. It has been useful, for instance, in the computation of homotopy groups of spheres and of the symmetric square of spheres, and in obtaining certain homology relations on regular orientable manifolds.

We have, furthermore, the opinion of Hurewicz that "perhaps the present trend of research does not put enough emphasis on tools that could be provided by the geometrical structure from the point of view of differential geometry, like properties of geodesic lines, study of critical points, etc."

It would seem reasonable, therefore, to seek concepts and methods through Morse theory which are inherently topological. Motivation for the attempt made in this direction here may be given as follows:

Let S be a (metric) space and f a mapping of S into the reals. Now if one succeeds in characterizing the critical points of f by means of a certain relative homology group, then one has an invariant of the space and the function. How can one get from this an invariant of the space alone?

In his paper on "Homology relations on regular orientable manifolds", Morse deals with special types of spaces and subspaces thereof which enable him to state that there exists a function of a certain prescribed type and thus by use of its critical groups he is able to prove homology relations that do not refer to the function at all. This technique cannot be used in general spaces.

As an alternative, one may try to follow the model of homotopy or cohomotopy theory by introducing some type of equivalence classes on the set of all mappings of S into the reals and attempting to introduce addition among these equivalence classes. (One kind of equivalence would be to say that mappings f, g are equivalent at level c if their "critical k-groups at level c" are isomorphic for each k.) For homotopy or cohomotopy, the resulting algebraic structure turns out to be a group. The algebraic structure resulting from a decomposition of real-valued functions according to the critical groups and from the addition defined on these classes is an add in the sense of Baer (introduced, significantly, in his study of free groups).

These adds, to be called function-generated adds, are topological invariants of an algebraic character and are capable of distinguishing certain spaces undistinguished by their homology or homotopy groups.

A deeper study reveals, however, that the process of attaching homology groups to a critical value of a function is precisely that of defining a group on a special filter of subsets of S. A more basic approach is found by considering equivalence classes of filters and introducing addition between these equivalence classes. To make the situation precise, the notion of multiplicative systems of filters becomes necessary. Thus we are led to the add $A(S, \Sigma, H)$.

1. Multiplicative Systems Σ of Filters, Category of Σ

1.1. Consider a class Σ of filters of S. Σ is a multiplicative system of filters if the product of each pair of filters of Σ belongs to Σ, cf. [E-S, 108] . For example let Σ_N be the collection of neighborhood filters of S, Σ_n the collection of neighborhood filters of finite collections of points,≠ Σ_F the collection of finite sums of function-generated filters, and Ω the collection of all filters of S. These multiplicative systems may be partially ordered by inclusion.

1.2. The category of a multiplicative system Σ consists of the lattice [E-S, 4] of pairs $(S, \mathcal{F})$ and all inclusion maps of the resulting pairs for each F in each $\mathcal{F}$ of Σ. If $\{\mathcal{B}\} = \Sigma^b$ is a collection of bases, one for each filter of Σ, we term the associated category a category base of Σ. For Σ_N, the system of neighborhood filters, a category base would be the class of pairs (S, C), C closed, and the associated maps.

1.3. We say that a homology theory H is admissible on the multiplicative system Σ if it is admissible on some category base of Σ. Note that singular theory or partially exact Čech theory [E-S, IX] is admissible on an arbitrary multiplicative system.

2. The Add $A(S, \Sigma, H)$

2.1. Let $\mathcal{F}$ be a filter of the multiplicative system Σ and let $H^k(\mathcal{F})$ be the principal k-group of $\mathcal{F}$. Filters $\mathcal{F}, \mathcal{G}$ of Σ are said to be k-equivalent if their principal k-groups are isomorphic.

≠ Another way of expressing this: Σ_n is the multiplicative system generated by the set of neighborhood filters of points.

If $\mathcal{F}$ and $\mathcal{G}$ are k-equivalent for each k, they will be called H-equivalent; another way of stating this equivalence is to say that the following sequences, to be called H-sequences,

2.1a $\quad \cdots \to 0 \to H^k(\mathcal{F}) \to H^k(\mathcal{F}) \to 0 \to H^{k-1}(\mathcal{F}) \to H^{k-1}(\mathcal{F}) \to 0 \cdots$

$\quad \cdots \to 0 \to H^k(\mathcal{G}) \to H^k(\mathcal{G}) \to 0 \to H^{k-1}(\mathcal{G}) \to H^{k-1}(\mathcal{G}) \to 0$

are isomorphic.

2.1c. Let the H-sequence 2.1a be denoted $H(\mathcal{F})$. Note that $H(\mathcal{F})$ and $HS(\mathcal{F})$ (I 3.5) are quite distinct sequences.

2.2. Either k-equivalence or H-equivalence effects a class decomposition of the filters of Σ. We shall concern ourselves principally with H-equivalences; the algebraic processes which we shall define on these H-equivalences are definable in exactly the same way for k-equivalences.

Let $\mathcal{F}^\#$ denote the H-equivalence class of Σ containing $\mathcal{F}$. Let $A(S, \Sigma, H)$ or simply A denote this set of H-equivalence classes.

2.3. The sum $\mathcal{F}^\# + \mathcal{G}^\#$ of elements $\mathcal{F}^\#$ and $\mathcal{G}^\#$ in $A(S, \Sigma, H)$ is defined if and only if there exist representatives, $\mathcal{F}, \mathcal{G}$ of $\mathcal{F}^\#$, $\mathcal{G}^\#$ such that $H(\mathcal{F} \cdot \mathcal{G})$ is isomorphic to the cartesian direct sum $H(\mathcal{F}^\#) \oplus H(\mathcal{G}^\#)$. This condition may be symbolized by the equation

$$\mathcal{F}^\# + \mathcal{G}^\# = (\mathcal{F} \cdot \mathcal{G})^\# .$$

Thus the sum of two elements $\mathcal{F}^\#, \mathcal{G}^\#$ of $A(S, \Sigma, H)$ is either undefined or is a uniquely determined element $\mathcal{F}^\# + \mathcal{G}^\#$ in A.

2.3a. Example. Suppose S is a closed 2-cell and $\Sigma = \Sigma_N$ is the multiplicative system of neighborhood filters of subsets of S. Then the neighborhood filters $\mathcal{N}(x)$ of points in the interior will lie in a single equivalence class $\mathcal{N}^\#(x)$, and the neighborhood filters $\mathcal{N}(b)$ of points on the boundary will lie in an equivalence class $\mathcal{N}^\#(b)$. The sum

$\mathcal{U}^{\#}(b) + \mathcal{U}^{\#}(x)$ can be formed: The sum is the class of neighborhood filters of pairs $b \cup x$, where b is a boundary and x an interior point. Likewise the sum $\mathcal{U}^{\#}(b) + \mathcal{U}^{\#}(b)$ (as well as $\mathcal{U}^{\#}(x) + \mathcal{U}^{\#}(x)$) can be formed but the representative of the first summand must differ from that of the second. The element containing the neighborhood filter $\{S\}$ can likewise be added to itself. A representative of $\{S\}^{\#} + \{S\}^{\#}$ would be, for example, the neighborhood filter of two disjoint 2-cells in S.

2.3b. Example: On the other hand, if S were S^2, the 2-sphere, then the element of $A(S^2, \Sigma_N, H)$ containing the neighborhood filter S^2 (it contains only this single filter) could not be added to itself, or to any other element.

2.4. With this law of composition, the set $A(S, \Sigma, H)$ is seen to be an <u>add</u> as defined by Reinhold Baer [B, 625] in his investigations on the "free sums of groups and their generalizations":

"An <u>add</u> is a set A of elements subject to a composition called addition and which meets only the following requirement:

"<u>The sum of the elements a and b in A is either undefined or else it is a uniquely determined element a + b in A</u>."

2.5. In exactly the same manner, the set $A^k(S, \Sigma, H)$ of k-equivalences of Σ is converted into an add, called the <u>kth add of (S, Σ, H)</u>.

3. Properties of the Add

This add $A = A(S, \Sigma, H)$ is seen to have the following properties, denoting elements of A by a, b, c, ...,

3.1. <u>Commutativity</u>. a + b is defined if and only if b + a is defined, and in this event, they are equal:

$$a + b = b + a$$

3.2. If the sums a + b + c, (a + b) + c, and a + (b + c) exist in A, then (a + b) + c = a + (b + c).

3.3. If the sums a + b and b + c exist in A, then the existence of (a + b) + c is necessary and sufficient for the existence of a + (b + c). Conditions 3.2 and 3.3 are the "postulates of associativity" of Baer.

3.4. A zero element does not necessarily exist. For example, let S be the space consisting of a single point x. Attach to the filter $\{S\}$ the 0th (augmented) homology group: this filter is the only filter of the whole space; the add consists of this single element $(\{S\}, H^0(S))$ and is evidently not a zero element. In fact it cannot be added to itself.

3.5. There exist no inverses whatever.

3.6. There may exist elements a in A which cannot be added to themselves, i.e., for which a + a is not defined. Such elements will be of special importance.

3.6a. Definition. An element a of A will be termed sterile if a + a is undefined. These particular add elements will be used in Chapter III, "Applications to Manifolds".

3.7. The properties 3.1 - 3.6a are valid for an arbitrary space S, additive system Σ of S and homology theory H admissible on Σ. Henceforth in this paper the term add will always mean an add satisfying these properties.

3.8. Definition. A prime of A is an element a which cannot be written as the sum $a_1 + a_2$ of two elements of A.

3.9. If the n-th but not the (n+1)-th sum of an element with itself is defined, the element shall be said to have caste n. Sterile elements are said to have caste 0.

4. The Add Structure

An alternative description of the add $A(S, \Sigma, H)$ will first be given.

4.1. $a(S, \Sigma, H)$. Let Σ be a multiplicative system of filters of a space S and H a homology admissible on Σ. We may say that two elements $\mathcal{F}, \mathcal{G} \in \Sigma$ can be added if $H(\mathcal{F} \cdot \mathcal{G}) \cong H(\mathcal{F}) + H(\mathcal{G})$; the filter $\mathcal{F} + \mathcal{G}$ of the sum is $\mathcal{F} \cdot \mathcal{G}$ and its group is $H(\mathcal{F} \cdot \mathcal{G})$. This definition induces an add structure on Σ, which we shall denote

$$a(S, \Sigma, H).$$

It might be expected that each element of $a(S, \Sigma, H)$ would be sterile, but this is not necessarily the case. It is not for example, if $H(\mathcal{F}) = 0$ for some $\mathcal{F}$, or if $H(\mathcal{F})$ is isomorphic to the cartesian product of a given sequence (or group) a countable infinity of times.

4.2. A mapping h of an add A into an add B is an add-homomorphism if in A, $c = a + b$ implies that $h(c) = h(a) + h(b)$ in B. Note however that if the sum $a + b$ is not defined in A, it might be the case that $h(a) + h(b)$ is defined. Thus an add-homomorphism does not preserve sterility. The homomorphic image of an add, however is always an add.

4.2a. A subadd C of an add A is a subset which under the composition induced by A is itself an add. We have the injection homomorphism of C into A and, as noted above, sterility of an element is not necessarily preserved under such a mapping. Thus it would be appropriate to use the terminology "sterile in C". If an element a is not sterile, i.e., if $a + a$ is defined, we shall say that a is fertile. Thus if C is a subadd of A it is possible for a to be sterile in C but fertile in A.

4.3. The natural homomorphism. In $a(S, \Sigma, H)$ let us identify all elements having isomorphic H-sequences. Let $\mathcal{F}^{\#}$ denote such

identification. We say (as in 2.3) that elements $\mathcal{f}^{\#}$, $\mathcal{g}^{\#}$ can be added if and only if there are elements in the identifications, $\mathcal{f}^{\#}$, $\mathcal{g}^{\#}$ such that $\mathcal{f} \cdot \mathcal{g}$ is defined in $a(S, \Sigma, H)$. Under this composition, the collection of the elements $\mathcal{f}^{\#}$ is seen to coincide with the add $A(S, \Sigma, H)$ defined in 2.3. We now have the natural homomorphism

$$\mathcal{f} \longrightarrow \mathcal{f}^{\#}$$

$$a(S, \Sigma, H) \longrightarrow A(S, \Sigma, H)$$

Under this mapping sterile elements frequently become fertile.

4.4. Suppose that Σ' is a multiplicative subsystem of the multiplicative system Σ, and suppose that H is admissible on Σ, hence on Σ'. Then we have a natural injection of $A(S, \Sigma', H)$ into $A(S, \Sigma, H)$, which is an add homomorphism.

4.5. Let S be a space and H a homology theory. Let $\mathcal{a} = \{\Sigma_\nu\}$ be the collection of multiplicative systems on which H is admissible; they are partially ordered by inclusion. They are inductively ordered [NB1, 36], for if $[\Sigma_\nu]$ is a simply ordered subcollection then the union of this collection belongs to $\mathcal{a}$. We may therefore apply Zorn's lemma to deduce the existence of a maximal additive system $\Sigma(H)$ upon which H is admissible. There seems to be no guarantee in general, however, of its uniqueness. Nevertheless for a homology theory (e.g., singular theory or Čech theory [E-S, IX]) valid for arbitrary pairs, the maximal element is unique: It is $\mathcal{\Omega}$, the additive system of all filters of S.

5. Topological Invariance of $A(S, \Sigma, H)$

Throughout, E, E' denote arbitrary sets; Ω, Ω' multiplicative systems of all filters of E, E'; and Σ, Σ' arbitrary multiplicative subsystems of E, E'.

5.1. Lemma. If f is an arbitrary mapping of a set E onto a set E', then (1) there is induced a mapping $\vec{f}$ of Ω onto Ω' which is a homomorphism of the multiplicative systems, and (2) there is induced a mapping $\overleftarrow{f}$ of Ω' into Ω, also a homomorphism.

Proof. (1) Let $\mathcal{F} \in \Omega$, $\mathcal{F} = \{F_\nu\}$. Consider the collection $\{f(F_\nu)\}$ of images of elements of $\mathcal{F}$. To show that this collection generates a filter in E', it is necessary and sufficient that each finite intersection of these sets is nonempty; Consider a set $f(F_{\nu_i})$, $i = 1, \ldots, n$; by well known properties of arbitrary maps of one set into another,

$$\cap f(F_{\nu_i}) \supset f(\cap F_{\nu_i}) \neq \emptyset.$$

Thus f induces a map $\vec{f}$ of filters of E into filters of E',

$$\vec{f}: \Omega \longrightarrow \Omega'.$$

This mapping is onto, for if $\mathcal{F}' \in \Omega'$, $\mathcal{F}' = \{F'_\nu\}$, then $\{f^{-1}(F'_\nu)\}$ generates a filter in E, for $\cap f^{-1}(F'_{\nu_i}) \neq \emptyset$. Furthermore, it is a homomorphism of the additive systems, for if $\mathcal{F} \cdot \mathcal{G} = \mathcal{H}$, then $\vec{f}(\mathcal{H})$ is generated by the same sets: $f(F_\nu \cup G_\mu) = f(F_\nu) \cup f(G_\mu)$ as $\vec{f}(\mathcal{F}) \cdot \vec{f}(\mathcal{G})$, $F_\nu \in \mathcal{F}$, $G_\mu \in \mathcal{G}$; thus

$$\vec{f}(\mathcal{F} \cdot \mathcal{G}) = \vec{f}(\mathcal{F}) \cdot \vec{f}(\mathcal{G}).$$

(2) As shown above, f induces a well defined mapping $\overleftarrow{f}$ of Ω' into Ω. This mapping is also a homomorphism, following from the equation $f^{-1}(F'_\nu \cup G'_\mu) = f^{-1}(F'_\nu) \cup f^{-1}(G'_\mu)$, Q. e. d.

5.1a. The following example shows that $\overleftarrow{f}$ is not in general onto: Let $E = \{x, y\}$, $E' = \{x\}$, $f(x) = f(y) = x$. Then $\overrightarrow{f}$ maps each of the filters $\mathfrak{F}_0 = \{x \cup y\}$, $\mathfrak{F}_1 = \{x, x \cup y\}$, $\mathfrak{F}_2 = \{y, x \cup y\}$ onto $\mathfrak{F}'_0 = \{x\}$, but $\overrightarrow{f}(\mathfrak{F}'_0) = \mathfrak{F}_0$, and $\Omega' = \{\mathfrak{F}'_0\}$.

5.2. Corollary. If f is 1-1 onto, then $\overrightarrow{f}$ is an isomorphism onto and $\overrightarrow{f} \cdot \overleftarrow{f}$ = identity. This says that each set of the same cardinality has the "same" multiplicative system of filters; a multiplicative system is thus a purely set-theoretic property.

5.3. Corollary. If Σ is a multiplicative subsystem of Ω, then $\overrightarrow{f}(\Sigma)$ is a multiplicative system of Ω' and $\overleftarrow{f} \cdot \overrightarrow{f}(\Sigma) = \Sigma$; if f is 1-1 onto, $\Sigma \cong \overrightarrow{f}(\Sigma)$.

5.4. Let S, S' denote topological spaces.

Let $\mathfrak{F}$ be a filter of S. The principal k-group $H^k(\mathfrak{F})$ is the direct limit group of the directed system $\{H^k(S, \mathfrak{F}_\nu), i^\nu_\mu\}$, which by the axioms [E-S, 11] is invariant under homeomorphisms of S. Hence f maps H-equivalences (k-equivalences) into H-equivalences (k-equivalences),

$$\mathfrak{F}^\# \longrightarrow (\overrightarrow{f}(\mathfrak{F}))^\#.$$

Hence by 5.3, this mapping $f^\#$ is an add-isomorphism of $A(S, \Sigma, H)$ onto $A(S', \Sigma', H)$; similarly for $A^k(S, \Sigma, H)$.

5.5. Invariance theorem. Let f be a homeomorphism of S on S'; let Σ be a multiplicative system of S; let H be a homology theory admissible on Σ; and let $\Sigma' = \overrightarrow{f}(\Sigma)$ be the multiplicative system of S' which is the isomorphic image of Σ under the induced $\overrightarrow{f}$. Then H is admissible on Σ' and

$$A(S, \Sigma, H) \cong A(S', \Sigma', H)$$

$$A^k(S, \Sigma, H) \cong A^k(S', \Sigma', H)$$

$$a(S, \Sigma, H) \cong a(S', \Sigma', H)$$

$$a^k(S, \Sigma, H) \cong a^k(S', \Sigma', H).$$

CHAPTER III

APPLICATIONS TO MANIFOLDS

1. Examples in General Spaces

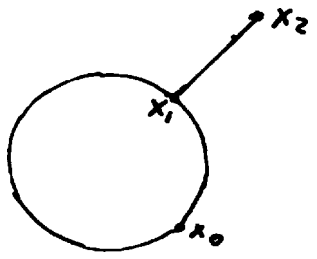
We shall first give a few simple examples of add-structures of general spaces to illustrate how their abnormalities are frequently non-trivially represented in an appropriate add-structure.

We shall here restrict ourselves to the subadds $A(S, \Sigma_n, H)$, where Σ_n is the multiplicative system of neighborhood filters of points of S, and H is, say, nonaugmented Čech theory.

1.1. For the 1-sphere S^1, this add has a single prime element $\mathcal{U}^{\#}(x_o)$, where x_o is any point of S^1; this element has caste ∞, for $\mathcal{U}^{\#}(x_o)$ can be added to itself "indefinitely" many times. We may say that $A(S^1, \Sigma_n, H)$ is generated by this prime element, denoting this fact by the equation

$$A(S^1, \Sigma_n, H) = \{\mathcal{U}^{\#}(x_o)\} .$$

1.2. Consider the space M shown in the figure below consisting of a 1-sphere S^1 to which has been adjoined an arc I meeting S^1 only at one of its endpoints



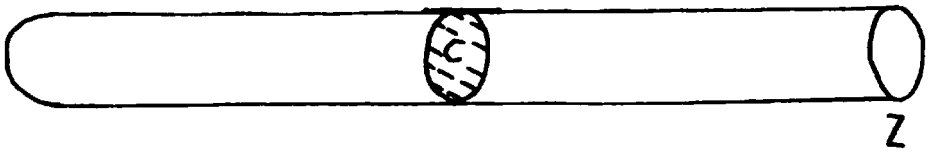
Then $A(M, \Sigma_n, H)$ has exactly three distinct primes; denoting the elements $\mathcal{U}^{\#}(x_i)$ by a_i, a_o has caste infinity; a_1 and a_2 have caste zero, i.e.,

they are sterile. Thus

$$A(M, \Sigma_n, H) = \{a_0, a_1, a_2\} .$$

1.3. Let M* be the union of two such spaces M, M' disjoint; then $A(M^*, \Sigma_n, H)$ will be generated by three elements, a_1, a_2, of caste 1 and a_0 of caste ∞.

1.4. Let Z be the surface of a cylinder in E^3 to which has been adjoined a 2-cell C whose boundary is a circle on the cylinder; see the figure below. Let us consider the additive system Σ_N of neighborhood filters of (closed) sets of Z. If $\mathcal{N}(C)$ is the neighborhood filter of C, $\mathcal{N}^{\#}(C)$ will be a sterile-prime, of caste 0. The subadd $A(Z, \Sigma_n, H)$ of $A(Z, \Sigma_N, H)$ has exactly 2 generators, of caste ∞.



1.5. If Z* is the union of k such spaces Z_i pairwise disjoint, then $\mathcal{N}^{\#}(C_i)$ will be a prime of caste k.

2. Manifold Classifications

It will now be our purpose to indicate how various add structures can be utilized in the problem of manifold classification.

2.1. The van Kampen example is described in Wilder's book [W, 245] . It is formed by deleting from a 3-sphere an ω-sequence of open 3-cells converging to a point p, and replacing each of these by a Poincaré space, identifying the boundary of each "void" in S^3 with the boundary of the Poincaré space inserted. The resulting space vK is a generalized closed manifold, a 3-gcm, orientable [W, 244] , but it has the property

that each neighborhood U of p has an ω-sequence of homotopically distinct simple closed curves through p in U, each linking one of the Poincaré spaces, that are not contractible to a point in U.

The group $\pi_1(U_\nu, p)$, $U_\nu \in \mathcal{U}(p)$, is a vector space of infinite dimension, and so is $P_1(\mathcal{U}(p)) = \varprojlim \pi_1(U_\nu, P)$. Likewise it may be shown that $P_2(\mathcal{U}(p))$ is also infinite dimensional. The higher groups vanish. Now we may define an add based on Σ_n with equivalences defined in terms of the single group $P_k(\mathcal{U}(x))$, denoting this add as, say,

$$A^k(S, \Sigma_n, P_k) .$$

On the other hand, it would be more convenient to deal with a sequence, and this sequence is

$$\cdots \longrightarrow 0 \longrightarrow P_n(\mathcal{F}) \longrightarrow P_n(\mathcal{F}) \longrightarrow 0 \longrightarrow P_{n+1}(\mathcal{F}) \longrightarrow \cdots$$

for a filter $\mathcal{F}$ belonging to the additive system under consideration. Let

$$HT(\mathcal{F})$$

denote this sequence. In particular, when $\mathcal{F} = \mathcal{U}(x)$, then $P_n(\mathcal{U}(x))$ shall be

$$\varprojlim \pi(V_\nu, x), \; V_\nu \in \mathcal{U}(x) .$$

For p the limit point of Poincaré spaces in vK, the element $\mathcal{U}^{\#}(p)$ of the add $A(S, \Sigma_n, HT)$ has an extraordinary property: It is not sterile, but it is <u>idempotent</u>:

$$\mathcal{U}^{\#}(p) + \mathcal{U}^{\#}(p) = \mathcal{U}^{\#}(p).$$

For S^3, the corresponding homotopy-type add, $a(S^3, \Sigma_n, HT)$, no such elements exist, since the neighborhood filter of each point in S^3 is in

particular homotopically like any other one, so that $\mathcal{W}^{\#}(x) + \mathcal{W}^{\#}(x)$ is always defined but is never equal to $\mathcal{W}^{\#}(x)$: $A(S^3, \Sigma_n, HT)$ has no idempotents.

The following two examples are "manufactured" from wild arcs in S^3 (cf. Fox and Artin [F-A]). These manifolds have both the same homology and homotopy groups, local and global, as S^3; as we shall see, however, it is possible to define add structures which distinguish them from S^3 and from each other.

2.2. The following example has been discussed by M. L. Curtis [C]. In S^3 let X be the knotted arc of example 1.1 of [F-A] , and let M^3 be the space obtained from S^3 by identifying all points of X. Let $f:S^3 \longrightarrow M^3$ be the induced mapping, and let $x^* = f(X)$. Then by Wilder's theorem [W1] , M^3 is a generalized 3-manifold, for f is 2-monotone.

The add structure based on the homotopy-type groups of the preceeding section will not distinguish these spaces, but the following one to be constructed does.

Let S be an arbitrary Hausdorff space and $\mathcal{W}_x$ a neighborhood filter of a point x of S. For each $V_\nu \in \mathcal{W}_x$, $y \in \phi V_\nu$ the group

$$\pi_1(\phi V_\nu, y)$$

is defined. Let y denote the arcwise connected component containing y in ϕV_ν and let $\pi_1(\phi V_\nu, y)$ denote any one of the isomorphic groups associated with y. Let $P_1(\phi V_\nu)$ be the direct product of these groups over all y in ϕV_ν. Then for $V_\nu \subset V_\mu$, the natural injections of $\pi_1(\phi V_\mu, y)$ into $\pi_1(\phi V_\nu, y)$ induce the injection

$$i_\mu^\nu : P_1(\phi V_\mu) = P_1(\phi V_\nu).$$

These groups and mapping define a direct limit system : Let

$$P_1(\emptyset\ \mathcal{W}_x)$$

denote the direct limit group.

For points x of S^3 and for "ordinary" points of M^3, this group will be trivial. For the point x^* of M however the situation is quite otherwise. As a base $\mathcal{L}$ for $\mathcal{W}_{x^*}$ we take the following system of "tubular" neighborhoods: For each integer n, V_n shall consist of the following parts (using Fox-Artin notation): The "endpoint" q of x^* will have a cubical neighborhood with q as center so as to contain D_n but not D_{n-1}. Similarly the endpoint p will have a cubical neighborhood containing D_{-n} but not D_{-n-1}. The sections of the arc from D_{-n} to D_n shall lie in a tubular neighborhood of sufficiently small radius r_n such that the tube may be contracted on the arc segments, $r_n < r_{n-1}$, with r_n converging to zero.

Then the computation of $P_1(\emptyset\ \mathcal{W}_{x^*})$ will be seen to coincide with the computation of $\pi_1(S-X)$ in S^3, which has been shown [F-A]to be non-trivial by exhibition of a nontrivial representation in the group of permutations of five letters.

Thus, for S^3 the add $A(S^3,\ \Sigma_n,\ P(\emptyset\ \mathcal{W}))$ will contain only the zero element, but for M^3, the add

$$A(M^3,\ \Sigma_n,\ P(\emptyset\ \mathcal{W})) = \{0,\ a\}$$

where $a = \mathcal{W}_{x^*}$.

2.3. The following generalized 3-manifold is constructed from example 1.3 of [F-A] of a wildly embedded arc whose complement is simply connected but is not an open 3-cell. Let Z denote this arc in S^3 and let N denote the space obtained from S^3 by identifying all points of Z. Let

g: $S^3 \longrightarrow N$ be the induced mapping and let $z = g(Z)$. Again, by [W1], N is a generalized 3-manifold. The local and global homology and homotopy groups of N are isomorphic to those of S^3, and by virtue of the simple connectedness of S^3-Z and the reasoning of the second last paragraph of 2.2 above, it follows that the add structure of $A(N, \Sigma_n, P_1(\mathcal{U}))$ is also trivial. We shall however construct a distinguishing add structure.

We now restrict ourselves to locally arcwise-connected spaces S satisfying the first axiom of countability. Actually we require somewhat more restrictive assumptions, namely that for every neighborhood V_{n-1} of a point x there exists a neighborhood V_n of x such that V_n-x is arcwise connected in $V_{n-1}-x$. For y in V_n-x, the group $\pi_1(V_n-x, y)$ and the injection homomorphism

$$i: \quad \pi_1(V_n-x, y) \longrightarrow \pi_1(V_{n-1}-x, y)$$

is defined [S, 75]. The <u>image</u> π_1^n under i is by the assumptions <u>independent of y</u>, up to an isomorphism. To proceed further we need the following lemma:

2.3.a. <u>Lemma</u>. Let A and B be abstract groups with normal subgroups $\bar{A}$ and $\bar{B}$ respectively. Let θ be a homomorphism of A into B such that $\theta(\bar{A}) \subset (\bar{B})$. Then θ induces a homomorphism θ^* of $A/\bar{A}$ into $B/\bar{B}$.

<u>Proof</u>. Let $a \in A$, $\bar{a} \in \bar{A}$ and $a_* = a\bar{a}$. Then, denoting the image of a under θ by a^θ, we have $a_*^\theta = a^\theta \cdot \bar{a}^\theta$. But since $\bar{a}^\theta \in \bar{B}$, $a_*^\theta \equiv a^\theta \mod \bar{B}$. Thus the mapping θ^* of cosets of A mod $\bar{A}$ into B mod $\bar{B}$ is well defined. That θ^* is a homomorphism follows from the equation

$$[(a\bar{a})\ (c\bar{c})]^\theta = (a\bar{a})^\theta \cdot (c\bar{c})^\theta .$$

Thus the mappings i above induce homomorphisms

$$i_m^n : \pi_1^n \longrightarrow \pi_1^m \qquad m < n$$

and the groups and mappings i_m^n are seen to form an inverse system of groups: Let

$$Q_1(x)$$

denote the inverse limit group.

For each point of S^3 and for each point of N-z, this group will be trivial. To compute $Q_1(z)$ we use a system of the same type of "tubular" neighborhoods V_n of z as for x^* in 2.2 above. It may be seen directly that $Q_1(z)$ is nontrivial, having at least one free element, corresponding to a simple closed curve in V_n-z encircling a segment of the arc exactly once: furthermore such an element a_n lies in each group $\pi_1^n(z)$ and this collection $\{a_n\}$ determines a non-zero element of the limit group $Q_1(z)$. Thus $A^1(N, \Sigma_n, Q) = \{0, a\}$ whereas $A^1(S^3, \Sigma_n, Q_1) = 0$.

3. Adds and Defining Manifolds

We begin with a few additional definitions and facts about adds.

3.1. An element a of an add A is said to be a prime if there do not exist elements b, c $\in$ A, neither 0 if 0 exists in A, such that b + c is defined and a = b + c.

3.2. As previously defined, the caste of an add element is the integer r if the rth but not the (r+1)th sum of a with itself is defined. If the rth sum is defined for every r, we say a has caste ∞ and is free; if it is not free, it is said to be bound. Accordingly primes shall be termed as bound or free. Thus a sterile prime is bound, of caste 0.

3.3. Let P denote the collection of primes of A; let B denote the collection of bound primes of an add A, and F the collection of free primes: $P = B \cup F$, $B \cap F = \emptyset$. An isomorphism between two adds, A, A'

sets up a 1-1 correspondence between their collections B, B' and F, F'. Consequently, the cardinalities β, γ of B, F, respectively of an add A are invariant under (add)-isomorphisms. Likewise, the caste n of any prime is an invariant under isomorphisms.

Every element of A can be written as a (finite) sum of prime elements; this consequence is "built in" to the definition of the collection P of primes. Examples show that the terms of this sum are not in general unique.

If an element a of A is such that

$$ka = a \quad ,$$

where ka denotes the sum of a, k times, then if n is the smallest integer for which this occurs, we say that a is <u>n-potent</u>. In this terminology idempotent reads "1-potent". Clearly the collection I_n of n-potent elements is contained in F.

3.4. As a consequence of the topological invariance of the adds $A(S, \Sigma, H)$ of a space, it follows that the algebraic invariants of this add discussed above, P, B, F, β, γ, and I_n are all topological invariants.

We now return to the study of manifolds.

3.5. <u>Definition of the local homological properties</u> (of a manifold). The ordinary definition of local homological properties of a manifold are given by specifying [W, 244] that the local augmented co-Betti or Betti numbers all vanish except for the dimension n of the manifold: $p^n(x) = \delta_k^n$. If we denote the homology groups corresponding to $p^k(x)$ by $H^k(x)$, an equivalent statement can obviously be made in terms of the sequence

3.5a. $\longrightarrow 0 \longrightarrow H^k(x) \longrightarrow H^k(x) \longrightarrow 0 \longrightarrow H^{k-1}(x) \longrightarrow$

Let $H^*(\mathcal{V}(x))$ denote this sequence. Then a possible definition of the local homological properties would be to state that the add

$$A\ [S,\ \Sigma_n,\ H^*(\mathcal{V}(x))]$$

has a single prime $\mathcal{V}^{\#}(x)$, which is free. Whether one would have to add that the sequence $H^*(\mathcal{V}(x))$ has zeros everywhere except for $n = k$ is related to the problem of Alexandroff, whose general solution is unknown. For S^1 the solution is known.

3.6. Definition of global homological properties of a manifold. Consider the filter $\{S\}$. A (closed) n-dimensional generalized manifold [W, 244, 250] is orientable if and only if the dimension of $p^n(S)$ of $H^n(S)$ is 1; if it is nonorientable, $p^n(S)$ is 0. Consider the subadd generated by the element containing the filter $\{S\}$. Let this element be denoted by a(S).

Proposition. A generalized manifold is orientable if and only if a(S) is sterile (caste 0).

In view of Wilder's important results concerning his property $(P, Q)_n$ [W, 244], it appears hopeful that for a generalized manifold, the subadd $A(S, \Sigma_n, H)$ and that generated by a(S) may be sufficient to compute the properties of the remaining add structure of the manifold.

CHAPTER IV

MORSE THEORY IN GENERAL SPACES

Introduction

Ab natura rei, the assumptions of Morse theory must be made simultaneously on the space S, the function f, and the homology theory H. We state a set of axioms for an arbitrary triple (S, f, H). Essentially algebraic in nature, they are stated in terms of properties of the homology theory H on the category of pairs arising from the function f on S. In the language of Chapter II, they are assumptions of Σ_f and H. In a sense they are minimal axioms for a nontrivial Morse theory.

This approach generalizes previous presentations and serves to clarify the relations between various levels of and approaches to Morse theory; in the last section a detailed comparison is made. It also settles in particular the problem of determining whether Morse critical point theory can be developed in terms of a particular homology theory; cf. for example Yang [Y].

In 1 we discuss various concrete definitions of critical groups, based on singular and Čech theory. In particular for Čech theory we define a group each element of which is in 1-1 correspondence with a k-cap with cap-height c of Morse; it is thus called the k-cap group at level c.

In 2 we define critical "H-sequences"; a real number is termed critical if its corresponding H-sequence is nontrivial.

Article 3 treats critical sets homologically.

Theorem 3.3 states that if $S_c = \{x \mid f(x) \leq c\}$ is compact Hausdorff and H is a continuous homology theory then each critical value determines a homological critical set at level c. This theorem is false if singular theory is used. Theorem 3.6 shows that a homological critical set is a homotopic critical set.

Article 4 defines a homology sequence of f at level c, the basic tool of the subsequent development. Axiom 0 (article 5) states that we are given a triple (S, f, H) consisting of a topological space S, a mapping f of S into E^1 and a partially exact homology theory H which is "admissible" on S and f. It is not assumed, e.g., that f is continuous.

In 6, axiom 1 is stated: The set of numbers c such that S_c carries a nonzero element u of $H^r(S)$ shall have a minimum. Axiom 0' specifies that the homology theory H on the category defined by S and f is exact. On the basis of these axioms the weak Morse relations are derived: $p^r(S) \leq m^{r,f}(S)$.

Axiom 2 (article 7) specifies that if u is a nonzero element of $H^r(S_c)$ which bounds on $H^r(S)$, then there is a minimum value b for which u lies in the kernel of the injection of $H^r(S_c)$ into $H^r(S_b)$. Axiom 3 states the condition of axiom 1 for nonzero elements of $H^r(S_c)$ for all c. On the basis of these axioms alone the strong Morse relations, in the form of isomorphisms, are derived; the finiteness of the Betti numbers and type numbers is not required.

In article 8, general conditions are given under which the axioms hold. Theorem 8.3: Let (S, f, H) be a triple consisting of a Hausdorff space S, a mapping f of S into the reals and a homology theory H continuous

on (S, F) such that (1) each $u \in H^r(S)$ is carried by some S_c, (2) S_c is compact for each c. Then A) all the axioms are satisfied so that B) the Morse relations hold and C) there is a critical set at each critical value.

Several consequences are deduced from this theorem. It has e.g. application to the theory of optimum trajectories of guided missiles. Section 9 provides a detailed comparison of various axiomatizations of Morse theory. Applications are made to minimal surface theory and to the space of geodesics on a Riemannian manifold.

1. Various Definitions of Critical Groups

In Chapter I a general definition is given of the critical group at level c of an arbitrary real-valued function defined on a topological space S. Here we shall give some specific definitions based either on Čech or singular theory.

1.1. Čech theory on compact pairs. Suppose that for each real c, the set S_c is compact, a frequent Morse axiom [M, 147], [M1, 52], [ML, 431]. Then for each e ($e > 0$), (S_c, S_{c-e}) is a compact pair, so that the Čech homology group based on finite coverings defines the group

1.1a. $$H^k(S_c, S_{c-e})$$

(Note, in comparing with definition of I 2.4, that $S_{c-e} = S_c - S_{c,e}$.) We may form the direct limit

1.1b. $$\varinjlim_{e>0} H^k(S_c, S_{c-e}),$$

which group we denote

1.1c. $$H^k(S, f, c).$$

1.1d. A cofinal system of groups can evidently be defined by considering pairs (S_c, K), where K is a compact subset of S_{c-}, hence the direct limit $\varinjlim H^k(S_c, K)$, where the compact subsets K, ordered by inclusions, $K_\nu > K_\mu \Leftrightarrow K_\nu \subset K_\mu$, form the directed sets, will be isomorphic to $H^k(S, f, c)$. (cf. [M, 147]).

1.1e. Lemma. u is a nonzero element of $H^k(S, f, c)$ if and only if u is a k-cap with cap-height c (Morse terminology [M, 147]).

Proof. If $u \in H^k(S, f, c)$, then [E-S, 221] there must exist an e such that $H^k(S_c, S_{c-e})$ contains an element u_e for which

$$i_e(u_e) = u .$$

Here i_e denotes the projection of $H^k(S_c, S_{c-e})$ into the limit group $H^k(S,f, c)$. We may thus identify u with u_e, and u_e satisfies the definition of a k-cap with cap-height c.

Conversely, suppose that v is a k-cap with cap-height c. By definition there exists $e > 0$ and a compact subset $K \subset S_{c-e}$ such that v is an element of a nonzero homology class of $H^k(S_c, K)$. Now $S_c - K$ is clearly an element of the filter of sets $S_{c,e}$ and so is a "coordinate" of the direct limit $H^k(S, f, c)$. Suppose that $i_e v = 0$; then there would exist some coordinate e' for which $i_{e'} v = 0$, which would mean that v was homologous to zero mod a compact subset of S_{c-}, contrary to the definition of k-cap. Q. e. d.

1.2. Čech theory on arbitrary pairs. Since Čech theory is now defined for arbitrary pairs, it would be natural to investigate a definition of critical group at level c for generalized Čech theory on pairs (S_c, S_{c-}):

1.2a. $H(S_c, S_{c-})$,

in the same way in which the critical group is defined with singular theory (cf. 1.3, [S-T, 22] , [M1, 57]). The advantage of this definition is of course its simplicity, but its associated homology sequence, unlike that of $H^k(S, f, c)$, is in general only partially exact. This suggests the use of generalized Čech cotheory on the pair (S_c, S_{c-}), which is always exact. The advantage of the groups of 1.1 is, however, that they satisfy the continuity axiom; cf. Theorem 3.3 below.

1.3. Singular critical groups. Singular theory in defining the critical groups is used by Morse for examining, e.g., critical points of "a positive lower semicontinuous, nondegenerate function on a metric space". The "topological nondegeneracy of the critical points" under these assumptions makes the singular theory adequate [M1, 48] .

Morse has given however a simple example of a compact metric space and continuous real-valued function defined on it which satisfies the hypothesis of "f-accessibility" for Čech and Vietoris theory but not singular theory [M4, 11] and this property is really a sine qua non of critical point theory, (cf. 6.2a and 8 below).

Singular theory is valid on the category of arbitrary pairs [E-S, vii] and so for the pair (S_c, S_{c-}) the group

1.3a. $H(S_c, S_{c-})$

is defined (cf. [ST, 221]); we term it "the singular critical group at level c of f". Its dimension is the kth type-number of f at the value c. The singular homology sequence of (S_c, S_{c-}) is exact.

We also have the critical group formed from the singular groups

1.3b. $$H^k(S_c, S_{c-e})$$

by taking their direct limit

1.3c. $$\varinjlim_{e} H^k(S_c, S_{c-e})$$

as in the definition I given for an arbitrary homology theory. The analogue of this group for singular cohomology theory has been considered by Deheuvels [D] . This group, whether generated from singular homology or cohomology groups, does not have the continuity property, however. Without this property, as Morse's example above shows [M3] , even a function of class C^n on a compact differentiable C^n-manifold, does not satisfy the hypothesis of f-accessibility, and thus our axiom 1. (cf. section 8 below.)

2. Critical H-sequences

For each "level" c we have defined various critical k-groups of f, and now we define critical H-sequences for each of these groups (cf. II 2.1). Suppose then that $H^k(\mathcal{F}_c(f))$ is such a group: The sequence

$$\cdots \longrightarrow 0 \longrightarrow H^k(\mathcal{F}_c(f)) \longrightarrow H^k(\mathcal{F}_c(f)) \longrightarrow 0 \longrightarrow H^{k-1}(\mathcal{F}_c(f)) \longrightarrow$$
$$\longrightarrow H^{k-1}(\mathcal{F}_c(f)) \longrightarrow 0 \cdots$$

is the critical H-seqeunce of f at level c; let it be denoted

$$H(\mathcal{F}_c(f)) .$$

I f this sequence is nontrivial, c is said to be a critical value of f.

3. Critical Sets

3.1. In order to define the critical set of f at level c we shall introduce another operation, called multiplication of filters: If $\mathcal{F}$ and $\mathcal{G}$ are filters, consider the sets $\{F_\mu \cap G_\nu \mid F_\mu \in \mathcal{F},\ G_\nu \in \mathcal{G}\}$;

if these sets satisfy the finite intersection condition they generate a filter, which we shall denote $\mathcal{F} \circ \mathcal{G}$. If $\mathcal{F} \circ \mathcal{G}$ is defined, i.e., if it is a filter, then its terminus is the intersection of the termini of $\mathcal{F}$ and $\mathcal{G}$.

3.2. In particular, consider the filter $\mathcal{F}_c(f)$; let σ be a subset of $S^c = \{x | f(x) = c\}$ and $\mathcal{U}^\sigma$ the neighborhood filter of σ in S_c. Then $\mathcal{U}^\sigma \circ \mathcal{F}_c(f)$ is defined. If the injection $i: H(\mathcal{U}^\sigma \circ \mathcal{F}_c) \longrightarrow H(\mathcal{F}_c)$ is an isomorphism onto, then we say that σ contains a critical set at level c. Our conditions are too general to guarantee the existence of a minimal critical set; however,

3.3. Theorem. If S_c is a compact Hausdorff for each c and the homology theory H is continuous (i.e., Čech theory), then if c is a critical value there exists a minimal critical set at level c.

Proof. Let $\{\sigma_\nu\} = \Sigma$ be the collection of subsets of S^c containing critical sets. We shall assume that the σ_ν are closed in S_c, and hence compact. Let Σ' be a subcollection containing critical sets. We shall assume that the σ_ν are closed in S_c, and hence compact. Let Σ'' be a subcollection of Σ' simply ordered by inclusion. By compactness, the intersection σ of sets in Σ'' is nonempty. Now σ may be considered as the inverse limit of the system Σ'', the mappings of the inverse system being inclusion mappings [E-S, 217]. By the continuity of the homology theory [E-S, 260] (specifically we may use the groups of 1.1),

$$H(\mathcal{U}^\sigma \circ \mathcal{F}_c) \cong \varinjlim H(\mathcal{U}^{\sigma_\nu} \circ \mathcal{F}_c)$$

and the isomorphisms $i : H(\mathcal{U}^{\sigma_\nu} \circ \mathcal{F}_c) \longrightarrow H(\mathcal{F}_c)$ define an isomorphism of $\lim H(\mathcal{U}^{\sigma_\nu} \circ \mathcal{F}_c)$ onto $H(\mathcal{F}_c)$. Thus (cf. [E-S, 259f]) σ is also an element of Σ'.

Thus Σ is inductively ordered [B1, 36] and so by Zorn's lemma there exists a minimal element of Σ, which we shall term a (minimal) critical set at level c.

3.4. An example may be helpful: Let S be the arc in E^2 in the sketch below; let f(x) be the height of x above the baseline. The number c will be a critical value, $S^c = x_o \cup x_1$, $H^1(f^c(f))$ has dimension 1, and all other groups of $f^c(f)$ vanish. Here $\mathcal{W}_c^{x_o} \circ f^c(f) = \mathcal{W}_c^{x_o}$, and $H^1(\mathcal{W}_c^{x_o}) = H^1(f^c(f))$; $x_o = \sigma(c)$.

3.5. Let C be a subset of S^c and U a neighborhood of C in S_c. A local f-deformation [M, 146] is a continuous mapping

$$D: \bar{U} \times I \longrightarrow S_c$$

such that $D(u, 0) = u$ and $f(D(u, t)) \leq f(u)$, $u \in \bar{U}$, $t \in I$. This deformation is called proper if $D(u, t) \neq u$ implies that $f(D(u, t)) < f(u)$. We say that C is homotopically critical if C is fixed under every proper local f-deformation.

3.6. Theorem. A minimal critical set at level c is homotopically critical.

Proof. Let σ be a critical set at level c. Suppose that D is a proper f-deformation of a neighborhood U of σ under which σ is not fixed.

Now D is obviously a homotopy between the identity mapping D(u, 0) and the terminal mapping D(u, 1) of $\bar{U}$ into S_c. By the homotopy axiom [E-S, 12] $H(\mathcal{W}^{\sigma} \circ f) = H(\mathcal{W}^{\sigma_1} \circ f)$, where $\sigma_1 = D(\sigma, 1)$, and if σ is not fixed under this map then it is not minimal, contrary to hypothesis. Q. e. d.

4. The Homology Sequence of f at Level c

In I 3 we defined the homology sequence of a filter f over a homology theory H. Here we deal with the space S_c and the filter $f_c(f)$ on S_c. The sequence we write in the form

2a $$\ldots \xrightarrow{\partial} H^k(\emptyset f_c) \xrightarrow{i} H^k(S_c) \xrightarrow{j} H^k(S_c, \emptyset f_c) \xrightarrow{\partial} H^{k-1}(\emptyset f) \xrightarrow{i} \ldots$$

Let

$$HS(S_c, f) \text{ or } HS(f_c(f))$$

denote this sequence; we assume that it is partially exact, that is, at each stage the image of the incoming homomorphism is contained in the kernel of the outgoing homomorphism.

To simplify our notation, let the groups $H^k(\emptyset f_c)$, $H^k(S_c)$, and $H^k(S_c, \emptyset f_c)$ be denoted H^k_{c-}, H^k_c, and $H^{k,f}_c$ respectively.

For singular theory, as in IV 1.3, H^k_{c-} is simply the group $H^k(S_{c-})$; H^k_c is $H^k(S_c)$, and $H^{k,f}_c$ is $H^k(S_c, S_{c-})$.

For Čech theory as in IV 1.1, H^k_{c-} is the direct limit group of groups $H^k(S_{c-e})$, H^k_c is $H^k(S_c)$ and $H^{k,f}_c$ is the direct limit group of the groups $H^k(S_c, S_{c-e})$.

The sequences corresponding to the other groups introduced in the preceding sections can likewise be identified easily. The above sequence is now written

$$\longrightarrow H^{k+1,f}_c \xrightarrow{\partial} H^k_{c-} \xrightarrow{i} H^k_c \xrightarrow{j} H^{k,f}_c \xrightarrow{\partial} H^{k-1}_{c-} \xrightarrow{i}$$

5. Axiom O

The principal feature on which we wish to fix our attention is not any particular manner of attaching the critical groups or sequences to a function at any level c. The result itself, the homology sequence of the f at level c, will be the basic consideration.

Axiom O is that we are given a triple (S, f, H) consisting of a topological space S, a mapping f of S into E^1, and a partially exact homology theory H which is admissible on the system of filters $\mathcal{F}(S_c)$ in S generated by f. We do not assume, e.g., that S is metric or that f is continuous.

6. Axioms 1, O', and the Weak Morse Relations

6.1. **Definition.** A subset M of S is said to carry an element u of $H^r(S)$ if u lies in the image of the injection $H^r(M) \longrightarrow H^r(S)$, [E-S, 255] .

6.2. Axiom 1. If u is a nonzero element of $H^r(S)$ and C(u) is the set of all c's such that S_c carries u, then the greatest lower bound c of C(u) belongs to C(u); i.e., S_c carries u, [S-T, 24] .

6.2a. Necessity of such an axiom for Morse theory: If a purpose of Morse theory is to relate the critical sets of a numerical function on a space S to the connectivity numbers of S, in the words of Morse [M, 144] , "to associate a critical set σ with each nontrivial homology class" of each homology group of S, it would seem that such an axiom is a sine qua non.

The following example indicates why. Let S be any space and I the open interval $0 < x < 1$ of E^1. Then for any homology theory, the

Betti numbers of S and $S \times I$ are equal:

$$p^r(S) = p^r(S \times I).$$

Now let $F(s, t) = t$, $(s, t) \in S \times I$. There are no critical points on this function, and this is true <u>regardless</u> of the Betti numbers of S. On the other hand, for $S \times \bar{I}$ the situation is otherwise.

As we show, axiom 1 is implied by the axioms for causal theory, the weakest assumptions made by Morse [M, 145]. Morse has also shown, however, that the f-accessibility hypothesis is not always satisfied for singular theory even when S is a compact C^n manifold and f is a function of class C^n.

6.2b. <u>Alternative statement of axiom 1</u>: If u is a nonzero element of $H^r(S)$ and if c is the greatest lower bound of numbers c such that u is in the image of the injection

$$H^r(S_c) \xrightarrow{i} H^r(S), \qquad c = \underline{c} + e, \quad e > 0$$

then u is also in the image of the injection

$$H^r(S_{\underline{c}}) \xrightarrow{i} H^r(S).$$

6.2c. <u>Lower homology bound</u>: $\underline{c} = \underline{c}(u)$ is called a lower homology bound of type r of u, or simply a <u>lower homology bound</u>. It is minimal in the sense that S_c is the smallest set of the class of sets $\{S_c\}$ carrying u.

6.2d. <u>$\underline{c} = \underline{c}(u)$ is unique, since it is a greatest lower bound.</u>

Henceforth we shall assume the homology theory exact, and so we state

6.3. <u>Axiom O'</u>: We are given a triple (S, f, H) consisting of a space S, a mapping f of S into E^1 and a homology theory H exact on the category arising from S and f.

6.4. Theorem. If the homology theory is exact (axiom 0'), then each lower homology bound is critical.

Proof. Reverting to the notation of section 4 we consider the homology sequence

$$\cdots \longrightarrow H_{c-}^{r} \xrightarrow{i_c} H_c^r \xrightarrow{j_c} H_c^{r,f} \xrightarrow{\partial c} H_{c-}^{r-1} \longrightarrow \cdots$$

Let u_c^r be an element of H_c^r which projects on u^r in $H^r(S)$. Suppose that $j_c u_c^r = 0$, i.e., u_c^r is in the kernel of j_c; then by exactness, there would be an element v_{c-}^r in H_{c-}^r whose image under i_c would be in u_c^r. Consequently, S_{c-} would also be a carrier, contrary to hypothesis. Thus $j_c u_c^r$ determines a nonzero element of $H_c^{r,f}$ and so c is a critical value. Q. e. d.

6.5. The weak Morse relations. Consider the r-group

$$H_c^{r,f}$$

of f at level c (arbitrary exact homology theory). Let

$$H^{r,f}$$

denote the cartesian direct sum of these r-groups at all level c,

6.5a. $$H^{r,f} = \oplus\sum_c H_c^{r,f} ,$$

this is clearly isomorphic to the cartesian direct sum of all critical r-groups of f at level c, for "critical" merely means nonzero. Let

$$\underline{H}^{r,f}$$

denote the cartesian direct sum of all groups of f at level c which are lower homology bounds of elements of $H^r(S)$. By theorem 6.4 these values are critical

$$\underline{H}^{r,f} \subset H^{r,f} .$$

6.5b. The rth type-number of f is the dimension

$$m^{r,f}$$

of $H^{r,f}$. If $m_c^{r,f}$ denotes the dimension of the rth (critical) group $H_c^{r,f}$ at level c, then from 6.5a

$$m^{r,f} = \sum_c m_c^{r,f}$$

6.5c. We now consider the diagram

$$\begin{array}{ccccccccc} & & & & H^r(S) & & & & \\ & & & & \uparrow I_c & & & & \\ \longrightarrow & H_{c-}^r & \xrightarrow{i_c} & H_c^r & \xrightarrow{j_c} & H_c^{r,f} & \longrightarrow & H_{c-}^{r-1} & \longrightarrow \end{array}$$

Here I_c denotes the injection homomorphism of H_c^r into $H^r(S)$. We now use the fact that the coefficient group K is a field so that the homology groups are vector spaces; let

$$H_c^r = H_{c*}^r + H_{c,2}^r + H_{c,3}^r ,$$

where $H_{c*}^r = i_c H_{c-}^r$, $I_c H_{c,2}^r = 0$, and $H_{c,3}^r$ is injected isomorphically into $H^r(S)$ under I_c. By exactness, j_c injects $H_{c,2}^r$ and $H_{c,3}^r$ isomorphically into $H_c^{r,f}$. Let

$$I_c H_{c,3}^r = \Delta_c^r(S).$$

6.5d. Lemma: $H^r(S)$ is the direct sum of the groups $\Delta_c^r(S)$,

$$H^r(S) = \dot{\sum}_c \Delta_c^r(S).$$

Proof. By axiom 1, each element of $H^r(S)$ has a lower homology bound c and $H^r(S)$ equals the sum $\sum_c \Delta_c^r(S)$. This sum is direct, for suppose u is an element of $\Delta_c^r(S)$ and $\Delta_{c'}^r(S)$. By 6.2c., $c(u) = c = c'$ so that $\Delta_c^r(S) = \Delta_{c'}^r(S)$. Q. e. d.

6.5e. Let

$$p_c^r(S) = \dim \Delta_c^r(S) .$$

Then by 6.5d,

$$p^r(S) = \sum_c p_c^r(S) .$$

Since $\Delta_c^r(S) \cong H_{c,3}^r$ and since $H_{c,3}^r$ is injected isomorphically into $H_c^{r,f}$, it follows that $p_c^r(S) \leq m_c^{r,f}$ and hence (6.5, 6.5b)

$$p^r(S) \leq m^{r,f} .$$

6.5f. Weak Morse relations: If (S, f, H) is an arbitrary triple consisting of a space S, a mapping f of S into E^1, and a homology theory H admissible on (S, f), satisfying axioms 0' and 1, then

$$p^r(S) \leq m^{r,f},$$

where $p^r(S)$ is the rth Betti number of S and $m^{r,f}$ is the rth type-number of f and S. In more abbreviated notation,

6.5g. $$p^r \leq m^r .$$

7. Axioms 2, 3 and the Strong Morse Relations

For the strong Morse relations two additional axioms are necessary.

7.1. Axiom 2. Let u be a nonzero element of $H^r(S_c) = H_c^r$ for some c and suppose that $I_c u = 0$; let $B(u) = \left\{ b \mid i_c^b u = 0 \right\}$. Then the greatest lower bound b of B(u) is an element of B(u). We term b the upper homology bound of u.

7.1a. Proposition. Each upper homology bound is critical for an exact homology theory (Axiom 0'). If b is the upper homology bound of an element of H_c^r, then b is critical of type r+1. Further, each upper homology bound is unique.

For consider the following portion of the homology sequence of f at level b

$$\longrightarrow H_{\underline{b}}^{r+1,f} \longrightarrow H_{\underline{b}-}^{r} \xrightarrow{i} H_{\underline{b}}^{r} \xrightarrow{j} .$$

If $\underline{b}$ is the upper homology bound of u, then u defines a nonzero element of $H_{\underline{b}-}^{r}$ and lies in the kernel of i. Then by exactness u is in the image of $H_{\underline{b}}^{r+1,f}$ so that $H_{\underline{b}}^{r+1,f}$ is nontrivial. Q. e. d.

7.1b. If an element of $H_c^{r,f}$ lies in the image of j_c, it is said to be of linking type [M1, 61]. The following theorem is the basic tool employed in computing Betti numbers of spaces by means of critical point theory [M1, 66].

7.1c. Theorem. When each critical value is of linking type, then under the assumptions of axioms 0', 1, and 2, the Betti numbers of S and the type-numbers of (S, f) are equal,

$$p^r(S) = m^{r,f}.$$

The proof of this theorem will be deferred to 7.6d, when our notation is more fully developed.

7.2. Axiom 3. Let u be a nonzero element of $H_c^r = H^r(S_c)$ for any c; let $A(u) = \{a \mid S_a \text{ carries } u\}$. Then the greatest lower bound a of A(u) is an element of A(u). Axiom 1 refers only to elements of $H^r(S)$, while axiom 3 refers to elements of H_c^r for any c. These axioms are in fact independent. The number a is called the lower homology bound of u.

7.2a. Proposition. Each lower homology bound a is critical (assuming always axiom 0'); if a is the lower homology bound of u^r having dimension r, then a is critical of type r. Furthermore the lower homology bound a of u^r is unique. For let u^r be the nonzero element with

lower homology bound $\underline{a}$. Then u defines an element of $H^r_{\underline{a}}$. Since $\underline{a}$ is minimal, it is not in the image of the injection $i_a: H^r_{\underline{a}-} \longrightarrow H^r_{\underline{a}}$. Then by exactness it is injected into a nonzero element of $H^{r,f}_a$ by j_a.

Henceforth, $H^r(S)$ will be denoted

$$H^r_\infty .$$

7.3. We begin with a detailed examination of the homology sequence of f at various levels c and mappings from one level to another. We write this sequence HS(S, f, c) in the form

7.3a. $$\cdots \xrightarrow{\partial} H^r_{c-} \xrightarrow{i} H^r_c \xrightarrow{j} H^{r,f}_c \xrightarrow{\partial} H^{r-1}_{c-} \xrightarrow{i} \cdots$$

We must in fact deal with the whole linear continuum of sequences HS(S, f, c), i.e., a sequence HS(S, f, c) for each c and mappings from one to the other. The injection mappings of H^r_a into H^r_c, $a < c$, will be denoted I^c_a:

$$I^c_a : H^r_a \longrightarrow H^r_c .$$

Likewise mappings I^{c-}_{a-} and I^{c-}_a are defined.

Consider thus the spectrum

7.3b.
$$\begin{array}{ccccccccc}
\longrightarrow & 0 & \longrightarrow & H^r_\infty & \longrightarrow & H^r_\infty & \longrightarrow & H^{r-1}_\infty & \longrightarrow \\
 & & & \uparrow I_{c-} & & \uparrow I_c & & \uparrow I_{c-} & \\
\longrightarrow & H^{r+1,f}_c & \xrightarrow{\partial c} & H^r_{c-} & \xrightarrow{i_c} & H^r_c & \xrightarrow{j_c} H^{r,f}_c & \xrightarrow{\partial c} & H^{r-1}_{c-} \longrightarrow \\
 & & & \uparrow I^{c-}_{a-} & & \uparrow I^c_a & & \uparrow I^{c-}_{a-} & \\
\longrightarrow & H^{r+1,f}_c & \xrightarrow{\partial a} & H^r_{a-} & \xrightarrow{i_a} & H^r_a & \xrightarrow{j_a} H^{r,f}_a & \xrightarrow{\partial a} & H^{r-1}_{a-} \longrightarrow
\end{array}$$

Let us first re-examine the homology sequence HS(S, f, c), at a given level c.

7.3c $$\cdots \longrightarrow H_c^{r+1,f} \xrightarrow{\partial c} H_{c-}^{r} \xrightarrow{i_c} H_c^r \xrightarrow{j_c} H_c^{r,f} \xrightarrow{\partial c} H_{c-}^{r-1} \xrightarrow{i_c} \cdots$$

7.3d. H_c^r can be written as the direct sum of four subgroups,

$$H_c^r = H_{c,0}^r + H_{c,1}^r + H_{c,2}^r + H_{c,3}^r$$

where $H_{c,0}^r + H_{c,1}^r$ is the kernel of j_c (hence image $i_c H_{c-}^r$) and $H_{c,0}^r + H_{c,2}^r$ is the kernel of I_c, the projection of H_c^r into H_∞^r. Thus $H_{c,2}^r + H_{c,3}^r$ is injected isomorphically into $H_c^{r,f}$ under j_c, while $H_{c,1}^r + H_{c,3}^r$ is injected isomorphically into H_∞^r under I_c.

7.3e. $\underline{H_c^{r,f}}$. Under j_c, $H_{c,2}^r$ and $H_{c,3}^r$ are injected isomorphically into $H_c^{r,f}$. Let

$$H_{c,0}^{r,f} = j_c H_{c,2}^r ,$$

$$H_{c,1}^{r,f} = j_c H_{c,3}^r .$$

Let $H_{c,2}^{r,f}$ be defined by the equation

$$H_c^{r,f} = H_{c,0}^{r,f} + H_{c,1}^{r,f} + H_{c,2}^{r,f} .$$

The subgroup $H_{c,2}^{r,f}$ is thus injected isomorphically into H_{c-}^{r-1} under ∂_c: Let $H_{c-,0}^{r-1}$ be the image of $H_{c,2}^{r,f}$ under ∂_c,

$$H_{c-,0}^{r-1} = \partial_c H_{c,2}^{r,f} .$$

7.3f. $\underline{H_{c-}^r}$. Let $H_{c-,2}^r$ and $H_{c-,3}^r$ be the subgroups of H_{c-}^r which are mapped by i_c isomorphically onto $H_{c,0}^r$ and $H_{c,1}^r$ respectively; then

$$i_c H_{c-,2}^r = H_{c,0}^r$$

$$i_c H_{c-,3}^r = H_{c,2}^r .$$

By exactness the subgroup $H_{c-,0}^r$ defined by the equation

$$H^r_{c-} = H^r_{c-,0} + H^r_{c-,2} + H^r_{c-,3}$$

is the image of $H_c^{r+1,f}$ under δ_c,

$$_cH_c^{r+1,f} = H^r_{c-,0}.$$

7.3g. Summarizing, the sequence 7.3c can now be written in the following expanded form:

7.3h. $$\ldots \xrightarrow{j_c} (H^{r+1,f}_{c,0} + H^{r+1,f}_{c,1}) + (H^{r+1,f}_{c,2}) \xrightarrow{\partial_c} (H^r_{c-,0}) + (H^r_{c-,2} + H^r_{c-,3}) \xrightarrow{i_c}$$

$$\xrightarrow{i_c} (H^r_{c,0} + H^r_{c,1}) + (H^r_{c,2} + H^r_{c,3}) \xrightarrow{j_c} (H^{r,f}_{c,0} + H^{r,f}_{c,1}) + (H^{r,f}_{c,2}) \xrightarrow{\partial_c} \ldots$$

At each stage, the subgroup in parentheses immediately preceding an arrow is isomorphic to the subgroup in parentheses immediately following the arrow; for example

$$(H^{r+1,f}_{c,2}) \cong (H^r_{c-,0})$$

$$(H^r_{c-,2} + H^r_{c-,3}) \cong (H^r_{c,0} + H^r_{c,1}).$$

Within the parentheses the groups are paired off, e.g.,

$$H^r_{c-,2} \cong H^r_{c,0}\ ,\ H^r_{c-,3} \cong H^r_{c,1}\ .$$

Note that if the second subscript of a group is i, then the group with which it is isomorphic has subscript congruent to i mod 2.

7.4. Examination of $H^{r,f}_c$. We first consider

7.4a. $H^{r,f}_{c,1}$: This group is isomorphic to $H^r_{c,3}$. As we have seen in section 6.5,

7.4b. $$\oplus\sum_c H^r_{c,3} \cong H^r_\infty.$$

Let $H_1^{r,f}$ denote the cartesian direct sum of the groups $H_{c,1}^{r,f}$ for all c,

$$H_1^{r,f} = \oplus\sum_c H_{c,1}^{r,f} .$$

Thus by 7.4b,

7.4c. $$H_1^{r,f} \cong H_\infty^r .$$

This is essentially a restatement of the weak Morse relations.

7.4d. Proof of theorem 7.1c: Since we have shown that 7.4c holds (without the use of axiom 3) we need only show that under the hypotheses of theorem 7.1c, $H_{c,0}^{r,f} = H_{c,2}^{r,f} = 0$. Now the assumption that every element of $H_c^{r,f}$ be of linking type is seen to imply that $H_{c,2}^{r,f}$ must be zero, since this group is disjoint from the image of j_c. It may be termed the nonlinking subgroup of $H_c^{r,f}$. This statement holds for each r. Suppose then that u is a nonzero element of $H_{c,0}^{r,f}$. Then $I_c u = 0$ and by axiom 2, for $b = b(u)$, the upper homology bound for u, $I_c^b u = 0$. Thus $I_c^{b-} u \in H_{b-,0}^r$. Then by exactness there would be a nonzero element u* of $H_{b,2}^{r+1,f}$ such that $\partial u^* = u$. But as we have seen this is the nonlinking subgroup of $H_b^{r+1,f}$ which is zero by hypothesis. Thus $H_{c,0}^r = 0$. Q. e. d.

7.4e. $H_{c,0}^{r,f}$: Let

$$H_0^{r,f} = \oplus\sum_c H_{c,0}^{r,f}$$

and

$$H_2^{r,f} = \oplus\sum_c H_{c,2}^{r,f} .$$

7.5. Theorem.

$$H_0^{r,f} \cong H_2^{r+1,f} .$$

Since $H_{c,0}^{r,f}$ and $H_{c,2}^{r+1,f}$ are isomorphic respectively to $H_{c,2}^r$, $H_{c,0}^r$ so that $H_0^{r,f} = \oplus\sum_c H_{c,2}^r$ and $H_2^{r+1,f} = \oplus\sum_b H_{b-,0}^r$, it will suffice to prove that

$$\oplus\sum_c H^r_{c,2} = \oplus\sum_b H^r_{b-,0} .$$

Consider then first the group $H^r_{c,2}$.

7.5a. Lemma. If u, u' are nonzero elements of $H^r_{c,2}$ and $u + u' \neq 0$, then $b(u + u') = \max(b(u), b(u'))$ and for λ a nonzero element of the coefficient field K, $b(u) = b(\lambda u)$.

7.5b. Lemma. If v, v' are nonzero elements of $H^r_{b-,0}$ and $v + v' \neq 0$, then $a(v + v') = \min(a(v), a(v'))$ and for $\lambda \in K - 0$, $a(v) = a(\lambda v)$.

Consider one of the summands

$$H^r_{c,2}$$

of $H^r_2 = \dot{\sum_c} H^r_{c,2}$. Let $u \in H^r_{c,2}$ and b be its upper homology bound. Let $K_{c,\nu}$ be the kernel of the injection $i_c^{b_\nu}$ of $H^r_{c,2}$ into $H^r_{b_\nu,0}$. Let $L_{c,\nu}$ be the subset of $K_{c,\nu}$ consisting of elements whose upper homology bound is less than b_ν; by 7.7a, $L_{c,}$ is a subgroup of $K_{c,\nu}$, and let

$$K_{c,\nu} = N_{c,\nu} + L_{c,\nu} .$$

Then $N_{c,\nu}$ is the subgroup each nonzero element of which has b_ν as its upper homology bound.

7.5c. Lemma. $H^r_{c,2} = \dot{\sum}_\nu N_{c,\nu}$; in words, $H^r_{c,2}$ is the direct sum of subgroups $N_{c,\nu}$ each nonzero element of which has upper homology bound b_ν, the sum running over the set of all upper homology bounds of elements of $H^r_{c,2}$.

Proof. The sum $\sum_\nu N_{c,\nu}$ is direct, by the uniqueness of the upper homology bound of each element of $H^r_{c,2}$. Let now

$$H^r_{c,2} = \sum_\nu N_{c,\nu} + Q$$

We shall show that $Q = 0$. Let $u \in Q - 0$ and b the upper homology bound of u. Then u can be written in the form

$$u = u_b - v_b, \qquad u_b \in N_b,\ v_b \in L_b.$$

Now since $u \in Q$ and $u_b \in N_b$ then v_b must belong to Q, so that $u - v_b \in Q$ and $u_b = u - u_b$; thus u_b must be 0. Then b cannot be the upper homology bound of u : Contradiction. Then $Q = 0$. Q. e. d.

7.5d. Then for each ν, $N_{c,\nu}$ is injected isomorphically into $H^r_{b_\nu -,0}$. Let $M_{b_\nu -,0,c}$ be the isomorphic image of $N_{c,\nu}$ in $H^r_{b_\nu -,0}$. Note that the lower homology bound of each nonzero element of $H^r_{b_\nu -,0,c}$ is c. Then the set $\left\{ i_c^{b_\nu -} \right\}$ of injections defines an injection isomorphism of $H^r_{c,2}$ onto $\oplus \sum M_{b_\nu -,0,c}$ and hence into $\oplus \sum_{b-} H^r_{b-,2}$; let k_c denote this isomorphism.

7.5e. Lemma. The set k_c defines an isomorphism of $\oplus \sum H^r_{c,2}$ onto $\oplus \sum H^r_{b-,0}$.

Proof. The set $\left\{ k_c \right\}$ certainly defines a homomorphism of $\oplus \sum H^r_{c,2}$ into $\oplus \sum H^r_{b-,2}$. for the image of any element u_c of $\oplus \sum H^r_{c,2}$ is $\sum k_c u_c$. This is first an isomorphism for the kernel is evidently zero. Likewise the mapping is onto, as is seen by (1) effecting a splitting of each level $H^r_{b-,0}$ of $\sum_{b-} H^r_{b-,0}$ into a direct sum of subgroups $N^r_{b-,\nu}$ (each element of which has lower homology bound b, exactly analogous to the decomposition process used above for the groups $H^r_{c,2}$) and (2) recalling that each element of $H^r_{b-,0}$ has a unique lower homology bound. Q. e. d.

These lemmas establish theorem 7.5.

7.6. The strong Morse relations. The principal results obtained thus far in this seacion are 7.4c and 7.5,

$$H_1^{r,f} \cong H_\infty^r$$

and

$$H_0^{r,f} \cong H_2^{r+1,f} .$$

Thus since

$$H^{r,f} = H_0^{r,f} + H_1^{r,f} + H_2^{r,f}$$

we have

$$H^{r,f} \cong H_\infty^r \oplus H_2^{r,f} \oplus H_2^{r+1,f}.$$

For notational convenience let

$$Q^r = H_2^{r,f}$$

and let

$$E^r = Q^r \oplus Q^{r+1},$$

the "excess group" (this name derives from the fact that $E^r \cong H^{r,f} \ominus H_\infty^r$). Note that $Q^0 = 0$. Then from the system of equations

7.6a.
$$E^0 = Q^1$$
$$E^1 = Q^1 \oplus Q^2$$
$$\cdots\cdots\cdots\cdots$$
$$E^r = Q^r \oplus Q^{r+1}$$
$$\cdots\cdots\cdots\cdots$$

we obtain the following system of isomorphisms

7.6b. $$E^r \ominus E^{r-1} \oplus E^{r-2} - \cdots \pm E^0 \cong Q^{r+1}. \qquad r = 0, 1, \ldots$$

In particular, when the groups Q^r and hence E^r have finite dimension for each r, then these isomorphisms may be expressed in terms of the dimensions e^r of these groups:

7.6c. $$e^r - e^{r-1} + \dots (-1)^r e^0 = q^{r+1} \geq 0$$

where q^{r+1} is the dimension of Q^{r+1}. Note that the "excess numbers" e^r may be finite even when the Betti numbers are infinite.

We term 7.6b the Morse isomorphisms or the strong Morse relations. The system 7.6a also has the following consequences

$$E^r \oplus Q^{r-1} \oplus Q^{r+2} \cong E^{r-1} \oplus E^{r+1}$$

$$E^r \precsim E^{r-1} \oplus E^{r+1}$$

$$e^r \leq e^{r-1} + e^{r+1}$$

where "$\precsim$" means "is isomorphic to a subgroup of". Except for 7.6c none of these relations make any finiteness assumptions whatsoever on the dimensions of these groups.

8. A General Theorem

It will first be necessary to restate axiom 1; it is seen to be equivalent to the two following axioms:

8.1. Axiom 1a. For each nonzero element u of $H^r(S)$, for each r, the set C(u) of carrier values of u is not empty.

8.2. Axiom 1b. If C(u) is not empty then the greatest lower bound of C(u) (exists and) belongs to C(u).

8.3. Theorem. Let (S, f, H) be a triple consisting of a Hausdorff space S, a mapping f of S into the reals (not necessarily continuous), and H a continuous homology theory (e.g., Čech) such that (1) axiom 1a is satisfied and (2) S_c is compact for each c. Then A) all the axioms are satisfied so that B) the Morse relations hold and further C) there is a critical set at each critical value.

Proof. We shall first verify axiom 1b above. Let c be a lower homology bound of u, an element of $H^r(S)$. Then by hypothesis, for each $e > 0$, u is contained in the image of the injection $I_e : H^r(S_{c+e}) \longrightarrow H^r(S)$. Thus the collection $\left\{ I_e^{-1}(u) \,\middle|\, e > 0 \right\}$ is a nonempty subset of $\varprojlim H^r(S_{c+e})$. On the other hand, since $S_c = \varprojlim(S_{c+e})$, by continuity of the homology theory, there exists a natural isomorphism

$$p : \varprojlim H^r(S_{c+e}) \longrightarrow H^r(S_c).$$

Then $U = p\left\{ I_e^{-1}(u) \right\}$ is the nonzero coset of $H^r(S_c)$ which maps on u so that axiom 1b is satisfied.

Axioms 2 and 3 are verified in similar fashion: Let $v \in H_c^r - 0$ and let b be its upper homology bound, which is assumed to be finite. To verify axiom 2 we must show that $i_c^b(u) = 0$. The homomorphisms $i_e : H_c^r \longrightarrow H_{b+e}^r$ defines a homomorphism of the inverse system H_c^r, consisting of the single group $\left\{ H_c^r \right\}$ and the identity mapping into the inverse system $\left\{ H_{b+e}^r \right\}$ and thus of their inverse limits. Let k denote this mapping, $k : H_c^r \longrightarrow \varprojlim H_{b+e}^r$. Since $i_e(u) = 0$ for all (positive) e, $k(u) = 0$. By the continuity of H, however, there exists a natural isomorphism

$$1 : H_b^r \cong \varprojlim H_{b+e}^r$$

and thus $1^{-1}k(u) = i_c^b(u) = 0$.

Axiom 3 is verified in a manner strictly analogous to that of axiom 1b. Then under the hypotheses of the theorem, all the axioms hold and so by the results of section 7, the Morse relations 7.4c and 7.6b hold. The last conclusion of the theorem was proven in section 3.3. Q. e. d.

8.4. The following example shows the necessity of axiom 1a in the hypotheses of theorem 8.3: Let $S^1 = \left\{ (r, 0) \,\middle|\, r = 1 \;\&\; -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right\}$;

$-\pi \leq \theta \leq \pi$

let $f((r,\theta)) = \tan^2\theta$ for $-\pi/2 < \theta < \pi/2$ and $f((r,\theta)) = 0$ otherwise. Then for each c, the subspace S_c is compact, but axiom 1a is not satisfied and the Morse relations do not hold for $p^1 = 1$, but $m^1 = 0$.

8.5. Definition. A triple (S, f, H) is said to be f-reducible [M, 147] if each compact subset of S can be "f-deformed" into some subset S_c; this condition specifies that the sets S_c "control" the compact subsets of S.

8.6. Lemma. For a compact continuous homology theory [W,246] f-reducibility implies axiom 1a. This follows immediately from the homotopy axiom. Thus in theorem 8.3 axiom 1a can be replaced by the stronger condition of f-reducibility, which is one of the principal hypotheses of Morse's span theory. (cf. 9.2 below.)

8.7. Corollary. If S is compact Hausdorff, f is continuous and H is a continuous theory then all the axioms are satisfied and each critical value determines a critical set.

8.7a. Corollary. If S is compact Hausdorff of dimension n, and f and H are continuous, then there exist no critical points of type greater than n and $m^n = p^n$. In particular then if S is an n-gcm and f is continuous, the same conclusion holds.

8.8. Corollary. If S is Hausdorff, f is continuous with S_c compact for each c and H is compact Cech theory, then all the axioms and Morse relations are satisfied, and each critical value determines a critical set.

For the image of a compact set is compact in E^1 and hence bounded so that (S, f) are f-reducible and so the corollary follows from 8.6 and 8.3.

8.9. Upper reducibility is a fundamental condition [M, 146] used by Morse to establish the existence of critical sets.

8.9a. Proposition. If (S, f, H) satisfy axiom 1 and the condition of upper reducibility, then the weak Morse relations hold and each critical value determines a critical set.

This proposition follows from 6.4 and theorem 8.1 of Morse [M4, 38].

9. Comparisons of various axiomatizations of critical point theory

Since Morse theory has been developed at various levels, it is appropriate in making comparisons of different axiomatizations to talk in terms of these various levels.

9.1. The Causal Theory as defined by Morse is based on the principal assumptions of f-accessibility and upper reducibility, Vietoris homology theory being used.

9.1a. Definition. S is f-accessible if the following condition holds: If a nonzero element u of $H^r(S)$ is in the kernel of each natural homomorphism

$$H^r(S) \xrightarrow{je} H^r(S, S_{c-e})$$

for each positive e, then u lies in the image of the injection

$$i_c : H^r(S_c) \longrightarrow H^r(S),$$

that is, S_c carries u.

9.1b. Lemma. For a partially exact homology theory (axiom 0), f-accessibility implies axiom 1b (cf. 8.2); for an exact homology theory (axiom 0') these properties are equivalent.

Proof. Let u be a nonzero element of $H^r(S)$, and c the greatest lower bound of C(u). Consider the portions

9.1c. $$\longrightarrow H^r(S_{c+e}) \xrightarrow{i_e} H^r(S) \xrightarrow{j_e} H^r(S, S_{c+e})$$

of the "e-sequences", e > 0. Let u_e be an element of $H^r(S_{c+e})$ such that

$$i_e u_e = u.$$

We recall that our homology theory is assumed to be <u>partially exact</u>, so that $j_e i_e u_e = 0$; hence

$$j_e u = 0.$$

This relation holds for every "e-sequence". This by the hypothesis of f-accessibility implies that S_c carries u.

This is a generalization of the first part of Theorem 2.1 of Morse [M, 146].

We now suppose our homology theory is exact. We refer again to the "e-sequences" 9.1c; exactness implies that the image $i_e H^r(S_{c+e})$ <u>coincides</u> with the kernel $j_e^{-1}(0)$. Thus the "if parts" of the statements of axiom 1a and of f-accessibility are equivalent, and their conclusions are the same. Q. e. d.

In view of the results of section 8 we may introduce what we may call the

9.2. <u>Strong causal theory</u>. Under this theory, S is Hausdorff, the sets S_c are compact for each c, axiom 1a holds and H is a continuous homology theory. See theorem 8.3.

9.3. <u>The span theory</u>. [M, 147] is based on the following assumptions: (S, f, H) is a triple such that (1) S is metric, locally f-connected, f-reducible, f is lower semicontinuous, $0 \leqslant f(x) < 1$, and H is Vietoris theory, while the sets S_c are compact for $c < 1$. These axioms

of course imply the axioms of the "strong causal theory" discussed above and so by theorem 8.3 all of our axioms and the Morse relations (in the form of isomorphisms) are satisfied.

Morse in this theory introduces the notion of the span of a k-cap, which e.g., for an element of some H_c^r which "eventually" bounds would equal $\underline{b}(u) - \underline{c}(u)$; Then he considers groups of k-caps of span greater than $e > 0$ denoting their dimensions by m_e^r. He then proves the remarkable result that under the span assumptions these numbers are finite and that relations of the type exist between the numbers m_e^r and p^r. We shall consider the span theory in a later paper.

9.3a. In their fundamental study of <u>minimal surfaces</u> of general critical types, Morse and Tompkins [M-T] consider a triple (Λ, F, H), where Λ is a class of surfaces in E^n bounding a simple closed curve admitting a representation with derivatives satisfying a Lipschitz condition, this class made into a metric space by an appropriate distance function; F is the Douglas-Dirichlet integral, and H is Vietoris theory. Under the assumptions that the Betti numbers and type numbers are finite, they show that the axioms for the span theory are satisfied. As we have shown, however, to establish the Morse relations it is only necessary to verify the axioms for the "strong causal theory" described above, and thus, e.g., the finiteness assumptions may be dispensed with.

9.4. <u>The nondegenerate theory</u> is the level of Morse theory on which applications have been made to topology. An axiomatization of this theory is given by Morse in [M1] , [M2] . These axioms are:

<u>Axiom I</u>. The metric space S shall be arcwise connected, the

function f positive, and the sets S_c compact for each c.

Axiom II. f shall be reducible at infinity in the sense that each compact subset of S shall admit a weak f-deformation into some subset S_c of S.

A point g of S will be said to have property C if there exists a proper f-deformation D_g of some neighborhood N(g) of g such that: (1) D_g leaves g invariant and deforms N(g) into a topological r-disc K(g), which contains g as an interior point and is below f(g) except for g when $r \neq 0$ and which reduces to g when r = 0; (2) The terminal mapping of D_g, restricted to $K(g) \cap N(g)$ is deformable to the identity on K(g), holding g fast and deforming K(g) - g on K(g) - g.

Axiom III. There shall exist a set (η) of points with the following properties: (1) Each point η shall have Property C. (2) The number of points η below any given constant c shall be finite. (3) Corresponding to any finite constant c there shall exist a proper weak-f-deformation Δ_c of S_c in which the points of (η) in S_c are invariant and all other points of S_c are displaced, while the value of f(E(p)) at the terminal image E(p) of p in S_c varies continuously with p in S_c.

Singular theory is used. Thus axiom 0' is verified.

Our axioms 1, 2, 3, appear as theorems in this theory and are proven by lemmas 28.2 and 28.3 of [M2, 135-139].

Morse has shown[M2] that these axioms, hence a fortiori our axioms, are satisfied for the space W(A, B) of arcs (not necessarily rectifiable) joining two points, not necessarily distinct, on a compact

differentiable manifold, and f is the "Peano J-length" of the arcs. Morse also shows that if $A \neq B$ and $A' \neq B'$ then the spaces W(A, B) and W(A', B') are homeomorphic [M2, 121]. Thus our axioms are seen to be satisfied for the special case of geodesics on a Riemannian manifold.

9.5. The axiomatization of Seifert and Threlfall employ singular theory and thus satisfy our axiom O'. Their axioms I and II are essentially identical to our axioms 1 and 2. For the strong Morse relations, however, they assume that the function J is of class C^2, that S is an n-dimensional manifold with a C^1 Riemannian metric, and that the type numbers and Betti numbers are finite. These hypotheses of course imply all of our axioms.

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