

UNIVERSITY MICROFILMS, INC., ANN ARBOR, MICH.

L. C. Card no. Mic 58-7772

MITCHELL, JR., Walter Edmund.

THE CENTER-LIMB VARIATION OF
THE INTENSITIES OF SELECTED
SOLAR LINES.

University of Michigan, Ph. D., 1958
Astronomy

Negative microfilm received
from University.

THE CENTER-LIMB VARIATION OF THE INTENSITIES
OF SELECTED SOLAR LINES

by
Walter E. Mitchell, Jr.

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1958

Doctoral Committee:

Professor Leo Goldberg, Chairman
Professor Lawrence H. Aller
Professor Orren C. Mohler
Associate Professor William C. Parkinson

THE CENTER-LIMB VARIATION OF THE INTENSITIES OF SELECTED SOLAR LINES

by Walter E. Mitchell, Jr.

The chief aim of this research is to employ center-limb observations of Fraunhofer lines, selected largely for their sensitivity to temperature and electron pressure, in an attempt to refine the model of the solar photosphere derived from limb-darkening observations in the continuum.

Center-limb observations of 14 lines - transitions in Si I and II, Mg I and II, Ca I and II, OI, Fe I, and Sr II between $\lambda 5780$ and $\lambda 21226$ - were obtained at the infrared spectrometer of the McMath-Hulbert Observatory, attached to the Snow horizontal solar telescope of the Mount Wilson Observatory.

The telescope objective is a 24-inch paraboloid of 60-foot focal length which is fed sunlight by a coelostat and second flat and forms an image of the sun 18.2 cm in diameter. The spectrograph, also of speed F/30, is of Pfund design and employed, for this research, a 15000 lines/inch Babcock grating. A predisperser placed in front of the entrance slit reduced scattered light and contamination by overlapping orders. With the exception of the prism in the predisperser and a field lens before the entrance slit, the telescope and spectrograph comprise an all-reflecting system, aluminum surfaces being used throughout.

The observations consist of tracings of intensity versus wave length obtained at several points on the solar disc with the following receivers: 1P21 for $\lambda\lambda 5700-6600$, CsO-Ag for $\lambda\lambda 7000-8000$, and PbS for $\lambda\lambda 10000-22000$. The receivers were used in combination with their several amplifying circuits and a continuous chart recorder. The total absorption in each line was obtained by measuring its area with a planimeter and reducing it to equivalent width with the aid of the dispersion measured on the tracing.

The temperature model to be tested was derived from the limb-darkening measures in the continuum carried out by Pierce at the same installation in 1953 and which extend to within one-half per cent of the limb. It was shown that the limb-darkening may be as well represented by any of a number, or combination, of the exponential integral forms. Though they yield a nearly unique temperature distribution over the observed range, they show some ambiguity outside that range. It is suggested that the deduced variation of boundary temperature with wave length may result from the use of a single exponential integral form for the limb-darkening representation at all wave lengths.

The electron pressure distribution of Aller, Pierce, and Elste was employed, with the empirically determined temperature distribution, to predict the center-limb behavior of the observed lines. The lines were assumed to be formed in pure absorption, and the method of weighting functions was used, in a 48-layer model, to calculate the dip at seven points in the emergent profile. Equivalent widths were obtained from integrated profiles, and the contribution functions of selected points in the line profiles

were computed. The calculations were carried out on an IBM 650 magnetic drum computer.

Isotropic and anisotropic large-scale motions were incorporated into the predictions, and, in all cases, the predicted strength fitted to the observed at the center of the disc. In general, the predicted line strengths exceeded those observed toward the limb, suggesting, alternatively, that the temperature be corrected upward 50° to 230° (increased line absorption) or the electron pressure downward by approximately one-half (decreased continuous absorption) in the outer layers. Consistency of sign in the corrections favors the latter. Consequently, a hydrostatic model was integrated at $\lambda\lambda 8000$, 11000 , and 16000 , using the metals abundance ($\log A = 4.28$) recently determined at the University of Michigan Observatory and heeding the suggested pressure corrections to the Aller-Pierce-Elste model.

A simple anisotropy (1.8 km/sec vertical, 2.9 km/sec horizontal) appears to account rather well for the magnitude but not the character of the center-limb change in certain lines. The character of the change may reflect the influence, on saturated lines, of microturbulence increasing with height.

ACKNOWLEDGMENTS

It is with great pleasure that I express thanks to Dr. Robert R. McMath for the opportunity, afforded me as observer with the Michigan Infrared Project on Mount Wilson, to embark on the observing program for this thesis. I am especially grateful to Dr. Ira S. Bowen, Director of the Mount Wilson and Palomar Observatories, and through whom the loan of the Snow telescope to the McMath-Hulbert Observatory was originally made possible, for the very cordial relations and generous provision accorded me during my stay as observer with the Project on Mount Wilson. I am particularly indebted to Dr. Orren C. Mohler, who gave me invaluable instruction in the maintenance and operation of the solar telescope; to Dr. Gunther Elste, who passed on invaluable experience in high-speed computing methods in astrophysics; and to Mr. Thomas Key Jones, who made important additions to my observations after I left the Infrared Project.

Enlightenment on many of the solar problems touched upon in this work came out of talks with Drs. M. Minnaert, Leo Goldberg, Lawrence H. Aller, Jesse Greenstein, J.-C. Pecker, Orren C. Mohler, A. Keith Pierce, Edith A. Muller, Gunther Elste, Edward A. Spiegel, and John H. Waddell.

The very extensive computations were initiated in the Statistical Laboratory of the University of Michigan, where the staff gave generously of their time, but the opportunity to continue the calculations, while I was teaching at Brown University, came through Mr. Robert S. Dines of the IBM Service Bureau in Boston. Mr. Dines very agreeably put at my disposal, for a period of eight months, an instruction board and its

wiring for the IBM 650. Where schedule permitted, Mr. Dines very cordially allowed me to use the Service Bureau 650. Thus, within a few weeks of my departure from Ann Arbor, I was able to proceed with the calculations. Mr. Phil Hankins of the M. I. T. Instrumentation Laboratory very kindly devoted several hours to the clearing up of technical problems encountered in checking out my wiring of the instruction board.

Subsequently, considerable time on IBM 650 machines was made available through the efforts of Messrs. John O. Martin, Marcel Benjamin, and Brad Schultz, of the American Optical Company, Southbridge, Massachusetts, and Mr. John E. Alman, of the Boston University Statistical Research Laboratory, Boston. I believe I was especially fortunate to have been permitted to carry on pure, though unsponsored, research at these several computing facilities. The great help each of these persons afforded me made the computing venture a pleasant one, indeed.

In the preparation of the thesis, Mr. Charles F. Mitchell gave needed help in the typing of an early draft, and Mrs. Delores Chambers and Mrs. Cellastein Ramey contributed very much to the preparation of tables.

Finally, I am grateful to the members of my committee who have taken great pains to go over this work, in just a few weeks' time, so that it may be acceptable, both scientifically and editorially.

TABLE OF CONTENTS

	Page
ACKNOWLEDGMENTS	ii
LIST OF TABLES	viii
LIST OF ILLUSTRATIONS	ix
INTRODUCTION	1
A. Historical	1
B. The Problem of Solar Structure	2
Chapter	
I. THE SOLAR ATMOSPHERE	6
A. The Equation of Transfer	6
B. The Transition from the Simplified to a More Exact Photospheric Model	8
C. Review of Studies Based on the Continuum	9
D. Preliminary Remarks on Studies of Fraunhofer Lines	11
II. THE OBSERVATIONS	13
A. The Suitability of Solar Lines for This Investigation	13
B. The Sensitivity of Solar Lines	14
1. To electron pressure	14
2. To temperature	14
C. Choice of Lines for This Investigation	16
1. The infrared survey	16
2. The adopted lines	17
D. Instrumentation	17
1. Telescope	17
2. Spectrograph	19
3. Monochromator	19
4. Image rotator	20
5. Guide telescope	20
6. Radiation receivers and associated electronic equipment	20
7. Grating	22
8. General performance of the spectrograph	22
9. Scattered light	23
10. The relation between deflection and intensity	25
E. The Observations	26
1. The method of observation	28

Chapter

II. (Cont.)

	Page
E. 2. Dispersion	28
3. The reduction to equivalent widths	28
III. THE LIMB-DARKENING AND THE SOLAR MODEL	30
A. Discussion of the Problem: the Parameters	
Temperature and Electron Pressure	30
B. Remarks on Studies of the Electron	
Pressure Distribution	31
C. The Method of Limb-Darkening Representation	32
1. The integral equation of transfer	32
2. The integro-exponential functions	34
D. On Obtaining a Temperature Model for Use in	
This Investigation	35
1. The limb-darkening observations	35
2. Solutions of the integral equation for the	
limb-darkening coefficients	42
3. The temperature distribution	43
E. Aspects of the Limb-Darkening Representation by	
Integro-Exponential Forms	46
1. Use of the $(d\tau/dT)$'s in an empirical approach	
to the continuous absorption, electron	
pressure, and chemical composition	47
2. Bichromatic electron pressures	48
3. The use of other integro-exponential forms in	
the limb-darkening representation	50
F. Summary	53
IV. THEORETICAL LINE STRENGTHS FOR THE ALLER-PIERCE-	
ELSTE MODEL	54
A. Theory and Equations	54
B. Local Thermodynamic Equilibrium	56
C. Departures from Local Thermodynamic Equilibrium	57
1. Scattering	57
2. Temperature inhomogeneities	58
3. Differences between radiative and kinetic	
temperatures	59
4. Excitation temperature	61
D. The Assumption of Local Thermodynamic Equilibrium	
in This Investigation	63
1. Scattering	63
a) O I 7772-7774-7775 ($3^5S^0-3^5P$)	65
b) Ca I 6573 ($4^1S-4^3P^0$)	65
2. Temperature inhomogeneities	66
3. Differences between radiative and kinetic	
temperatures	67

Chapter
IV. (Cont.)

	Page
E. The Calculation of Theoretical Equivalent Widths . . .	68
1. Damping	71
2. Turbulence	72
3. Variation in $\Delta\lambda_D$ resulting from the temperature gradient	72
a) Variation in $\Delta\lambda_D$ over the path length of line-formation	72
b) Variation in the mean $\Delta\lambda_D$ across the solar disc.	72
4. The continuous absorption	73
5. The data required for the calculations	74
a) Partition functions	74
b) The atmosphere data and the parameters of the Aller-Pierce-Elste Model	75
(1) Temperature (T)	75
(2) Electron pressure (P_e)	80
(3) The absorption coefficient of the negative hydrogen ion ($\alpha(H^-)$)	80
(4) The weighting functions for pure absorption ($g(x, \mu)$)	80
6. Extent of the calculations	94
7. Contribution functions	94
V. COMPARISON OF THEORY AND OBSERVATION	97
A. Temperature	101
B. Pressure	101
C. On the Neglect of Damping	107
D. On the Choice of the Interpretation of the Observations	108
E. Large-Scale Motions	110
VI. DERIVATION OF A NEW THEORETICAL-EMPIRICAL MODEL . . .	116
A. Solar Pressure Models	117
B. The Influence of Chemical Composition on the Gas and Electron Pressure Distributions	117
C. The Chemical Composition of the Solar Atmosphere . . .	118
D. Data for Construction of Model	120
E. The Calculations	120
1. Uniqueness of the solar model	121
2. On constructing a model from an intermediate depth	122
3. Trial calculations	122
4. The calculation in the boundary interval	123
5. The calculations through the photosphere	124
6. The results	124

Chapter		
VI. (Cont.)		Page
E. 7. Derivation of a "trichromatic" boundary temperature		124
F. Conclusion		127
VII. REMARKS AND SUMMARY OF CONCLUSIONS		129
A. Remarks on Recent Results Concerning:		
1. The empirical determination of departures from LTE		129
2. The empirical determination of the velocity field		130
3. The possibility of back-radiation from the chromosphere		132
4. The location of a temperature inversion in the high photosphere		133
B. Some By-Products of This Investigation		133
1. Auto-ionization		133
2. On the role of the gas and electron pressure distributions in solar electrodynamics		135
3. On the abundances of O, Mg, Si, Ca, Fe, and Sr		135
C. Summary of Conclusions		136
APPENDIX I. A METHOD FOR INTRODUCING SPHERICITY INTO THE ANALYTIC SOLUTION OF THE INTEGRAL EQUATION OF LIMB-DARKENING FOR THE PLANE-PARALLEL ATMOSPHERE		140
APPENDIX II. COMPUTING PROGRAMS CARRIED OUT WITH THE IBM 650 INTERPRETIVE ROUTINE MITILAC		142
REFERENCES		146

LIST OF TABLES

Table		Page
1.	The Adopted Lines	18
2.	Observed Equivalent Widths	27
3.	Limb-Darkening Data	37
4.	Limb-Darkening Coefficients, 3-Term Solution Using E_2 .	44
5.	Data for Calculation of Bichromatic Electron Pressure	49
6.	Behavior of Normalized Source Function in Limb-Darkening Solutions With Integro-Exponential Forms E_1, E_2 , and E_3	51
7.	Weights of the Limb-Darkening Coefficients From the Gaussian Elimination	51
8.	The Mechanism of Line-Formation	64
9.	Partition Functions As $f(\theta)$	75
10.	Temperature Distributions	79
11.	Electron Pressure for Aller-Pierce-Elste Model	81
12.	Absorption Coefficient of Negative Hydrogen Ion	82
13.	Monochromatic Weighting Functions	83
14.	Temperature Sensitivity of Adopted Lines	102
15.	Corrections to Temperature and Pressure in the Photosphere	102
16.	Theoretical Equivalent Widths for O I 7772 With and Without Damping (Anisotropic)	109
17.	Alternative Atmosphere Corrections	109
18.	Abundances	119
19.	Gas and Electron Pressure Distributions	125
20.	Observed Strength Parameters	137

LIST OF ILLUSTRATIONS

Figure		Page
1.	The Exponential Integral Forms E_1 , $E_{1.5}$, and E_2	36
2.	Limb-Darkening Coefficients, 3-term Solution Using E_2 .	76
3.	Comparison of Theory and Observation	98
4.	Contribution Functions	103
5.	Electron Pressure Distributions	126
6.	On the Use of the Eddington-Barbier Relation	131
7.	Flow Diagram for Line Profiles, Equivalent Widths, and Contribution Functions	145

INTRODUCTION

A. Historical

The quest for a full, exact, and consistent account of conditions in the sun commenced with the telescopic observation of sunspots by Fabricius, Galileo, and others in 1611. There followed gradual recognition that the sun shared the characteristics of objects far more remote-- that it was, in fact, one star among many. The structure and evolution of all the stars then became the object of the quest. The sun and stars have since played active and complementary roles in the advance of astrophysics, the nearness of the one providing rich and detailed observations to match the others' great diverseness.

The study of the continuous solar spectrum may be said to have started with its discovery by Newton in 1663, but almost 150 years passed before the solar spectrum was further shown, by Wollaston in 1802, to contain a number of dark lines.

On the other hand, as early as 1752, when Melvill observed bright line spectra in a laboratory flame, the science of terrestrial spectrum analysis began beating an independent path to the conclusion that spectral lines are to be associated with specific substances. This possibility was stressed by Melvill, Sir John Herschel, and Wheatstone, to mention a few, and finally culminated in demonstrations, by Foucault in 1829, and later by Kirchhoff, Bunsen, and others, that substances have characteristic spectra.

If those are the achievements which unified laboratory and

astronomical spectral analysis, then Kirchhoff's announcement of the laws of emission and absorption, in 1859, perhaps may be regarded as the "birth" of astrophysics itself. The first half-century of astrophysics thereafter found solar astronomers chiefly engaged in the measurement and identification of spectral lines by comparison with laboratory sources, and in the observing of solar phenomena, particularly at the time of eclipse. A fairly comprehensive account of the astronomical achievements of that period may be found in Miss Clerke's well-known volume, The History of Astronomy during the 19th Century (Clerke 1893).

The picture of the solar atmosphere at the beginning of the 20th century was as follows: clouds of condensed vapors suspended in a gaseous medium gave rise to the continuous spectrum while an overlying cooler layer of gas produced the absorption lines. At that time the foundations for the theoretical study of stellar atmospheres were firmly laid in the pioneering investigations of Schuster (1905) and Schwarzschild (1906, 1914). The principles of the continuous scattering, absorption, and transfer of radiation in a stellar gas were formulated by them and proved capable of accounting rather well for the existing spectrum observations of the sun and stars.

An excellent review of the work, particularly of Schuster and Schwarzschild, and the chief contributors who have followed, viz., Milne, Eddington, Bohr, Saha, Fowler, and Chandrasekhar, appears in Stromgren's article in Astrophysics (Hynek 1951) and more recent developments are reviewed by Minnaert in The Sun (Kuiper 1953).

B. The Problem of Solar Structure

The detailed structure of the sun is given in terms of the variation of temperature and pressure throughout the atmosphere and interior.

These two parameters are related in a complex and possibly unique way to the chemical composition and the methods of energy generation and transport, so that their precise specification throughout the sun constitutes the solar model. In their influence on temperature and pressure, other properties of the sun, such as magnetic fields and large-scale motions, are implicitly present in a refined empirical solar model, though they may by no means be easily deduced therefrom. The problems of solar hydromagnetics are reviewed by Cowling (1953) and briefly referred to later in this thesis (Section VII-B-2).

The solution of the interior problem has been considerably accelerated by the calculation of theoretical opacities and the determination of nuclear cross-sections, while the gross quantities: luminosity, mass, radius, and chemical composition (and possibly granulation) serve as the only observables. However far the interior problem may progress, and the same may be said of the somewhat controversial chromosphere problem, the intermediate photospheric region provides boundary conditions which are prescribed by an overabundance of direct observations of the line and continuous spectrum. There has been, nevertheless, a disquieting failure to achieve consistent refinements in the model of the photospheric layers themselves.

It occurred to me that some refinement of the solar model might be achieved through the study of the center-limb behavior of selected Fraunhofer lines. The opportunity to explore the observational side of the problem came during my employ as the observer with the University of Michigan Infrared Project on Mt. Wilson, from September 1953 to December 1955. During that time the problem was formulated, the lines chosen, and most of the center-limb observations obtained.

The primary purpose of this investigation is, then, to determine to what extent an empirical model of the solar atmosphere, embracing current views on large-scale motions, can predict the center-limb behavior of Fraunhofer lines selected for their sensitivity to temperature and electron pressure.

This study of the solar atmosphere begins, in Chapter I, with a brief recapitulation of radiative transfer. The transition from the simplified to a more exact photospheric model is described, and a broad view is presented of the considerable work that has been done on both the line and continuous radiation from the photospheric regions. In Chapter II the suitability of Fraunhofer lines for a detailed study of the atmosphere is discussed and the criteria for the choice of lines are determined. There follows, then, a description of: (1) the instrumentation with which center-limb observations of those lines were obtained; (2) the method of observation; and (3) the measurement of equivalent widths for comparison with theory. The exact choice of temperature model to be tested is described in Chapter III, where Pierce's 1953 limb-darkening observations are re-analyzed and methods of limb-darkening representation are compared. In Chapter IV the theory of formation of absorption lines under conditions of local thermodynamic equilibrium is presented and followed by a review of recent work on just that question in the solar atmosphere, and the exact calculation of theoretical equivalent widths for the Aller-Pierce-Elste model is described. The comparison of the line observations of Chapter II and the predictions of Chapter IV is made in Chapter V. The possibility of changing the temperature and/or electron pressure in certain layers to reduce discrepancies between theory and observation is proposed, and the influence of large-scale motions on the shape and

strength of a line is examined qualitatively. In Chapter VI I derive a theoretical-empirical solar model based on the recent determination of the metals abundance, at the University of Michigan Observatory, and in keeping with the electron pressure corrections of Chapter V. Finally, Chapter VII contains, in addition to a summary of conclusions, some remarks on recent work on special problems in the solar atmosphere.

CHAPTER I

THE SOLAR ATMOSPHERE

In this chapter the elements of radiative transfer are reviewed, and a brief survey is given of the chief theoretical and observational contributions to our knowledge of the solar atmosphere.

A. The Equation of Transfer.

Basic to both theoretical and empirical study of the solar atmosphere is the expression which describes the flow of radiation, the equation of transfer. Radiation of monochromatic intensity, I_ν , on passing through an elemental cylinder of thickness dx and cross-section $d\Omega$ suffers a diminution: $I_\nu k_\nu \rho dx / \mu$, where k_ν is the monochromatic mass absorption coefficient of the material, ρ its density, μ the cosine of the angle of emergence of the beam with respect to the outward normal of the star, and x is measured from the surface inward.

The ratio, j_ν/k_ν , of the coefficient of emission to the coefficient of absorption in the elemental volume $d\Omega dx$, determines how much radiation is restored to the beam and is called, appropriately, the source function, which we denote by $\mathcal{J}_\nu$: $j_\nu/k_\nu = \mathcal{J}_\nu$, and $\mathcal{J}_\nu k_\nu \rho dx / \mu$ is the intensity restored to the original beam by emission. With the substitution $d\tau_\nu = -k_\nu \rho dx$, the equation of transfer may then be written:

$$(I-1) \quad \mu dI_\nu / d\tau_\nu = I_\nu - \mathcal{J}_\nu$$

where τ_ν is the monochromatic optical depth measured inward from the surface of the star.

While the amount absorbed from the beam depends on the opacity of the elemental volume and on the intensity of the incident beam, the amount restored to the beam depends on how much is abstracted by whatever means (scattering, absorption, extinction [Houtgast 1942]) from radiation incident on the volume from all directions.

The equation of transfer is, strictly speaking, therefore, an integro-differential equation in I_ν : the intensity restored to the beam is found by averaging the incident intensity over all directions, the coefficient of absorption being regarded as isotropic:

$$(I-2) \quad \frac{\rho dx}{\mu} \left(\frac{1}{2} \int_{-1}^{+1} k_\nu I_\nu d\mu \right) = \frac{k_\nu \rho dx}{\mu} \frac{1}{2} \int_{-1}^{+1} I_\nu d\mu = \frac{d\tau}{\mu} \mathcal{J}_\nu$$

In most treatments of the solar atmosphere the generality (I-2) is not retained, and local thermodynamic equilibrium is assumed (Chandrasekhar 1950, pp. 288-89); that is, in any elemental volume the ratio of emissivity to absorption is assumed to be given explicitly by the Planck function rather than by the integral form above. In short:

$$(I-3) \quad j_\nu/k_\nu = B_\nu(T)$$

and $\mathcal{J}_\nu$, the general source function, is replaced by the Planckian, B_ν . This assumption will be discussed in greater detail in the treatment of line radiation (Section IV-B).

Any volume within which the radiation field has attained equilibrium with the particles through detailed balance may be regarded as a "black-body" enclosure. In such a volume the requirements for a Planckian description of the radiation field, viz., $B(T)$, are met; it is to this volume that "local" refers in the description of local thermodynamic equilibrium (LTE). Implicit in (I-3), the so-called Kirchhoff relation, is the importance of the frequency of collisions in insuring that the Planckian temperature, T in $B(T)$, is also the temperature which describes

the distribution of particle velocities in the gas (Kourganoff 1952, p. 8). This may be regarded as a special and very useful case of the conditions described above where "the amount restored to the beam depends on how much is abstracted by whatever means (absorption, scattering, or extinction) from radiation incident on the volume from all directions" (p. 7, 11. 3-5); the entire radiation field becomes describable by means of the single parameter T (Chandrasekhar 1950, p. 289).

B. The Transition from the Simplified to a More Exact Photospheric Model.

The earliest astrophysical view of the solar atmosphere had it composed of a hot source of continuous radiation, the photosphere, with the absorption lines being formed in an overlying cooler stratum of gas, the "reversing layer". This nomenclature for the region of line-formation, which derives from the striking eclipse phenomenon of the flash spectrum, has helped to perpetuate the somewhat simplified view which is itself embodied in the so-called Schuster-Schwarzschild model.

The earliest measurements of limb-darkening in the continuum, however, were shown by Schuster to require that there be continuous absorption and emission in the gaseous layers overlying the photosphere (Stromgren 1951, p. 173), a view which later became the basis of the Milne-Eddington model.

A compromise between these models came in the form of Eddington's proposal for a layer model in which the selective absorbing region occupies an arbitrary range of depth through the photosphere. A model of the layer type recognizes the fact that the region in which any line is formed depends on the nature of the transition involved: its excitation potential, the ionization potential of the atom, and the strength of the

line. It is, of course, also clear that the region is not sharply bounded, for, in fact, all the temperature-and pressure-dependent quantities influencing line-formation vary continuously throughout the layers of the atmosphere. We will use throughout this paper, therefore, the term "line-forming" to refer to the layers formerly idealized as the "reversing layer". The extent to which they are enmeshed in the continuum-forming region hinges largely on the excitation characteristics of the line.

C. Review of Studies Based on the Continuum.

The study of the structure of the solar atmosphere has been chiefly aided on the theoretical side by the concepts of local thermodynamic equilibrium, radiative equilibrium, and the theory of the continuous opacity, and on the observational side by the measurement of the solar constant, the wave length distribution of energy at the center of the disc, and the monochromatic limb-darkening. Let us survey briefly the highlights of the theoretical and empirical investigations that have been based on these concepts and measurements in recent years (since 1946).

The equation of transfer for integrated radiation may be written as an integral equation in B and a solution for B found in terms of the optical depth in integrated light. It has been solved exactly by Ambarzumian (1944), Chandrasekhar (1947), Menzel (1949) and Placzek (1947a). Implicit is the assumption characterizing the so-called gray atmosphere; namely, that the continuous absorption coefficient is independent of wave length. The source function so derived, when used to calculate the limb-darkening and the emergent net flux, leads to a fair prediction of

both the limb-darkening and the energy-wave length curve (Minnaert 1953, p. 109).

After a suggestion by R. Wildt in 1939 (Wildt 1939a) it was realized that the negative hydrogen ion was the principal source of opacity in the solar atmosphere. This discovery gave enormous impetus to the theoretical investigation of stellar atmospheres in general.

With the introduction of the non-gray solar model, theory was then able to achieve considerably better agreement with observation. In such a model the variation of absorption coefficient with wavelength is deduced from the discrepancy between the observed flux or intensity and that computed in the gray model. This comparison yields throughout the observable spectral region ($\lambda\lambda 4000-24000$) an estimate of the ratio of the monochromatic absorption coefficient to the mean absorption coefficient. Over most of that range the behavior of the ratio corresponds, within the errors of observation, to the wave length variation of H^- absorption at the effective temperature of the sun deduced from the solar constant (Munch 1945, 1946). The ratio, however, turns out nearly everywhere to be less than one, indicating that outside the observed spectral range the absorption coefficient must be considerably greater than the mean.

We have reviewed briefly how a broad analysis has succeeded in predicting rather well the continuous spectrum of the sun using little more than the theory of the gray atmosphere and the absorption of the negative hydrogen ion. The success of the latter has been increasingly well emphasized by a succession of empirical studies of the continuum in which the negative hydrogen ion has been shown to describe rather well the continuous solar spectrum from $\lambda 4000$ to $\lambda 24000$ (Chalonge and

Kourganoff 1946; Barbier 1946; Aller and Pierce 1951, 1952; Peyturaux 1952; Michard 1953).

D. Preliminary Remarks on Studies of Fraunhofer Lines.

The interpretation of Fraunhofer lines, however, has not led to an entirely consistent picture of the conditions in the upper layers of the solar atmosphere. This circumstance results from difficulties of both theory and observation. As we turn our attention from the continuum to the line-forming region, we find that the rapid drop in temperature and density further complicates some problems at the same time that it introduces others: e.g., large-scale motions and possible departures from LTE. There is little wonder, then, that photospheric models, both empirical and theoretical, have been vying in recent years in the interpretation of Fraunhofer lines. This research was, in fact, prompted by the ambiguity among the empirical models themselves. See, for example, Fig. 32 in Claas (1951) and Fig. 12 in Michard (1953).

The practice has been to modify or construct a model which adequately describes the continuum so that in its outer layers it predicts satisfactorily the shapes or strengths of certain Fraunhofer lines. Notable among the models derived in this way are those by De Jager (1952), Bohm (1954), and Pagel (1955). The recent investigations which have a bearing on the present research are reviewed separately in Section (IV-C).

A large number of investigations have been directed at particular aspects of the atmosphere that influence line-formation but not the observed continuum (Bohm 1954, p. 201). The conclusions which may be drawn from these investigations are, very generally, that: (a) the damping is one to two orders of magnitude that of the classical (Minnaert 1953,

p. 159; Rogerson 1957); (b) departures from local thermodynamic equilibrium in the outer layers allow of differences between excitation and ionization temperature (Bohm 1954); and (c) large-scale motions exist with velocities of the order 2 km/sec and a probable anisotropy (Allen 1949, Waddell 1957, Suemoto 1957).

CHAPTER II

THE OBSERVATIONS

In this chapter the atomic lines in the solar spectrum are examined to see to what extent observations of them may be used to refine a temperature-pressure model of the solar atmosphere. Lines are chosen for the investigation in a number of wave length regions; and the instrumentation, method of observation, and reduction of the observations are described.

A. The Suitability of Solar Lines for This Investigation

The suitability of solar lines for the determination of abundances was investigated in detail by Stromgren (1940) in his classic paper, "The Composition of the Solar Atmosphere". For weak and strong lines of neutral and singly ionized metal atoms, he shows the dependence on: (a) the electron pressure, (b) the gas pressure, (c) the hydrogen-metals ratio, (d) the metal-metals ratio, and (e) the continuous absorption coefficient.

In addition to the relationships pointed out by Stromgren, and to which we return in Chapter VI, we are interested in the criteria which indicate to what extent observations of a line at the limb of the sun refer to higher layers (than do observations at the center of the disc). The center-limb behavior of a line constitutes a direct test of the temperature and pressure gradients in the line-forming region: (1) insofar as there is a shift upward in the mean depth of line-formation, and

(2) given that the continuum-forming region is well described by LTE and the observations of limb-darkening. The assumption (2) is made throughout this investigation. We preface remarks on (1) with a discussion on the sensitivity of solar lines to changes in the temperature and pressure model.

The vast majority of solar lines are of atomic origin and correspond, typically, to transitions of low or medium excitation potential in the neutral stage of metal atoms, i.e., atoms which are, for the most part, in the first stage of ionization and have low or medium first ionization potentials.

B. The Sensitivity of Solar Lines:

1. To electron pressure.--For transitions in the neutral stage of metal atoms, the Saha relation shows that the strength of a line is directly dependent on the electron pressure. This dependence is exactly masked, however, by the proportionality of the continuous absorption to the electron pressure.

2. To temperature.--The percentage repopulation, R_X , of a level with respect to all the neutral atoms for an increment in θ is given by:

$$(II-1) \quad R_X \sim 2.3X\Delta\theta \quad (\text{Boltzmann; } X = \text{excitation potential})$$

while the percentage depopulation, R_I , of the level by ionization is given by:

$$(II-2) \quad R_I \sim (2.3I + 2.5/\theta)\Delta\theta \quad (\text{Saha; } I = \text{ionization potential in e.v.})$$

For metal lines of medium and high excitation potential the population (or Boltzmann) factors do not play a critical role in their center-limb behavior for two reasons: (1) any change of temperature affects in opposite ways the populations of level (Boltzmann) and stage (Saha) so that the change is somewhat masked; and (2) the change of temperature

in the optical path, center versus limb, seldom exceeds 15 per cent.

For metal lines of low excitation potential, however, we note that $R_{\chi} \sim 0$ and the depopulation according to (II-2) has full effect; hence, low excitation and resonance lines of metals are in principle very sensitive to changes in temperature.

We return now to the consideration of assumption (1) above, i.e., the upward shift in the mean depth of line-formation from center to limb. The factors determining the location of the mean depth of line-formation are excitation potential and strength. They act oppositely to one another as follows: a greater excitation potential corresponds to greater mean depth of line-formation, and a greater strength to a smaller mean depth.

However, the magnitude of the center-limb change of a given line depends on the gradients of the parameters, temperature and/or pressure, over the depths in which the line is formed.

It is now possible to return to the subject of the highly temperature-sensitive neutral lines of metal atoms mentioned above. Being of zero or low excitation potential, their mean depth of formation is high and in the region where the temperature gradient is small (Plaskett 1955, p. 645). Consequently, though their temperature sensitivity may be high and produce an upward shift, the magnitude of the center-limb change should be small.

The systematic effect of temperature on the continuous absorption is such that a steeper temperature gradient enhances the center-limb change in a line that weakens toward the limb and reduces the center-limb change in a line that strengthens toward the limb.

C. Choice of Lines for This Investigation

It appears, then, that the vast majority of solar lines show little sensitivity to the temperature and pressure distributions (Minnaert 1953, p. 154). For our purpose it seemed advisable to avoid as far as possible the choice of: (1) strong lines, on account of uncertainties of damping and of chromospheric contribution; and (2) resonance and other low-lying lines, on account of the uncertain role of scattering. It was necessary, therefore, to select for this study principally lines of ionized metals and high excitation lines of non-metals. My position as observer with the Michigan Infrared Project entailed principally the re-mapping of the infra-red solar spectrum from $\lambda 7000$ to $\lambda 30000$; hence, I was provided an excellent opportunity to survey that region in detail for features which would be useful in an investigation of the type proposed.

1. The infrared survey.--When the survey was commenced in 1954, the available compilations of already identified lines were those of Mohler (1953), $1.45\text{-}2.5\mu$, and Babcock and Moore (1947), $0.66\text{-}1.3495\mu$. In addition, Dr. Mohler very kindly provided a microfilm copy of his extensive infrared wave length tables (Mohler 1955) in advance of publication. Published maps of the spectral region were available in the photometric atlases of the solar spectrum (Minnaert 1940, Mohler et al. 1950). The near infrared atlas was supplemented with improved maps obtained at the Michigan infrared project after 1950.

The search for hitherto unidentified lines of interest was greatly aided by the infrared multiplet tables of Swensson (1951), the multiplet tables of Miss Moore (1945), and also the atomic energy levels compiled by her (1949, 1952). Attention was paid here principally to the term

schemes of neutral and ionized magnesium and calcium. Wave lengths were calculated for all the likely transitions in Mg and Ca not listed in Moore or Swensson. For these lines, an extensive search was carried out on several hundred high dispersion tracings of the spectral region $\lambda\lambda 5000-25000$. It was felt, in the early stages of the problem, that if a sufficiently wide variety of lines could be found in the spectra of these two atoms, the theoretical strengths of the lines would provide an independent test of the satisfactoriness of the model (Stromgren 1951, p. 227). The hydrogen-like behavior of their ions suggests that the Coulomb approximation (Bates and Damgaard 1949) would be a good one in the calculation of f values of transitions in the ions and probably a usable one in the case of the neutral atoms (Claas 1951, p.35; Woolley and Stibbs 1953, p. 193).

It very soon became clear that the scarcity of lines satisfactory from the standpoint of sensitivity to the solar model would require the use of lines chosen over a broad wave length range and from a number of elements besides Mg and Ca.

2. The adopted lines.--The finally adopted set of lines is listed in Table 1: in column (1), the atom; column (2), the stage of ionization; column (3), the wave length; column (4), the source from which the wave length is taken; column (5), the lower and upper term descriptions; column (6), the lower and upper excitation potentials, χ_l and χ_u ; column (7), the ionization potential, I ; and in column (8), the difference between the ionization potential and the lower excitation potential, $I - \chi_l$. χ and I are expressed always in electron volts in this thesis.

D. Instrumentation

1. Telescope.--The Snow telescope of the Mount Wilson Observatory

TABLE 1
THE ADOPTED LINES

Element	Stage	λ	λ Source ^a	Terms		X_l	X_u	I	I-X _l
				Lower	Upper				
Ca	I	6572.781	1	4 ¹ S	4 ³ P ^o	0.00	1.88	6.11	6.11
		16150.97	2	5p ³ P ^o	5d ³ D	4.53	5.10	6.11	1.58
	II	(8912.1) (8927.4)	3	4 ² D	4 ² F ^o	7.02	8.40	11.87	4.85
Mg	I	15879.54	2	3d ³ D	5p ³ P ^o	5.94	6.72	7.64	1.70
		21225.80	2	5p ³ P ^o	8s ³ S	6.72	7.30	7.64	1.70
	II	7896.378	3	4 ² P ^o	4 ² D	9.96	11.52	15.03	5.07
Si		10914.25	3	3 ² D	4 ² P ^o	8.83	9.96	15.03	6.20
	I	5780.452	1	4s ³ P ^o	5p ³ D	4.90	7.03	8.11	3.21
	II	6371.359	1	4 ² S	4 ² P ^o	8.09	10.02	16.27	8.18
O	I	(7771.954) (7774.177) (7775.395)	3	3 ⁵ S ^o	3 ⁵ P	9.11	10.69	13.61	4.50
Sr	II	10914.88	3	4 ² D	5 ² P ^o	1.80	2.93	10.98	9.18
Fe	I	15878.50	2	e ⁵ D ₂	t ⁵ D ₂ ^o	5.54	6.32	7.89	2.35

^a (1) Moore (1945); (2) Mohler (1955); (3) Babcock and Moore (1947).

was the first installation set up on the Mt. Wilson site by Hale, in 1904, and is described by him in an early paper (Hale 1906) and more briefly by Pierce (1954a). It is a horizontal solar telescope, $F/30$, comprised of a coelostat and second flat each of 24 inches aperture. The beam is delivered a distance of approximately 120 feet to the focusing element of the telescope, a paraboloid of 29-inch aperture and 60-foot focal length. This mirror forms an achromatic image of the sun 18.2 cm in diameter which may be directed, through an aperture in the observing room, onto the face of the spectrograph.

The observing room houses the duplicate infrared spectrometer of the McMath-Hulbert Observatory and associated optical and electronic equipment. This installation in its original form is described by Mohler (1950). Its occasional modifications are described in the annual reports of the University of Michigan Observatory (Michigan 1949-1956).

2. Spectrograph.--In brief, the spectrometer is of Pfund (1927) design consisting of entrance slit, 15-foot paraboloidal collimator, and grating. The collimated beam passes to the grating and is focussed on the exit slit by a decollimator of design identical to that of the collimator. At the exit slit the beam may undergo one additional reflection, when the monochromatic image of the coelostat is focussed, by a small off-axis paraboloid, onto the sensitive surface of the lead sulfide receiver described below.

The mirror surfaces throughout the telescope and spectrograph are of aluminum.

3. Monochromator.--A small prism monochromator is mounted before the entrance slit and provides predispersion of the order of $1000\text{\AA}/\text{mm}$ in the visual. A small field lens is also introduced here in order to provide

full illumination of the collimator.

Calcium fluoride, a crystal with excellent transmission properties throughout the infrared, has been used in the transmitting optical parts of the instrument, namely, the predispersing prism and field lens.

4. Image rotator.--An image rotator is mounted before the monochromator whenever it is desired to make observations with the slit on a particular chord of the solar disc. It was decided to dispense with the use of the image rotator on account of the undesirable loss of light caused by its three internal reflections. Consequently, the observations are confined to an equatorial band, approximately 80° in width, which during the period of the observations was carried across the slit, principally by the changing orientations of the telescope mirrors.

5. Guide telescope.--A one-inch aperture telescope objective was mounted before the slit in such a way as to cast on a white cardboard, opposite the face of the spectrograph, a much enlarged image of the entrance slit. This served the dual purpose of facilitating careful guiding during the recording of the tracings and of locating, with the aid of fiducial markings, the exact point on the disc at which the observation was to be made.

6. Radiation receivers and associated electronic equipment.--The achromatic character of the spectrograph makes it readily adaptable to observation over a broad wavelength range. At the exit slit the beam may be delivered directly to any one of a number of radiation detectors that have been designed for use with the instrument:

- a) for the wavelength range: ultraviolet cut-off to $\lambda 6600$, a 1P21,
- b) $\lambda \lambda 6600-10000$: a cesium-oxide-on-silver (CsO-Ag) gas multiplier,
- c) $\lambda \lambda 10000-32000$: a lead sulfide (PbS) photoconductive cell, and

d) $\lambda\lambda 32000-50000$: a lead telluride (PbTe) photoconductive cell.

Receivers (a) and (b) use respectively voltage and gas multiplication to enhance a flow of electrons which have been freed by incident radiation. The former is of the type developed during World War II which has achieved very wide use in astronomical photometry. The signal from the photomultiplier is amplified by a d. c. amplifier before being fed through an adjustable time constant to a Leeds and Northrup Speedomax recorder. These last two elements are used in the circuit of each of the detectors. The 1P21 proved, in this case, to be most dependable without any refrigeration owing to instability from the condensation of moisture.

The amplifier for (b) was constructed after the design proposed and tested by Kron (1952). The CsO-Ag cell is continuously evacuated and cooled with dry ice during use. Its d. c. signal is first fed to a contact-modulated direct-coupled a. c. amplifier and then to a d. c. amplifier where the signal is made suitable for driving the Speedomax recorder.

The receivers (c) and (d) are photoconductive cells produced by Cashman (1946). The former, which was the one used in this investigation, receives a signal modulated at 1080 cps by a mechanical chopper mounted before the entrance slit of the monochromator. Its output is amplified by a Wilson a. c. amplifier (Kuiper 1947) which in turn feeds the signal to a matching network with adjustable time constant. The cell is cooled with a dry ice-acetone mixture.

Continuous chart tracings of the intensity fluctuations at the exit slit were obtained at any of a number of paper speeds on the Speedomax.

7. Grating.--In the history of the Snow project a succession of several gratings ruled in the Mount Wilson shops in Pasadena, principally under the direction of H. W. Babcock, has been used for the predominantly infrared work. These occasional changes of grating, which are recorded along with other changes in the Observatory reports (1946-1956), have led to systematic increases in resolving power and reduction of scattered light.

The principal dispersing element of the spectrograph during the course of this investigation was a 600 lines/mm Babcock grating (75B) with a ruled face 5 x 8 inches in size and hence capable of accommodating substantially the full collimated beam up to an angle of incidence of 45°.

An 1800 rpm synchronous motor is supported behind and outside of the spectrograph room. By means of belt, shaft, and reduction gears, the grating may thus be made to turn at any of a number of speeds: 0°6, 0°3, and 0°15 per hour. The spectrum is driven across the exit slit along the direction of dispersion.

8. General performance of the spectrograph.--With the present grating a resolving power of 200,000 at 1.2μ is attainable when the mass of air in the spectrograph is reasonably stable (Michigan 1955). By this criterion the quality of observations obtained with grating 75B is fourfold that of the first Michigan atlas (Mohler et al. 1950). The performance of the grating in the very near infrared, i. e. $\lambda\lambda 6500-8500$, has resulted in tracings notably superior to the photographically obtained records in the Utrecht atlas; a number of times, observation of lines formed in a tube of iodine vapor in the region $\lambda\lambda 5500-6000$ has shown separation of features corresponding to a resolving power of 100,000-115,000. The most striking improvement over the Utrecht records,

however, is achieved in the ultraviolet where the atlas was based on provisional observations.

9. Scattered light.--The effect of scattered light, from whatever source, is to cause in the observed intensity an increase which is usually dependent on wave length. If the increment is weakly dependent on wave length, it is called general scattering; if it is strongly dependent on wave length, it is called local scattering. The wave length response characteristics of the receiver set, of course, natural bounds on the breadth of wave length region that can produce contamination. To eliminate stray light from external sources the spectrograph has been completely enclosed in asbestos sheeting. The Cashman cells detect only properly modulated light signals, so that stray light from any other source than the entrance slit is of no consequence. Finally, the prism monochromator reduces considerably the total light admitted to the spectrograph and, thereby, light which may enter into the general scattering.

Estimates of the amount of general scattered light were made by means of repeated tests carried out over the whole accessible wave length range with the aid of the prism monochromator. The purpose of the tests was to ascertain to what extent radiation from all those wave lengths outside the narrow band delivered by grating to receiver contaminates the spectrum at the wave length of interest. The procedure used in estimating the degree of general scattering at a particular wave length consisted of the following: the grating is set so that the wave length of interest falls on the receiver; with the spectrograph slits at their normal size the prism angle in the monochromator is very slowly changed so that the admitted spectral band is replaced by a neighboring

spectral band. None of this light would be expected to be diffracted by the grating to the receiver until light from the next higher or lower order is admitted by the monochromator. On the other hand, general scattered light would persist in giving a deflection for any adjustment of the monochromator. Throughout the near infrared region down to $\lambda 7000$ no general scattered light was detected under these conditions, but contamination at shorter wave lengths rose rather gradually becoming most serious in the blue-green, the maximum of the 1P21 response curve. In this region extreme care had to be used in adjustment of monochromator and slits in order to limit contamination of this type to less than one per cent of the continuum.

A method for estimating the degree of local scattering in the absence of general scattering is now described. Wherever opaque features occur somewhat isolated in a spectral region, local scattering may be regarded as negligible over spectral bands half the width of such a feature. In the infrared region $\lambda\lambda 9000-25000$, there are numerous features, due principally to water vapor and carbon dioxide, which provide an excellent test for the local scattered light. It was found to be negligible for slits even considerably wider than normal. Owing to the absence of opaque features at shorter wave lengths it was necessary, in order to apply the same test, to observe strong features with the sun at large zenith distance. Under these conditions it was possible to estimate with the aid of nearly opaque features the maximum contamination by local scattering. Features with central intensities of only a few per cent, e.g., the oxygen A and B band, did not exhibit the prematurely flattened cores that one would expect with local scattering. For the region $\lambda\lambda 7000-9000$ it was considered to be negligible.

Tests for local scattering in the visual region were aided by the abundance of solar energy at these wave lengths and the high sensitivity of the 1P21 receiver. Slit widths up to 25 times narrower than for the infrared region were able to be used here. The following experiment indicated that, relative to general scattering, the magnitude of local scattering was negligible: in the neighborhood of a strong line, such as H α or Mgb, the purity of the light admitted to the spectrograph was altered by widening or narrowing the entrance slit over the range 2.5 to 25 microns. At the largest of these slit widths the very broad features are completely resolved, so that on going to higher and higher purity, an increasing amount of the surrounding spectral band is removed as a possible contributor to local scattering. The profiles of the well-resolved strong features showed no dependence on slit width.

10. The relation between deflection and intensity.--For the circuit associated with the 1P21 photomultiplier, the linear relation between deflection and intensity has been established a number of times; hence, for the system employed, the assumption was made that the observed deflection was directly proportional to the intensity received.

For the circuit associated with the CsO-Ag gas multiplier, the use of an inverse feedback amplifier assures over-all linearity of the circuit up to and including large signals (Kron 1952, p. 6; 1955, p. 23).

For the circuit associated with the PbS photoconductive cell, the non-linearity of the Wilson amplifier (Mohler 1950) may be offset by supplying to the cell a constant supplementary source of light to boost the zero point up and beyond the toe of the response curve (Pierce 1954b, p. 222; Pagel 1955, p. 495). The equivalent of this may be achieved, however, by applying a constant correction to the measures in the course

of reduction (Mohler 1950). According to Pierce (1954b) the remaining non-linearity amounts to approximately one per cent.

By the method of rotating sectors (Pierce 1954b) the constant to be added to the deflection to place it on a linear scale was found to agree substantially with the value 1.5 cm suggested by Dr. Mohler (personal communication). I added this amount as a constant correction to the observations made with the PbS cell.

Finally, a set of neutral density filters calibrated with the aid of the PbS cell were in turn used to look for non-linear behavior in the CsO-Ag and 1P21 response; none was detected within the one per cent reliability of this method.

E. The Observations

The center-limb observations of the features in Table 1 are summed up in Table 2, where given in column (1) is the wave length of the line and in columns (2)-(8) its observed equivalent width and the probable error associated with a single observation, where more than one observation was made. The number of observations is given in parentheses.

The bulk of the observations were obtained at the Snow telescope by me in the period August-December 1955; these were augmented by a number of observations obtained at the same installation by Mr. Thomas Key Jones in February 1956 and April 1957.

1. The method of observation.--Line observations were made at from 3 to 7 points on the solar disc, from $\mu=1.00$ to $\mu=0.20$. On the 18.2 cm image formed by the Snow telescope, the slit height was limited to 0.6 mm at $\mu=0.20$. At this farthest limb point the geometrical smearing or contamination is always governed by the seeing conditions. Consequently, for the observations near the limb, at $\mu=0.4$, seeing of quality 3 or

TABLE 2
OBSERVED EQUIVALENT WIDTHS^a

Atom	Stage	λ	$\mu = 1.0$	0.8	0.6	0.4	0.3	0.25	0.20
Ca	I	6573	23.4 ± 2.0(6)	26.0 ± 1.4(3)	22.7 ± 1.1(3)	24.7 ± 0.9(3)	24.2 ± 1.5(4)	28.4 ± 0.7(3)	26.0 ± 0.8(4)
	II	16151	74.8 ± 3.3(16)	75.5 (1)	79.2 ± 1.8(8)	88.7 ± 2.2(9)	95.6 ± 2.7(4)	89.8 (1)	98.6 ± 5.4(10)
Mg		8912	95.5 ± 4.7(4)	94.9 ± 0.0(2)	-	-	80.3 ± 1.9(4)	-	77.1 ± 3.5(4)
	I	8927	128.9 ± 6.3(3)	-	-	-	-	-	-
		15880	119.8 ± 8.9(6)	-	144.8 ± 1.1(3)	152.5 ± 17.0(4)	136.1 ± 1.3(3)	144.8 ± 1.9(2)	135.9 ± 2.0(5)
	II	21226	25.7 ± 0.6(2)	-	31.0 ± 1.7(3)	41.6 ± 1.3(4)	37.7 ± 4.2(6)	35.9 ± 1.8(3)	35.3 ± 5.4(2)
Si		7896	23.6 ± 0.6(3)	17.6 (1)	-	10.1 ± 2.1(3)	8.0 ± 0.7(6)	-	-
	I	10914	47.3 ± 3.0(11)	-	37.0 ± 3.4(5)	36.3 ± 4.2(2)	28.2 ± 2.5(8)	27.4 ± 3.9(6)	31.0 ± 4.4(7)
	II	5780	20.9 ± 2.1(6)	-	-	-	28.3 (1)	-	26.6 ± 1.4(4)
O		6371	30.8 ± 1.6(3)	-	-	22.3 ± 0.3(3)	19.4 ± 1.2(2)	-	-
	I	7772	77.9 ± 5.2(6)	-	65.9 ± 3.3(2)	53.8 (1)	56.0 (1)	52.9 (1)	51.5 (1)
		7774	69.4 (1)	-	52.7 ± 1.7(2)	52.1 (1)	42.3 (1)	40.4 (1)	32.7 (1)
Sr		7775	50.9 (1)	-	37.3 ± 2.1(2)	37.5 (1)	33.3 (1)	32.7 (1)	29.0 (1)
	II	10915	122.0 ± 3.7(11)	122.9 ± 4.7(5)	124.2 ± 2.9(5)	126.7 ± 4.2(5)	132.5 ± 2.4(10)	135.3 ± 3.2(8)	145.2 ± 8.4(9)
Fe		15878	84.2 ± 7.0(6)	-	103.7 ± 1.3(3)	116.6 ± 13.9(4)	109.4 ± 2.0(4)	115.9 ± 10.6(2)	102.4 ± 3.7(5)

^a In milliangstroms.

better on the Mt. Wilson solar seeing scale (0-10) was required. With seeing of this quality, oscillations of the limb of the 18.2 cm image are just detectible to the unaided eye at a distance of approximately one meter. The corresponding uncertainty in μ at $\mu=0.2$ is about 10 per cent.

Certain lines were observed in more than one order of the grating, though in general the orders used were as follows: (75B)

$\lambda\lambda 10000-15000$	(-II)	$\lambda\lambda 15000-25000$	(-I)
$\lambda\lambda 7000-1000$	(-III)	$\lambda\lambda 5000-7000$	(-IV)

The line tracings were made of sufficient length to establish with a reasonable degree of certainty the position of the continuum and of the zero signal on both sides of the line. It was usually possible to record two or more features with well-defined central dips for which well-determined observed wave lengths are given and with which the dispersion on the tracing could be measured.

2. Dispersion.--Since various combinations of grating speed, paper speed, and observed grating order were made use of, there resulted upwards of two dozen values for the dispersion measured on the tracings. These values covered the range 0.138 to 1.925 A/mm with one tracing recorded at 3.849 A/mm. In a few instances it was necessary to find a tracing dispersion from the well-defined grating speed or paper speed ratios. For tracings of a given spectral region, variations in the dispersion exceeding a fraction of a per cent were able to be detected when the tracings were examined in overlay. In this respect records made over a period of time on a given observing day showed a high degree of reproducibility; hence, the paper dispersion for such a set was taken as constant.

3. The reduction to equivalent widths.--The equivalent widths in Table 2 were found by running over the area of the trace ten times with

a Lasico desk planimeter and taking 1/10th the reading. The position of zero deflection was usually very well-defined by identical dark readings at beginning and end of trace. In any case, it was linearly interpolated between the ends of the trace. The height of the continuum was measured from the interpolated zero deflection to the point of its intersection with the central axis of the Fraunhofer line.

All blends were resolved graphically on the original trace by bisecting the lines at their central minima and reconstructing their blended wings by symmetry with the unblended sides. Pagel (1955) raises the objection to this practice of the Michigan workers that it is difficult to draw accurately the central axis of a faint line. The resolution in the present records, however, has with few exceptions given a well-defined central minimum to even the weakest lines observed. A point was determined midway between the continuum and the point of maximum intensity in the blended wings; through this point the reconstructed wing was required to pass, with its curvature determined by symmetry with the side opposite the central axis of the line. The following expectation was very well satisfied: when the intensity dips in symmetric profiles reconstructed under these conditions are recombined additively, the observed profile is recovered. This practice is sufficiently accurate where the dip in the blended portion of the profiles is only a few per cent. Judged on the basis of the symmetrical way in which the intensities apportioned themselves out, it appears to provide a perfectly satisfactory resolution of the blends.

CHAPTER III

THE LIMB-DARKENING AND THE SOLAR MODEL

This chapter is primarily concerned with the choice of an empirical temperature model to be used in the calculation of theoretical line strengths. Following a short outline on the state of affairs in observational models, the most complete continuum observations available are analyzed with appropriate theoretical forms to derive a temperature model, and a number of ramifications of the limb-darkening problem are discussed.

A. Discussion of the Problem: the Parameters Temperature and Electron Pressure

The ambiguity among empirical temperature models is well illustrated in a diagram by C. de Jager (1952, Fig. 32) where he compares his own model VII with those of Voigt, Barbier, and Ten Bruggencate, Gollnow, and Jager. The difference among these several models amount to 150° at $\tau = 1.0$ and 350° at $\tau = 0.1$, with the best agreement occurring at $\tau = 0.22$ where the temperature is $5450^\circ \pm 20^\circ$.

Pressure models based on different temperature distributions would be expected, of course, to differ, but the ambiguity among pressure models exists primarily as a result of the difficulty in determining the relative abundances of hydrogen, helium, and the metals. It is customary to include as a metal any atom which should be counted as a contributor of electrons, both by virtue of its low ionization potential and its

large abundance relative to the remaining atoms.

The influence of the chemical composition on the pressure model is discussed in Chapter VI, where the model is obtained at three wave lengths employing abundances derived recently at Michigan (April 1956).

It is clear from a diagram by Michard (1953, Fig. 12) what pressure differences exist among current observational models quite apart from differences in composition. Among the models of De Jager, Pierce, and Aller, and Michard, for example, the differences increase in the following way from the layer at 5000° to the boundary layer: in the gas pressure, 15 per cent to 250 per cent; and in the electron pressure, 5 per cent to 170 per cent.

Well in anticipation of the problem proposed in this thesis, Evans (1947, p. 422) has pointed out that the study of Fraunhofer lines ought to provide a method of attack on the ambiguity which he found in the source function on using, in the limb-darkening representation, a boundary constraint and different degree polynomial expressions in μ . It is the same result that we find below with the exponential integral solutions except for the important difference that they (the exponential integral forms) may be employed to give a nearly unique source function over the observed range together with almost any desired boundary condition. However, we return to that aspect of the problem in Section (III-D-3) when we derive the empirical temperature model to be tested in this thesis.

B. Remarks on Studies of the Electron Pressure Distribution

A few investigations have been made with the aim to get at the electron pressure distribution directly through observation rather than

through integration of the solar model. When continuum observations are so used, the problem is intimately linked up with the method of limb-darkening representation. I have postponed to Section (III-E-3) below, therefore, remarks on continuum studies of this type and discuss now results obtained recently from observations of hydrogen lines.

De Jager (1952, p. 62) has corrected the continuum model of Barbier (1946) with the aid of observed hydrogen line profiles. From their wings he has determined corrections to the temperature distribution in the deeper layers, and from their cores corrections to the temperature distribution in the upper layers. Then, regarding the temperature distribution well established, he has credited the difference between the observed and computed Balmer discontinuity to a small error in the electron pressure. The correction found in this way amounts to only 14 per cent at the center of the disc and is equivalent to a temperature correction of 40° . It was in the region of the discontinuity, however, that his computed hydrogen line profiles differed very greatly from the observed, possibly on account of the difficulty of disentangling the lines in that region.

C. The Method of Limb-Darkening Representation

1. The integral equation of transfer.--The usefulness of the equation of transfer in its integral form (Kourganoff 1952, pp. 25-27):

$$(III-1) \quad I_\lambda(0, \mu) = \int_0^\infty B_\lambda e^{-\tau_\lambda/\mu} d\tau_\lambda/\mu$$

derives from two facts: (1) it is free from the assumption that radiative equilibrium exists in the solar atmosphere, and (2) there exists a variety of analytic forms whose Laplace transforms are known.

To express the second of these in another way: there are a

number of fairly simple analytic forms which one can use for the source function, J_v , and thereby relate it to the observed intensity at any point of the solar disc. In contrast to the situation with more distant stars, the possibility of direct observation of limb-darkening thus frees the study of the sun from an assumption not easily dispensed with in the theoretical model.

In Appendix I will be found a method for modifying in practice the plane-parallel approximation in equation (III-1) for the case of the spherical sun.

In the derivation of an empirical model, the line of attack of H. H. Plaskett (1936, 1941) and his co-workers (e. g., Sykes 1953) has been to choose a form for the source function that in direct numerical integration of the expression (III-1) yields the emergent intensity at each observed value of μ .

A larger number of investigators (Barbier, Chalonge and Kourganoff, Peyturaux, Michard, Pierce and Aller), commencing with Milne (1922) and Lundblad (1923), have made use of known inverse Laplace transforms to obtain the source function. With a simple polynomial expansion for the left-hand side of expression (III-1), the inverse Laplace transform allows one to express the corresponding source function analytically as a function of optical depth. Probably the most familiar among these analytic approaches are the following forms for the emergent intensity:

$$(III-2) \quad I_\lambda(0, \mu) = A + B\mu + C\mu^2 + \dots$$

$$(III-3) \quad I_\lambda(0, \mu) = A + B\mu + C[1 - \mu \ln(1 + \mu^{-1})] \\ + D[1/2 - \mu \ln(1 - \mu \ln(1 + \mu^{-1}))] + \dots$$

whose inverse Laplace transforms are, respectively:

$$(III-4) \quad B_\lambda(T) = A + B\tau + \frac{C\tau^2}{2} + \dots$$

and (III-5) $B_{\lambda}(T) = A + B\tau + CE_2(\tau) + DE_3(\tau) + \dots$

The first (equations (III-2) and (III-4)) we owe to Milne and to Lundblad, the second (equations (III-3) and (III-5)) to Kourganoff (1949).

Eddington (1926) has shown that the exponential integral forms, $E_n(\tau)$, arise naturally in the integration of the equation of transfer, while Kourganoff (1952, p. 198) has demonstrated their close relation to its exact solution. The special suitability and great flexibility of these forms recommend their use in the analysis of limb-darkening measures. For these reasons the expansion suggested by Kourganoff (expressions (III-3) and (III-5)) were used in the treatment of the limb-darkening observations of Pierce (1954b).

2. The integro-exponential functions.--Let us now define, after Kourganoff (1952, p. 253) the integro-exponential function of order n :

$$(III-6) \quad E_n(\tau) = \int_0^{+1} e^{-\tau/\mu} \mu^{n-2} d\mu$$

The E_n are tabulated in Kourganoff (1952, p. 266) to $n = 7$, $\tau = 3.50$, as well as in a number of other sources (Placzek 1947, Jahnke-Emde 1945, New York 1940). Pecker (1951, V-14, p. 158) has written this definition for half-integral subscripts:

$$(III-7) \quad E_{n+1/2}(\tau) = \int_0^{+1} e^{-\tau/\mu} \mu^{n-3/2} d\mu$$

and has tabulated the function up to $n = 4$, $\tau = 10.0$ (Pecker 1953). We would expect, therefore, the definition for q of any value in $E_{n+q}(\tau)$ to be:

$$(III-8) \quad E_{n+q}(\tau) = \int_0^{+1} e^{-\tau/\mu} \mu^{n+q-2} d\mu$$

where n is reserved for positive integral values, and q is positive and sub-integral. The principal advantage that may accrue from writing the subscript in integral and sub-integral parts is in the calculation of the function itself. As Pecker (1953) has indicated, the function for higher $n+1/2$ than $n = 0$ may be got from the recursion formula:

$$(III-9) \quad E_{n+1/2}(\tau) = \frac{1}{n-1/2} [e^{-\tau} - \tau E_{n-1/2}(\tau)]$$

where it is supposed that we may generalize to:

$$(III-10) \quad E_{n+q}(\tau) = \frac{1}{(n+q-1)} [e^{-\tau} - \tau E_{n+q-1}(\tau)]$$

Since the behavior of the $E_n(\tau)$ are of considerable interest here, a graph of the function for $n = 1, 1.5, 2$, and 3 is shown in Figure 1. The $E_n(\tau)$ are characterized by boundary values of $1/(n-1)$ at $\tau = 0$; their behavior at τ greater than zero is logarithmic in τ (Kourganoff 1952, p. 254) with more and more an exponential character appearing with larger n (note recursion formula (12) and Figure 1 in Kourganoff 1952).

D. On Obtaining a Temperature Model for Use in This Investigation

1. The limb-darkening observations.--The observations of Pierce (1954b) were adopted as the most recent, complete, and homogeneous set available. Furthermore, they have the advantage of having been made in May 1953, near time of minimum solar activity. Probably the only infrared observations made with instrumentation and technique comparable to those of Pierce's 1953 infrared observations are those of Peyturaux (1952) and Pierce (Pierce et al. 1950). The latter were definitely superseded by the 1953 measures; as for the former, the spectral purity attained by Peyturaux is less than 1/30 that by Pierce, say, at $\lambda 15500$.

It seems to me unwise, therefore, to risk vitiating the material by combining it with any earlier observations. The fully corrected measures for his 26 published infrared wave lengths are given in Table 3. Dr. Pierce has very generously made available his unpublished limb-darkening measures for 19 additional wave lengths, 2 in the infrared and 17 extending into the visual region, together with data for 6 ultraviolet wave lengths interpolated from Peyturaux (1955). These have been included

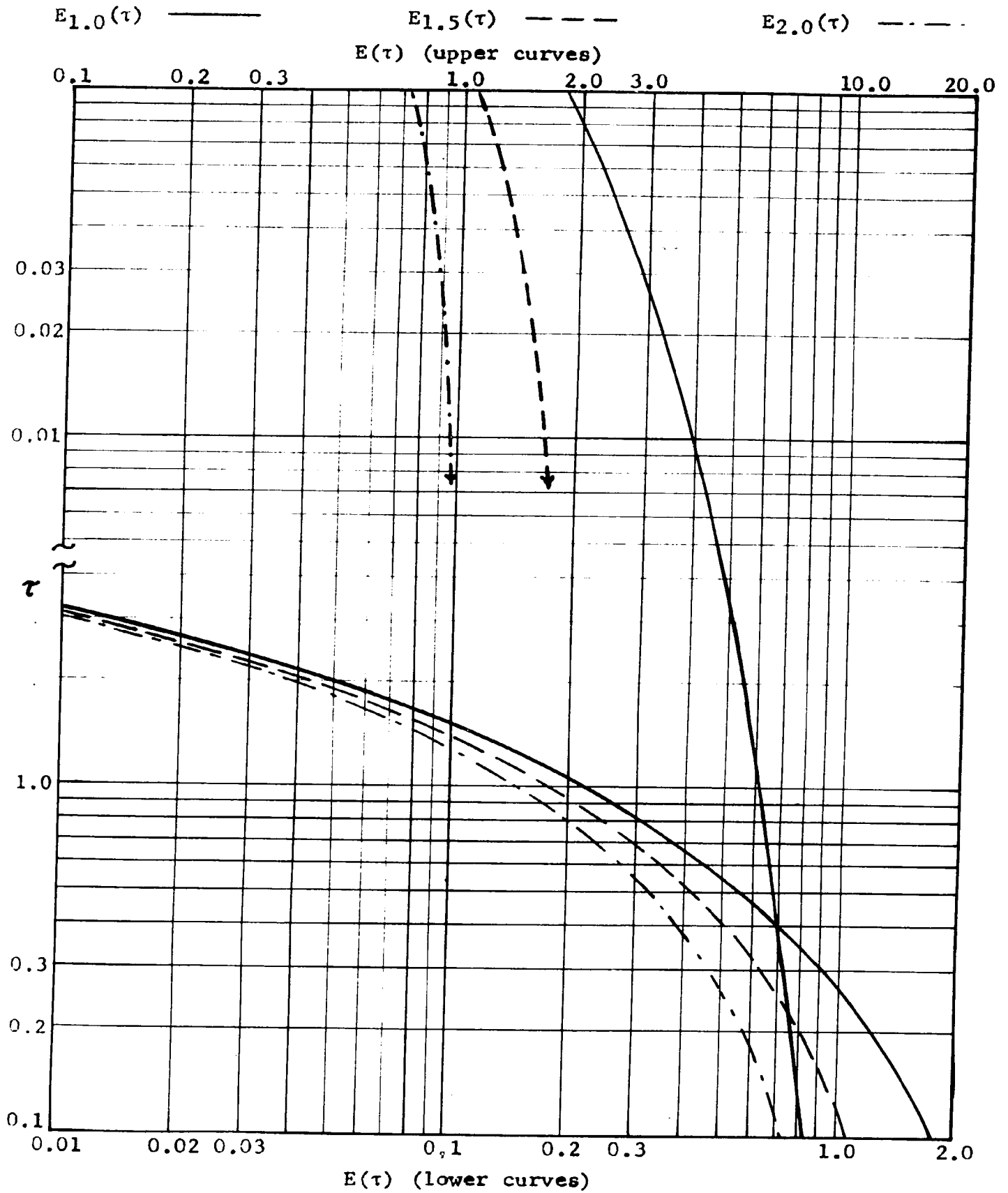


Fig. 1.-The exponential integral forms E_1 , $E_{1.5}$, and E_2 .

TABLE 3

LIMB-DARKENING DATA^a

μ	$I_{\lambda}(0, \mu)/I_{\lambda}(0, 0)$									
	$\lambda 3811.5$	$\lambda 4151.3$	$\lambda 4499.8$	$\lambda 4573.5$	$\lambda 4747.0$	$\lambda 4879.5$	$\lambda 4960.0$	$\lambda 5190.0$	$\lambda 5305.0$	$\lambda 5450.0$
1.0000	1.0000	1.0000	1.0000	1.0000	1.0030	0.9990	1.0000	1.0000	1.0000	1.0000
9992	9980	9990	9990	9990	9980	1.0000	9990	9990	1.0000	9990
9501	9580	9610	9640	9660	9690	9710	9700	9700	9720	9740
8999	9150	9200	9280	9340	9360	9430	9400	9400	9430	9470
8499	8770	8810	8940	9000	9050	9120	9080	9140	9150	9190
8000	8370	8430	8590	8650	8710	8820	8760	8840	8850	8910
7504	7940	8040	8250	8320	8380	8510	8450	8550	8580	8630
7001	7500	7640	7910	7979	8060	8180	8120	8240	8260	8330
6499	7060	7230	7550	7640	7700	7850	7780	7930	7930	8040
6000	6660	6840	7190	7260	7370	7530	7450	7600	7610	7720
5548	6240	6460	6830	6940	7050	7230	7140	7310	7350	7450
5000	5750	6010	6400	6520	6650	6840	6750	6920	6970	7090
4490	5300	5560	6010	6140	6290	6490	6390	6570	6630	6740
4012	4880	5140	5600	5740	5930	6130	6020	6210	6280	6400
3493	4400	4680	5180	5310	5500	5710	5590	5810	5880	6020
2998	3970	4240	4760	4880	5080	5270	5180	5400	5490	5630
2509	3530	3790	4310	4430	4640	4840	4750	4980	5080	5220
1990	3050	3320	3860	3930	4140	4320	4250	4510	4600	4730
4490	5240	5530	5980	6150	6240	6450	6360	6560	6670	6660
4012	4820	5100	5520	5790	5900	6080	5980	6170	6250	6330
3493	4310	4650	5130	5310	5440	5610	5560	5820	5920	5890
2998	3930	4220	4670	4900	4970	5210	5150	5300	5470	5410
2509	3450	3800	4280	4500	4550	4760	4720	4950	5040	5160
1990	3060	3360	3810	3970	4060	4260	4190	4520	4610	4610
1750	2800	3170	3580	3780	3860	4000	3980	4290	4400	4430
0.1506	0.2620	0.2960	0.3350	0.3510	0.3600	0.3680	0.3710	0.4030	0.4080	0.4190

^aPierce 1954b, revised.

TABLE 3 - Continued

$I_{\lambda}(0, \mu) / I_{\lambda}(0, 0)$										
μ	$\lambda 5656.5$	$\lambda 5818.5$	$\lambda 6110.0$	$\lambda 6198.0$	$\lambda 6263.0$	$\lambda 6423.5$	$\lambda 6600.5$	$\lambda 7793.0$	$\lambda 8475.0$	$\lambda 8853.0$
1.0000	0.9990	1.0000	1.0000	1.0000	1.0010	1.0000	1.0000	0.9996	1.0000	0.9977
9992	1.0000	9990	1.0000	9990	9990	1.0000	1.0000	9994	9994	1.0003
9501	9750	9740	9770	9760	9770	9760	9760	9810	9851	9825
8999	9490	9490	9530	9500	9520	9520	9560	9632	9659	9679
8499	9220	9220	9280	9260	9270	9290	9310	9454	9486	9518
8000	8950	8960	9020	9010	9020	9060	9100	9255	9304	9336
7504	8680	8690	8750	8750	8760	8790	8870	9061	9118	9168
7001	8400	8410	8490	8490	8500	8550	8600	8852	8935	8988
6499	8110	8140	8210	8220	8240	8300	8350	8637	8715	8772
6000	7790	7840	7930	7950	7960	8040	8100	8408	8501	8583
5548	7580	7580	7650	7690	7720	7770	7850	8207	8315	8423
5000	7180	7240	7340	7360	7400	7480	7540	7923	8046	8156
4490	6840	6910	7010	7070	7090	7160	7240	7671	7808	7935
4012	6510	6590	6700	6760	6790	6860	6950	7426	7562	7686
3493	6160	6240	6350	6400	6450	6500	6610	7122	7295	7410
2998	5770	5860	5980	6040	6100	6150	6280	6822	7012	7138
2509	5350	5470	5610	5670	5710	5790	5900	6496	6716	6858
1990	4870	5020	5140	5210	5240	5300	5460	6103	6353	6496
4490	6910	6900	6990	7110	7060	7150	7320	7690	7828	7941
4012	6510	6600	6670	6730	6800	6850	6970	7423	7584	7679
3493	6110	6250	6330	6380	6430	6530	6630	7155	7331	7452
2998	5720	5830	5920	6040	6100	6200	6270	6860	7038	7148
2509	5250	5350	5540	5640	5740	5780	5870	6516	6724	6895
1990	4890	4980	5070	5190	5270	5320	5470	6162	6358	6548
1750	4590	4760	4820	4960	5040	5050	5190	5942	6170	6371
1506	0.4280	0.4500	0.4590	0.4670	0.4780	0.4840	0.4900	5763	6016	6177
1246	-	-	-	-	-	-	-	5558	5774	5948
0.0999	-	-	-	-	-	-	-	0.5194	0.5445	0.5660

TABLE 3 - Continued

39

μ	$I_{\lambda}(0, \mu) / I_{\lambda}(0, 0)$									
	$\lambda 9057$	$\lambda 9955$	$\lambda 10135$	$\lambda 10478$	$\lambda 10910$	$\lambda 11760$	$\lambda 12515$	$\lambda 13014$	$\lambda 14998$	$\lambda 15575$
1.0000	0.9991	0.9997	0.9998	1.0002	0.9992	0.9997	0.9995	0.9993	0.9999	0.9996
9992	9997	9995	9995	9994	9998	9997	9998	9999	1.0000	9997
9501	9822	9841	9841	9873	9858	9868	9885	9887	9903	9908
8999	9658	9692	9691	9706	9723	9732	9751	9759	9800	9814
8499	9515	9540	9536	9547	9477	9598	9617	9640	9707	9711
8000	9331	9366	9380	9390	9440	9451	9478	9509	9579	9607
7504	9143	9204	9213	9228	9279	9312	9347	9366	9456	9498
7001	8954	9040	9034	9054	9112	9157	9193	9217	9341	9380
6499	8764	8862	8874	8880	8954	8993	9032	9069	9203	9255
6000	8554	8671	8679	8693	8773	8830	8877	8916	9066	9129
5548	8376	8483	8484	8508	8606	8651	8690	8776	8910	9001
5000	8152	8243	8266	8289	8399	8450	8473	8570	8751	8831
4490	7900	8002	8044	8073	8189	8237	8280	8379	8575	8670
4012	7629	7795	7836	7855	7981	8043	8114	8190	8387	8492
3493	7386	7531	7599	7611	7733	7805	7844	7963	8181	8291
2998	7136	7271	7321	7349	7484	7544	7591	7719	7944	8055
2509	6847	6981	7068	7089	7218	7292	7321	7445	7689	7824
1990	6523	6659	6750	6715	6887	6962	6973	7120	7365	7505
4490	7936	8047	8068	8090	8213	8219	8335	8327	8586	8746
4012	7771	7793	7863	7871	7973	8010	8111	8127	8397	8522
3493	7387	7509	7594	7636	7733	7785	7891	7922	8157	8364
2998	7124	7207	7359	7361	7500	7538	7647	7713	7960	8156
2509	6857	6975	7058	7073	7216	7269	7383	7424	7675	7904
1990	6457	6659	6726	6735	6862	6939	7038	7112	7355	7538
1750	6251	6450	6570	6546	6685	6752	6835	6936	7180	7365
1506	6093	6285	6402	6365	6508	6617	6648	6709	7031	7112
1246	5879	6024 ^b	6194	6084	6254	6247	6414	6518	6826	6916
0.0999	0.5578	0.6068 ^b	0.5904	0.5784	0.5959	0.5956	0.6215	0.6325	0.6450	0.6630

^bvalue is evidently in error and, in 28-point solution, accounts for the large [pvv] (cf. Table 4);
 25-point solution is unaffected.

TABLE 3 - Continued

μ	$I_{\lambda}(0, \mu) / I_{\lambda}(0, 0)$									
	$\lambda 15950$	$\lambda 16265$	$\lambda 16577$	$\lambda 17020$	$\lambda 17185$	$\lambda 17587$	$\lambda 21028$	$\lambda 22480^c$	$\lambda 21285$	$\lambda 21800$
1.0000	0.9996	0.9992	1.0000	1.0005	1.0002	1.0004	0.9985	1.0003	1.0003	1.0001
9992	9998	1.0000	9997	9994	9995	9994	1.0005	9996	9995	9997
9501	9911	9911	9912	9900	9912	9922	9933	9924	9920	9928
8999	9818	9828	9816	9819	9830	9837	9854	9828	9843	9833
8499	9724	9734	9728	9733	9736	9762	9758	9766	9761	9746
8000	9611	9634	9625	9598	9635	9665	9684	9671	9679	9683
7504	9517	9539	9529	9543	9538	9571	9595	9598	9586	9594
7001	9400	9420	9415	9430	9443	9459	9499	9511	9514	9508
6499	9281	9308	9305	9316	9326	9356	9394	9426	9392	9414
6000	9153	9181	9182	9199	9213	9248	9288	9317	9289	9305
5548	9037	9063	9067	9087	9102	9141	9183	9226	9187	9197
5000	8870	8915	8927	8940	8955	9034	9059	9092	9059	9086
4490	8721	8760	8763	8750	8807	8847	8924	8978	8929	8952
4012	8551	8599	8604	8592	8637	8694	8790	8831	8795	8822
3493	8363	8397	8418	8459	8468	8524	8613	8682	8648	8661
2998	8137	8191	8217	8238	8281	8305	8446	8505	8459	8489
2509	7908	7989	7994	8024	8047	8110	8242	8328	8238	8297
1990	7610	7664	7697	7697	7744	7820	7957	8019	7990	8017
4490	8689	8750	8778	8797	8794	8822	8953	8934	8929	8903
4012	8545	8596	8628	8608	8634	8691	8836	8813	8788	8768
3493	8340	8408	8453	8433	8436	8468	8640	8642	8615	8600
2998	8083	8197	8248	8237	8236	8305	8462	8459	8452	8418
2509	7855	7979	8004	8005	8011	8119	8254	8298	8246	8236
1990	7511	7697	7694	7735	7705	7799	7979	8036	7969	7955
1750	7306	7511	7525	7554	7551	7635	7800	7908	7855	7811
1506	7091	7361	7360	7343	7370	7404	7611	7651	7667	7594
1246	6896	7077	7069	7007	7174	7188	7297	7379	7448	7416
0.0999	0.6631	0.6758	0.6705	0.6633	0.6938	0.7029	0.6826	0.7060	0.7238	0.7120

^c Column should appear between $\lambda 22115$ and $\lambda 23075$ on next page of table.

TABLE 3 - Continued

	$I_{\lambda}(0, \mu)/I_{\lambda}(0, 0)$										
μ	$\lambda 22115$	$\lambda 23075$	$\lambda 23125$	$\lambda 23595$	$\lambda 24388$	$\lambda 3190^d$	$\lambda 3300^d$	$\lambda 3400^d$	$\lambda 3539^d$	$\lambda 3714^d$	$\lambda 4000^d$
1.0000	1.0000	1.0004	1.0011	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9992	9998	9997	9992	9997	9997	9990	9990	9990	9990	9990	9990
9501	9923	9923	9928	9923	9930	9540	9540	9570	9590	9640	9590
8999	9835	9848	9842	9856	9856	9060	9090	9130	9180	9270	9170
8400	9758	9773	9756	9780	9774	8590	8650	8690	8760	8890	8750
8000	9676	9701	9661	9685	9685	8130	8190	8260	8340	8510	8330
7504	9598	9602	9573	9601	9608	7670	7760	7830	7920	8110	7900
7001	9496	9521	9492	9513	9517	7170	7330	7390	7510	7700	7460
6499	9409	9429	9400	9421	9423	6680	6860	6940	7080	7300	7040
6000	9303	9323	9293	9327	9323	6230	6420	6500	6630	6910	6600
5548	9206	9231	9197	9227	9241	5810	6000	6110	6260	6530	6230
5000	9065	9106	9074	9098	9109	5270	5480	5610	5780	6060	5750
4490	8937	8986	8951	8985	9000	4800	5030	5150	5310	5640	5320
4012	8819	8863	8814	8849	8861	4370	4610	4700	4860	5200	4880
3493	8657	8704	8645	8693	8711	3910	4150	4240	4400	4730	4400
2998	8495	8546	8468	8526	8554	3470	3670	3820	3950	4260	4010
2509	8309	8354	8278	8344	8370	3060	3230	3380	3530	3810	3580
1990	8079	8097	7998	8084	8133	0.2680	0.2820	0.2990	0.3170	0.3360	0.3200
4490	8933	9035	8893	8945	9003	-	-	-	-	-	-
4012	8805	8923	8753	8814	8849	-	-	-	-	-	-
3493	8632	8742	8554	8665	8697	-	-	-	-	-	-
2998	8456	8574	8408	8516	8529	-	-	-	-	-	-
2509	8291	8407	8193	8304	8332	-	-	-	-	-	-
1990	8010	8198	7927	8064	8084	-	-	-	-	-	-
1750	7853	8075	7734	7935	7938	-	-	-	-	-	-
1506	7700	7943	7541	7772	7751	-	-	-	-	-	-
1246	7598	7765	7373	7693	7658	-	-	-	-	-	-
0.0999	0.7464	0.7533	0.7167	0.7918 ^e	0.7637	-	-	-	-	-	-

^dInterpolated by Pierce from Peyturaux (1955).^evalue is in error and, in 28-point solution, accounts for the large [pvv] (cf. Table 4); 25-point solution unaffected.

in Table 3.

2. Solutions of the integral equation for the limb-darkening coefficients.--The availability of an electronic computer (IBM 650) made it possible to carry out somewhat more extensive calculations on this and other problems connected with the research than would have otherwise been possible. In the Appendix II are described the several problems written with the aid of the first interpretive routine devised for the 650 (Mitilac).

With normalization to the intensity at the center of the disc, $I_\lambda(0,0)$, the integral equation of limb-darkening may be written:

$$(III-11) \quad \phi_\lambda(0,\mu) = \frac{I_\lambda(0,\mu)}{I_\lambda(0,0)} = \int_0^\infty \frac{B_\lambda(T)}{I_\lambda(0,0)} e^{-\tau/\mu} d\tau$$

where the quantity ϕ_λ is the quantity tabulated in Table 1. ϕ_λ , the normalized emergent intensity is expanded as in equation (III-3). The inverse transform, i.e., the normalized source function, is then:

$$(III-12) \quad S_\lambda = \frac{B_\lambda(T)}{I(0,0)} = A + B\tau + CE_2(\tau)$$

We refer to the constants A, B, and C, as the limb-darkening coefficients and obtain their values by the method of least squares.

Pierce's limb-darkening measures refer to 22 points on the solar disc. Owing to the manner in which the observations were carried out in the infrared, there exist in one group ϕ_λ values for 18 values of μ (1.0 to 0.1990) given unit weight; and in another group, ϕ_λ values for 10 values of μ (0.449 to 0.0999) given weight 1/3. There was, then, a system of 28 equations of condition for the unknown limb-darkening coefficients. Multiplying the last ten conditional equations by the factor 1/3 gives to them a weight, after the theory of least squares (Chauvenet 1906, p. 521) of 1/9th. The limb-darkening coefficients were obtained from the normal equations by the Gaussian method of elimination (Chauvenet 1906, p. 530)

and are tabulated with their residuals in Table 4. Included also in Table 4 are the coefficients and residuals obtained in a 25-point solution, in which the observations at the extreme limb ($\mu=0.1506, 0.1246,$ and 0.0999) were omitted. In columns (2)-(4) are listed the coefficients A, B, and C, for the several solutions; in columns (5) and (6), the extrapolation of ϕ_λ to the limb ($\tau=0$), $A + C$; and in column (7), the residuals $[pVV]$. Appended to Table 4 are the limb-darkening coefficients and monochromatic intensities emergent at the center of the disc, $I_\lambda(0,0)$, which were finally adopted for calculating the temperature distributions used in this thesis.

Details of the 28-point least-squares solution may be summed up as follows: the probable error for the coefficient A varies from ± 0.0020 at $\lambda 21028$ to ± 0.0005 at $\lambda 13014$, with the weights of the coefficients A, B, and C, respectively: 3.26, 6.61, and 8.18. In other words, B and C at each wave length have probable errors 70 per cent and 63 per cent of that of A. As examination of Table 4 will show, the coefficient B is smallest at $\lambda 21028$, the wave length at which all of the coefficients are least well determined. It is, however, positive and six times the size of the probable error.

3. The temperature distribution.--For the temperature, T, at any optical depth, we obtain from Planck's law the expression:

$$(III-13) \quad T = (c_2/\lambda) [\ln(1 + P/S_\lambda)]^{-1}$$

$$\text{in which: } (III-14) \quad P = c_1/I_\lambda(0,0)\lambda^5$$

$$\text{and: } (III-15) \quad S_\lambda = B_\lambda(T)/I_\lambda(0,0) = A + B\tau + CE_2(\tau)$$

c_1 and c_2 are radiation constants (Allen 1955, p. 100); B_λ , the Planckian, and $I_\lambda(0,0)$, the emergent intensity at the center of the disc, both in wave length units. The I_λ 's for the center of the disc were taken from

TABLE 4

LIMB-DARKENING COEFFICIENTS
3-TERM SOLUTION USING E_2

λ		A	B	C	A+C	[p _{vv}] $\times 10^4$
3190.0 ^a	(18-point)	-0.03829	+0.98828	+0.16297	+0.12468	0.098
3300.0 ^a		+0.10314	89845	-0.00521	09793	047
3400.0 ^a		13721	87083	02618	11103	078
3539.0 ^a		22871	81037	12735	10136	195
3714.0 ^a		44097	67615	38170	05928	095
4000.0 ^a		17470	83783	04083	13386	062
3811.5	(26-point)	24880	80158	16419	08461	191
4151.3		32841	73952	22136	10705	122
4499.8		50432	51205	37992	12510	124
4573.5		56452	57007	43861	12591	069
4747.0		62149	52874	48959	13190	119
4879.5		73983	45157	62375	11608	111
4960.0		66310	49868	52724	13586	030
5190.0		70425	46092	53828	16597	077
5305.0		70535	45501	52261	18273	044
5450.0		76873	41352	59392	17481	263
5656.5		80655	38338	61897	18758	109
5818.5		77258	39513	54656	22602	063
6110.0		82117	36134	59476	22641	082
6198.0		80467	36588	55579	24888	057
6263.0		79843	36611	53622	26221	043
6423.5		84279	33768	58811	25468	056
6600.5		86631	31748	59896	26735	077
7793.0	(28-point)	80953	25655	54124	36830	057
	(25-point)	91667	25293	55271	36396	027
8475.0		92245	23731	52064	40181	050
		92217	23751	52039	40179	028
8853.0		93941	21905	51640	42301	062
		94310	21719	52236	42074	052
9057.0		93797	22182	52076	41721	123
		92583	22808	50159	42425	102
9955.0		94204	20963	49430	44774	214
		96511	19769	53054	43457	044
10135		-----	-----	-----	-----	---
		93670	20866	47373	46297	020
10478		98684	18133	54804	43879	075
		+0.96709	19152	51690	45019	020
10910		+1.01301	15825	55812	45489	051
		+0.99549	16729	53047	46502	011
11760		+1.04014	13956	58565	45450	148
		+1.01262	15378	54229	47033	020
12515		+1.04017	13512	57124	46893	067
		+1.04343	+0.13344	-0.57638	+0.46704	0.064

^a Interpolated from Peyturaux (1955).

TABLE 4 - Continued

λ	A	B	C	A+C	[p _{vv}] $\times 10^4$
13014.0 (28-point)	+1.05596	+0.12025	-0.57427	+0.48169	0.043
(25-point)	1.04913	12372	56332	48581	030
14998	1.11967	07049	61973	49994	062
	1.10963	07572	60402	50561	013
15575	1.13926	05148	62159	51767	159
	1.11361	06465	58094	53267	092
15950	1.16453	03525	65107	51346	254
	1.12586	05508	58969	53618	107
16265	1.13652	04523	59230	54422	142
	1.10696	96949	54569	56127	170
16577	1.14299	04096	59949	54350	276
	1.09933	06348	53058	56875	031
17020	1.15944	03165	62274	53670	499
	1.09900	06278	52725	57175	068
17185	1.13406	04279	57632	55773	089
	1.11281	05370	54263	57018	045
17587	1.14960	03048	58685	56275	124
	1.12260	04430	54392	57868	045
21028	1.17744	00861	60632	57112	723
	1.10471	04611	49150	61321	051
21285	1.12042	03773	51542	60500	100
	1.09360	05153	47293	62066	028
21800	1.14718	02277	55387	59331	332
	1.09977	04714	47875	62102	028
22115	1.08437	05457	45280	63157	091
	1.07206	06084	43308	63897	056
22480	1.15894	01492	56660	59234	493
	1.09508	04778	46555	62952	068
23075	1.07093	05702	41696	65396	101
	1.06586	05967	40909	65677	081
23125	1.14389	02584	55312	59077	337
	1.10312	04674	48837	61475	172
23595	1.04925	07027	38951	65974	481
	1.08380	05232	44359	64021	032
24388	1.08387	05098	43947	64440	131
	+1.07856	+0.05360	-0.43069	+0.64788	0.048
5780 ^b	+0.782	+0.392	-0.568	3.63 ^c $\times 10^{14}$	
6370	+0.828	+0.350	-0.568	3.09	
6575	+0.864	+0.320	-0.598	2.95	
7772 (7793)	+0.9095	+0.2565	-0.5412	2.10	
7896	+0.912	+0.257	-0.536	2.04	
8912	+0.940	+0.178	-0.560	1.59	
10914 (10910)	+1.0130	+0.1583	-0.5581	0.99	
15880	+1.164	+0.035	-0.652	0.388	
16150	+1.142	+0.044	-0.602	0.373	
21225	+1.135	+0.034	-0.538	0.1377	

^b Thesis wave lengths.^c Intensities at $\mu=1.0$ in ergs/cm²-sec-angstrom.

Allen (1955, p. 140) for $\lambda < 10000$ and from Pierce (1954a) for $\lambda > 10000$. The relative energy distribution of Pierce fits rather well onto that given by Allen (1955) for the region $\lambda 10000-16000$.

Calculation of the temperature distribution with the IBM 650 is described in Appendix II (Program 2). Very useful for spot calculations of the temperature for a wide range of frequencies and values of the Planckian, B_ν , were the tables of Wilkinson (unpublished).

A systematic influence of the points observed near the limb appears in a comparison between the 28-point and the 25-point solutions. For 21 out of 27 wave lengths the extrapolation to boundary ($\tau=0$) yields a lower value when the extreme limb observations are included, even with the very low weight $1/9$ th. This systematic effect is, however, somewhat overshadowed by the wave length variation of the boundary temperature. It has a range of 200° - 300° on either side of 4650° .

The (T, τ) distribution finally adopted for the calculation of line strengths (Chapter IV) was the one deriving from the limb-darkening coefficients in Table 4 obtained with E_2 , i. e., with the expression (III-3) to 3 terms.

Our discussion of the behavior of the solutions in extrapolation (Section (III-E)) should be prefaced by the strong reminder that there is absolutely no way, at a single wave length, to employ the observations in a particular range of the parameter μ , to obtain unambiguous information on its inverse transform τ , outside that range (Evans 1947, p. 435).

E. Aspects of the Limb-Darkening Representation by Integro-Exponential Forms

In the use of a 4-term expansion for ϕ , Pierce (1950) and Michard (1953), and earlier Menzel and Sen (Kourganoff 1952, p. 199), found an

indeterminacy in the limb-darkening coefficients. Its cause may be recognized in the matrix of the normal equations: the columns representing the limb-darkening coefficients C and D are very closely alike; the unknowns may not, therefore, be expected to be well separated.

Michard circumvented the indeterminacy by constraining the solution in such a way as to give a boundary temperature of 4500° . The constraint appears to have had its effect on the values of $(d\tau/dT)$ got by him from a 3-term and a 4-term expansion. In his Table II we see that the 3-term and 4-term $(d\tau/dT)$'s are alike at only a single depth, where the temperature is 6000° or greater; at other depths the difference may become as large as 25 per cent. The imposition of a boundary constraint was demonstrated by Evans (1947), using polynomial expansions in τ , to affect the source function at all depths without in any way altering the limb-darkening over the observed region.

1. Use of the $(d\tau/dT)$'s in an empirical approach to the continuous absorption, electron pressure, and chemical composition.--Conclusions on the helium abundance based on Michard's 3-term solution are rejected because they are not corroborated by the 4-term solution. Michard finally adopted for the helium-hydrogen (B) and hydrogen-metals (A) ratios, respectively: $B = 0$, and $A = 10^4$, where the ratio refers to the relative number of atoms.

The residuals, theoretical absorption minus observed, in both solutions appear, however, to describe unknown absorption at wave lengths shorter than $\lambda 4000$. The infrared observations used by Michard (Peyturaux 1952, p. 334) led to negative values of the limb-darkening coefficient B, which implies the physically impossible situation of a source function diminishing at large optical depths. This difficulty does not arise with

Pierce's 1953 observations.

2. Bichromatic electron pressures.--A more direct application of the $(d\tau/dT)$'s to determine "observational" electron pressures was made by Pierce and Waddell (unpublished). The (τ, T) gradient at some temperature and wave length may be written as:

$$(III-16) \quad (d\tau/dT)_\lambda = (p ds/dT) [(1-x)/m_H] [P_e \alpha(H^-) + \alpha(H)]$$

where ds is the element of geometrical depth, x the fraction of hydrogen ionized, m_H the mass of the hydrogen atom, and $\alpha(H^-)$ and $\alpha(H)$ the absorption coefficients of the negative hydrogen ion (per dyne per neutral hydrogen atom) and neutral hydrogen atom respectively.

By using the temperature sensitive absorption coefficients, $\alpha(H^-)$ and $\alpha(H)$, and the observed $(d\tau/dT)$'s for two wave lengths, one may solve for the electron pressure:

$$(III-17) \quad P_e = [R\alpha_{\lambda_1}(H) - \alpha_{\lambda_2}(H)] / [\alpha_{\lambda_2}(H^-) - R\alpha_{\lambda_1}(H^-)]$$

where: (III-18) $R = (d\tau/dT)_{\lambda_2} / (d\tau/dT)_{\lambda_1}$

and all the quantities are, of course, taken for the same temperature.

The difficulty of this ingeniously simple method may be best illustrated by one detailed numerical example, where we have extracted data from Pierce and Waddell's unpublished tables. The (τ, T) gradient may be written:

$$(III-19) \quad d\tau/dT = [c_2 S_\lambda (S_\lambda + P)] / [\lambda P (B - CE_1) T^2]$$

in which B and C are the limb-darkening coefficients, c_2 the radiation constant (1.4387 cm-deg.), and P and S_λ are defined by equations (III-14) and (III-15). The calculation is carried out for the layer at a temperature of 6300° at $\lambda 10000$ and $\lambda 15000$. The data required in equation (III-17) are assembled in Table 5.

Substituted in equation (III-17), the ratio, R , gives for the

TABLE 5

DATA FOR CALCULATION OF BICHROMATIC ELECTRON PRESSURE
($T = 6300^\circ$, $\theta = 0.80$)

	$\lambda_1=10000$	$\lambda_2=15000$		$\lambda_1=10000$	$\lambda_2=15000$
A	+0.933	+1.104	P	9.52	8.82
B	+0.216	+0.080	$\alpha(H)$	1.67×10^{-26}	2.31×10^{-26}
C	-0.486	-0.598	$\alpha(H^-)$	4.4×10^{-26}	2.4×10^{-26}
E ₁	0.18	0.42	$d\tau/dT$	14.93×10^{-4}	8.14×10^{-4}
E ₂	0.125	0.26	R_{λ_2/λ_1}	0.546	

electron pressure:

$$\begin{aligned}
 \text{(III-20)} \quad P_e &= [(0.546)(1.69) - (2.31)] / [(2.390) - (0.546)(4.401)] \\
 &= -1.39 / -0.01 = 139 \text{ dynes} \quad (\text{cf. } 50 \text{ dynes, Minnaert } 1953, \text{ p. } 127)
 \end{aligned}$$

The Pierce-Waddell tabulated values of $(d\tau/dT)_{\lambda_1}$ and $(d\tau/dT)_{\lambda_2}$ give, however, a ratio: $R_{\lambda_2/\lambda_1} = 0.63$, which in the above becomes:

$$\begin{aligned}
 \text{(III-21)} \quad P_e &= [(0.63)(1.69) - (2.31)] / [(2.390) - (0.63)(4.40)] \\
 &= -1.24 / -0.38 = 3.3 \text{ dynes}
 \end{aligned}$$

Here a 13 per cent change in the ratio of the (τ, T) gradients has introduced a factor of 42 into the observed electron pressure.

It is already clear from Michard's results that the (τ, T) gradient is sensitive to the method of limb-darkening representation. From equation (III-17) it must also be apparent that probably the greater difficulty lies in the differencing of the absorption coefficients. For the deeper layers ($\tau \approx 1.0$) the $\alpha(H^-)$'s are of the order 10^{-26} , but for the upper layers ($\tau \approx 0.1$) $\alpha(H)$ has diminished by 3 orders of magnitude while $\alpha(H^-)$ has not increased by one order of magnitude. With R usually not far from unity, P_e would be expected to drop off roughly in the

manner of the neutral hydrogen absorption; in other words, the wave length variation of either $\alpha(H^-)$ or $\alpha(H)$ is very much less pronounced than the change of $\alpha(H)$ with depth. The observed pressures result essentially from the depth dependence of the ratio of the absorption coefficient difference for any pair of wave lengths, i.e.: $\Delta\alpha(H)/\Delta\alpha(H^-)$. We are then faced with the paradox of having derived the electron pressure, or rather its gradient, from the depth behavior of an absorber the contribution of which has been regarded negligible over most of the depths in question (Munch 1945).

3. The use of other integro-exponential forms in the limb-darkening representation.--Regarded as a problem in curve-fitting, the description of the limb-darkening has interesting aspects which have been overlooked in the haste to interpret results got with the forms E_2 and E_3 .

As suggested earlier, the behavior of the source function in extrapolation and, in addition, its gradient everywhere are sensitive to the method of limb-darkening representation. The first of these characteristics was strongly demonstrated in two additional 3-term solutions for the limb-darkening coefficients using E_1 and E_3 . The behavior of S_λ for $\lambda 7793$ and $\lambda 14998$ for a number of optical depths in parallel solutions using E_1 , E_2 , and E_3 is shown in Table 6 and is based on the unrevised limb-darkening measures (Pierce 1954b).

In terms of residuals each of these integro-exponential forms gives an equally satisfactory description of the observations. On the other hand, the coefficients themselves are more precisely determined with the simpler integro-exponential forms. This interesting fact appears from a comparison of the weights obtained for A, B, and C in the Gaussian elimination. The results with E_2 were mentioned earlier

TABLE 6

BEHAVIOR OF NORMALIZED SOURCE FUNCTION IN LIMB-DARKENING SOLUTIONS
WITH INTEGRO-EXPONENTIAL FORMS E_1 , E_2 , AND E_3

$x = \log \tau_\lambda$	$S_\lambda=7793$			$S_\lambda=14998$		
	E_1	E_2	E_3	E_1	E_2	E_3
-4.0	-0.182	+0.362	+0.404	-0.185	+0.497	+0.548
-3.0	+0.057	+0.365	+0.404	+0.111	+0.500	+0.548
-2.0	+0.300	+0.390	+0.419	+0.412	+0.528	+0.562
-1.0	+0.562	+0.529	+0.523	+0.714	+0.675	+0.667
0.0	+1.052	+1.080	+1.093	+1.064	+1.098	+1.113
+1.0	+4.303	+3.785	+3.467	+2.503	+1.876	+1.494

(Section III-D-3)). The weights of the unknowns in all three solutions are given in Table 7.

TABLE 7

WEIGHTS OF THE LIMB-DARKENING COEFFICIENTS
FROM THE GAUSSIAN ELIMINATION

	A	B	C
E_1	178.6	378.2	185.2
E_2	3.26	6.61	8.18
E_3	0.394	0.785	1.612

Aside from its impossible behavior near the boundary, the E_1 expansion yields limb-darkening coefficients, judged by the size of their probable error, 5 to 8 times superior to those obtained with E_2 ; and E_2 , 2 to 3 times those with E_3 . With the best determination given by the function most ill-behaved in extrapolation, it seems then very difficult to determine what criterion should be used in the choice of the method of limb-darkening representation.

Our attention now turns to the continuum of integro-exponential

forms. We find it possible, by means of their interpolation or superposition, to obtain almost any desired boundary temperature. Though the generalization of their inverse Laplace transform may be easily derived (Kourganoff 1952, p. 256), it is not feasible to derive limb-darkening coefficients for many E_n 's with non-integral subscripts. It is possible, however, to interpolate the limb-darkening coefficients for $E_{1.5}$ from the separate E_1 , E_2 , E_3 solutions and to use them with Pecker's tabulated values of $E_{1.5}$ to obtain a source function which extrapolates to a boundary temperature, say, of 4000° . If this artifice is insufficient (for obtaining a sufficiently low boundary temperature, e.g.) the observations may be re-solved with the inverse Laplace transform for E_n where its boundary value $1/(n-1)$ is 3 or larger.

Since all of the E_n forms are perfectly adequate representations of the source function between $\tau = 0.1$ and 1.00 , they may be fractionally superposed in any fashion to give the desired source function at a particular depth outside the range of observation.

It is clear, then, that any one of, or combination of, many integro-exponential forms can in principle be used to represent the limb-darkening. It seems altogether possible, therefore, that if we restrict the limb-darkening representation at all wave lengths to one integro-exponential form, say E_2 , we may expect that the suitability of that particular form will vary from wave length to wave length. In other words, the limb-darkening curves at two wave lengths differ by more than a simple scale factor and in a way that may be reflected in the wave length variation of boundary temperature. The problem of curve-fitting here may be likened to that encountered by the double-star astronomer who tries to deduce the apparent orbit of a wide binary from observations

scattered along only a limited arc.

F. Summary

The integro-exponential forms are recommended for the representation of the source function for the reasons that: (1) they arise naturally in the integration of the equation of transfer; and (2) they give a more nearly unique temperature distribution over the observed range ($\tau = 1.0$ to $\tau = 0.1$) than does the simple polynomial expansion in μ .

An empirical temperature model for the calculation of line strengths in Chapter IV is derived from the limb-darkening measures by Pierce (1954b), using the integro-exponential form E_2 (equation (III-12)).

It is suggested that the wave length variation of boundary temperature results from the use of the same integro-exponential form for the source function at all wave lengths.

CHAPTER IV

THEORETICAL LINE STRENGTHS FOR THE ALLER-PIERCE-ELSTE MODEL

This chapter contains a discussion of the mechanism of line formation, in the way of introduction to our assumption of local thermodynamic equilibrium (LTE) in the calculation of theoretical line strengths for the Aller-Pierce-Elste model. Other assumptions bearing on the exactness of the calculations are described, and the data used are given in detail.

A. Theory and Equations

The equation of transfer for the case of line radiation of frequency ν may be written:

$$(IV-1) \quad \mu dI_\nu/d\tau = (1 + \eta_\nu)I_\nu - (1 + \epsilon\eta_\nu)B_\nu - (1 - \epsilon)\eta_\nu J_\nu$$

in which we define the following:

$$(IV-2) \quad \eta_\nu = k_\nu/k$$

$$(IV-3) \quad J_\nu = (1/2) \int_{-1}^{+1} I_\nu d\mu$$

$$(IV-4) \quad d\tau = -k\rho dx$$

and the Planck function is given in frequency units:

$$(IV-5) \quad B_\nu = (2h\nu^3/c^2)(e^{h\nu/k'T} - 1)^{-1}$$

k_ν is the mass absorption coefficient for line radiation of frequency ν ; k is the mass absorption coefficient for continuous radiation, also ν -dependent; ϵ describes the relative importance of the processes of absorption and scattering in the formation of the line; J_ν is the mean intensity; and k' distinguishes Boltzmann's constant from the continuous

absorption coefficient.

In the absence of line absorption, equation (IV-1) reduces to equation (I-1) with the source function given by the Planckian B_ν . Exact solutions of (IV-1) have been found by Chandrasekhar (1947) for η_ν constant with optical depth and a linear source function, by Stromgren (1937) for a limited variation of η_ν with depth, and by Tuberg (1946) for an arbitrary variation of η_ν with depth.

In order to evaluate the roles of absorption and scattering in the solution of (IV-1), it is appropriate to discuss now in some detail the mechanism of formation of Fraunhofer lines. In this connection our interest is focussed on the validity of the assumption that the lines of this study are formed under conditions of local thermodynamic equilibrium. Since this assumption is an important starting point for the conclusions arrived at in the study, it will be stated in detail and later examined semi-quantitatively with the aid of the term schemes of the atoms and ions involved. The conclusions which may be reached from the use of a term diagram and the knowledge of some atomic parameters refer, however, only to phenomena on a microscopic level.

Conditions which invalidate the use of a single-valued run of temperature and pressure with depth in the solar atmosphere, e.g., the presence of up- and down-traveling columns of gas, may be expected to cause the observed center-limb variation of lines to differ from that predicted on the basis of LTE. Consequently, in the discussion of LTE which follows, we review both the microscopic and macroscopic aspects of the problem. Where the former are treated semi-quantitatively in the final disposition of the problem, the latter are treated qualitatively.

B. Local Thermodynamic Equilibrium

The existence of local thermodynamic equilibrium requires that every atomic process going on in a gas be exactly balanced by its inverse process (the principle of detailed balancing; see, e. g., Aller 1953, p. 73): photoionization from a given level is balanced by recapture on that level and return of a similar photon to the radiation field; radiative or collisional excitation from a lower to an upper level, is balanced by radiative or collisional de-excitation from the upper to the lower level; and so on for all possible atomic processes.

The degree of coupling between any atomic excitation level and the continuum, directly or via other levels, is a very good measure of whether LTE may be assumed to describe the population of that level. This fact may be expressed exactly in terms of the transition probabilities and radiative intensities governing the level populations in the term scheme of interest.

Letting p be the upper level, q the lower, we may write the ratio of absorption, k , to scattering, σ , in terms of the atomic processes governing the populations of p and q :

$$(IV-6) \quad \frac{k_{\nu}}{\sigma_{\nu}} = \left\{ \sum_{\substack{m=1 \\ m \neq q}}^{\infty} [A_{pm} + B_{pm} J_{pm} (1 + \frac{g_p}{g_m})] + C_{pk} + C_{pf} \right\} / [A_{pq} + B_{pq} J_{pq} (1 + \frac{g_p}{g_q})]$$

in which the numerator includes all possible transitions from p to levels other than q , and the denominator expresses the coupling between the levels p and q . A_{nm} and B_{nm} are, respectively, the Einstein coefficients for spontaneous transition and induced emission or absorption; J_{pm} is the mean intensity of radiation of frequency $\underline{pm}$; g , the statistical weight; and C_{pk} is the probability that the electron may be removed by collision from level p . The possibility of photoionization from p and

the corresponding inverse process is expressed by C_{pf} , which depends on the conditions of temperature and pressure and the collisional cross-section of the atom. An expression similar to (IV-6) appears in an early paper by Stromgren (1935).

In Chapter I it was pointed out that, in the case of the solar atmosphere, local thermodynamic equilibrium is commonly assumed to exist in the continuum-forming region, a condition favored by the high temperature and density of those layers. The lower region of the photosphere is marked, however, by a steep temperature and density gradient (Plaskett 1955, p. 645; Minnaert 1953, p. 158), so the question arises as to what degree is the assumption of LTE a safe one in the higher or line-forming region.

This rather difficult problem has been injected into the equation of transfer for line radiation (equation (IV-1)) in the guise of the quantity ϵ (Eddington 1929), which expresses what proportion of the total energy absorbed in a line frequency is converted into kinetic energy by superelastic collisions (i. e., excitation energy converted to kinetic energy of passing electrons). It is now of interest to look into the interpretation of ϵ and to review some recent attacks on this problem.

C. Departures from Local Thermodynamic Equilibrium

Phenomena which have been investigated as causes for, or results of, departures from LTE are: (1) scattering, (2) temperature inhomogeneities, and (3) differences between ionization and excitation, and kinetic and electron temperatures.

1. Scattering.--The first of these constitutes the classical approach to departures from LTE and embraces the subjects of coherent and non-

coherent scattering. The observational and theoretical results obtained by Houtgast (1942) and others on the importance of the latter process in line-formation have been summed up by Minnaert (1953). Its most complete formulation, however, may be found in The Outer Layers of a Star (Woolley and Stibbs 1953). Non-coherence produced by complete redistribution, the widening of energy levels, and the Doppler effect are incorporated into the equation of transfer, and an approximate solution is obtained for the residual intensity everywhere in the line but at its center.

Aller (1953, p. 278) has shown that for the profile of the Ca K line the theoretical center-limb variation is in better agreement with that observed when corrections for non-coherent scattering are made. But of particular interest is the result obtained by Savedoff (1952), that profiles of lines produced in non-coherent scattering differ from pure absorption profiles only in the line core and at small μ .

It is interesting to express ϵ in terms of the ratio k_ν/σ_ν and to show how for the case of complete redistribution the quantity, ϵ , may be related to radiative quantities:

$$(IV-7) \quad \epsilon = 1/(1 + \sigma_\nu/k_\nu) = (|\bar{J} - J_\nu|)/(B_\nu - J_\nu)$$

in which J_ν is defined by equation (IV-3), B_ν by equation (IV-4); $\bar{J}$ is the intensity averaged, with respect to the absorption coefficient, over the line.

2. Temperature inhomogeneities.--Bohm (1954) has invoked temperature inhomogeneities of $\pm 500^\circ$ in an attempt to account for the center-limb behavior of the wings of strong Fe lines. He has calculated the wavelength variation of the mean intensity for a number of depths in the non-gray radiative equilibrium model and has shown how it departs considerably from the Planck intensity over that wavelength region in which the

ionizing radiation for the neutral metals is produced. The granulation is regarded as the observational counterpart of the inhomogeneities.

The Bohm model was employed by Marianne Bretz (1956) in the study, with the aid of exact curves of growth, of the center-limb variation of a large number of lines in the visual region. The run of ionization temperature is taken to be different from that of excitation temperature in such a way that the center-limb changes may be accounted for; this is the view adopted by her in preference to the possibility, suggested in a footnote, that the same effects may be accounted for by anisotropic large-scale motions of a sense opposite that found by Allen (1949) and others.

3. Differences between radiative and kinetic temperatures.--Departures from LTE have also been discussed in terms of the temperature qualified in a number of ways, such as ionization, excitation, radiation, kinetic, and electron temperature (Bhatnagar 1955, Plaskett 1955). This represents an attempt to retain the thermodynamic variable T , so useful in characterizing a Maxwellian velocity distribution and the wave length variation of the source function.

Plaskett reasons from a theory of Eddington (1926, p. 376) that since the kinetic and ionization temperatures are very nearly equal to the mean radiation temperature, the temperature distribution determined for any wave length at which the specific intensity of the radiation equals that mean radiative intensity, is the distribution which properly describes the ionization and particle velocities in the atmosphere:

$$T(B_{\lambda}, \lambda) = \bar{T}_{\text{rad}} \sim T_{\text{kin}} \sim T_{\text{ion}}$$

In a discussion of chromospheric jets, Bhatnagar shows that if the kinetic temperature of the atoms is increased, say by the intrusion

of a jet, the temperature of the electron gas will, through electron-atom and electron-ion collisions, rapidly follow suit despite the cooling effects of radiative capture and collisional excitation followed by ultimate radiation. With the considerably greater density of the photosphere, kinetic and electron temperatures may be considered virtually equal over disturbances of a scale larger than that of jets.

Be that as it may, there remains the question of ionization and excitation temperature, which has already appeared under "Temperature Inhomogeneities".

Whether LTE obtains in a given volume or not, we define ionization temperature as that temperature which gives in the Saha relation the correct ratio for the number of atoms in the various stages of ionization.

Bohm and Aller (1947) have shown that even under nebular conditions, electron-electron collisions very quickly establish a Maxwellian distribution of velocities among the electrons. Their calculation may be scaled from nebular to photospheric densities with the results that, even though the mean lifetime of an electron in the photosphere may be on the order of one second, the time of relaxation of the electron gas is of the order of only 10^{-8} seconds. More recently, Woolley and Stibbs (1953, p. 60) have shown that, for conditions appropriate to the sun, electron-electron collisions occur at 10^4 times the rate of recapture.

The type of departure from LTE of principal interest in the photosphere is that corresponding to an excess or deficit in the ionizing part of the radiation field. Since the ionizing portion of the radiation field lies in the poorly observed rocket ultraviolet, then the larger the part that photoionization plays relative to collisional ionization, the greater the uncertainty in the ionization temperature.

Insofar as ultimate radiation tends to be restored to the field on electron recapture (Milne 1930, p. 153), there exists a sort of feedback between the field of particle velocities and at least the energetic portion of the radiation field. The feedback is, of course, slightly distorted by the dependence of the electron capture cross-section on the electron energy. So long as the feedback exists, however, the ionization temperature will correspond exactly neither to the electron temperature nor to the radiation temperature in the ultraviolet.

Woolley and Stibbs (1953, p. 60) extend their argument on behalf of LTE even to conditions of low pressure. They show that its assumption is valid anywhere in the photospheric regions, where either hydrogen is mostly ionized or the negative hydrogen ion is the main source of opacity.

4. Excitation temperature.--The excitation temperature may be thought of as arising from another portion of the radiation field, since the average energy required for excitation is less than that for ionization.

Pecker (1957a, b) has recently attacked the problem in an ingenious empirical way. His scheme is a highly contrived one using essentially simple notions and, of course, partly hinges on the adopted model.

For lines in selected multiplets of Ti I (Fe I in progress) he has integrated the equation of transfer to total optical depth unity for the center of the line. To the corresponding optical depth in the continuum he assigns an "excitation" temperature for that multiplet using the Eddington-Barbier relation (Unsold 1948):

$$(IV-8) \quad I_{\lambda}(0, \mu) \sim S(\mu)$$

Equation (IV-8) is exact for a source function linear in τ . For the emergent intensity on the left-hand side of equation (IV-8) he has used central intensities from the Utrecht Atlas corrected for instrumental

profile. From the wide range of intensities in a given multiplet, a run of excitation temperature can be obtained for continuous optical depth. The temperature distribution derived in this way Pecker uses as the basis for computing a new weighting function, which now must be regarded as pertaining to one multiplet only. The new weighting function is used to repeat in higher approximation the calculation described above.

For the low-lying multiplets studied the excitation temperature is everywhere greater by several hundred degrees than the temperature derived from limb-darkening (cf. Bohm's contention with respect to temperature inhomogeneities that the excitation temperature would not be influenced beyond 150° (Bohm 1954, p. 192)).

For his calculation the f values are required, of course, as well as a value for the abundance of the element. Concerning the latter, Pecker's work seems to indicate that low-lying multiplets of an atom tend to give a "non-equilibrium" value for the abundance, that may be brought into agreement with the determination from high-lying multiplets by the rather complicated method he proposes. Further remarks on Pecker's method will be found in Section (VII-A-1).

An interesting aspect of his approach is that, while Bohm was led to a non-equilibrium model by the behavior of line wings, and the observed phenomenon of granulation, Pecker has used the central intensities, without the assumption of temperature inhomogeneities, to investigate departures from LTE.

We will now look at the lines of this investigation in the light of the foregoing topics, viz., scattering, temperature inhomogeneities, and differences between radiative (ionization and excitation) and kinetic temperatures.

D. The Assumption of Local Thermodynamic Equilibrium in This Investigation

1. Scattering.--For most of the lines of this study we may, with the aid of term diagrams, estimate maximum values for the ratio of scattering to absorption. An examination of equation (IV-6) shows that:

$$(IV-11) \quad \sigma_v/k_v < A_{pq} / \sum_{\substack{m=1 \\ m \neq q}}^{\infty} A_{pm}$$

The inequality arises when we omit those terms involving radiative intensities and the parameters describing the escape of an electron from the upper level p by collision or photoionization.

We further confine our attention to the transitions which are possible between the upper level p and all lower levels other than q . The spontaneous transition probabilities from p to lower levels m are related approximately as the cubes of the frequencies of the transitions pm (Aller 1953, p. 127; Page 1951, p. 52):

$$(IV-12) \quad A_{pq}/A_{pm} \approx (\nu_{pq}/\nu_{pm})^3$$

An estimate of σ_v/k_v is now made using one term, preferably the largest ($m=1$), in the denominator of (IV-11) and the ratio (IV-12). Such a calculation is possible for ten out of thirteen transitions in Table 1, the exceptions being Ca I $4^1S-4^3P^0$, Si I $4^3P^0-5^3D$, and O I $3^5S^0-3^5P$, which do not share upper levels with strong downward transitions. The results of this calculation for the other lines of Table 1, however, are listed in Table 8. The element, stage of ionization, and wavelength are given in columns (1)-(3); the lower terms of the transitions being compared are given in column (4) and their common upper term in column (5). Column (6) contains the estimated maximum value of σ_v/k_v . Column (7) contains a useful reference for the energy level diagram or term levels.

The possibility of photoionization from the level p , given by C_{pf}

THE MECHANISM OF LINE-FORMATION

Element	Stage	λ	Lower Terms		Common Upper Term	σ_v/k_v Upper Limit	Refer- ence ^a	I-X _u	Ionizing Radiation
Ca	I	6573	4 ¹ S	-----	4 ³ P ^o	-----	1	-----	-----
		16151	5 ³ P ^o	4 ³ P ^o	5 ³ D	0.012	1	1.01	< λ 12300
	II	8912, (8927)	4 ² D	3 ² D	4 ² F ^o	0.008	2	-----	-----
Mg	I	15880	3 ³ D	4 ³ S	5 ³ P ^o	0.113	3	0.92	< λ 13400
		21226	5 ³ P ^o	(3 ¹ S)	8 ³ S	0.0015	3	intercombination line)	
	II	7896	4 ² P ^o	3 ² P ^o	4 ² D	0.0019	3	0.34	< λ 36300
Si		10914	3 ² D	3 ² S	4 ² P ^o	0.009	2	-----	-----
	I	5780	4 ³ P ^o	-----	5 ³ D	0.0017	2	-----	-----
	II	6371	4 ² S	3 ² D	4 ² P ^o	-----	4,5	1.08	< λ 11400
O	I	7772, (7774) (7775)	3 ⁵ S ^o	-----	3 ⁵ P	0.224	5	-----	-----
						-----	6,7	(2.92)	< λ 4220)
Sr	II	10915	4 ² D	5 ² S	5 ² P ^o	0.053	2	-----	-----
Fe	I	15878	e ⁵ D ₂	b ³ F ₄	t ⁵ D ^o	0.0084	5	1.57	< λ 7860

^a (1) White (1935), pp. 172, 182; (2) *ibid.*, pp. 97-98; (3) *ibid.*, p. 179; (4) page (1951), p. 75;
 (5) Moore (1945), Part I; (6) White (1935), p. 266; (7) Condon and Shortley (1935), p. 218.

in equation (IV-6), becomes large where p is close to the ionization limit, i. e., the atom becomes ionizable by radiation from an appreciable range of the energetic frequency spectrum. In Table 8, therefore, I have included, for transitions whose upper levels differ from the ionization limit by less than 1.6 electron volts, the size of that difference, $(I - X_u)$, in column (8); and the wave length range capable of ionization, in column (9).

The results shown in column (6) indicate that for eight of the transitions, the influence of scattering is entirely negligible, while for two transitions the scattering might attain 10 to 25 per cent of the magnitude of the absorption. The results shown in column (9), the radiation capable of producing ionization from the upper level, lend some weight to the assumption of LTE for five lines.

The lines for which the assumption of LTE has not been justified to a fairly high degree are Ca I 6573 and the triplet of O I.

a) O I 7772-7774-7775 ($3^5S^0 - 3^5P$).--These lines originate from the so-called resonance potential of O I at 9.11 e. v. (Hopfield 1924). Their upper level is 2.92 e. v. removed from the ionization limit of O I, consequently only radiation of wave lengths less than $\lambda 4220$ is capable of ionization. Their high excitation potential and low strength indicate that even at the limb these lines are formed relatively deep in the atmosphere. The region of their formation, as a matter of fact, appears to coincide with that of the continuum, so that the conditions of high temperature and pressure support the assumption of LTE for these lines in particular.

b) Ca I 6573 ($4^1S - 4^3P^0$).--This line originates from the resonance level of Ca I. Being an intercombination line its transition probability

is very small compared to most of the transitions near it in the term scheme. It is on this account that I, along with Claas (1951, p. 40), regard the line as being formed in pure absorption.

Finally, there is evidence that the scattering in Fraunhofer lines is predominantly non-coherent (Aller 1953, p. 287). Now the demonstration by Savedoff (1952), that non-coherent scattering profiles resemble pure absorption profiles to a large degree, suggests that predictions based on the latter ought to suffice, if only phenomenally.

2. Temperature inhomogeneities.--As to the influence which large-scale temperature inhomogeneities may have on the observed center-limb variation of equivalent widths, a qualitative prediction can be made at this point. The prediction is made with the view that hotter ascending volumes of gas may contribute a different proportion of the radiation observed at the center of the disc compared with what they contribute to the radiation observed at the limb. If the proportional contribution of the hotter elements diminishes toward the limb, the temperature gradient obtained from the observed limb-darkening would appear somewhat steeper than a simple temperature distribution got by averaging over the hot and cold elements. In any case, the use of an empirically determined (T, τ) distribution insures that the mean temperature assigned to any depth has been weighted with due regard for the sizes, temperatures, and opacities of all the contributing volumes of gas. One would expect that such a temperature distribution would be equally valid for the description of lines arising in the continuum-forming region.

For the calculation of level populations there arises the question of whether one should use a mean temperature or, instead, average the level populations computed separately for the hot and cold elements

(Bohm 1954, p. 192). A simple calculation for the extreme case of the oxygen triplet indicates that the results got in both ways differ by less than 10 per cent for a temperature difference as large as 500°.

3. Differences between radiative and kinetic temperatures.--This topic deals with (indirectly) observable, gross characteristics of the microscopic phenomena discussed under "Scattering" (Section (IV-C-I)). It logically occupies, therefore, the middle ground between scattering and the macroscopic phenomena of temperature inhomogeneities. But the arguments we would like to state here overlap considerably those in Section (IV-D-1) and, furthermore, have already been anticipated in Section (IV-C-3). Not yet discussed, however, is back-radiation from the chromosphere and corona. Since I regard that subject as being somewhat more speculative than the subjects so far touched upon, I have reserved for Chapter VII (Section (A-3)) a discussion of possible effects of back-radiation from those regions.

Since the contribution functions introduced in Chapter V lend much clarity to the explanation, I have also postponed, until Chapter VII (Section (A-1)), critical remarks on Pecker's method for determining excitation temperature.

For lines formed in regions not entirely coincident with the continuum-forming region, the effect of departures from LTE and temperature inhomogeneities may be looked upon as a differential one, since only the center-limb behavior of the lines is being examined. On this account, if the differences between the observed center-limb variations and the equilibrium predictions are the result of departures from LTE or temperature inhomogeneities, those differences must themselves refer to center-limb variation in departures from LTE or in the inhomogeneities.

The detection or interpretation of higher order effects of that kind are beyond the scope of this thesis.

In summary: the mechanism of pure absorption is assumed, with the varying degrees of justification outlined above, to account for the formation of the lines in this study (Unsöld 1948; 1955, pp. 285, 373; Page 1951; Woolley and Stibbs 1953, pp. 57-60; Pagel 1955, 1956; Minnaert 1953, p. 130).

E. The Calculation of Theoretical Equivalent Widths

For the calculation of the emergent intensity of line radiation at a point on the solar disc, we again employ the equation of transfer in its integral form:

$$(IV-9) \quad I_V(0, \mu) = (1/\mu) \int_0^\infty \mathcal{J}(\tau_V) e^{-\tau_V/\mu} d\tau_V$$

Adoption of numerical methods at this point is deemed not only advisable owing to the limitations in the form and applicability of the exact solutions, but also desirable on account of the possibility of calculating in detail, with the aid of the observational model, the variation with depth of the quantities determining both line and continuous absorption.

The line depth may be written, after Pecker (1951), as:

$$(IV-10) \quad r_V = (I_C - I_V)/I_C = (1/\mu) \int_0^\infty \eta_V \exp\left(-\frac{1}{\mu} \int_0^\tau \eta_V d\tau\right) g(\tau) d\tau \quad (\eta_V = k_V/k)$$

where I_C and I_V are, respectively, the intensity in the continuum and the point in the line profile; k_V and k , the line and continuous mass absorption coefficients; and τ , the optical depth in the continuum. $g(\tau)$, the weighting function for pure absorption, represents the efficiency with which each layer contributes to the dip in the line profile. The weighting function bears precisely the same relation to the line dip

as the source function bears to the intensity in the continuum; hence, by analogy it may be called the "sink" function. Mathematically stated: where B is the inverse Laplace transform of unit intensity, g is the inverse Laplace transform of zero intensity. It is given exactly by the expression (Pecker 1951, Unsold 1955):

$$(IV-13) \quad g_{\lambda}(\tau, \mu) = \frac{[(1/\mu) \int_{\tau}^{\infty} B_{\lambda}(\tau) e^{-\tau/\mu} d\tau - B_{\lambda}(\tau) e^{-\tau/\mu}]}{(1/\mu) \int_0^{\infty} B_{\lambda}(\tau) e^{-\tau/\mu} d\tau}$$

Throughout, we denote the monochromatic optical depth in the continuum, by τ without subscript and the monochromatic line optical depth by τ_{λ} or τ_{ν} . No use is made of mean optical depth and no wave length is adopted as standard. The monochromatic character of the Planckian and weighting functions may be denoted by a subscript τ or ν .

Following the suggestion of P. ten Bruggencate (1955) and G. Elste (1955) we transform the τ -scale to the $\log_{10}\tau$ -scale. The use of the latter scale is advantageous for two reasons: (1) a change of scale during the integration of (IV-10) is obviated, and (2) the formation of an absorption line is much more readily visualized in the $\log \tau$ -scale owing to the fact that it is approximately proportional to the geometrical depth (e. g., see Pagel 1956, Fig. 1). Equation (IV-13) may then be written, with $x = \log_{10}\tau$:

$$(IV-14) \quad g_{\lambda}(x, \mu) = [(1/\mu) \int_x^{+\infty} B_{\lambda} A dx - B_{\lambda} C] / (1/\mu) \int_{-\infty}^{+\infty} B_{\lambda} A dx$$

in which we define:

$$(IV-15) \quad C = e^{-e^{z/\text{mod}}}$$

$$(IV-16) \quad A = C e^{z/\text{mod}} / \text{mod}$$

$$(IV-17) \quad z = x - \log \mu$$

and mod is the modulus 0.4343.

For the calculation of η_{ν} we require the following (Aller 1953,

pp. 252ff.): for the mass absorption coefficient in the line:

$k_V = N_n \alpha_V / N_H m_H$, in which α_V is the atomic absorption coefficient, N_H the number of hydrogen atoms per gram of stellar material, and m_H the mass of the hydrogen atom; and where N_n , the number of atoms in a level $\underline{n}$ which can participate in the absorption $\underline{n}$ to $\underline{n}'$, is given by the Boltzmann relation:

$$(IV-18) \quad N_n = N_i g_n e^{-X_n/kT} / B_i(T) = N_i g_n 10^{-\theta X_n / B_i(T)}$$

N_i is the number of atoms of the element which are found in a given stage of ionization, $\underline{i}$; g_n , the statistical weight of the level $\underline{n}$; B_n , the partition function, which describes the level populations in a non-vacuous universe and is, at solar temperatures, slightly temperature dependent; X_n , the excitation energy of the lower level $\underline{n}$; and θ is given by:

$$(IV-19) \quad \theta = 5040/T$$

For the atoms of interest, calculation has shown for the atmosphere to be tested that the second stage of ionization ('zero'th stage is the neutral atom) is found to be completely negligible, so that we may write, for a given atom:

$$(IV-20) \quad N_0 + N_1 = AN_H$$

where A is the abundance of the element in number of atoms with respect to hydrogen.

The distribution of atoms between the first two stages of ionization is given by the Saha relation:

$$(IV-21) \quad \log(N_1/N_0) = -\log P_e - \theta I - 2.5 \log \theta + \log[2B_1(T)/B_0(T)] + 8.779$$

which we find useful to express as:

$$(IV-22) \quad s = e^{[-\theta I + \log(2B_1/B_0) - \log P_e + 8.779] / \log e} \theta^{-5/2}$$

For a neutral lines, then, the number of participating atoms is given by:

$$(IV-23) \quad N_n = [AN_H g_1 10^{-\theta x_1}] / [(s + 1) B_0(T)]$$

and for an ion line, by:

$$(IV-24) \quad N_n = [s/(s+1)] AN_H g_1 10^{-\theta x_1} / B_1(T)$$

α_v , the atomic absorption coefficient, is most conveniently expressed in terms of α_0 , the fictitious absorption coefficient for line center and zero damping, and in units of the Doppler width, Δv_D . These quantities are given by:

$$(IV-25) \quad \alpha_0 = (\pi^{1/2} e^2 / mc) (f / \Delta v_0)$$

with f the classical oscillator strength, and:

$$(IV-26) \quad \Delta v_0 = (v_0 / c) (2k'T/M)^{1/2}$$

in which v_0 is the frequency of line center, T the temperature, k' Boltzmann's constant, M the mass of the atom, and the remaining symbols have their familiar meanings.

Expressed as a power series the quantity α_v / α_0 may be easily calculated with the aid of tables given by Harris (1948) and excerpted in Aller (1953, p. 254):

$$(IV-27) \quad \alpha_v / \alpha_0 = H_0(u) + aH_1(u) + a^2H_2(u) + \dots$$

where u is the distance from line center in units of the Doppler width and a , the ratio of the damping to the Doppler width.

1. Damping. -- One of the chief advantages gained by the study of weak lines is that the effects of damping are usually ignorable; consequently, poorly determined damping coefficients and the variation with depth of the parameters on which they depend are unable to influence equivalent widths (Minnaert 1953, p. 161; Elste 1955).

The ratio of damping to Doppler width is, therefore, taken equal to zero throughout the calculations in this investigation. In one instance an arbitrary finite damping is employed, however, in a test of

the qualitative prediction made for non-zero damping (Section (V-C)).

2. Turbulence.--We adopt the terminology, macroturbulence and microturbulence, to describe large-scale motions which are, respectively, large and small with respect to the layer of line formation. We also adopt the practice of trying to account for the effect of large-scale motions by including a turbulence parameter in the expression for the Doppler width as follows:

$$(IV-28) \quad \Delta v_D = (v_o/c) (2k'T/M + \beta^2)^{1/2}$$

$$\Delta \lambda_D = (\lambda_o/c) (2k'T/M + \beta^2)^{1/2}$$

where β may be thought of as the mean square velocity of the bodies of gas, both large and small (Huang 1950), that act "thermally" to increase the absorption.

3. Variation in $\Delta \lambda_D$ resulting from the temperature gradient.--Since the exact calculations discussed in Chapter V are based on a Doppler width taken constant with depth, we show here why this approximation is a very good one. For the lightest atom involved, namely, oxygen, the thermal portion of the Doppler width is approximately 2/3 of the total $(\Delta \lambda_D)^2$.

a) Variation in $\Delta \lambda_D$ over the path length of line-formation.--At $\mu=1.0$ the effective limits of formation of O I 7772, for example, are at optical depths $x=-0.7$ and $+0.4$, where the corresponding temperatures are 5356° and 7033° . In this range the Doppler width would be expected to change by 9 per cent due to temperature. We adopt then a mean $\Delta \lambda_D$ for the whole line-forming region with $T=6000^\circ$.

b) Variation in the mean $\Delta \lambda_D$ across the solar disc.--The effective depths of formation for the same line, O I 7772, at $\mu=1.0$ and $\mu=0.2$ are $x=+0.1$ and $x=-0.9$, where the corresponding temperatures are 6414° and

and 5177° . The mean Doppler width is thus diminished by 7 per cent, center to limb.

There are three circumstances that make the variation in the thermal portion of $\Delta\lambda_D$ just described an upper limit for the remaining lines under study: (1) the atoms Mg through Sr are from 1-1/2 to 5-1/2 times heavier than O; (2) lines of lower excitation potential are formed higher in the photosphere in a region of smaller temperature gradient, with the results that (b) is very much reduced and (a) slightly; and (3) where anisotropic large-scale motions are introduced into $\Delta\lambda_D$ for $\mu < 1.0$, the thermal fraction of the Doppler width is reduced.

4. The continuous absorption.--For the continuous mass absorption coefficient, we write down the sum of the contributions of the negative hydrogen ion and the neutral hydrogen atom:

$$(IV-29) \quad k = (1 - x)[P_e\alpha(H^-) + \alpha(H)]$$

where $\alpha(H^-)$ is the coefficient of absorption of the negative hydrogen ion per neutral hydrogen atom per dyne electron pressure given by Chandrasekhar and Breen (1946) and includes the stimulated emission, and $\alpha(H)$ is the absorption coefficient of neutral hydrogen, per neutral hydrogen atom. x stands for the fraction of hydrogen atoms which are ionized.

Since the degree of ionization of hydrogen in the continuum and line-forming regions certainly does not exceed one per cent, we set $x = 0$ in (IV-29).

In addition, the calculations are much simplified by the omission of the neutral hydrogen absorption. More important than the degree of ionization, this omission has the effect, in the most critical instance, of giving equivalent widths at the center of the disc which are too

large by 2.6 per cent (oxygen I).

5. The data required for the calculations.--For the case of pure absorption, the equivalent widths of the observed lines were calculated at 8 points on the solar disc. Owing to the restrictions imposed as well as the extra labor required in reducing all calculations to a standard wave length, it was decided to carry out the calculations at each of the wave lengths observed. Comparison of line-formation in the same geometrical layers but at different wave lengths may be accomplished by simply overlaying the graphs of the so-called contribution functions of the lines.

Aside from the constants π , ϵ , m , m_H , c , and k , which are common to all the calculations, we require: (1) for each atom, its atomic mass (M), abundance relative to hydrogen (A), ionization potential (I), and partition functions ($B_1(T)$); (2) for each transition, the excitation potential of its lower level (χ_1), the statistical weight of its lower level (g_n), the oscillator strength (f), and the frequency of line center (ν_0); and (3) for each level in the atmosphere, the temperature ($T=5040/\theta$), electron pressure (P_e), the continuous absorption coefficient per dyne electron pressure ($\alpha(H^-)$), and the weighting function for pure absorption ($g(x, \mu)$).

Since the Michigan abundances of April 1956 (Goldberg et al. 1957) were entered explicitly into the calculations, the parameter used in fitting the observed line strengths is:

$$(IV-30) \quad (\epsilon^2 \pi^{1/2} / mc^2) (\lambda^2) (gf) = g \Delta \lambda_D \alpha_0$$

a) Partition functions.--To attain exactness in the high-speed calculations where machine instruction storage permitted, the partition functions (Claas 1951) were introduced as analytic functions of θ . The

least-squares coefficients expressing $B_1(T)$ as a quadratic in θ are given in Table 9.

TABLE 9
PARTITION FUNCTIONS AS $f(\theta)$

	Element					
	Ca	Mg	Si	O	Sr	Fe
a	- 0.480	+ 0.14	+ 0.069	- 0.046	- 1.370	+ 0.0725
b	+ 1.617	+ 0.759	- 0.1098	0	+ 3.436	+ 0.742
c	- 0.585	- 0.310	+ 0.1207	0	- 1.585	- 0.329
a'	+ 6.619	+ 2.0	+ 5.75	-	+ 6.619	-
b'	- 8.654	0	0	-	- 8.654	-
c'	+ 4.256	0	0	-	+ 4.256	-
a''	+11.245	+ 2.203	+13.42	+ 9.0	-	+ 95.98
b''	-19.383	- 2.07	- 5.85	0	-	-121.65
c''	+ 9.390	+ 0.878	+ 1.890	0	-	+ 53.90

$$\log(2B_1/B_0) = a + b\theta + c\theta^2$$

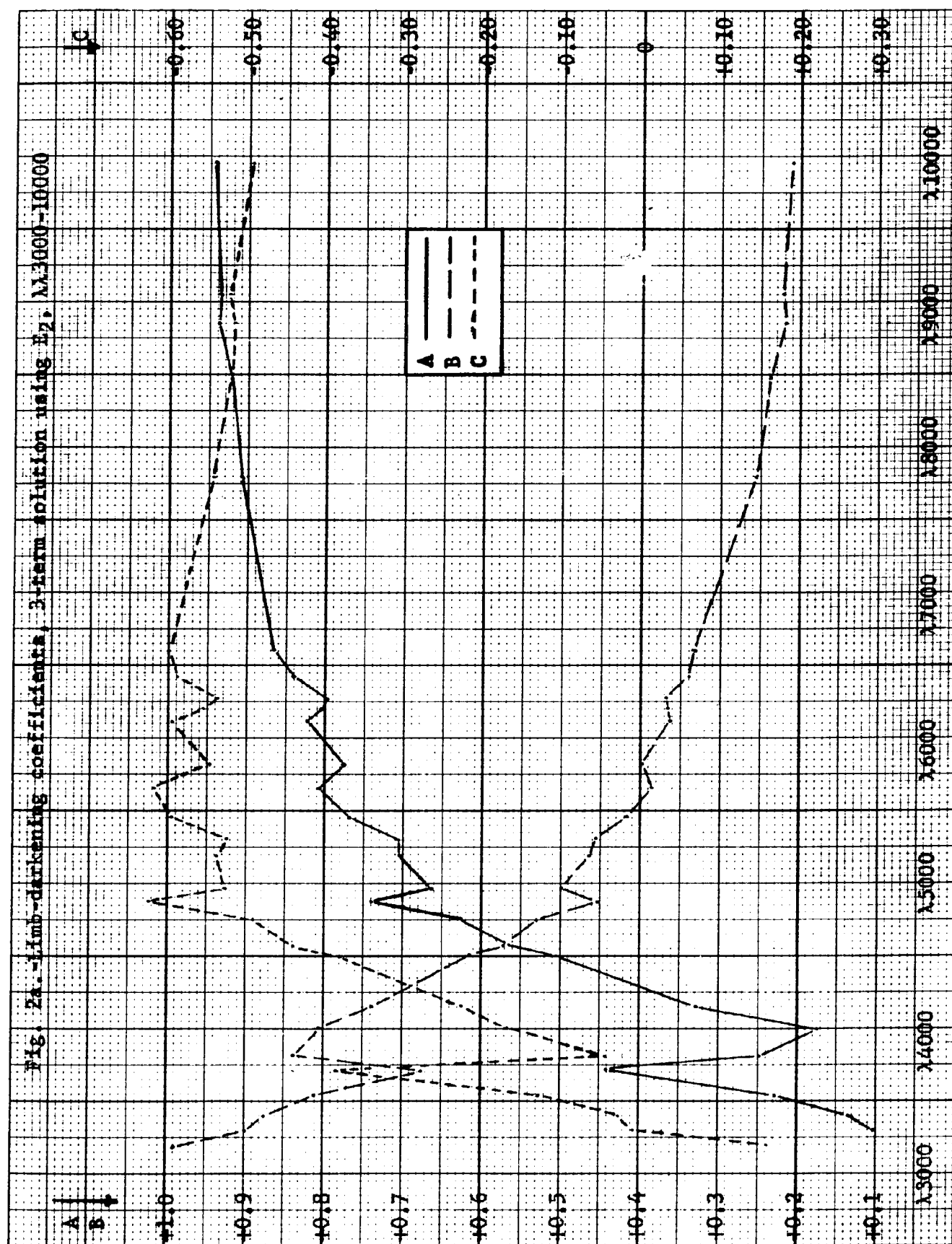
$$B_1 = a' + b'\theta + c'\theta^2$$

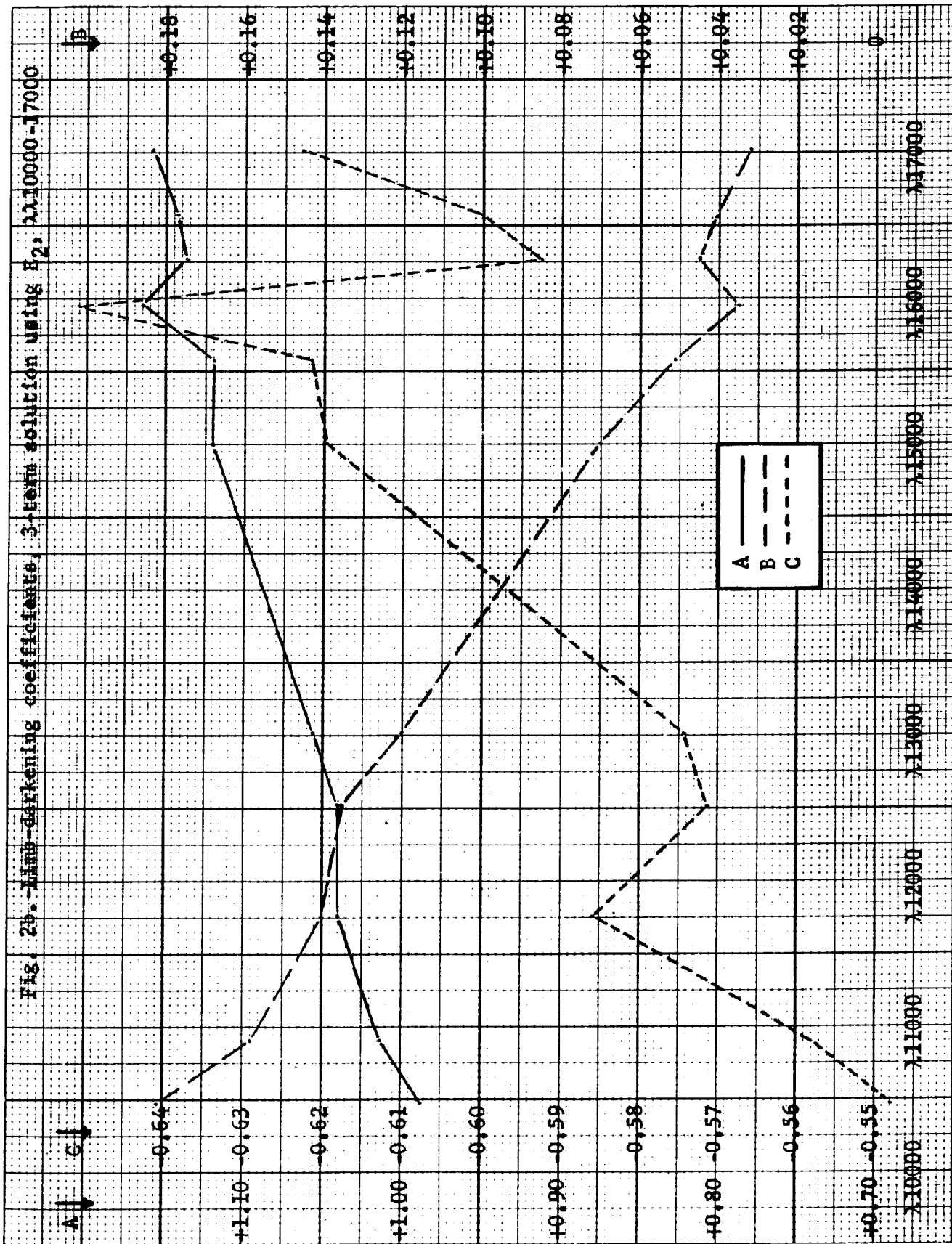
$$B_0 = a'' + b''\theta + c''\theta^2$$

b) The atmosphere data and the parameters of the Aller-Pierce-Elste model.--The parameters θ , P_e , $\alpha(H^-)$, and g for the ten wave lengths used in this investigation were obtained in the following way:

(1) Temperature (T).--The temperature distribution (T, τ and θ, τ) was calculated with the aid of equation (III-13) and the limb-darkening coefficients obtained in the 28-point E2 solution described in Section (III-D). For wave lengths $\lambda\lambda 7772, 10914, \text{ and } 15879$, the limb-darkening coefficients came directly from the observations, while for $\lambda\lambda 5780, 6371, 6575, 7896, 8912, 16151, \text{ and } 21225$, the coefficients were interpolated graphically from Figure 2.

Given in Table 10 are the calculated values of T ; the corresponding values of θ were entered into the machine calculations. Column (1) gives the $\log_{10}$ of the monochromatic optical depth, and





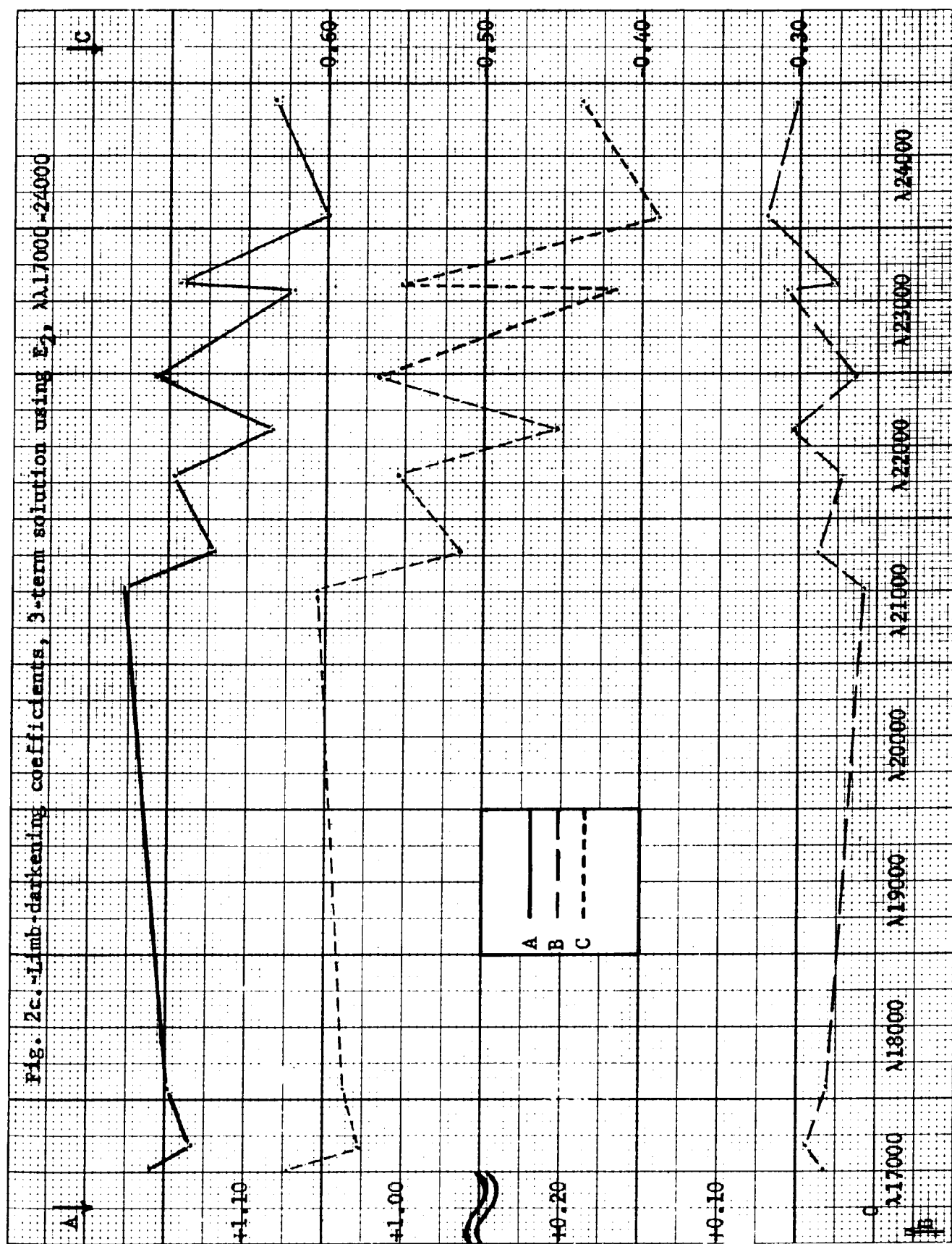


TABLE 10

TEMPERATURE DISTRIBUTIONS

x	$\lambda 5780$	$\lambda 6371$	$\lambda 6575$	$\lambda 7772$	$\lambda 7896$	$\lambda 8912$	$\lambda 10914$	$\lambda 15879$	$\lambda 16151$	$\lambda 21225$
-3.8	4549 ^o	4557 ^o	4538 ^o	4617 ^o	4622 ^o	4505 ^o	4554 ^o	4679 ^o	4807 ^o	4602 ^o
-3.4	4553	4562	4542	4620	4626	4599	4557	4684	4812	4606
-3.0	4563	4570	4551	4628	4633	4517	4565	4695	4822	4614
-2.5	4592	4597	4578	4649	4654	4540	4588	4727	4852	4641
-2.4	4602	4605	4588	4657	4661	4547	4596	4738	4862	4650
-2.3	4613	4616	4599	4666	4670	4557	4606	4751	4874	4661
-2.2	4628	4629	2612	4676	4681	4568	4617	4767	4889	4674
-2.1	4645	4644	4628	4689	4693	4581	4631	4786	4907	4690
-2.0	4665	4663	4647	4704	4708	4597	4647	4808	4927	4708
-1.9	4689	4684	4669	4772	4726	4616	4666	4835	4952	4731
-1.8	4717	4701	4696	4744	4747	4639	4689	4866	4982	4757
-1.7	4749	4740	4727	4769	4772	4665	4716	4903	5016	4788
-1.6	4787	4775	4763	4799	4802	4696	4747	4947	5057	4824
-1.5	4831	4815	4804	4833	4836	4732	4784	4997	5104	4867
-1.4	4881	4862	4852	4873	4876	4774	4827	5056	5160	4907
-1.3	4938	4915	4907	4920	4921	4882	4877	5124	5224	4974
-1.2	5003	4976	4970	4973	4974	4877	4934	5202	5297	5040
-1.1	5075	5045	5040	5033	5034	4940	4999	5291	5390	5116
-1.0	5156	5121	5118	5101	5102	5010	5073	5390	5475	5201
-0.9	5245	5206	5205	5177	5177	5089	5156	5502	5580	5297
-0.8	5342	5300	5301	5262	5262	5176	5248	5625	5693	5403
-0.7	5448	5403	5405	5356	5355	5272	5349	5749	5825	5520
-0.6	5563	5514	5518	5458	5457	5376	5460	5903	5963	5646
-0.5	5687	5634	5639	5568	5567	5487	5579	6056	6109	5780
-0.4	5818	5763	5768	5687	5686	5606	5707	6215	6262	5920
-0.3	5959	5900	5906	5815	5814	5731	5842	6377	6411	6064
-0.2	6108	6047	6051	5950	5949	5862	5983	6537	6576	6208
-0.1	6268	6203	6206	6094	6094	5998	6130	6693	6739	6348
0.0	6439	6371	6370	6248	6249	6140	6282	6839	6876	6482
+0.1	6624	6553	6547	6414	6416	6289	6442	6974	7014	6606
+0.2	6830	6754	6741	6596	6600	6448	6612	7095	7141	6720
+0.3	7058	6978	7967	6800	6806	6622	6798	7205	7261	6825
+0.4	7318	7234	7203	7034	7042	6818	7007	7306	7378	6925
+0.5	7618	7531	7488	7307	7318	7046	7250	7407	7499	7027
+0.6	7966	7879	7823	7630	7646	7317	7541	7515	7634	7139
+0.7	8374	8289	8218	8017	8039	7643	7894	7641	7794	7270
+0.8	8852	8775	8688	8483	8551	8039	8326	7794	7990	7431
+0.9	9414	9350	9247	9043	9080	8519	8857	7984	8233	7632

columns (2)-(11), the temperature. The variation in temperature along any horizontal line in Table 10 will serve to remind the reader that the log τ -scale in column (1) is a monochromatic scale. The same geometrical layer can be found for the different wave lengths by tracing an isotherm through the table.

(2) Electron pressure (P_e).--The electron pressure distribution of the modified Aller-Pierce model (Goldberg et al. 1957) was expressed in terms of the individual (T, τ) scales in the following straightforward way: the electron pressure-temperature distribution was taken to be unique, i. e., at those wave lengths for which the temperature distributions were calculated, the appropriate modified Aller-Pierce electron pressure distribution was interpolated. The values are given in Table 11.

(3) The absorption coefficient of the negative hydrogen ion ($\alpha(H^-)$).--Values for the H^- absorption coefficient have been calculated by Chandrasekhar and Breen (1946). However, their published tables give the coefficients for only 11 temperatures ($\theta=0.50$ to 2.0) and 18 wave lengths (0.4050μ to 9.113μ). Since extensive use was to be made of these theoretical quantities, the tables of Chandrasekhar and Breen were accurately interpolated for the wave lengths of interest and further extended by the same program (Program 5, Appendix II), to provide tables for the region $\lambda\lambda 5200-25000$ in 200 A. intervals. The values interpolated for a single wave length were in turn graphed against temperature and the coefficients for each log τ -scale read off. The adopted H^- coefficients are listed in Table 12.

(4) The weighting functions for pure absorption ($g(x,\mu)$).--The monochromatic weighting functions for pure absorption, $g_\lambda(x,\mu)$, are tabulated in Table 13 (a, b, ...j) for the wave lengths of this thesis.

TABLE 11

ELECTRON PRESSURE FOR ALLER-PIERCE-ELSTE MODEL

x	Log P _e (dynes)									
	$\lambda 5780$	$\lambda 6371$	$\lambda 6575$	$\lambda 7772$	$\lambda 7896$	$\lambda 8912$	$\lambda 19014$	$\lambda 15879$	$\lambda 16151$	$\lambda 21225$
-3.8	-0.39	-0.35	-0.47	-0.07	-0.05	-0.68	-0.38	+0.14	+0.44	-0.12
-3.4	-0.36	-0.30	-0.43	-0.06	-0.03	-0.64	-0.32	+0.15	+0.45	-0.10
-3.0	-0.30	-0.25	-0.38	-0.02	+0.00	-0.58	-0.28	+0.18	+0.46	-0.07
-2.9	-0.28	-0.24	-0.35	-0.01	+0.01	-0.56	-0.27	+0.19	+0.46	-0.06
-2.8	-0.25	-0.22	-0.32	+0.00	+0.02	-0.52	-0.25	+0.20	+0.47	-0.04
-2.7	-0.23	-0.20	-0.28	+0.01	+0.03	-0.51	-0.23	+0.22	+0.48	-0.02
-2.6	-0.20	-0.17	-0.25	+0.03	+0.05	-0.48	-0.21	+0.24	+0.49	+0.00
-2.5	-0.16	-0.14	-0.21	+0.05	+0.07	-0.41	-0.18	+0.26	+0.51	+0.03
-2.4	-0.11	-0.10	-0.18	+0.08	+0.09	-0.34	-0.15	+0.29	+0.53	+0.06
-2.3	-0.07	-0.07	-0.13	+0.10	+0.11	-0.27	-0.11	+0.32	+0.55	+0.09
-2.2	-0.02	-0.02	-0.08	+0.12	+0.13	-0.20	-0.07	+0.35	+0.58	+0.13
-2.1	+0.04	+0.04	-0.01	+0.16	+0.17	-0.13	-0.02	+0.39	+0.61	+0.17
-2.0	+0.10	+0.09	+0.05	+0.21	+0.22	-0.07	+0.04	+0.44	+0.64	+0.22
-1.9	+0.15	+0.15	+0.11	+0.26	+0.26	-0.03	+0.10	+0.48	+0.67	+0.27
-1.8	+0.23	+0.22	+0.18	+0.31	+0.31	+0.01	+0.16	+0.53	+0.71	+0.32
-1.7	+0.31	+0.29	+0.26	+0.36	+0.36	+0.09	+0.23	+0.59	+0.75	+0.39
-1.6	+0.39	+0.36	+0.35	+0.42	+0.42	+0.18	+0.31	+0.66	+0.79	+0.46
-1.5	+0.48	+0.44	+0.43	+0.49	+0.48	+0.27	+0.39	+0.72	+0.84	+0.54
-1.4	+0.56	+0.52	+0.51	+0.56	+0.55	+0.36	+0.47	+0.78	+0.90	+0.62
-1.3	+0.65	+0.61	+0.60	+0.62	+0.62	+0.45	+0.56	+0.86	+0.96	+0.69
-1.2	+0.73	+0.70	+0.69	+0.69	+0.70	+0.55	+0.65	+0.94	+1.03	+0.77
-1.1	+0.81	+0.78	+0.77	+0.76	+0.76	+0.64	+0.73	+1.02	+1.09	+0.85
-1.0	+0.89	+0.86	+0.86	+0.83	+0.83	+0.74	+0.81	+1.10	+1.16	+0.94
-0.9	+0.97	+0.94	+0.94	+0.91	+0.91	+0.83	+0.89	+1.18	+1.23	+1.03
-0.8	+1.06	+1.03	+1.03	+0.99	+1.00	+0.92	+0.98	+1.26	+1.30	+1.12
-0.7	+1.14	+1.11	+1.11	+1.07	+1.07	+1.00	+1.07	+1.35	+1.39	+1.20
-0.6	+1.22	+1.19	+1.19	+1.15	+1.15	+1.09	+1.16	+1.45	+1.48	+1.28
-0.5	+1.30	+1.27	+1.27	+1.22	+1.22	+1.17	+1.23	+1.55	+1.59	+1.37
-0.4	+1.39	+1.35	+1.35	+1.30	+1.30	+1.25	+1.31	+1.66	+1.70	+1.46
-0.3	+1.48	+1.44	+1.45	+1.39	+1.39	+1.33	+1.40	+1.79	+1.82	+1.55
-0.2	+1.59	+1.54	+1.54	+1.48	+1.48	+1.42	+1.50	+1.92	+1.95	+1.65
-0.1	+1.70	+1.66	+1.66	+1.58	+1.58	+1.51	+1.60	+2.04	+2.06	+1.76
0.0	+1.84	+1.78	+1.78	+1.68	+1.68	+1.61	+1.71	+2.14	+2.17	+1.87
+0.1	+1.98	+1.93	+1.93	+1.82	+1.75	+1.72	+1.84	+2.25	+2.27	+1.97
+0.2	+2.14	+2.09	+2.07	+1.97	+1.82	+1.84	+1.98	+2.33	+2.37	+2.06
+0.3	+2.32	+2.27	+2.24	+2.12	+1.97	+1.98	+2.13	+2.42	+2.45	+2.13
+0.4	+2.49	+2.44	+2.41	+2.29	+2.12	+2.13	+2.28	+2.49	+2.53	+2.21
+0.5	+2.76	+2.68	+2.63	+2.49	+2.49	+2.30	+2.45	+2.55	+2.65	+2.29
+0.6	+3.07	+2.99	+2.95	+2.76	+2.78	+2.49	+2.68	+2.65	+2.77	+2.37
+0.7	+3.44	+3.37	+3.30	+3.13	+3.15	+2.78	+3.01	+2.78	+2.91	+2.46
+0.8	+3.85	+3.79	+3.72	+3.55	+3.57	+3.15	+3.40	+2.91	+3.09	+2.58
+0.9	+4.30	+4.25	+4.21	+4.02	+4.04	+3.57	+3.85	+3.09	+3.33	+2.76

TABLE 12

ABSORPTION COEFFICIENT OF NEGATIVE HYDROGEN ION^a

x	$\alpha(H^-) \times 10^{+26}$									
	$\lambda 5780$	$\lambda 6371$	$\lambda 6575$	$\lambda 7772$	$\lambda 7896$	$\lambda 8912$	$\lambda 10914$	$\lambda 15879$	$\lambda 16151$	$\lambda 21225$
-3.8	14.32	15.20	15.90	16.02	15.90	17.55	14.40	4.13	3.78	5.28
-3.4	14.28	15.14	15.75	16.02	15.84	17.47	14.37	4.12	3.76	5.27
-3.0	14.15	15.04	15.60	15.80	15.68	17.35	14.29	4.11	3.74	5.26
-2.9	14.09	14.99	15.55	15.80	15.62	17.30	14.26	4.10	3.73	5.26
-2.8	14.02	14.93	15.49	15.70	15.55	17.35	14.23	4.08	3.73	5.26
-2.7	13.95	14.87	15.42	15.70	15.48	17.20	14.20	4.07	3.72	5.25
-2.6	13.88	14.80	15.34	15.60	15.40	17.10	14.05	4.05	3.72	5.24
-2.5	13.75	14.68	15.25	15.60	15.29	17.00	13.90	4.03	3.71	5.23
-2.4	13.61	14.55	15.03	15.50	15.18	16.86	13.82	4.01	3.70	5.22
-2.3	13.46	14.41	14.78	15.54	15.07	16.71	13.66	3.99	3.68	5.20
-2.2	13.29	14.26	14.56	15.20	14.94	16.55	13.60	3.96	3.66	5.18
-2.1	13.10	14.10	14.34	15.00	14.80	16.38	13.40	3.92	3.63	5.15
-2.0	12.82	13.88	14.11	14.80	14.60	16.10	13.18	3.88	3.59	5.13
-1.9	12.62	13.60	13.87	14.60	14.40	15.80	12.98	3.83	3.56	5.10
-1.8	12.34	13.30	13.62	14.30	14.10	15.49	12.78	3.78	3.53	5.07
-1.7	11.96	12.94	13.22	13.90	13.80	15.15	12.46	3.72	3.47	5.03
-1.6	11.58	12.58	12.84	13.60	13.50	14.69	12.06	3.66	3.41	4.96
-1.5	11.06	12.14	12.36	13.20	13.05	14.19	11.78	3.57	3.35	4.90
-1.4	10.62	11.72	11.92	12.80	12.60	13.70	11.34	3.48	3.29	4.83
-1.3	10.10	11.10	11.34	12.30	12.10	13.16	11.11	3.38	3.21	4.75
-1.2	9.56	10.62	10.78	11.70	11.60	12.54	10.38	3.28	3.13	4.67
-1.1	9.06	9.98	10.16	11.00	11.00	11.90	9.83	3.17	3.04	4.60
-1.0	8.50	9.30	9.54	10.90	10.50	11.28	9.24	3.07	2.94	4.48
-0.9	7.90	8.78	8.96	9.86	9.52	10.54	8.64	2.93	2.84	4.36
-0.8	7.24	8.18	8.34	9.28	9.00	9.70	8.12	2.80	2.73	4.26
-0.7	6.73	7.54	7.68	8.67	8.02	9.16	7.52	2.68	2.61	4.12
-0.6	6.26	6.92	7.04	7.91	7.64	8.48	6.94	2.57	2.52	3.98
-0.5	5.74	6.40	6.46	7.42	7.38	7.74	6.40	2.45	2.40	3.85
-0.4	5.24	5.86	5.92	6.83	6.76	7.16	5.88	2.34	2.30	3.73
-0.3	4.80	5.37	5.42	6.26	6.22	6.58	5.45	2.22	2.21	3.60
-0.2	4.36	4.88	4.94	5.73	5.70	6.03	5.01	2.13	2.12	3.48
-0.1	3.96	4.44	4.48	5.24	5.16	5.51	4.56	2.04	2.04	3.37
0.0	3.56	3.98	4.06	4.76	4.84	5.06	4.21	1.96	1.97	3.28
+0.1	3.19	3.60	3.68	4.32	4.30	4.62	3.84	1.90	1.90	3.20
+0.2	2.86	3.24	3.34	3.90	3.88	4.19	3.44	1.84	1.85	3.11
+0.3	2.54	2.84	2.96	3.49	3.46	3.82	3.18	1.79	1.79	3.04
+0.4	2.26	2.51	2.60	3.08	3.04	3.46	2.88	1.75	1.75	2.97
+0.5	1.95	2.21	2.24	2.73	2.67	3.08	3.58	1.72	1.71	2.91
+0.6	1.67	1.88	1.90	2.30	2.29	2.72	2.31	1.68	1.66	2.84
+0.7	1.40	1.56	1.63	1.90	1.92	2.53	2.02	1.63	1.61	2.76
+0.8	1.16	1.30	1.38	1.50	1.62	1.98	1.70	1.58	1.54	2.68
+0.9	0.96	1.08	1.14	1.10	1.31	1.64	1.42	1.52	1.46	2.58

^aper neutral hydrogen atom and per unit electron pressure.

TABLE 13a

MONOCHROMATIC WEIGHTING FUNCTION FOR $\lambda 5780$

x	$g_{\lambda}(x, \mu)$							
	$\mu=1.00$	0.80	0.60	0.40	0.30	0.25	0.20	0.15
-3.8	0.7845	0.7601	0.7257	0.6722	0.6305	0.6026	0.5668	0.5187
-3.4	7834	7588	7243	6705	6286	6004	5645	5162
-3.0	7809	7560	7211	6666	6242	5958	5594	5105
-2.9	7799	7549	7198	6651	6225	5939	5574	5083
-2.8	7786	7535	7182	6632	6204	5916	5549	5055
-2.7	7771	7518	7163	6609	6178	5889	5519	5022
-2.6	7753	7498	7140	6581	6147	5855	5483	4982
-2.5	7731	7474	7112	6548	6109	5815	5439	4933
-2.4	7704	7444	7078	6507	6064	5766	5386	4875
-2.3	7672	7408	7037	6459	6009	5707	5332	4804
-2.2	7633	7365	6987	6400	5943	5637	5246	4720
-2.1	7586	7313	6928	6329	5864	5552	5154	4620
-2.0	7530	7250	6857	6245	5770	5451	5045	4500
-1.9	7463	7176	6772	6144	5657	5331	4916	4359
-1.8	7382	7087	6671	6025	5524	5190	4764	4194
-1.7	7287	6981	6551	5884	5367	5023	4585	4001
-1.6	7174	6856	6409	5717	5183	4828	4377	3779
-1.5	7041	6708	6242	5523	4969	4602	4138	3525
-1.4	6884	6535	6048	5297	4722	4342	3864	3238
-1.3	6701	6334	5821	5036	4439	4046	3556	2921
-1.2	6488	6100	5561	4739	4119	3715	3215	2576
-1.1	6243	5832	5264	4404	3763	3350	2845	2210
-1.0	5963	5528	4928	4032	3374	2955	2451	1834
-0.9	5646	5184	4554	3624	2956	2538	2045	1459
-0.8	5292	4803	4142	3188	2518	2110	1640	1104
-0.7	4899	4385	3699	2731	2075	1686	1253	0784
-0.6	4472	3936	3231	2267	1640	1284	0903	0516
-0.5	4014	3461	2748	1811	1234	0922	0605	0308
-0.4	3533	2971	2264	1382	0874	0616	0371	0164
-0.3	3039	2479	1799	0997	0575	0377	0204	0075
-0.2	2544	2000	1365	0672	0346	0207	0098	0029
-0.1	2060	1549	0981	0418	0186	0099	0040	0009
-0.0	1603	1142	0660	0234	0087	0040	0013	0.0002
+0.1	1187	0792	0409	0116	0035	0014	0.0004	-
+0.2	0825	0508	0228	0049	0011	0.0004	-	-
+0.3	0527	0295	0112	0017	0.0003	-	-	-
+0.4	0300	0150	0046	0005	-	-	-	-
+0.5	0143	0063	0015	0.0001	-	-	-	-
+0.6	0049	0020	0.0004	-	-	-	-	-
+0.7	0.0002	0.0002	-	-	-	-	-	-

TABLE 13b

MONOCHROMATIC WEIGHTING FUNCTION FOR $\lambda 6371$

x	$g_{\lambda}(x, \mu)$							
	$\mu=1.00$	0.80	0.60	0.40	0.30	0.25	0.20	0.15
-3.8	0.7395	0.7128	0.6756	0.6190	0.5758	0.5472	0.5112	0.4634
-3.4	7384	7115	6743	6174	5740	5453	5091	4611
-3.0	7359	7088	6711	6137	5699	5409	5044	4560
-2.9	7349	7076	6699	6122	5683	5392	5025	4539
-2.8	7337	7063	6684	6105	5663	5371	5002	4514
-2.7	7322	7047	6665	6083	5639	5345	4975	4484
-2.6	7304	7027	6643	6057	5610	5314	4941	4448
-2.5	7282	7003	6616	6025	5575	5277	4901	4403
-2.4	7256	6974	6583	5987	5532	5231	4852	4350
-2.3	7224	6939	6544	5941	5481	5177	4794	4287
-2.2	7186	6897	6497	5885	5420	5112	4724	4210
-2.1	7140	6846	6440	5819	5346	5034	4640	4119
-2.0	7085	6786	6372	5739	5258	4940	4540	4012
-1.9	7019	6714	6290	5645	5154	4839	4432	3884
-1.8	6941	6627	6194	5532	5030	4699	4283	3735
-1.7	6847	6525	6078	5399	4884	4545	4121	3561
-1.6	6737	6404	5943	5243	4713	4365	3930	3360
-1.5	6607	6261	5784	5060	4515	4157	3711	3132
-1.4	6454	6094	5598	4849	4286	3919	3463	2875
-1.3	6276	5900	5383	4605	4024	3648	3183	2590
-1.2	6069	5675	5135	4327	3730	3345	2874	2281
-1.1	5832	5418	4853	4015	3402	3011	2538	1954
-1.0	5561	5126	4535	3699	3044	2651	2183	1617
-0.9	5255	4798	4182	3291	2660	2272	1817	1284
-0.8	4913	4434	3795	2887	2261	1883	1453	0968
-0.7	4537	4037	3379	2466	1856	1500	1107	0686
-0.6	4129	3612	2941	2039	1463	1138	0794	0449
-0.5	3694	3165	2492	1622	1096	0814	0530	0268
-0.4	3239	2706	2046	1232	0772	0541	0323	0141
-0.3	2774	2248	1616	0884	0505	0329	0176	0064
-0.2	2312	1804	1220	0593	0302	0180	0084	0025
-0.1	1864	1391	0872	0366	0162	0086	0034	0008
0.0	1444	1020	0584	0204	0075	0035	0011	0.0002
+0.1	1065	0704	0360	0100	0030	0012	0.0003	-
+0.2	0737	0450	0200	0042	0010	0.0003	-	-
+0.3	0470	0260	0098	0015	0.0002	-	-	-
+0.4	0266	0132	0040	0.0004	-	-	-	-
+0.5	0127	0056	0013	-	-	-	-	-
+0.6	0043	0018	0.0003	-	-	-	-	-
+0.7	0.0002	0.0001	-	-	-	-	-	-

TABLE 13c

MONOCHROMATIC WEIGHTING FUNCTION FOR $\lambda 6575$

x	$g_{\lambda}(x, \mu)$							
	$\mu=1.00$	0.80	0.60	0.40	0.30	0.25	0.20	0.15
-3.8	0.7327	0.7065	0.6705	0.6153	0.5730	0.5451	0.5097	0.4628
-3.4	7315	7053	6690	6136	5712	5431	5076	4604
-3.0	7289	7024	6658	6098	5670	5386	5028	4552
-2.9	7278	7013	6645	6083	5653	5369	5009	4531
-2.8	7266	6999	6629	6065	5633	5347	4986	4506
-2.7	7250	6982	6610	6043	5609	5321	4958	4475
-2.6	7232	6961	6587	6016	5579	5289	4924	4438
-2.5	7209	6937	6560	5983	5543	5251	4882	4393
-2.4	7182	6907	6526	5944	5500	5205	4883	4339
-2.3	7149	6870	6485	5897	5447	5150	4774	4274
-2.2	7109	6827	6437	5840	5385	5083	4702	4197
-2.1	7061	6775	6378	5772	5310	5004	4617	4105
-2.0	7004	6712	6308	5691	5220	4909	4516	3996
-1.9	6936	6637	6224	5594	5114	4796	4396	3867
-1.8	6854	6548	6125	5480	4988	4664	4255	3716
-1.7	6758	6443	6007	5344	4840	4508	4090	3540
-1.6	6644	6318	5869	5186	4667	4326	3899	3338
-1.5	6509	6172	5706	5000	4466	4116	3678	3108
-1.4	6352	6000	5516	4785	4235	3875	3428	2849
-1.3	6168	5801	5297	4538	3971	3602	3147	2563
-1.2	5956	5572	5045	4258	3673	3297	2836	2254
-1.1	5713	5309	4759	3943	3344	2963	2500	1926
-1.0	5436	5012	4437	3594	2985	2602	2145	1591
-0.9	5124	4679	4081	3215	2602	2224	1780	1260
-0.8	4777	4311	3692	2811	2204	1838	1419	0947
-0.7	4396	3912	3275	2392	1803	1458	1076	0668
-0.6	3985	3485	2838	1970	1414	1101	0769	0435
-0.5	3549	3040	2394	1559	1054	0783	0510	0258
-0.4	3097	2585	1954	1177	0738	0517	0309	0135
-0.3	2638	2135	1534	0839	0480	0312	0167	0061
-0.2	2185	1703	1150	0558	0284	0169	0079	0023
-0.1	1751	1304	0817	0342	0151	0080	0032	0007
0.0	1349	0951	0542	0189	0070	0032	0010	0.0002
+0.1	0989	0652	0332	0092	0027	0010	0.0003	-
+0.2	0682	0415	0184	0038	0009	0.0003	-	-
+0.3	0432	0239	0089	0013	0.0002	-	-	-
+0.4	0244	0121	0037	0.0004	-	-	-	-
+0.5	0116	0051	0012	-	-	-	-	-
+0.6	0.0039	0.0016	0.0003	-	-	-	-	-

TABLE 13d

MONOCHROMATIC WEIGHTING FUNCTION FOR $\lambda 7772$

x	$g_{\lambda}(x, \mu)$							
	$\mu=1.00$	0.80	0.60	0.40	0.30	0.25	0.20	0.15
-3.8	0.6302	0.6007	0.5610	0.5026	0.4599	0.4324	0.3985	0.3550
-3.4	6292	5995	5597	5012	4584	4308	3968	3531
-3.0	6268	5970	5569	4981	4549	4272	3930	3490
-2.9	6259	5960	5558	4968	4536	4257	3914	3474
-2.8	6247	5948	5545	4953	4519	4240	3896	3454
-2.7	6233	5933	5528	4934	4499	4219	3874	3431
-2.6	6217	5915	5508	4812	4474	4193	3847	3402
-2.5	6196	5893	5484	4885	4445	4162	3814	3367
-2.4	6172	5866	5455	4852	4410	4125	3775	3325
-2.3	6142	5834	5420	4812	4367	4080	3727	3275
-2.2	6107	5796	5378	4765	4316	4027	3671	3215
-2.1	6064	5750	5328	4708	4254	3962	3603	3143
-2.0	6013	5695	5268	4640	4181	3886	3523	3058
-1.9	5952	5629	5196	4560	4094	3796	3428	2959
-1.8	5880	5551	5110	4464	3992	3689	3317	2842
-1.7	5793	5459	5009	4352	3871	3563	3186	2706
-1.6	5693	5350	4890	4219	3730	3417	3055	2550
-1.5	5572	5221	4751	4065	3567	3249	2861	2372
-1.4	5432	5072	4589	3886	3378	3056	2664	2172
-1.3	5269	4898	4401	3682	3164	2837	2442	1952
-1.2	5081	4698	4186	3450	2924	2594	2198	1714
-1.1	4866	4469	3942	3189	2657	2326	1935	1462
-1.0	4622	4211	3669	2902	2367	2040	1657	1206
-0.9	4347	3923	3366	2590	2059	1739	1372	0952
-0.8	4042	3605	3037	2258	1739	1433	1090	0714
-0.7	3709	3261	2686	1915	1418	1133	0825	0502
-0.6	3350	2895	2320	1571	1108	0852	0587	0326
-0.5	2972	2514	1947	1238	0822	0603	0387	0192
-0.4	2582	2128	1582	0930	0572	0396	0233	0100
-0.3	2188	1748	1235	0659	0370	0238	0126	0045
-0.2	1803	1387	0920	0436	0218	0128	0059	0017
-0.1	1436	1056	0649	0265	0114	0060	0023	0005
0.0	1100	0765	0428	0145	0052	0024	0008	0.0001
+0.1	0802	0522	0261	0070	0020	0008	0.0002	-
+0.2	0550	0330	0143	0029	0006	0.0002	-	-
+0.3	0347	0189	0069	0010	0.0002	-	-	-
+0.4	0195	0095	0028	0.0003	-	-	-	-
+0.5	0092	0040	0009	-	-	-	-	-
+0.6	0.0030	0.0012	0.0002	-	-	-	-	-

TABLE 13e
MONOCHROMATIC WEIGHTING FUNCTION FOR $\lambda 7896$

	$g_{\lambda}(x, \mu)$							
x	$\mu=1.00$	0.80	0.60	0.40	0.30	0.25	0.20	0.15
-3.8	0.6242	0.5944	0.5544	0.4959	0.4531	0.4256	0.3918	0.3486
-3.4	6232	5933	5532	4945	4516	4240	3902	3468
-3.0	6209	5908	5504	4914	4482	4205	3864	3428
-2.9	6200	5898	5944	4901	4469	4191	3849	3412
-2.8	6188	5886	5480	4886	4452	4174	3831	3392
-2.7	6175	5871	5464	4868	4432	4153	3809	3369
-2.6	6158	5853	5444	4846	4408	4128	3783	3341
-2.5	6138	5832	5421	4819	4380	4097	3750	3306
-2.4	6114	5806	5392	4787	4345	4061	3712	3265
-2.3	6085	5774	5358	4748	4303	4017	3666	3216
-2.2	6050	5737	5316	4702	4252	3964	3610	3157
-2.1	6008	5691	5267	4646	4192	3901	3544	3087
-2.0	5957	5637	5207	4579	4120	3826	3465	3004
-1.9	5897	5572	5136	4499	4035	3737	3372	2906
-1.8	5825	5495	5052	4405	3934	3632	3262	2791
-1.7	5740	5404	4952	4294	3815	3509	3134	2658
-1.6	5640	5296	4835	4164	3676	3365	2986	2505
-1.5	5522	5169	4698	4012	3516	3199	2815	2330
-1.4	5384	5022	4538	3836	3330	3010	2621	2134
-1.3	5223	4850	4353	3635	3120	2795	2403	1918
-1.2	5037	4652	4141	3406	2883	2555	2164	1684
-1.1	4824	4427	3900	3149	2620	2292	1904	1438
-1.0	4583	4172	3630	2866	2335	2010	1631	1185
-0.9	4311	3887	3332	2558	2031	1714	1351	0936
-0.8	4010	3573	3007	2232	1716	1413	1074	0702
-0.7	3680	3233	2660	1893	1400	1118	0812	0494
-0.6	3326	2871	2298	1553	1094	0841	0578	0321
-0.5	2952	2495	1930	1224	0812	0595	0382	0189
-0.4	2565	2113	1568	0921	0566	0391	0230	0099
-0.3	2175	1736	1225	0652	0366	0235	0124	0044
-0.2	1793	1378	0914	0432	0215	0126	0058	0017
-0.1	1429	1050	0645	0263	0113	0059	0023	0005
0.0	1095	0761	0426	0144	0052	0024	0008	0.0001
+0.1	0799	0519	0259	0070	0020	0008	0.0002	-
+0.2	0548	0329	0142	0029	0006	0.0002	-	-
+0.3	0346	0189	0069	0010	0.0002	-	-	-
+0.4	0194	0095	0028	0.0003	-	-	-	-
+0.5	0091	0040	0009	-	-	-	-	-
+0.6	0.0030	0.0012	0.0002	-	-	-	-	-

TABLE 13f

MONOCHROMATIC WEIGHTING FUNCTION FOR $\lambda 8912$

x	$g_{\lambda}(x, \mu)$							
	$\mu=1.00$	0.80	0.60	0.40	0.30	0.25	0.20	0.15
-3.8	0.5969	0.5699	0.5335	0.4796	0.4398	0.4140	0.3821	0.3410
-3.4	5958	5687	5322	4782	4382	4124	3804	3392
-3.0	5933	5660	5293	4750	4348	4087	3766	3351
-2.9	5923	5659	5282	4737	4334	4073	3750	3335
-2.8	5910	5637	5267	4721	4317	4055	3732	3315
-2.7	5896	5621	5250	4702	4296	4034	3710	3291
-2.6	5878	5602	5230	4679	4272	4008	3682	3262
-2.5	5856	5579	5205	4651	4242	3977	3650	3227
-2.4	5830	5551	5175	4618	4206	3940	3610	3186
-2.3	5799	5518	5139	4578	4163	3895	3563	3136
-2.2	5761	5478	5095	4530	4111	3841	3507	3076
-2.1	5716	5430	5043	4472	4050	3777	3439	3005
-2.0	5662	5372	4981	4403	3976	3700	3360	2921
-1.9	5598	5304	4907	4332	3889	3610	3265	2823
-1.8	5522	5223	4820	4225	3786	3503	3155	2707
-1.7	5431	5127	4716	4111	3666	3379	3026	2574
-1.6	5324	5014	4594	3978	3525	3234	2876	2420
-1.5	5200	4881	4453	3824	3363	3067	2705	2246
-1.4	5054	4727	4228	3646	3176	2877	2512	2052
-1.3	4885	4549	4099	3442	2965	2662	2295	1837
-1.2	4691	4345	3882	3212	2729	2424	2058	1067
-1.1	4470	4113	3638	2955	2468	2164	1802	1365
-1.0	4220	3852	3366	2672	2186	1886	1534	1119
-0.9	3941	3563	3066	2368	1888	1597	1262	0878
-0.8	3634	3247	2743	2048	1582	1305	0995	0652
-0.7	3301	2908	2401	1720	1276	1022	0745	0454
-0.6	2948	2551	2049	1393	0985	0759	0524	0291
-0.5	2579	2185	1696	1082	0720	0530	0341	0169
-0.4	2205	1820	1354	0799	0493	0342	0202	0087
-0.3	1836	1467	1037	0555	0312	0201	0106	0038
-0.2	1483	1140	0756	0358	0179	0105	0048	0014
-0.1	1157	0848	0521	0212	0092	0048	0019	0004
0.0	0867	0600	0335	0113	0041	0018	0006	0.0001
+0.1	0619	0401	0199	0053	0015	0006	0.0002	-
+0.2	0416	0248	0107	0022	0005	0.0002	-	-
+0.3	0258	0140	0051	0007	0.0001	-	-	-
+0.4	0143	0070	0020	0.0002	-	-	-	-
+0.5	0066	0029	0007	-	-	-	-	-
+0.6	0.0020	0.0009	0.0002	-	-	-	-	-

TABLE 13g

MONOCHROMATIC WEIGHTING FUNCTION FOR $\lambda 10914$

x	$g_{\lambda}(x, \mu)$							
	$\mu=1.00$	0.80	0.60	0.40	0.30	0.25	0.20	0.15
-3.8	0.5437	0.5170	0.4812	0.4289	0.3907	0.3663	0.3363	0.2981
-3.4	5426	5158	4840	4276	3893	3648	3348	2965
-3.0	5403	5133	4773	4246	3862	3615	3314	2929
-2.9	5393	5123	4762	4234	3849	3602	3300	2914
-2.8	5382	5111	4749	4220	3834	3586	3283	2897
-2.7	5368	5097	4734	4203	3816	3568	3263	2876
-2.6	5351	5079	4714	4182	3793	3544	3239	2850
-2.5	5331	5058	4692	4157	3766	3516	3210	2820
-2.4	5307	5032	4664	4126	3734	3483	3175	2783
-2.3	5277	5001	4631	4090	3696	3443	3133	2739
-2.2	5242	4964	4591	4046	3649	3395	3083	2686
-2.1	5220	4919	4543	3994	3594	3337	3023	2624
-2.0	5149	4866	4486	3932	3528	3269	2953	2550
-1.9	5089	4802	4418	3858	3449	3188	2869	2463
-1.8	5018	4727	4338	3770	3357	3093	2771	2362
-1.7	4933	4638	4243	3667	3249	2982	2656	2244
-1.6	4834	4533	4131	3547	3123	2853	2524	2110
-1.5	4717	4410	4001	3407	2977	2704	2373	1957
-1.4	4582	4286	3850	3246	2810	2534	2201	1786
-1.3	4424	4103	3677	3062	2621	2344	2010	1598
-1.2	4244	3915	3479	2854	2410	2132	1800	1396
-1.1	4038	3701	3256	2622	2177	1901	1575	1185
-1.0	3806	3461	3007	2368	1925	1654	1339	0970
-0.9	3547	3195	2735	2095	1660	1398	1099	0759
-0.8	3263	2904	2440	1808	1387	1140	0866	0563
-0.7	2956	2594	2131	1514	1117	0890	0646	0391
-0.6	2631	2268	1812	1223	0859	0660	0452	0250
-0.5	2293	1934	1494	0946	0626	0458	0293	0145
-0.4	1951	1604	1187	0695	0426	0294	0173	0073
-0.3	1616	1286	0904	0480	0268	0172	0090	0032
-0.2	1297	0993	0655	0308	0153	0090	0041	0012
-0.1	1005	0734	0448	0181	0078	0040	0016	0.0004
0.0	0747	0515	0286	0096	0034	0016	0005	-
+0.1	0530	0341	0168	0045	0013	0005	0.0001	-
+0.2	0353	0210	0090	0017	0004	0.0001	-	-
+0.3	0218	0118	0042	0006	0.0001	-	-	-
+0.4	0120	0058	0017	0.0002	-	-	-	-
+0.5	0054	0024	0006	-	-	-	-	-
+0.6	0.0016	0.0007	0.0001	-	-	-	-	-

TABLE 13h

MONOCHROMATIC WEIGHTING FUNCTION FOR $\lambda 15879$

x	$g_{\lambda}(x, \mu)$							
	$\mu=1.00$	0.80	0.60	0.40	0.30	0.25	0.20	0.15
-3.8	0.4861	0.4673	0.4405	0.3985	0.3663	0.3450	0.3186	0.2842
-3.4	4849	4660	4392	3971	3648	3435	3170	2825
-3.0	4822	4633	4363	3940	3615	3401	3134	2788
-2.9	4812	4622	4351	3928	3602	3388	3120	2773
-2.8	4799	4608	4337	3912	3586	3372	3104	2756
-2.7	4783	4592	4320	3894	3567	3352	3083	2734
-2.6	4764	4573	4300	3872	3544	3328	3058	2708
-2.5	4742	4550	4275	3846	3516	3299	3029	2677
-2.4	4714	4521	4246	3814	3483	3265	2993	2640
-2.3	4682	4487	4210	3776	3443	3224	2950	2596
-2.2	4642	4447	4167	3731	3395	3175	2900	2542
-2.1	4596	4398	4116	3676	3338	3116	2839	2480
-2.0	4540	4340	4056	3612	3271	3047	2767	2406
-1.9	4473	4272	3984	3535	3191	2965	2683	2319
-1.8	4394	4190	3899	3445	3097	2869	2584	2218
-1.7	4302	4095	3800	3339	2987	2757	2470	2101
-1.6	4193	3983	3683	3216	2860	2627	2338	1968
-1.5	4066	3852	3548	3074	2714	2479	2188	1817
-1.4	3919	3701	3391	2911	2547	2310	2019	1650
-1.3	3750	3528	3213	2726	2359	2122	1832	1467
-1.2	3558	3331	3011	2519	2151	1915	1628	1272
-1.1	3340	3109	2785	2290	1923	1690	1410	1069
-1.0	3097	2863	2535	2041	1680	1454	1185	0865
-0.9	2830	2594	2265	1777	1426	1210	0959	0668
-0.8	2540	2304	1978	1503	1170	0969	0741	0487
-0.7	2233	1999	1681	1228	0920	0740	0541	0331
-0.6	1914	1687	1383	0961	0687	0532	0369	0206
-0.5	1593	1376	1093	0715	0482	0356	0230	0105
-0.4	1280	1079	0824	0499	0313	0218	0130	0056
-0.3	0986	0807	0586	0324	0185	0120	0064	0023
-0.2	0724	0571	0390	0191	0097	0058	0027	0008
-0.1	0503	0379	0240	0101	0044	0024	0009	0.0002
0.0	0328	0234	0134	0047	0017	0008	0.0002	-
+0.1	0200	0133	0068	0019	0005	0.0002	-	-
+0.2	0113	0070	0030	0006	0.0001	-	-	-
+0.3	0058	0033	0012	0.0002	-	-	-	-
+0.4	0025	0014	0004	-	-	-	-	-
+0.5	0.0007	0.0005	0.0001	-	-	-	-	-

TABLE 13i

MONOCHROMATIC WEIGHTING FUNCTION FOR $\lambda 16151$

x	$g_{\lambda}(x, \mu)$							
	$\mu=1.00$	0.80	0.60	0.40	0.30	0.25	0.20	0.15
-3.8	0.4594	0.4399	0.4126	0.3706	0.3389	0.3182	0.2926	0.2597
-3.4	4582	4387	4113	3693	3375	3168	2912	2582
-3.0	4558	4362	4087	3664	3345	3137	2880	2548
-2.9	4548	4352	4076	3653	3333	3125	2867	2535
-2.8	4536	4339	4063	3639	3319	3110	2851	2519
-2.7	4522	4324	4048	3622	3301	3092	2833	2500
-2.6	4504	4307	4029	3602	3280	3070	2810	2476
-2.5	4484	4285	4006	3578	3254	3044	2783	2448
-2.4	4458	4259	3979	3549	3224	3012	2751	2414
-2.3	4428	4227	3946	3514	3187	2975	2712	2374
-2.2	4392	4190	3907	3472	3143	2930	2665	2325
-2.1	4348	4145	3860	3422	3091	2876	2610	2268
-2.0	4296	4091	3804	3362	3029	2812	2544	2201
-1.9	4235	4028	3737	3292	2956	2737	2468	2122
-1.8	4162	3953	3659	3209	2870	2649	2378	2030
-1.7	4076	3864	3567	3112	2768	2547	2273	1923
-1.6	3976	3760	3459	2998	2652	2428	2152	1802
-1.5	3858	3640	3334	2867	2518	2292	2015	1665
-1.4	3722	3500	3189	2717	2364	2137	1860	1512
-1.3	3565	3339	3024	2546	2191	1964	1689	1346
-1.2	3386	3156	2836	2354	2000	1774	1502	1168
-1.1	3184	2950	2627	2143	1790	1568	1303	0982
-1.0	2958	2721	2396	1913	1566	1350	1096	0796
-0.9	2709	2470	2144	1668	1331	1126	0888	0615
-0.8	2439	2200	1877	1414	1094	0903	0688	0450
-0.7	2152	1916	1600	1158	0862	0691	0503	0306
-0.6	1853	1624	1321	0910	0646	0499	0344	0191
-0.5	1552	1332	1050	0680	0455	0335	0216	0107
-0.4	1256	1052	0796	0478	0297	0206	0122	0052
-0.3	0978	0795	0572	0312	0177	0114	0061	0022
-0.2	0728	0570	0385	0186	0094	0056	0026	0007
-0.1	0515	0385	0241	0100	0044	0023	0009	0.0002
0.0	0344	0243	0138	0048	0017	0008	0.0002	-
+0.1	0216	0143	0072	0019	0006	0.0002	-	-
+0.2	0127	0077	0034	0007	0.0002	-	-	-
+0.3	0068	0038	0014	0.0002	-	-	-	-
+0.4	0032	0017	0005	-	-	-	-	-
+0.5	0.0010	0.0006	0.0002	-	-	-	-	-

TABLE 13j

MONOCHROMATIC WEIGHTING FUNCTION FOR $\lambda 21225$

x	$g_{\lambda}(x, \mu)$							
	$\mu=1.00$	0.80	0.60	0.40	0.30	0.25	0.20	0.15
-3.8	0.4041	0.3857	0.3600	0.3209	0.2917	0.2728	0.2497	0.2202
-3.4	4031	3847	3590	3198	2905	2716	2484	2189
-3.0	4009	3824	3566	3173	2879	2689	2457	2160
-2.9	4000	3815	3557	3163	2869	2679	2446	2149
-2.8	3990	3804	3545	3151	2856	2666	2432	2135
-2.7	3977	3791	3532	3136	2841	2650	2416	2119
-2.6	3962	3775	3515	3119	2823	2632	2397	2099
-2.5	3943	3756	3495	3098	2801	2609	2374	2075
-2.4	3920	3733	3471	3072	2774	2582	2346	2046
-2.3	3893	3705	3442	3042	2743	2550	2313	2012
-2.2	3861	3672	3408	3005	2705	2511	2273	1971
-2.1	3823	3632	3366	2962	2660	2465	2226	1922
-2.0	3776	3585	3317	2910	2606	2410	2170	1865
-1.9	3722	3529	3259	2849	2543	2346	2104	1798
-1.8	3657	3462	3190	2776	2468	2270	2027	1720
-1.7	3581	3384	3109	2692	2381	2182	1937	1629
-1.6	3491	3292	3014	2593	2280	2079	1834	1526
-1.5	3387	3185	2904	2479	2164	1962	1717	1410
-1.4	3266	3062	2777	2348	2032	1830	1585	1280
-1.3	3126	2920	2632	2200	1882	1681	1438	1139
-1.2	2968	2758	2468	2034	1717	1517	1278	0988
-1.1	2788	2576	2284	1850	1536	1340	1108	0830
-1.0	2588	2374	2081	1650	1343	1153	0932	0672
-0.9	2367	2153	1861	1437	1141	0961	0755	0520
-0.8	2128	1915	1628	1217	0937	0770	0584	0379
-0.7	1874	1665	1385	0995	0737	0588	0427	0258
-0.6	1610	1408	1141	0781	0552	0424	0291	0161
-0.5	1344	1152	0904	0582	0388	0284	0182	0090
-0.4	1084	0907	0684	0408	0252	0175	0103	0044
-0.3	0839	0682	0489	0265	0150	0096	0051	0018
-0.2	0620	0486	0328	0158	0079	0046	0021	0006
-0.1	0435	0325	0203	0084	0036	0019	0007	0.0002
0.0	0287	0203	0115	0039	0014	0006	0.0002	-
+0.1	0177	0118	0059	0016	0005	0.0002	-	-
+0.2	0102	0062	0027	0005	0.0001	-	-	-
+0.3	0053	0030	0011	0.0002	-	-	-	-
+0.4	0024	0013	0004	-	-	-	-	-
+0.5	0.0007	0.0004	0.0001	-	-	-	-	-

They were calculated with the same source function employed in (1) above by numerical integration, using the classical form (equation (IV-13) and Program 6, Appendix II) rather than after the suggestion of Elste (private communication) to use the analytic form. The latter method is ideally suited for finding readily any value of $g(x, \mu)$, but in the systematic calculation of g for many x and several μ , it requires frequent regeneration of the exponential integral and, hence, is more consumptive of machine time.

The former method provides an excellent confirmation of the limb-darkening solution, for the emergent intensity at several points on the disc is automatically obtained in the denominator of equation (IV-13). The integral: $(1/\mu) \int_{-\infty}^{+\infty} B_{\lambda} A dx$ was normalized with respect to its value at $\mu=1.0$, and its center-limb change was found to reproduce the limb-darkening very closely indeed from $\mu=1.0$ to 0.15.

The limits of the integration of (IV-13) were in practice taken at $x=-4.0$ and $+0.9$, since the exponential character of A and C and the diminishing difference in the numerator prevent any contribution to g to come from outside this range. Elste's objection (private communication) to the numerical method, that the differencing in the numerator leads to inaccuracies, would seem to apply only when the interval of integration is several times larger than the $\Delta x=0.1$ used here. The difference appears well-determined except where it is effectively equal to zero; in any case, occasional negative values for g at $x=+0.7$, $+0.8$, and $+0.9$ are essentially zero.

One additional advantage to the numerical or classical method in the calculation of g accrues from its versatility, an advantage only realizable when the problem has been adapted to high-speed

computing methods. In the program employed (Program 6, Appendix II), the source function could be varied arbitrarily in an analytic or synthetic way with the possibility to obtain easily the corresponding weighting function and limb-darkening.

6. Extent of the calculations.--Equivalent widths: for the theoretical center-limb variation of the lines of interest, the expression on the right side of (IV-10) was evaluated (Program 7, Appendix II) at 7 points on the line profile at each of 8 points on the solar disc. The integration, which extended in steps of 0.1 in $\log \tau$ from -3.8 to +0.9, was carried out for one point on the line at a time and the equivalent width obtained by integration over the computed points. The running time of the equivalent width program on the IBM 650 was approximately 80 minutes.

When a number of trial runs had been made at the center of the disc, it was possible to choose a value for $g\Delta\lambda_D\alpha_0$ (see expression (IV-30)) which would produce a theoretical equivalent width as close as desired to that observed. The program was then allowed to run its course with the adopted $g\Delta\lambda_D\alpha_0$ and to calculate the full center-limb variation of the line.

The integration was carried out for the following points on the solar disc (μ) and in the line profile (v):

μ : 1.0, 0.8, 0.6, 0.4, 0.3, 0.25, 0.20, 0.15

v : 0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.1

7. Contribution functions.--The depth variation of the contribution to the emergent intensity in the point of a line profile was obtained by substantially the same program as the equivalent width, since it is merely the integrand in equation (IV-10).

The theoretical and observed equivalent widths, together with some contribution functions, are compared and discussed in Chapter V.

CHAPTER V

COMPARISON OF THEORY AND OBSERVATION

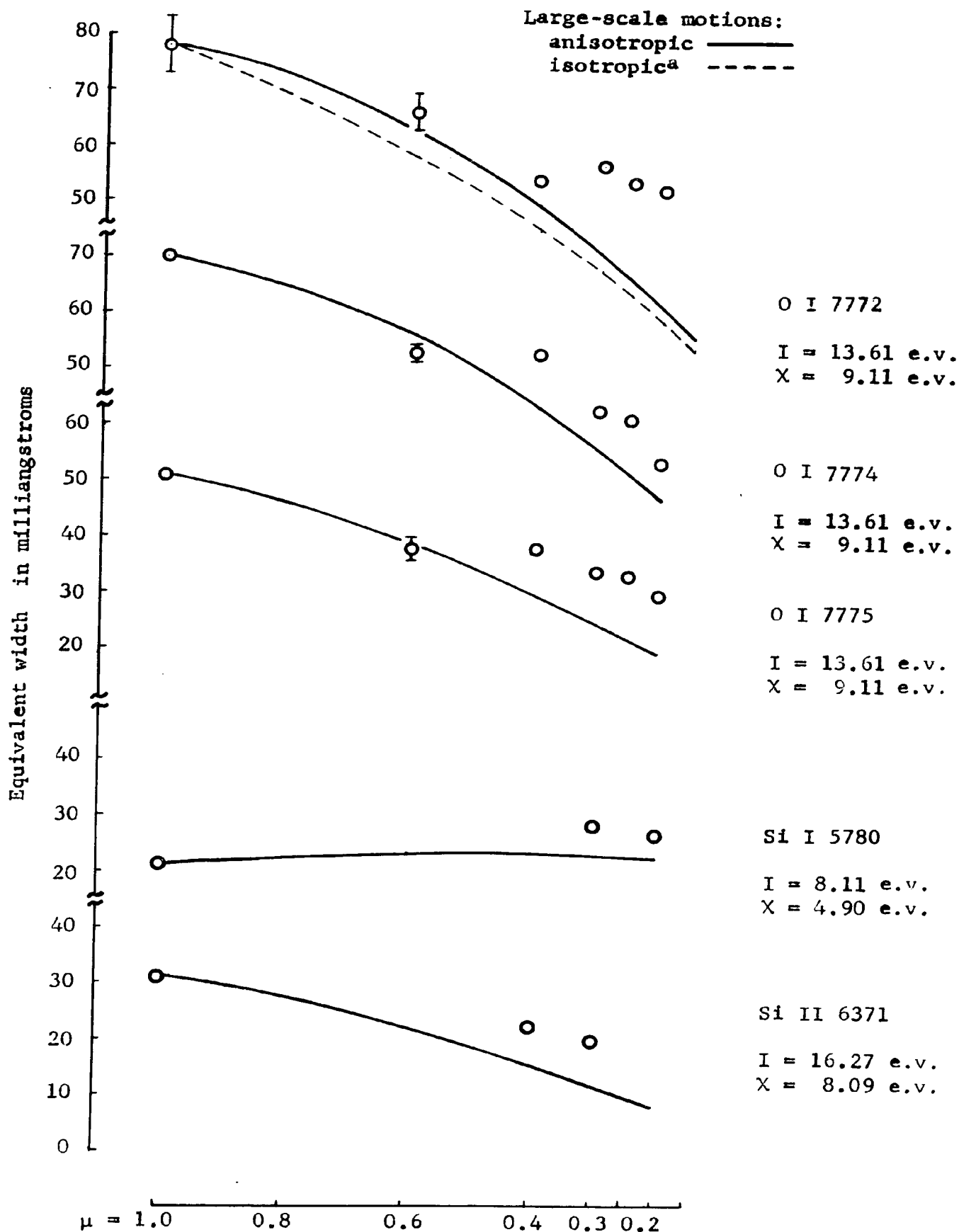
In this chapter a comparison is made between the results of the calculations described in Chapter IV and the observations presented in Chapter II (Table 2). Proposals are made for reducing the discrepancies between theory and observation by modifying the temperature and/or pressure in the outer layers of the Aller-Pierce-Elste model. The complications introduced into the analysis of line shapes and strengths by a velocity field of large-scale motions are discussed, and the groundwork is laid for a theoretical analysis of line shapes and strengths in terms of a simple turbulent spectrum.

In view of the success achieved by Waddell (1957) in describing the center-limb behavior of weak-line profiles, we include predictions for large-scale motions which are not only isotropic (1.8 km/sec) but also anisotropic (1.8 km/sec vertical, 2.9 km/sec horizontal).

In a study of laminar flow, Unno (1957) has cast some interesting light on the theoretical likelihood of large-scale motions, but we defer discussion of the observed parameters until the equivalent widths have been analyzed.

The theoretical and observed center-limb changes for the 14 lines of this study are shown in Figure 3 (a,b,etc.) for the cases of isotropic and anisotropic large-scale motions. A cursory examination of the results allows one to come to two conclusions immediately: (1) in the case of isotropic turbulence, the lines, with one exception (Ca I 6573),

Fig. 3a.-Comparison of theory and observation



^aIsotropic prediction shown only for stronger lines, where it differs noticeably from the anisotropic prediction.

Fig. 3b.-Comparison of theory and observation

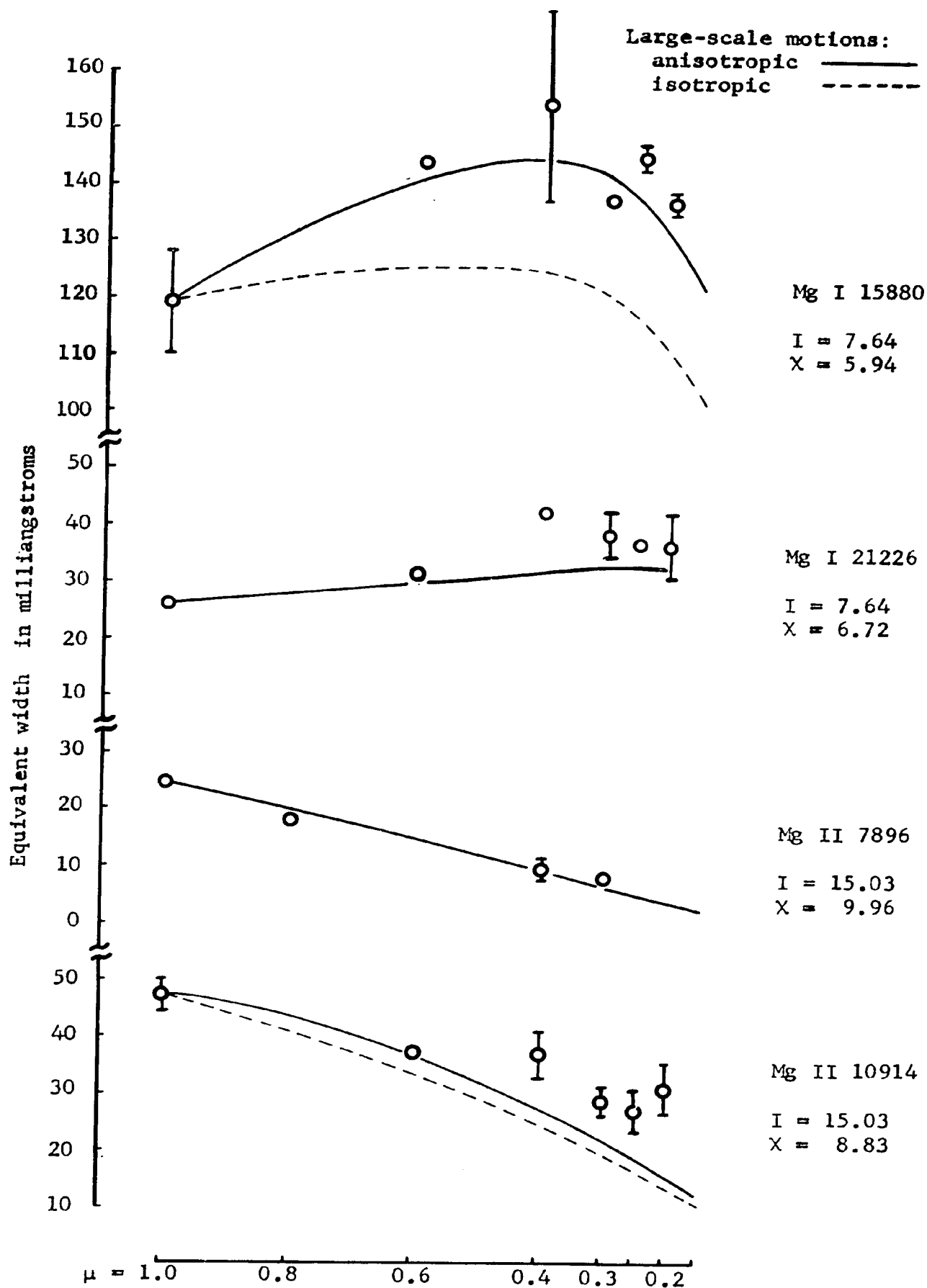
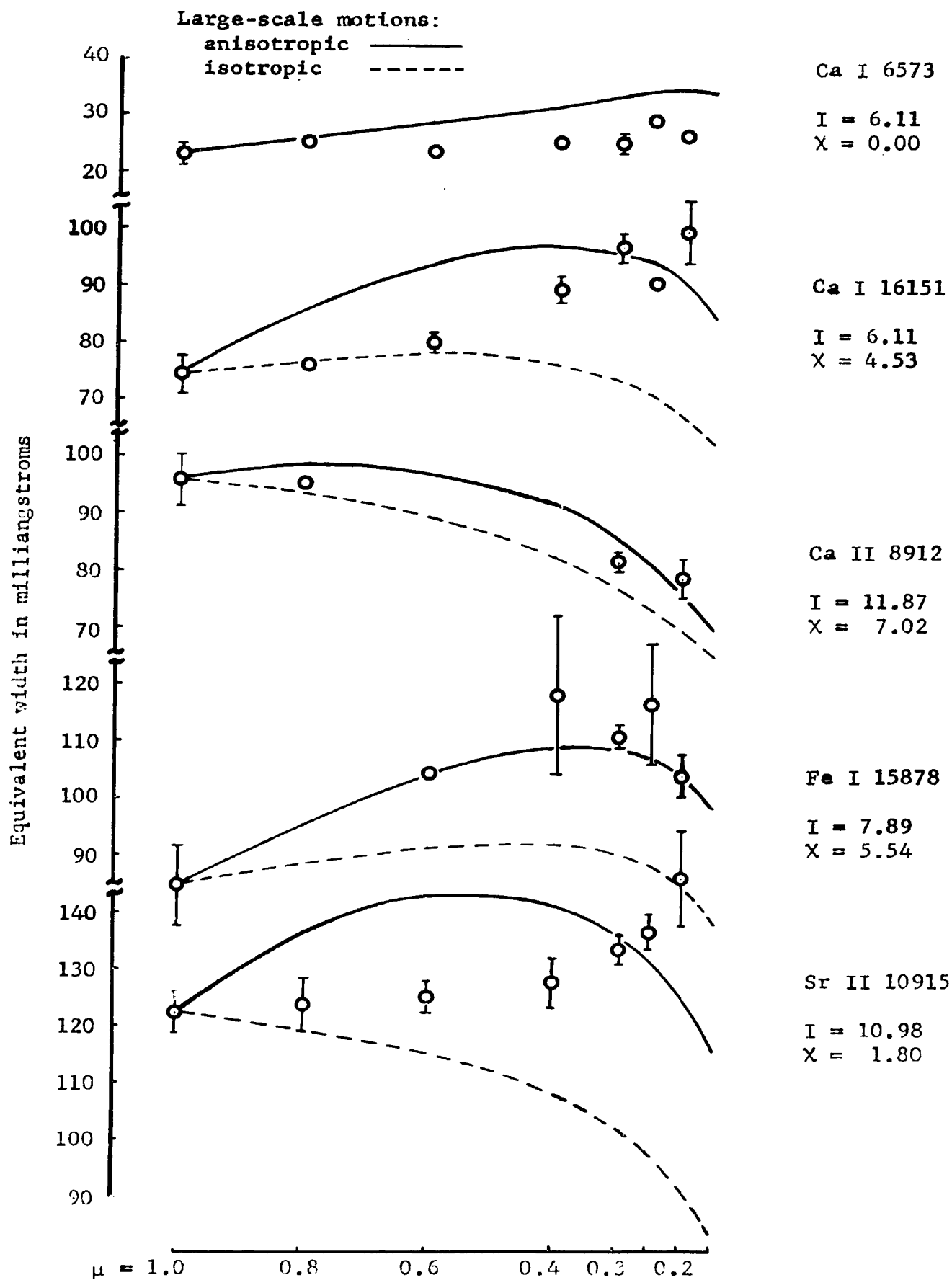


Fig. 3c.-Comparison of theory and observation



are stronger than their predicted values toward the limb; and (2) the center-limb variation in the equivalent width of a sufficiently weak line is completely insensitive to the magnitude of the Doppler width.

It is the first of these conclusions which has the greatest bearing on the parameters, temperature and pressure, which are the primary object of this investigation. We now explore in a semi-quantitative way the changes of temperature and pressure suggested by the observed center-limb behavior of the lines.

A. Temperature

For this purpose we have listed in Table 14 the transitions in the order of the sensitivity of their lower level population to a one per cent increase of temperature at $T=5000^{\circ}$, as determined from the Saha and Boltzmann relations. With the aid of Table 14, the ratios of observed to computed (anisotropic) equivalent widths at $\mu=0.2$ for the more sensitive lines were converted to temperature corrections which are listed in Table 15. In column (1) is the atom, (2) the stage, (3) the wave length, (4) change in temperature, (6) the relative strength of the line, W/λ , and (7) an estimate of depths, in terms of T , to which the corrections apply. The estimates in column (7) were obtained with the aid of the contribution functions plotted in Figure 4 (a,b,etc.).

In the determination of each correction, the temperature dependence of the negative hydrogen ion absorption coefficient has been taken into account.

B. Pressure

For the lines supposedly most sensitive to the electron pressure, corrections to that quantity have also been calculated on the straightforward assumption that the disparity between observed and computed

TABLE 14

TEMPERATURE SENSITIVITY OF ADOPTED LINES

Atom	Stage	λ	Percentage Change ^a	Atom	Stage	λ	Percentage Change ^a
Mg	II	7896	+ 26 %	Si	I	5780	- 9 %
O	I	7772-4-5	+ 23 %	Fe	I	15878	- 7 %
Mg	II	10915	+ 23 %	Ca	I	16151	- 6 %
Si	II	6371	+ 21 %	Mg	I	15880	- 5 %
Ca	II	8912	+ 18 %	Sr	II	10915	+ 4 %
Ca	I	6573	- 17 %	Mg	I	21226	- 3 %

^aPercentage change in the number of absorbing atoms for a one per cent increase in temperature at $T = 5000^\circ$.

TABLE 15

CORRECTIONS TO TEMPERATURE AND PRESSURE IN THE PHOTOSPHERE

Atom	Stage	λ	ΔT	$\Delta \log P_e$	W/λ	Depth
Ca	I	6573	+ 50°	-	0.356	4670°
Mg	II	7896	+ 50°	-0.108	0.299	5355°
		10914	+130°	-0.231	0.434	5250°
Si	I	5780	-200°	-	0.362	4787°
	II	6371	+180°	-0.28	0.483	5300°
O	I	7772	+230°	-0.445	1.000	5180°
		7774	"	"	0.892	"
		7775	"	"	0.655	"
Sr	II	10915	-	-0.19	1.11	4620°

limb ($\mu=0.2$) strengths of these lines may be resolved by increasing the efficiency of each of the line absorbers, i. e., to reduce the continuous

Fig. 4a. -Contribution Function: O I 7772

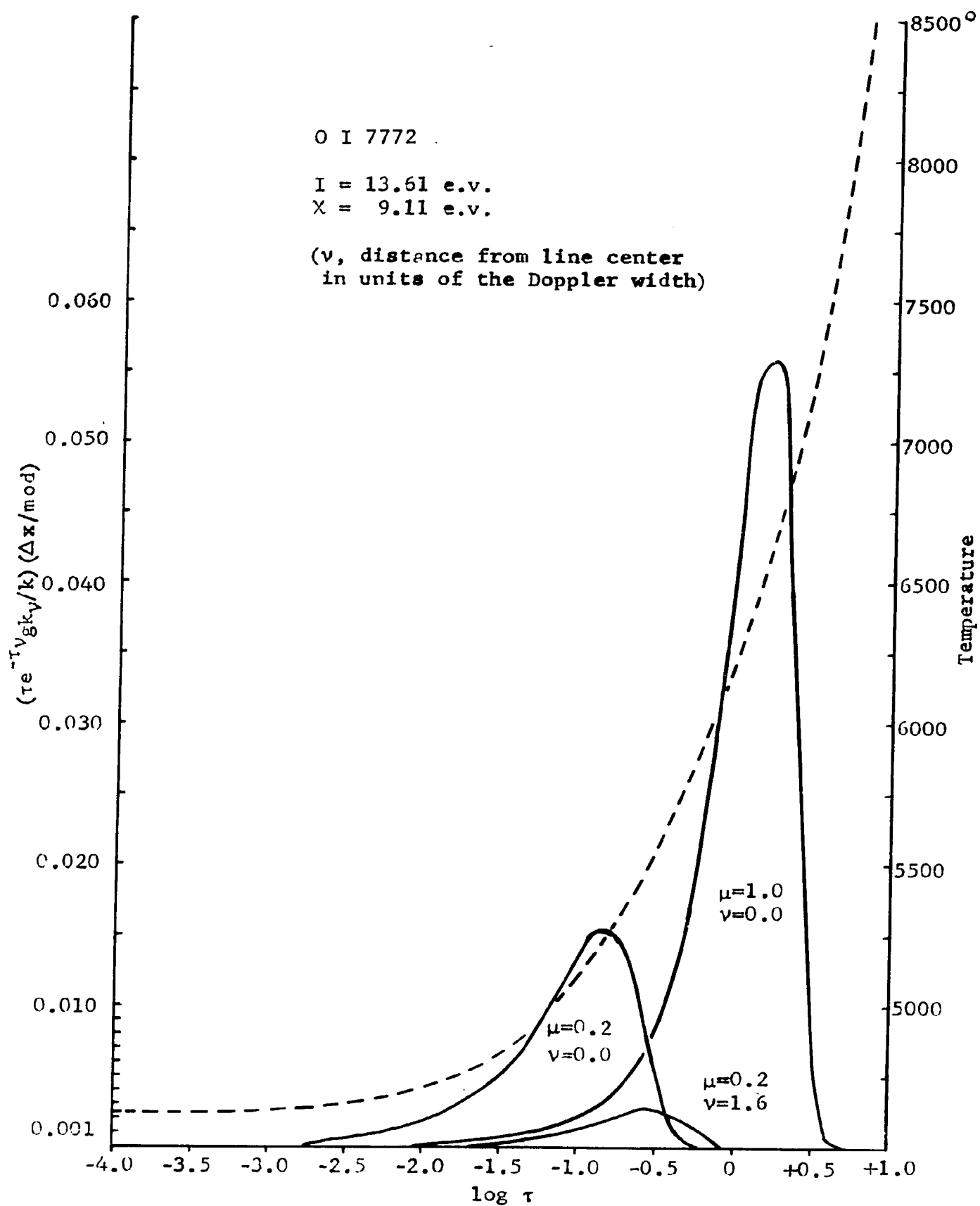


Fig. 4b.-Contribution Function: Sr II 10915

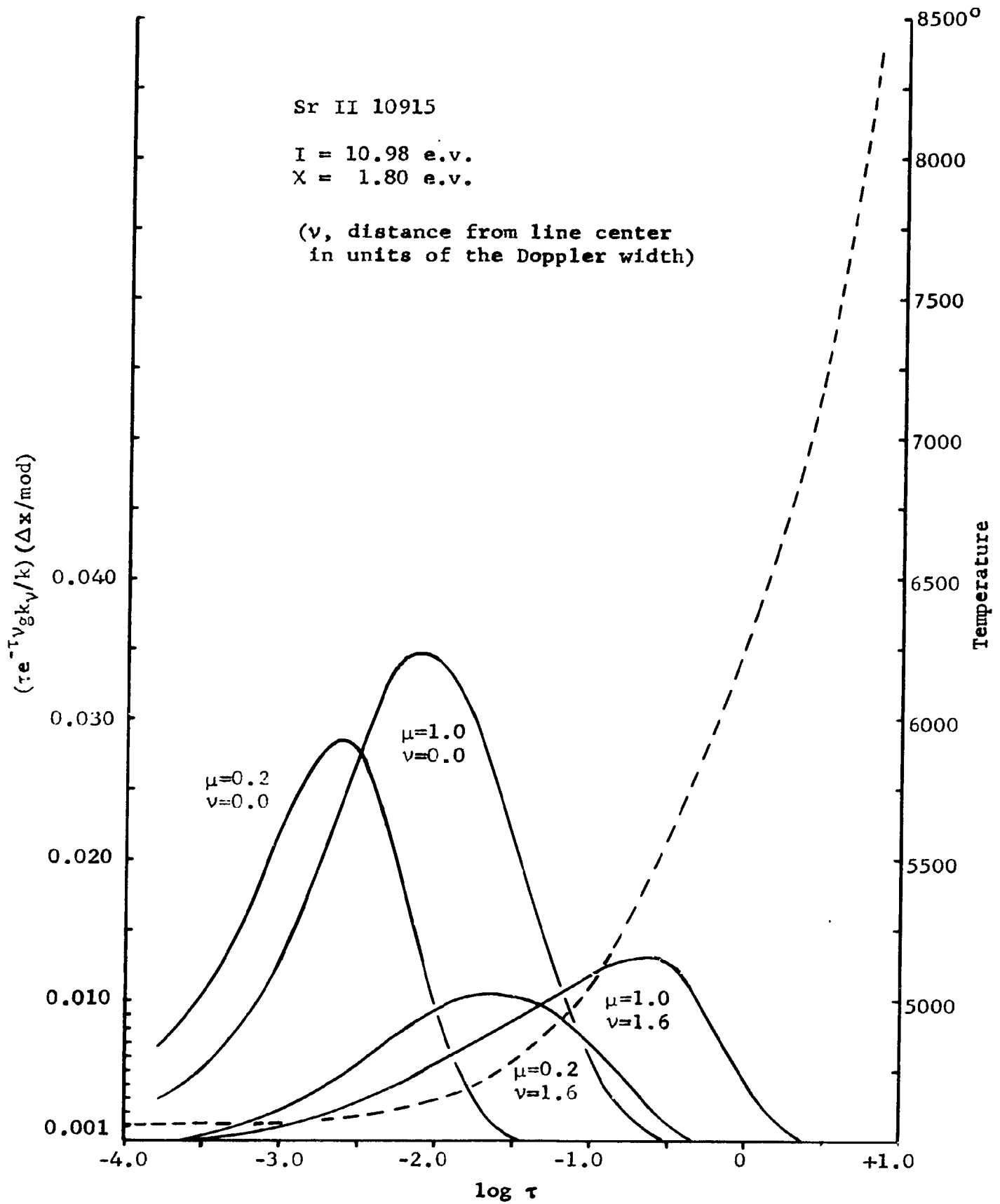


Fig. 4c. -Contribution Function: Mg II 7896

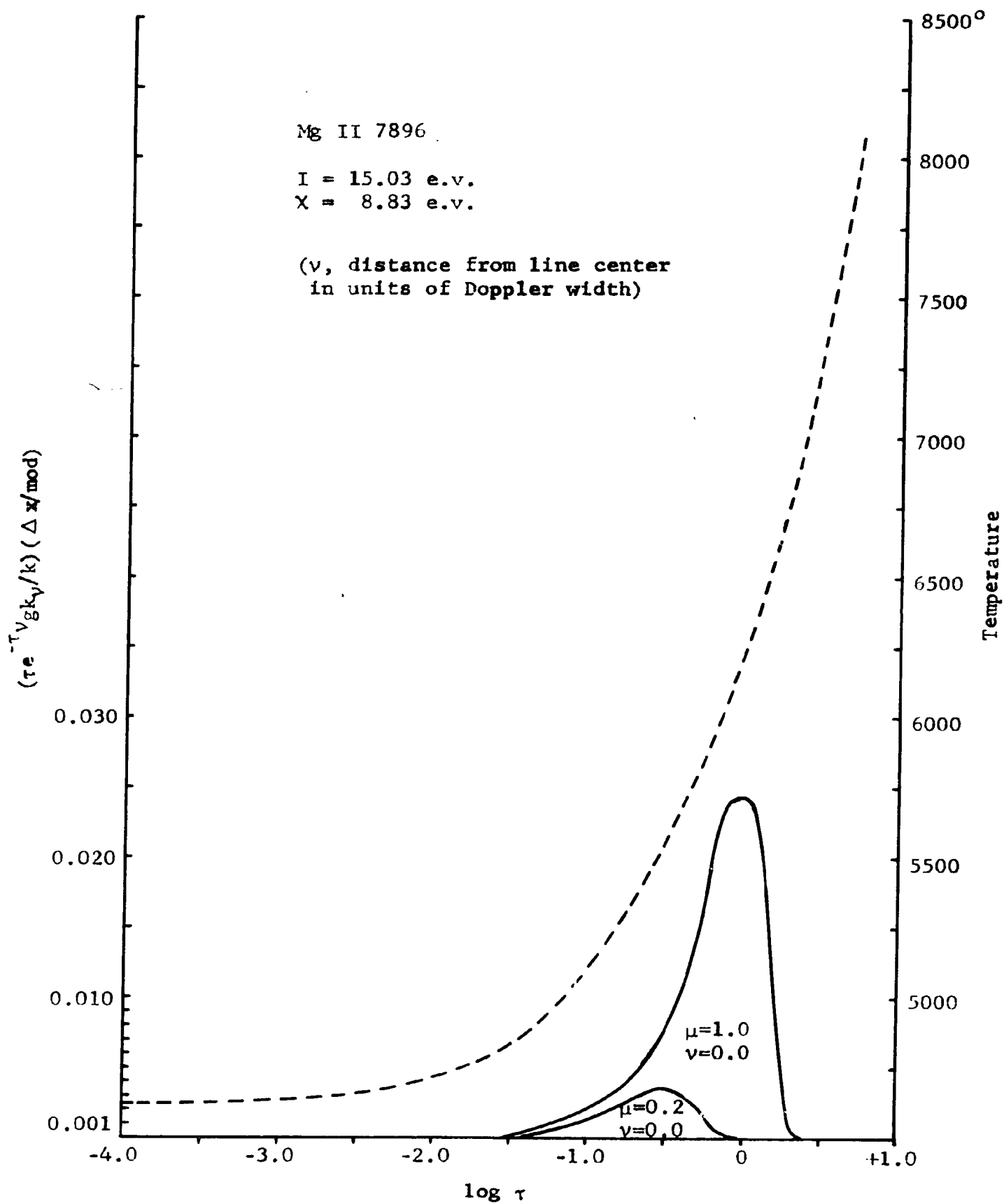
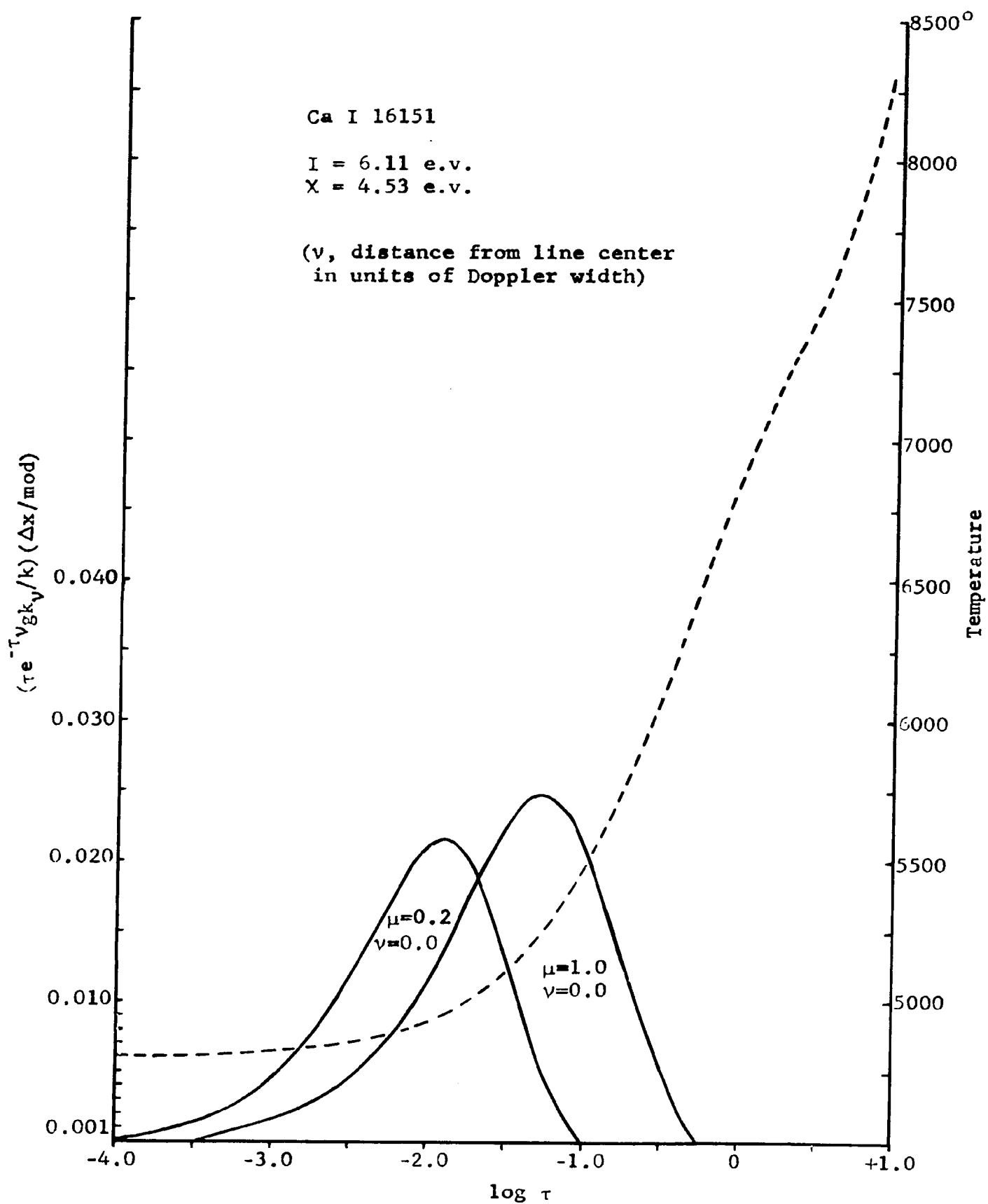


Fig. 4d.-Contribution function: Ca I 16151



absorption a proportionate amount by lowering the electron pressure. Strictly true only for the unsaturated lines, this provides a lower limit for the $|\Delta \log P_e|$ in the case of the strongest line, Sr II 10915. The electron pressure corrections so computed are listed as $\Delta \log P_e$ in column (5) of Table 15.

C. On the Neglect of Damping

It was thought adequate for our purposes to argue qualitatively the effects of damping, but it is now possible better to justify its neglect with the aid of some calculations. The inclusion of damping in any of the theoretical calculations requires that the saturation in the line be less in order to produce the same equivalent width as for no damping. Consequently, the strengthening of a line toward the limb would be greater on account of a "delayed" onset of saturation. Since the saturation of any line at the limb is already greater than for the same line at the center of the disc, a percentage change in strength (for a given change in the number of absorbing atoms) for the line at limb can never attain that for the line at center, whether or not damping is taken into account. This was borne out by calculation of the center-limb behavior of O I 7772 with $a=0$ (see equation (IV-27)) and $a=0.1$ (corresponding to 350 times the classical damping) and for 3 points on the disc. The results are shown in Table 16. In column (1) is listed the position on the disc; in columns (2) and (3), the equivalent widths without and with damping, respectively; and in column (4), their percentage difference. Since the number of absorbing atoms was not reduced for the calculation with damping, the line with damping appears stronger than the line without damping.

However, our procedure has been to fit the theoretical equivalent

width to that observed at the center of the disc. The calculations including damping, nevertheless, may be used to show that, of lines of the same strength at the center of the disc, the one with the greater damping has reduced saturation at the limb and, hence, greater strength relative to center, as is borne out in Table 16. In order to show the effect of damping on the center-limb behavior of lines equated (to the observed) at the center of the disc, the percentage changes in equivalent width brought about by damping are taken of opposite sign, reduced to zero for disc center, in column (5), and finally applied to the equivalent width without damping. The result, in column (6), may be compared with the unsmoothed observations in column (7).

The inclusion of damping, even of this large magnitude, evidently leads to no significant improvement between theory and observation. Furthermore, the taking into account of damping variable with depth may be seen to affect the center-limb behavior of a line in a manner opposed to the "desaturation" effect described above.

We have already seen that when the theoretical and observed strengths of a weak line have been made to agree at disc center, the adopted turbulence parameters cannot alter the center-limb behavior of the total absorption. By the same token, the center-limb behavior of a weak line is unaffected by the magnitude of the damping parameter.

D. On the Choice of the Interpretation of the Observations

In the preceding sections our results suggest the application of temperature or pressure corrections to the Aller-Pierce-Elste model as independent means of harmonizing theory and observation. We find, from Table 15, that the temperature and $\log P_e$ corrections vary with

TABLE 16

THEORETICAL EQUIVALENT WIDTHS FOR O I 7772
WITH AND WITHOUT DAMPING
(ANISOTROPIC)

μ	Theoretical Equivalent Widths in Milliangstroms					
	(a=0)	(a=0.1)	Percentage Change	Reduced to $\mu=1.0$	Final Computed with Damping	Observed
1.0	76.9	81.3	+5.4	0	76.9	77.9
0.4	47.9	49.7	+4.6	+0.8	48.2	53.8
0.25	34.8	35.7	+2.8	+2.6	35.7	52.9

decreasing temperature in the following way:

TABLE 17

ALTERNATIVE ATMOSPHERE CORRECTIONS

Depth as f(T)	ΔT	$\Delta \log P_e$
5355°	+ 50°	- 0.11
5300°	+ 180°	- 0.28
5250°	+ 130°	- 0.23
5177°	+ 230°	- 0.44
4787°	- 200°	--
4670°	+ 50°	--
4623°	--	- 0.19

Of greatest interest in these results is not the magnitude of the corrections but their near unanimity in sign. The general conclusion seems, therefore, inescapable that the observed ratio of line to continuous absorption is greater than predicted, for the case of pure absorption, over the range of depths in at least the upper half of Table 17.

It has already been shown in Section (III-E-3) that the observed

limb-darkening of the continuum is not incompatible with higher (or lower) temperatures at $\tau < 0.1$. The considerable extent to which the contribution functions overlap one another, however, makes the assignment of a correction of given magnitude to a particular layer a crude one indeed. On this account I feel that the negative temperature correction found near 4800° throws doubt on interpreting the other discrepancies as due to temperature rather than pressure. Consequently, I have dealt at greater length, in Chapter VI, with the inference to be drawn from the existence of a steeper electron pressure gradient, namely, a reduced metals abundance.

E. Large-Scale Motions

Regarding the conclusion that the equivalent width of a sufficiently weak line is completely insensitive to the magnitude of the Doppler width (Chapter V, introduction), it is clear that the actual scale of large-scale motions cannot, in the absence of a satisfactory theory, be got from either the equivalent width or profile of such a line.

Of very great interest are the nature and magnitude of the strengthening toward the limb of Ca I 16151, Mg I 15880, Fe I 15878, and the pressure-dependent line Sr II 10915. The nature of the center-limb change is not satisfactorily accounted for by either isotropy or a simple anisotropy in the large-scale motions, though its magnitude at $\mu=0.3$ is rather well represented by the latter. The observed behavior of Ca I 16151 and Sr II 10915 from $\mu=1.0$ to 0.8 (observations of Mg I 15880 and Fe I 15878 lacking) suggests that a sufficient degree of saturation in the line contradicts the notion of a simple anisotropy, or at least its expression as turbulence parameters in the absorption

coefficient. The degree of saturation intended as "sufficient" in this case is characterized by a substantial difference between the predictions for isotropy and anisotropy, say 10 per cent or greater.

The chief theoretical exploration of the effects of large-scale motions on equivalent widths and profiles has been carried out by Wrubel (1950). He defines a critical eddy size in the following way: eddies of size larger than the critical size affect the shape of the line profile without contributing to the equivalent width; eddies of size smaller than the critical size affect both profile and equivalent width. The factor, $s(k)$, which Wrubel proposes for the weighting of eddies by size (eddy diameter = π/k , where k is the wave number) has been evaluated for several turbulent models and variation of η_v with depth. The results indicate that where the critical size is about 2.5 times the scale height, a Maxwellian distribution of turbulent velocities can produce at most a 16 per cent difference between the velocities determined from equivalent widths and profiles. Wrubel concludes, then, that if an appreciable difference is observed, a larger proportion of the energy must be tied up in the motion of large eddies. This is particularly a possibility in the case of supergiant stars.

For the case of an increasing turbulent velocity with height, Wrubel points out (1950, p. 430) that even waves of infinite length make contributions to the curve of growth. This fact, together with the finite upward shift exhibited by all lines in their mean depth of formation, limb versus center, raises the possibility of accounting in this way for the center-limb behavior of Ca I 16151, Mg I 15880, Fe I 15878, and Sr II 10915. Elste (1955, p. 211), Suemoto (1957, p. 16), and Voigt (1956) have made the suggestion, in the way of accounting for

center-limb profile changes, that the microturbulent velocity varies with height in the photosphere.

However, the significance of Wrubel's theoretical investigation does not seem to have been explored for the case of the center-limb variation of equivalent widths across the disc. The problem is as follows: in terms of the familiar notions of macro- and microturbulence it must be clear that the smaller end of the macro-turbulent scale, for a line at the center of the disc, must become, for the same line toward the limb, the larger end of the microturbulent scale. [Though expressed in a different way, this fact was recognized by Elste (1950, p. 211).] On this account one would conclude qualitatively, from the observed trend in equivalent widths, that there exist isotropic motions with a dispersion 2.5 to 3.0 km/sec and a scale 2 to 3 times the scale height. This, however, is controverted by an observed dispersion, from line profiles at the center of the disc, of 1.8 km/sec (Waddell 1957).

Microturbulent vertical motions increasing up to 2.5 to 3.0 km/sec, however, may well be concealed in the emergent profile at the center of the disc for the simple reason that the sum of a succession of Voigt profiles over continuous optical depth may be very likely able, it seems, to be fitted by a mean Voigt profile, where the separate profiles are weighted by $g(\tau)e^{-\tau}v$.

Finally, let us examine qualitatively a semi-theoretical approach to the interpretation of the variation of equivalent widths and line profiles across the disc, in which something akin to the physically plausible Kolmogoroff spectrum of turbulence (Chandrasekhar 1949) is assumed to exist in the photosphere.

We imagine a hierarchy of eddies in which the velocity distribution

is Maxwellian, and of which the largest may be of the order of size of the granulation. A velocity dispersion may be assigned to each eddy size. Since the hierarchy is constructed from the superposition of smaller and smaller eddies, we should distinguish between the velocity dispersion to be assigned to a given eddy size, L_n , in a frame at rest in the photosphere and the velocity dispersion which characterizes an eddy size with respect to the next larger eddy size, L_{n-1} . For the purposes of this discussion we call the absolute velocity dispersion, β_n , and the relative, $\beta_n/(n-1)$. They are related in precisely the same way as a smeared function is related to the successive functions by which it is smeared:

$$(V-1) \quad \beta_n^2 = \sum_{i=1}^n \beta_i^2 / (i-1)$$

The representation on the right is of considerable aid in our approach for it leaves open both ends of the turbulence scale. In other words, the relative dispersions enable us to view the small eddies as being smeared by larger and larger ones, or the large eddies being smeared by smaller and smaller ones.

The appearance of granulation suggests that at the large end of the spectrum the scale of eddy sizes is not continuous, but for smaller L it probably rapidly becomes continuous. In any case, we are now interested only in qualitatively disentangling the effects of a hierarchy of motions on equivalent widths and profiles.

Consider the space velocities of all those atoms with a given absolute value of radial velocity. Call the absolute velocity dispersion describing that assemblage, β_n . To each radial velocity in a line may be assigned an absolute velocity dispersion, β_n , which we have already associated with a given eddy size, L_n , and which is always greater than

the Doppler displacement of that point from line center. For points more and more distant from line center, then, the absolute velocity dispersion of the absorbing atoms becomes larger and larger, approaching as a limit in the extreme wing, the actual Doppler displacement from line center. The lower velocity dispersions are, of course, to be associated with the larger eddies or macroturbulence, the higher velocity dispersions with the smaller eddies or microturbulence. Since the effects of saturation start at line center and diminish outward into the wings, it is now possible to visualize the changing relative importance of macro- and microturbulence in the center-limb variation of a line. With the onset of saturation, the strength of the line is increased by a smaller proportion in the line core or macroturbulent portion, than it is in the line wings or microturbulent part. Since the role of macroturbulence is more pronounced in the line core, and microturbulence in the wings, it is evident that the latter has a greater influence on equivalent widths, while both influence the line profile. Struve and Huang (1952) have employed a simple correlation between these quantities in evaluating the scale of motion from lines observed in stellar atmospheres.

Finding the spectrum of turbulence, however, involves determining as well the relative velocity dispersion of each eddy size with respect to the next larger eddy size and the assignment of wave numbers to eddies of all velocities.

To return to the observations, the somewhat slower-than-anisotropic growth in the center-limb variation of Ca I 16151 and Sr II 10915 suggests that the Doppler width adopted at the center of the disc was too large. The line, in computation, was made too weak throughout and was, consequently, able to increase in equivalent width proportionately more than

if a smaller Doppler width had been adopted. The corollary to this fact--that for sufficiently weak lines the center-limb variation of Doppler width does not alter the equivalent width--has already been demonstrated by isotropic and anisotropic calculations for weak lines.

The equivalent width, then, is weighted more by the smaller, faster eddies, though inherent in their velocity dispersion is the velocity dispersion of the very largest eddies. For this reason precise equivalent widths in a more enlightened analysis of the spectrum of turbulence ought to supplement results obtained for the entire spectrum from line profiles.

CHAPTER VI

DERIVATION OF A NEW THEORETICAL-EMPIRICAL MODEL

Since the interpretation of the line observations in Chapter V has some degree of ambiguity, we shall now investigate independent evidence for one of the implications of that chapter; namely, an atmospheric model in which the electron pressure gradient is steeper than that of the Aller-Pierce-Elste model.

It was first pointed out by Schwarzschild (1905) that the pressure may, through the equation of hydrostatic equilibrium, be related to either τ or T :

$$(VI-1) \quad dP/d\tau = g/k$$

in which P is the gas pressure and g the value of gravity for the solar atmosphere. Since that time a great many models have been computed with the aid of the relation (IV-1) in which the chief variant has been the continuous absorption coefficient. The greatest strides have been made, however, since the demonstration by Wildt in 1939 (Wildt 1939b) that the negative hydrogen ion could much more satisfactorily account for the observed wave length variation of the absorption coefficient than could pure hydrogen and metal absorption. Though absorption by metals over all of the spectral range but the ultraviolet is now known to be of no consequence, the abundance of those atoms continues to play an important role in the continuous absorption owing to their contribution to the electron pressure. In the region of line-formation, where hydrogen and the non-metals are essentially un-ionized, all of the free electrons

derive from the relatively few more abundant metals like Mg, Fe, Ca, and Si.

A. Solar Pressure Models

Minnaert (1953, p. 102) has reviewed the connection between theoretical and empirical models and compared a number of them which have been derived in recent years.

From theory or observation we can obtain a relation $T = T(\tau)$, where τ for theory refers to the optical depth in integrated light and for observation refers to monochromatic optical depth. The two are able to be related only through the theoretical absorption coefficient.

The gas pressure, P , in equation (VI-1) is a function only of the temperature, T , and the chemical composition, which we shall denote in terms of the helium-hydrogen and hydrogen-metals ratios, respectively A and B : $P = P(T, A, B)$.

With the continuous absorption rather well-accounted for over most of the observed spectral range, the calculation of models by equation (VI-1) has been directed chiefly toward resolving the problem of chemical composition, i. e., the values of A and B . Insofar as this investigation is concerned with the electron pressure distribution in the line-forming region, it also has bearing on the value of A .

B. The Influence of Chemical Composition on the Gas and Electron Pressure Distributions

Models for numerous compositions have been calculated, notably by Strongren (1941), Munch (1947), De Jager (1952), Aller and Pierce (1952), and Michard (1953). With the aid of these and calculations of his own, Hubenet (1956) has illustrated the influence of changes in chemical

composition on the gas and electron pressures. For one and the same temperature distribution he has reduced the pressure distributions for six models to the same composition, $\log A = 3.8$ and $B = 0.2$, and tabulated the differential corrections required to convert a model of that composition to any with $\log A$ from 3.0 to 4.2 and B from 0.00 to 0.40.

In summary, the effects of ΔA are as follows: decrease in metals:

(1) steepens the electron pressure gradient above the region where its absolute value is essentially fixed by the incipient ionization of hydrogen, and (2) diminishes very slightly the gas pressure gradient while increasing the absolute value of the gas pressure. For a change in helium content the pressure changes are independent of θ , i. e., the gradient is not altered. For example: (from Hubenet's Table III) in the layers $\theta = 0.7$ and 1.1 ($T=7200^\circ$ to 4580°) for $\Delta \log A = +0.4$, the electron pressure changes by + 10 per cent and - 60 per cent, respectively, the gas pressure by + 20 per cent and + 45 per cent. For $\Delta B = +0.2$, the electron pressure increases by 20 per cent and the gas pressure by 55 per cent.

C. The Chemical Composition of the Solar Atmosphere

With the model that has been tested in this thesis, the chemical composition of the solar atmosphere has been redetermined at the University of Michigan Observatory (Goldberg et al. 1957). Part of a larger program for the determination of abundances in the sun and stars (University of Michigan Observatory reports 1955, 1956), the investigation employs Pecker's theory for the exact curve of growth (Elste 1957, Pecker 1951). Dr. Edith A. Muller made available to me, in April 1956, the solar abundance determinations already obtained by that time.

The April 1956 abundances, together with their subsequently revised values (also supplied by Dr. Muller), are listed in Table 18.

TABLE 18
ABUNDANCES^a

Element	April 1956	Revised ^b
Oxygen ^c	8.90	9.00
Carbon ^c	8.98	8.98
Silicon	7.40	7.60
Magnesium	7.30	7.28
Sulphur ^c	7.30	7.14
Iron	6.4	6.76
Calcium	6.30	6.38
Sodium	6.18	6.30
Aluminum	6.24	6.21
Chromium	5.8	6.00
Potassium	4.96	4.96
Barium	2.12	2.26

^aGiven in logarithm to the base 10, where hydrogen is 12.00.

^bGoldberg et al. 1957.

^cNon-metals.

They indicate a continuation of the trend over the years toward a reduction in the total abundance of metals relative to hydrogen. Owing to its direct influence on the electron pressure, we have derived a solar model for the April 1956 abundances and the Pierce limb-darkening (1954b).

Since it was of further interest to ascertain the satisfactoriness of the temperature transform for P_e , described in Section (IV-E-5-b-2),

at all depths, the model was constructed at three infrared wave lengths: $\lambda\lambda$ 7793, 10914, and 16150. The first and last of these are, respectively, the regions of least and greatest transparency in the solar atmosphere.

D. Data for Construction of Model

Required for the construction of a hydrostatic model after equation (VI-1) are: (1) a (T, τ) relation; (2) a (P_g, P_e, T) relation; (3) the theoretical absorption coefficient; and (4) the value of gravity, g , in the solar atmosphere.

1) The (T, τ) relation is, as before, taken from the 28-point solution for the coefficients of the revised limb-darkening measures (Tables 2 and 9).

2) The relation, $\log(P_g/P_e)$ vs. $\log P_g$ for a number of θ 's was calculated for $\log A = 4.28$ (April 1956 metals) and $B = 0$. The calculation was detailed to the extent that the abundances, ionization potentials, and temperature-dependence of the partition functions of Si, Fe, Mg, Ca, Al, and Na, as well as the ionization of hydrogen, were taken into account. The foregoing list of elements is by decreasing ionization potential; by decreasing abundance it becomes: Si, Mg, Fe, Ca, Al, and Na.

3) The theoretical absorption coefficient was taken from the extended tables described in Section (IV-E-5-b-3).

4) The value of gravity in the solar atmosphere is $2.753 \times 10^4 \text{ cm/sec}^2$.

E. The Calculations

It is desirable to rewrite the equation of hydrostatic equilibrium with the variables $\log P_g$ and $\log \tau$ in order to avert a change of scale

during integration. Let $y = \log P_g$ and $x = \log \tau$, and then write equation (VI-1) as:

$$(VI-2) \quad dy/dx = (g/k) 10^{(x-y)} = [g 10^{(x-y)} / N_0 P_e \alpha(H^-)]$$

in which k is the absorption coefficient per gram of solar material:

$$(VI-3) \quad k = N_0 P_e \alpha(H^-)$$

and N_0 is Avogadro's number, 6.0251×10^{23} .

1. Uniqueness of the solar model.--That the character of the limb-darkening may be uniquely associated with the metals and helium abundance is suggested by the possibility of deriving an expression relating the limb-darkening, electron pressure, and absorption coefficient (Section (III-E-2)). The logical extension of this notion seems, to me at least, to be that for a given (T, τ) distribution and theoretical absorption coefficient, one and only one hydrostatic model should be obtainable, namely, that model with the true composition. True though this may be in principle, in practice the gas pressure distribution is less strictly prescribed than the electron pressure distribution: while the limb-darkening may give the electron pressure distribution, the electron pressure is itself not very sensitive to the abundance of helium. But at the same time the helium abundance influences the gas pressure considerably more than does the metals abundance.

In line with this view of uniqueness, the use of (VI-2) rather than (VI-1) makes the solution more sensitive to the pressure distribution in the line-forming region (a surmise not yet confirmed by calculation). In (VI-1), P_g is obtained by successive integrations in which the $(n-1)$ st result is used as an approximation for obtaining the n th result. A second method (Aller 1952, p. 227), which also converges rapidly, employs (VI-1), but the values of P_g obtained in the $(n-1)$ st

distribution are multiplied into (VI-1), thus weighting the integrand by P_g in the n th determination of P_g . Finally, the method proposed here employs (VI-2) and thereby introduces a weighting factor $1/P_g$ into the integrand. By the way of comparison, the three methods involve, respectively, weighting factors 1, P_g , and $1/P_g$, only the last of which diminishes with depth.

2. On constructing a model from an intermediate depth.--In the early stages of the calculation of a new model, it seemed worth while to look into the advisability of avoiding altogether the uncertainties of temperature and pressure in the outer layers, and of constructing the model starting at the most reliably observed depth in the photosphere. This is analogous to the approach of Sykes (1953) in his analysis of limb-darkening. By transforming variables in the integral equation of transfer, he was able to derive limb-darkening coefficients which do not, as in the method of Milne and Lundblad, rely on the limb intensity and its derivatives but rather on the intensity observed at the center of the disc.

3. Trial calculations.--For the solution of (VI-2), two numerical methods for the solution of differential equations were tried: (1) Euler's method, and (2) the method of approximating polynomials employing Newton's formula for backward interpolation and horizontal differences (Scarborough 1955, pp. 235-246). A number of attempts were made to start the integration at the level $x = -0.3$ ($\tau \approx 0.5$) which is approximately the mean optical depth to which the limb-darkening observations refer. At this depth, a value was adopted for the electron pressure and the composition, the local gradient was computed, and y at neighboring intervals obtained through one or more successive approximations. The behavior of the

gradient with depth proved to be highly sensitive to the initial conditions. No satisfactorily stable result was obtained by this method, though a number of initial pressures and compositions were tried.

It is suggested that the instability, which persisted to one degree or another in the trials carried out, may result from the weighting factor $1/P_g$ mentioned above rather than from the numerical methods used. (This, of course, should be checked by attempting integration from mid-point in an existing model.)

The method of integration of (VI-2) from the boundary inward was finally adopted.

4. The calculation in the boundary interval.--The pressure near the boundary may be found by repeated integration of the following expression:

$$(VI-4) \quad P_g^{(n)} = \left\{ (2g/N_0) \int_0^{\tau^{(n-1)}} [P_g/P_e \alpha(H^-)] d\tau \right\}^{1/2}$$

assuming at the outset ($n=0$) that the metals are completely ionized, i.e., $P_g/P_e = A$, entering the ionization tables for $n = 1$, and adopting a mean temperature for the interval.

Since the extrapolated limb temperatures for $\lambda\lambda 7793$, 10914 , and 16150 were, respectively, 4616° , 4553° , and 4806° , a "boundary temperature" of 4500° was assigned to $\tau_\lambda = 0.0001$ for each of these wave lengths. At $\tau_\lambda = 0.01$, the limb-darkening temperature was adopted, and for $\tau_\lambda = 0.001$, the temperature was linearly interpolated on the log τ -scale between $T(\tau_\lambda=0.01)$ and $T(\tau_\lambda=0.0001)$. In this context, the λ -subscript is used to denote continuous rather than line optical depth.

The pressures at $\tau_\lambda = 0.0001$, 0.001 , and 0.01 were then obtained from (VI-4) with the initial interval 0 to τ_λ in each case. The smaller the value of τ , the quicker was the convergence, but in any case n did not exceed 4 .

Certainly, the principal variant in this part of the problem is the quantity A , for it is highly unlikely that a similar percentage uncertainty exists in either the coefficient $\alpha(H^-)$ or the temperature to which it refers.

5. The Calculations through the photosphere.--For P_g and P_e at depths greater than $\tau_\lambda = 0.01$, the Runge-Kutta method was found useful in the solution of equation (VI-2) (Scarborough 1955, p. 299). With $\Delta x = 0.2$, the integration was carried out in this way from $x = -2.0$ to $x = +0.8$ ($\tau_\lambda = 6.3$) for $\lambda\lambda 10914$ and 16150 , and to $x = +0.4$ ($\tau_\lambda = 2.5$) for $\lambda 7793$.

6. The results.--The calculated gas and electron pressure distributions are listed in Table 19, and the latter are compared in Figure 5.

Taken with respect to temperature, the distributions are indistinguishable at all T greater than 5600° , ($\tau_{10914} = 0.37$) and differ not more than 5 per cent down to 4950° ($\tau_{10914} = 0.067$). The differences between the models outside the limb-darkening range may well arise from our treatment of the initial interval.

7. Derivation of a "trichromatic" boundary temperature.--On the assumption that the hydrogen-metals ratio, $\log A = 4.28$, is correct, an effort was made to resolve the difference in pressure distributions by noting their behavior in the initial interval for a temperature of 4940° at $\tau_\lambda = 0.0001$. Use of the higher temperature in the boundary interval had the effect of changing the sign, but not the magnitude, of the difference between the boundary interval pressures for $\lambda\lambda 7793$ and 16150 . Since the $\alpha(H^-)$'s for $\lambda\lambda 7793$ and 10914 have a similar dependence on temperature, a boundary interval temperature of about 4700° would be expected to yield a pressure distribution for these three wave lengths

TABLE 19

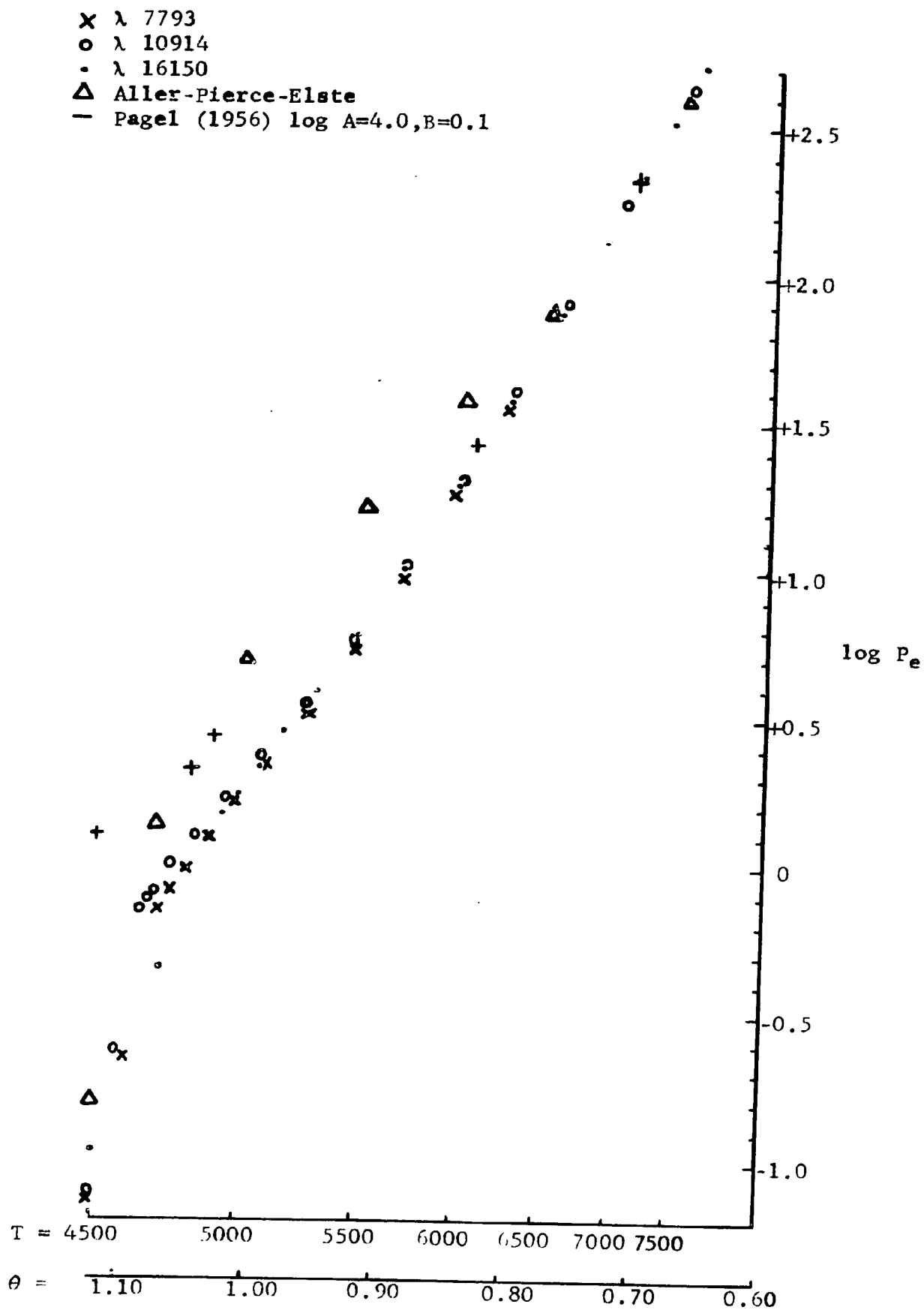
GAS AND ELECTRON PRESSURE DISTRIBUTIONS^a

125

x	$\lambda 7793$				$\lambda 10914$				$\lambda 16150$			
	θ	$\alpha(H^-) \times 10^{+25}$	Log P _g	Log P _e	θ	$\alpha(H^-) \times 10^{+25}$	Log P _g	Log P _e	θ	$\alpha(H^-) \times 10^{+25}$	Log P _g	Log P _e
-2.0	1.071	1.480	4.193	-0.157	1.085	1.318	4.233	-0.151	1.023	0.359	4.473	+0.073
-1.8	1.052	1.430	4.256	-0.091	1.075	1.278	4.295	-0.084	1.012	0.353	4.536	+0.241
-1.6	1.050	1.360	4.332	-0.010	1.062	1.205	4.371	+0.003	0.997	0.341	4.610	+0.330
-1.4	1.034	1.280	4.418	+0.095	1.044	1.134	4.455	+0.101	0.977	0.329	4.689	+0.458
-1.2	1.014	1.170	4.508	+0.218	1.021	1.038	4.587	+0.227	0.952	0.313	4.767	+0.589
-1.0	0.988	1.090	4.603	+0.353	0.993	0.924	4.642	+0.372	0.921	0.294	4.844	+0.784
-0.8	0.958	0.928	4.697	+0.521	0.960	0.812	4.737	+0.547	0.885	0.273	4.912	+1.015
-0.6	0.923	0.791	4.786	+0.735	0.923	0.694	4.831	+0.767	0.845	0.252	4.969	+1.289
-0.4	0.885	0.683	4.868	+0.978	0.883	0.588	4.909	+1.023	0.805	0.230	5.014 ^b	+1.576
-0.2	0.847	0.573	4.938	+1.258	0.842	0.501	4.975	+1.311	0.766	0.212	5.048	+1.873
0.0	0.807	0.475	4.997 ^b	+1.549	0.802	0.421	5.030 ^b	+1.600	0.733	0.197	5.077	+2.124
+0.2	0.764	0.390	5.044	+1.877	0.762	0.344	5.076	+1.913	0.706	0.185	5.103	+2.341
+0.4	0.717	0.308	5.081	+2.241	0.719	0.288	5.114	+2.251	0.683	0.175	5.127	+2.521
+0.6	0.651	0.230	-	-	0.668	0.231	5.137	+2.637	0.660	0.166	5.151	+2.709
+0.8	0.594	0.150	-	-	0.605	0.170	5.145	+3.115	0.631	0.154	5.173	+2.936

^a Log A = 4.28, B = 0.^b The level above which the contribution of atomic hydrogen to the total continuous absorption was taken into account.

Fig. 5.-Electron pressure distributions



which is unambiguous over the whole range of temperature.

F. Conclusion

The principal significance of the excellent agreement among the pressure distributions for all $T > 5600^{\circ}$ is the satisfactoriness, in describing the absorption in those layers, of the theoretical absorption coefficient of the negative hydrogen ion at the wave lengths in question.

In this connection, Chandrasekhar (Chandrasekhar and Breen 1946, p. 445) has shown the relative contribution of bound-free and free-free transitions to the continuous absorption coefficient of the negative hydrogen ion. He suggests that taking into account the exchange and polarization for the slow s-electrons, which would necessitate very extended calculation, may have the effect of reducing the tabulated free-free contribution. While the influence of the bound-free transitions is almost 8 times greater than the free-free at $\lambda 8000$, at $\lambda 16000$ the situation is reversed. Consequently, one would expect that if, in fact, the free-free contribution is much in error, the atmospheric model we have derived from the limb-darkening observations and the theoretical absorption coefficient at the longer wave length would exhibit a marked difference from the models at $\lambda 8000$ and $\lambda 11000$.

I gather from an earlier investigation by Chandrasekhar (1954b, p. 401) that the calculation of the bound-free contribution to the continuous absorption coefficient, in which the plane-wave approximation for the ejected electron is replaced by a Hartree field approximation, has reached a stage in approximation not likely to be seriously revised in the near future. At that, the higher approximation differs from the earlier one (Chandrasekhar 1945a) by only five per cent.

However, returning to the question of uniqueness of the solar

model for a given (T, τ) distribution and chemical composition, I would say that it remains a possibility not very susceptible either to proof or to disproof.

CHAPTER VII

REMARKS AND SUMMARY OF CONCLUSIONS

A. Remarks on Recent Results Concerning:

1. The empirical determination of departures from LTE.--Described in Section (IV-C-4) is recent work by Pecker in which he uses the Eddington-Barbier relation to derive excitation temperatures at different optical depths in the continuum. With the aid of contribution functions, of which examples are shown in Figure 4, and the run of calculations on which they are based, we may illustrate the problem that arises on using the Eddington-Barbier relation on lines of different strength.

Pecker assigns the temperature got from $I(0) = S(\tau=1)$, where $I(0)$ is the central intensity of a line, to the continuous optical depth at which the integration of the equation of transfer reaches unity in total optical depth. This corresponds to advancing from left to right along the $\log \tau$ axis until the line optical depth equals $(1-\tau)$. Since every element of area under the curve contributes equally to the emergent intensity, the effective depth of formation corresponds to the abscissa of the centroid of the figure. Justification for this assignment of effective depth of line-formation appears on differentiation of the ordinate in Figure 4, which may be written as:

$$(VII-1) \quad X \sim (1/\mu) (k_v/k) g e^{-\tau_v/\mu}$$

Then: (VII-2) $dx/x = dk/k + dk_v/k_v - a/\mu$, where $a = \rho ds$,
and percentage changes in continuous or line absorption are seen to

produce like changes in the contribution function, a/μ remaining constant for the function at a given disc point.

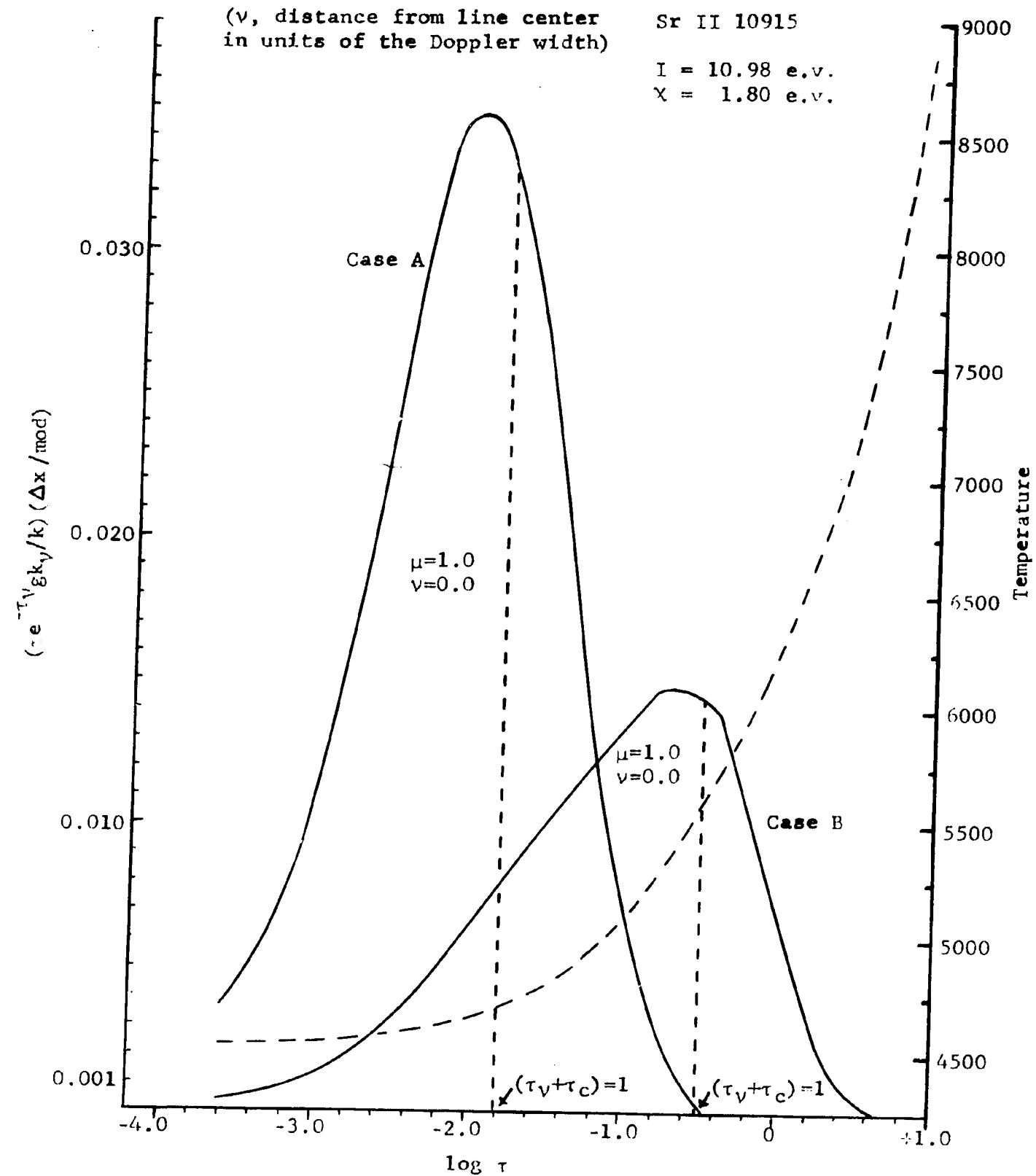
The Eddington-Barbier relation would be most nearly satisfied, then, by a line whose central intensity has a contribution function peaked at total optical depth unity and is fairly symmetrical about that point. Figure 6 illustrates this notion for the Sr II line $\lambda 10915$, calculated at two different strengths. In Case A there are ten times as many absorbing atoms as in Case B, and the central dips are, respectively, 0.54 and 0.27. In the figure, the optical depths in the continuum are indicated to which Pecker would assign the "excitation" temperature, from an observed central intensity, on integrating inward.

The figure also helps to show that for a smaller central intensity (greater dip), the effective depth of formation may be deeper than the layer to which the excitation temperature is assigned, and that for a line of greater central intensity (lesser dip), the effective depth of formation may be higher than the layer to which the excitation temperature is assigned. The net effect is a distortion of scale such that the excitation temperature gradient is reduced.

Pecker (1957a) reports that further work, to be published, involving the use of a non-linear source function for the central intensity confirms the high run of excitation temperature. It is not clear, however, how a refinement of that type removes the objection raised above.

2. The empirical determination of the velocity field.--Suemoto (1957) has also employed the Eddington-Barbier relation to fix the depths of formation of lines formed at different points on the solar disc. From the observed relation between half half-widths and effective

Fig. 6.-On the use of the Eddington-Barbier relation

Case A: 33% of line formed at $\tau > (\tau_v + \tau_c) = 1$ Case B: 21% of line formed at $\tau > (\tau_v + \tau_c) = 1$ 

optical depth of formation (of the intensity at the half half-width), Suemoto deduces an anisotropy at around $\tau = 0.4$, the horizontal motions being greater, and increasing microturbulence with depth.

The remarks made in Section (A-1) above may be more significant here owing to the dependence of his results on the distinctions made concerning depth. Suffice it to point out that his $(\Delta\lambda_D - \tau)$ scale probably suffers the same distortion as Pecker's $(T_{\text{exc.}} - \tau)$ -scale.

3. The possibility of back-radiation from the chromosphere.--In order to account for the center-limb variation in the central depth of the oxygen triplet, H.-H. Voigt (1956) has found it necessary to introduce "re-radiation" from the chromosphere. He assigns an excitation temperature of 4700° to the upper photosphere. For the equivalent width to be affected in the same manner as the central depth, the back-radiation must, of course, penetrate to somewhat greater depths in the photosphere.

The 9.1-volt excitation energy of the oxygen triplet lower term corresponds to $\lambda 1355$ and is less than the energy of the Lyman continuum or any of the Lyman series. An excess of ultraviolet radiation at that wave length is not very likely. Voigt points out that the cause of overpopulation of the $O\ I\ 3^5S^0$ level more likely arises, therefore, from ionization followed by recapture on that level from the continuum. We have already pointed out, however, in Section (IV-C-4), that the electrons exchange energy many times more rapidly than is necessary to maintain LTE (Bohm and Aller 1947, Woolley and Stibbs 1953).

In a study of the center-limb variation of carbon monoxide lines in the region $\lambda\lambda 23100-24200$, Newkirk (1957) considered the effects of hyperdissociation of the CO molecule by back-radiation. He adopted,

however, the interpretation given by a composite De Jager-Aller-Pierce model proposed by Minnaert (1953), without hyperdissociation.

If Bohm's contention (Bohm 1954) is correct, that radiative processes are dominant in the upper photosphere, then the behavior of lines formed largely in that region will have to be predicted on the basis of: (a) atomic parameters most of which are not easily obtained, and (b) the radiation field (Oster 1956, Bohm-Vitense 1955). By making some judicious-looking estimates, Pagel (1956) has shown that below $\tau \sim 0.01$, collisional processes predominate; above that level, photo-ionization and collision dissociation are of the same order of magnitude over a considerable range.

4. The location of a temperature inversion in the high photosphere.

--Our result obtained with the Si I line $\lambda 5780$ brings to mind the conclusion of Pagel (1955, 1956) and of Athay, Menzel, Pecker, and Thomas (1955) that the temperature distribution passes through a minimum in the high photosphere at $\tau \sim 0.015$. The existence of a minimum has long been recognized as a necessity to explain high chromospheric temperatures, but it had not been previously assigned to the familiar line-forming region. It is of interest to note, therefore, that whether we interpret as temperature corrections all our results, or just those from Si I 5780, a temperature minimum at $\tau \approx 0.02$ is suggested.

B. Some By-products of This Investigation

1. Auto-ionization.--Certain transitions in Ca and Sr, as well as in Ba and Cu, are subject to auto-ionization (White 1934, p. 397), a phenomenon which has so far not seemed to merit any practical attention in the astrophysical literature. Auto-ionization may take place in

a two-electron system when the second electron has acquired sufficient energy to allow the escape of the jumping electron from a high bound level to a level of the same energy in the continuum. Condon and Shortley (1935, p. 370) point out that the spectroscopic effects on lines whose initial levels are subject to the phenomenon are: (1) a broadening, and (2) an intensity varying with electron pressure.

The affected transitions ought, then, to offer a direct means of estimating the electron pressure in stellar atmospheres where the populations in the neutral levels, of Ca, Sr, Ba, and Cu, are otherwise independent of electron pressure. This situation is found in late-type stellar atmospheres, where the atoms in question are predominately neutral. On this account, neutral lines arising from levels subject to auto-ionization might be expected, by virtue of dependence on electron pressure, to behave like metallic ion lines in the solar spectrum. If the continuous absorption varies directly with the electron pressure, as it does in the solar atmosphere, then the observational data of interest would be the ratios of the intensities of lines subject to auto-ionization to the intensities of lines not subject to auto-ionization. In any case, the ratios should supplement the luminosity criteria already in use for late-type stars.

According to Condon and Shortley, little has been done on quantitative calculations of the amount of auto-ionization. Late-type stars of different luminosity might be employed, therefore, to ascertain first, whether the phenomenon shows a variation from luminosity class to luminosity class, and second, whether there exists an empirical quantitative relation between the line ratios and conditions in the stellar atmospheres. Providing auto-ionization is observable, it would be desirable,

from an astrophysical point of view, of course, to invert the second process and use this phenomenon as an aid in determining the conditions in late-type stellar atmospheres.

2. On the role of the gas and electron pressure distributions in solar electrodynamics.--An improved knowledge of the gas and electron pressure distributions has been one of the aims of this thesis. Whether or not electric currents may be set up in the atmosphere, with the consequent production of a magnetic field, depends on the gas and electron pressure gradients (Cowling 1953): the electron pressure gradient produces the equivalent of an electric force, which may or may not be balanced by the equivalent electric field produced by the gas pressure gradient.

Finally, solar rotation adds its influence to the pressure gradients so as to determine: (1) whether there may exist an azimuthal magnetic field, and (2) whether Coriolis force can produce an interaction (between the equivalent electric forces in rising and sinking elements) such as might give rise to the poloidal magnetic field generally assumed for the sun.

3. On the abundances of O, Mg, Si, Ca, Fe, and Sr.--At the outset of this investigation it was hoped that the use of transitions whose theoretical f values are calculable would offer additional means of checking the satisfactoriness of the model. In particular, for Ca, Mg, and Si, it would offer a check on the degree of ionization, since abundance determinations from neutral atom and ion of a given element should agree. On the possibility that the results may prove to be only a commentary on the reliability of the theoretical f values, I have omitted such a determination and list, for the lines studied, the raw

data with which such a comparison could be carried out. In Table 20 may be found, therefore, the values of the parameter, $(\epsilon^{2\pi^{1/2}/mc^2})\lambda^2_{gf}$, in equation (IV-30), with which the observed equivalent widths at the center of the disc were reproduced in calculation.

TABLE 20

OBSERVED STRENGTH PARAMETERS

Atom	Stage	λ	$(\epsilon^{2\pi^{1/2}/mc^2})\lambda^2_{gf}$	Atom	Stage	λ	$(\epsilon^{2\pi^{1/2}/mc^2})\lambda^2_{gf}$
Ca	I	6572	1.10×10^{-25}	Si	I	5780	1.47×10^{-23}
		16151	1.32×10^{-20}			6371	3.08×10^{-21}
Mg	II	8912	8.30×10^{-20}	O	I	7772	7.20×10^{-21}
		15880	1.56×10^{-21}			7774	5.22×10^{-21}
	I	21226	1.2×10^{-21}			7775	3.08×10^{-21}
		7896	2.71×10^{-20}	Sr	II	10915	1.78×10^{-21}
		10914	1.63×10^{-20}			15878	4.30×10^{-20}

C. Summary of Conclusions

A number of Fraunhofer lines particularly sensitive to the run of temperature and pressure in the solar atmosphere have been observed at several points on the solar disc. Their equivalent widths have been compared with predictions of center-limb variation obtained from the Aller-Pierce-Elste model with the lines assumed to be formed in pure absorption.

Analysis of the method of limb-darkening representation, for the purpose of deriving an empirical (T, τ) distribution, reaffirms the ambiguity in a source function obtained from limb-darkening observations over a limited range. The exponential integrals are found to exaggerate the ambiguity outside that range at the same time that they help to

reduce it inside the range. Further, it is suggested that the wave length variation of boundary temperature may result from the use of the same integro-exponential form for the source function at all wave lengths.

The method of weighting functions has been employed in a 48-layer model to determine as closely as possible the layers relevant to line-formation for each of the lines in question. Very weak lines which are formed deep in the atmosphere show good agreement with theory. Weak lines sensitive to temperature and/or pressure show, with the exception of Ca I 6573, a center-limb behavior which implies that, relative to the center of the disc, the observed line absorption exceeds that predicted for anisotropic turbulence. The discrepancies between theory and observation have been interpreted as corrections to be applied to the temperature or the pressure in the model used. Of these alternatives, which, by the way, are not mutually exclusive, consistency of sign favors the adoption of the pressure corrections.

Several weak to middle-strong lines, not particularly sensitive to temperature or electron pressure, exhibit a center-limb behavior which may be fully accounted for only with the aid of a more satisfactory representation of the influence of the velocity field on line shape and strength. The probability exists that microturbulence increases with height.

A hydrostatic model with reduced metals abundance, following recent determinations at the University of Michigan Observatory, has been proposed which heeds the corrections implied by the line observations. Integration of the model at three wave lengths, where the bound-free and free-free transitions of the negative hydrogen ion play proportionately quite different roles, gives pressure-temperature distributions

which differ slightly in the outer layers but are identical for the deeper layers. The present theoretical values for the free-free contribution to the negative hydrogen ion absorption (Chandrasekhar and Breen 1946) are probably not in error, therefore, by any significant amount. The most satisfactory agreement among the pressure models calculated at three wave lengths, with $\log A = 4.28$, is obtained with a temperature of 4700° in the boundary interval.

I believe that the limited results achieved here do not at all reflect on the quality of the observations actually used, either of the continuum or of the lines, but rather on: (1) the practical limit on the disc to which limb observations can be made, and (2) the difficulty of taking into more detailed account the influence of the velocity field.

Though we may look with some hope toward the possibility of making extra-terrestrial observations of the solar disc (see, e. g., Scientific American, 169, Nov. 1957, p. 68; and Sky and Telescope, 17, Jan. 1958, p. 112), some time may pass before high-resolution solar spectroscopy can be so realized. On the other hand, it is probably fortunate, after all, that the principal channel now open to us, in the study of the photosphere and low chromosphere, is the examination of the details of the solar surface under the best possible conditions of spectral and geometrical resolution. By determining the characteristics of size and velocity, and the run of temperature and pressure, in the discrete, granular-type elements in the solar atmosphere, we will make an important step toward the synthesizing of the entire solar line spectrum.

Simultaneously, efforts should be made to improve the theoretical

side of the problem in such a way that important theoretical refinements may be practically incorporated into the solar model. This step is particularly urgent for the mechanism of line-formation, which includes, of course, the difficult problem of the influence of the velocity field on line shape and strength. In this connection, a now nearly-indispensable link between precise observations and their theoretical analysis is the modern high-speed computer. Its arrival on the scene is very timely in view of the complex observational and theoretical picture evolving for the solar atmosphere.

In conclusion, it is my hope that this research has delimited the problem of the solar photospheric model on a level commensurate with the use of observations of moderate accuracy, calculations of high precision, and theory of reasonable, if somewhat restricted, applicability.

APPENDIX I

A METHOD FOR INTRODUCING SPHERICITY INTO THE ANALYTIC SOLUTION OF THE INTEGRAL EQUATION OF LIMB-DARKENING FOR THE PLANE-PARALLEL ATMOSPHERE

From Unsold (1948) we know that the path length h/μ in the plane case is for the sun: $(h/\mu)[1 - (h/2R)(\mu^{-2} - 1)]$ where h is the geometrical depth of the atmosphere and is, strictly speaking, dependent on wave length; μ , the cosine of the angle of emergence θ ; and R , the radius of the sun. Let us call the quantity $(h/2R)(\mu^{-2} - 1)$, Δ , and regard it as a small perturbation of the path length in the plane-parallel atmosphere.

Where the limb-darkening coefficients derive from a set of equations in μ , it is possible to use effective rather than observed values of μ . The observed μ 's refer to the spherical sun, but in the limb-darkening equation, to the plane-parallel case. Let us then perturb the μ 's in the limb-darkening equation by this small amount, Δ , and regard them as effective μ 's to be used in the reduction of the measures:

$$\mu_{pp} = \mu_{sph}(1 - \Delta)$$

where pp stands for plane-parallel, sph for spherical.

The second exponential integral form for the limb-darkening equation would then be (cf. equation (III-3)):

$$A + B\mu(1 - \Delta) + C[1 - \mu(1 - \Delta)\ln[1 + 1/\mu(1 - \Delta)]] = I_{\lambda}(0, \mu)/I_{\lambda}(0, 0)$$

This perturbation affects primarily the value of the third limb-darkening coefficient, C . Consequently, it becomes possible to reduce the distortion of the τ -scale for optical depths less than 0.2 which may

result from use of the plane-parallel approximation. The use of the method implies, of course, an assumption concerning the wave length dependence of the quantity, h/R .

APPENDIX II

COMPUTING PROGRAMS CARRIED OUT WITH THE IBM 650 INTERPRETIVE ROUTINE MITILAC

Since its introduction several years ago, the IBM 650 Magnetic Drum Computer has had a very large number of programming schemes devised for it, for the purpose of making ready as well as efficient use of the machine. The very rapid evolution of programming methods in the latter direction made the first interpretive routine, Mitilac, obsolete rather quickly. This routine was, however, ideally suited for programming most of the calculations described in this thesis. It incorporated several useful functional sub-routines, e. g., $\log x$, e^{-x} , $\sin x$, etc., and provided a moderate amount of storage space. Address modification was simple though relatively time-consuming. Only in the program for equivalent widths and contribution functions was the storage completely exhausted. This necessitated a very careful ordering of the calculations, so that the earlier stages could be obliterated and the storage space re-used.

Since the detailed flow charts for these programs reflect to some degree the limitations of Mitilac, it was felt that they would be of no use, as well as of little interest. Aside from giving a very summary flow diagram for the equivalent width program, I have described below, therefore, only the output obtained in some of these programs.

1. Limb-darkening representation.--The limb-darkening coefficients A, B, and C, their weights, the limb value (A+C), and the [pvv], all

found in Table 4, were obtained after the Gaussian method of elimination (Chauvenet 1906), from the revised limb-darkening measures by Pierce (1954b), found, in their revised form, in Table 3; for 51 wave lengths, and alternative solutions for 27 wave lengths; using the Laplace transforms of the integro-exponential forms E_1 , E_2 , and E_3 , viz., equation (III-3) to 3 terms.

2. (T, τ) distribution.-- $T(\tau)$ and $\theta(\tau)$ were calculated at the 10 thesis wave lengths, as well as a number of other wave lengths: $\lambda\lambda 6000$, 7000, 8000, 9000, and 10000; from $x_\lambda (= \log \tau_\lambda) = -4.0$ to $+0.9$, with $\Delta x_\lambda = 0.1$; using equations (III-13, 14, 15).

3. $P_g - P_e$ relation.--This relation was computed for $\log A = 4.28$ and $B = 0$; $\log P_g/P_e$ and $\log P_g$ were tabulated for $\log P_e = -2.6$ to $+3.2$ with $\Delta \log P_e = 0.2$, and for $\theta = 1.50$ to 0.60 with $\Delta \theta = 0.05$; from equation (IV-21) and the relations in Aller (1953, p. 89).

The individual abundances and ionization potentials of Si, Mg, Fe, Ca, Al, and Na, and the temperature dependence of their partition functions were entered into the calculations.

4. The exponential integrals $E_1(x)$, $E_2(x)$, and $E_3(x)$.--Calculation of the (T, τ) distribution employs the exponential integral function. For the values of x in program (2) above, the exponential integral $E_1(x)$ was calculated using an expansion given by Kourganoff (1952, see his equation 36.11). From his expression 36.6, $E_2(x)$ was obtained from $E_1(x)$, and $E_3(x)$ in turn from $E_2(x)$, over the same range. These functions have not been tabulated by Kourganoff between $x = -\infty$ and $x = -2.0$.

5. Interpolation of $\alpha(H^-)$ tables.--A six-point Lagrangian interpolation formula (Scarborough 1955) was used to interpolate, in the tables of Chandrasekhar and Breen (1946), the values of $\alpha(H^-)$ from $\lambda 5200$ to

$\lambda 25000$ in 200 Å. intervals, and from $\theta = 0.70$ to $\theta = 1.80$ with $\Delta\theta = 0.02$.

Interpolation was also carried out for the 10 wave lengths of this thesis and for 8 wave lengths at which observations of limb-darkening have been made (Pierce 1954b): $\lambda\lambda 6475, 8853, 9955, 10135, 11760, 12515, 15950, \text{ and } 17020$.

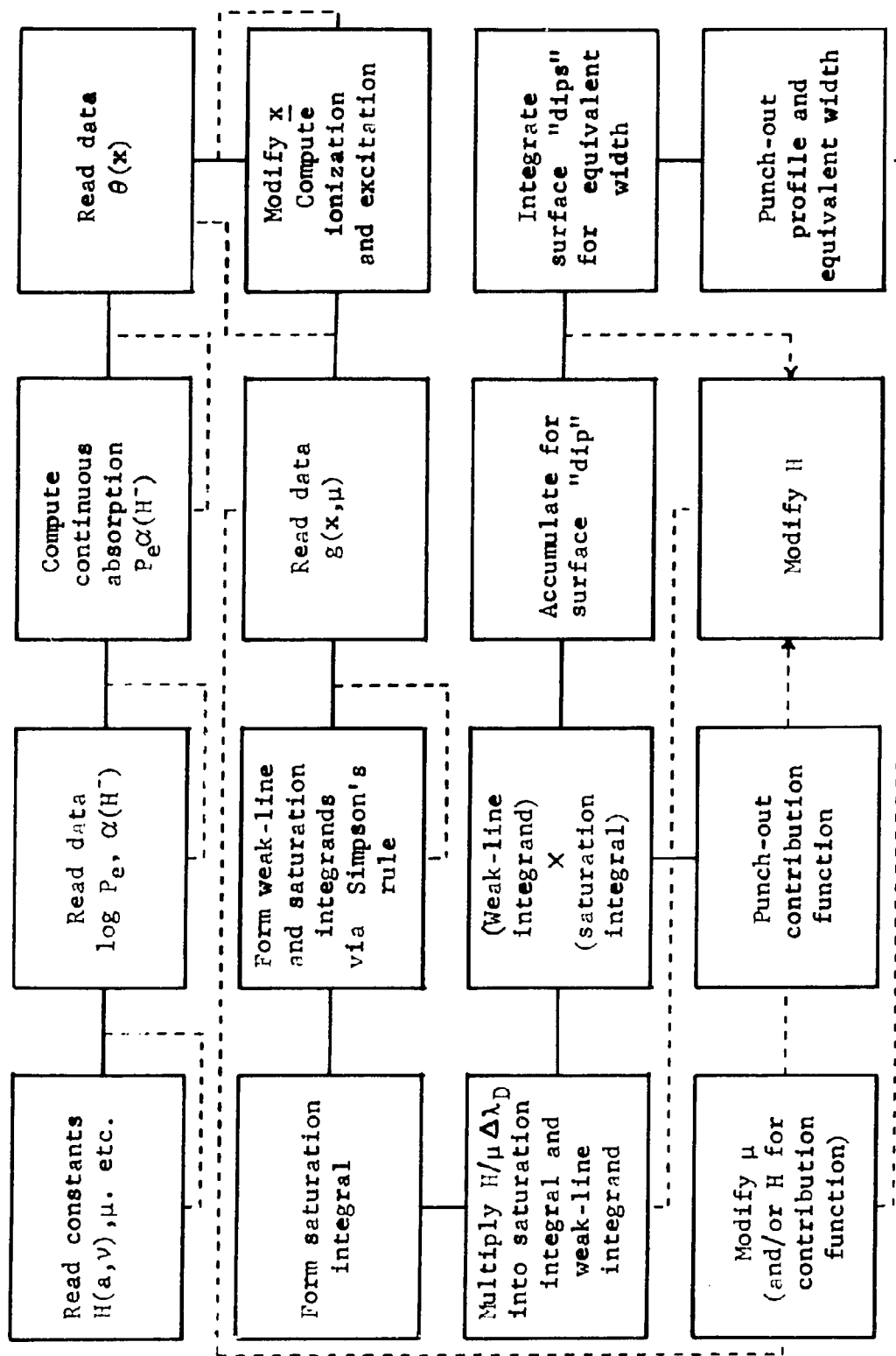
6. Weighting functions, $g_\lambda(x, \mu)$. -- The weighting functions were computed, after (IV-13), for $x = -3.8$ to $+0.9$ with $\Delta x = 0.1$, and for $\mu = 1.0, 0.8, 0.6, 0.4, 0.3, 0.25, 0.2, \text{ and } 0.15$; for the 10 wave lengths of this thesis. They are given in Table 13, from which some values have been omitted where g varies negligibly.

7. Line profiles, equivalent widths, and contribution functions. -- For the lines of interest, the dip was computed, after (IV-10), at line center and at 6 points located symmetrically on each side of line center. This profile was then integrated to give the equivalent width, the calculations being carried out for 8 points on the solar disc.

For representative line strengths, and points on the disc and in the line profile, the contribution function, $g(k_\nu/k)e^{-\tau_\nu}$, was also computed. Data for the computations were entered for $x_\lambda = -3.8$ to $+0.9$, with $\Delta x = 0.1$. Consequently, the contribution functions, as well as equivalent widths, refer to a 48-layer model. The program is outlined schematically below. It consisted of 198 instructions and took into storage 36 constants, and the values of $\log P_e$, $\alpha(H^-)$, $\theta(x)$, and $g(x, \mu)$ as they were required. Cycling loops are indicated by dotted lines.

The 56 profile points and 8 equivalent widths required a total operating time of approximately 80 minutes. Calculation of one profile point, in the same detail, with the aid of a desk calculator requires 10 to 12 hours.

Fig. 7.-Flow diagram for line profiles, contribution functions, and equivalent widths



REFERENCES

- Allen, C. W. 1949, M. N. 109, 343.
- _____. 1955, Astrophysical Quantities (London: University of London; Athlone Press).
- Aller, L. H. 1953, The Atmospheres of the Sun and Stars (New York: the Ronald Press).
- _____. and Pierce, A. K. 1952, Ap. J. 116, 176.
- Athay, R. G., Menzel, D. H., Pecker, J.-C., and Thomas, R. N. 1955, Ap. J. Suppl. 1, 505
- Babcock, H. D. and Moore, C. E. 1947, The Solar Spectrum, $\lambda\lambda 6600$ to 13495 ; Carnegie Institution of Washington Publ. 579, Washington, D. C.
- Barbier, D. 1946, Ann. d'Astrophys. 9, 173.
- Bates, D. R., and Damgaard, A. 1949, Phil. Trans. R. Soc. London, A, 242, 101.
- Beer, A. 1955, Vistas in Astronomy (London: Pergamon Press), 1.
- Bell, B. 1951, A Study of Doppler and Damping Effects in the Solar Atmosphere; Harv. Univ. Spec. Report 35, Jan. 12.
- Bhatnagar, P. L., Krook, M., Menzel, D. H., and Thomas, R. N. 1955, "'Turbulence', Kinetic Temperature, and Electron Temperature in Stellar Atmospheres"; Beer (1955), p. 296.
- Bohm, D., and Aller, L. H. 1947, Ap. J. 105, 131.
- Bohm, K.-H. 1954, Zs. f. Ap. 35, 179.
- Bohm-Vitense, E. 1955, Zs. f. Ap. 36, 145.
- Bretz, M. 1956, Zs. f. Ap. 38, 259.
- Cashman, R. J. 1946, Jour. Opt. Soc. Amer. 36, 356.
- Chalonge, D., and Kourganoff, V. 1946, Ann d'Astrophys. 9, 69.
- Chandrasekhar, S. 1945a, Ap. J. 102, 395.
- _____. 1945b, Ap. J. 102, 223.

- Chandrasekhar, S. 1947, Ap. J. 106, 145.
- _____. 1949, Ap. J. 110, 329.
- _____. 1950, Radiative Transfer (Oxford: at the Clarendon Press).
- _____. and Breen, F. 1946, Ap. J. 104, 430.
- Chauvenet, W. 1906, A Manual of Spherical and Practical Astronomy (5th edition; Philadelphia: J. B. Lippincott Co.), 2.
- Claas, W. J. 1951, "The Composition of the Solar Atmosphere"; Rech. Astron. d'Obs. d'Utrecht, 12, Part 1.
- Clerke, A. M. 1893, History of Astronomy during the 19th Century (3rd edition; London: Adam and Charles Black).
- Condon, E. U. and Shortley, G. H. 1935, The Theory of Atomic Spectra (Cambridge: at the University Press).
- Cowling, T. G. 1953, "Solar Electrodynamics", Kuiper (1953), p. 532.
- Eddington, A. S. 1926, The Internal Constitution of the Stars (Cambridge: at the University Press).
- _____. 1929, M. N. 89, 621.
- Elste, G. 1955, Zs. f. Ap. 37, 184.
- _____. 1957, A. J. 62, 14 (Abstr.).
- Evans, D. S. 1947, M. N. 107, 433.
- Goldberg, L. 1950, Reports on Progress in Physics, 13, 24.
- _____. , Muller, E. A., and Aller, L. H. 1957, A. J. 62, 15 (Abstr.).
- Hale, G. E. 1906, Ap. J. 23, 6.
- Harris, D. L. 1948, Ap. J. 108, 114.
- Hopfield, J. J. 1924, Ap. J. 59, 114.
- Houtgast, J. 1942, The Variations in the Profile of Strong Fraunhofer Lines along a Radius of the Solar Disk; dissertation, Univ. of Utrecht.
- Huang, S. 1950, Ap. J. 112, 418.
- _____. and Struve, O. 1952, Ap. J. 116, 410.
- Hubenet, H. 1956, Koninkl. Nederl. Akademie van Wetenschappen Proceedings, B, 59, No. 3.

- Hynek, J. A. 1951, Astrophysics--A Topical Symposium (New York: McGraw-Hill).
- De Jager, C. 1952, "The Hydrogen Spectrum of the Sun"; Rech. Astron. d'Obs. d'Utrecht, 13, Part 1.
- Jahnke, E. and Emde, F. 1946, Tables of Functions (New York: Dover).
- Kourganoff, V. 1949, Comptes Rendus 228, 2011.
- _____. 1952, Basic Methods in Transfer Problems (Oxford: at the Clarendon Press).
- Kron, G. E. 1952, Ap. J. 115, 1.
- Kuiper, G. P. 1953, The Sun (Chicago: University of Chicago Press).
- _____. , Wilson, W., and Cashman, R. J. 1947, Ap. J. 106, 243.
- Lundblad, R. 1923, Ap. J. 58, 113.
- Michard, R. 1953, Ann. d'Astrophys. 16, 217.
- Michigan, Univ. of, Obs. annual report 1949, A. J. 54, 213.
1950, A. J. 55, 195.
1951, A. J. 56, 162.
1952, A. J. 57, 182.
1953, A. J. 58, 257.
1954, A. J. 59, 351.
1955, A. J. 60, 293.
1956, A. J. 61, 334.
- Milne, E. A. 1922, Phil. Trans. R. Soc. London, A, 223, 214.
- _____. 1930, Hdb. d. Ap. 3, Part 1, 65.
- Minnaert, M., Mulders, G. F. W., and Houtgast, J. 1940, Photometric Atlas of the Solar Spectrum $\lambda 3612$ to $\lambda 8771$ (Amsterdam: Schnabel; Kampert and Helm).
- Minnaert, M. 1953, "The Photosphere", Kuiper (1953), p. 88.
- Mohler, O. C. 1955, A Table of Solar Spectrum Wave Lengths, 11984A to 25578A (Ann Arbor: Univ. of Michigan Press).
- _____. , Pierce, A. K., McMath, R. R., and Goldberg, L. 1950, Photometric Atlas of the Near Infrared Solar Spectrum $\lambda 8465$ to $\lambda 25242$ (Ann Arbor: Univ. of Michigan Press).
- _____. , Pierce, A. K., McMath, R. R., and Goldberg, L. 1953, Ap. J. 117, 41.
- Moore, C. E. 1945, Contr. Princeton Univ. Obs., No. 20.
- _____. 1949, Atomic Energy Levels; Circ. Nat. Bur. Stand. 467, 1.

- Moore, C. E. 1952, Atomic Energy Levels; Circ. Nat. Bur. Stand. 467, 2.
- Munch, G. 1945, Ap. J. 102, 385.
- Newkirk, G. A. 1957, Ap. J. 125, 571.
- New York. 1940, Tables of Sine, Cosine, and Exponential Integrals, i and ii; Fed. Works Agency, Work Projects, Admin. for the City of New York.
- Oster, L. 1956, Zs. f. Ap. 40, 28.
- Page, R. M. 1951, Dissertation, Univ. of Michigan.
- Pagel, B. E. J. 1955, M. N. 115, 493.
- _____. 1956, M. N. 116, 608.
- Pecker, J.-C. 1950, Ann. d'Astrophys. 13, 294 (Parts 1 and 2), 433 (Part 3).
- _____. 1951, Ann. d'Astrophys. 14, 115, (Parts 4 and 5).
- _____. 1953, Contr. l'Inst. d'Astrophys. Paris, A, No. 132.
- _____. 1957a, Comptes Rendus 245, 639.
- _____. 1957b, Comptes Rendus 245, 499.
- Peyturaux, R. 1952, Ann. d'Astrophys. 15, 302.
- _____. 1955, Ann. d'Astrophys. 18, 36.
- Pfund, A. H. 1927, Jour. Opt. Soc. Amer. 14, 337.
- Pierce, A. K. 1954a, Ap. J. 119, 312.
- _____. 1954b, Ap. J. 120, 221.
- _____. , McMath, R. R., Goldberg, L., and Mohler, O. C. 1950, Ap. J. 112, 289.
- Pierce, A. K. and Aller, L. H. 1951, Ap. J. 114, 145.
- Placzek, G. 1947a, The Neutron Density near a Plane Surface; Nat. Res. Council of Canada, Div. of Atomic Energy, M.T.-16, Chalk River, Ontario.
- _____. 1947b, The Functions $E_n(x)$; Nat. Res. Council of Canada, Atomic Energy Proj., Chalk River, Ontario.
- Plaskett, H. H. 1936, M. N. 96, 402.
- _____. 1941, M. N. 101, 3.

- Plaskett, H. H. 1955, "Physical Conditions in the Solar Photosphere"; Beer (1955), p. 637.
- Redman, R. O. 1943, M. N. 103, 173.
- Rogerson, J. B. 1957, Ap. J. 125, 275.
- Savedoff, M. P. 1952, Ap. J. 115, 509.
- Scarborough, J. B. 1955, Numerical Mathematical Analysis (3rd edition; Baltimore: Johns Hopkins Press).
- Schuster, A. 1905, Ap. J. 21, 1.
- Schwarzschild, K. 1906, Nachr. v. d. k. Ges. d. Wiss. zu Gottingen, 1.
 _____. 1914, Sitzgsber. Preuss. Ak. d. Wiss., 2, 1183.
- Stromgren, B. 1935, Zs. f. Ap. 10, 237.
 _____. 1937, Ap. J. 86, 1.
 _____. 1940, Festschrift fur Elis Stromgren (Copenhagen: Munksgaard), p. 218.
 _____. 1944, Publ. Copenhagen Obs., No. 138.
 _____. 1951, "The Growth of our Knowledge of the Physics of the Stars"; Hynek (1951), p. 172.
- Suemoto, Z. 1957, M. N. 117, 2.
- Swensson, J. W. 1951, Mem. Soc. Roy. Sci. Liege, 4th series, 11, No. 1.
- Sykes, J. B. 1953, M. N. 113, 189.
- Tuberg, M. 1946, Ap. J. 103, 145.
- Unno, W. 1957, Ap. J. 126, 259.
- Unsold, A. 1943, Zs. f. Ap. 22, 356.
 _____. 1948, Ann. d. Physik 3, 124.
 _____. 1955, Physik der Sternatmospharen (2d edition; Berlin: Springer).
 _____. and Weidemann, V. 1955, "Fraunhofer Lines and the Structure of Stellar Atmospheres"; Beer (1955), p. 249.
- Voigt, H.-H. 1956, Zs. f. Ap. 40, 157.
- Waddell, J. H. 1957, Dissertation, Univ. of Michigan.
- White, H. E. 1934, Introduction to Atomic Spectra (New York: McGraw-Hill).

Wildt, R. 1939a, Ap. J. 89, 295.

_____. 1939b, Ap. J. 90, 611.

Wilkinson, J. K., Formulae for the Computation of the Planck Function,
 $B_{\nu}(T)d\nu$ (mimeographed).

Woolley, R. v. d. R., and Stibbs, D. W. N. 1953, The Outer Layers of a
Star (Oxford: at the Clarendon Press).

Wrubel, M. H. 1950, Ap. J. 112, 424.