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# THE COMPRESSIBLE FLOW ABOUT A POINTED BODY OF REVOLUTION OF CURVED PROFILE WITH ATTACHED SHOCK WAVE

by
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# ABSTRACT

## THE COMPRESSIBLE FLOW ABOUT A POINTED BODY OF REVOLUTION OF CURVED PROFILE WITH ATTACHED SHOCK WAVE

by Guilford L. Spencer II

The main problem considered is the constructive solution of the equations for the steady flow between the surface of a pointed body of revolution immersed in a uniform flow without yaw, and its attached shock wave, in a plane through the axis of revolution. The determination of the shock wave is an integral part of this problem.

In the plane through the axis of revolution, polar coordinates $(r,\theta)$ are introduced; pole at the point of the body, and initial ray coincident with the body axis. Consideration of the $(r,\theta)$ coordinates as Cartesian coordinates has the effect of separating the shock and body curves, and of removing the singularity caused by the difference between the velocity on the back of the shock and the velocity on the body at the tip. The boundary conditions and equations for $r = 0$ lead to two point boundary value problems for ordinary equations. The solutions of these problems yield functions of $\theta$ which can be used as coefficients in power series in r, which express the flow variables as functions of r and $\theta$. These solutions also yield a power series in r for the shock curve. If these power series converge, it is proved that they represent a solution.

The major results obtained are:

(A) a new proof of the existence of solutions of the Taylor-Maccoll equations for conical flow and associated inequalities.

(B) a procedure for the calculation of a non-yaw flow about "cone-like" bodies. (A body is "cone-like" if its profile curve in polar coordinates has all derivatives of odd order with respect to $r$ equal to zero for $r = 0$.)

(C) a procedure for the calculation of shock-body curvature ratios at the nose.

(D) a method of solution of singular boundary value problems for quasi-linear hyperbolic systems of partial differential equations.

The solutions of the two point boundary value problems are obtained in terms of auxiliary functions which depend only on the Mach number of the undisturbed flow and the body nose angle. For practical use, these functions could be tabulated. The determination of the coefficients of the series could then be made fairly easily.

## PREFACE

The problem which is the main subject of this dissertation was first brought to my attention during the summer of 1950 by Dr. John H. Giese and Dr. William C. Carter* of the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland. At the time of their presentation of the problem, I was engaged in reading recent literature on hyperbolic systems of partial differential equations. The simultaneous occurrence of these events led me to the basic ideas developed here.

During my employment at the Ballistic Research Laboratories (the summers of 1950 and 1951), I attempted to expand these ideas. I am grateful for the helpful advice Dr. Giese and Dr. Carter gave to me while I was there.

The work was continued at the University of Michigan under the able direction of Professor Charles L. Dolph. I have appreciated very much his helpful suggestions and constructive criticism, and want to take this opportunity to thank him for them. I also wish to thank Professor R. F. C. Bartels for similar assistance.

My appreciation is expressed to those friends who have assisted in checking the numerous calculations. In addition, the support given by the Office of Naval Research (Contract Number N8-ONR 71400) and the Office of Army

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*Dr. Carter is currently employed by the Raytheon Manufacturing Company, Waltham, Massachusetts.

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# INTRODUCTION

A problem which has attracted considerable interest in recent years is that of determining the supersonic flow about a pointed body of revolution with attached shock wave. A solution for a right circular cone without yaw has been known for some time[1]. However, a solution for a curved body profile is not known and the latter problem has been the key motivation for this research. The problem is not completely solved, but some ideas of intrinsic mathematical interest and of practical application are developed. The "cone-like" (defined on p. 78) body profiles without yaw are treated in detail, and the ideas developed for these profiles can be extended to other problems.

The major results obtained are:

(A) a proof of the existence of solutions of the Taylor-Maccoll equations for conical flow and associated inequalities. (A proof by a different method has been announced by Shü[2].)

(B) a new procedure for the calculation of a non-yaw flow about "cone-like" bodies. This procedure consists of two steps: (1) the calculation of certain auxiliary functions

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[1] G. I. Taylor and J. W. Maccoll, "The Air Pressure on a Cone Moving at High Speeds," Proc. Roy. Soc. A139(1939), 278-311.

[2] S.S. Shü, "On the Solution in the Large of a Cauchy Problem (With Special References to the Compressible Flow After a Stationary Shock)," Proceedings of the International Congress of Mathematicians, I(1950),637.

of body nose angle. Since only the nose angle is involved here, tables similar to those of Kopal[3] may be constructed. (2) the determination of the coefficients of a power series from the body shape and the auxiliary functions. If this power series converges, it is shown to be a solution of the flow problem.

(C) a method of solution of singular boundary value problems for quasi-linear hyperbolic systems of partial differential equations in two independent variables. The types of singularity which can be treated occur frequently in the field of fluid dynamics.

(D) a procedure for the calculation of shock-body curvature ratio at the nose. (C. C. Lin[4] has computed such ratios from a linearization of the flow equations; comparison of results would be interesting and informative.)

The central mathematical problem of this research is the solution of a singular boundary value problem for a quasi-linear hyperbolic system of partial differential equations in two independent variables. This is discussed in Chapter I, where a method of solution is presented together with an illustration. The particular flow problem is then stated precisely in Chapter II, followed by its solution in Chapters III and IV. (In Chapter III the assumption of irrotationality is made, but

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[3]Zdenek Kopal, Tables of Supersonic Flow Around Cones (Cambridge, Massachusetts, 1946).

[4]C. C. Lin, "Some Problems of Flow Behind Shocks of Varying Strength," Symposium on Shock-Wave Phenomena, Naval Ordnance Laboratory Report NOLR 1135 (White Oak, Maryland, 1949).

this restriction is removed in Chapter IV.) Chapter V contains subsidiary results, comments on other applications of the method of solution, and supplementary remarks. In addition, some of the more cumbersome details are collected in four appendices.

# CHAPTER I

# THE SOLUTION OF A CLASS OF SINGULAR BOUNDARY VALUE PROBLEMS OF QUASI-LINEAR HYPERBOLIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS IN TWO INDEPENDENT VARIABLES

## Section 1: Introduction to the Singular Problem.

A quasi-linear system of partial differential equations in two independent variables x and y is of the form:

$$1.1 \quad \sum_{j=1}^{n} (a^{ij} u_x^j + b^{ij} u_y^j) + c^i = 0 \qquad i = 1, \ldots, n \ ,$$

where the $u^j$ are the dependent variables, the subscripts x and y denote partial differentiation, and the quantities $a^{ij}$, $b^{ij}$ and $c^i$ depend upon x, y and the vector $\bar{u}$ whose components are $u^j$. The expression:

$$1.2 \quad a^{ij} u_x^j + b^{ij} u_y^j \qquad \text{fixed } i \text{ and } j$$

can be interpreted as a directional derivative of $u^j$. The system 1.1 is termed hyperbolic if there exists a linearly equivalent system such that for each equation all the expressions 1.2 can be interpreted as derivatives in the same real direction. These directions are called the characteristic directions of the system.

Courant and Lax[5] have given an elegant treatment of quasi-linear hyperbolic systems in two independent variables. Among the problems which they consider is the following:

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[5] Richard Courant and Peter Lax, "On Non-Linear Partial Differential Equations in Two Independent Variables," Communications on Applied Mathematics, II(1949) 2 and 3, 255-273.

Find $\bar{u}$ satisfying 1.1 in a wedge-shaped neighborhood D, formed by two intersecting curves, 1 and 2. See Figure 1. Curve 1 is given, $\bar{u}$ is known on curve 1, and relations involving $\bar{u}$ are known on curve 2. The number and form of these relations are not arbitrary. Curve 2 itself may be given, or, if an additional relation of a certain kind is given, its determination may be included as part of the problem.

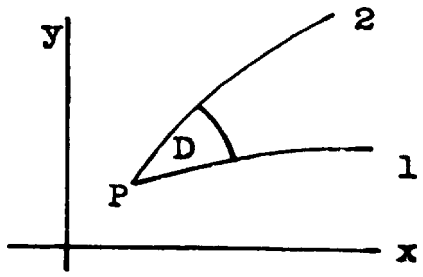



Figure 1.

Courant and Lax's solution depends upon the assumption that the relations involving $\bar{u}$ on curve 2 are consistent with the data on curve 1 at P, the point of intersection of curves 1 and 2. In the case that the relations and data are not consistent, the problem is termed singular. The singular problem is treated here in a generalized form; on curve 1 knowledge of $\bar{u}$ is replaced by knowledge of relations involving $\bar{u}$.

## Section 2: Method of Solution of the Singular Problem.

The point P is assumed to lie on the origin of the (x,y)-plane. The system 1.1 may be stated in polar coordinates as follows:

$$2.1 \quad \sum_{j=1}^{n} [(-a^{ij}\sin\theta + b^{ij}\cos\theta)\, u_\theta^j + r(a^{ij}\cos\theta + b^{ij}\sin\theta)\, u_r^j] + rc^i = 0 \qquad i = 1, \ldots, n .$$

A solution, $\bar{u}(r,\theta)$, of 2.1 becomes a solution, $\bar{u}(x,y)$, of 1.1 upon change of variables.

If the new coordinates r and $\theta$ are regarded as Cartesian coordinates, the analog of Figure 1 is shown in Figure 2.

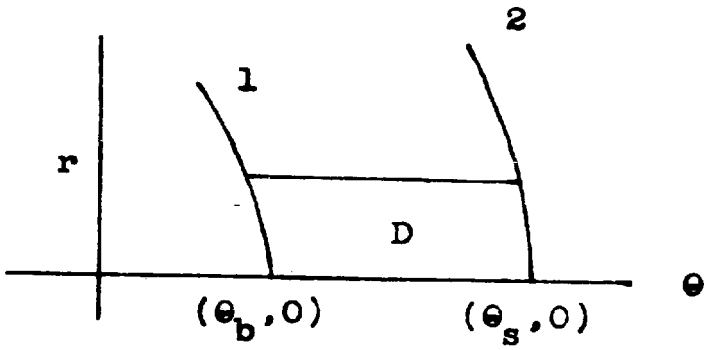



Figure 2 .

The tangents of $\theta_b$ and $\theta_s$ are the slopes of curves 1 and 2 respectively in (x,y)-coordinates at P. If the coefficients $a^{ij}, b^{ij}$ and $c^i$ are reasonably well behaved in the sense that

2.2 $$\lim_{r \to o} a^{ij} = a_o^{ij}$$

$$\lim_{r \to o} b^{ij} = b_o^{ij}$$

$$\lim_{r \to o} rc^i = c_o^i ,$$

then 2.1 reduces to

2.3 $$\sum_{j=1}^{n} (-a_o^{ij} \sin\theta + b_o^{ij} \cos\theta)\, u_\theta^j + c_o^i = 0 \qquad i = 1, \ldots, n$$

on the interval $I_{b,s}$:

2.4 $$I_{b,s} = [(\theta,r) \mid \theta_b \leq \theta \leq \theta_s,\ r = 0] .$$

The boundary conditions associated with the system 2.3 of ordinary differential equations are derived from the relations on curves 1 and 2 by taking their limits as $r \to 0$ along the curves. This limit process gives conditions at

$(\theta_b, 0)$ and at $(\theta_s, 0)$. The problem of determining $\bar{u}$ on $I_{b,s}$ is a two point boundary value problem for a system of ordinary differential equations. If curve 2 is not known, the point $(\theta_s, 0)$ is to be determined as a part of the solution.

The solution of the two point boundary value problem on $I_{b,s}$ yields $\bar{u}(0,\theta)$ and $(\theta_s, 0)$. It does not give any information about r-derivatives of $\bar{u}$ on $I_{b,s}$. In this sense $I_{b,s}$ is a characteristic curve of multiplicity n. Indeed, the conditions 2.2 imply that $I_{b,s}$ is a limiting member of each family of characteristic curves.

The differentiation of 2.1 with respect to r gives:

$$2.5 \quad \sum_{j=1}^{n} \{ (-a^{ij}\sin\theta + b^{ij}\cos\theta)(u_r^j)_\theta + (a^{ij}\cos\theta + b^{ij}\sin\theta)u_r^j + (-a^{ij}\sin\theta + b^{ij}\cos\theta)_r\, u_\theta^j + r[(a^{ij}\cos\theta + b^{ij}\sin\theta)u^j]_r \} + c^i + rc_r^i = 0 \qquad i = 1, \ldots, n$$

where it is assumed not only that the functions involved are differentiable, but that the order of differentiation can be interchanged. If the limiting values of the coefficients exist as $r \to 0$, the system 2.5 on $I_{b,s}$ is written as :

$$2.6 \quad \sum_{j=1}^{n} [(-a_o^{ij}\sin\theta + b_o^{ij}\cos\theta)(u_r^j)_\theta + (a_o^{ij}\cos\theta + b_o^{ij}\sin\theta)u_r^j + (-a_{o,r}^{ij}\sin\theta + b_{o,r}^{ij}\cos\theta)u_\theta^j] + c_o^i + c_{o,r}^i = 0 \qquad i = 1, \ldots, n$$

where the limiting values are denoted by the subscript o. The coefficients in this system are functions of $\theta$ alone since $\bar{u}$ has been determined as a function of $\theta$, the solution of 2.3 on $I_{b,s}$. Therefore, 2.6 is a linear system of ordinary differential equations in $\bar{u}_r$ on $I_{b,s}$.

The boundary conditions associated with the system 2.6 are derived from the relations on curves 1 and 2 by differentiation with respect to r along the curve on which each holds, followed by evaluation of the limit of each as $r \to 0$. These limits are linear relations in the components of $\bar{u}_r$ at $(\theta_b, 0)$ and $(\theta_s, 0)$. In the case that curve 2 is not known, the relations at $(\theta_s, 0)$ also involve the unknown slope of curve 2. The solution of this two point boundary value problem yields $\bar{u}_r$ on $I_{b,s}$ and the slope of curve 2 at $(\theta_s, 0)$.

In similar fashion, a second differentiation and a limit operation yield a two point boundary value problem for a linear first order system of ordinary differential equations. Its solution gives $\bar{u}_{r2}$ on $I_{b,s}$ and the second derivative of curve 2 at $(\theta_s, 0)$. Inductively, the process gives $\bar{u}_{rn}$ (the nth partial derivative of $\bar{u}$ with respect to r) provided that, at each stage, the two point boundary value problem is well-defined and has a unique solution. It also gives the nth derivative of curve 2 at $(\theta_s, 0)$. On the basis of this induction, it may be possible to express the desired solution $\bar{u}$ of 2.1 in the form:

2.7 $$\bar{u}(r,\theta) = \bar{u}(0,\theta) + \bar{u}_r(0,\theta)\, r + \bar{u}_{r2}(0,\theta)\, r^2/2! + \ldots,$$

and the curve 2 as a power series in r. This possibility is the basis of the proposed method of solution, which consists of two major steps. The first of these is the inductive definition of the coefficients of the series 2.7, on a suitable interval of the $\theta$-axis. The second is the proof that the series represents a solution in D.

Section 3: An Example of the Method of Solution.

The following example illustrates the method of solution which has been presented. It is closely related to the problem of determining the supersonic flow about a right circular cone without yaw with attached shock wave. The details of this relationship may be found in Chapter III. Justification of the steps in this outline of the solution may also be found there. This example is simple in that the series 2.7 developed for the solution terminates after the first term. In spite of this degeneracy, the example illustrates very well the ideas outlined in the preceding section. Consider the system:

3.1 $v_x - u_y = 0$

3.2 $(c^2 - u^2)u_x - uv(u_y + v_x) + (c^2 - v^2)v_y + \frac{c^2 v}{y} = 0.$

Curve 1 is:

3.3 $y = kx$, k is a given positive constant.

Curve 2 is unknown, passes through the origin and is denoted by:

3.4 $y = g(x)$.

On curve 1 the relation:

3.5 $v = ku$

is given. On curve 2, the relations:

3.6 $q_o - u = vg'(x)$

3.7 $v^2(b - u) = (q_o - u)^2(u - d)$

are given. The quantities $c^2$, b and d are given by:

3.8 $c^2 = \frac{\gamma-1}{2}(1 - u^2 - v^2)$,

$b = \frac{\gamma-1}{\gamma+1}\frac{1}{q_o} + \frac{2}{\gamma+1}q_o$, and

$d = \frac{\gamma-1}{\gamma+1}\frac{1}{q_o}$,

where:

(i) $\gamma$ is a given constant such that $1 < \gamma < 1.76$

(ii) $q_o$ is a given constant such that $\sqrt{\frac{\gamma-1}{\gamma+1}} < q_o < 1$.

The problem is to find $\bar{u} = (u,v)$ and curve 2, such that $\bar{u}$ satisfies 3.1 and 3.2 in D (see Fig. 1, Sec. 1) and relations 3.5 - 3.7 on curves 1 and 2.

In polar coordinates r and $\theta$, with the convenient change of variable

$$U = u\cos\theta + v\sin\theta$$

$$V = -u\sin\theta + v\cos\theta,$$

3.1 and 3.2 may be written as:

3.9 $U_\theta = rV_r + V$

3.10 $V_\theta = r\left(-\frac{c^2 - U^2}{c^2 - V^2} U_r + \frac{2UV}{c^2-V^2} V_r\right) - U - c^2\left(\frac{U + V\cot\theta}{c^2-V^2}\right)$ .

Curve 1 becomes:

3.11 $\theta = \arctan k = \theta_b$ .

Curve 2 becomes:

3.12 $\theta = S(r)$.

The relation on curve 1 becomes:

3.13 $V = 0$ .

The relations on curve 2 become:

3.14 $U + rS'V = q_o(\cos\theta - rS'\sin\theta)$

3.15 $U\cos\theta - V\sin\theta = d + \frac{2q_o(\cos\theta - rS'\sin\theta)^2}{(\gamma+1)(1 + r^2S'^2)}$ .

The quantity $c^2$ becomes:

3.16 $c^2 = \frac{\gamma-1}{2}(1 - U^2 - V^2)$

and the quantities b, d, $\gamma$ and $q_o$ remain as in 3.8.

When r = 0 in 3.9 through 3.15, the following system results:

3.17 $U_\theta = V$

3.18 $V_\theta = -U - c^2\left(\frac{U + V\cot\theta}{c^2 - V^2}\right)$

3.19 $V = 0$ at $(\theta_b, 0)$

3.20 $U = q_o\cos\theta_s$ at $(\theta_s, 0)$

3.21 $U\cos\theta_s - V\sin\theta_s = d + \frac{2q_o\cos^2\theta_s}{\gamma+1}$ at $(\theta_s, 0)$.

For suitable given values of $\theta_b = \arctan k$, it is shown in Chapter III that a solution (U, V) to 3.17 and 3.18 exists, and that a value $\theta_s = S(0)$ can be determined so that 3.17-3.21 are satisfied.

The next step, as outlined in Section 2, is the determination of $U_r$, $V_r$ on $I_{b,s}$ and $S'(0)$. Differentiation of 3.9 and 3.10 with respect to r gives, for r = 0:

3.22 $(U_r)_\theta = 2V_r$

3.23 $(V_r)_\theta = D_{1,0}U_r + E_{1,0}V_r$ ,

where:

3.24 $D_{1,0} = -1 - \frac{c^2 - U^2}{c^2 - V^2} + (\gamma - 1)U\frac{U + V\cot\theta}{c^2 - V^2} - \frac{c^2}{c^2 - V^2}$
$- c^2U(\gamma - 1)\frac{U + V\cot\theta}{(c^2 - V^2)^2}$

3.25 $E_{1,0} = \frac{2UV}{c^2 - V^2} + (\gamma-1)V\frac{U + V\cot\theta}{c^2 - V^2} - \frac{c^2\cot\theta}{c^2 - V^2}$
$-(\gamma + 1)c^2V\frac{U + V\cot\theta}{(c^2 - V^2)^2}$ .

Differentiation of the boundary conditions with respect to r along curves 1 and 2 gives, for r = 0:

3.26 $V_r = 0$ at $(\theta_b, 0)$,

3.27 $U_r = -(2V + 2U\tan\theta)S'(0)$ at $(\theta_s, 0)$

3.28 $V_r = \left(\frac{2V^2 - c^2}{c^2 - V^2}(U + V\cot\theta) + \frac{8U}{\gamma + 1}\right)S'(0)$ at $(\theta_s, 0)$.

Elimination of $S'(0)$ between 3.27 and 3.28 gives:

3.29 $V_r = K\, U_r$ at $(\theta_s, 0)$

where:

3.30 $$K = \left[ -\frac{2V^2 - c^2}{c^2 - V^2}\,\frac{\cot\theta}{2} - \frac{4U\cot\theta}{(\gamma+1)(U + V\cot\theta)} \right]_{\theta=\theta_s}.$$

The unique solution (see Chapter III) of the two point boundary value problem 3.22, 3.23, 3.26, 3.29 is $U_r \equiv 0$, $V_r \equiv 0$ on $I_{b,s}$. Since $U_r$ and $V_r$ are zero at $(\theta_s, 0)$, $S'(0) = 0$.

In general, to find the nth partial derivatives of $(U, V)$ with respect to r, the nth r derivatives of 3.9 and 3.10 must be taken. For $r = 0$, this gives:

3.31 $$(U_{rn})_\theta = (n+1)V_{rn}$$

3.32 $$(V_{rn})_\theta = \left( -(n-1)\frac{c^2 - U^2}{c^2 - V^2} + D_{1,0} \right)U_{rn} + \left( +(n-1)\frac{2UV}{c^2 - V^2} + E_{1,0} \right)V_{rn}.$$

The boundary conditions for 3.31 and 3.32 are obtained by differentiation of the relations n times along the curves 1 and 2. This yields:

3.33 $V_{rn} = 0$ at $(\theta_b, 0)$,

3.34 $U_{rn} + rS'V_{rn} = N_{n,0}S^{(n)}(0)$ at $(\theta_s, 0)$,

3.35 $U_{rn}\cos\theta_s - V_{rn}\sin\theta_s = P_{n,0}S^{(n)}(0)$ at $(\theta_s, 0)$,

where, for $n = 1, 2, \ldots$

$$N_{n,0} = -(n+1)(V + q_0\sin\theta_s)S^{(n)}(0)$$

$$P_{n,0} = -(\sin\theta_s)\left( \frac{c^2}{c^2 - V^2}(U + V\cot\theta_s) - \frac{8+4n}{\gamma+1}q_0\cos\theta_s \right)S^{(n)}(0).$$

Elimination of $S^{(n)}(0)$ between 3.34 and 3.35 gives:

3.36 $$\left( \frac{V^2 \cot\theta_s}{(c^2 - V^2)(n+1)} - \frac{n}{n+1}\cot\theta_s + \frac{8 + 4(n-1)}{\gamma+1} \frac{U}{(n+1)(V + q_o \sin\theta_s)} \right) U_{rn} + V_{rn} = 0 \quad \text{at } (\theta_s, 0).$$

The unique solution to the two point boundary value problem, 3.31, 3.32, 3.33 and 3.36, is $U_{rn} \equiv 0$, $V_{rn} \equiv 0$ on $I_{b,s}$. Since $U_{rn}$ and $V_{rn}$ are zero at $(\theta_s, 0)$, $S^{(n)}(0) = 0$. The system 3.31 - 3.35, as expressed above, has been simplified by dropping all terms which contain $U_{rj}$, $V_{rj}$, or $S^{(j)}(0)$ for $j = 1, \ldots, n-1$ as a factor.

The power series for this example are:

3.37 $U(r, \theta) \equiv U(0, \theta)$

3.38 $V(r, \theta) \equiv V(0, \theta)$

3.39 $S(r) \equiv \theta_s$.

The functions $U(0, \theta)$, $V(0, \theta)$, and $S(0) = \theta_s$ satisfy 3.17 - 3.21 and it is easily verified that 3.37 - 3.39 represent a solution to the original problem.

# CHAPTER II

## FORMULATION OF THE FLOW PROBLEM

### Section 1: Notation and Assumptions.

The assumptions which lead to the mathematical formulation of the aerodynamic problem of determining the flow about a pointed body with attached shock wave are presented in this section. The theory which leads to the formulation may be found in the work of Courant and Friedrichs[6], to which the reader is referred.

The body is assumed to be in Cartesian (X,Y,Z)-space, with nose at the origin. It is immersed in a gas which is moving at uniform speed in the direction of the Z axis, in such a manner that an attached shock wave is formed at the nose. The shock wave and the flow between the shock wave and the body must be determined. The following notation is adopted:

1.1 p--pressure

T--temperature

$\tau$--specific volume

$\rho$--density ($\rho\tau = 1$)

E--specific entropy

e--specific internal energy

i--specific enthalpy (defined by $i = e + p\tau$)

c--sound speed

$C_v$-specific heat at constant volume ,

[6]Richard Courant and K.O. Friedrichs, Supersonic Flow and Shock Waves (New York, 1948).

and the following assumptions are made:

1.2

1. The gas is perfect; i.e. $p\tau = RT$ where R is the gas constant.
2. The gas is polytropic; i.e. the internal energy is proportional to T.[7] Therefore:

   $p = A\rho^{\gamma}$

   where:

   $\gamma = 1 + RC_v^{-1}$

   $E_o$ is a constant

   $A = (\gamma - 1)\exp[\frac{\gamma - 1}{R}(E - E_o)]$

   and:

   $c^2 = \gamma A\rho^{\gamma-1} = \gamma p\tau = \gamma RT = \frac{\partial}{\partial\rho}(A\rho^{\gamma}) = (\gamma - 1)i$ .
3. The gas upstream from the shock surface has constant supersonic speed $q_o$ in the direction of the Z-axis.
4. The flow is steady; i.e. the velocity $\bar{u}$ at each point is independent of time.
5. The flow satisfies the Euler equation of motion:

   $$\rho\frac{d\bar{u}}{dt} = -\text{ grad } p .$$

   Friction, gravity, and other external forces are neglected.
6. The flow satisfies the equation of continuity:

   $$\text{div}\,\rho\,\bar{u} = 0 .$$
7. Changes of state are adiabatic:

   $$\bar{u}\cdot\text{grad } E = 0.$$

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[7] Ibid., p. 6.

8. If $\overline{N}$ is the normal component of velocity, $\overline{L}$ is the tangential component of velocity, q is the magnitude of velocity, and the subscripts o and 1 refer to the front and back of the shock respectively:

   (a) $\overline{N}_o \rho_o = \overline{N}_1 \rho_1$

   (b) $\rho_o \overline{N}_o^2 + p_o = \rho_1 \overline{N}_1^2 + p_1$

   (c) $\overline{L}_o = \overline{L}_1$

   (d) $\frac{1}{2} q_o^2 + i_o = \frac{1}{2} q_1^2 + i_1$ .

9. The quantity $\frac{1}{2} q^2 + i$ is constant throughout the flow. It may be expressed as $\frac{1}{2} \hat{q}^2$, where $\hat{q}$ is the Bernoulli limit speed.

10. The flow is tangential to the body.

11. The shock is attached to the nose.

Courant and Friedrichs discuss the physical basis for these assumptions, and their interrelations. It is sufficient to say here that air, at moderate temperatures, can be considered to be polytropic. For air, $\gamma$ is approximately 1.4.

## Section 2: Formulation of the Problem in Cylindrical Coordinates.

The determination of the flow between a pointed body of revolution without yaw in a uniform flow, and its attached shock wave is the problem to be studied. The symmetry about the axis of the body immediately suggests using the assumptions 1.2 to describe the flow in cylindrical coordinates.

The flow problem consists in finding functions which satisfy this description. The general form of the description, which provides the basis for some remarks in Chapter V, is worked out. It is then simplified by use of the symmetry assumptions, and the problem is stated precisely.

For cylindrical coordinates, y, $\theta$ and x, where:

2.1 $X = y\cos\theta$

$Y = y\sin\theta$

$Z = x$

$\bar{u} = i_1 v + i_2 w + i_3 u$ ($i_1$, $i_2$, and $i_3$ are unit vectors in the y, $\theta$, and x directions respectively, with the usual right-handed orientation),

the equations 5-7 of 1.2 may be written as five scalar equations:

2.2
$$vv_y + uv_x + v_\theta \frac{w}{y} - \frac{w^2}{y} = -\frac{p_y}{\rho}$$

$$vw_y + uw_x + w_\theta \frac{w}{y} + \frac{vw}{y} = -\frac{p_\theta}{y\rho}$$

$$vu_y + uu_x + u_\theta \frac{w}{y} = -\frac{p_x}{\rho}$$

$$\frac{v}{y} + v_y + \frac{w_\theta}{y} + u_x = -v\frac{\rho_y}{\rho} - \frac{w}{y}\frac{\rho_\theta}{\rho} - u\frac{\rho_x}{\rho}$$

$$vE_y + \frac{w}{y}E_\theta + uE_x = 0 .$$

These involve the six dependent variables v, w, u, p, $\rho$ and E. From assumption 2 of 1.2:

2.3 $$p = (\gamma-1)\rho^\gamma \exp[\tfrac{\gamma-1}{R}(E-E_o)],\quad c^2 = \gamma(\gamma-1)\rho^{\gamma-1}\exp[\tfrac{\gamma-1}{R}(E-E_o)] .$$

Therefore:

2.4
$$p_y = \frac{\gamma-1}{\gamma R}c^2\rho E_y + c^2\rho_y$$

$$p_\theta = \frac{\gamma-1}{\gamma R}c^2\rho E_\theta + c^2\rho_\theta$$

$$p_x = \frac{\gamma-1}{\gamma R}c^2\rho E_x + c^2\rho_x .$$

Elimination of p from the system 2.2 by use of 2.4 yields:

2.5 (a) $$vv_y + uv_x + v_\theta \frac{w}{y} - \frac{w^2}{y} = -\frac{\gamma-1}{\gamma R} c^2 E_y - c^2 \frac{\rho_y}{\rho}$$

(b) $$vw_y + uw_x + w_\theta \frac{w}{y} + v \frac{w}{y} = -\frac{\gamma-1}{\gamma R} c^2 \frac{E_\theta}{y} - c^2 \frac{\rho_\theta}{y\rho}$$

(c) $$vu_y + uu_x + u_\theta \frac{w}{y} = -\frac{\gamma-1}{\gamma R} c^2 E_x - \frac{c^2 \rho_x}{\rho}$$

(d) $$v_y + u_x + \frac{w_\theta}{y} + \frac{v}{y} = -v \frac{\rho_y}{\rho} - \frac{w}{y} \frac{\rho_\theta}{\rho} - u \frac{\rho_x}{\rho}$$

(e) $$vE_y + uE_x + \frac{w}{y} E_\theta = 0 \quad .$$

The linear combination of equations, v[2.5(a)] + w[2.5(b)] + u[2.5(c)] - $c^2$[2.5(d)], is:

$$v^2 v_y + uvv_x + vv_\theta \frac{w}{y} - \frac{vw^2}{y} + wvw_y + uww_x + \frac{w^2}{y} w_\theta + v \frac{w^2}{y}$$
$$+ uvu_y + u^2 u_x + \frac{uw}{y} u_\theta - \frac{c^2 v}{y} - c^2 v_y - c^2 \frac{w_\theta}{y} - c^2 u_x$$
$$= -\frac{\gamma-1}{\gamma R} c^2 (vE_y + \frac{w}{y} E_\theta + uE_x) \quad .$$

Application of 2.5 (e) gives:

2.6 $$(c^2 - v^2)v_y - uv(u_y + v_x) + (c^2 - u^2)u_x + \frac{c^2 v}{y}$$
$$+ \frac{c^2 - w^2}{y} w_\theta - uw(w_x + \frac{u_\theta}{y}) - vw(w_y + \frac{v_\theta}{y}) = 0 \quad .$$

The subtraction of $uu_y + ww_y + vv_y$, $\frac{vv_\theta + ww_\theta + uu_\theta}{y}$, and $vv_x + ww_x + uu_x$ from both sides of equations 2.5 (a), (b) and (c) respectively yields:

2.7 $$u(v_x - u_y) - ww_y + \frac{w}{y} v_\theta - \frac{w^2}{y} = -\frac{\gamma-1}{\gamma R} c^2 E_y - \frac{c^2 \rho_y}{\rho} - uu_y - ww_y - vv_y$$

$$\frac{vv_\theta}{y} - \frac{uu_\theta}{y} + vw_y + uw_x + v \frac{w}{y} = -\frac{\gamma-1}{\gamma R} \frac{c^2 E_\theta}{y} - \frac{c^2 \rho_\theta}{y\rho} + \frac{-uu_\theta - ww_\theta - vv_\theta}{y}$$

$$v(u_y - v_x) - ww_x + \frac{w}{y} u_\theta = -\frac{\gamma-1}{\gamma R} c^2 E_x - c^2 \frac{\rho_x}{\rho} - vv_x - ww_x - uu_x \quad .$$

With the introduction of a new quantity H,

2.8 $$H = \exp\left[ -\frac{\gamma-1}{R}(E - E_o) \right] ,$$

and the use of assumptions 2 and 9 of 1.2 to obtain:

2.9 $$c^2 = \frac{\gamma-1}{2}( \hat{q}^2 - v^2 - w^2 - u^2) ,$$

the system 2.7 may be written in the form

2.10 $$u(v_x - u_y) - ww_y + \frac{w}{y} v_\theta - \frac{w^2}{y} = \frac{c^2}{\gamma(\gamma-1)} \left\{\frac{\partial}{\partial y}\left[\ln \frac{H^{\gamma-1}(c^2)^\gamma}{\rho^{\gamma(\gamma-1)}}\right]\right\}$$

$$\frac{vv_\theta}{y} - \frac{uu_\theta}{y} + vw_y + uw_x + v\frac{w}{y} = \frac{c^2}{\gamma(\gamma-1)y} \left\{\frac{\partial}{\partial \theta}\left[\ln \frac{H^{\gamma-1}(c^2)^\gamma}{\rho^{\gamma(\gamma-1)}}\right]\right\}$$

$$v(u_y - v_x) - ww_x + \frac{w}{y} u_\theta = \frac{c^2}{\gamma(\gamma-1)} \left\{\frac{\partial}{\partial x}\left[\ln \frac{H^{\gamma-1}(c^2)^\gamma}{\rho^{\gamma(\gamma-1)}}\right]\right\} .$$

The argument of the logarithm occurring in 2.10 is introduced now as a convenient function, h,

2.11 $$h = H^{\gamma-1}\left(\frac{c^2}{\rho^{\gamma-1}}\right)^\gamma .$$

From assumption 2 of 1.2:

2.12 $$h = [\gamma(\gamma-1)]^\gamma \exp\left[\frac{\gamma-1}{R}(E - E_o)\right] .$$

Thus, h is seen to be a function of entropy alone, and it follows from 2.5 (e) that:

2.13 $$vh_y + \frac{w}{y} h_\theta + uh_x = 0.$$

The equations of the system 2.2 may now be rewritten in terms of four dependent variables v, w, u and h. The four resulting independent equations are:

2.14 (a) $$u(v_x - u_y) - ww_y + \frac{w}{y} v_\theta - \frac{w^2}{y} = \frac{c^2}{\gamma(\gamma-1)} \frac{h_y}{h}$$

(b) $$v(u_y - v_x) - ww_x + \frac{w}{y} u_\theta = \frac{c^2}{\gamma(\gamma-1)} \frac{h_x}{h}$$

$$(c)\quad (c^2 - v^2)v_y - uv(u_y + v_x) + (c^2 - u^2)u_x + \frac{c^2 v}{y}$$
$$+ \frac{c^2 - w^2}{y} w_\theta - uw(w_x + \frac{u_\theta}{y}) - vw(w_y + \frac{v_\theta}{y}) = 0$$

$$(d)\quad vh_y + \frac{w}{y} h_\theta + uh_x = 0 ,$$

where:

$$c^2 = \frac{\gamma - 1}{2} ( \hat{q}^2 - v^2 - w^2 - u^2) .$$

The body surface may be represented by:

2.15 $$y = B(\theta, x) .$$

The unit normal to this surface is

2.16 $$\frac{i_1 - i_2 \frac{B_\theta}{y} - i_3 B_x}{\sqrt{1 + \frac{B_\theta^2}{y^2} + B_x^2}}$$

and therefore, the condition that the flow be tangential, assumption 9 of 1.2, is:

2.17 $$v - \frac{w}{y} B_\theta - uB_x = 0 .$$

The shock surface may be represented by:

2.18 $$y = S(\theta, x) .$$

The unit normal to this surface is:

2.19 $$\frac{i_1 - i_2 \frac{S_\theta}{y} - i_3 S_x}{\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}}$$

and two mutually orthogonal unit vectors tangent to it are:

2.20 $$\frac{i_1 S_x + i_3}{\sqrt{1 + S_x^2}} \quad \text{and} \quad \frac{i_1 \frac{S_\theta}{y} + i_2(1 + S_x^2) - i_3 \frac{S_\theta S_x}{y}}{\sqrt{1 + S_x^2}\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}} .$$

Therefore, the vectors $\bar{N}_o$, $\bar{L}_o$, $\bar{N}_1$ and $\bar{L}_1$, as defined in assumption 8 of 1.2, are:

2.21

$$\bar{N}_o = \frac{-q_o S_x}{\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}} \quad \frac{i_1 - i_2 \frac{S_\theta}{y} - i_3 S_x}{\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}}$$

$$\bar{L}_o = \frac{q_o}{\sqrt{1 + S_x^2}} \left(\frac{i_1 S_x + i_3}{\sqrt{1 + S_x^2}}\right)$$

$$- \frac{q_o \frac{S_\theta}{y} S_x}{\sqrt{1 + S_x^2}\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}} \quad \frac{i_1 \frac{S_\theta}{y} + i_2(1 + S_x^2) - i_3 \frac{S_\theta S_x}{y}}{\sqrt{1 + S_x^2}\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}}$$

$$\bar{N}_1 = \frac{v - \frac{w}{y} S_\theta - u S_x}{\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}} \quad \frac{i_1 - i_2 \frac{S_\theta}{y} - i_3 S_x}{\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}}$$

$$\bar{L}_1 = \frac{v S_x + u}{\sqrt{1 + S_x^2}} \quad \frac{i_1 S_x + i_3}{\sqrt{1 + S_x^2}}$$

$$+ \frac{v \frac{S_\theta}{y} + w(1 + S_x^2) - u \frac{S_\theta}{y} S_x}{\sqrt{1 + S_x^2}\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}} \quad \frac{i_1 \frac{S_\theta}{y} + i_2(1 + S_x^2) - i_3 \frac{S_\theta S_x}{y}}{\sqrt{1 + S_x^2}\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}} .$$

The conditions 8 and 9 of 1.2 can be written:

2.22 (a) $$-q_o S_x \rho_o = (v - \frac{w}{y} S_\theta - u S_x)\rho_1 ,$$

(b) $$\left(\frac{-q_o S_x}{\sqrt{1 + S_x^2}}\right)^2 \rho_o + p_o = \left(\frac{v - \frac{w}{y} S_\theta - u S_x}{\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}}\right)^2 \rho_1 + p_1 ,$$

(c) 1. $q_o = vS_x + u$

2. $\frac{-q_o S_\theta S_x}{y} = v\frac{S_\theta}{y} + w(1 + S_x^2) - u\frac{S_\theta}{y}S_x$ ,

(d) $\frac{1}{2}q_o^2 + \frac{c_o^2}{\gamma-1} = \frac{1}{2}q_1^2 + \frac{c_1^2}{\gamma-1} = \frac{1}{2}\hat{q}^2$ .

It is possible to express the above conditions, 2.22, in terms of v, w, u, and h alone. The next few pages are devoted to doing just that. If the notation

2.23 $$N_o = \frac{-q_o S_x}{\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}}$$

$$N_1 = \frac{v - \frac{w}{y}S_\theta - uS_x}{\sqrt{1 + \frac{S_\theta^2}{y^2} + S_x^2}}$$

is introduced, 2.22 can be written as:

2.24 (a) $N_o\rho_o = N_1\rho_1$

(b) $N_o^2\rho_o + p_o = N_1^2\rho_1 + p_1$

(c) $\frac{q_o - u}{v} = S_x$

(d) $-\frac{w}{v} = \frac{S_\theta}{y}$

(e) $\mu^2 q_o^2 + (1-\mu^2)c_o^2 = \mu^2 q_1^2 + (1-\mu^2)c_1^2 = \mu^2\hat{q}^2$

where:

$$\mu^2 = \frac{\gamma-1}{\gamma+1} .$$

Division of (b) by (a) and multiplication by $N_oN_1$ gives:

2.25 $$N_oN_1(N_o - N_1) = N_o\frac{p_1}{\rho_1} - N_1\frac{p_o}{\rho_o} .$$

Assumption 2 of 1.2 implies:

2.26 $$\frac{p}{\rho} = \frac{c^2}{\gamma} \quad .$$

Use of 2.24 (e) and 2.26 in 2.25 yields:

2.27 $$N_o N_1 (N_o - N_1) = \frac{1 - \mu^2}{1 + \mu^2}(N_o - N_1)c_o^2 + \frac{\mu^2}{1 + \mu^2} N_o (q_o^2 - q_1^2) \quad .$$

The equalities:

2.28 $$q_1^2 = \bar{N}_1^2 + \bar{L}_1^2$$
$$q_o^2 = \bar{N}_o^2 + \bar{L}_o^2$$
$$\bar{L}_1 = \bar{L}_o$$

imply that

2.29 $$q_o^2 - q_1^2 = N_o^2 - N_1^2 \quad .$$

The result

2.30 $$\left\{ \frac{w^2}{q_o - u} \left[ \frac{\mu^2}{q_o - u}\left(\frac{\hat{q}^2}{q_o} - q_o\right) + 1 \right] + \mu^2 \frac{\hat{q}^2}{q_o} - u \right\} S_x^2$$
$$= -\mu^2 \frac{q^2}{q_o} - (1 - \mu^2)q_o + u$$

is obtained from 2.27 by use of 2.29 to cancel the factor $(N_o - N_1)$ from both sides of the equation; from definition 2.23; and finally from 2.24 (c) and (d) which eliminate $\frac{S_\Theta}{y}$ and v.

From 2.12 and assumption 2 of 1.2, it follows that

$$h = [\gamma(\gamma-1)]^{\gamma-1} \frac{c^2}{\rho^{\gamma-1}} \quad .$$

Use of 2.24(a) and (e) gives:

$$h_1 = [\gamma(\gamma-1)]^{\gamma-1} \frac{\gamma-1}{2} (\hat{q}^2 - v^2 - w^2 - u^2)\left(\frac{N_o \rho_o}{N_1}\right)^{1-\gamma}$$

and

$$h_o = [\gamma(\gamma-1)]^{\gamma-1} \frac{\gamma-1}{2} (\hat{q}^2 - q_o^2) \rho_o^{1-\gamma} \quad .$$

Therefore,

$$2.32 \qquad \frac{h_1}{h_o} = \frac{\hat{q}^2 - v^2 - w^2 - u^2}{\hat{q}^2 - q_o^2} \left( \frac{-q_o S_x}{v - \frac{wS_\theta}{y} - uS_x} \right)^{1-\gamma} .$$

To summarize, the flow may be described in cylindrical coordinates with v, w, u and h as the dependent variables. For convenience, the equations and boundary conditions are collected below. The equations (2.14) are:

$$2.33 \quad (a) \quad u(v_x - u_y) - ww_y + \frac{w}{y} v_\theta - \frac{w^2}{y} = \frac{c^2}{\gamma(\gamma-1)} \frac{h_y}{h}$$

$$(b) \quad v(u_y - v_x) - ww_x + \frac{w}{y} u_\theta = \frac{c^2}{\gamma(\gamma-1)} \frac{h_x}{h}$$

$$(c) \quad (c^2 - v^2) v_y - uv(u_y + v_x) + (c^2 - u^2) u_x + \frac{c^2 v}{y} + \frac{c^2 - w^2}{y} w_\theta - uw(w_x + \frac{u_\theta}{y}) - vw(w_y + \frac{v_\theta}{y}) = 0 .$$

$$(d) \quad vh_y + \frac{w}{y} h_\theta + uh_x = 0$$

where

$$c^2 = \frac{\gamma-1}{2} (\hat{q}^2 - v^2 - w^2 - u^2) .$$

The boundary condition (2.17) on $y = B(\theta,x)$ is:

$$2.34 \qquad v - \frac{w}{y} B_\theta - uB_x = 0,$$

and the boundary conditions (2.30, 2.32, 2.24 (c) and (d) ) on $y = S(\theta, x)$ are:

$$2.35 \qquad \left\{ \frac{w^2}{q_o - u} \left[ \frac{\mu^2}{q_o - u} \left( \frac{\hat{q}^2}{q_o} - q_o \right) + 1 \right] + \mu^2 \frac{\hat{q}^2}{q_o} - u \right\} S_x^2$$

$$= -\mu^2 \frac{\hat{q}^2}{q_o} - (1 - \mu^2) q_o + u$$

$$h_1 = \left( \frac{\hat{q}^2 - v^2 - w^2 - u^2}{\hat{q}^2 - q_o^2} \right) \left( \frac{-q_o S_x}{v - \frac{wS_\theta}{y} - uS_x} \right)^{1-\gamma} h_o$$

$$\frac{q_o - u}{v} = S_x$$

$$-\frac{w}{y} = \frac{S_\theta}{y} \quad .$$

In the particular case that the body is one of revolution without yaw, the description of the flow can be considerably simplified. The shock surface is then a surface of revolution about the Z-axis, the flow is independent of $\theta$, and $w \equiv 0$. The equations 2.33 become:

2.36 (a) $$u(v_x - u_y) = \frac{c^2}{\gamma(\gamma-1)} \frac{h_y}{h}$$

(b) $$(c^2 - v^2)v_y - uv(u_y + v_x) + (c^2 - u^2)u_x + \frac{c^2 v}{y} = 0$$

(c) $$vh_y + uh_x = 0$$

where

$$c^2 = \frac{\gamma-1}{2}(\hat{q}^2 - v^2 - u^2)$$

and 2.33(b) for $w \equiv 0$ is dependent on 2.36 (a) and (c) .

The boundary condition 2.34 on $y = B(x)$ becomes:

2.37 $$v - uB_x = 0 \quad ,$$

and the boundary conditions 2.35 on $y = S(x)$ become:

2.38 $$\left[\mu^2 \frac{\hat{q}^2}{q_o} - u\right] S_x^2 = \left[u - \mu^2 \frac{\hat{q}^2}{q_o} - (1 - \mu^2)q_o\right]$$

$$h_1 = \left(\frac{\hat{q}^2 - v^2 - u^2}{\hat{q}^2 - q_o^2}\right)\left(\frac{-q_o S_x}{v - uS_x}\right)^{1-\gamma} h_o$$

$$\frac{q_o - u}{v} = S_x \quad .$$

The equations and conditions can be put in dimensionless form by setting

$$u^* = \frac{u}{\hat{q}}\ ,\ v^* = \frac{v}{\hat{q}},\ w^* = \frac{w}{\hat{q}},\quad q_o^* = \frac{q_o}{\hat{q}} \quad \text{and} \quad h^* = \frac{h}{h_o}\ .$$

This change of form can be accomplished by replacing $\hat{q}$ by 1, $h_o$ by 1, and interpreting u, v, w, $q_o$ and h as the same as the quantities starred. This form is used throughout Chapters III and IV. However, the stars are omitted for convenience in notation.

The flow problem is now stated in this dimensionless form. The quantities $q_o$, $\gamma$, and the curve $y = B(x)$ are given; the problem is to find a solution $(u(x,y),\ v(x,y),\ h(x,y))$ of the equations:

2.39 (a) $$u(v_x - u_y) = \frac{c^2}{\gamma(\gamma-1)}\ \frac{h_y}{h}$$

(b) $$(c^2 - v^2)v_y - uv(u_y + v_x) + (c^2 - u^2)u_x + \frac{c^2 v}{y} = 0$$

(c) $$vh_y + uh_x = 0,$$

where:

$$c^2 = \frac{\gamma-1}{2}(1 - u^2 - v^2),$$

such that, on $y = B(x)$:

2.40 $$v = uB_x$$

and to find a curve $y = S(x)$ such that on this curve:

2.41 $$\left[\frac{\gamma-1}{\gamma+1}\ \frac{1}{q_o} - u\right] S_x^2 = \left[u - \frac{\gamma-1}{\gamma+1}\ \frac{1}{q_o} - \frac{2}{\gamma+1}\ q_o\right]$$

2.42 $$h = \frac{1 - u^2 - v^2}{1 - q_o^2}\ \left(\frac{-q_o S_x}{v - uS_x}\right)^{1-\gamma}$$

2.43 $$\frac{q_o - u}{v} = S_x\ .$$

The solution is to be valid in a region D:

2.44 $$D = \left[\ (x,y)\ \middle|\ 0 < x^2 + y^2 \leq \delta,\ B(x) \leq y \leq S(x)\ \right]\ .$$

Because of the physical background of the problem, certain additional assumptions can reasonably be made. Those which concern the given data: $\gamma$ , $q_o$ and $y = B(x)$ are listed below.

2.45 (a) $1 < \gamma < 1.76$. Theoretically, $\gamma = 1 + \frac{2}{n}$ for an ideal gas[8], where n is the number of degrees of freedom of the molecules of the gas. For a monatomic gas, n = 3; diatomic, n = 5; triatomic, n = 7; etc. For air under normal conditions $\gamma$ is approximately 1.4 .

(b) The Bernoulli limit speed, $\hat{q}$, is never attained. Therefore, all velocities in the dimensionless form have magnitude less than 1.

(c) $1 \geq q_o > \sqrt{\frac{\gamma-1}{\gamma+1}}$ . The first inequality is a consequence of (b) above. The second is due to the fact that $q_o$ is supersonic. The square of the sound speed in the uniform flow is $\frac{\gamma-1}{2}(1 - q_o^2)$. Therefore, $q_o^2 > \frac{\gamma-1}{2}(1 - q_o^2)$ is the condition that $q_o$ be supersonic and this is equivalent to $q_o > \sqrt{\frac{\gamma-1}{\gamma+1}}$ .

(d) The following assumptions are made on the body curve, $y = B(x)$:

1. $y = B(x)$ is analytic.

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[8] M. Y. Colby, Sound Waves and Acoustics (New York, 1934), p. 133.

2. The first derivative of $B(x)$ is positive at $x = 0$. The next non-zero derivative of $B(x)$ at $x = 0$ is negative. Essentially, this says the body is concave.

3. $B(0) = 0$ and $B(x) > 0$ for $0 < x \leq \delta$.

It should be noted that the assumption 2.45 (b) not only concerns the given data, but places a requirement on (u,v) as well. There are other requirements which must be placed on the solution to insure its being a physically reasonable solution as well as a mathematical one. These are listed below.

2.46 (a) $1 > u^2 + v^2 - c^2 > 0$ in D. This includes 2.45 (b), and requires the solution to be supersonic. [ It is not strictly necessary that the solution be supersonic but it is convenient that it be so.]

(b) $u > 0$ in D.

(c) $v > 0$ in D. The requirements (b) and (c) cause the velocity vector to lie in the first quadrant.

(d) $S(0) = B(0)$ and $S(x) > B(x)$ for $\delta \geq x > 0$. These require that the shock profile lie outside the body and be attached.

The flow problem is singular because

$$\lim_{x \to 0} u(x, B(x)) \neq \lim_{x \to 0} u(x, S(x))$$

$$\lim_{x \to 0} v(x, B(x)) \neq \lim_{x \to 0} v(x, S(x)) \quad .$$

This is not readily apparent from the formulation 2.39-2.43

and the assumptions. However, it is easily seen when the system is formulated in polar coordinates, which is done in the following section.

## Section 3: Formulation of the Problem in Spherical Coordinates.

To carry out the procedure of Chapter I in solving the flow problem, it is necessary to reformulate it in polar coordinates. This is accomplished in this section by describing the flow in spherical coordinates and then specializing the description by use of the symmetry assumption.

For spherical coordinates r, $\theta$ and $\varphi$

3.1
$$X = r \sin\theta \cos\varphi$$
$$Y = r \sin\theta \sin\varphi$$
$$Z = r \cos\theta$$
$$\bar{u} = i_1 u + i_2 v + i_3 w$$ ($i_1$, $i_2$, and $i_3$ are unit vectors in the r, $\theta$, and $\varphi$ directions respectively, with the usual right-handed orientation),

3.2
$$uu_r + \frac{v}{r} u_\theta + \frac{w}{r\sin\theta} u_\varphi - \frac{v^2}{r} - \frac{w^2}{r} = -\frac{p_r}{\rho}$$
$$uw_r + \frac{v}{r} v_\theta + \frac{w}{r\sin\theta} v_\varphi + \frac{uv}{r} - \frac{w^2\cot\theta}{r} = -\frac{p_\theta}{r\rho}$$
$$uw_r + \frac{v}{r} w_\theta + \frac{w}{r\sin\theta} w_\varphi + \frac{uw}{r} + \frac{vw\cot\theta}{r} = -\frac{p_\varphi}{r\rho\sin\theta}$$
$$u_r + \frac{v_\theta}{r} + \frac{w_\varphi}{r\sin\theta} + \frac{2u}{r} + \frac{v}{r}\cot\theta = -u\frac{\rho_r}{\rho} - \frac{v}{r}\frac{\rho_\theta}{\rho} - \frac{w}{r\sin\theta}\frac{\rho_\varphi}{\rho}$$
$$uE_r + \frac{v}{r} E_\theta + \frac{w}{r\sin\theta} E_\varphi = 0.$$

These involve the six dependent variables u, v, w, p, $\rho$ and E. From the assumption 2 of 1.2

3.3 $$p_r = \frac{\gamma-1}{\gamma R} c^2 \rho E_r + c^2 \rho_r$$

$$p_\theta = \frac{\gamma-1}{\gamma R} c^2 \rho E_\theta + c^2 \rho_\theta$$

$$p_\varphi = \frac{\gamma-1}{\gamma R} c^2 \rho E_\varphi + c^2 \rho_\varphi \ .$$

Elimination of p from the system 3.2 by use of 3.3 yields:

3.4 (a) $$uu_r + \frac{v}{r} u_\theta + \frac{w}{r \sin\theta} u_\varphi - \frac{v^2}{r} - \frac{w^2}{r} = - \frac{\gamma-1}{\gamma R} c^2 E_r - \frac{c^2 \rho_r}{\rho}$$

(b) $$uv_r + \frac{vv_\theta}{r} + \frac{uv}{r} + \frac{wv_\varphi - w^2 \cos\theta}{r \sin\theta} = - \frac{\gamma-1}{\gamma R} \frac{c^2}{r} E_\theta - \frac{c^2 \rho_\theta}{r\rho}$$

(c) $$uw_r + \frac{vw_\theta}{r} + \frac{w}{r}\left(\frac{w_\varphi}{\sin\theta} + u + v \cot\theta\right) = \frac{-c^2}{r \sin\theta} \left(\frac{\gamma-1}{\gamma R} E_\varphi + \frac{\rho_\varphi}{\rho}\right)$$

(d) $$u_r + \frac{v_\theta}{r} + \frac{w_\varphi}{r \sin\theta} + \frac{2u}{r} + \frac{v}{r} \cot\theta = - u \frac{\rho_r}{\rho} - \frac{v}{r} \frac{\rho_\theta}{\rho} - \frac{w}{r \sin\theta} \frac{\rho_\varphi}{\rho}$$

(e) $$uE_r + \frac{v}{r} E_\theta + \frac{w}{r \sin\theta} E_\varphi = 0 \ .$$

As in the previous section, these may be combined to give:

3.5 $$(c^2 - u^2) u_r + \frac{c^2 - v^2}{r} v_\theta + \frac{c^2 - w^2}{r \sin\theta} w_\varphi + \frac{u}{r}(c^2 - vu_\theta)$$

$$+ \frac{c^2}{r}(u + v \cot\theta) - uvv_r - \frac{w}{r \sin\theta} (uv_\varphi + vu_\varphi) + uww_r + \frac{vw}{r} w_\theta = 0.$$

Addition of the proper quantities to both sides of 3.4(a), (b) and (c) gives:

3.6 $$\frac{v}{r} u_\theta + \frac{w}{r \sin\theta} u_\varphi - \frac{v^2}{r} - \frac{w^2}{r} - vv_r - ww_r = \frac{c^2}{\gamma(\gamma-1)} \left[ - \frac{(\gamma-1)^2}{R} E_r \right.$$

$$\left. -\gamma(\gamma-1) \frac{\rho_r}{\rho} - \gamma(\gamma-1) \frac{uu_r + vv_r + ww_r}{c^2} \right]$$

$$uv_r + \frac{w}{r \sin\theta} v_\varphi + \frac{uv}{r} - \frac{w^2 \cot\theta}{r} - \frac{uu_\theta}{r} - \frac{ww_\theta}{r}$$

$$= \frac{c^2}{\gamma(\gamma-1)r} \left[ - \frac{(\gamma-1)^2}{R} E_\theta - \gamma(\gamma-1) \frac{\rho_\theta}{\rho} - \gamma(\gamma-1) \frac{uu_\theta + vv_\theta + ww_\theta}{c^2} \right]$$

$$uw_r + \frac{v}{r} w_\theta + \frac{uw}{r} + \frac{vw\cot\theta}{r} - \frac{uu_\varphi + vv_\varphi}{r\sin\theta}$$

$$= \frac{c^2}{\gamma(\gamma-1)\, r\sin\theta}\left[-\frac{(\gamma-1)^2}{R}E_\varphi - \gamma(\gamma-1)\frac{\rho_\varphi}{\rho} - \gamma(\gamma-1)\frac{uu_\varphi+vv_\varphi+ww_\varphi}{c^2}\right].$$

If h, as defined by 2.11, is introduced, the four resulting independent equations in the dependent variables u, v, w and h are:

3.7 (a) $$\frac{v}{r} u_\theta + \frac{w}{r\sin\theta} u_\varphi - \frac{v^2}{r} - \frac{w^2}{r} - vv_r - ww_r = \frac{c^2}{\gamma(\gamma-1)} \frac{h_r}{h}$$

(b) $$uv_r + \frac{w}{r\sin\theta} v_\varphi + \frac{uv}{r} - \frac{w^2\cot\theta}{r} - \frac{uu_\theta + ww_\theta}{r} = \frac{c^2}{\gamma(\gamma-1)r} \frac{h_\theta}{h}$$

(c) $$(c^2 - u^2)u_r + \frac{c^2 - v^2}{r} v_\theta + \frac{c^2 - w^2}{r\sin\theta} w_\varphi + \frac{u}{r}(c^2 - vu_\theta)$$

$$+ \frac{c^2}{r}(u + v\cot\theta) - uvv_r - \frac{w}{r\sin\theta}(uv_\varphi + vu_\varphi)$$

$$+ uww_r + \frac{vw}{r} w_\theta = 0$$

(d) $$uh_r + \frac{v}{r} h_\theta + \frac{w}{r\sin\theta} h_\varphi = 0$$

where:

$$c^2 = \frac{\gamma-1}{2}(\hat{q}^2 - u^2 - v^2 - w^2) .$$

The body surface may be represented by:

3.8 $$\theta = B(r, \varphi) .$$

The unit normal to this surface is

3.9 $$\frac{i_1 B_r - i_2 \frac{1}{r} + i_3 \frac{B_\varphi}{r\sin\theta}}{\sqrt{B_r^2 + \frac{1}{r^2} + \left(\frac{B_\varphi}{r\sin\theta}\right)^2}}$$

and therefore, the condition that the flow be tangential, assumption 9 of 1.2, is :

3.10 $$uB_r - \frac{v}{r} + \frac{wB_\varphi}{r\sin\theta} = 0 .$$

The shock surface may be represented by:

3.11 $$\theta = S(r, \varphi) \quad .$$

The unit normal to this surface is:

3.12 $$\frac{i_1 S_r - i_2 \frac{1}{r} + i_3 \frac{S_\varphi}{r \sin\theta}}{\sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r \sin\theta})^2}}$$

and two mutually orthogonal unit vectors tangent to it are:

3.13 $$\frac{i_1 \frac{1}{r} + i_2 S_r}{\sqrt{\frac{1}{r^2} + S_r^2}} \text{ and } \frac{i_1 \frac{S_r S_\varphi}{r \sin\theta} - i_2 \frac{S_\varphi}{r^2 \sin\theta} - i_3 (S_r^2 + \frac{1}{r^2})}{\sqrt{\frac{1}{r^2} + S_r^2} \sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r \sin\theta})^2}} \quad .$$

Therefore:

$$\bar{N}_o = \frac{q_o S_r \cos\theta + \frac{q_o \sin\theta}{r}}{\sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r \sin\theta})^2}} \frac{i_1 S_r - i_2 \frac{1}{r} + i_3 \frac{S_\varphi}{r \sin\theta}}{\sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r \sin\theta})^2}}$$

$$\bar{L}_o = \frac{\frac{q_o \cos\theta}{r} - q_o S_r \sin\theta}{\sqrt{\frac{1}{r^2} + S_r^2}} \frac{i_1 \frac{1}{r} + i_2 S_r}{\sqrt{\frac{1}{r^2} + S_r^2}}$$

$$+ \frac{q_o \frac{S_r S_\varphi}{r \sin\theta} \cos\theta + \frac{q_o S_\varphi}{r^2}}{\sqrt{1 + S_r^2} \sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r \sin\theta})^2}} \frac{i_1 \frac{S_r S_\varphi}{r \sin\theta} - i_2 \frac{S_\varphi}{r^2 \sin\theta} - i_3 (S_r^2 + \frac{1}{r^2})}{\sqrt{1 + S_r^2} \sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r \sin\theta})^2}}$$

$$\bar{N}_1 = \frac{u S_r - \frac{v}{r} + \frac{w S_\varphi}{r \sin\theta}}{\sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r \sin\theta})^2}} \frac{i_1 S_r - i_2 \frac{1}{r} + i_3 \frac{S_\varphi}{r \sin\theta}}{\sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r \sin\theta})^2}}$$

$$\bar{L}_1 = \frac{\frac{u}{r} + vS_r}{\sqrt{\frac{1}{r^2} + S_r^2}} \quad \frac{i_1\frac{1}{r} + i_2 S_r}{\sqrt{\frac{1}{r^2} + S_r^2}}$$

$$+ \frac{\frac{uS_rS_\varphi}{r\sin\theta} - \frac{vS_\varphi}{r^2\sin\theta} - w(S_r^2 + \frac{1}{r^2})}{\sqrt{\frac{1}{r^2}+S_r^2}\sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r\sin\theta})^2}} \quad \frac{i_1\frac{S_rS_\varphi}{r\sin\theta} - i_2\frac{S_\varphi}{r^2\sin\theta} - i_3(S_r^2 + \frac{1}{r^2})}{\sqrt{\frac{1}{r^2}+S_r^2}\sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r\sin\theta})^2}}.$$

The conditions 8 and 9 of 1.2 can be written:

3.15 (a) $$(q_o\cos\theta\, S_r + \frac{q_o\sin\theta}{r})\rho_o = (uS_r - \frac{v}{r} + \frac{wS_\varphi}{r\sin\theta})\rho_1$$

(b) $$\left(\frac{q_o\cos\theta\, S_r + \frac{q_o\sin\theta}{r}}{\sqrt{S_r^2 + \frac{1}{r^2} + (\frac{S_\varphi}{r\sin\theta})^2}}\right)^2\rho_o + p_o = \left(\frac{uS_r - \frac{v}{r} + \frac{wS_\varphi}{r\sin\theta}}{\sqrt{S_r^2+\frac{1}{r^2} + (\frac{S_\varphi}{r\sin\theta})^2}}\right)^2\rho_1 + p_1$$

(c) 1. $$\frac{q_o\cos\theta}{r} - q_o\sin\theta\, S_r = \frac{u}{r} + vS_r$$

2. $$q_o\frac{S_rS_\varphi}{r\sin\theta}\cos\theta + \frac{q_oS_\varphi}{r^2}$$

$$= u\frac{S_rS_\varphi}{r\sin\theta} - v\frac{S_\varphi}{r^2\sin\theta} - w(S_r^2 + \frac{1}{r^2})$$

(d) $$\frac{1}{2}q_o^2 + \frac{c_o^2}{\gamma-1} = \frac{1}{2}q_1^2 + \frac{c_1^2}{\gamma-1} = \frac{1}{2}\hat{q}^2 \quad .$$

It is possible to express the above conditions, 3.15, in terms of u, v, w and h alone. A procedure similar to that used in the preceding section yields:

3.16 $$u\cos\theta - v\sin\theta$$

$$= \mu^2\frac{\hat{q}^2}{q_o} + \frac{(1-\mu^2)q_o\left[(\cos\theta - rS_r\sin\theta)^2 + \frac{w^2}{(v + q_o\sin\theta)2}\right]}{1 + r^2S_r^2 + \frac{w^2}{(v + q_o\sin\theta)^2}}$$

and

$$3.17 \quad h_1 = \frac{\hat{q}^2 - u^2 - v^2 - w^2}{\hat{q}^2 - q_o^2} \left( \frac{q_o \cos\theta \, S_r + \frac{q_o \sin\theta}{r}}{u S_r - \frac{v}{r} + \frac{w S_\varphi}{r \sin\theta}} \right)^{1-\gamma} h_o \quad .$$

To summarize, the flow may be described in spherical coordinates with u, v, w and h as the dependent variables. For convenience, the equations and boundary conditions are collected below. The equations (3.7) are:

$$3.18 \quad (a) \; v(u_\theta - v) + \frac{w}{\sin\theta} u_\varphi - w^2 - r(v v_r + w w_r) = \frac{r c^2}{\gamma(\gamma-1)} \frac{h_r}{h}$$

$$(b) \; u(u_\theta - v) - \frac{w}{\sin\theta} v_\varphi + w^2 \cot\theta + w w_\theta - r u v_r = - \frac{c^2}{\gamma(\gamma-1)} \frac{h_\theta}{h}$$

$$(c) \; (c^2 - v^2) v_\theta = r\left[-(c^2 - u^2) u_r + 2uv v_r + \frac{c^2 u h_r}{\gamma(\gamma-1)h}\right] - c^2(u + v\cot\theta) - (c^2 - w^2)\frac{w_\varphi}{\sin\theta} + uw^2 + \frac{vw}{\sin\theta} u_\varphi - vww_\theta - (c^2 - v^2)u$$

$$(d) \; r u h_r + v h_\theta + \frac{w}{\sin\theta} h_\varphi = 0$$

where

$$c^2 = \frac{\gamma-1}{2}(\hat{q}^2 - u^2 - v^2 - w^2) \quad ,$$

and 3.18 (c) is a linear combination of 3.7 (a) and (c) . The boundary condition (3.10) on $\theta = B(r, \varphi)$ is:

$$3.19 \quad v = ruB_r + \frac{wB_\varphi}{\sin\theta} \, ,$$

and the boundary conditions (3.15 (c), 3.16, 3.17) on $\theta = S(r, \varphi)$ are:

$$3.20 \quad u + rS_r v = q_o(\cos\theta - rS_r \sin\theta)$$

$$S_\varphi = - \frac{w \sin\theta}{v + q_o \sin\theta}$$

$$u\cos\theta - v\sin\theta = \frac{\mu^2 q^2}{q_o} + \frac{(1-\mu^2) q_o \left[(\cos\theta - rS_r \sin\theta)^2 + \frac{w^2}{(v + q_o \sin\theta)^2}\right]}{1 + r^2 S_r^2 + \frac{w^2}{(v + q_o \sin\theta)^2}}$$

$$h = \frac{\hat{q}^2 - u^2 - v^2 - w^2}{\hat{q}^2 - q_o^2} \left( \frac{q_o \cos\theta \, S_r + \frac{q_o \sin\theta}{r}}{uS_r - \frac{v}{r} + \frac{wS_\varphi}{r \sin\theta}} \right)^{1-\gamma} h_o \quad .$$

In the particular case that the body is of revolution without yaw, the description of the flow can be simplified. The shock surface is then a surface of revolution about the Z-axis, the flow is independent of $\varphi$, and $w \equiv 0$. The equations 3.18 become:

3.21 (a) $$v(u_\theta - v - rv_r) = -\frac{c^2}{\gamma(\gamma-1)} \frac{h_\theta}{h}$$

(b) $$(c^2 - v^2)v_\theta = r[-(c^2 - u^2)u_r + 2uvv_r + \frac{c^2 u h_r}{\gamma(\gamma-1)h}] - (c^2 - v^2)u - c^2(u + v\cot\theta)$$

(c) $$ruh_r + vh_\theta = 0$$

where

$$c^2 = \frac{\gamma-1}{2}(\hat{q}^2 - u^2 - v^2) \quad ,$$

and 3.18 (a) for $w \equiv 0$ is dependent on 3.21 (a) and (c). The boundary condition 3.19 on $\theta = B(r)$ becomes:

3.22 $$v = ruB_r \quad ,$$

and the boundary conditions 3.20 on $\theta = S(r)$ become:

3.23 $$u + rS_r v = q_o(\cos\theta - rS_r \sin\theta)$$

$$u\cos\theta - v\sin\theta = \frac{\mu^2 \hat{q}^2}{q_o} + \frac{(1-\mu^2)q_o(\cos\theta - rS_r\sin\theta)^2}{1 + r^2 S_r^2}$$

$$h = \frac{\hat{q}^2 - u^2 - v^2}{\hat{q}^2 - q_o^2} \left( \frac{rq_o\cos\theta \, S_r + q_o \sin\theta}{ruS_r - v} \right)^{1-\gamma} h_o \quad .$$

The equations and conditions can be put in dimensionless form by setting:

$$u^* = \frac{u}{\hat{q}},\quad v^* = \frac{v}{\hat{q}},\quad w^* = \frac{w}{\hat{q}},\quad q_o^* = \frac{q_o}{\hat{q}},\quad h^* = \frac{h}{h_o}\ .$$

This change of form is accomplished by replacing $\hat{q}$ by 1, $h_o$ by 1, and interpreting the u, v, w, $q_o$ and h as the same quantities starred. This form is used throughout Chapters III and IV. However, the stars are omitted for convenience in notation.

The flow problem may now be stated in dimensionless form in polar coordinates. The quantities $q_o$, $\gamma$ and the curve $\theta = B(r)$ are given; the problem is to find a solution $(u(r,\theta),\ v(r,\theta),\ h(r,\theta))$ of the equations:

3.24 (a) $$u(u_\theta - v - rv_r) = \frac{-c^2}{\gamma(\gamma-1)}\frac{h_\theta}{h}$$

(b) $$(c^2 - v^2)v_\theta = r\left[-(c^2-u^2)u_r + 2uvv_r + \frac{c^2 u h_r}{\gamma(\gamma-1)h}\right] - (c^2 - v^2)u - c^2(u + v\cot\theta)$$

(c) $$ruh_r + vh_\theta = 0$$

where:

$$c^2 = \frac{\gamma-1}{2}(1 - u^2 - v^2),$$

such that on $\theta = B(r)$

3.25 $$v = ruB_r\ ;$$

and to find a curve $\theta = S(r)$ such that on this curve

3.26 $$u + rS_r v = q_o(\cos\theta - rS_r \sin\theta)$$

3.27 $$u\cos\theta - v\sin\theta = \frac{\gamma-1}{\gamma+1}\frac{1}{q_o} + \frac{2q_o(\cos\theta - rS_r\sin\theta)^2}{(\gamma+1)(1 + r^2S_r^2)}$$

3.28 $$h = \frac{1 - u^2 - v^2}{1 - q_o^2}\left(\frac{rq_oS_r\cos\theta + q_o\sin\theta}{ruS_r - v}\right)^{1-\gamma}\ .$$

The solution is to be valid in a region $D(\delta)$:

3.29 $D(\delta) = [(r,\theta) \mid B(r) \leq \theta \leq S(r),\ 0 \leq r \leq \delta,\ \delta > 0]$.

The following assumptions, which were discussed in 2.45 and 2.46, are made. Those which concern $\gamma$, $q_o$ and $\theta = B(r)$ are:

3.30 (a) $1 < \gamma < 1.76$ .

(b) All velocities in dimensionless form have magnitude less than 1.

(c) $1 > q_o > \sqrt{\frac{\gamma-1}{\gamma+1}}$ .

(d) The following assumptions are made on the body curve $\theta = B(r)$:

1. $\theta = B(r)$ is analytic.
2. The first non-zero derivative of $B(r)$ at $r = 0$ is negative.
3. $\frac{\pi}{2} > \theta_b = B(0) > 0$ .
4. $\theta_b \geq B(r) \geq 0$ when $\delta \geq r \geq 0$ .

The other assumptions are:

3.31 (a) $1 \geq u^2 + v^2 - c^2 \geq 0$ in $D(\delta)$.

(b) $u\cos\theta - v\sin\theta \geq 0$ in $D(\delta)$.

(c) $u\sin\theta + v\cos\theta \geq 0$ in $D(\delta)$.

(d) $\frac{\pi}{2} > \theta_s = S(0) \geq \theta_b \geq 0$ .

(e) $S(r) \geq B(r)$ for $0 \leq r \leq \delta$.

(f) $c^2 - v^2 \geq 0$ in $D(\delta)$ . The analog of this assumption was not discussed in Section 2. It is introduced as a mathematical convenience.

The problem is easily seen to be singular. Elimination of $q_o$ between equations 3.26 and 3.27 for $r = 0$ gives:

3.32 $$v = \frac{u^2-1}{u}\frac{\gamma-1}{\gamma+1}\cot\theta_s \quad .$$

Since $1 \geq u \geq 0$, the quantity v above is non-zero at $(\theta_s, 0)$. On the other hand 3.25 gives:

3.33 $v = 0$

at $(\theta_b, 0)$. In the (x,y)-plane of Section 2, both $(\theta_s, 0)$ and $(\theta_b, 0)$ lie at the origin.

## Section 4: The Simplified Flow Problem.

The flow problem, 3.24-3.31, is treated in Chapter IV. Its solution is based on that of a simplified problem, obtained from 3.24-3.31, which is developed in Chapter III. The simplified flow is assumed to be irrotational, i.e. $u_\theta = v + rv_r$, and the condition 3.28 on h is omitted. The quantities $q_o, \gamma$ and the curve $\theta = B(r)$ are given; the problem is to find a solution $(u(r,\theta), v(r,\theta))$ of the equations:

4.1 $$u_\theta = v + rv_r$$

4.2 $$(c^2 - v^2)u_\theta = r[-(c^2 - u^2)u_r + 2uvv_r] - (c^2 - v^2)u - c^2(u + v\cot\theta)$$

where:

$$c^2 = \frac{\gamma-1}{2}(1 - u^2 - v^2) \ ,$$

such that on $\theta = B(r)$

4.3 $$v = ruB_r \ ;$$

and to find a curve $\theta = S(r)$ such that on it:

4.4 $$u + rS_r v = q_o(\cos\theta - rS_r\sin\theta)$$

4.5 $$u\cos\theta - v\sin\theta = \frac{\gamma-1}{\gamma+1}\frac{1}{q_o} + \frac{2q_o(\cos\theta - rS_r\sin\theta)^2}{(\gamma+1)(1 + r^2S_r^2)} \quad .$$

The solution is to be valid in a region $D(\delta)$:

4.6 $$D(\delta) = [(r,\theta) \mid B(r) \leq \theta \leq S(r),\ 0 \leq r \leq \delta,\ \delta > 0] .$$

The conditions 3.30 and 3.31 are also assumed to be valid.

The assumption of irrotationality implies that h is constant between the body and the shock profiles, $\theta = B(r)$ and $\theta = S(r)$ respectively. Thus, a solution to 4.1 - 4.5 is a solution to 3.24 - 3.28 except that the condition

4.7 $$h = \frac{1 - u^2 - v^2}{1 - q_o^2} \left( \frac{q_o(\sin\theta + rS_r\cos\theta)}{ruS_r - v} \right)^{1-\gamma}$$

on $\theta = S(r)$ may not be fulfilled. Therefore, 4.7 may be regarded as a condition that the shock profile be physically admissible. It is shown in this section that the only physically admissible shock profiles associated with the simplified flow problem are those for which S(r) is identically constant. The body is then a right circular cone. In spite of the fact that the simplified problem is physically significant only in a trivial case, its solution for general curves $\theta = B(r)$ provides the mathematical groundwork on which a solution of the physical problem 3.24 - 3.28 can be based.

It is shown now that if (u,v) and S(r) are a solution of the simplified flow problem, S(r) must be identically constant to be physically admissible. The equations 4.4 and 4.5 give:

4.8 $$u = \frac{1}{q_o(\sin\theta + rS_r\cos\theta)} \left[ q_o^2 \sin\theta(\cos\theta - rS_r\sin\theta) + \mu^2 rS_r + \frac{(1-\mu^2)q_o^2(\cos\theta - rS_r\sin\theta)^2}{1 + r^2S_r^2} rS_r \right]$$

$$v = \frac{1}{q_o(\sin\theta + rS_r\cos\theta)} \left[ q_o^2\cos\theta(\cos\theta - rS_r\sin\theta) - \mu^2 - \frac{(1-\mu^2)q_o^2(\cos\theta - rS_r\sin\theta)^2}{1 + r^2S_r^2} \right]$$

An angle, A, is introduced:

4.9 $$\sin A = \frac{\sin\theta + rS_r\cos\theta}{\sqrt{1 + r^2S_r^2}}$$

$$\cos A = \frac{\cos\theta - rS_r\sin\theta}{\sqrt{1 + r^2S_r^2}} \quad .$$

From 4.8 and 4.9 the expressions:

4.10 $$1 - u^2 - v^2 = \frac{1-q_o^2\cos^2 A}{q_o^2(1-\cos^2 A)}[q_o^2 - \mu^4 - (1-\mu^4)q_o^2\cos^2 A]$$

$$\frac{ruS_r - v}{q_o\sin\theta + rS_r\cos\theta} = \frac{\mu^2(1-q_o^2\cos^2 A)}{q_o^2(1 - \cos^2 A)}$$

are formed. Use of 4.10 in 4.7 yields:

4.11 $$h = \frac{(\mu^2)^{\gamma-1}}{(1-q_o^2)(q_o^2)^{\gamma}} [q_o^2 - {}^2 - (1-\mu^4)q_o^2\cos^2 A] \left(\frac{1-q_o^2\cos^2 A}{1-\cos^2 A}\right)^{\gamma} .$$

If h is to be constant on $\theta = S(r)$ then

4.12 $$\frac{dh}{d(\cos^2 A)} \frac{d(\cos^2 A)}{dr} \equiv 0 \quad .$$

Therefore, either

4.13 $$\frac{dh}{d(\cos^2 A)} \equiv 0$$

or

4.14 $$\frac{d\cos^2 A}{dr} \equiv 0 \quad .$$

The expression 4.13 in terms of $\cos^2 A$ is:

4.15 $$-\left(\frac{1-q_o^2\cos^2 A}{1-\cos^2 A}\right)^{\gamma-1}\frac{\gamma}{(1-\cos^2 A)^2}\{[(1-\mu^2)q_o^2\cos^2 A]^2$$

$$-2(1-\mu^2 q_o^2)[(1-\mu^2)q_o^2\cos^2 A] + (q_o^2-\mu^2)^2\} \equiv 0 \quad .$$

For $0 \leq r \leq \delta$ , $\cos^2 A \neq 1$. There are two possible cases. Either

4.16 $$1-q_o^2\cos^2 A \equiv 0$$

or

4.17 $$[(1-\mu^2)q_o^2\cos^2 A]^2 - 2(1-\mu^2 q_o^2)[(1-\mu^2)q_o^2\cos^2 A] + (q_o^2-\mu^2)^2 \equiv 0 \quad .$$

The case 4.16 is impossible since $1 > q_o^2$. The case 4.17 reduces to

4.18 $$\cos^2 A \equiv \frac{1-\mu^2 q_o^2 \pm \sqrt{(1-\mu^4)(1-q_o^4)}}{(1-\mu^2)q_o^2} \quad ,$$

from which it follows that $\cos^2 A$ is constant. Hence 4.13 implies 4.14.

The expression 4.14 in terms of r and S(r) is:

4.19 $$\frac{-2(\cos\theta - rS_r\sin\theta)(\sin\theta + rS_r\cos\theta)}{(1+r^2S_r^2)^2}\{S_r(1+r^2S_r^2)$$

$$+ (S_r + rS_{r2})\} \equiv 0 \quad .$$

For $0 \leq r \leq \delta$, the factors $(\cos\theta - rS_r\sin\theta)$ and $(\sin\theta + rS_r\cos\theta)$ do not vanish. The curvature of $\theta = S(r)$ in polar coordinates is

4.20 $$\frac{r^2S_r^3 + 2S_r + rS_{r2}}{(1+r^2S_r^2)^{3/2}} \; S_{r2} \quad .$$

Thus, the factor $\{S_r(1+r^2S_r^2) + (S_r + rS_{r2})\}$ vanishes identically if and only if $S(r) \equiv$ constant.

The fact that $S(r) \equiv$ constant implies that the body is a right circular cone. This is a consequence of the relations which exist between the derivatives of $S(r)$ at $r = 0$ and the derivatives of $B(r)$ at $r = 0$. The relations are developed in Chapter III.

# CHAPTER III

## SOLUTION OF THE SIMPLIFIED FLOW PROBLEM

### Section 1: Detailed Plan of Attack.

Chapter III is devoted to the simplified singular boundary value problem which was stated in Section 4 of Chapter II. The problem is studied in detail, and a formal solution is obtained in a constructive manner. For convenience the problem is now restated. The quantities $q_o$, $\gamma$ and the curve $\theta = B(r)$ are given; the problem is to find a solution $(u(r,\theta), v(r,\theta))$ of the equations:

1.1 $u_\theta = v + rv_r$

1.2 $(c^2 - v^2)v_\theta = r[ - (c^2 - u^2)u_r + 2uvv_r] - (c^2 - v^2)u$
$-c^2(u + v\cot\theta)$,

where:

$$c^2 = \frac{\gamma-1}{2}(1 - u^2 - v^2) \text{ and } \mu^2 = \frac{\gamma-1}{\gamma+1}$$

such that on $\theta = B(r)$,

1.3 $v = ruB(r)$;

and to find a curve $\theta = S(r)$ such that on it:

1.4 $u + rS_r v = q_o(\cos\theta - rS_r\sin\theta)$

1.5 $$u\cos\theta - v\sin\theta = \frac{\mu^2}{q_o} + \frac{(1-\mu^2)q_o(\cos\theta - rS_r\sin\theta)^2}{1 + r^2S_r^2}.$$

The solution is to be valid in a region $D(\delta)$:

1.6 $D(\delta) = [\ (r,\theta) \mid B(r) \leq \theta \leq S(r),\ 0 \leq r \leq \delta,\ \delta > 0\ ]$.

The assumptions concerning the given data: $q_o$, $\gamma$ and $\theta = B(r)$

are listed below.

1.7 (a) $1 < \gamma < 1.76$.

(b) All velocities have magnitude less than 1.

(c) $1 > q_o > \sqrt{\frac{\gamma - 1}{\gamma + 1}}$ .

(d) The following assumptions are made on the body curve, $\theta = B(r)$:

(1) $\theta = B(r)$ is analytic.

(2) The first non-zero derivative of $B(r)$ is negative at $r = 0$.

(3) $\frac{\pi}{2} > \theta_b = B(0) > 0$ .

(4) $\theta_b \geq B(r) \geq 0$ when $\delta \geq r \geq 0$ .

The other assumptions are now listed below.

1.8 (a) $1 > u^2 + v^2 - c^2 > 0$ in $D(\delta)$.

(b) $u \cos\theta - v \sin\theta > 0$ in $D(\delta)$.

(c) $u \sin\theta + v \cos\theta > 0$ in $D(\delta)$.

(d) $\frac{\pi}{2} > \theta_s = S(0) > \theta_b > 0$ .

(e) $S(r) > B(r)$ for $0 \leq r \leq \delta$ .

(f) $c^2 - v^2 > 0$ in $D(\delta)$ .

In accord with the general description of the method, as outlined in Chapter I, the first step in the development of a formal solution is the determination of $\theta_s$ and $(u,v)$ on $I_{b-\epsilon, s+\epsilon}$ (the interval $\theta_b - \epsilon \leq \theta \leq \theta_s + \epsilon$, $\epsilon > 0$). For this purpose, it is necessary to find a solution of the system

1.9 $u_\theta = v$ and

$$v_\theta = - u - \frac{c^2}{c^2 - v^2} (u + v \cot\theta)$$

such that at $\theta = \theta_b$ :

1.10 $v = 0$

and at $\theta = \theta_s$ :

1.11 $u = q_o \cos\theta_s$

$$u \cos\theta_s - v \sin\theta_s = \frac{\mu^2}{q_o} + (1 - \mu^2) q_o \cos^2\theta_s \quad .$$

The solution of this problem is the solution of the flow about an infinite right circular cone which is studied by G. I. Taylor and J. W. Maccoll[9] . They do not, nor does Kopal[10], discuss the existence of a solution, although the latter provides extensive computational data. A proof that there does actually exist a solution to the problem is given in Appendix I. The existence theorem stated there not only guarantees a solution which meets the end conditions, but also a solution which meets the physical restrictions, 1.7 and 1.8. Only a short description of the technique involved is given here; the bulk of detail is left to Appendix I.

Briefly, a value $u_o$ is assumed, such that $1 > u_o > \sqrt{\frac{\gamma-1}{\gamma+1}}$ and this value assigned to u at $\theta = \theta_b$. It is then noted that 1.9 satisfies a Lipschitz condition in (u,v) in a neighborhood of $\theta_b$; therefore a unique solution, (u,v), exists

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[9] Taylor and Maccoll, op. cit.

[10] Kopal, op. cit.

in this neighborhood such that $u = u_o$ and $v = 0$ at $\theta = \theta_b$. The argument then shows that there are values $u_o$ for which the neighborhood of $\theta_b$ (containing the unique solution $(u,v)$) includes a value of $\theta$ such that:

1.12 $$\frac{\pi}{2} > \theta > \theta_b$$

and

1.13 $$v = \frac{1 - u^2}{-u} \frac{\gamma-1}{\gamma+1} \cot\theta \ .$$

This $\theta$ is denoted by $\theta_s$. The last equation is derived from 1.11 by elimination of $q_o$. From the equation:

1.14 $$q_o = \frac{u(\theta_s)}{\cos\theta_s} \ ,$$

$q_o$ is computed. The value of $q_o$ is dependent upon $u_o$. It is further shown that if the pair $(u_o, \theta_b)$ lies in a certain set, the $q_o$ defined by 1.14 satisfies 1.7 (c) and throughout $I_{b,s}$, all the conditions are met. For these values $(u_o, \theta_b)$ the problem is solved. The inequalities established in Appendix I, however, are very obviously rough, and therefore the set of pairs $(q_o, \theta_b)$ allowed there does not include all the possible pairs as indicated by Kopal[11] for the supersonic case. The following theorem is proved in Appendix I:

Theorem 1: There is a non-empty subset, $D^*(\gamma)$, of $(q_o, \theta_b)$-space defined for all $\gamma$, $1 < \gamma < 1.76$, such that if $(q_o, \theta_b)$ belongs to $D^*(\gamma)$ then there exists a solution $(u,v)$ of 1.9 - 1.11 of class $C^\omega$ on $I_{b-\epsilon, s+\epsilon}$ for suitable $\epsilon > 0$. This solution is such that on $I_{b-\epsilon, s+\epsilon}$ :

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[11] Ibid.

(a) $1 > u^2 + v^2 - c^2 > 0,$

(b) $c^2 - v^2 > 0$

(c) $u\cos\theta - v\sin\theta > 0$

(d) $u\sin\theta + v\cos\theta > 0$

and such that:

$$1 > q_o > \sqrt{\frac{\gamma-1}{\gamma+1}}$$

$$\frac{\pi}{2} > \theta_s > \theta_b > 0.$$

The second step in the development of the formal solution is the inductive proof of the unique existence of $(u_{rn}, v_{rn})$, $n = 1, 2, \ldots$ on $I_{b-\epsilon, s+\epsilon}$ which satisfy the systems obtained from 1.1 - 1.5 by differentiation with respect to r the appropriate number of times. The proof depends upon certain inequalities which are established in Appendix II. The remainder of the proof and discussion are found in the following section. The argument also yields the $S_{rn}(0)$, $n = 1, 2, \ldots$, and shows them to be unique.

The formal series which follow are now well defined

1.15
$$u(r,\theta) = u(\theta) + ru_r(\theta) + \frac{r^2}{2!}u_{r2}(\theta) + \ldots$$

$$v(r,\theta) = v(\theta) + rv_r(\theta) + \frac{r^2}{2!}v_{r2}(\theta) + \ldots$$

$$S(r) = \theta_s + rS_r(0) + \frac{r^2}{2!}S_{r2}(0) + \ldots,$$

and the third step in the development of the formal solution is the proof that if they converge in $D(\delta)$ then they represent a solution to the problem. This justifies the term "formal solution" in connection with the series.

Section 2: Existence and Uniqueness of $u_{rn}(\theta)$, $v_{rn}(\theta)$, and $S_{rn}(0)$.

The differentiation of 1.1 - 1.2 with respect to r yields:

2.1 $(u_{rn})_\theta = A_n u_{rn} + B_n v_{rn} + C_n$

2.2 $(v_{rn})_\theta = D_n u_{rn} + E_n v_{rn} + F_n$ .

The differentiation of 1.3 with respect to r along the curve $\theta = B(r)$ yields:

2.3 $G_n u_{rn} + H_n v_{rn} = I_n$ .

The differentiation of 1.4 and 1.5 with respect to r along the curve $\theta = S(r)$ yields:

2.4 $u_{rn} + rS_{rn}(r) v_{rn} = M_n + N_n S_{rn}(r)$

2.5 $u_{rn} \cos\theta - v_{rn} \sin\theta = O_n + P_n S_{rn}(r)$.

The quantities $A_n, \ldots, P_n$ used in 2.1 - 2.5 are defined for $n = 1, \ldots$ as follows:

2.6 $A_n = 0$

$B_n = n+1$

$C_n = r v_{rn+1}$

$$D_n = -n\frac{c^2 - u^2}{c^2 - v^2} - 1 - \frac{c^2}{c^2 - v^2} - (\gamma-1)\, u(u + v\cot\theta)\frac{v^2}{(c^2 - v^2)^2}$$

$$E_n = \frac{2nuv}{c^2 - v^2} - \frac{c^2}{c^2 - v^2}\cot\theta + (\gamma-1)v\,\frac{u+v\cot\theta}{c^2 - v^2}$$

$$- (\gamma+1)\frac{c^2 v}{(c^2 - v^2)^2}\,(u + v\cot\theta)$$

$$F_1 = r\left[-\frac{c^2 - u^2}{c^2 - v^2} u_{rr} + \frac{2uv}{c^2 - v^2} v_{rr} - u_r \frac{\partial}{\partial r}\left(\frac{c^2 - u^2}{c^2 - v^2}\right) + v_r \frac{\partial}{\partial r}\left(\frac{2uv}{c^2 - v^2}\right)\right]$$

$$F_{n+1} = u_{rn} \frac{\partial}{\partial r} D_n + v_{rn} \frac{\partial}{\partial r} E_n + \frac{\partial}{\partial r} F_n + \frac{c^2 - u^2}{c^2 - v^2} u_{rn+1}$$

$$- \frac{2uv}{c^2 - v^2} v_{rn+1}$$

$$G_n = -rB_r(r)$$

$$H_n = 1$$

$$I_1 = - v_\theta B_r + uB_r + ruB_{rr} + ru_\theta (B_r)^2 + ru_r B_r$$

$$I_{n+1} = u_{rn} B_r + ru_{rn} B_{rr} + \frac{d}{dr} I_n$$

$$M_1 = - r(S_r)^2 (v_\theta + q_o \sin\theta) - rS_{r2}(v + q_o \sin\theta)$$

$$M_{n+1} = \frac{d}{dr} M_n + (v + q_o \sin\theta) S_{rn} - v_{rn}(S_r + rS_{r2}) - rv_{rn\theta}(S_r)^2$$

$$-u_{rn\theta} S_r + S_{rn} \frac{d}{dr} N_n$$

$$N_n = -(u_\theta + nv + (n+1)q_o \sin\theta)$$

$$O_1 = -rS_{r2} \left(\frac{4q_o}{\gamma+1} \frac{\cos\theta \sin\theta}{1 + r^2 S_r^2}\right) - \frac{4q_o \cos^2\theta}{\gamma+1} \frac{rS_r^2}{1 + r^2 S_r^2}$$

$$- \frac{4q_o \sin\theta \, rS_r}{(\gamma+1)(1 + r^2 S_r^2)} (- 2S_r \sin\theta - rS_{rr} \sin\theta - rS_r^2 \cos\theta)$$

$$- \frac{4q_o}{\gamma+1} \frac{rS_r^2 + r^2 S_r S_{rr}}{(1 + r^2 S_r^2)^2} (\cos\theta - rS_r \sin\theta)^2$$

$$O_{n+1} = \frac{d}{dr} O_n + S_{rn} \frac{d}{dr} P_n - (u_{rn\theta} \cos\theta - u_{rn} \sin\theta) S_r$$

$$+ (v_{rn\theta} \sin\theta + v_{rn} \cos\theta) S_r$$

$$- r \frac{d}{dr} \left[ S_{rn+1} \left(\frac{4q_o}{\gamma+1} \frac{\cos\theta \sin\theta}{1 + r^2 S_r^2}\right) \right]$$

$$P_n = \left(-u_\theta \cos\theta + \tfrac{1}{2} v \cos\theta + \tfrac{1}{2} v \cos\theta + u \sin\theta + v_\theta \sin\theta \right.$$

$$\left. - \frac{8 + 4(n-1)}{\gamma + 1} \frac{q_o \sin\theta \cos\theta}{1 + r^2 S_r^2} \right).$$

Let $A_{n,o}, \ldots, P_{n,o}$ denote $A_n, \ldots, P_n$ for $r = 0$ on $I_{b-\epsilon, s+\epsilon}$. The equations and conditions 2.1 - 2.5 then become, for $r = 0$,

2.7 $(u_{rn})_\theta = (n+1)\, v_{rn}$

2.8 $(v_{rn})_\theta = D_{n,o}\, u_{rn} + E_{n,o}\, v_{rn} + F_{n,o}$

where for $\theta = \theta_b$:

2.9 $v_{rn} = I_{n,o}$

and for $\theta = \theta_s$:

2.10 $u_{rn} = M_{n,o} + N_{n,o} S_{rn}(0)$

2.11 $u_{rn} \cos\theta - v_{rn} \sin\theta = O_{n,o} + P_{n,o}\, S_{rn}(0)$.

In Appendix II it is established that:

2.12 $N_{n,o} < 0$ at $\theta = \theta_s$.

Therefore $S_{rn}(0)$ may be eliminated between 2.10 and 2.11. The existence and uniqueness of a solution of 2.7 - 2.11 is then equivalent to the existence and uniqueness of a solution of the system:

2.13 $(u_{rn})_\theta = (n + 1)\, v_{rn}$

2.14 $(v_{rn})_\theta = D_{n,o}\, u_{rn} + E_{n,o}\, v_{rn} + F_{n,o}$

where, for $\theta = \theta_b$:

2.15 $v_{rn} = I_{n,o}$

and for $\theta = \theta_s$:

2.16 $\left(\frac{P_{n,o}}{N_{n,o} \sin\theta} - \cot\theta\right) u_{rn} + v_{rn} = \frac{-1}{\sin\theta} \left(O_{n,o} - \frac{M_{n,o}}{N_{n,o}} P_{n,o}\right)$.

The quantity $S_{rn}(0)$ is determined as follows:

$$2.17 \quad S_{rn}(0) = \left. \frac{u_{rn} - M_{n,o}}{N_{n,o}} \right]_{\theta = \theta_s} .$$

The unique existence of solutions of these systems is asserted in the following theorem.

Theorem 2. Let (u,v) be a solution to 1.9 - 1.11 as described in Theorem 1, and let $1 < \gamma < 1.5$. If $\theta = B(r)$ meets the requirements of 1.7 (d), then there exist unique functions $u_{rn}$, $v_{rn}$, n = 1, 2, ..... of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ for suitable $\epsilon > 0$ and $S_{rn}(0)$, n = 1, 2, ..... such that the $u_{rn}$, $v_{rn}$, and $S_{rn}(0)$ satisfy 2.7 - 2.11.

Proof: For n = 1 the coefficients in 2.7 - 2.11 are of class $C^{\omega}$ in $\theta$ on $I_{b-\epsilon,s+\epsilon}$ since u and v are of this class and the inequalities given in Theorem 1 are valid on the interval. Hence, if $u_r$, $v_r$ and $S_r(0)$ exist and are the unique solution of 2.7 - 2.11 for n = 1, then $u_r$ and $v_r$ are of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$. Similarly, if $u_{rn}$, $v_{rn}$ and $S_{rn}(0)$ exist, are unique, and are of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ for n = 1, ..., k, then the coefficients in 2.7 - 2.11 for n = k + 1 are of class $C^{\omega}$ in $\theta$ on $I_{b-\epsilon,s+\epsilon}$. Moreover, if $u_{rk+1}$, $v_{rk+1}$ and $S_{rk+1}(0)$ exist and form the unique solution of 2.7 - 2.11 on $I_{b-\epsilon,s+\epsilon}$ for n = k + 1, then they are of class $C^{\omega}$ in $\theta$ on the interval.

It remains to show that a unique solution of 2.7 - 2.11 exists for n = 1, and that if unique solutions of 2.7 - 2.11 exist for n = 1, ..., k, then one will exist for n = k + 1.

If $u_r$, $v_r$, $S_r(0)$ and $u_r^*$, $v_r^*$, $S_r^*(0)$ are assumed to be two solutions to 2.7 - 2.11, then $x = u_r - u_r^*$ and $y = v_r - v_r^*$ satisfy the equations:

2.18 $$\frac{dx}{d\theta} = 2y$$

2.19 $$\frac{dy}{d\theta} = D_{1,o}x + E_{1,o}y$$

and the conditions:

2.20 $y = 0$ at $\theta = \theta_b$

2.21 $$\left[\frac{P_{1,o}}{N_{1,o}\sin\theta} - \cot\theta\right] x + y = 0 \text{ at } \theta = \theta_s .$$

The coefficients depend only on u, v, and $\theta$. The inequalities:

2.22 $D_{1,o} > 0$ on $I_{b-\epsilon,s+\epsilon}$

and

2.23 $$\frac{P_{1,o}}{N_{1,o}\sin\theta} - \cot\theta > 0 \text{ for } \theta = \theta_s$$

are established in Appendix II. From 2.18 and 2.19 it follows that x satisfies the second order equation:

$$\frac{d^2x}{d\theta^2} - E_{1,o}\frac{dx}{d\theta} - 2D_{1,o}x = 0$$

on $I_{b-\epsilon,s+\epsilon}$ . This may be written in the form:

2.24 $$\frac{d}{d\theta}\left(\frac{dx}{d\theta}\exp\int_{\theta_b}^{\theta} - E_{1,o}(t)dt\right) = 2xD_{1,o}\exp\int_{\theta_b}^{\theta} - E_{1,o}(t)dt .$$

At $\theta = \theta_s$, x and $\frac{dx}{d\theta} = 2y$ are of opposite sign unless $x = 0$, from 2.21 and 2.23. If, at $\theta = \theta_s$, $x > 0$ and $\frac{dx}{d\theta} < 0$, then from 2.22 and 2.24 as $\theta$ decreases from $\theta = \theta_s$ the quantity

$$\frac{dx}{d\theta}\exp\int_{\theta_b}^{\theta} - E_{1,o}(t)dt$$

decreases and x increases. This leads to the conclusion that if $x > 0$ at $\theta = \theta_s$, then at $\theta = \theta_b$, $x > 0$ and $\frac{dx}{d\theta} = 2y < 0$. On the other hand, if $x < 0$ at $\theta = \theta_s$, then at $\theta = \theta_b$, $x < 0$ and $\frac{dx}{d\theta} = 2y > 0$. Both of these cases are incompatible with 2.20. Therefore, 2.18 - 2.21 admits only the identically zero solution. This implies the uniqueness of the solution $u_r$, $v_r$ and $S_r(0)$ for $n = 1$.

The assumption that the solutions of 2.7 - 2.11 exist for $n = 1, \ldots, k, k+1$ and are unique for $n = 1, \ldots, k$ implies that the solution for $n = k + 1$ is unique. The argument is essentially the same as that given for the case $n = 1$ above. The key inequalities

2.25 $\quad D_{n,o} > 0$ on $I_{b-\epsilon, s+\epsilon} \quad n = 1, 2, \ldots$

and

2.26 $\quad \frac{P_{n,o}}{N_{n,o} \sin\theta} - \cot\theta > 0$ for $\theta = \theta_s \quad n = 1, 2, \ldots$

are established in Appendix II.

The existence proof is based on the properties of certain auxiliary functions, $\alpha_j$, $\beta_j$, $\alpha_j^*$ and $\beta_j^*$. The functions $\alpha_j$ and $\beta_j$ satisfy the system:

2.27 $\quad \frac{d\alpha_j}{d\theta} = - D_{j,o} \beta_j$

2.28 $\quad \frac{d\beta_j}{d\theta} = - (j + 1)\alpha_j - E_{j,o} \beta_j$

and the conditions at $\theta = \theta_b$ :

2.29 $\quad \alpha_j = 0$

$\quad \beta_j = 1$ .

This system is well defined for $j = 1, \ldots$ on $I_{b-\epsilon, s+\epsilon}$

since $D_{j,o}$ and $E_{j,o}$ depend only on u, v and $\theta$ . Furthermore, the coefficients are of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ . Since the system is linear with a one point boundary condition, it has a unique solution of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$. The functions $\alpha_j^*$ and $\beta_j^*$ satisfy the system:

2.30 $$\frac{d\alpha_j^*}{d\theta} = - D_{j,o}\beta_j^*$$

2.31 $$\frac{d\beta_j^*}{d\theta} = - (j + 1)\alpha_j^* - E_{j,o}\,\beta_j^*$$

and the conditions at $\theta = \theta_s$:

2.32 $$\alpha_j^* = \frac{P_{j,o}}{N_{j,o}\sin\theta} - \cot\theta$$

$$\beta_j^* = 1 \quad .$$

This system is well defined for j = 1, ..... on $I_{b-\epsilon,s+\epsilon}$ since $D_{j,o}, E_{j,o}$, $P_{j,o}$ and $N_{j,o}$ depend only upon u, v and $\theta$ . Furthermore, the coefficients are of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ . Since the system is linear with a one point boundary condition, it has a unique solution of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ .

The following argument shows that $\alpha_j \beta_j^* - \alpha_j^* \beta_j \neq 0$ on $I_{b-\epsilon,s+\epsilon}$. This fact will be important in later work. From 2.27, 2.28, 2.30 and 2.31 it follows that:

$$\frac{d}{d\theta}(\alpha_j \beta_j^* - \alpha_j^* \beta_j) = -(\alpha_j \beta_j^* - \alpha_j^* \beta_j)\, E_{j,o} \quad .$$

Therefore:

2.33 $$\alpha_j \beta_j^* - \alpha_j^* \beta_j = - \alpha_j^*(\theta_b) \exp \int_{\theta_b}^{\theta} - E_{j,o}(t)dt \quad .$$

From 2.25, 2.30 and 2.31, $\alpha_j^*(\theta)$ satisfies:

$$2.34 \quad \frac{d^2\alpha_j^*}{d\theta^2} + \left(E_{j,o} - \frac{1}{D_{j,o}} \frac{dD_{j,o}}{d\theta}\right) \frac{d\alpha_j^*}{d\theta} = (j+1) D_{j,o} \alpha_j^*$$

Hence:

$$2.35 \quad \frac{d}{d\theta}\left[\frac{d\alpha_j^*}{d\theta} \exp \int_{\theta_b}^{\theta} \left(E_{j,o} - \frac{1}{D_{j,o}} \frac{dD_{j,o}}{d\theta}\right)dt\right]$$

$$= (j+1) \alpha_j^* D_{j,o} \exp \int_{\theta_b}^{\theta} \left(E_{j,o} - \frac{1}{D_{j,o}} \frac{dD_{j,o}}{d\theta}\right)dt \quad .$$

From 2.26 and 2.32, $\alpha_j^*(\theta_s) > 0$; and from 2.25, 2.26 and 2.32, $\frac{d\alpha_j^*(\theta)_s}{d\theta} < 0$. It then follows as in the uniqueness proof that $\alpha_j^*(\theta) > 0$ on $I_{b-\epsilon,s+\epsilon}$. This fact, combined with 2.33, gives the inequality:

$$2.36 \quad \alpha_j(\theta)\beta_j^*(\theta) - \alpha_j^*(\theta)\beta_j(\theta) < 0 \text{ on } I_{b-\epsilon,s+\epsilon} \quad .$$

If $f_1$, $f_2$, $f_3$ and $f_4$ are functions of class $C^{\omega}$ in $\theta$ on $I_{b-\epsilon,s+\epsilon}$, it is formally true that:

$$2.37 \quad \int_{\theta_b}^{\theta} f_1 (u_{rj})_\theta dt \equiv f_1 u_{rj} \Big]_{\theta_b}^{\theta} - \int_{\theta_b}^{\theta} u_{rj} \frac{df_1}{d\theta} dt$$

$$\int_{\theta_b}^{\theta} f_2 (v_{rj})_\theta dt \equiv f_2 v_{rj} \Big]_{\theta_b}^{\theta} - \int_{\theta_b}^{\theta} v_{rj} \frac{df_2}{d\theta} dt$$

$$\int_{\theta}^{\theta_s} f_3 (u_{rj})_\theta dt \equiv f_3 u_{rj} \Big]_{\theta}^{\theta_s} - \int_{\theta}^{\theta_s} u_{rj} \frac{df_3}{d\theta} dt$$

$$\int_{\theta}^{\theta_s} f_4 (v_{rj})_\theta dt \equiv f_4 v_{rj} \Big]_{\theta}^{\theta_s} - \int_{\theta}^{\theta_s} v_{rj} \frac{df_4}{d\theta} dt \quad .$$

Addition of these identities and use of 2.13 and 2.14 yields:

$$2.38 \quad \int_{\theta_b}^{\theta} \Big[ u_{rj}\left(\frac{df_1}{d\theta} + D_{j,o} f_2\right) + v_{rj}\left(\frac{df_2}{d\theta} + (j+1) f_1 + E_{j,o} f_2\right) + F_{j,o} f_2 \Big] dt$$

$$+ \int_{\theta}^{\theta_s} \Big[ u_{rj}\left(\frac{df_3}{d\theta} + D_{j,o} f_4\right) + v_{rj}\left(\frac{df_4}{d\theta} + (j+1) f_3 + E_{j,o} f_4\right) + F_{j,o} f_4 \Big] dt$$

$$= (f_3 u_{rj} + f_4 v_{rj})]_{\theta=\theta_s} + (f_1 - f_3)u_{rj}]_{\theta=\theta} + (f_2 - f_4)v_{rj}]_{\theta=\theta}$$

$$- (f_1 u_{rj} + f_2 v_{rj})]_{\theta=\theta_b} \quad .$$

If $f_1$, $f_2$, $f_3$ and $f_4$ are chosen so that:

2.39
$$f_1 = \mathfrak{r}\alpha_j$$
$$f_2 = \mathfrak{r}\beta_j$$
$$f_3 = k\alpha_j^*$$
$$f_4 = k\beta_j^*$$

where $\mathfrak{r}$ and k are constants, 2.38 reduces, with the aid of 2.15, 2.16, 2.27-2.32, and 2.39, to the following:

2.40
$$\mathfrak{r} \int_{\theta_b}^{\theta} F_{j,o}\,\beta_j dt + k\int_{\theta}^{\theta_s} F_{j,o}\,\beta_j^* dt$$
$$= -\frac{k}{\sin\theta_s}\left[ O_{j,o} - \frac{M_{j,\varrho}}{N_{j,o}} P_{j,o}\right]_{\theta=\theta_s} - \mathfrak{r}[I_{j,o}]_{\theta=\theta_b}$$
$$+ (\mathfrak{r}\alpha_j - k\alpha_j^*)\, u_{rj}]_{\theta=\theta} + (\mathfrak{r}\beta_j - k\beta_j^*)v_{rj}]_{\theta=\theta} \quad .$$

The constants, $\mathfrak{r}$ and k, may be chosen to satisfy

2.41
$$\mathfrak{r}\,\alpha_j(\theta) - k\,\alpha_j^*(\theta) = 1$$
$$\mathfrak{r}\,\beta_j(\theta) - k\,\beta_j^*(\theta) = 0 \quad ;$$

that is:

2.42
$$\mathfrak{r} = \beta_j^*(\theta)\,\Delta_j(\theta)$$
$$k = \beta_j(\theta)\,\Delta_j(\theta)$$

where:

2.43
$$\Delta_j(\theta) = (\alpha_j \beta_j^* - \alpha_j^* \beta_j)^{-1} \quad .$$

Then:

$$2.44 \quad u_{r\,j}(\theta) = \Delta_j(\theta)\ \{\beta_j^*(\theta) \int_{\theta_b}^{\theta} F_{j,o}\,\beta_j\,dt + \beta_j(\theta) \int_{\theta}^{\theta_s} F_{j,o}\,\beta_j^*\,dt$$

$$+ \frac{\beta_j(\theta)}{\sin\theta_s}\ [\ O_{j,o} - \frac{M_{j,o}}{N_{j,o}}\ P_{j,o}]_{\theta = \theta_s}$$

$$+ \beta_j^*(\theta)\ [I_{j,o}]_{\theta = \theta_b}\ \}$$

and:

$$2.45 \quad v_{r\,j}(\theta) = -\Delta_j(\theta)\ \{\alpha_j^*(\theta) \int_{\theta_b}^{\theta} F_{j,o}\,\beta_j\,dt + \alpha_j(\theta) \int_{\theta}^{\theta_s} F_{j,o}\,\beta_j^*\,dt$$

$$+ \frac{\alpha_j(\theta)}{\sin\theta_s}\ [\ O_{j,o} - \frac{M_{j,o}}{N_{j,o}}\ P_{j,o}]_{\theta = \theta_s}$$

$$+ \alpha_j^*(\theta)\ [\ I_{j,o}\ ]_{\theta = \theta_b}\ \} \quad .$$

For $j = 1$, the expressions 2.44 and 2.45 are well defined and are the unique solutions of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ of the system 2.13 - 2.16 for $n = 1$. That they are solutions can be verified by direct substitution in 2.13 - 2.16. If the expressions 2.44 and 2.45 for $j = 1, \ldots, k$ are the unique solutions of the system 2.13 - 2.16 for $n = 1, \ldots, k$ of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$, then 2.44 and 2.45 for $j = k + 1$ are, as may be verified by direct substitution, the unique solutions of the system 2.13 - 2.16 for $n = k + 1$ of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$. The quantity $S_{r\,j}(0)$, $j = 1, \ldots\ldots$, is defined by the equation:

$$2.46 \quad S_{rj}(0) = \left. \frac{u_{rj} - M_{j,o}}{N_{j,o}} \right]_{\theta=\theta_s} .$$

Thus, the unique solutions of the system 2.7 - 2.11 for $n = 1, \ldots$ of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ are given by 2.44 - 2.46.

The auxiliary functions $\alpha_j$, $\beta_j$, $\alpha_j^*$, and $\beta_j^*$ are of great interest because of the following corollaries to Theorem 2.

Corollary 1 to Theorem 2: Let (u,v) be a solution to 1.9 - 1.11 as described in Theorem 1 and $1 < \gamma < 1.5$. The auxiliary functions $\alpha_j$, $\beta_j$, $\alpha_j^*$ and $\beta_j^*$ defined for $j = 1, \ldots$ as solutions of the systems 2.27 - 2.29 and 2.30 - 2.32, are unique and of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ and depend only on (u,v).

This corollary shows that it is possible to construct tables of the auxiliary functions for $j = 1, \ldots, k$ similar to those that exist for the Taylor-Maccoll solutions. Such tables, together with the formulae 2.44 - 2.46, would make it possible to compute by hand the quantities $u_{rj}$, $v_{rj}$ and $S_{rj}(0)$, $j = 1, \ldots k$ for any desired case.

Corollary 2 to Theorem 2: Let (u,v) be a solution to 1.9 - 1.11 as described in Theorem 1 and $1 < \gamma < 1.5$. If $K_s$ and $K_b$ are the curvatures of the shock and body profiles respectively at the nose then

$$2.47 \quad \frac{K_s}{K_b} = \frac{-3u(\theta_b)\Delta_1(\theta_s)\cot\theta_s}{2(u(\theta_s) + v(\theta_s)\cot\theta_s)} .$$

Furthermore this ratio is positive and depends only on (u,v).

Proof: It is apparent that:

$$2.48 \quad \frac{K_s}{K_b} = \frac{S_r(0)}{B_r(0)} \quad .$$

From 2.6 and 2.46:

$$2.49 \quad S_r(0) = \frac{u_r(\theta_s)}{-2(v(\theta_s) + u(\theta_s)\tan \theta_s)} \quad .$$

From 2.6, 2.32 and 2.44:

$$2.50 \quad u_r(\theta_s) = 3u(\theta_b)B_r(0)\Delta_1(\theta_s) \quad .$$

Substitution of 2.50 in 2.49 yields 2.47 . From Theorem 1 and condition 2.36, it follows that the ratio is positive. The dependence of the ratio on (u,v) alone is a consequence of Corollary 1.

The following is an immediate consequence of Theorem 2.

Theorem 3. Let (u,v) be a solution to 1.9 - 1.11 as described in Theorem 1 and let $1 < \gamma < 1.5$. Let $\theta = B(r)$ meet the requirements of 1.7 (d). The power series:

$$2.51 \quad u(\theta) + u_r(\theta)\, r + u_{r2}(\theta)\, \frac{r^2}{2!} + \cdots\cdots$$

$$2.52 \quad v(\theta) + v_r(\theta)\, r + v_{r2}(\theta)\, \frac{r^2}{2!} + \cdots\cdots$$

$$2.53 \quad \theta_s + S_r(0)\, r + S_{r2}(0)\, \frac{r^2}{2!} + \cdots\cdots$$

are well defined on $I_{b-\epsilon,s+\epsilon}$ where $u_{rn}$, $v_{rn}$ and $S_{rn}(0)$ $n = 1, \cdots\cdots$ are the unique solutions of 2.7 - 2.11 of class $C^{\omega}$ on this interval.

## Section 3: The Series Representation as a Solution.

This section is devoted to a proof of the following theorem:

**Theorem 4.** Let $(u,v)$ be a solution to 1.9 - 1.11, as described in Theorem 1, on $I_{b-\epsilon,s+\epsilon}$ and let $1 < \gamma < 1.5$. Let $\theta = B(r)$ meet the requirements of 1.7 (d) for $0 \leq r \leq \delta_1$, $\delta_1 > 0$. If

(a) the formal power series 2.51 - 2.53 of Theorem 3 represent analytic functions of $r$ and $\theta$ denoted by $u(r,\theta)$, $v(r,\theta)$ and $S(r)$ respectively in the region $D^*(\delta_2,\epsilon_2)$:

$$3.1 \quad D^*(\delta_2,\epsilon_2) = [(r,\theta) \mid 0 < \theta_b - \epsilon_2 \leq \theta \leq \theta_s + \epsilon_2 < \frac{\pi}{2}, \quad 0 \leq r \leq \delta_2,\ \delta_2 > 0,\ \epsilon_2 > 0]$$

(b) the derivatives of $u(r,\theta)$, $v(r,\theta)$ and $S(r)$ can be found by termwise differentiation of the series 2.46 - 2.48 respectively in $D^*(\delta_2, \epsilon_2)$, and

(c) $B_r(0) \neq 0$ implies $B_{r2}(0)$ is such that:

$$3.2 \quad \left[\left(3v_r + u_r \frac{c^2}{c^2 - v^2} \cot\theta\right) \frac{-u_r}{q_o \sin\theta + v} + u_{rr}\right]_{(\theta,r) = (\theta_s,0)} > 0,$$

then $u(r,\theta)$, $v(r,\theta)$ and $S(r)$ are solutions of 1.1 - 1.5 in $D(\delta)$ for suitable $\delta > 0$ and the conditions of 1.7 and 1.8 are met in $D(\delta)$.

**Proof:** The theorem is proved in a sequence of five lemmas. The first lemma makes use of the continuity of the functions $u(r,\theta)$, $v(r,\theta)$ and $S(r)$ and their properties on

$I_{b-\epsilon, s+\epsilon}$ to show that the conditions 1.7 and 1.8 are met. The second lemma makes use of the Schauder Fixed Point Theorem to establish that there is a curve on which condition 1.4 is met. This curve is then shown to coincide with $\theta = S(r)$. The third lemma uses the Implicit Function Theorem and the result of the second lemma to establish the analogous result for condition 1.5. The last two lemmas use certain auxiliary functions to show that $u(r,\theta)$ and $v(r,\theta)$ satisfy equations 1.1 - 1.2 and also to prove that condition 1.3 is met on $\theta = B(r)$.

Lemma 1: There exist $\delta > 0$ and $\epsilon > 0$ such that $u(r,\theta)$, $v(r,\theta)$ and $S(r)$ satisfy the conditions 1.7 and 1.8 in $D^*(\delta, \epsilon)$ and such that $[(r,B(r)) \mid 0 \leq r \leq \delta\ ]$ and $[(r,S(r)) \mid 0 \leq r \leq \delta\ ]$ lie in $D^*(\delta, \epsilon)$.

Proof: Condition 1.7 (a) is met since $1 < \gamma < 1.5$ by hypothesis. Condition 1.7 (c) is met since (u,v) is a solution of 1.9 - 1.11 as described in Theorem 1. Condition 1.7 (d) is met by hypothesis for $0 < \delta \leq \delta_1$. The quantity $\theta_s$ is such that $\frac{\pi}{2} > \theta_s = S(0) > \theta_b$, for it was obtained with (u,v) from Theorem 1. By hypothesis, B(r) and S(r) are continuous in r; therefore there exists $\delta_3 > 0$ such that $\delta_3 \leq \min(\delta_1, \delta_2)$ and the conditions 1.8 (d) and (e) are satisfied.

If $\epsilon_3 = \frac{1}{2} \min(\epsilon, \epsilon_2)$, then on the closed interval $I_{b-\epsilon_3,\ s+\epsilon_3}$

(i) $1 > u^2 + v^2 - c^2 > 0$

(ii) $u\cos\theta - v\sin\theta > 0$

(iii) $u\sin\theta + v\cos\theta > 0$

(iv) $c^2 - v^2 > 0$

(v) $u > 0$ .

The set $D^*(\delta_3, \epsilon_3)$ is compact; therefore $u(r,\theta)$ and $v(r,\theta)$ are uniformly continuous on it. Thus, there exists $\delta_4 > 0$, $\delta_4 \leq \delta_3$ such that the inequalities (i) - (v) are satisfied on $D^*(\delta_4, \epsilon_3)$. Therefore conditions 1.8 (a), (b), (c) and (f) are satisfied.

The functions $B(r)$ and $S(r)$ are continuous on $0 \leq r \leq \delta_4$. Hence, there exists $\delta_5 > 0$, $\delta_5 \leq \delta_4$ such that $[(r,B(r)) \mid 0 \leq r \leq \delta_5]$ and $[(r,S(r)) \mid 0 \leq r \leq \delta_5]$ lie in $D^*(\delta_5, \epsilon_3)$. The quantities $\delta = \delta_5$ and $\epsilon = \epsilon_3$ are those required.

Throughout the remainder of the proof of Theorem 4, it is to be understood that each new choice of $\delta$ and $\epsilon$ is made so that the new set $D^*(\delta, \epsilon)$ is contained in the old set $D^*(\delta, \epsilon)$ which just precedes it. This fact will not always be indicated explicitly.

Lemma 2: There exists $\delta > 0$ such that for $0 \leq r \leq \delta$ the equation:

$$3.3 \quad rS_1'(r) = \frac{q_o\cos(S_1(r)) - u(r,S_1(r))}{v(r, S_1(r)) + q_o\sin(S_1(r))}$$

has a unique solution $\theta = S_1(r)$ for which $S_1(0) = \theta_s$ and $S_1'(0) = S_r(0)$. Further, $S_1(r) \equiv S(r)$ for $0 \leq r \leq \delta$ .

Proof: The function $q_o \sin\theta + v(r,\theta)$ is continuous over $D^*(\delta, \epsilon)$. Theorem 1 and Lemma 1 imply that, at the point $(\theta_s, 0)$,

3.4 $$q_o \sin\theta_s + v(0,\theta_s) = \frac{1}{\cos\theta_s}\,(u(\theta_s)\sin\theta_s + v(\theta_s)\cos\theta_s) \neq 0.$$

Thus, there exist $\delta_1 > 0$, $\delta_2 > 0$ and a closed set $B(\delta_1, \delta_2)$ defined by:

3.5 $$B(\delta_1, \delta_2) = [(r,\theta) \mid 0 \leq r \leq \delta_1,\ \theta_s - \delta_2 \leq \theta \leq \theta_s + \delta_2]$$

such that the inequality:

$$q_o \sin\theta + v \geq \epsilon_1 > 0$$

is valid in $B(\delta_1, \delta_2)$. Therefore, the function $W(r,\theta)$,

3.6 $$W(r,\theta) = \frac{q_o\cos\theta - u(r,\theta)}{q_o\sin\theta + v(r,\theta)},$$

is well defined and analytic over $B(\delta_1, \delta_2)$. At the point $(\theta_s, 0)$ direct calculation gives:

3.7 $$\frac{\partial W}{\partial \theta} = -1 .$$

By continuity, there exist $\delta_3 > 0$ and $\delta_4 > 0$ such that the inequality

3.8 $$\frac{\partial W}{\partial \theta} < 0$$

is valid over $B(\delta_3, \delta_4)$. Application of the Implicit Function Theorem to $W(r,\theta) = 0$ in $B(\delta_3, \delta_4)$ shows that there exists $t_o(r) = \theta$ of class $C^\omega$ in r for which

$$W(r, t_o(r)) \equiv 0 \qquad 0 \leq r \leq \delta_3 .$$

Calculation of $t_o'(0)$ yields:

3.9 $$t_o'(0) = \frac{-u_r(\theta_s)}{q_o \sin\theta_s + v(\theta_s)} = 2S_r(0) \quad .$$

From hypothesis 1.7 (d) on $\theta = B(r)$ and Corollary 2 to Theorem 2,

3.10 $$S_r(0) \leq 0 \quad .$$

The subset F of $B(\delta_3, \delta_4)$ is defined:

3.11 $$F = [(r,\theta) \mid 0 \leq r \leq \delta \ , \ \delta > 0, \ t_o(r) \leq \theta \leq \theta_s]$$

where $\delta > 0$ is so chosen that $\delta \leq \delta_3$, $[(r, t_o(r)) \mid 0 \leq r \leq \delta\ ]$ lies in $B(\delta, \delta_4)$, and $t_o(r) \leq rS_r(0) + \theta_s \leq \theta_s$ for $0 \leq r \leq \delta$. It is not trivial to show that such a $\delta > 0$ can be chosen when $S_r(0) = 0$. The details of the argument may be found in Appendix III. The key fact established there is that the first non-zero derivative of $t_o(r)$ evaluated for r=0 is negative.

The inequality

3.12 $$0 \geq \frac{1}{r} W(r,\theta) \geq \frac{1}{r} W(r,\theta_s)$$

is valid over F for $0 < r \leq \delta$ since 3.8 holds in F.

By direct calculation:

3.13 $$\lim_{r \to +0} \frac{1}{r} W(r,\theta_s) = 2S_r(0) \quad .$$

The function $\frac{1}{r} W(r,\theta_s)$ is continuous for $0 \leq r \leq \delta$ when its value at r = 0 is defined to be this limit. The set $0 \leq r \leq \delta$ is compact. Therefore, $\operatorname{lub}_{0 \leq r \leq \delta} \left|\frac{1}{r} W(r,\theta)\right|$ exists. It is denoted by N.

3.14 $$N = \operatorname{lub}_{0 \leq r \leq \delta} \left| \frac{1}{r} W(r, \theta_s) \right| .$$

A set of functions, G is defined:

3.15 G: all functions $\theta = t(r)$ of class $C^0$ defined for $0 \leq r \leq \delta$ such that:

(a) $(r, t(r))$ lies in F for $0 \leq r \leq \delta$

(b) $|t(r_1) - t(r_2)| \leq N|r_1 - r_2|$ for $0 \leq r_i \leq \delta$, $i = 1, 2$ .

This set may be topologized by the metric, d, defined by

3.16 $$d(t_1, t_2) = \operatorname{lub}_{0 \leq r \leq \delta} |t_1(r) - t_2(r)| .$$

It is apparent that G is a subset of the Banach space J, of continuous functions on $[0, \delta]$. The space G is non-empty since $\theta = rS_r(0) + \theta_s$ lies in G. It is convex; for if $t_1(r)$ and $t_2(r)$ belong to G, then the function $\lambda t_1(r) + (1 - \lambda)t_2(r)$ for each $\lambda$, $0 \leq \lambda \leq 1$, is continuous and satisfies 3.15 (a) and (b). The space F is separable metric and compact, and 3.15 (a) and (b) imply that G is a uniformly bounded equicontinuous class of functions. Therefore, by Ascoli's theorem every infinite subset of G contains a sequence of functions converging uniformly to a function, t, of J. Clearly, this limit function must satisfy 3.15 (a) and (b). Therefore G is compact.

A map f: $G \to J$, $f(t) = t^*$ is defined by the equation:

3.17 $$t^*(r) = \theta_s + \int_0^r \frac{1}{\xi} W(\xi, t(\xi))d\xi .$$

Since

$$3.18 \quad \left|t^*(r_1) - t^*(r_2)\right| = \left|\int_{r_2}^{r_1} \frac{1}{\xi} W(\xi, t(\xi))d\xi\right| \leq N\left|r_1 - r_2\right|,$$

then $t^*(r)$ is not only continuous but also satisfies 3.15 (b) for $0 \leq r \leq \delta$. To show that $t^*(r)$ also satisfies 3.15 (a), the curve $t_2(r)$ defined by:

$$3.19 \quad t_2(r) = \theta_s + \int_o^r \frac{1}{\xi} W(\xi, \theta_s)d\xi \qquad 0 \leq r \leq \delta$$

is considered. From 3.8, it follows that

$$3.20 \quad t_2(r) \leq t^*(r) \leq \theta_s \qquad 0 \leq r \leq \delta .$$

If

$$3.21 \quad t_2'(r) - t_o'(r) \geq 0 \qquad 0 \leq r \leq \delta ,$$

then

$$3.22 \quad t_o(r) \leq t_2(r) \qquad 0 \leq r \leq \delta$$

and consequently

$$3.23 \quad t_o(r) \leq t^*(r) \leq \theta_s \qquad 0 \leq r \leq \delta .$$

The argument leading to a proof of inequality 3.21 may be found in Appendix III. It is thus concluded that $f(G) \subset G$.

It is shown now that the map $f: G \to G$ is continuous. If $t_1$ and $t_2$ belong to G,

$$3.24 \quad \left|t_1^*(r) - t_2^*(r)\right| \leq \int_o^r \frac{1}{\xi} \left|W(\xi, t_1(\xi)) - W(\xi, t_2(\xi))\right| d\xi$$

$$0 \leq r \leq \delta .$$

With $\epsilon_o > 0$ given, it is required to find $\delta_o > 0$ such that $d(t_1, t_2) < \delta_o$ implies $d(t_1^*, t_2^*) < \epsilon_o$. The quantity 2N bounds the integrand in 3.24; hence there exists $r_1$, $0 < r_1 \leq \delta$

such that $2Nr_1 < \frac{\epsilon_o}{2}$ . Thus:

$$3.25 \quad \left|t_1^*(r) - t_2^*(r)\right| \leq \int_{r_1}^{r} \frac{1}{\xi}\left|W(\xi, t_1(\xi)) - W(\xi, t_2(\xi))\right| d\xi + \frac{\epsilon_o}{2} .$$

The function $\frac{1}{r} W(r,\theta)$ is uniformly continuous over the closed subset of F for which $r_1 \leq r \leq \delta$ . Therefore there exists $\delta_o(r_1) > 0$ such that $\frac{1}{r}\left|W(r,\theta_1) - W(r,\theta_2)\right| < \frac{\epsilon_o}{2\delta}$ for $|\theta_1 - \theta_2| < \delta_o(r_1)$. This is the required $\delta_o$.

The Schauder Theorem[12] asserts that a continuous map of a compact convex non-empty subset of a Banach space into itself has a fixed point. These hypotheses are satisfied by the set G and the map, f, defined above. If $\theta = S_1(r)$ is a fixed point of the map f, then

$$3.26 \quad S_1(r) = \theta_s + \int_o^r \frac{1}{\xi} W(\xi, S_1(\xi))d\xi \quad .$$

It is clear that $S_1(r)$ is continuously differentiable on $0 < r \leq \delta$ . Application of L'Hospital's rule gives:

$$3.27 \quad \lim_{r \to +0} S_1'(r) = \lim_{r \to +0} [W_r(r,S_1(r)) + W_\theta(r,S_1(r))S_1'(r)],$$

and hence

$$3.28 \quad \lim_{r \to +0} S_1'(r) = \lim_{r \to +0} \frac{W_r(r,S_1(r))}{1 - W_\theta(r,S_1(r))} = S_r(0) \quad .$$

Thus $S_1(r)$ is continuously differentiable on $0 \leq r \leq \delta$ and is a solution of 3.3.

This solution, $S_1(r)$, is unique. If it is assumed that two distinct solutions $t_5(r)$ and $t_6(r)$ exist, then there

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[12] Solomon Lefschetz, Topics in Topology (Princeton University, 1942) .

exists $\delta_3$ , $0 < \delta_3 \leq \delta$ such that

3.29 $t_5(r) < t_6(r) \qquad 0 < r \leq \delta_3$

(where the subscripts 5 and 6 are chosen appropriately). From 3.3 and 3.8,

3.30 $t_5'(r) > t_6'(r) \qquad 0 < r \leq \delta_3$

and

$$t_5'(0) = t_6'(0) \ .$$

Hence,

3.31 $t_5'(r) \geq t_6'(r) \qquad 0 \leq r \leq \delta_3$ .

Integration of the above yields:

3.32 $t_5(r) \geq t_6(r) \qquad 0 \leq r \leq \delta_3$

which contradicts 3.29 for $0 < r \leq \delta_3$. Thus, the two solutions coincide for $0 \leq r \leq \delta_3$. The function $\frac{1}{r} W(r,\theta)$ is analytic on F for $0 < \delta_3 \leq r \leq \delta$ . Therefore, the solutions which agree at $r = \delta_3$ agree for $\delta_3 \leq r \leq \delta$ , and the solution $S_1(r)$ is unique.

The function $\frac{1}{r} W(r,\theta)$ is analytic for $0 < r \leq \delta$ ; hence $S_1(r)$ is analytic for $0 < r \leq \delta$ . The second derivative of $S_1(r)$ for $r = 0$ is evaluated using L'Hospital's rule as follows:

$$rS_1^{(1)}(r) = W(r, S_1(r))$$

$$S_1^{(1)}(r) + rS_1^{(2)}(r) = W_r(r, S_1(r)) + W_\theta(r, S_1(r))\, S_1^{(1)}(r)$$

$$S_1^{(2)}(r) = \frac{W_r(r, S_1(r)) + (W_\theta - 1) S_1^{(1)}(r)}{r}$$

$$\lim_{r \to +0} S_1^{(2)}(r) = \lim_{r \to +0} (W_{rr} + 2W_{r\theta}\, S_1^{(1)}(r) + W_{\theta\theta}((S_1^{(1)}(r))^2 + (W_\theta - 1) S_1^{(2)}(r))$$

3.33 $$\lim_{r \to +0} S_1^{(2)}(r) = \lim_{r \to +0} \frac{W_{rr} + 2W_{r\theta} S_1^{(1)}(r) + W_{\theta\theta}(S_1^{(1)}(r))^2}{2 - W_\theta} .$$

In general, the $n^{th}$ derivative of $S_1(r)$ at $r = 0$ may be computed in the same way, since the denominator on the right will be $n - W_\theta$ which equals $n + 1$ at $(\theta_s, 0)$. This $n^{th}$ derivative is $S_{rn}(0)$ by condition 2.17. The series 2.53 has as coefficients $S_{rn}(0)$ and has a positive radius of convergence greater than or equal to $\delta$. Thus $S_1(r)$ is analytic on $0 \leq r \leq \delta$ and coincides with $S(r)$.

Lemma 3. There exists $\delta > 0$ such that the implicit equation

3.34 $$u(r,\theta)\cos\theta - v(r,\theta)\sin\theta - \frac{\gamma-1}{\gamma+1}\frac{1}{q_o} + 2q_o\left(\cot\theta - \frac{q_o\cos\theta - u(r,\theta)}{v(r,\theta) + q_o\sin\theta}\right)^2 \sin^2\theta - \frac{}{(\gamma+1)\left[1 + \left(\frac{q_o\cos\theta - u(r,\theta)}{v(r,\theta) + q_o\sin\theta}\right)^2\right]} = 0$$

has a unique solution $\theta = S_2(r)$ for $0 \leq r \leq \delta$ of class $C^\omega$ in $r$, such that $\theta_s = S_2(0)$. Also, $S_2(r) \equiv S(r)$ for $0 \leq r \leq \delta$.

Proof: If $Z(r,\theta)$ denotes the left hand side of equation 3.34 then $Z(r,\theta)$ is a well-defined and analytic function over the set $B(\delta_1, \delta_2)$. (See 3.5). By direct calculation:

3.35 $$\frac{\partial Z}{\partial \theta} = \frac{c^2}{c^2 - v^2}(u + v\cot\theta)\sin\theta\Big]_{(\theta,r)=(\theta_s,0)} > 0 .$$

Therefore, $\delta_5 > 0$ and $\delta_6 > 0$ exist such that

3.36 $\frac{\partial Z}{\partial \theta} > 0$ over $B(\delta_5, \delta_6)$ .

Since $Z(0,\theta_s) = 0$, the Implicit Function Theorem asserts the existence of a unique analytic solution $\theta = S_2(r)$ for $0 \leq r \leq \delta$ for some $\delta > 0$ such that $\theta_s = S_2(0)$ .

To verify that $S_2^{(n)}(0) = S_{rn}(0)$, $n = 1, 2, \ldots\ldots$, it is noted that condition 2.5 upon $S_{rn}(0)$ can be written in the following manner:

3.37 $$\frac{d^n}{dr^n}\Big[\, u(r,S(r))\cos(S(r)) - v(r,S(r))\sin(S(r)) - \frac{\gamma-1}{(\gamma+1)q_o} - \frac{2q_o\left(\cos(S(r)) + rS_r(r)\sin(S(r))\right)^2}{(\gamma+1)(1 + r^2S_r^2(r))}\,\Big]\Big]_{r=0} = 0 \ .$$

From Lemma 2,

3.38 $$rS_r(r) = \frac{q_o\cos\theta - u(r,\theta)}{q_o\sin\theta + v(r,\theta)}\Big]_{\text{on } \theta = S(r)}$$

and substitution of this value for $rS_r(r)$ in the above (3.37) shows that $S_2^{(n)}(0) = S_{rn}(0)$ for $n = 1, 2, \ldots\ldots$ . It then follows from the Identity Theorem that $S_2(r) \equiv S(r)$.

Lemma 4. There is a $\delta > 0$ such that over $D^*(\delta, \epsilon)$ there exists a unique solution $({}^ou_r, {}^ov_r)$ of class $C^\omega$ of the system of equations:

3.39 $$({}^ou_r)_\theta = rv_{r2} + v_r + {}^ov_r$$

$$3.40 \quad ({}^{o}v_r)_\theta = -\frac{c^2 - u^2}{c^2 - v^2} u_r + \frac{2uv}{c^2 - v^2} v_r$$

$$+ r\left[\frac{-(c^2 - v^2)(-(\gamma-1)(uu_r + vv_r) - 2uu_r)}{(c^2 - v^2)^2} u_r\right.$$

$$- \frac{(c^2 - u^2)(-(\gamma-1)(uu_r + vv_r) - 2vv_r)}{(c^2 - v^2)^2} u_r$$

$$- \frac{c^2 - u^2}{c^2 - v^2} u_{r2} + \frac{2uv}{c^2 - v^2} v_{r2} + \frac{2uv_r}{c^2 - v^2} v_r$$

$$\left. + \frac{(c^2 - v^2)(2u_r v) + 2uv(+(\gamma-1)(uu_r + vv_r) + 2vv_r)}{(c^2 - v^2)^2} v_r\right]$$

$$+ \left[1 + \frac{c^2}{c^2 - v^2} + (\gamma-1)u(u + v\cot\theta)\frac{v^2}{(c^2 - v^2)^2}\right]{}^{o}u_r$$

$$+ \left[-(\gamma-1)v\frac{u + v\cot\theta}{c^2 - v^2} + \frac{c^2}{c^2 - v^2}\cot\theta\right.$$

$$\left. + \frac{c^2}{(c^2 - v^2)^2}(\gamma+1)v(u + v\cot\theta)\right]{}^{o}v_r$$

$$+ \left[-2 + 2(\gamma-1)u\frac{u + v\cot\theta}{c^2 - v^2} - \frac{2c^2}{c^2 - v^2}\right.$$

$$\left. - \frac{2c^2}{(c^2 - v^2)^2}(\gamma-1)u(u + v\cot\theta)\right] u_r$$

$$+ \left[2(\gamma-1)\frac{u + v\cot\theta}{c^2 - v^2} - \frac{2c^2}{c^2 - v^2}\cot\theta\right.$$

$$\left. - \frac{2c^2}{(c^2 - v^2)^2}(\gamma+1)v(u + v\cot\theta)\right] v_r$$

such that on $\theta = B(r)$:

3.41 $$ {}^{o}v_r + rB_r\,{}^{o}u_r = B_r[\, rB_r(rv_r + v) + u $$

$$ + r\left(\frac{c^2 - u^2}{c^2 - v^2}\,u_r + \frac{-2uv}{c^2 - v^2}\,v_r\right) + u $$

$$ + \frac{c^2}{c^2 - v^2}\,(u + v\cot\theta)] + ruB_{r2} + 2ru_rB_r $$

and on $\theta = S(r)$ :

3.42 $$ {}^{o}u_r\left(\cos\theta + rS_r\,\frac{c^2 - u^2}{c^2 - v^2}\right) + {}^{o}v_r\left(\sin\theta + rS_r\cos\theta + r\,\frac{2uv}{c^2 - v^2}\,S_r\right) $$

$$ = 2v_r\sin\theta - 2vS_r\cos\theta - 2uS_r\sin\theta - \frac{c^2 S_r}{c^2 - v^2}(u + v\cot\theta)\sin\theta $$

$$ + \frac{2q_o}{\gamma + 1}\,\frac{d}{dr}\left[\frac{\left(\cos(S(r)) - \dfrac{q_o\cos(S(r)) - u(r,S(r))}{v(r,S(r)) + q_o\sin(S(r))}\,\sin(S(r))\right)^2}{1 + \left(\dfrac{q_o\cos(S(r)) - u(r,S(r))}{v(r,S(r)) + q_o\sin(S(r))}\right)^2}\right]. $$

Furthermore, on $D^*(\delta, \epsilon)$

3.43 $$ {}^{o}u_r(r,\theta) \equiv u_r(r,\theta) $$

$$ {}^{o}v_r(r,\theta) \equiv v_r(r,\theta) $$

Proof: From Lemma 1, the coefficient of ${}^{o}u_r$ in 3.40 is positive on $D^*(\delta, \epsilon)$. There is a $\delta > 0$ such that for $0 \leq r \leq \delta$ the coefficients of ${}^{o}u_r$ and ${}^{o}v_r$ in 3.42 are positive. For $r = 0$, the method of proof of Theorem 2 applies directly. For fixed $r \geq 0$, the quantity

$$ \alpha(r,\theta)\beta^*(r,\theta) - \alpha^*(r,\theta)\beta(r,\theta) $$

may vanish. The $\beta$ , $\alpha$ , $\beta^*$ and $\alpha^*$ are the auxiliary functions for the system. However, at $(\theta_b, 0)$ it does not vanish. Hence, there exists $\delta > 0$ such that it does not vanish for $0 \leq r \leq \delta$ , $\theta = \theta_b$. For this $\delta$ , the method of proof of Theorem 2 assures the unique existence of an analytic solution of 3.39- 3.42 over $D^*(\delta, \epsilon)$. On $I_{b-\epsilon, s+\epsilon}$ it is clear that:

3.44.. $$ {}^{o}u_r(0,\theta) \equiv u_r(0,\theta) $$

$$ {}^{o}v_r(0,\theta) \equiv v_r(0,\theta) $$

since both satisfy 3.39 - 3.42 and the solution is unique. The system 3.39 - 3.42 may be regarded as formed from the system 2.1 - 2.5 by adding and subtracting equal quantities--and then labeling some of the quantities with superscripts. Differentiation of 3.39 - 3.42 with respect to r and the use of 3.44 yields a system that could be formed from the system 2.7 - 2.11 for n = 2 by a similar process. Since the solution is unique,

3.45 $$ {}^{o}u_{r2}(0,\theta) \equiv u_{r2}(0,\theta) $$

$$ {}^{o}v_{r2}(0,\theta) \equiv v_{r2}(0,\theta) \quad . $$

If the identities

3.46 $$ {}^{o}u_{rj}(0,\theta) \equiv u_{rj}(0,\theta) $$

$$ {}^{o}v_{rj}(0,\theta) \equiv v_{rj}(0,\theta) $$

are valid for j = 1, ..., k, then the same argument shows that they are valid for j = k + 1. Therefore the identities 3.46 hold for j = 1, ..... and 3.43 holds on $D^*(\delta, \epsilon)$.

Lemma 5. There is a $\delta > 0$ such that there exists a unique solution $(u^*, v^*)$ of class $C^\omega$ over $D^*(\delta, \epsilon)$ of the system:

3.47 $$u^*_\theta = rv_r + v^*$$

3.48 $$v^*_\theta = r\left(-\frac{c^2 - u^2}{c^2 - v^2} u_r + \frac{2uv}{c^2 - v^2} v_r\right) - 2u - 2c^2 \frac{u + v\cot\theta}{c^2 - v^2}$$
$$+ u^* + \frac{c^{*2}}{c^{*2} - v^{*2}} (u^* + v^* \cot\theta)$$

such that:

3.49 $$u^*(r, B(r)) = u(0, \theta_b) + \int_0^r [u_\theta(\xi, B(\xi)) B_r(\xi) + {}^0u_r(\xi, B(\xi))] d\xi$$

3.50 $$v^*(r, B(r)) = rB_r(r)\, u^*(r, B(r)) .$$

Furthermore, on $D^*(\delta, \epsilon)$:

3.51 $$u^*(r, \theta) \equiv u(r, \theta)$$
$$v^*(r, \theta) \equiv v(r, \theta)$$

and

3.52 $$v(r, B(r)) = ru(r, B(r)) B_r(r) .$$

Proof: For $r = 0$ it is clear that $u^* \equiv u$, $v^* \equiv v$ is a solution. The one point boundary conditions 3.49 - 3.50 ($r$ regarded as a parameter) are analytic in $r$. Therefore, there exists $\delta > 0$ such that 3.47 - 3.50 have a unique solution of class $C^\omega$ on $D^*(\delta, \epsilon)$. Differentiation of 3.49 with respect to $r$ yields:

3.53 $$u^*_\theta(r, B(r)) B_r(r) + u^*_r(r, B(r)) = u_\theta(r, B(r)) B_r(r)$$
$$+ {}^0u_r(r, B(r)) .$$

Since $u^*(0,\theta) \equiv u(0,\theta)$, it follows that $u^*_\theta(0,\theta) \equiv u_\theta(0,\theta)$. Hence, for $r = 0$, equation 3.53 and Lemma 4 give:

3.54 $$u^*_r(0,\theta_b) = {}^0u_r(0,\theta_b) = u_r(0,\theta_b).$$

Differentiation of 3.50 with respect to r yields:

3.55 $$\begin{aligned} v^*_r(r,B(r)) + v^*_\theta(r,B(r))B_r(r) = {} & B_r(r)u^*(r,B(r)) \\ & + rB_{r2}(r)u^*(r,B(r)) \\ & + rB_r(r)[u^*_r(r,B_r(r)) \\ & + u^*_\theta(r,B(r))B_r(r)\ ]\ . \end{aligned}$$

The result

3.56 $$v^*_r(0,\theta_b) = {}^0v_r(0,\theta_b) = v_r(0,\theta_b)$$

is obtained from 3.41, 3.54 and 3.55 for $r = 0$. Differentiation of the equations 3.47 and 3.48 with respect to r yields

3.57 $$(u^*_r)_\theta = rv_{r2} + v_r + v^*_r$$

3.58 $$\begin{aligned} (v^*_r)_\theta = {} & -\frac{c^2-u^2}{c^2-v^2}u_r + \frac{2uv}{c^2-v^2}v_r \\ & + r\Bigg[\frac{-(c^2-v^2)(-(\gamma-1)(uu_r+vv_r)-2uu_r)}{(c^2-v^2)^2}u_r \\ & - \frac{(c^2-u^2)(-(\gamma-1)(uu_r+vv_r)-2vv_r)}{(c^2-v^2)^2}u_r \\ & - \frac{c^2-u^2}{c^2-v^2}u_{r2} + \frac{2uv}{c^2-v^2}v_{r2} \\ & + \frac{(c^2-v^2)(2u_rv+2uv_r)-2uv(-(\gamma-1)(uu_r+vv_r)-2vv_r)}{(c^2-v^2)^2}v_r\Bigg] \end{aligned}$$

$$+ \left[ 1 + \frac{c^2}{c^2 - v^2} + (\gamma-1)\, u\, (u + v\cot\theta)\, \frac{v^2}{(c^2 - v^2)^2} \right] u_r^*$$

$$+ \left[ -(\gamma-1)\, v\, \frac{u + v\cot\theta}{c^2 - v^2} + \frac{c^2}{c^2 - v^2}\cot\theta \right.$$

$$\left. + \frac{c^2}{(c^2 - v^2)^2}\, (\gamma+1)\, v\, (u + v\cot\theta) \right] v_r^*$$

$$+ \left[ -2 + 2(\gamma-1)\, u\, \frac{u + v\cot\theta}{c^2 - v^2} - \frac{2c^2}{c^2 - v^2} \right.$$

$$\left. - \frac{2c^2}{(c^2 - v^2)^2}\, (\gamma-1)\, u\, (u + v\cot\theta) \right] u_r$$

$$+ \left[ 2(\gamma-1)\, v\, \frac{u + v\cot\theta}{c^2 - v^2} - \frac{2c^2}{c^2 - v^2}\cot\theta \right.$$

$$\left. - \frac{2c^2}{(c^2 - v^2)^2}\, (\gamma+1)\, v\, (u + v\cot\theta) \right] v_r \quad .$$

From 3.39, 3.40, 3.57 and 3.58, it is apparent that on $I_{b-\epsilon;s+\epsilon}$, $u_r^* - {}^{o}u_r$ and $v_r^* - {}^{o}v_r$ satisfy the equations:

3.59 $$(u_r^* - {}^{o}u_r)_\theta = (v_r^* - {}^{o}v_r)$$

3.60 $$(v_r^* - {}^{o}v_r)_\theta = \left[ 1 + \frac{c^2}{c^2 - v^2} \right.$$

$$\left. + (\gamma-1)u(u + v\cot\theta)\, \frac{v^2}{(c^2 - v^2)^2} \right](u_r^* - {}^{o}u_r)$$

$$+ \left[ -(\gamma-1)v\, \frac{u + v\cot\theta}{c^2 - v^2} + \frac{c^2}{c^2 - v^2}\cot\theta \right.$$

$$\left. + \frac{c^2}{(c^2 - v^2)^2}\, (\gamma+1)v(u + v\cot\theta) \right](v_r^* - {}^{o}v_r) \quad .$$

From 3.54 and 3.56:

3.61 $u_r^* - {}^o u_r = 0$

$v_r^* - {}^o v_r = 0 \qquad$ at $(0, \Theta_b)$ .

The system 3.59 - 3.60 is linear and homogeneous. Therefore, it has the unique solution $u_r^* - {}^o u_r \equiv 0$ and $v_r^* - {}^o v_r \equiv 0$. Lemma 4 together with this result gives:

3.62 $u_r^*(0,\Theta) \equiv {}^o u_r(0,\Theta) \equiv u_r(0,\Theta)$

$v_r^*(0,\Theta) \equiv {}^o v_r(0,\Theta) \equiv v_r(0,\Theta)$ .

If the identities

3.63 $u_{rn}^*(0,\Theta) \equiv {}^o u_{rn}(0,\Theta) \equiv u_{rn}(0,\Theta)$

$v_{rn}^*(0,\Theta) \equiv {}^o v_{rn}(0,\Theta) \equiv v_{rn}(0,\Theta)$

are valid for $n = 1, \ldots, k$, then the same argument shows that they are valid for $n = k + 1$. Thus, 3.51 holds on $D^*(\delta, \epsilon)$. Lemma 4 and the identities 3.51 combined with 3.49 and 3.50 imply 3.52.

The proof of Theorem 4 is now complete. The existence of $\delta > 0$ and $\epsilon > 0$ have been established such that in $D^*(\delta, \epsilon)$ and therefore in $D(\delta)$:

(a) the conditions 1.7 and 1.8 are satisfied (Lemma 1)

(b) $u(r,\Theta)$ and $v(r,\Theta)$ satisfy 1.1 - 1.3 (Lemma 5)

(c) $u(r,\Theta)$, $v(r,\Theta)$ and $S(r)$ satisfy 1.4 (Lemma 2)

(d) $u(r,\Theta)$, $v(r,\Theta)$ and $S(r)$ satisfy 1.5 (Lemma 3) .

# CHAPTER IV

## SOLUTION OF THE FLOW PROBLEM

### Section 1: Introduction.

Chapter IV is devoted to the singular boundary value problem which was stated in Section 3 of Chapter II. This problem is studied in detail, and, with the technique developed in Chapter II, a formal solution is obtained in a constructive manner for the class of "cone-like" bodies.

**Definition:** *A body profile $\theta = B(r)$ is cone-like if and only if it is analytic and*

1.1 $$B_{r^{2n-1}}(0) = 0 \qquad n = 1, 2, \ldots .$$

The reason for the restriction of the argument to cone-like bodies is given in the next section.

The dimensionless form of the problem is used. The quantities $q_o, \gamma$ and the curve $\theta = B(r)$ are given; the problem is to find a solution $(u(r,\theta), v(r,\theta), h(r,\theta))$ of the equations:

1.2 $$u(u_\theta - v - rv_r) = \frac{-c^2}{\gamma(\gamma-1)h} h_\theta$$

1.3 $$v_\theta = r\left[-\frac{c^2-u^2}{c^2-v^2} u_r + \frac{2uv}{c^2-v^2} v_r + \frac{c^2 u}{c^2-v^2} \frac{1}{\gamma(\gamma-1)h} h_r\right]$$

$$- u - \frac{c^2}{c^2-v^2}(u + v\cot\theta)$$

1.4 $$ruh_r + vh_\theta = 0$$

where:

1.5 $$c^2 = \frac{\gamma-1}{2}(1 - u^2 - v^2) \quad \text{and} \quad \mu^2 = \frac{\gamma-1}{\gamma+1} ,$$

such that on $\theta = B(r)$:

1.6 $\quad v = ruB_r$,

and to find a curve $\theta = S(r)$ such that on it:

1.7 $\quad u + rS_r v = q_o(\cos\theta - rS_r \sin\theta)$

$$1.8 \quad u\cos\theta - v\sin\theta = \frac{\mu^2}{q_o} + \frac{(1-\mu^2)q_o(\cos\theta - rS_r\sin\theta)^2}{1 + r^2S_r^2}$$

$$1.9 \quad h = \left(\frac{1 - u^2 - v^2}{1 - q_o^2}\right)\left(\frac{q_o(\sin\theta + rS_r\cos\theta)}{ruS_r - v}\right)^{1-\gamma} .$$

The solution is to be valid in a region $D(\delta)$:

1.10 $\quad D(\delta) = [(r,\theta) \mid B(r) \leq \theta \leq S(r),\ 0 \leq r \leq \delta\ ,\ \delta > 0]$ .

The physical stipulations of Chapter II, 3.30 and 3.31 are also made.

The first step in the development of a formal solution is the determination of $\theta_s$ and $(u,v,h)$ on $I_{b-\epsilon,s+\epsilon}$ . For this purpose, it is necessary to find a solution of

$$1.11 \quad u(u_\theta - v) = -\frac{c^2}{\gamma(\gamma-1)h}h_\theta$$

$$1.12 \quad v_\theta = -u - \frac{c^2}{c^2 - v^2}(u + v\cot\theta)$$

1.13 $\quad vh_\theta = 0$ ,

such that at $\theta = \theta_b$ :

1.14 $\quad v = 0$

while at $\theta = \theta_s$ :

1.15 $\quad u = q_o\cos\theta$

$$1.16 \quad u\cos\theta - v\sin\theta = \frac{\mu^2}{q_o} + (1 - \mu^2)q_o\cos^2\theta$$

1.17 $$h = \left(\frac{1 - u^2 - v^2}{1-q_o^2}\right)\left(\frac{q_o \sin\theta}{-v}\right)^{1-\gamma} .$$

The Taylor-Maccoll solution, (u,v), together with a constant, h, determined by 1.17, form a solution of this system. Therefore, from Theorem 1:

Theorem 5. There exists a non-empty subset $D^*(\gamma)$, of $(q_o, \theta_b)$-space defined for all $\gamma$, $1 < \gamma < 1.76$ such that if $(q_o, \theta_b)$ belongs to $D^*(\gamma)$, there exists a solution $(u,v,h)$ of class $C^\omega$ of 1.11 - 1.17 on $I_{b-\epsilon, s+\epsilon}$ for suitable $\epsilon > 0$. This solution is such that on $I_{b-\epsilon, s+\epsilon}$ :

(a) $1 > u^2 + v^2 - c^2 > 0$

(b) $c^2 - v^2 > 0$

(c) $u\cos\theta - v\sin\theta > 0$

(d) $u\sin\theta + v\cos\theta > 0$

(e) $h > 0$,

and also such that :

$$1 > q_o > \sqrt{\frac{\gamma-1}{\gamma+1}}$$

$$\frac{\pi}{2} > \theta_s > \theta_b > 0 .$$

The second step in the development of the formal solution is the inductive proof of the unique existence of $(u_{r,n}, v_{r,n}, h_{r,n})$ $n = 1, 2, \ldots$ on $I_{b-\epsilon, s+\epsilon}$ satisfying the systems obtained from 1.2 - 1.9 by differentiation with respect to r the appropriate number of times. This induction is carried out only for cone-like bodies. The determination

of $(u_r, v_r, h_r)$ is carried out without the restriction to cone-like bodies, since these quantities are needed to determine the shock-body curvature ratio at the nose and to make clear the reason for the discussion of cone-like bodies. The determination of $(u_{rn}, v_{rn}, h_{rn})$ on $I_{b-\epsilon, s+\epsilon}$ also involves the determination of $S_{rn}(0)$.

The third step in the development is the proof that if the series

1.18 $$u(r,\theta) = u(\theta) + ru_r(\theta) + \frac{r^2}{2!} u_{r2}(\theta) + \cdots\cdots$$

1.19 $$v(r,\theta) = v(\theta) + rv_r(\theta) + \frac{r2}{2!} v_{r2}(\theta) + \cdots\cdots$$

1.20 $$h(r,\theta) = h(\theta) + rh_r(\theta) + \frac{r2}{2!} h_{r2}(\theta) + \cdots\cdots$$

1.21 $$S(r) = \theta_s + rS_r(0) + \frac{r^2}{2!} S_{r2}(0) + \cdots\cdots$$

converge in $D(\delta)$, they represent a solution to the problem.

Section 2: Existence and Uniqueness of $u_{rn}(\theta)$, $v_{rn}(\theta)$, $h_{rn}(\theta)$ and $S_{rn}(0)$.

The equations 1.2 - 1.4 are differentiated n times with respect to r to obtain the system:

2.1 $$(u_{rn})_\theta = A_n u_{rn} + B_n v_{rn} + Q_n h_{rn\theta} + R_n$$

2.2 $$(v_{rn})_\theta = D_n u_{rn} + E_n v_{rn} + T_n h_{rn} + W_n$$

2.3 $$v(h_{rn})_\theta = X_n h_{rn} + Y_n \quad .$$

The differentiation of 1.6 with respect to r along the curve $\theta = B(r)$ yields:

2.4 $$G_n u_{rn} + H_n v_{rn} = I_n \quad .$$

The differentiation of 1.7 - 1.9 with respect to r along the curve $\theta = S(r)$ yields:

2.5 $u_{rn} + rS_r v_{rn} = M_n + N_n S_{rn}$

2.6 $u_{rn} \cos\theta - v_{rn} \sin\theta = O_n + P_n S_{rn}$

2.7 $h_{rn} = Z_n S_{rn} + J_n u_{rn} + K_n v_{rn} + L_n$ .

The quantities:

2.8 $A_n$, $B_n$, $C_n$, $D_n$, $E_n$, $F_n$, $G_n$, $H_n$, $I_n$, $M_n$, $N_n$, $O_n$ and $P_n$ used in 2.1-2.7 are as defined in Chapter III, 2.6. The additional definitions:

2.9 $$Q_n = - \frac{c^2}{\gamma(\gamma-1)hu}$$

$$R_1 = rv_{rr} + h_\theta \frac{\partial}{\partial r} (Q_1)$$

$$R_{n+1} = \frac{\partial}{\partial r} [A_n u_{rn} + B_n v_{rn} + Q_n h_{rn\theta}] - A_n u_{rn+1} - B_n v_{rn+1} - Q_n h_{rn+1\theta}$$

$$T_n = \frac{nc^2 u}{(c^2 - v^2)\gamma(\gamma-1)h}$$

$$W_1 = F_1 + r \frac{\partial}{\partial r} (T_1)$$

$$W_{n+1} = F_{n+1} + \frac{\partial}{\partial r}(W_n - F_n) - T_1 h_{rn+1} + h_{rn} \frac{\partial}{\partial r} T_n$$

$$X_n = nu$$

$$Y_1 = - v_r h_\theta - r u_r h_r + u h_{r2}$$

$$Y_{n+1} = \frac{\partial}{\partial r} ( - nuh_r + Y_n) + (n + 1) u h_{rn+1}$$

are also made. This leaves $Z_n$, $J_n$, $K_n$ and $L_n$ still undefined.

The use of 1.7 to eliminate $rS_r$ in 1.9 yields:

2.10 $$h = \frac{1 - u^2 - v^2}{1 - q_o^2} \left( \frac{q_o(u \cos\theta - v \sin\theta) - u^2 - v^2}{q_o^2 - q_o(u \cos\theta - v \sin\theta)} \right)^{\gamma-1} .$$

Differentiation of h as expressed in 2.10 with respect to r along $\theta = S(r)$ gives:

2.11 $$h_r = Z_1 S_r + J_1 u_r + K_1 v_r + L_1 \ ,$$

where:

2.12 $$Z_1 = \frac{\partial}{\partial\theta}\left[\frac{1-u^2-v^2}{1-q_o^2}\left(\frac{q_o(u\cos\theta - v\sin\theta) - u^2 - v^2}{q_o^2 - q_o(u\cos\theta - v\sin\theta)}\right)^{\gamma-1}\right] - h_\theta$$

$$J_1 = -\frac{2u}{1-q_o^2}\left(\frac{q_o(u\cos\theta - v\sin\theta) - u^2 - v^2}{q_o^2 - q_o(u\cos\theta - v\sin\theta)}\right)^{\gamma-1}$$

$$+ (\gamma-1)\left[\frac{1-u^2-v^2}{1-q_o^2}\left(\frac{q_o(u\cos\theta - v\sin\theta) - u^2 - v^2}{q_o^2 - q_o(u\cos\theta - v\sin\theta)}\right)^{\gamma-2}\right.$$

$$\left.\left(\frac{q_o(q_o^2 - u^2 - v^2)\cos\theta}{[q_o^2 - q_o(u\cos\theta - v\sin\theta)]^2}\right)\right]$$

$$K_1 = -\frac{2v}{1-q_o^2}\left(\frac{q_o(u\cos\theta - v\sin\theta) - u^2 - v^2}{q_o^2 - q_o(u\cos\theta - v\sin\theta)}\right)^{\gamma-1}$$

$$- (\gamma-1)\left[\frac{1-u^2-v^2}{1-q_o^2}\left(\frac{q_o(u\cos\theta - v\sin\theta) - u^2 - v^2}{q_o^2 - q_o(u\cos\theta - v\sin\theta)}\right)^{\gamma-2}\right.$$

$$\left.\left(\frac{q_o(q_o^2 - u^2 - v^2)\sin\theta}{[q_o^2 - q_o(u\cos\theta - v\sin\theta)]^2}\right)\right]$$

$$L_1 = 0 \ .$$

The above expressions define $Z_1$, $J_1$, $K_1$ and $L_1$. From the $n^{th}$ derivative of h along $\theta = S(r)$ the definitions:

2.13 $$Z_n = Z_1$$

$$J_n = J_1$$

$$K_n = K_1$$

$$L_{n+1} = \frac{d}{dr}[Z_n S_{rn} + J_n u_{rn} + K_n v_{rn} + L_n] - Z_{n+1} S_{rn+1}$$

$$- J_{n+1} u_{rn+1} - K_{n+1} v_{rn+1}$$

are obtained.

If $A_{n,o}, \ldots, Z_{n,o}$ denote $A_n, \ldots, Z_n$ respectively on $I_{b-\epsilon, s+\epsilon}$, the equations and conditions 2.1 - 2.7 then become for $r = 0$:

2.14 $(u_{rn})_\theta = (n+1)\, v_{rn} + Q_{n,o}\, h_{rn\theta} + R_{n,o}$

2.15 $(v_{rn})_\theta = D_{n,o} u_{rn} + E_{n,o}\, v_{rn} + T_{n,o} h_{rn} + W_{n,o}$

2.16 $v(h_{rn})_\theta = X_{n,o} h_{rn} + Y_{n,o}$ ,

where at $\theta = \theta_b$ :

2.17 $v_{rn} = I_{n,o}$

and at $\theta = \theta_s$:

2.18 $u_{rn} = M_{n,o} + N_{n,o}\, S_{rn}(0)$

2.19 $u_{rn} \cos\theta - v_{rn} \sin\theta = O_{n,o} + P_{n,o} S_{rn}(0)$

2.20 $h_{rn} = Z_{n,o} S_{rn}(0) + J_{n,o} u_{rn} + K_{n,o} v_{rn} + L_{n,o}$ .

Since

2.21 $N_{n,o}\,]_{\theta=\theta_s} < 0$ ,

the system 2.14 - 2.20 is equivalent to the system :

2.22 $(u_{rn})_\theta = (n+1) v_{rn} + Q_{n,o} h_{rn\theta} + R_{n,o}$

2.23 $(v_{rn})_\theta = D_{n,o} u_{rn} + E_{n,o} v_{rn} + T_{n,o} h_{rn} + W_{n,o}$

2.24 $v(h_{rn})_\theta = X_{n,o} h_{rn} + Y_{n,o}$ .

where at $\Theta = \Theta_b$ :

2.25 $\quad v_{rn} = I_{n,o}$

and at $\Theta = \Theta_s$ :

2.26 $$\left(\frac{P_{n,o}}{N_{n,o}\sin\Theta} - \cot\Theta\right) u_{rn} + v_{rn}$$
$$= -\frac{1}{\sin\Theta}\left[O_{n,o} - \frac{M_{n,o}P_{n,o}}{N_{n,o}}\right]$$

2.27 $$h_{rn} - \left(\frac{Z_{n,o}}{N_{n,o}} + J_{n,o}\right) u_{rn} - K_{n,o}v_{rn} = L_{n,o} - \frac{Z_{n,o}M_{n,o}}{N_{n,o}} \quad .$$

A solution of 2.22 - 2.27 is a solution of 2.14 - 2.20 where $S_{rn}(0)$ is defined as:

2.28 $$S_{rn}(0) = \frac{u_{rn} - M_{n,o}}{N_{n,o}}\Bigg]_{\Theta = \Theta_s} \quad .$$

The study of the system 2.22 - 2.27 depends upon the fact that equation 2.24 involves the single unknown $h_{rn}$ if $(u_{rj}, v_{rj}, h_{rj})$ are known for $j = 1, \ldots, n-1$. For $n = 1$, this equation is:

2.29 $\quad v(h_r)_\Theta = -uh_r \quad .$

<u>Theorem 6.</u> <u>Let</u> $(u,v,h)$ <u>be a solution of</u> 1.11 - 1.17 <u>as described in Theorem 5. The function</u> $h^*_{rn}(\Theta)$ :

2.30 $$h^*_{rn}(\Theta) = \left(\frac{\Theta-\Theta_b}{\Theta_s-\Theta_b}\right)^{\frac{n}{2}} \exp\left[n\int_\Theta^{\Theta_s}\left(\frac{u}{v} + \frac{1}{2(t-\Theta_b)}\right)dt\right]$$

<u>is of class</u> $C^\omega$ <u>on</u> $I_{b-\epsilon,s+\epsilon}$ <u>if</u> n <u>is even and it is of class</u> $C^\omega$ <u>on</u> $I_{b,s+\epsilon}$ , <u>except at</u> $\Theta = \Theta_b$ <u>where it is continuous, if</u> n <u>is odd. In the case that</u> n <u>is even, it is the unique solution of class</u> $C^\omega$ <u>of the reduced equation</u> 2.24 <u>on</u> $I_{b-\epsilon,s+\epsilon}$ <u>such that</u>

$h^*_{rn}(\theta_s) = 1$. *In the case that n is odd, it is the unique solution of the reduced equation 2.24 on $I_{b,s+\epsilon}$ such that* $h^*_{rn}(\theta_s) = 1$.

*Proof*: The function

2.31 $$\frac{u}{v} + \frac{1}{2(\theta - \theta_b)}$$

is of class $C^\omega$ on $I_{b-\epsilon,s+\epsilon}$ since $\frac{u}{v}$ has a pole of order one at $\theta = \theta_b$ with residue $-\frac{1}{2}$. Therefore, the function $h^*_{rn}(\theta)$ is of class $C^\omega$ on $I_{b-\epsilon,s+\epsilon}$ if n is even and is of class $C^\omega$ on $I_{b,s+\epsilon}$, except at $\theta = \theta_b$ where it is continuous, if n is odd. By direct calculation, $h^*_{rn}(\theta)$ is a solution of the reduced equation 2.24, which is

2.32 $$v(h_{rn})_\theta = -nuh_{rn} ,$$

such that $h^*_{rn}(\theta_s) = 1$. In every open interval of $I_{b-\epsilon,s+\epsilon}$ containing $\theta = \theta_s$ but not $\theta = \theta_b$, the solution of 2.32 is unique for given value at $\theta = \theta_s$. The difference of two solutions having the value one at $\theta = \theta_s$ is identically zero in an open interval containing $\theta = \theta_s$. Therefore, these solutions coincide on $I_{b-\epsilon,s+\epsilon}$ if n is even and coincide on $I_{b,s+\epsilon}$ if n is odd. Thus the solution $h^*_{rn}(\theta)$ is unique.

By Theorem 6, any solution of 2.29 on $I_{b,s+\epsilon}$ can be expressed in the form:

2.33 $$k_1 h^*_r(\theta)$$

where $k_1$ is a constant. If this expression is used for $h_r$ in the system 2.22 - 2.26 (for the case n = 1), the auxiliary

functions $\alpha_1$, $\beta_1$, $\alpha_1^*$ and $\beta_1^*$ introduced in Chapter III may be used to express a solution of this system, dependent on the parameter $k_1$. The condition 2.27 may then be used to determine $k_1$ and thus to obtain a solution of 2.22 - 2.27, or, equivalently, a solution of the system 2.14 - 2.20 for n = 1.

In the following discussion the abbreviation $\Phi_n$, where

$$2.34 \quad \Phi_n = \left[\frac{Z_{n,o}}{N_{n,o}} + J_{n,o} - K_{n,o}\left(\frac{P_{n,o}}{N_{n,o}\sin\theta} - \cot\theta\right)\right]_{\theta=\theta_s} \Delta_n(\theta_s) \int_{\theta_b}^{\theta_s} (\alpha_n h^*_{rn\theta} Q_{n,o} + \beta_n h^*_{rn} T_{n,o})\, d\theta$$

is convenient, because the quantity occurs frequently.

Theorem 7. Let (u,v,h) be a solution of 1.11 - 1.17 as described in Theorem 5 and $1 < \gamma < 1.5$. Let $\theta = B(r)$ meet the requirements 3.30 (d) of Chapter III. If the inequality

$$2.35 \quad \Phi_1 \neq 1$$

is valid, the system 2.14 - 2.20 has a unique solution $(u_r, v_r, h_r)$ on $I_{b,s+\epsilon}$ of class $C^\omega$, except possibly at $\theta = \theta_b$ where it is at least continuous.

Proof: If $(u_r, v_r, h_r)_1$ and $(u_r, v_r, h_r)_2$ are two solutions of the system 2.22 - 2.27 for n = 1 on $I_{b,s+\epsilon}$, their respective differences (x, y, z) satisfy the system:

$$2.36 \quad x_\theta = 2y + Q_{1,o} z$$

$$2.37 \quad y_\theta = D_{1,o} x + E_{1,o} y + T_{1,o} z$$

$$2.38 \quad vz_\theta = X_{1,o} z$$

where at $\theta = \theta_b$:

2.39 $\quad y = 0$

and at $\theta = \theta_s$:

2.40 $\quad \left(\frac{P_{1,o}}{N_{1,o}\sin\theta} - \cot\theta\right) x + y = 0$

2.41 $\quad z - \left(\frac{Z_{1,o}}{N_{1,o}}\right) + J_{1,o})x + K_{1,o}y = 0$ .

The general solution of 2.38 is

2.42 $\quad z = mh_r^*(\theta)$

where m is a constant. The use of the auxiliary functions $\alpha_1$, $\beta_1$, $\alpha_1^*$ and $\beta_1^*$, as in Chapter III, yields:

2.43 $\quad x = m\Delta_1(\theta)\ [\ \beta_1^*(\theta) \int_{\theta_b}^{\theta} (\alpha_1\, h_{r\theta}^*\, Q_{1,o} + \beta_1\, h_r^*\, T_{1,o})\, dt$

$$+ \beta_1(\theta) \int_{\theta}^{\theta_s} (\alpha_1^*\, h_{r\theta}^*\, Q_{1,o} + \beta_1^*\, h_r^*\, T_{1,o})\, dt\ ]$$

2.44 $\quad y = -\, m\Delta_1(\theta)\ [\ \alpha_1^*(\theta) \int_{\theta_b}^{\theta} (\alpha_1 h_{r\theta}^* Q_{1,o} + \beta_1 h_r^*\, T_{1,o})\, dt$

$$+ \alpha_1(\theta) \int_{\theta}^{\theta_s} (\alpha_1^*\, h_{r\theta}^*\, Q_{1,o} + \beta_1^*\, h_r^*\, T_{1,o})\, dt\ ]\ .$$

The integrals involved in 2.43 and 2.44 exist since their integrands are continuous on $I_{b,s+\epsilon}$, with the possible exception of the point $\theta = \theta_b$ near which they may behave as badly as the function $(\theta - \theta_b)^{-\frac{1}{2}}$. The functions x, y and z defined in 2.42 - 2.44 satisfy the system 2.36 - 2.40 . From 2.42 - 2.44 it follows that:

$$2.45 \qquad z - \left(\frac{Z_{1,o}}{N_{1,o}} + J_{1,o}\right)x + K_{1,o}y \Big]_{\theta=\theta_s} = m(1 - \bar{\Phi}_1) \quad .$$

Thus, condition 2.41 is met if and only if m = 0. Therefore, if a solution $(u_r, v_r, h_r)$ of 2.22 - 2.27 exists, it is unique.

If 2.35 is valid, the unique solution of 2.14 - 2.20 does exist for n = 1 on $I_{b,s+\epsilon}$ . It is

$$2.46 \qquad u_r = \Delta_1(\theta) \Big[ k_1 \beta_1^*(\theta) \int_{\theta_b}^{\theta} (\alpha_1 h_{r\theta}^* Q_{1,o} + \beta_1 h_r^* T_{1,o})dt$$

$$+ k_1 \beta_1(\theta) \int_{\theta}^{\theta_s} (\alpha_1^* h_{r\theta}^* Q_{1,o} + \beta_1^* h_r^* T_{1,o})dt$$

$$+ \beta_1^*(\theta) \left( I_{1,o} \right)_{\theta=\theta_b} \Big]$$

$$2.47 \qquad v_r = -\Delta_1(\theta) \Big[ k_1 \alpha_1^*(\theta) \int_{\theta_b}^{\theta} (\alpha_1 h_{r\theta}^* Q_{1,o} + \beta_1 h_r^* T_{1,o})dt$$

$$+ k_1 \alpha_1(\theta) \int_{\theta}^{\theta_s} (\alpha_1^* h_{r\theta}^* Q_{1,o} + \beta_1^* h_r^* T_{1,o})dt$$

$$+ \alpha_1^*(\theta) \left( I_{1,o} \right)_{\theta=\theta_b} \Big]$$

$$2.48 \qquad h_r = k_1 h_r^*$$

$$2.49 \qquad S_r(0) = \frac{u_r}{N_{1,o}} \Big]_{\theta=\theta_s} \quad ,$$

where $k_1$ is the constant given by the equation:

$$2.50 \qquad k_1 = \left[ I_{1,o} \right]_{\theta=\theta_b} \Delta_1(\theta_s) \frac{\left[\frac{Z_{1,o}}{N_{1,o}} + J_{1,o}\right]_{\theta=\theta_s} \beta_1^*(\theta_s)}{1 - \bar{\Phi}_1}$$

$$- \frac{(K_{1,o})_{\theta=\theta_s} \alpha_1^*(\theta_s)}{1 - \bar{\Phi}_1} \quad .$$

It is clear that this solution has the required properties on $I_{b,s+\epsilon}$ .

Corollary 1 to Theorem 7. Let (u, v,h) be a solution of 1.11 - 1.17 as described in Theorem 5 and $1 < \gamma < 1.5$. If the inequality 2.35 is valid, the ratio of $K_s$ and $K_b$, the curvatures of the shock and body profiles respectively at the nose, is:

$$2.51 \quad \frac{K_s}{K_b} = \frac{-3u(\theta_b)\Delta_1(\theta_s)\cot\theta_s}{2(u(\theta_s) + v(\theta_s)\cot\theta_s)} \frac{1}{1 - \Phi_1} .$$

Furthermore, this ratio depends only on (u,v, h) .

Theorem 7 would be strengthened if it could be established without having to make the assumption 2.35 . The following inequalities are established in Appendix IV :

$$2.52 \quad \frac{Z_{1,o}}{N_{1,o}} + J_{1,o} - K_{1,o}\left( \frac{P_{1,o}}{N_{1,o}\sin\theta} - \cot\theta\right)\Big]_{\theta=\theta_s} < 0$$

$$2.53 \quad \int_{\theta_b}^{\theta_s} (\alpha_1 h_{r\theta}^* Q_{1,o} + \beta_1 h_r^* T_{1,o})d\theta > 0 .$$

These, together with the result:

$$2.54 \quad \Delta_1(\theta_s) < 0$$

from Chapter III, imply that

$$2.55 \quad \Phi_1 > 0.$$

This is the only result obtained in this work in the attempt to establish the assumption. However, not only is it probable that 2.35 holds, but it is also probable that the stronger inequality

2.56 $0 < \Phi_1 < 1$

is valid; for the above is equivalent (from 2.50) to the reasonable supposition :

2.57 $\frac{K_s}{K_b} > 0$ .

In considering the system 2.14 - 2.20 for $n > 1$, it is important to notice that $I_{n,o}$ will contain multiple $\Theta$-derivatives of $u_r$ and $v_r$. Thus, if $h_r \not\equiv 0$, $I_{n,o}$ is undefined, in general, for $n > 1$. Indeed, if $h_{rj} \not\equiv 0$, j odd, this situation will arise for some $n > j$. For this reason it is natural to consider those body profiles for which $h_{rn} \equiv 0$ for odd index n. The next theorem reveals the consequences of this consideration.

Theorem 8. Let (u,v,h) be a solution of 1.11 - 1.17 as described in Theorem 5 and $1 < \gamma < 1.5$. If the functions $u_{rn}$, $v_{rn}$, $h_{rn}$ and $S_{rn}(0)$ form a unique family of solutions of the system 2.14 - 2.20 on $I_{b,s+\epsilon}$ of class $C^{\omega}$ such that:

2.58 $h_{r2n-1} \equiv 0 \qquad n = 1, 2, \ldots\ldots ,$

then

2.59 $B_{r2n-1}(0) = 0$

$u_{r2n-1}(\Theta) \equiv 0$

$v_{r2n-1}(\Theta) \equiv 0$

$S_{r2n-1}(0) = 0 \qquad n = 1, 2, \ldots\ldots .$

Proof: The proof is inductive.

Case n = 1. From 2.26 and 2.27 :

$$2.60 \quad \left(\frac{P_{1,o}}{N_{1,o}\sin\theta} - \cot\theta\right) u_r + v_r = 0$$

$$\left(\frac{Z_{1,o}}{N_{1,o}} + J_{1,o}\right) u_r + K_{1,o} v_r = 0$$

at $\theta = \theta_s$. Therefore, with use of 2.52:

$$2.61 \quad u_r(\theta_s) = v_r(\theta_s) = 0 \quad .$$

The system of equations 2.22 - 2.23 is homogeneous, hence:

$$2.62 \quad u_r(\theta) \equiv v_r(\theta) \equiv 0$$

on $I_{b,s+\epsilon}$ . From 2.25, 2.28, and the above :

$$2.63 \quad B_r(0) = S_r(0) = 0 \quad .$$

It is assumed that 2.59 is valid for n = 1, ..., k .

Case n = k + 1. From 1.7,

$$2.64 \quad u + rS_r v = q_o(\cos\theta - rS_r \sin\theta)$$

on $\theta = S(r)$. The $(2k + 1)^{th}$ derivative of 2.64 with respect to r along $\theta = S(r)$ evaluated for r = 0 will involve only the derivatives $u_{r2k+1}$ or $S_{r2k+1}$ since 2.59 is valid for n = 1, ... , k. For instance, in the expansion of $\frac{d^{2k+1}}{d_{r2k+1}} (u(r,S(r)))$ every term involves r derivatives of u on S(r), and the sum of the orders of differentiation in r in each term is odd. Hence, the induction assumption implies every term is zero for r = 0 except those which involve $u_{r2k+1}$ or $S_{r2k+1}$ . The same argument applies to the other terms of 2.64 with the modification that if the sum

of the orders of differentiation in r is even, then the term contains r as a factor. The same type of reasoning leads to the general conclusions:

2.65 $$R_{2k+1,o} \equiv W_{2k+1,o} \equiv Y_{2k+1,o} \equiv 0$$

on $I_{b,s+\epsilon}$ ,

2.66 $$I_{2k+1,o} = - v_\theta B_{r\,2k+1} + (2k+1)uB_{r\,2k+1} \Big]_{\theta=\theta_b} ,$$

and at $\theta = \theta_s$:

2.67 $$O_{2k+1,o} = M_{2k+1,o} = L_{2k+1,o} = 0 .$$

It is established in Appendix IV that

2.68 $$\frac{Z_{n,o}}{N_{n,o}} + J_{n,o} - K_{n,o}\left(\frac{P_{n,o}}{N_{n,o}\sin\theta} - \cot\theta\right) \Big]_{\theta = \theta_s} < 0$$

for all n. Therefore, from 2.26, 2.27, 2.67, and 2.68:

2.69 $$u_{r\,2k+1}(\theta_s) = v_{r\,2k+1}(\theta_s) = 0 .$$

From 2.65, the system of equations 2.22 - 2.23 is homogeneous; hence:

2.70 $$u_{r\,2k+1} \equiv v_{r\,2k+1} \equiv 0$$

on $I_{b,s+\epsilon}$. From 2.25, 2.28, 2.66, 2.67 and 2.70,

2.71 $$B_{r\,2k+1}(0) = S_{r\,2k+1}(0) = 0 .$$

This completes the induction.

The reason for the consideration of cone-like body profiles (see 1.1) is evident from Theorem 8 and the remarks which precede it. The next theorem establishes the analog of Theorem 2 for cone-like bodies.

Theorem 9. Let $(u,v,h)$ be a solution of 1.11 - 1.17 as described in Theorem 5 and $1 < \gamma < 1.5$. If $\Theta = B(r)$ meets the requirements 3.30 (d) of Chapter III and is cone-like, then there exist unique functions $u_{rn}$, $v_{rn}$, $h_{rn}$, and $S_{rn}(0)$, $n = 1, \ldots$ of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ which satisfy the system 2.14 - 2.20, provided that for all n the inequality:

2.72 $$\Phi_n \neq 1$$

and the equality:

2.73 $$\frac{d^n}{d\Theta^n}\left[\frac{-X_{2n,o}(\Theta)\frac{Y_{2n,o}(\Theta_b)}{X_{2n,o}(\Theta_b)} + Y_{2n,o}(\Theta)}{(\frac{1}{\Theta_s-\Theta_b})^n \frac{v}{\Theta-\Theta_b} \exp\left[2n\int_{\Theta}^{\Theta_s}(\frac{u}{v} + \frac{1}{2(t-\Theta_b)})dt\right]}\right]_{\Theta=\Theta_b} = 0$$

are valid. Further, the functions $u_{rn}$, $v_{rn}$, $h_{rn}$ and $S_{rn}(0)$ are such that for odd index n

2.74 $$S_{rn} = 0$$

$$u_{rn} \equiv 0$$

$$v_{rn} \equiv 0$$

$$h_{rn} \equiv 0$$

on $I_{b-\epsilon,s+\epsilon}$. Also, the series

2.75 $$u(\Theta) + u_r(\Theta)r + u_{r2}(\Theta)\frac{r^2}{2!} + \ldots$$

2.76 $$v(\Theta) + v_r(\Theta)r + v_{r2}(\Theta)\frac{r^2}{2!} + \ldots$$

2.77 $$h(\Theta) + h_r(\Theta)r + h_{r2}(\Theta)\frac{r^2}{2!} + \ldots$$

2.78 $$\Theta_s + S_r(0) + S_{r2}(0)\frac{r^2}{2!} + \ldots$$

are well defined on $I_{b-\epsilon,s+\epsilon}$.

Proof: The proof is inductive. The general scheme of the induction at the $n^{th}$ stage consists in using the technique of Theorem 7 to show that the solution is unique. If n is odd, the method of Theorem 8 shows the system is homogeneous; therefore a solution exists and is identically zero. If n is even and the system is not homogeneous, condition 2.73 is used to show the existence of a particular solution $h^{o}_{rn}(\theta)$ of 2.16 of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ such that $h^{o}_{rn}(\theta_s) = 0$. The general solution of 2.16 is then written in the form:

2.79 $$k_n h^{*}_{rn}(\theta) + h^{o}_{rn}(\theta) ,$$

where $k_n$ is a parameter. The auxiliary functions $\alpha_n$, $\beta_n$, $\alpha^{*}_n$ and $\beta^{*}_n$ and the technique of Theorem 7 are then used to establish the existence of a solution.

In the case n = 1, the system 2.22 - 2.27 is homogeneous since $B_r(0) = 0$. From Theorem 7, the solution is unique on $I_{b,s+\epsilon}$ and is identically zero. Hence, the unique solution of class $C^{\omega}$ on $I_{b-\epsilon,s+\epsilon}$ is as given by 2.74 for n = 1.

In the case n = 2, the system 2.22 - 2.27 is homogeneous with the possible exception of condition 2.25. The homogeneity is a consequence of 2.74. The argument applied to the system 2.36 - 2.41 in the proof of Theorem 7 (adapted by change of index from 1 to 2), the use of Theorem 6, and the hypothesis 2.72 prove the uniqueness. A similar adaptation of the method of Theorem 7 gives the existence of a

solution and the formulas:

$$2.80 \quad u_{r2} = \Delta_2(\theta)\Big[k_2\,\beta_2^*(\theta)\int_{\theta_b}^{\theta}(\alpha_2 h_{r2\theta}^* Q_{2,o} + \beta_2 h_{r2}^*\,T_{2,o})\,dt + k_2\,\beta_2(\theta)\int_{\theta}^{\theta_s}(\alpha_2^* h_{r2\theta}^* Q_{2,o} + \beta_2^* h_{r2}^*\,T_{2,o})\,dt + \beta_2^*(\theta)\,(I_{2,o})_{\theta=\theta_b}\Big]$$

$$2.81 \quad v_{r2} = -\Delta_2(\theta)\Big[k_2\,\alpha_2^*(\theta)\int_{\theta_b}^{\theta}(\alpha_2 h_{r2\theta}^* Q_{2,o} + \beta_2 h_{r2}^*\,T_{2,o})\,dt + k_2\,\alpha_2(\theta)\int_{\theta}^{\theta_s}(\alpha_2^* h_{r2\theta}^* Q_{2,o} + \beta_2^* h_{r2}^*\,T_{2,o})\,dt + \alpha_2^*(\theta)\,(I_{2,o})_{\theta=\theta_b}\Big]$$

$$2.82 \quad h_{r2} = k_2 h_{r2}^*$$

$$2.83 \quad S_{r2}(0) = \frac{u_{r2}}{N_{2,o}}\Big]_{\theta=\theta_s}$$

where:

$$2.84 \quad k_2 = (I_{2,o})_{\theta=\theta_b}\,\Delta_2(\theta_s)\Big[\Big(\frac{Z_{2,o}}{N_{2,o}} + J_{2,o}\Big)_{\theta=\theta_s}\beta_2(\theta_s) - (K_{2,o})_{\theta=\theta_s}\,\alpha_2^*(\theta_s)\Big]\frac{1}{1-\Phi_2}\,.$$

In the case n = 3, the equations 2.74 for n = 1 and the argument of Theorem 8 with $B_{r3}(0) = 0$ show the system 2.22 - 2.27 to be homogeneous. The inequality 2.72 shows the identically zero solution to be unique as in the case n = 1. Thus 2.74 is valid for n = 3.

In the case n = 4, the equation 2.24 is

2.85 $v(h_{r4})_\theta = -4uh_{r4} - 12u_{r2}h_{r2} - 6v_{r2}h_{r2\theta}$ .

The first step is to find a particular solution of class $C^\omega$ on $I_{b-\epsilon,s+\epsilon}$ . The homogeneous solution is known from Theorem 6. Adaptation of the method of variation of parameters leads to the consideration of:

2.86 $$h_{r4} = -\frac{6v_{r2}(\theta_b)h_{r2\theta}(\theta_b)}{4u(\theta_b)} + \zeta_4(\theta)h^*_{r4}(\theta)$$

where $\zeta_4(\theta)$ is to be determined so that 2.86 satisfies 2.85. Substitution in 2.85 yields:

2.87 $$vh^*_{r4}\frac{d\zeta_4}{d\theta} = 4\left(\frac{6v_{r2}(\theta_b)h_{r2\theta}(\theta_b)}{4u(\theta_b)}\right)u - 12u_{r2}h_{r2} - 6v_{r2}h_{r2\theta} .$$

The function $vh^*_{r4}$ can be written in the form

2.88 $$(\theta - \theta_b)^3 \frac{v}{(\theta-\theta_b)(\theta_s-\theta_b)^2} \exp\left[4\int_\theta^{\theta_s}\left(\frac{u}{v} + \frac{1}{(2(t-\theta_b))}\right)dt\right] .$$

The coefficient of $(\theta - \theta_b)^3$ is non-zero and of class $C^\omega$ on $I_{b-\epsilon,s+\epsilon}$. Equation 2.87 may now be written in the form:

2.89

$$(\theta-\theta_b)^3\frac{d\zeta_4}{d\theta} = \frac{4\left(\frac{6v_{r2}(\theta_b)h_{r2\theta}(\theta_b)}{4u(\theta_b)}\right)u - 12u_{r2}h_{r2} - 6v_{r2}h_{r2\theta}}{\frac{v}{(\theta-\theta_b)(\theta_s-\theta_b)(\theta_s-\theta_b)}\exp\left[4\int_\theta^{\theta_s}\left(\frac{u}{v} + \frac{1}{2(t-\theta_b)}\right)dt\right]} .$$

The right-hand member of the above equation is analytic and can be expressed in the form:

2.90 $$a_1(\theta - \theta_b) + a_2\frac{(\theta - \theta_b)^2}{2!} + (\theta - \theta_b)^3 f(\theta)$$

where $a_1$ and $a_2$ are its first and second derivatives evaluated at $\theta = \theta_b$. Thus,

$$2.91 \qquad \zeta_4 = -\frac{a_1}{\theta-\theta_b} + \frac{a_2}{2}\ln|\theta - \theta_b| + \int f(\theta)\, d\theta \ .$$

Condition 2.73 implies that $a_2 = 0$. Thus, formula 2.91 combined with 2.86 represents a solution of 2.85 of class $C^\omega$ on $I_{b-\epsilon,s+\epsilon}$. From Theorem 6, there is a unique solution $h^o_{r4}$ such that $h^o_{r4}(\theta_s) = 0$. The general solution of 2.16 is now written in the form

$$2.92 \qquad k_4 h^*_{r4}(\theta) + h^o_{r4}(\theta) \ .$$

The technique of Theorem 7 using 2.92 shows that the unique solution of the system 2.14 - 2.20 exists and can be expressed as:

$$\begin{aligned} 2.93 \qquad u_{r4} = \Delta_4(\theta)\Big[ & k_4\beta^*_4(\theta)\int_{\theta_b}^{\theta} (\alpha_4 h^*_{r4\theta}Q_{4,o} + \beta_4 h^*_{r4}T_{4,o})\, dt \\ & + \beta^*_4(\theta)\int_{\theta_b}^{\theta} (\alpha_4(h^o_{r4\theta}Q_{4,o}+R_{4,o}) + \beta_4(h^o_{r4}T_{4,o}+W_{4,o}))\, dt \\ & + k_4\beta_4(\theta)\int_{\theta}^{\theta_s} (\alpha^*_4 h^*_{r4\theta}Q_{4,o} + \beta^*_4 h^*_{r4}T_{4,o})\, dt \\ & + \beta_4(\theta)\int_{\theta}^{\theta_s} (\alpha^*_4(h^o_{r4\theta}Q_{4,o}+R_{4,o}) + \beta^*_4(h^o_{r4}T_{4,o}+W_{4,o}))\, dt \\ & + \frac{\beta_4(\theta)}{\sin\theta_s}\left[ O_{4,o} - \frac{M_{4,o}}{N_{4,o}} P_{4,o} \right]_{\theta=\theta_s} + \beta^*_4(\theta)\left[ I_{4,o} \right]_{\theta=\theta_b}\Big] \end{aligned}$$

$$2.94\quad v_{r4} = -\Delta_4(\theta)\Bigg[k_4\alpha_4^*(\theta)\int_{\theta_b}^{\theta}(\alpha_4 h^*_{r4\theta}Q_{4,o} + \beta_4 h^*_{r4}T_{4,o})\,dt$$

$$+\alpha_4^*(\theta)\int_{\theta_b}^{\theta}(\alpha_4(h^o_{r4\theta}Q_{4,o}+R_{4,o}) + \beta_4(h^o_{r4}T_{4,o}+W_{4,o}))\,dt$$

$$+k_4\alpha_4(\theta)\int_{\theta}^{\theta_s}(\alpha_4^* h^*_{r4\theta}Q_{4,o} + \beta_4^* h_{r4}T_{4,o})\,dt$$

$$+\alpha_4(\theta)\int_{\theta}^{\theta_s}(\alpha_4^*(h^o_{r4\theta}Q_{4,o}+R_{4,o}) + \beta_4^*(h^o_{r4}T_{4,o}+W_{4,o}))\,dt$$

$$+\frac{\alpha_4(\theta)}{\sin\theta_s}\left[O_{4,o} - \frac{M_{4,o}}{N_{4,o}}P_{4,o}\right]_{\theta=\theta_s} + \alpha_4^*(\theta)\left[I_{4,o}\right]_{\theta=\theta_b}\Bigg]$$

$$2.95\quad h_{r4} = k_4 h^*_{r4} + h^o_{r4}$$

$$2.96\quad S_{r4}(0) = \left.\frac{u_{r4} - M_{4,o}}{N_{4,o}}\right]_{\theta=\theta_s},$$

where:

$$2.97\quad k_4 = \Bigg[L_{4,o} - \frac{Z_{4,o}M_{4,o}}{N_{4,o}} + \Delta_4(\theta_s)\left(\frac{Z_{4,o}}{N_{4,o}}\right) + J_{4,o}\Bigg)$$

$$\Bigg(\int_{\theta_b}^{\theta_s}(\alpha_4(h^o_{r4\theta}Q_{4,o} + R_{4,o}) + \beta_4(h^o_{r}T_{4,o} + W_{4,o}))\,d\theta$$

$$+\frac{\beta_4(\theta_s)}{\sin\theta_s}\left[O_{4,o} - \frac{M_{4,o}}{N_{4,o}}P_{4,o}\right]_{\theta=\theta_s} + \left[I_{4,o}\right]_{\theta=\theta_s}\Bigg)$$

$$-K_{4,o}\Delta_4(\theta_s)\Bigg(\alpha_4^*(\theta_s)\int_{\theta_b}^{\theta_s}(\alpha_4(h^o_{r4\theta}Q_{4,o} + R_{4,o})$$

$$+\beta_4(h^o_{r4}T_{4,o} + W_{4,o}))\,dt + \frac{\alpha_4(\theta_s)}{\sin\theta_s}\left[O_{4,o} - \frac{M_{4,o}}{N_{4,o}}P_{4,o}\right]_{\theta=\theta_s}$$

$$+\left[I_{4,o}\right]_{\theta=\theta_s}\Bigg)\Bigg]\frac{1}{1-\Phi_4}$$

The induction continues as described. For odd n, the solutions are given by 2.74. For even n, the solutions are given by 2.93 - 2.97 with suitable change of index.

It would be desirable to establish the results of Theorem 9 without having to make the assumptions 2.72 and 2.73. In Appendix IV it is shown that

2.98 $$\Phi_n > 0 \qquad n = 1, 2, \ldots.$$

This is the best result obtained in this work in the attempt to prove the validity of 2.72. The assumption is, however, fairly credible. The equality 2.73 enters the argument essentially only for even $n \geq 4$. The next theorem establishes it for n = 4 and thus makes it also fairly credible, although no further result in this direction is obtained in this work.

Theorem 10. If the inequality 2.72 of Theorem 9 is valid for n = 1, 2, and 3, then the equality 2.73 of Theorem 9 holds for n = 2.

Proof: Since the inequality 2.72 of Theorem 9 is valid for n = 1, 2, and 3, the quantities involved in 2.73 for n = 2 are well defined and of class $C^{\omega}$ on $I_{b-\epsilon, s+\epsilon}$. The validity of 2.73 for n = 2 (see 2.89) is equivalent to showing that:

2.99 $$WZ'' - 2W'Z' \,]_{\theta=\theta_b} k_2 = 0,$$

where:

2.100 $$W = \frac{v}{\theta-\theta_b} \exp\left[ 2 \int_{\theta}^{\theta_s} \left( \frac{2u}{v} + \frac{1}{t-\theta_b} \right) dt \right.$$

$$Z = 4 \frac{6v_{r2}(\theta_b)h^*_{r2\theta}(\theta_b)}{4u(\theta_b)} u - 12\, u_{r2}h^*_{r2} - 6\, v_{r2}h^*_{r2\theta} \quad .$$

By direct calculation:

2.101 $$W(\theta_b) = - 2u(\theta_b) \exp\left[ 2 \int_{\theta_b}^{\theta_s} \left( \frac{2u}{v} + \frac{1}{\theta-\theta_b} \right) d\theta \right]$$

$$W'(\theta_b) = - u(\theta_b)\cot\theta_b \exp\left[2 \int_{\theta_b}^{\theta_s} \left( \frac{2u}{v} + \frac{1}{\theta-\theta_b} \right) d\theta \right]$$

$$Z'(\theta_b) = -12u_{r2}(\theta_b)h^*_{r2\theta}(\theta_b) - 6v_{r2\theta}(\theta_b)h^*_{r2\theta}(\theta_b)$$

$$- 6v_{r2}(\theta_b)h^*_{r2\theta 2}(\theta_b)$$

$$Z''(\theta_b) = - 12v_{r2}(\theta_b)h^*_{r2\theta}(\theta_b) - 24u_{r2\theta}(\theta_b)h^*_{r2\theta}(\theta_b)$$

$$-12u_{r2}(\theta_b)h^*_{r2\theta 2}(\theta_b) - 6\, v_{r2\theta 2}(\theta_b)h^*_{r2\theta}(\theta_b)$$

$$-12v_{r2\theta}(\theta_b)h^*_{r2\theta 2}(\theta_b) - 6v_{r2}(\theta_b)h^*_{r2\theta 3}(\theta_b)$$

$$h^*_{r2\theta 2}(\theta_b) = h^*_{r2\theta}(\theta_b)\cot\theta_b$$

$$h^*_{r2\theta 3}(\theta_b) = - \left(3 + \frac{2u^2(\theta_b)}{c^2(\theta_b)}\right) h^*_{r2\theta}(\theta_b)$$

$$u_{r2\theta}(\theta_b) = 3v_{r2}(\theta_b) - \frac{k_2 c^2(\theta_b)}{\gamma(\gamma-1)hu(\theta_b)} h^*_{r2\theta}(\theta_b)$$

$$v_{r2\theta}(\theta_b) = \left( -4 + \frac{2u^2(\theta_b)}{c^2(\theta_b)} \right) u_{r2}(\theta_b) - v_{r2}(\theta_b)\cot\theta_b$$

$$v_{r2\theta 2}(\theta_b) = \left[ \csc^2\theta_b + \frac{2u^2(\theta_b)}{c^2(\theta_b)} - 12 + \cot^2\theta_b \right] v_{r2}(\theta_b)$$

$$+ \frac{4\, c^2(\theta_b) k_2}{\gamma(\gamma-1)hu(\theta_b)}\, h^{*}_{r2\theta}(\theta_b)$$

$$-\left(-4 + \frac{2u^2(\theta_b)}{c^2(\theta_b)}\right)\cot\theta_b\, u_{r2}(\theta_b)$$

$$v_{\theta 3} = -6u\cot^2\theta - \frac{8u^3}{c^2} \quad ,$$

and therefore, 2.99 holds .

## Section 3: The Series Representation as a Solution for a Cone-Like Body.

The proof that the series 2.75 - 2.78 of Theorem 9 represent a solution of the flow problem for a cone-like body follows both the conditions and proof of Theorem 4 so closely that only a sketch of the proof will be given. The theorem is formulated as follows:

**Theorem 11.** Let $(u,v,h)$ be a solution of 1.11 - 1.17 as described in Theorem 5 and $1 < \gamma < 1.5$. Let $\theta = B(r)$ meet the requirements 3.30 (d) of Chapter III and be cone-like for $0 \leq r \leq \delta_1$, $\delta_1 > 0$. If:

(a) the inequality 2.72 and the equality 2.73 are valid for all n ,

(b) the formal power series 2.75 - 2.78 of Theorem 9 represent analytic functions of r and $\theta$ denoted by $u(r,\theta)$, $v(r,\theta)$, $h(r,\theta)$ and $S(r)$ respectively in the region $D^*(\delta_2,\epsilon_2)$:

3.1 $$D^*(\delta_2,\epsilon_2) = [(r,\theta) \mid 0 < \theta_b - \epsilon_2 \leq \theta \leq \theta_s + \epsilon_2 < \frac{\pi}{2},$$

$$0 < r \leq \delta_2,\ \delta_2 > 0,\ \epsilon_2 > 0\ ],$$

(c) the derivatives of $u(r,\theta)$, $v(r,\theta)$, $h(r,\theta)$ and $S(r)$ can be found by termwise differentiation of the series 2.75 - 2.78 in $D^*(\delta_2,\epsilon_2)$, and

(d) the inequality

3.2 $$1 - \Phi_n > 0$$

is valid for the least integer n such that

3.3 $$B_{rn}(0) \neq 0,$$

then $u(r,\theta)$, $v(r,\theta)$, $h(r,\theta)$ and $S(r)$ are solutions of the system 1.2 - 1.9 in $D(\delta)$ for suitable $\delta > 0$, $\epsilon > 0$ and the conditions 3.30 and 3.31 of Chapter III are met in $D(\delta)$.

Sketch of Proof. The proof is parallel to that given for Theorem 4. From continuity considerations and Theorem 5, the functions may be shown to satisfy 3.30 and 3.31 of Chapter III in $D^*(\delta,\epsilon)$ for suitable $\delta > 0$ and $\epsilon > 0$. The details are similar to those given in the proof of Lemma 1. Lemma 2 is established in exactly the same way, in this instance, as in the proof of Theorem 4, after it is noted that the first non-zero quantity $S_{rn}(0)$ is negative. This

is established by formation of the ratio $\frac{S_{rn}(0)}{B_{rn}(0)}$, by use of the hypothesis 3.2 to prove that it is positive, and finally, by use of the hypothesis that $B_{rn}(0) < 0$. Lemma 3 is also established in the same manner as in the proof of Theorem 4. It may be shown directly that the series satisfy 1.4. The analog of Lemmas 4 and 5 follows if the systems of these lemmas are slightly modified to include $h(r,\theta)$ as a known function. At this stage of the argument, the functions $u(r,\theta)$, $v(r,\theta)$, $h(r,\theta)$ and $S(r)$ are known to satisfy 1.2 - 1.8 in $D^*(\delta,\epsilon)$ for some $\delta > 0, \epsilon > 0$. The function $h(r,\theta)$ satisfies condition 1.8 along $\theta = S(r)$ since it and all its derivatives are the same at $(\theta_s, 0)$ as those of function:

$$\left(\frac{1 - u^2 - v^2}{1-q_o^2}\right)\left(\frac{q_o(\sin\theta - rS_r\cos\theta)}{ruS_r - v}\right)^{1-\gamma}\Bigg]_{\theta=S(r)} .$$

Therefore, the two functions coincide along $\theta = S(r)$. Since the results which have been established hold in $D^*(\delta,\epsilon)$, they hold in $D(\delta)$ and the theorem is proved.

# CHAPTER V

## APPLICATIONS AND REMARKS

### Section 1: Computation of Flows and Shock-Body Curvature Ratios.

Some comments were made in Chapter III concerning the computation of solutions of the simplified flow problem. Those remarks also apply to the computation of flow about cone-like bodies, using the procedure and formulas of Chapter IV. For given values of $\gamma$, $\theta_b$ and $q_o$, the functions: (a) the Taylor-Maccoll solutions, (u,v), for conical flow, (b) the auxiliary functions, $\alpha_n$, $\beta_n$, $\alpha_n^*$ and $\beta_n^*$, and (c) the homogeneous solutions, $h_p^*n$; could be computed. The resulting data could be tabulated in a manner similar to that used by Kopal[13]. The determination of the first few terms of the series 2.75 - 2.78 for a particular body could then be made by hand computation, using the tabulated data and formulas 2.93 - 2.97 for suitable index n.

The first few terms of the series would only be expected to yield accurate information near the nose, but this is precisely the information needed to start numerical procedures for continuing the flow downstream. Since errors propogate downstream in supersonic flow, the data should be

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[13] Kopal, op. cit.

as accurate as possible. The use of the series has the advantage of avoiding the practice of fitting the body with a small conical tip in order to obtain initial data to start a numerical procedure. The latter practice is of dubious merit because as the conical tip approaches zero in size with vertex fixed, the approximations do not approach the actual solution uniformly. This may be seen since, if uniform, the order of limit processes could be interchanged and it would follow that the radial derivatives vanish at the nose. The theory developed in this work shows that the radial derivatives vanish only if the body is a cone.

For non-cone-like bodies of revolution, Theorem 6 may be of use in obtaining better initial data than that obtained by fitting a conical tip. The functions:

1.1
$$u(r,\theta) = u(\theta) + r u_r(\theta)$$
$$v(r,\theta) = v(\theta) + r v_r(\theta)$$
$$h(r,\theta) = h(\theta) + r h_r(\theta)$$

are defined on $I_{b,s+\epsilon}$. If it is assumed that these functions describe the flow in the region G:

1.2
$$G(\delta) = [(r,\theta) \mid 0 \leq r \leq \delta,\ \theta_b \leq \theta \leq S(r),\ \delta > 0\ ],$$

then it may be possible to extend the description to include the region $D(\delta) - G(\delta)$. There are difficulties in extending $h(r,\theta)$ as given in 1.1 since equation 2.29 of Chapter 4 is singular at $\theta = \theta_b$. However, it may be possible to find an approximate solution in $D(\delta) - G(\delta)$ by the method of Courant and Lax, using the values of the functions 1.1 on the

line $\theta = \theta_b$ as initial data.

Two formulas for the shock-body curvature ratio, $\frac{K_s}{K_b}$, have been derived. That of Chapter III, equation 2.47, neglects rotation; while that of Chapter IV, equation 2.51, takes it into account. These ratios differ by a factor of $\frac{1}{1 - \bar{\Phi}_1}$. Since $\bar{\Phi}_1 > 0$ and presumably $\bar{\Phi}_1 < 1$, the ratio including the rotational effect is larger than the other. Determination of $\frac{1}{1 - \bar{\Phi}_1}$ and comparison should be interesting; as should also a comparison of both results with those of Lin[14], who has computed shock-body curvature ratios on the basis of a linearization of the flow problem. Preliminary computations for a particular case indicate that $\bar{\Phi}_1$ is less than one and that the ratio neglecting rotation is close to the result of Lin.

## Section 2: Comments on the Theory.

In Chapter 1, the general class of singular boundary value problems was described. The simplified flow problem of Chapter III is one of this type. As far as the mathematics is concerned, this particular problem is solved except for the question of convergence of the series. In general, each $n^{th}$ coefficient depends upon the earlier coefficients of all the series; e. g., $u_{rh}(\theta)$ depends upon $u_{rj}(\theta)$, $v_{rj}(\theta)$, $S_{rj}(0)$ and $B_{rj}(0)$, $j = 1, \ldots, n - 1$. The growth of the magnitude of the coefficients with respect to

[14] Lin, op. cit.

is therefore very complicated. In the case that the body is a cone, the coefficients are simple (see Chapter I, Section 3) and the series do converge. The solution obtained in this case is the well-known Taylor-Maccoll flow. The fact that the Taylor-Maccoll flow is recovered, together with the physical background of the simplified flow problem, makes plausible the convergence of the series in general. The convergence of the series for the conical case depends on certain inequalities, which are sub-results of the existence proof in Appendix I. These sub-results are also vital in the formation of the series in general. A proof independent of that announced by Shü[15] is given, in order to obtain the sub-results.

The singular boundary value problem considered in Chapter IV also admits the Taylor-Maccoll solution. However, in general, not only the convergence, but also the validity of conditions 2.72 and 2.73 is unsettled. The convergence is of the same complicated nature as that in the simplified flow. With respect to condition 2.72 on $\Phi_n$, it is suspected that $\Phi_n \leq \Phi_1$. If this could be shown, it would indicate, at least for the particular case mentioned in the preceding section, that the condition is valid. With respect to the equality 2.73, the class of cone-like bodies for which it is valid is non-empty, since it holds for a cone. Theorem 10 provides some basis for believing that there are other

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[15] Shü, op. cit.

such cone-like bodies. These indications, together with the physical background of the problem, make the assumptions 2.72, 2.73 and the convergence plausible.

The method used in solving the singular boundary value problems considered, may also be applied to other hydrodynamic problems. In view of this, the next two sections outline initial procedures for two such problems.

## Section 3: An Introduction to Flow about a Yawed Pointed Body of Revolution with Attached Shock Wave.

The flow for a yawed body of revolution with attached shock wave was described by the system of equations and conditions 3.18 - 3.20 of Chapter II. It differs from the flow in the non-yaw case by the dependence on $\varphi$ and the appearance of the third velocity component w. For r = 0, the equations and conditions in dimensionless form are:

3.1 $$v(u_\theta - v) + \frac{w}{\sin\theta} u_\varphi - w^2 = 0$$

3.2 $$u(u_\theta - v) - \frac{w}{\sin\theta} v_\varphi + w^2\cot\theta + w_\theta = \frac{-c^2}{\gamma(\gamma-1)h} h_\theta$$

3.3 $$(c^2 - v^2)v_\theta = -(c^2 - v^2)u - c^2(u + v\cot\theta) - (c^2 - w^2)\frac{w_\varphi}{\sin\theta} + uw^2 + \frac{vw}{\sin\theta} u_\varphi - vww_\theta$$

3.4 $$vh_\theta + \frac{w}{\sin\theta} h_\varphi = 0$$

where:

3.5 $$c^2 = \frac{\gamma-1}{2}(1 - u^2 - v^2 - w^2);$$

the boundary condition on $\theta = B(0, \varphi)$ is:

3.6 $$v = ruB_r + \frac{w}{\sin\theta} B_\varphi \ .$$

and those on $\theta = S(0, \varphi)$ are:

3.7 $$u = q_o \cos\theta$$

3.8 $$S_\varphi = - \frac{w \sin\theta}{v + q_o \sin\theta}$$

3.9 $$u \cos\theta - v \sin\theta = \frac{\mu^2}{q_o} + \frac{(1-\mu^2)q_o(\cos^2\theta + \frac{w^2}{(v + q_o\sin\theta)^2})}{1 + \frac{w^2}{(v + q_o \sin\theta)^2}}$$

3.10 $$h = \left(\frac{1 - u^2 - v^2 - w^2}{1 - q_o^2}\right)\left(\frac{q_o \sin\theta(v + q_o\sin\theta)}{-w^2 - v(v+q_o\sin\theta)}\right)^{1-\gamma} \ .$$

The first step in the development of a solution to the flow problem is the finding of a solution to the above system of equations, 3.1 - 3.5, and a curve $\theta = S(0, \varphi)$ such that all the equations and conditions, 3.1 - 3.10, are met in the region R:

$$R = [(\theta, \varphi) \mid 0 \leq \varphi \leq 2\pi, \ B(0, \varphi) \leq \theta \leq S(0, \varphi)] \ .$$

This solution would then play the same role in the development as the Taylor-Maccoll solution for the problems of Chapters III and IV. Since the region R corresponds to a point in physical space, it is reasonable to require that h is constant on R. Thus, all derivatives of h appearing in the system 3.1 - 3.10 are zero. It should also be noted that it would be convenient to replace $\frac{w}{\sin\theta}$ by a single variable.

## Section 4: An Introduction to the Extension of the Flow Past the Shoulder of a Cone-Cylinder.

Another aerodynamic problem of interest is that of extending the flow past the shoulder of a cone-cylinder. It is assumed that the body is without yaw and has an attached shock wave. In Figure 3, the curve obc is the

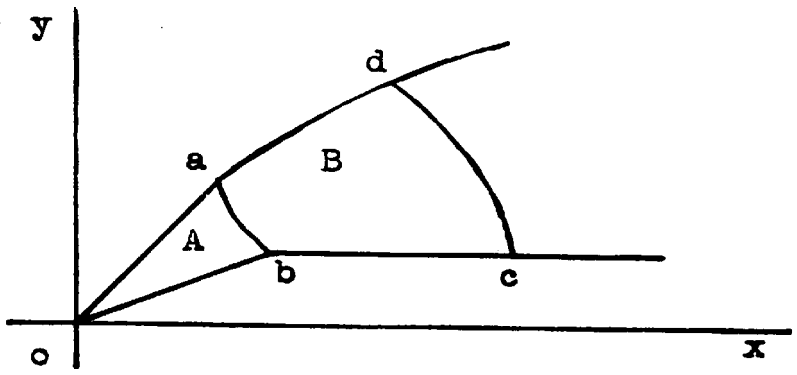



Figure 3.

profile of the cone-cylinder, oa is the known portion of the shock profile, ad is the unknown portion of the shock profile, and ab is a characteristic curve. The solution is known in the region A since it is the Taylor-Maccoll solution. The problem is to determine a solution in B, and to determine the unknown portion of the shock profile, ad, such that these join the Taylor-Maccoll flow smoothly. The velocity components are known on the curve ab, the slope of the shock is given at a, and the boundary conditions on the curves ad and bc are specified.

At the point b there is a singularity which appears to be of the same type as that at the point o. This suggests the transformation to polar coordinates, r and $\theta$, with pole at the point b and initial ray along the line segment bc. The transformed region B in the $(r,\theta)$ - plane is indicated in Figure 4, with the corresponding points labeled as in Figure 3.

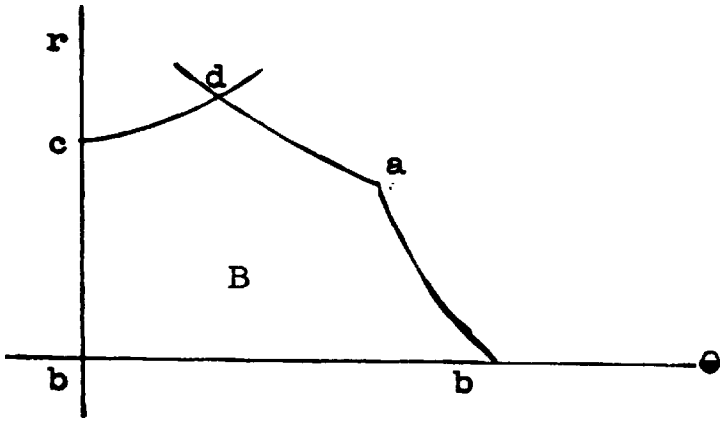



Figure 4 .

The segment bb of Figure 4 should yield two point boundary value problems determining the normal derivatives of the velocity components, among other quantities, at the point b on ab. Since ab is a characteristic curve, these will determine the normal derivatives along ab. These, in turn, will determine the curvature of the shock profile extension, ad, at a . By breaking B into sub-regions bounded by suitable characteristics, the solution throughout B should be obtainable.

The problems in the last two sections have been introduced to illustrate possible applications of the technique.

Although this technique was developed for the main problems of this research, it may be useful in many other situations. The ideas presented in the last two sections indicate, in part, how such applications might be made.

# APPENDIX I

## EXISTENCE OF TAYLOR-MACCOLL FLOW AND RELATED INEQUALITIES

The existence of Taylor-Maccoll Flow for certain given values of $q_o$ and $\theta_b$ is the content of Theorem 1 of the text. In the course of the proof presented here several useful inequalities are obtained. Their use is explained after a restatement of Theorem 1.

Theorem: There is a non-empty subset, $D^*(\gamma)$, of $(q_o, \theta_b)$-space defined for all $\gamma$, $1 < \gamma < 1.76$, such that if $(q_o, \theta_b)$ belongs to $D^*(\gamma)$ then there exists a solution $(u,v)$ of:

1.1 $$u_\theta = v$$

1.2 $$v_\theta = - u - \frac{c^2}{c^2 - v^2} (u + v \cot\theta)$$

where:

1.3 $$c^2 = \frac{\gamma-1}{2} (1 - u^2 - v^2) \ ,$$

such that at $\theta = \theta_b$:

1.4 $$v = 0$$

and at $\theta = \theta_s$:

1.5 $$u = q_o \cos \theta$$

1.6 $$u \cos\theta - v \sin\theta = \frac{\gamma-1}{\gamma+1} \frac{1}{q_o} + \frac{2 q_o \cos^2\theta}{\gamma+1} \ ,$$

of class $C^\omega$ on $I_{b-\epsilon, s+\epsilon}$ for suitable $\epsilon > 0$. This solution is such that on $I_{b-\epsilon, s+\epsilon}$ :

1.7 (a) $1 > u^2 + v^2 - c^2 > 0$

(b) $c^2 - v^2 > 0$

(c) $u\cos\theta - v\sin\theta > 0$

(d) $u\sin\theta + v\cos\theta > 0$

and such that:

1.8 (a) $1 > q_o > \sqrt{\frac{\gamma-1}{\gamma+1}}$

(b) $\frac{\pi}{2} > \theta_s > \theta_b > 0$ .

It would suffice from the abstract point of view and be very natural to define $D^*(\gamma)$, $1 < \gamma < 1.76$, to consist of all pairs $(q_o, \theta_b)$ which satisfy the inequalities:

1.9 (a) $1 > q_o > \sqrt{\frac{\gamma-1}{\gamma+1}}$

(b) $\frac{\pi}{2} > \theta_b > 0$

and for which a solution $(u,v)$ of class $C^\omega$ of the system 1.1 - 1.6 exists and satisfies the inequalities 1.7 - 1.8 on $I_{b,s}$. The proof could then be made by exhibiting one pair $(q_o, \theta_b)$ (and some $\gamma$, $1 < \gamma < 1.76$) for which such a solution exists. As a practical matter, this result would be almost useless, for there would be now way of knowing whether or not such a solution existed for given $q_o$, $\theta_b$ and $\gamma$, unless the given set of values happened to be the particular set used in the proof. By contrast, the definition of $D^*(\gamma)$, $1 < \gamma < 1.76$, given in the proof below, is of practical use. Corresponding to each pair $(q_o, \theta_b)$ there is a pair $(u_o, \theta_b)$ for which $u(\theta_b) = u_o$ and conversely. Advantage of

this correspondence is taken to define $D^*(\gamma)$ in terms of a subset $D(\gamma)$ of $(u_o, \theta_b)$-space. This subset, $D(\gamma)$, is explicitly defined in terms of inequalities. If these inequalities are satisfied by a pair $(u_o, \theta_b)$, $1 < \gamma < 1.5$, then a solution exists and the theory developed in Chapters III and IV is applicable. Since Kopal's Tables[16] for $\gamma = 1.405$ (the most widely used value in practice) permit easy determination of $(u_o, \theta_b)$, this mode of definition is practical. To make the inequalities easier to use, a tabulation is provided, from which the curves defining the set D(1.4) can easily be constructed. It should be noted, however, that the inequalities produced are not as sharp as possible. Nevertheless, the set $D(\gamma)$ is fairly large.

Proof: In $(u,v,\theta)$ - space, the equation $c^2 - v^2 = 0$ represents the right elliptical cylinder,

$$1.9 \quad \frac{\gamma-1}{2} u^2 + \frac{\gamma+1}{2} v^2 = \frac{\gamma-1}{2} ,$$

whose axis of symmetry is the $\theta$-axis. The semi-major axis of the ellipse is of length 1 and is parallel to the u-axis. The semi-minor axis is of length $\sqrt{\frac{\gamma-1}{\gamma+1}}$ and is parallel to the v-axis.

The subset $K(\epsilon_1)$:

$$1.10 \quad K(\epsilon_1) = [\, (u,v,\theta) \mid c^2 - v^2 \geq \epsilon_1,\ \epsilon_1 \leq \theta \leq \frac{\pi}{2} - \epsilon_1,\ \epsilon_1 > 0 \,]$$

for $\epsilon_1$ sufficiently small represents a section of an elliptical cylinder which is contained in and similar to the

---

[16] Kopal, op. cit.

cylinder $c^2 - v^2 = 0$. On $K(\epsilon_1)$, the functions:

$$v, \qquad -u - \frac{c^2}{c^2 - v^2}(u + v\cot\theta)$$

which appear in 1.1 - 1.2 are continuously differentiable in u and v. The set $K(\epsilon_1)$ is convex and compact; thus, these functions satisfy a uniform Lipschitz condition in (u,v) on $K(\epsilon_1)$. Through a given interior point of $K(\epsilon_1)$, there is therefore a unique solution of 1.1 -.1.2 which can be extended to the boundary of $K(\epsilon_1)$. This holds for all $\epsilon_1 > 0$. Since the functions are analytic in u, v and $\theta$, the solutions (u,v) are of class $C^{\omega}$ in $\theta$ on $K(\epsilon_1)$.

If $(u_o, 0, \theta_b)$ is a point in $(u,v,\theta)$-space such that:

1.11 (a) $1 > u_o > \sqrt{\frac{\gamma-1}{\gamma+1}}$

(b) $0 < \theta_b < \frac{\pi}{2}$ ,

then there exists $\epsilon_1 > 0$ such that $(u_o, 0, \theta_b)$ is interior to $K(\epsilon_1)$. Therefore, there is $\alpha > \theta_b$ such that on $I_{b,\alpha}$:

1.12 $$I_{b,\alpha} = [\ \theta \mid \theta_b \leq \theta \leq \alpha < \frac{\pi}{2}\ ]$$

there exists a unique integral curve of 1.1 - 1.2 which is contained in $K(\epsilon_1)$ and passes through the point. Further, from continuity considerations, it is apparent that $\alpha > \theta_b$ may be chosen so that on $I_{b,\alpha}$ the solution satisfies the conditions:

1.13 (a) $(u,v) = (u_o, 0)$ at $\theta = \theta_b$

(b) $c^2 - v^2 > 0$

(c) $1 > u > \sqrt{\frac{\gamma-1}{\gamma+1}}$

(d) $u + v \cot\theta > 0.$

In the remainder of the proof, the only integral curves considered are those which satisfy the conditions 1.13 on $I_{b,\alpha}$ (although this may not always be mentioned explicitly). The $\alpha$ depends, of course, on the particular integral curve considered. The next major step consists of proving that the $\alpha$ associated with certain integral curves may be taken large enough, so that there exists a value $\theta = \theta_s \leq \alpha$ such that at $\theta = \theta_s$:

1.14 $$v = \frac{u^2 - 1}{u} \frac{\gamma-1}{\gamma+1} \cot\theta.$$

The equation above is obtained by elimination of $q_o$ between 1.5 and 1.6.

An integral curve on $I_{b,\alpha}$ satisfying 1.13 also satisfies 1.4, 1.7 (b) and (d). From 1.13 and 1.2 it is apparent that on $I_{b,\alpha}$ :

1.15 (a) $v_\theta < 0$

(b) $v \leq 0$

(c) $u_\theta \leq 0.$

Hence, 1.7(c) holds on $I_{b,\alpha}$. At $\theta = \theta_b$,

1.16 $$u^2 + v^2 - c^2 = \frac{\gamma+1}{2} u_o^2 - \frac{\gamma-1}{2} = \frac{\gamma+1}{2}\left( u_o^2 - \frac{\gamma-1}{\gamma+1} \right) > 0 .$$

Now,

1.17 $$\frac{d}{d\theta}(u^2+v^2-c^2) = -(\gamma+1)\frac{c^2 v}{c^2-v^2}(u+v\cot\theta) \geq 0$$

on $I_{b,\alpha}$. Thus:

1.18 $$u^2+v^2-c^2 > 0$$

on $I_{b,\alpha}$. In addition,

$$1 > u^2 > u^2-(c^2-v^2) = u^2+v^2-c^2$$

on $I_{b,\alpha}$. Therefore, 1.7 (a) holds on $I_{b,\alpha}$.

The equations $u^2+v^2-c^2=0$ and $u^2+v^2-c^2=1$ are equivalently:

1.19 $$u^2+v^2 = \frac{\gamma-1}{\gamma+1}$$

$$u^2+v^2 = 1 \quad .$$

They represent right circular cylinders with axes of symmetry coincident with the $\theta$-axis and of radii $\sqrt{\frac{\gamma-1}{\gamma+1}}$ and 1 respectively. Since 1.7 (a) and (b) and 1.15 (b) hold on $I_{b,\alpha}$, the integral curves on $I_{b,\alpha}$ lie interior to the quarter elliptical cylinder and exterior to the circular cylinder, as shown in Figure 5.

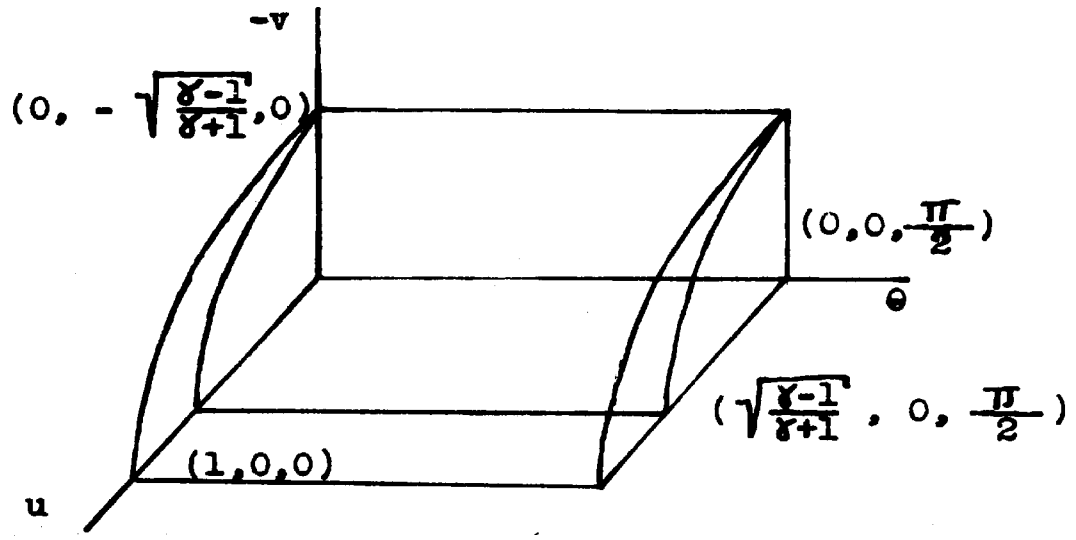



Figure 5 .

The information obtained above on the location of the integral curves is now made more precise. Initially (i.e. at $\theta = \theta_b$):

$$1.20 \quad u^2 + v^2 - c^2 = u_o^2 - c_o^2$$

where:

$$1.21 \quad c_o^2 = c^2 \Big]_{\theta=\theta_b} .$$

From 1.16 and 1.17, it follows that each integral curve lies exterior to the right circular cylinder

$$1.22 \quad u^2 + v^2 - c^2 = u_o^2 - c_o^2$$

of radius $u_o$. It follows from 1.15 that u is monontone decreasing. Therefore, each integral curve considered has its projection in the region B (dependent upon the curve)

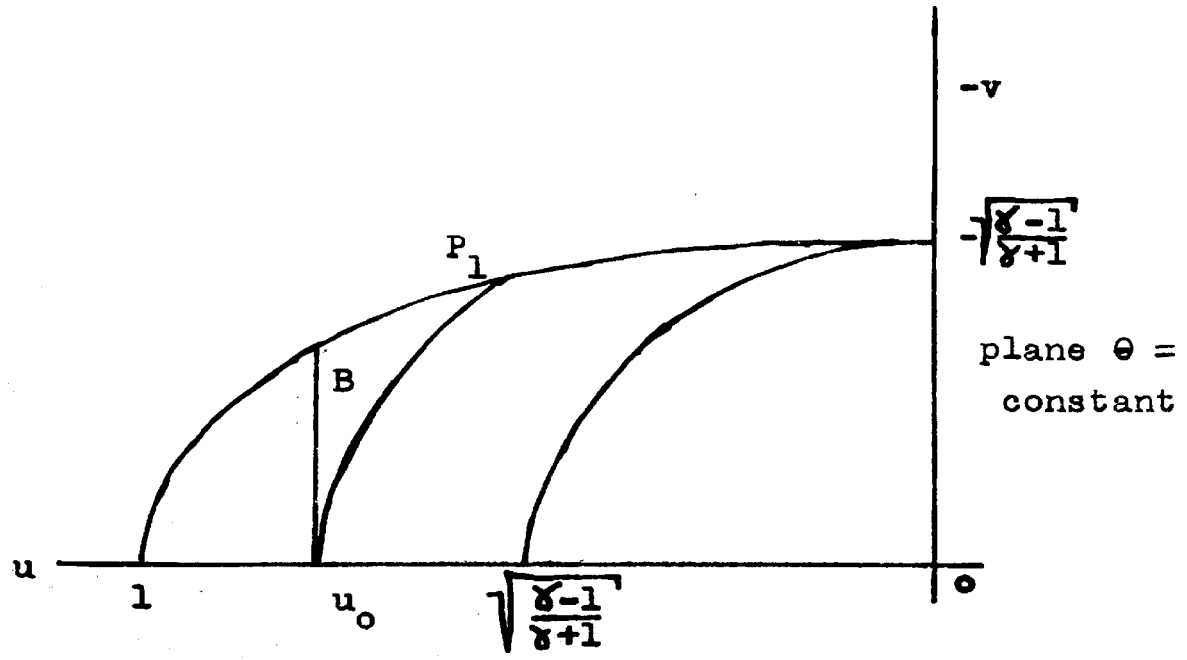



Figure 6 .

of Figure 6. The coordinates of $P_1$ in this plane are:

1.23 $$P_1 = \left(\sqrt{u_o^2 - c_o^2}\ ,\ -\sqrt{c_o^2}\right) .$$

It is then seen that the inequalities:

1.24 $$u_o \geq u > \sqrt{u_o^2 - c_o^2}$$
$$0 \geq v > -\sqrt{c_o^2}$$

hold on $I_{b,\alpha}$.

To study the shock conditions 1.5 - 1.6, it is conveninet to eliminate $q_o$ between them, which gives:

1.25 $$v = \frac{u^2 - 1}{u} \frac{\gamma - 1}{\gamma + 1} \cot\theta ,$$

and to regard this as a surface in $(u,v,\theta)$-space. The trace of this surface in the $\theta = \beta$ plane, $\beta$ a constant and $0 < \beta < \frac{\pi}{2}$, is a hyperbola. The branch which passes through the point $(1,0)$ in the plane $\theta = \beta$ has the v-axis as an asymptote and is concave downward. The point of intersection, $P_2$, of the hyperbola with the trace of the elliptical cylinder $c^2 - v^2 = 0$ has as coordinates:

1.26 $$P_2 = \left(\sqrt{\frac{\nu^2 \cot^2\beta}{1+\nu^2\cot^2\beta}}\ ,\ -\sqrt{\frac{\nu^2\cot^2\beta}{1+\nu^2\cot^2\beta}} \tan\beta\right).$$

In the plane $\theta = \frac{\pi}{2}$, the hyperbola degenerates, becoming the two axes.

The surface $u+v\cot\theta=\theta$ has as its trace in the $\theta = \beta$ plane, the straight line through the origin of slope $-\tan\beta$. From 1.26, it is seen that $P_2$ also lies on this line. As $\theta$ increases from $\theta = \theta_b$, the slope of this line in the $\theta$-plane

decreases; hence, $P_2$ moves toward the v-axis. These curves and the curves of Figure 6 are combined in Figure 7.

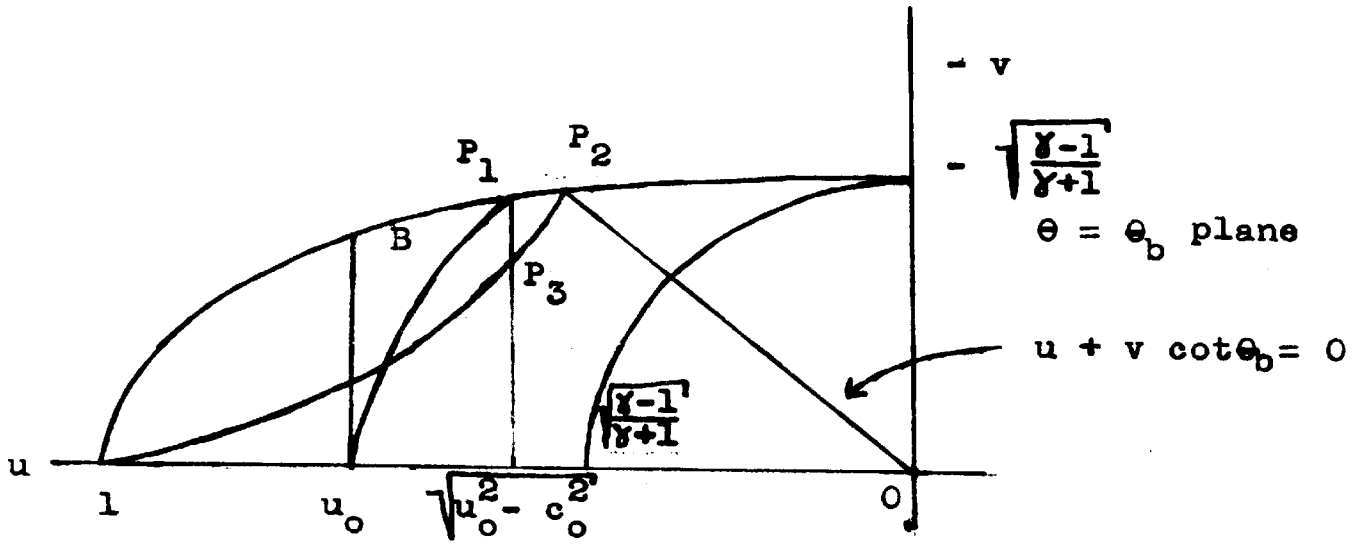



Figure 7 .

In Figure 7, the point $P_2$ has been placed to the right of $P_1$. This is not necessarily the case. However, if $P_2$ is to the right of $P_1$ or coincident with it in the $\theta = \theta_b$ plane for an integral curve, then it is always to the right in the $\theta = \beta$ plane, $\beta > \theta_b$. Thus, the integral curve can be extended to intersect the surface 1.25. The value of $\theta$ for the point of intersection is denoted by $\theta_s$. On the interval $I_{b,s}$ :

1.27 $$I_{b,s} = [\, \theta \mid \theta_b \leq \theta \leq \theta_s \,] ,$$

such an integral curve satisfies the conditions 1.13. Analytically, the condition that $P_2$ is to the right of or coincident with $P_2$ in the $\theta = \theta_b$ plane may be expressed in terms of the slopes of the lines $OP_1$ and $OP_2$ :

1.28 $$-\frac{\sqrt{c_o^2}}{\sqrt{u_o^2 - c_o^2}} \leq -\tan\theta_b \ .$$

This is equivalent to the inequality:

1.29 $$u_o^2 \geq \frac{\gamma-1}{2} \frac{1}{\frac{\gamma-1}{2} + \sin^2\theta_b} \ .$$

If an integral curve on $I_{b,\alpha}$ satisfies the above inequality, then the interval $I_{b,\alpha}$ may be extended to include a value $\theta = \theta_s$ for which 1.14 and 1.8 (b) hold. At $\theta = \theta_s$, the value $q_o$ is assigned by the equation:

1.30 $$q_o = \frac{u(\theta_s)}{\cos\theta_s} \ .$$

For this value, $q_o$, the conditions 1.5 and 1.6 are met at $\theta = \theta_s$.

For integral curves satisfying 1.29, the inequality 1.8 (a) is now considered. Since

$$\frac{d}{d\theta}\frac{u}{\cos\theta} = \frac{v\cos\theta + u\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{\cos^2\theta}(u + v\cot\theta) > 0$$

on $I_{b,s}$, the function $\frac{u}{\cos\theta}$ is increasing in $\theta$ on this interval. Therefore, using 1.13 (c):

$$\frac{u(\theta_s)}{\cos\theta_s} > \frac{u_o}{\cos\theta_b} > \sqrt{\frac{\gamma-1}{\gamma+1}}\ \frac{1}{\cos\theta_b} > \sqrt{\frac{\gamma-1}{\gamma+1}} \ .$$

That is:

1.31 $$q_o > \sqrt{\frac{\gamma-1}{\gamma+1}} \ .$$

To establish the other half of 1.8 (a), namely $1 > q_o$, requires an additional restriction on $(u_o, \theta_b)$. From 1.2 and 1.24,

$$1.32 \qquad -v_\theta > \sqrt{u_o^2 - c_o^2}$$

on $I_{b,\alpha}$. Hence,

$$1.33 \qquad -v \geq \sqrt{u_o^2 - c_o^2}\ (\theta - \theta_b)$$

on $I_{b,\alpha}$. From Figure 7, if $-v$ for an integral curve on $I_{b,\alpha}$ exceeds or equals the height of the point $P_3$, then the value $\theta = \theta_s$ has already been attained. The $-v$ coordinate of $P_3$ in the $\theta$-plane is:

$$\frac{c_o^2}{\sqrt{u_o^2 - c_o^2}} \cot \theta .$$

Thus, if

$$1.34 \qquad \sqrt{u_o^2 - c_o^2}\ (\theta - \theta_b) \geq \frac{c_o^2}{\sqrt{u_o^2 - c_o^2}} \cot\theta$$

for some $\theta$, $\theta_b \leq \theta \leq \alpha$, then $\theta_s$ lies in the interval $\theta_b < \theta_s < \theta$. If this $\theta < \arccos u_o$, then:

$$\cos\theta_s > \cos\theta > u_o ,$$

which implies that:

$$1.35 \qquad \frac{u(\theta_s)}{\cos\theta_s} < \frac{u(\theta_s)}{\cos\theta} < \frac{u(\theta_s)}{u_o} < 1.$$

Therefore, if 1.35 holds with $\theta = \arccos u_o$, it follows that $1 > q_o$.

Explicitly, this condition is:

$$\sqrt{u_o^2 - c_o^2}\,(\arccos u_o - \theta_b) \geq \frac{c_o^2}{\sqrt{u_o^2 - c_o^2}} \quad \frac{u_o}{\sqrt{1 - u_o^2}} \;;$$

that is:

$$1.36 \quad \arccos u_o - \frac{c_o^2}{u_o^2 - c_o^2} \quad \frac{u_o}{\sqrt{1 - u_o^2}} \geq \theta_b \;.$$

If an integral curve on $I_{b,\alpha}$ satisfies both inequalities 1.29 and that above, then there exist $\theta_s$ and $q_o$ such that on $I_{b,s}$ the equations and conditions 1.1 - 1.8 are satisfied.

In order to apply the theory of the text, it is important[17] to know that the inequality

$$1.37 \quad u^2 + v^2 - 3c^2 - (\gamma-1)\, u\, (u + v \cot\theta)\, \frac{v^2}{c^2 - v^2} > 0$$

holds along integral curves of the system 1.1 - 1.8 in the interval $I_{b,s}$. Conditions on $(u_o, \theta_b)$ are now derived to insure the validity of this inequality. By differentiation and use of the preceding results, it follows that $u^2 + v^2 - 3c^2$ is an increasing function of $\theta$ on $I_{b,s}$. Therefore,

$$u^2 + v^2 - 3c^2 \geq u_o^2 - 3c_o^2 = \frac{3\gamma-1}{2}\, u_o^2 - \frac{3(\gamma-1)}{2}$$

on $I_{b,s}$; thus the inequality:

$$1.38 \quad \frac{3\gamma-1}{2}\, u_o^2 - (\gamma-1)\, u_o^2 \quad \frac{v^2}{c^2 - v^2} - \frac{3(\gamma-1)}{2} > 0$$

on $I_{b,s}$ is sufficient to guarantee that 1.37 holds.

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[17] The inequality is used to establish that $D_{n,o} > 0$ for all n. See the proof of Theorem 2.

Bounds on $\frac{v^2}{c^2 - v^2}$ are now considered. Certainly,

$$\frac{v^2}{c^2 - v^2} \geq 0$$

on $I_{b,s}$. The surfaces $c^2 - v^2 =$ constant form a family of similar right elliptical cylinders. The equation of the member of the family which passes through the point $P_3$ (see Figure 7) is:

1.39 $$c^2 - v^2 = \frac{\gamma+1}{2} c_o^2 - \frac{\gamma+1}{2} \frac{c_o^4}{u_o^2 - c_o^2} \cot^2\theta_b \ .$$

The trace of this cylinder intersects the trace of the surface 1.25 in the plane $\theta = \theta_b$, in points whose coordinates satisfy the system:

1.40 $$\frac{\gamma+1}{2} v^2 = \frac{\gamma-1}{2} - \frac{\gamma-1}{2} u^2 - \frac{\gamma+1}{2} c_o^2 + \frac{\gamma+1}{2} \frac{c^4}{u_o^2 - c_o^2} \cot^2\theta_b$$

$$\frac{\gamma+1}{2} v^2 = \frac{\gamma+1}{2} \left(\frac{u^2-1}{u}\right)^2 \left(\frac{\gamma-1}{\gamma+1}\right)^2 \cot^2\theta_b \ .$$

Elimination of $v^2$ leads to a quadratic equation in $u^2$, which is easily factored since the root $u^2 = u_o^2 - c_o^2$ is already known. The result is:

$$(u^2 - [\, u_o^2 - c_o^2 \,]) \left(u^2 - \frac{(\gamma-1)\cot^2\theta_b}{(\gamma-1)\cot^2\theta_b + \gamma + 1} \frac{1}{u_o^2 - c_o^2}\right) = 0 \ .$$

The first factor is non negative in the region B. (See Figure 7.) If it is true that

1.41 $$\frac{(\gamma-1)\cot^2\theta_b}{(\gamma-1)\cot^2\theta_b + \gamma + 1} \frac{1}{u_o^2 - c_o^2} \geq u_o^2 \ ,$$

then the trace of the cylinder 1.39 is above that of the shock surface 1.25 in B in Figure 7. Therefore, if 1.41 is satisfied,

$$1.42 \quad c^2 - v^2 \geq \frac{\gamma+1}{2} c_o^2 - \frac{\gamma+1}{2} \frac{c_o^4}{u_o^2 - c_o^2} \cot^2\theta_b$$

on $I_{b,s}$, since the integral curve is interior to the cylinder 1.39. The condition 1.41 can be written in the form:

$$\frac{\gamma+1}{2} u_o^4 - \frac{\gamma-1}{2} u_o^2 - \frac{(\gamma-1)\cot^2\theta_b}{(\gamma-1)\cot^2\theta_b + \gamma + 1} \leq 0 ;$$

or, equivalently, as:

$$1.43 \quad u_o^2 \leq \frac{1}{2}\frac{\gamma-1}{\gamma+1} + \frac{1}{\gamma+1} \sqrt{\left(\frac{\gamma-1}{2}\right)^2 + \frac{2(\gamma+1)(\gamma-1)}{(\gamma-1) + (\gamma+1)\tan^2\theta_b}} .$$

The function $v^2$ cannot exceed the square of the ordinate of $P_3$ on $I_{b,s}$.

Hence,

$$1.44 \quad v^2 \leq \frac{c_o^4}{u_o^2 - c_o^2} \cot^2\theta_b .$$

Upon assuming that 1.43 is valid and combining 1.39 and 1.44 to obtain an upper bound for $\frac{v^2}{c^2 - v^2}$ on $I_{b,s}$, the condition 1.38 becomes:

$$1.45 \quad \frac{3\gamma-1}{2} u_o^2 - \frac{3(\gamma-1)}{2} > (\gamma-1) u_o^2 \frac{\frac{c_o^4}{u_o^2 - c_o^2} \cot^2\theta_b}{\frac{\gamma+1}{2} c_o^2 - \frac{\gamma+1}{2} \frac{c_o^4}{u_o^2 - c_o^2} \cot^2\theta_b} .$$

This inequality can be written in the form:

$$1.46 \quad [\, 2(\gamma+1)^2\tan^2\theta_b + (\gamma-1)(3\gamma^2 + 6\gamma - 5)\,]\, u_o^4$$

$$- [\, (3\gamma-1)(\gamma^2-1)\tan^2\theta_b - (\gamma-1)(6\gamma^2 + 6\gamma - 8)\,]\, u_o^2$$

$$+ [\, 3(\gamma-1)(\gamma^2-1)\tan^2\theta_b + (\gamma-1)(3\gamma^2 - 3)\,] > 0.$$

If both 1.43 and 1.46 hold, the condition 1.37 is valid on $I_{b,s}$ .

The set $D(\gamma)$ is defined for $1 < \gamma < 1.76$ to consist of all pairs $(u_o, \theta_b)$, $0 < u_o < 1$, $0 < \theta_b < \frac{\pi}{2}$ , such that :

(a) if $1 < \gamma < 1.76$, they satisfy the inequalities 1.29 and 1.36, and

(b) if $1 < \gamma < 1.5$, they satisfy the inequalities 1.43 and 1.46 .

To show that $D(\gamma)$ is non-empty, the inequality signs in 1.29, 1.36, 1.43 and 1.46 are replaced by equality signs, and the resultant curves in the $(u_o, \theta_b)$-plane are considered. The point (1,0) lies on each. On the curve for 1.29:

$$1.47 \quad \left.\frac{du_o}{d\theta_b}\right]_{(1,0)} = 0 \qquad \left.\frac{d^2u_o}{d\theta_b^2}\right]_{(1,0)} = -\frac{2}{\gamma-1} \ .$$

On the curve for 1.36:

$$1.48 \quad \left.\frac{du_o}{d\theta_b}\right]_{(1,0)} = 0 \qquad \left.\frac{d^2u_o}{d\theta_b^2}\right]_{(1,0)} = -\frac{4}{(\gamma-3)^2} \ .$$

On the curve for 1.43:

$$1.49 \quad \left.\frac{du_o}{d\theta_b}\right]_{(1,0)} = 0 \qquad \left.\frac{d^2u_o}{d\theta_b^2}\right]_{(1,0)} = -\frac{2(\gamma+1)}{(\gamma-1)(\gamma+3)} \ .$$

On the curve for 1.46:

$$1.50 \quad \left.\frac{du_o}{d\theta_b}\right]_{(1,0)} = 0 \qquad \left.\frac{d^2u_o}{d\theta_b^2}\right]_{(1,0)} = -\frac{2(\gamma+1)}{(\gamma-1)(3\gamma-1)} .$$

For the ranges of $\gamma$ specified, the following inequalities hold:

$$1.51 \quad \frac{2}{\gamma-1} > \frac{2(\gamma+1)}{(\gamma-1)(3\gamma-1)} > \frac{2(\gamma+1)}{(\gamma-1)(\gamma+3)} > \frac{4}{(\gamma-3)2} , \quad 1 < \gamma < 1.5$$

$$\frac{2}{\gamma-1} \quad \frac{4}{(\gamma-3)2} , \quad 1 < \gamma < 1.76 .$$

These inequalities, together with 1.47 - 1.50 show that the curves associated with the inequalities 1.29, 1.36, 1.43 and 1.46 spread apart in such a manner that the set $D(\gamma)$ as defined is non-empty. For each $(u_o, \theta_b)$ in $D(\gamma)$, there corresponds an integral curve of 1.1 - 1.2 and a pair $(q_o, \theta_s)$ such that on $I_{b,s}$, the system 1.1 - 1.8 is satisfied. This curve can be extended in both directions. Then, by a continuity argument, there exists $\epsilon > 0$ such that not only 1.1 - 1.6 and 1.8 are satisfied on $I_{b-\epsilon, s+\epsilon}$, but 1.7 as well. Corresponding to a point $(u_o, \theta_b)$ in $D(\gamma)$, there is a point $(q_o, \theta_b)$ where $q_o$ is determined by the curve associated with the point $(u_o, \theta_b)$. The set of such points $(q_o, \theta_b)$ corresponding to points in $D(\gamma)$ form, by definition, the set $D^*(\gamma)$. It is non-empty since $D(\gamma)$ is non-empty.

The region D (1.4) may easily be graphed, or pairs $(u_o, \theta_b)$, tested by use of the following table. To test

a pair $(u_o, \theta_b)$, enter the table using the argument $\theta_b$; then:

1.52 (a) $u_o > u_1$ means 1.28 is satisfied by the pair

(b) $u_o > u_2$ means 1.46 is satisfied by the pair

(c) $u_o < u_3$ means 1.43 is satisfied by the pair

(d) $u_o < u_4$ means 1.30 is satisfied by the pair ;

hence, to belong to D (1.4) all the above inequalities must be satisfied. The points $(\theta_b, u_1)$, $(\theta_b, u_2)$, $(\theta_b, u_3)$ and $(\theta_b, u_4)$ lie on the curves associated with the inequalities 1.28, 1.46, 1.43 and 1.30 respectively, as indicated in 1.52. The inequalities derived in the previous proof are not the best possible; therefore, failure of a pair $(u_o, \theta_b)$ to belong to D(1.4) does not necessarily mean that the general theory does not apply.

TABLE 1

DATA FOR DETERMINATION OF D (1.4)

| $\theta_b$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|---|---|---|---|---|
| 0° | 1.0000 | 1.000 | 1.000 | 1.000 |
| 5 | .9815 | .986 | .993 | ... |
| 8.78 | ... | ... | ... | .9815 |
| 10 | .9322 | .948 | .963 | ... |
| 15 | .8655 | .896 | .926 | ... |
| 16.63 | ... | ... | ... | .9322 |
| 20 | .7943 | .839 | .882 | ... |
| 22.95 | ... | ... | ... | .8665 |
| 25 | .7268 | .785 | .835 | ... |
| 27.48 | ... | ... | ... | .7943 |
| 30 | .6667 | .739 | .788 | ... |
| 30.20 | ... | ... | ... | .7268 |
| 31.11 | ... | ... | ... | .6667 |
| 35 | .6149 | .701 | .745 | ... |

## APPENDIX II

### INEQUALITIES FOR THE QUANTITIES $D_{n,o}$, $N_{n,o}$ AND $\frac{P_{n,o}}{N_{n,o}\sin\theta} - \cot\theta$

In Chapters III and IV, the inequalities:

1.1 $D_{n,o} > 0$ on $I_{b-\epsilon, s+\epsilon}$

1.2 $N_{n,o} < 0$ at $\theta = \theta_s$

1.3 $\frac{P_{n,o}}{N_{n,o}\sin\theta} - \cot\theta > 0$ at $\theta = \theta_s$

for all n are used. These are now established. The inequalities are used in the theory only when the functions (u,v) are as described in Theorem 1 and $1 < \gamma < 1.5$. These limitations are assumed here; consequently, the inequalities derived in the proof of Theorem 1 may be used in the arguments.

The quantity $D_{n,o}$ was defined in Section 2 of Chapter III to be:

$$1.4 \quad D_{n,o} = -n\frac{c^2-u^2}{c^2-v^2} - 1 - \frac{c^2}{c^2-v^2} - (\gamma-1)\,u(u+v\cot\theta)\frac{v^2}{(c^2-v^2)^2} .$$

Therefore:

$$1.5 \quad D_{1,o} = -\frac{c^2-u^2}{c^2-v^2} - 1 - \frac{c^2}{c^2-v^2} - (\gamma-1)\,u\,(u+v\cot\theta)\frac{v^2}{(c^2-v^2)^2} ;$$

that is:

$$1.6 \quad D_{1,o} = \frac{1}{c^2 - v^2} \left[ u^2 + v^2 - 3c^2 \right.$$

$$\left. - (\gamma-1)\, u \left( u + v \cot\theta \right) \frac{v^2}{c^2 - v^2} \right] .$$

The first factor of the above is positive. In Appendix I, the second factor was shown to be positive. (See Appendix I, 1.37 et seq.) Therefore, $D_{1,o} > 0$ on $I_{b-\epsilon, s+\epsilon}$. Hence,

$$1.7 \quad - \frac{c^2 - u^2}{c^2 - v^2} > 1 + \frac{c^2}{c^2 - v^2}$$

$$+ (\gamma-1)\, u \left( u + v \cot\theta \right) \frac{v^2}{(c^2 - v^2)^2} > 0 .$$

Since

$$1.8 \quad D_{n,o} = - (n-1) \frac{c^2 - u^2}{c^2 - v^2} + D_{1,o} ,$$

the inequality 1.1 is established.

The quantity $N_{n,o}$ was defined in Section 2 of Chapter III to be:

$$1.9 \quad N_{n,o} = - \left( u_\theta + nv + (n+1) q_o \sin\theta \right) .$$

Since $u_\theta = v$ and $u(\theta_s) = q_o \sin\theta_s$ ,

$$1.10 \quad N_{n,o}]_{\theta=\theta_s} = -(n+1)(v + u \tan\theta)]_{\theta=\theta_s} .$$

Since

$$v + u \tan\theta = (u \sin\theta + v \cos\theta) \sec\theta,$$

and since both factors on the right are shown in Theorem 1

to be positive, the inequality 1.2 is established.

The quantity $P_{n,o}$ was defined in Section 2 of Chapter III to be:

$$1.11 \quad P_{n,o} = - u_\theta \cos\theta + v \cos\theta + u \sin\theta + v_\theta \sin\theta - \frac{8 + 4(n-1)}{\gamma+1} q_o \sin\theta \cos\theta \ .$$

Since

$$u_\theta = v$$

$$v_\theta = - u - \frac{c^2}{c^2 - v^2} (u + v \cot\theta)$$

$$u(\theta_s) = q_o \cos\theta_s \ ,$$

it follows that at $\theta = \theta_s$:

$$1.12 \quad \frac{P_{n,o}}{N_{n,o}\sin\theta} - \cot\theta = \left[ \frac{v^2}{c^2 - v^2} (u + v \cot\theta) + \frac{4+4n}{\gamma+1} u - n(u + v \cot\theta) \right] \frac{\cot\theta}{(n+1)(u+v\cot\theta)}$$

$$= \left[ \frac{v^2}{c^2 - v^2}(u + v \cot\theta) + \left(\frac{4+4n}{\gamma+1} - n\right) u - nu \cot\theta \right] \frac{\cot\theta}{(n+1)(u + v \cot\theta)} \ .$$

From 1.13 and 1.15 of Appendix I, this is positive if

$$1.13 \quad \frac{4+4n}{\gamma+1} - n = \frac{(3-\gamma)n + 4}{\gamma+1} > 0 \ .$$

Therefore, inequality 1.3 is established, since the above holds for $1 < \gamma < 1.5$.

## APPENDIX III

### SUPPLEMENT TO LEMMA 2, CHAPTER III: PROOF THAT F IS NON-EMPTY AND $f(G) \subset G$

In Lemma 2, the subset F of $B(\delta_3, \delta_4)$ was defined to be:

1.1 $F = [\, (r,\theta) \mid 0 \leq r \leq \delta \ , \ \delta > 0, \ t_o(r) \leq \theta \leq \theta_s \,]$,

where $\delta > 0$ is so chosen that $\delta \leq \delta_3$, $[\, (r, t_o(r)) \mid 0 \leq r \leq \delta \,]$ lies in $B(\delta, \delta_4)$, and $t_o(r) \leq \theta_s$ for $0 \leq r \leq \delta$. The curve $\theta = t_o(r)$ is the unique curve for which:

1.2 $q_o \cos(t_o(r)) - u(r, t_o(r)) \equiv 0$

$t_o(0) = \theta_s$ .

(This is equivalent to $W(r, t_o(r)) \equiv 0$.)

The argument showing that such a $\delta$ can be chosen is based on the hypothesis that the first non-zero derivative $B_{rn}(0)$ is negative. If $B_r(0) < 0$, then from Corollary 2 of Theorem 2, $S_r(0) < 0$. Consequently $t_o'(0) < 0$. If $B_r(0) = 0$, then from 2.44 - 2.46 of Chapter III: $u_r(\theta) \equiv 0$, $v_r(\theta) \equiv 0$ and $S_r(0) = 0$. If

1.3 $B_{rj}(0) = 0$,

$u_{rj}(\theta) \equiv 0$,

$v_{rj}(\theta) \equiv 0$ and

$S_{rj}(0) = 0 \qquad j = 1, \ldots, n$,

then the equations 2.13 - 2.14 and the condition 2.16 of Chapter III are homogeneous for $j = n + 1$, and $I_{n+1,o}(\theta_b) = (n+3)\, u(\theta_b)\, B_{rn+1}(0)$. Hence, from 2.36, 2.44 - 2.46 of Chapter III, it is evident that if $B_{rn+1}(0) < 0$,

then $u_{rn+1}(\theta_s) > 0$. Further, if $B_{rn+1}(0) = 0$, then 1.3 holds for $j = n + 1$. Therefore, if $B_{rn}(0) = 0$ for all n, it is established that

$$u(r,\theta) \equiv u(\theta), \quad v(r,\theta) \equiv v(\theta), \quad S(r) \equiv \theta_s$$

(i.e., the Taylor-Maccoll solution is obtained); hence $t_o(r) \equiv \theta_s$, and a $\delta > 0$ can be chosen. If $B_{rn}(0) \neq 0$ for some n, let k denote the least integer such that $B_{rk}(0) \neq 0$. If $k = 1$, then as previously shown, $t_o'(0) < 0$. If $k > 1$, then by implicit differentiation of 1.2 and use of 1.3 for $j = 1, \ldots, k - 1$, it follows that:

$$t_o^{(j)}(0) = 0 \qquad j = 1, \ldots, k - 1$$

$$t_o^{(k)}(0) = \frac{-u_{rk}(\theta_s)}{q_o \sin\theta_s + v(\theta_s)} \; .$$

It may then be concluded that $t_o^{(k)}(0) < 0$. Thus, in general, the first non-zero derivative $t_o^{(k)}(0)$ is negative. From the continuity of $t_o^{(k)}(r)$, there exists $\delta^* > 0$, $\delta^* < \delta_3$, such that $t_o^{(k)}(r) < 0$, $0 \leq r \leq \delta^*$; hence, $t_o^{(k-1)}(r) \leq 0$ on $0 \leq r \leq \delta^*$ and by induction, $t_o^{(1)}(r) \leq 0$ on $0 \leq r \leq \delta^*$. This implies that $t_o(r) \leq \theta_s$ for $0 \leq r \leq \delta^*$. From the continuity of $t_o(r)$, there then exists $\delta > 0$, $\delta < \delta^*$ such that $[\,(r, t_o(r)) \mid 0 \leq r \leq \delta\,]$ lies in $B(\delta, \delta_4)$. Therefore the set F is non-empty.

In Lemma 2, the argument that $f(G) \subset G$ is complete except for the proof that:

1.4 $$t_2'(r) - t_o'(r) \geq 0 \qquad 0 \leq r \leq S$$

where:

1.5 $$t_2(r) = \theta_s + \int_o^r \frac{1}{\xi} W(\xi, \theta_s)\, d\xi \ .$$

The difference $t_2'(r) - t_o'(r)$ can be written as:

1.6 $$t_2'(r) - t_o'(r) = \frac{1}{r} \frac{q_o \cos\theta_s - u(r,\theta_s)}{v(r,\theta_s) + q_o \sin\theta_s} + \frac{u_r(r, t_o(r))}{q_o \sin(t_o(r)) + u_\theta(r, t_o(r))} \ .$$

By use of the series 2.51 and 2.52 of Chapter III, this may be expressed in the form:

1.7 $$t_2'(r) - t_o'(r) = \frac{\sum_{n=1}^{\infty} u_{rn}(t_o(r)) \frac{r^{n-1}}{(n-1)!}}{q_o \sin\theta_s + q_o (t_o(r) - \theta_s) N(t_o(r)) + v(t_o(r)) + \sum_{n=1}^{\infty} v_{rn}(t_o(r)) \frac{r^n}{n!}(n+1)} - \frac{\sum_{n=1}^{\infty} u_{rn}(\theta_s) \frac{r^{n-1}}{n!}}{q_o \sin\theta_s + v(\theta_s) + \sum_{n=1}^{\infty} v_{rn}(\theta_s) \frac{r^n}{n!}} \ ,$$

where $N(\theta)$ is defined by:

1.8 $$N(\theta) = \frac{\sin\theta - \sin\theta_s}{\theta - \theta_s}$$

$$N(\theta_s) = \cos\theta_s \ .$$

If $B_{rn}(0) = 0$ for all n, then $u_{rn}(\theta) \equiv 0$, $v_{rn}(\theta) \equiv 0$, and $t_o(r) \equiv \theta_s$ . Therefore, from 1.7

1.9 $t_2'(r) - t_o'(r) \equiv 0$ .

If $B_{rn}(0) \neq 0$ for some n, let k denote the least such integer n. Then,

1.10 $t_o^{(j)}(0) = 0 \qquad j = 1, \ldots, k-1$

$$t_o^{(k)}(0) = - \frac{u_{rk}(\theta_s)}{q_o \sin\theta_s + v(\theta_s)}$$

$u_{rj}(\theta) \equiv 0$ for $j = 1, \ldots, k-1$

$v_{rj}(\theta) \equiv 0$ for $j = 1, \ldots, k-1$

$u_{rk}(\theta_s) > 0$

$v_{rk}(\theta_s) < 0$ .

If $k > 1$,

1.11
$$\lim_{r \to +0} \frac{1}{r^{k-1}} (t_2'(r) - t_o'(r)) = \lim_{r \to +0}$$

$$\left[ \frac{\sum_{n=k}^{\infty} u_{rn}(t_o(r)) \frac{r^{n-k}}{(n+1)!}}{q_o \sin\theta_s + q_o (t_o(r) - \theta_s) M(t_o(r)) + v(t_o(r)) + \sum_{n=k}^{\infty} v_{rn}(t_o(r)) \frac{r^n}{n!}(n+1)} \right.$$

$$\left. - \frac{\sum_{n=k}^{\infty} u_{rn}(\theta_s) \frac{r^{n-k}}{n!}}{q_o \sin\theta_s + v(\theta_s) + \sum_{n=k}^{\infty} v_{rn}(\theta_s) \frac{r^n}{n!}} \right]$$

$$= \frac{(k-1) u_{rk}(\theta_s)}{k! \, (q_o \sin\theta_s + v(\theta_s))} \quad .$$

Since $q_o \sin\theta + v(\theta_s) > 0$, $u_{rk}(\theta_s) > 0$ and $k > 1$, it follows that this limit is positive. Therefore,

$$t_2'(r) - t_o'(r) > 0$$

for $0 < r \leq \delta^*$ for some $\delta^* > 0$, $\delta^* < \delta_3$. In this case, $\delta > 0$ is chosen equal to $\delta^*$.

If $k = 1$,

1.12 $$\lim_{r \to +0} \frac{1}{r} (t_2'(r) - t_o'(r))$$

$$= \frac{1}{(q_o \sin\theta_s + v(\theta_s))^2} \lim_{r \to +0} \frac{AD - BC}{r}$$

where:

1.13 $$A = \sum_{n=1}^{\infty} u_{rn}(t_o(r)) \frac{r^{n-1}}{(n-1)!}$$

$$B = q_o \sin\theta_s + q_o(t_o(r) - \theta_s) N(t_o(r)) + v(t_o(r)) + \sum_{n=1}^{\infty} v_{rn}(t_o(r)) \frac{r^n}{n!} (n+1)$$

$$C = \sum_{n=1}^{\infty} u_{rn}(\theta_s) \frac{r^{n-1}}{n!}$$

$$D = q_o \sin\theta_s + v(\theta_s) + \sum_{n=1}^{\infty} v_{rn}(\theta_s) \frac{r^n}{n!} .$$

Now,

1.14 $$\lim_{r \to +0} \frac{AD - BC}{r} = \lim_{r \to +0} \left(\frac{A - C}{r} D + \frac{D - B}{r} C\right)$$

and

1.15 $$\lim_{r \to +0} \frac{A - C}{r} = \lim_{r \to +0} \left[ \frac{u_r(t_o(r)) - u_r(\theta_s)}{r} + \frac{\sum_{n=2}^{\infty} [\, n u_{rn}(t_o(r)) - u_{rn}(\theta_s)] \frac{r^{n-1}}{n!}}{r} \right]$$

$$= u_{r\theta}(\theta_s)t_o'(0) + u_{rr}(\theta_s)$$

$$\lim_{r \to +0} D = q_o \sin\theta_s + v(\theta_s)$$

$$\lim_{r \to +0} C = u_r(\theta_s)$$

$$\lim_{r \to +0} \frac{D-B}{r} = \lim_{r \to +0} \left[ \frac{v(\theta_s) - v(t_o(r)) - q_o(t_o(r)-\theta_s)N(t_o(r))}{r} \right.$$

$$+ v_r(\theta_s) - 2u_r(t_o(r))$$

$$\left. + \sum_{n=2}^{\infty} v_r(\theta_s) \frac{r^{n-1}}{n!} - \sum_{n=2}^{\infty} v_{rn}(t_o(r)) \frac{r^{n-1}}{n!} (n+1) \right]$$

$$= -v_\theta(\theta_s)t_o'(0) - q_o \cos\theta_s \, t_o'(0) - v_r(\theta_s) \quad .$$

Hence, by use of 1.9, 1.11, 2.13, and 3.9 of Chapter III,

1.16 $$\lim_{r \to +0} \frac{AD - BC}{r} = \left[ -2v_r u_r + (q_o \sin\theta + v) u_{rr} \right.$$

$$\left. + \left( - \frac{c^2}{c^2 - v^2} \cot\theta \, u_r - v_r \right) u_r \right]_{\theta=\theta_s}$$

$$= (q_o \sin\theta_s + v(\theta_s)) \left[ \left( 3v_r + \right. \right.$$

$$\left. \left. + u_r \frac{c^2}{c^2 - v^2} \cot\theta \right) \frac{-u_r}{q_o \sin\theta + v} \right.$$

$$\left. + u_{rr} \right]_{\theta=\theta_s} \quad .$$

Thus, by hypothesis 3.2 of Theorem 3, it follows that

$$\lim_{r \to +0} \frac{1}{r}\left( t_2^{(1)}(r) - t_o^{(1)}(r)\right)$$

is positive in this case. Therefore,

$$t_2^{(1)}(r) - t_o^{(1)}(r) \geq 0 \qquad 0 \leq r \leq \delta^*$$

for some $\delta^* > 0$, $\delta^* > \delta_3$. In this case $\delta > 0$ is chosen equal to $\delta^*$. The condition 1.4 is therefore established.

## APPENDIX IV

## THE CONDITION $\Phi_n > 0$ AND RELATED INEQUALITIES

In Chapter IV, the quantities $Z_{n,o}$, $J_{n,o}$ and $K_{n,o}$ were introduced. It is shown first that these satisfy the condition:

$$1.1 \quad \frac{Z_{n,o}}{N_{n,o}} + J_{n,o} - K_{n,o}\left(\frac{P_{n,o}}{N_{n,o}\sin\theta} - \cot\theta\right)\Big]_{\theta=\theta_s} < 0$$

for all n. From the definitions 2.12 - 2.13 of Chapter IV and 2.6 of Chapter III and some manipulation,

$$1.2 \quad Z_{n,o} = \frac{2c^2(u + v\cot\theta)}{q_o \sin\theta(1 - q_o^2)}\left(\frac{-v}{q_o \sin\theta}\right)^{\gamma-2}$$

$$J_{n,o} = \frac{2u}{1-q_o^2}\left(\frac{-v}{q_o\sin\theta}\right)^{\gamma-2}\left(\frac{1}{q_o^2\sin^2\theta}\right)(vq_o\sin\theta - c^2)$$

$$K_{n,o} = \frac{-2(c^2 - v^2)}{(1-q_o^2)q_o\sin\theta}\left(\frac{-v}{q_o\sin\theta}\right)^{\gamma-2}$$

$$N_{n,o} = -(n+1)(q_o\sin\theta + v)$$

$$P_{n,o} - N_{n,o}\cos\theta = \Big[ -\frac{v^2}{c^2 - v^2}(u + v\cot\theta) + nv\cot\theta - \frac{4 + (3-\gamma)n}{\gamma+1}u \Big]\sin\theta$$

at $\theta = \theta_s$. Hence,

$$1.3 \quad \frac{Z_{n,o}}{N_{n,o}} + J_{n,o} - K_{n,o}\left(\frac{P_{n,o}}{N_{n,o}\sin\theta} - \cot\theta\right)$$

$$= \frac{2}{(1-q_o^2)q_o N_{n,o}\sin\theta}\left(\frac{-v}{q_o\sin\theta}\right)^{\gamma-2}[A + (n-1)B]$$

where:

$$A = [(c^2 - v^2)(u + v\cot\theta)$$

$$- \frac{2u}{q_o \sin\theta}(vq_o \sin\theta - c^2)(q_o \sin\theta + v)$$

$$- \frac{7-\gamma}{\gamma+1}(c^2 - v^2)u + (c^2 - v^2)v\cot\theta\,]_{\theta=\theta_s}$$

$$B = \left[ - \frac{2u}{q_o \sin\theta}(vq_o \sin\theta - c^2)(q_o \sin\theta + v)\right.$$

$$\left. + (c^2 - v^2)v\cot\theta + \frac{3-\gamma}{\gamma+1}(c^2 - v^2)u\,\right]_{\theta=\theta_s} .$$

From 1.14 of Appendix I,

$$1.5 \quad v(\theta_s) = \frac{u^2 - 1}{u}\frac{\gamma-1}{\gamma+1}\cot\theta\,]_{\theta=\theta_s} .$$

Therefore,

$$1.6 \quad c^2 - v^2]_{\theta=\theta_s} = -\frac{\gamma+1}{2}v(v + u\tan\theta)\,]_{\theta=\theta_s} .$$

By use of the above, A and B may be expressed as:

$$1.7 \quad A = (u + v\cot\theta)^2(-2\,v\tan\theta)]_{\theta=\theta_s}$$

$$B = (u\tan\theta + v)\left(-\frac{9+\gamma}{2}uv + \frac{1-3\gamma}{2}v^2\cot\theta\right)]_{\theta=\theta_s}$$

Since $v < 0$ at $\theta = \theta_s$, it is clear that $A > 0$. The second factor in the expression for B may be written as:

$$1.8 \quad -\frac{3\gamma-1}{2}v\left(\frac{-9\gamma^2 + 8\gamma + 17}{2(3\gamma-1)}u + \frac{3\gamma-1}{2}(u + v\cot\theta)\right) .$$

Since $-9\gamma^2 + 8\gamma + 17 > 0$ for $-1 < \gamma < \frac{17}{9}$, $u + v\cot\theta > 0$ and $v < 0$ at $\theta_s$; it is apparent that not only this factor, but B itself, is positive. From these results and the fact that $N_{n,o}]_{\theta=\theta_s} < 0$, the condition 1.1 is established.

The argument used in the proof of Theorem 2 in Chapter III to establish

1.9 $\alpha_n^*(\theta) > 0$

$\Delta_n(\theta) < 0 \qquad n = 1, 2, \ldots\ldots$

on $I_{b-\epsilon,s+\epsilon}$ may also be used to establish

1.10 $\alpha_n(\theta) \leq 0$

$\beta_n(\theta) > 0 \qquad n = 1, 2, \ldots\ldots$

on $I_{b,s}$. The function $h_{rn}^*(\theta)$ defined in Theorem 6 is such that

1.11 $h_{rn}^*(\theta) > 0$

$h_{rn\theta}^*(\theta) > 0 \qquad n = 1, 2, \ldots\ldots$

for $\theta_b < \theta \leq \theta_s$. From the definitions 2.9 of Chapter IV,

1.12 $Q_{n,o} = -\dfrac{c^2}{\gamma(\gamma-1)hu}$

$T_{n,o} = \dfrac{nc^2u}{(c^2 - v^2)\gamma(\gamma-1)h} \qquad n = 1, 2, \ldots\ldots .$

Thus,

1.13 $$Q_{n,o} < 0$$

$$T_{n,o} > 0 \qquad n = 1, 2, \ldots\ldots$$

on $I_{b,s}$ . Combination of 1.10, 1.11 and 1.13 yields the inequality:

1.14 $$\int_{\theta_b}^{\theta_s} ( \alpha_n h^*_{rn\theta} Q_{n,o} + \beta_n h^*_{rn} T_{n,o} ) \, d\theta > 0 \qquad n = 1, 2, \ldots\ldots$$

is established.

The results 1.1, 1.9 and 1.14 imply the inequality:

1.15 $$\Phi_n > 0 \qquad n = 1, 2, \ldots\ldots \quad .$$

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