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THERMAL RELAXATION IN CONTRACTING STARS

by
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CHAPTER I

INTRODUCTION

1. The physics of contracting stars, which was for many years a matter of quite limited interest, has recently attained a new degree of prominence in astronomy. In part this is due to the general advance of our understanding of stellar evolution, but it is more directly due to the identification of particular stars which appear to be contracting toward the main sequence at the present time. The main evidence for such identification has come from the study of young clusters by M. F. Walker (1956, 1957, 1959, 1961) and others with modern photometric techniques. Most of the fainter stars in these clusters lie to the right of the main sequence in a color-magnitude diagram, and thus they have larger radii than their main sequence counterparts if the normal color-temperature relationship is valid. Another line of evidence is found in the distribution of certain peculiar types of stars, such as the T Tauri variables and the dwarf stars with emission lines in their spectra. Their tendency to be associated with groups of massive and short-lived stars leads to the suspicion that they are young and are still contracting toward the main sequence. The nebulous objects discovered by G. H. Herbig (1952, 1955) and G. Haro (1952, 1955) are particularly suggestive of an early evolutionary stage. The accumulation of observational data on suspected contracting stars has now reached the point where it is one of the basic problems of astronomy to interpret

the observational picture. Before any detailed interpretation can be worked out, however, we will need a better understanding of stellar contraction than we have had up to now. It is the purpose of the present study to investigate a certain basic aspect of the contraction process: the question of thermal relaxation, or the convergence of stars with different initial conditions to a common path of evolution.

2. The development of the theory of stellar contraction has been a gradual process. It was possible for J. H. Lane (1870), A. Ritter (1879), and Lord Kelvin (1887) to show that a gaseous sphere would grow hotter as it contracted, and that if the contraction proceeded through a sequence of similar configurations -- a homologous sequence -- the temperature at every point would vary inversely as the radius of the star. At that time it was generally believed that the stars maintained their energy by gravitational contraction, but no one knew to what extent they could be considered gaseous. Furthermore, there was no way to tell what sort of density distribution might occur in a gaseous star. In the absence of any knowledge of the opacity of stellar material, one generally assumed that the energy was transported from the interior to the surface by convection. This assumption led to a definite model and to a definite evolutionary sequence, but it was impossible to say whether or not the assumption was justified. It was not possible, moreover, to predict the luminosities of convective stars -- a problem which remains unsolved to the present day.

Lane, Ritter, and Kelvin realized that their gaseous stars could

contract only by losing energy. The rate of energy loss could not be predicted, but it could be observed with fair accuracy in the case of the sun and a few other stars. It was possible therefore to compute the time scale for contraction of the sun, assuming that it is a perfect gas, that the rate of radiation is constant, and that the energy lost by radiation is not replaced. The result depends to a certain extent on the internal structure and on the thermodynamic properties of the gas, but with almost any assumptions one finds a time scale of a few times 10^7 years for the sun. Thus in the nineteenth century the basic order-of-magnitude theory for the contraction of gaseous stars was established, and this theory was combined with observation to find an approximate time scale for the contraction of stars of solar mass and luminosity. Still unanswered were the major questions which stars, if any, were now in the process of contraction, and what the sequences of configuration might be like if convection were not the right mechanism for heat transport.

The studies of R. Emden (1907) on gaseous bodies extended and improved the earlier work, but he did not add any new ideas about the contraction process. About this time K. Schwarzschild (1906) was pointing out the importance of radiative heat transport in stars, and interest in the convective models declined.

J. H. Jeans first took up the problem of a contracting star with radiative heat transport. In connection with his studies of stellar stability he formulated (1918, p. 118) the differential equations that

describe the complete hydrodynamic and thermodynamic behavior of a gas sphere once its initial state is given. Jeans pointed out that the solution of these equations for general initial conditions would not be a simple matter, but he showed that a homologous sequence is possible if the absorption coefficient is independent of temperature and density. Of special interest for the present study is a conclusion reached by Jeans (1918, p. 198) concerning the convergence of other sequences toward the homologous sequence:

> . . . it can be seen, although not by strict mathematical proof, that any configuration not on the permanent homologous series always moves towards that series as contraction proceeds. Thus a mass of gas which has been contracting for a sufficient length of time may be assumed to be on the permanent homologous series.

The arguments by which Jeans reached this conclusion were quite rough and qualitative, however, and they do not seem as convincing today as they did to him. It is the main point of the present investigation to test the validity of this convergence hypothesis of Jeans and to discover what constitutes "a sufficient length of time", a point which he himself did not attempt to settle.

Following this discussion of Jeans there was a general lapse of interest in contracting stars of any description, due to the gradually mounting evidence that the stars were much older than could be accounted for on the contraction hypothesis. The far-reaching analyses of A. S. Eddington, summarized in the monograph (1926) that has become the classic text in stellar structure, were decisive in showing

that most stars must be considered as bodies of perfect gas in a steady state and in turning the attention of astronomers toward the problems posed by these stars. Although Eddington did not concern himself particularly with contracting star theories, his researches had a great significance for practically every problem of stellar structure.

Although the main sequence stars were now removed from the ranks of possible contracting bodies, the early Hertzsprung-Russell diagrams suggested to Eddington and his contemporaries that the red giants might be contracting. A fresh theoretical attack on the contracting star problem was made by L. H. Thomas (1931a, b). He discussed the convergence question in more detail than Jeans had done and tried to formulate a convergence criterion, but he was unable to reach any conclusions on this point. His discussion was valuable however in showing that the convergence question was not an easy one to settle, and in demonstrating again what Emden had previously pointed out; namely, the usefulness of the entropy concept in discussions of contracting stars. Thomas also showed that the homologous sequences envisaged by Jeans are possible under more general conditions of opacity than Jeans had considered.

Since 1931 there has been practically no change in the basic theory of stellar contraction. (We leave for later consideration a recent suggestion that would restore the convective heat transport hypothesis for at least part of the contraction process.) On the observational side there has been another change in the identification of the particular

stars which are thought to be contracting. It is no longer the ordinary red giants, for these are now understood in quite different terms; but it is rather the faint stars in young clusters that now seem to fill this role.

As far as theoretical evolutionary tracks are concerned, the way has been clear to calculate them ever since the law of opacity in the stellar interior was worked out in approximate form by H. A. Kramers (1923). The work of calculation is considerable, however, and no tracks were computed until R. D. Levée (1953) found the structure and evolutionary track of a homologously contracting star with Kramers opacity. Two years later a more detailed study was made by L. G. Henyey, R. E. LeLevier and Levée (1955), in which they took into account deviations from the Kramers formula. Their sequences were not much different from the purely homologous ones of Levée, however, because they started each sequence with a suitably scaled version of the Levée model, and in the cases they studied the deviations from the Kramers formula were not too pronounced. The main point of interest in this work of Henyey, LeLevier and Levée was that it showed for the first time the transition from the homologously contracting model to the main sequence model as nuclear reactions come into play. They did not give any results concerning the question of convergence during contraction, except to say that ". . . some check calculations indicate that a considerable latitude in choice /of a sta..ting configuration/ leads after a few time steps to very nearly the same results."

The only other detailed calculations are the recent ones of R. R. Brownlee and A. N. Cox (1961) for the pre-main-sequence contraction of the sun. They found two possible sequences, corresponding to two different original chemical compositions. In one case the sun contracts directly to the main sequence, while in the other it experiences a short deuterium burning stage. In each case only one starting configuration was considered, and hence no estimate can be made as to the extent to which the assumed starting conditions determine the subsequent evolution.

Soon after the work of Henyey, LeLevier and Levée (HLL) appeared, attempts were made to interpret the color-magnitude diagrams of the young clusters in terms of the idea that stars approach the main sequence along the HLL tracks, a unique track being assigned to each mass. This interpretation encountered immediate difficulties, the details of which we need not go into here. A number of attempts have been made to account for the discrepancy between the observed color-magnitude diagrams and the sort of diagram that the theory would predict. Some suggestions involve a rejection or correction of the observations, while others look for a physical process not included in the theory. We shall not attempt to pass judgment on the extent to which these various suggestions may be correct. Our concern is whether the possibilities of contraction under the present theory have been sufficiently explored to permit a conclusive comparison with the observations. In making such a comparison we should be sure that we know what a given theory

predicts.

3. This brings us to the point of the present investigation. All studies of contraction, beginning with that of Jeans, have been concerned with finding the sequence of configurations through which a slowly contracting gaseous star will pass, assuming a certain standard theory of the basic processes involved. That standard theory assumes that there is no source of energy except gravity, and that the energy is carried from the interior of the star to the surface by radiation. (The work of Brownlee and Cox is not strictly limited by these two assumptions.) It takes no account of rotation or of large-scale magnetic fields. The star material is supposed to be a perfect gas. This standard theory has been found to permit homologous or nearly homologous sequences as one kind of solution, and a number of such sequences have been found.

Homologous contraction is a very special solution of the equations, however, and there has been no determination of what the non-homologous sequences are like. Some very fundamental questions remain unanswered. To what extent is there a unique evolutionary sequence for a contracting star, independent of its initial conditions? Do all possible sequences converge, as Jeans imagined? If they do, what is the time scale for convergence? In other words, how much departure does the theory allow from the tracks that have already been found? The present study is an exploration of these questions in terms of a sampling of particular cases. That is, we make up a variety of initial models

and follow their evolution by numerical calculation to see what the convergence properties are. While we should prefer to see a general analytic treatment of the subject, the unsuccessful efforts of Jeans and of Thomas do not lend encouragement to such an approach. A study by means of examples, on the other hand, presents no special difficulties, and one may feel a reasonable expectation that such a study will give meaningful results.

The plan for the presentation of the methods and results of this investigation is as follows. In Chapter II we present the physical and mathematical description of a contracting star. The physical properties that play an important role in determining a star's contractive evolution are listed. Certain standard approximations for the mathematical description of those properties are introduced, and the validity of the approximations is discussed.

In Chapter III we describe selected details of the method of numerical calculation that is employed. The details described are primarily those which have an important bearing on the accuracy of the results. Further details are shown schematically in an appendix.

In Chapter IV we present the results of a series of calculations for stars of 10 solar masses. These results confirm the convergence hypothesis of Jeans. Beyond that they establish the rate of convergence and the terms in which it is most naturally expressed. The convergence is found to be fast in terms of the contraction of an interstellar cloud to stellar dimensions, but slow in terms of evolution through the

range of normal stellar surface temperatures. Several aspects of the behavior of contracting stars are discussed.

Finally in Chapter V we consider to what extent our results can be carried over to other cases of stellar contraction, and we attempt to delimit the regions in the Hertzsprung-Russell diagram to which our conclusions can be applied.

CHAPTER II

PHYSICAL AND MATHEMATICAL DESCRIPTION OF A CONTRACTING STAR

4. Our general formulation of the contracting star problem is similar to that of Thomas (1931a). Our starting point, like his, is an object that has already contracted from interstellar to stellar density, but which has still some distance to go before it reaches the main sequence. For convenience we start with internal temperatures of the order of one tenth the main sequence value, corresponding to radii of the order of ten times the main sequence value. Special problems associated with the loss of angular momentum, the dissipation of magnetic energy, and the onset of ionization are considered to have been already solved when the present evolutionary stage begins. Thus we start with an object that is like a main sequence star in all its physical properties, except that it has not yet attained a high enough central temperature to start nuclear reactions going.

We assume that the star is initially in exact hydrostatic equilibrium, since if it were not so it would attain that equilibrium in a matter of hours or days. Our investigation is concerned not with such pressure adjustments on the hydrodynamic time scale, but with the secular contraction of the star brought about by the loss of energy through radiation at the surface. This contraction proceeds on the Kelvin or thermodynamic time scale, measured in thousands or millions of years. It may be called thermodynamic evolution, in contrast to situations of

mass motions and accelerations which are hydrodynamic in nature. We shall use the term "simple thermodynamic contraction" as a short name for the type of evolution with which this study is concerned.

At each stage of its thermodynamic contraction a star will find an exact hydrostatic equilibrium in which the gravitational forces are everywhere balanced by pressure gradients and no mass motions exist. This is expressed mathematically by saying that the mass and pressure distributions have to satisfy a pair of first order differential equations:

$$\frac{\partial P}{\partial r} = -\rho \frac{Gm}{r^2} \tag{4.1}$$

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho \tag{4.2}$$

where r = radial distance from the center

m = mass interior to the radius r

P = total pressure

ρ = density

G = gravitational constant.

For given mass and composition there is an infinite variety of structures which satisfy these equations of hydrostatic equilibrium together with appropriate boundary conditions. These different structures correspond to different temperature distributions. If the temperature or some other suitable thermal condition is specified at every point, then the infinite set of hydrostatic structures is narrowed down to a unique model. For example, in cases where nuclear energy sources make a thermally steady state possible, the steady state model is unique. This is the Vogt-Russell theorem.

In contraction there are no thermally steady states, and hence the Vogt-Russell theorem does not apply. What we have is a time-dependent heat flow problem in which the thermal state at any time depends on the state that preceded it. The appropriate thermal condition, therefore, is that the heat content at every point is the same as in the preceding model of the sequence, except for the small changes due to heat flow in the intervening time interval. Heat content in this context refers to the specific entropy, not the temperature, because the latter will in general undergo additional changes due to the compressions and expansions which accompany the flow of heat. For this reason the entropy distribution plays a more fundamental role than the temperature distribution in the thermodynamic evolution of a star. Given an initial entropy distribution, one can find the corresponding pressure and density distributions for hydrostatic equilibrium. The combination of entropy with pressure or density gives the temperature at each point. From the temperature gradient one can calculate the heat flow and the consequent changes of entropy over a short time interval, leading to the specification of a new model slightly further advanced in time.

The calculation of a time sequence for a contracting star is the same in principle as the calculation of a time sequence for any problem of non-stationary heat flow. In such calculations one may choose between what are generally known as explicit and implicit methods. An explicit method is used here. Its characteristic feature is that the changes of entropy between two successive evolutionary steps are cal-

culated on the basis of the heat flow pattern in the earlier model. The point of view may be expressed roughly as follows: "Here is a star with a certain temperature distribution. That distribution enables us to calculate the heat flux at every point, and the divergence of the flux tells us how the star will change over, say, the next 1000 years. After that time the fluxes will have changed slightly, so we will recalculate them on the basis of the new temperature distribution in order to tell what happens in the subsequent 1000 years." Thus the explicit method is one that extrapolates forward in time.

In contrast to this is the implicit approach, which has been used by Henyey, LeLevier, and Levée. In this method the changes of entropy between two successive models are made to be consistent with the heat flow pattern in the later model. But it is more to the point to say that the heat flow pattern in the later model is made to be consistent with the changes of entropy between the two models. The point of view is now the following: "This star was known to have a certain temperature distribution 1000 years ago. Let us make a guess as to what the temperature distribution is today. Then we will calculate the present-day heat flux at each point, and if our guess is correct we will find that the divergence of that flux is consistent with the entropy changes over the last 1000 years. We must expect to make several guesses before we get it right." This method is therefore one that replaces the forward extrapolation in time by a sort of backward extrapolation.

It may be debated which of the two methods is preferable for the present problem, but it is clear that either could be used. We have chosen the explicit method because of its directness. It makes the calculation of each model simpler than the implicit method, because the thermal specification is always known at the outset rather than having to be guessed. To offset this advantage, however, there is a more severe limitation with respect to the maximum time interval that can be used. The reason for this is that small-scale irregularities in temperature are suppressed in the implicit method, whereas in the explicit method they are found to oscillate with increasing amplitude unless one proceeds in time steps appropriate to the decay of these small-scale features. This limitation will be discussed in further detail in section 18.

For the actual calculation we introduce into the star a set of mesh points that move with the material -- i. e. a Lagrangian coordinate system. Because of spherical symmetry these mesh points take the form of concentric shells. The analysis then proceeds along the following lines.

a) The entropy at each mesh point is data originally specified.

b) Find the hydrostatic pressure distribution that is consistent with the known entropy values and the boundary conditions.

c) Now find other properties of the model which can be derived from entropy and pressure; in particular the temperature, heat flux, and divergence of flux at each mesh point.

d) From the flux divergence and the temperature, calculate the time derivative of entropy at each mesh point.

e) Taking a suitable step forward in time, calculate the new entropy value at each mesh point.

f) Repeat the sequence from step b.

The preceding is a very broad outline of the problem of simple thermodynamic contraction and how it is solved. The remainder of this chapter and the entirety of the next are devoted to the all- important details of the specification and solution of that problem.

The basic differential equations

5. In the years since the fundamental work of Eddington appeared, most studies of stellar structure have been concerned with steady state models. As a result the four differential equations which express the steady state conditions have become very well known, and have in fact come to be known as the four equations of stellar structure. These equations may be found in their most commonly used form in the recent text by M. Schwarzschild (1958, p. 96). It seems appropriate to discuss the equations used here in terms of their relation to the four given by Schwarzschild.

Schwarzschild's first two equations express the condition that the star is in hydrostatic equilibrium in its own gravitational field. These equations apply to the present problem and have already been introduced as (4. 1) and (4. 2). In keeping with our Lagrangian approach we find it convenient to reexpress these equations in terms of the

mass m as independent variable, so that they appear in the following form:

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \tag{5.1}$$

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{5.2}$$

Schwarzschild's third equation expresses the condition that the star is in a thermally steady state with divergence of heat flow equal to rate of nuclear energy generation. For the contracting star this is replaced by our equation (5.4) below, which relates the divergence of the heat flow to the rate of change of specific entropy. The fourth differential equation of Schwarzchild gives the relation between the heat flow and the temperature gradient. There are two alternative equations for this relation, corresponding to radiative and convective heat transport. Since we are not considering convective cases we use the radiative equation, again taking mass as the independent variable. This is equation (5.3) in our set.

$$H = -4\pi r^2 \rho \frac{c}{\kappa\rho} \frac{\partial P_r}{\partial m} \tag{5.3}$$

$$\frac{\partial S}{\partial t} = -\frac{1}{T} \frac{\partial}{\partial m}\left(4\pi r^2 H\right) \tag{5.4}$$

The notation followed here is close to that used by Eddington and subsequent writers. The meaning of symbols not previously defined is as follows:

T = temperature

S = entropy per gram (specific entropy)

H = energy current density (ergs/cm^2/sec)

t = time

κ = absorption coefficient per gram

c = velocity of light

P_r = radiation pressure.

These four differential equations contain eight dependent variables: r, T, P, P_r, ρ, S, H, and κ. Therefore we require in the actual integrations a sufficient number of auxiliary relations to reduce the number of dependent variables to four. These auxiliary relations are an opacity formula and a set of relations between the state variables which we may call the equations of state. These relations will be presented in sections 7 and 10.

There is an important difference between the interpretation of the third and fourth equations in steady state problems, and that given to them here. In steady state problems the entire set of four equations is regarded as a fourth order system which steady state stars must satisfy. To find the steady state structures, one has to integrate the four equations simultaneously, subject to four boundary conditions. A similar approach would be taken for a contracting star if the implicit method were used. In the present treatment by the explicit method, however, the first two equations are sufficient to determine the structure of the star because the thermal state is not a free quantity but is

known from the star's previous history. Thus we have a second order system to integrate instead of a fourth order one. Equations (5.3) and (5.4) are regarded merely as formulae for calculating H and $\frac{\partial S}{\partial t}$ after the model is found.

Boundary conditions

6. In the steady state stellar structure problem, with its four differential equations, there are four boundary conditions which the stars must satisfy and which must be invoked in order to find the unique models. In the contracting star problem the two differential equations which express the condition of a thermally steady state are absent, and hence the thermal boundary conditions are also absent. Corresponding to the radius and pressure differential equations which remain, there are two boundary conditions, one dealing with radius and one with pressure:

$$r = 0 \text{ at } m = 0 \tag{6.1}$$

$$P = 0 \text{ at } m = M \tag{6.2}$$

where M is the total mass of the star.

It is not convenient to extend the integration out to an actual zero of pressure, however, particularly when the specific entropy comes into the equations. In this problem it has been found better to terminate the integration at the photosphere of the star, this being defined as the place where the total optical thickness of the overlying material is 2/3. The boundary condition for pressure at this level, taken from the simplest theory of atmospheres in radiative equilibrium, is:

$$P = \frac{2}{3}\frac{GM}{KR^2} \qquad \text{at} \quad m = M \tag{6.3}$$

where R is the radius of the star. This boundary condition was used by Hoyle and Schwarzschild (1955) for part of the red giant sequence in their exploratory paper on that subject. It is more realistic physically than the more commonly used (6.2), but for many problems it makes no difference which is used.

Equations of state

7. We use equations of state that are appropriate to a mixture of fully ionized non-degenerate gases with radiation. Both the gas and the radiation contribute to the energy density, the pressure, and the entropy of the system. The contributions are additive:

$$U = U_g + U_r \qquad \text{(energy per unit mass)} \tag{7.1}$$

$$P = P_g + P_r \qquad \text{(pressure)} \tag{7.2}$$

$$S = S_g + S_r \qquad \text{(entropy per unit mass)} \tag{7.3}$$

Because the mean free path of a photon is so short compared with the distance over which the temperature varies significiantly, the radiation field at any point inside a star resembles a black body field to a very high degree of accuracy. This point was made very convincingly by Eddington (1926, p. 106) in a numerical example for a typical case, where he showed that the radiation temperature may be expected to vary by about 1 part in 10^9 over the mean free path of a photon. From this further discussion it becomes clear that the outer layer in which

the black body approximation fails is of such negligible mass that it can be neglected even when other surface effects such as incomplete ionization have to be taken in account. The region where the black body approximation fails is in fact the stellar atmosphere. Following all other investigators of the stellar interior we neglect the atmosphere and use the following relations throughout:

$$U_r = \frac{1}{\rho} a T^4 \tag{7.4}$$

$$P_r = \frac{1}{3} a T^4 \tag{7.5}$$

$$S_r = \frac{4}{3\rho} a T^3 \tag{7.6}$$

For the gas we specify the composition by the customary three mass fractions X (hydrogen), Y (helium), and Z (all other elements). The gas is assumed to be fully ionized and non-degenerate in all regions, so that the following formulae apply:

$$\mu = \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z} \tag{7.7}$$

$$U_g = \frac{3}{2} \frac{\mathcal{R} T}{\mu m_H} \tag{7.8}$$

$$P_g = \frac{\rho}{\mu m_H} \mathcal{R} T \tag{7.9}$$

$$S_g = \frac{\mathcal{R}}{\mu m_H} \left[\ln \left(\frac{T^{3/2}}{\rho} \right) - 10.788 \right] \tag{7.10}$$

In these equations μ stands for the mean molecular weight per particle and m_H is the mass of the hydrogen atom.

Wherever entropy is presented in our results we give not the entropy per gram, which is what equations (7.6) and (7.10)* refer to, but the entropy per particle in units of the Boltzmann constant; i.e. $S/\mathfrak{k}N$ where N is the number of electrons plus nuclei per gram.

$$N = \frac{1}{\mu m_H}$$

This is a more directly meaningful kind of entropy, from the point of view of statistical mechanics, than the entropy per gram. We have made a similar reduction where the total entropy of the star is presented, defining N in that case as the total number of electrons plus nuclei in the star.

The equations of state for the gas represent an idealization of the real stellar conditions which is by no means as accurate as the black body approximation is for the radiation field. It is nevertheless an idealization which is almost always made in stellar structure problems. It has been known since the early work of Eddington that for the conditions which prevail in normal stellar interiors the ionization of hydrogen

*The constant in the entropy formula could be given many different values in a gas of impure composition, depending on how precise a description we want. The value in (7.10) applies to our particular composition only, and it assumes that the Z fraction of the gas consists of a single element. Actually it makes no difference to our calculations what value we assign to the constant, as we are not concerned with an approach to zero temperature or to Fermi-Dirac degeneracy.

and helium is essentially complete, and while the heavier elements may retain some electrons they are not abundant enough to make an appreciable error in the formulae. Near the surface, however, where the assumption of complete ionization begins to break down for helium and more especially for hydrogen, the properties of the gas are very poorly described by these equations. For hydrogen the failure occurs when the temperature is about 10^4 °K; for helium, when the temperature is about 5×10^4 °K. These regions of incomplete ionization extend over a deeper region than can be called the atmosphere in most stars. Nevertheless, in our stars they encompass a good deal less than 1% of the mass, as our tables of physical properties in Chapter IV will show. In most of our cases the figure is about one tenth of 1%. Therefore, our equations of state are wrong in a region which is hardly more consequential than the atmosphere of the star. It is well established that in cases of radiative heat transport the outer layers of the star have a negligible influence on the main body. (See for example Eddington, 1930, and Schwarzschild, 1958 p. 89ff).

Surface convection zones

8. A serious error may be made, however, in cases where deep surface convection zones exist. The surface convection currents in the red giants and the lower main sequence stars appear to be closely bound up with the special properties of the gas in the zone of partial ionization of hydrogen. If one ignores these properties, of which the specific heat and the opacity are both important, one does not find any

outer convection zone, and the calculated model may be far from the real structure. All the early models of the sun suffered from this defect, and it was not until 1957 that an attempt was made to calculate a solar model that took this zone into account (Schwarzschild, Howard and Härm, 1957). The much more serious effect of ignoring the outer convection zone in red giants is well described by Hoyle and Schwarzschild (1955), and by Kippenhahn, Temesvary and Biermann (1958). The situation with which the investigator of stellar structure is faced is that he must know or guess in advance whether the surface convection zone is significant or not. If it is not, then he can safely use the equations for a highly ionized gas all the way to the surface. If on the other hand the surface convection zone is significant, it becomes necessary to treat the zone of partial ionization more accurately if the model is to be found from theory alone. Even then there will be some uncertainty in the model because of the inadequacy of convective heat transport theory. We shall see that this uncertainty is very great in some cases and relatively small in others.

Thus, by ignoring the special properties of partially ionized hydrogen we are automatically ruling out any possibility of a surface convection zone. We must accordingly list it as one of our fundamental assumptions that no such zone is present. Although it would be desirable to determine the extent to which contracting stars are affected by outer convection zones, the current knowledge of convective heat transport is insufficient to provide a satisfactory answer in the general case.

This is ultimately a question of considerable importance for the theory of stellar contraction, however, and we will therefore do well to consider somewhat further what are the limitations of present knowledge concerning these zones.

9. The convective transport theory, as it is used in stellar problems, was developed by L. Biermann (1932, 1935, 1938) on the basis of the mixing length concept of Prandtl. In this theory one deals with an average convective element and thus avoids all the complexities of the complete turbulence spectrum. The dimensions and lifetime of this average element are not given by the theory, however. It is customary to lump all the unknown factors together in a single quantity which has the dimensions of a length, and to refer to this unspecified length as the mixing length of the average element. General considerations indicate that this mixing length should be of the same order of magnitude as the scale height for pressure variations or density variations in the regions concerned. For an assumed ratio of mixing length to scale height, the convective heat transport can be approximately calculated as a function of the temperature gradient.

There have been a number of attempts to work out a more complete convection theory, but nothing has yet emerged which lends itself to the calculation of stellar models. Only the mixing length theory has been at all useful for that purpose. Therefore we shall examine the extent to which models calculated by this theory are uncertain. This subject has been reviewed in a number of recent papers

of which those of J. C. Pecker (1959), S. Temesvary (1959), and M. Schwarzschild (1961) are especially valuable. A consideration of the following four cases, drawn from recent works on steady state stars, will give a general idea of the varying extent to which the incomplete convection theory makes the models uncertain.

a) In the massive stars of the main sequence it is easily established that on account of the high temperature there is either no outer convection zone at all or at most a feeble one due to the onset of helium ionization. Consequently it is possible to calculate models of these stars, as R. Kushwaha (1957) and others have done, without any uncertainty due to surface convection.

b) For metal-deficient stars of 0.6 to 1.0 solar masses, P. Demarque (1960, 1961) has found that the outer convection zones comprise up to 30% of the mass. He is nevertheless able to calculate reliable models despite the vagueness of the theory, because in these stars the non-adiabatic part of the convection zone, where knowledge of the mixing length is required, is very thin and has little influence on the linkage between the photosphere and the adiabatic region.

c) In the case of the sun the degree of uncertainty is considerably greater. Schwarzschild, Howard and Härm (1957) found that the best approach was to use the measured solar radius to fix the extent of the convection zone, and from this they were able to deduce empirically the ratio of mixing length to pressure scale height in the non-abiabatic part of the zone. The fact that the mixing length can be

determined in this way shows that its influence on the solar radius is not negligible. The conditions in the deep interior are not seriously affected, however, as the solar convection zone is still a fairly shallow one.

d) In the case of the red dwarfs and of the inhomogeneous red giants of population II the convection zones are deep enough to have a significant influence on the entire stellar structure. This is most clearly brought out in the work of Kippenhahn, Temesvary and Biermann (1958), who showed that for red giants of about solar mass both the evolutionary track in the Hertzsprung-Russell diagram and the central temperature attained are sensitive to the unknown ratio of mixing length to scale height.

The aforementioned investigations of convection zones have all been based on studies of complete stellar models of the steady state variety. They suggest that the outer convection zone may always be extensive and influential when the surface temperature is below a certain limit, and that in most of these cases the convective heat transport theory (mixing length theory) is inadequate to predict the extent of the zone. In seeking to evaluate the applicability of this picture to contracting stars we should note that a somewhat different approach to the problem is made possible by the fact that the important region for convection -- the zone of partially ionized hydrogen -- always lies very close to the surface. One can therefore study the structure of this region in terms of the surface conditions as is cus-

tomary in the study of stellar atmospheres, without requiring knowledge of the conditions in the deep interior. In particular, one may investigate the occurrence of convectively unstable zones as a function of the surface temperature T_e and the gravitational field strength g. The most extensive studies of this kind have been those of C. Pecker (1953), E. Böhm-Vitense (1951, 1953) and S. Temesvary (1959). Their results confirm the view that all cool stars have outer convection zones. Our Figure 1, adapted from C. Pecker, illustrates the situation. The nearly vertical line divides the (log g, log T_e) diagram into two regions: one of low surface temperature where hydrogen convection zones occur, and one of high surface temperature where they do not occur. The location of this line is not very sensitive to composition. Criteria for the presence of helium convection zones are omitted from the diagram, as Pecker found that these zones are never very extensive.

Although the calculations of Pecker and Böhm-Vitense are strictly applicable only to steady state stars, we may feel safe in applying the criterion illustrated in Figure 1 to contracting stars as well; for the criterion is based on atmospheric layers, and these layers can hardly be affected by the star's overall balance or imbalance of energy. Because of their very small heat capacity, the atmospheric layers adjust themselves to accommodate the heat flowing out from the interior in a time which is very short compared to the contraction time scale of the star. Therefore they may be considered to be in a steady state

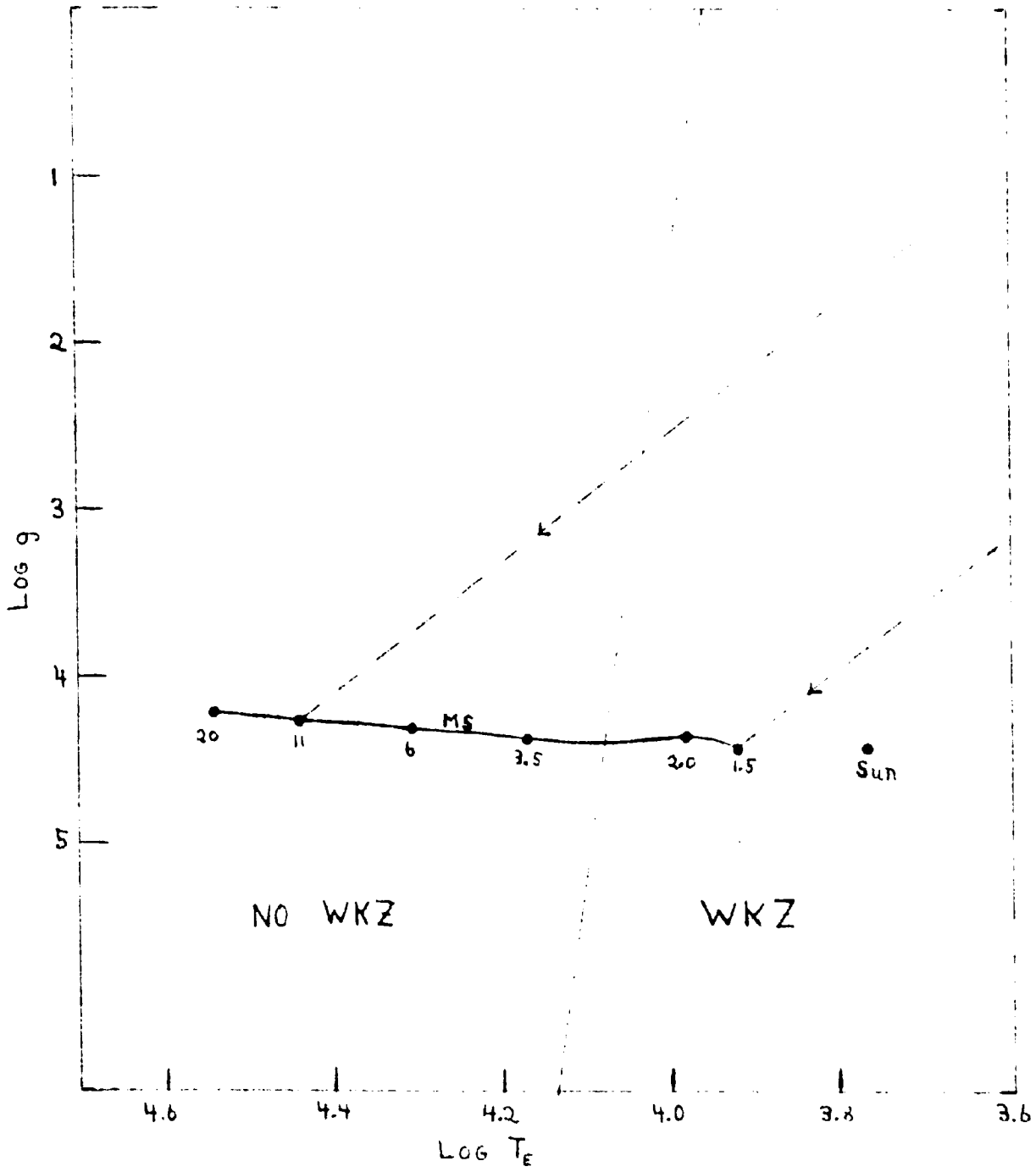


Figure 1. Criterion of C. Pecker for the existence of hydrogen convection zones (WKZ) in terms of effective temperature (T_E) and surface gravity (g). The six points with masses indicated are main sequence models of Henyey, Lelevier, and Levée. The dashed lines are discussed in the text.

even while the star as a whole is not. We shall have occasion to return to this point later on, for it introduces an extra complication into the calculation at the surface of the star. Some results will be presented that illustrate the rapidity with which these layers attain a steady state.

We have plotted some additional features in Figure 1 in order to see how the criterion of Pecker and Böhm-Vitense is related to contracting stars of different masses. The nearly horizontal curve is the main sequence according to Henyey, LeLevier and Levée (1959) for the composition X = 0.68, Y = 0.31, Z = 0.01. The masses of stars on this main sequence are indicated. The position of the sun is also shown. The diagonal lines represent hypothetical pre-main-sequence contraction tracks for two stars of different mass on the supposition that they contract at constant luminosity (g/T_e^4 = constant). These lines are meant only to indicate the general direction of contracting evolution in the diagram, and are not to be taken as actual evolutionary tracks of any stars. We shall see in Chapter IV how far some actual tracks deviate from constant luminosity. For the present these schematic tracks show us that all stars should be expected to have hydrogen convection zones for at least part of their pre-main-sequence history, and that stars of more than about 3 solar masses should lose their convection zones before they reach the main sequence. The criterion of Pecker and Bohm-Vitense is definite on these points, but it does not give any information as to the depth of the convection

zone when it does occur.

C. Hayashi has recently suggested (1962) that a contracting star will be completely convective as it approaches the main sequence, until it gets fairly close to the dividing line of Figure 1. An opposite conclusion was reached by Brownlee and Cox (1961) in their calculation of a contraction path for the sun. They allowed for convective heat transport, in a manner not specified in their preliminary publication of results, and found that the convective region never plays a major role in that evolutionary sequence. From what has been published it is not apparent how this conflict can be resolved, and we do not seek to resolve it here.

Our view will be that the question of the importance of convection zones in the right-hand region of Figure 1 awaits final settlement. We shall proceed on the assumption that the influence of such zones can be ignored, but shall exercise caution to the extent of staying out of that region of the diagram. We must recognize that we are investigating a limited problem, and that a number of factors are being ignored which may in the actual event cause the pre-main-sequence evolution of stars to be different from what we find here. Even the limited problem under consideration is one which has remained unsolved since it was originally formulated by Jeans. Its limitations are precisely those which apply to all the contraction sequences that have ever been computed in detail, if we discount the small outer convection zone in the Brownlee-Cox models. Our aim is to discover how much devia-

tion from these sequences is permitted within the limitations of the simple theory on which they are based. After examining our results on this question, we may consider in what way they are likely to be altered by the introduction of additional phenomena such as convection.

Opacity

10. The continuous absorption coefficient in the interior of stars depends in general on three processes: bound-free transitions in the few atoms which have attached electrons, free-free transitions involving all ions, and scattering by the free electrons. Kramers (1923) found for the first two processes the approximate formula:

$$\kappa = \kappa_0 \rho T^{-3.5} \tag{10.1}$$

where κ_0 depends on the composition of the material. The contribution of the free electrons, in case of complete ionization, is independent of temperature and density. A number of investigations since that of Kramers have shown that, while the detailed dependence of opacity on temperature and density may be quite complicated, the run of opacity in a given star or even in a homologous group of stars may be represented with fair accuracy by a formula of the form:

$$\kappa = \kappa_0 \rho^{\alpha} T^{-3.5-\beta} + \sigma \tag{10.2}$$

where the constants κ_0, α, and β are allowed to vary somewhat from one star or group of stars to another. This type of formula has been used very extensively in modern studies of stellar evolution, the

general practice being to adjust the exponents in the formula in such a way as to give a reasonable fit to the opacities obtained by exact calculation such as those published by Keller and Meyerott (1955).

For reasons which will be discussed later, we have carried out our main contraction study at 10 solar masses. Kushwaha (1957) calculated the main sequence position and the early nuclear evolution away from the main sequence for stars of this mass and a composition which would now be considered intermediate between very young and very old stars. In order to facilitate a comparison of our results with his, we adopt his composition and his opacity formula, which is of the type described in the preceding paragraph:

$$\kappa = 3.69 \times 10^{25} Z(1+X)\rho T^{-3.5} + 0.19(1+X). \qquad (10.3)$$

11. We have already remarked that our equations of state for the gas do not hold good in a small region near the stellar surface, and we have dismissed this failing as one of no consequence so long as the outer convection zone is not an essential factor. We must now note that the opacity formula (10.3) likewise fails in approximately this same outer region. The reason for the one failure is the same as the reason for the other: incomplete ionization. The Kramers formula is based on the assumption that the absorbing ions are kept for the most part dissociated. Beyond that, it is assumed that the electron density corresponds to complete ionization of the abundant elements. Eddington showed (1926, p. 218) that the simplest way of arriving at the Kramers

formula is to calculate the rate of recombinations under these conditions, and to equate it to the rate of photodissociations in accordance with the principle of detailed balancing. It is clear that in the outer layers, where the assumption of complete ionization breaks down, the Kramers formula will no longer give the true opacity. Other mechanisms of absorption come into play, such as the negative hydrogen ion, and these mechanisms do not follow the Kramers formula. In any given atmosphere the absorbing mechanism at work depends very much on the temperature, and no one simple formula will cover all conditions.

It is customary in studies of stellar interiors to ignore the special opacity properties of the atmospheric layers, for the same reason that one ignores the special equations of state: unless a deep surface convection zone is present, these layers simply are not significant in the overall structure of the star. It is therefore not a matter of concern to us that the opacity formula gives incorrect values in a region which comprises less than one per cent of the total mass. We might well use formula (10.3) right up to the surface of the star, as other investigators commonly do, except that in our method of calculation this can lead to awkward situations. The difficulty is not so much that the formula is inaccurate as that it is inaccurate in an unacceptable way; namely it gives an infinite opacity as $T \rightarrow 0$. This makes it difficult or impossible to locate the photosphere in some models, and the whole scheme of calculation breaks down.

To avert this procedural difficulty, which stems from an artificial

infinity in the formula, we use a different expression for the opacity when the temperature is below 10^5 °K:

$$\kappa = 3.69 \times 10^{25} Z(1+X) \rho \left(10^{-17.5}\right) + 0.19(1+X) \qquad (10.4)$$

This formula is not based on physical notions; it is a mathematical devise whose procedural advantage is that it is continuous with formula (10.3) at the changeover point and that instead of tending toward infinity as $T \to 0$ it tends toward the constant value for electron scattering. An approximate physical picture of what we are doing when we use this modification is the following. As we progress outward from the level where $T = 10^5$ °K to the photosphere we gradually drop out all sources of opacity except the scattering by electrons. This does not give us correct opacities in the affected region -- which comprises about 1% of the mass -- but it gives us well behaved opacities. In such an insignificant region we could in fact use any formula whatsoever, as long as it came within a few powers of 10 of the correct values, without altering the evolution that we find for the main body of the star.

The problem of continuity of flux in the outer regions

12. In steady state star problems the equations that express the thermally steady state condition guarantee at the same time that the flux is a smooth, continuous function in space. For a contracting star on the other hand the condition of thermal stationarity is removed, and as a result the flux is no longer smooth as a matter of course. How smooth it will be then depends on the smoothness of the specified

entropy distribution. A particularly critical situation arises at the photosphere, for there the emergent flux depends on T^4 while the rate of energy transport upward from the lower layers depends on grad T^4. In general the two methods of calculating the flux will not agree, unless the temperature distribution (or entropy distribution) has been selected in a special way. This poses for us something of a dilemma. On the one hand we are trying to see what will happen to an arbitrarily chosen entropy distribution; on the other, we have to recognize that an arbitrary distribution will not give a continuous flux across the photosphere.

It is evident of course that discontinuities of flux cannot really occur in stars. The situation referred to above would not look discontinuous if we used a detailed transfer theory in the region of the photosphere. There would then appear a concentrated heat source or sink of finite extension, instead. However, it would be of no value to our main problem to consider concentrated heat sources or sinks in the outer layers. A much more useful approach is to express the fact that the star will quickly spread out any sources or sinks over a considerable region. This has been done in the following way.

Instead of requiring the entire star to conform to a specified entropy distribution, which is in effect the star's memory of its past thermal history, we require it to conform only in the region $q = 0$ to $q = .99$. (The variable q stands for the fractional mass, m/M.) In the outer 1% of the stellar mass we consider the previous thermal history to be unimportant. This outermost shell is represented as a

smoothly distributed source or sink of energy, of whatever magnitude is necessary to absorb the flux difference between q = .99 and q = 1. As one might expect, it develops that after a few evolutionary time steps this flux difference becomes small, but it never vanishes altogether. This is in keeping with the general terms of the problem, for it is to be expected that every part of the star will contribute something to the energy radiated away. Figure 2 illustrates the smoothing out of the initially unbalanced flux distribution in one of our sequences. It is clear that after ten time steps -- which is a very short time on the contraction time scale -- the flux distribution has been very nicely smoothed out by a readjustment of the initially unbalanced outer layers.

As to the exact distribution of the energy source in the outer 1% shell, it appears on the basis of some experiment that any smooth function can be assumed. For example, we have tried the following three forms, each of which has a certain physical plausibility:

$$\text{div } H = C\rho \tag{12.1}$$

$$\text{div } H = C\rho T \tag{12.2}$$

$$\text{div } H = C\rho TS \tag{12.3}$$

In each case, C is an arbitrary coefficient. No matter which of these distributions is assumed, the source all but disappears after a few time steps, and from that point onward the difference between the various distributions becomes insignificant. The essential thing is to

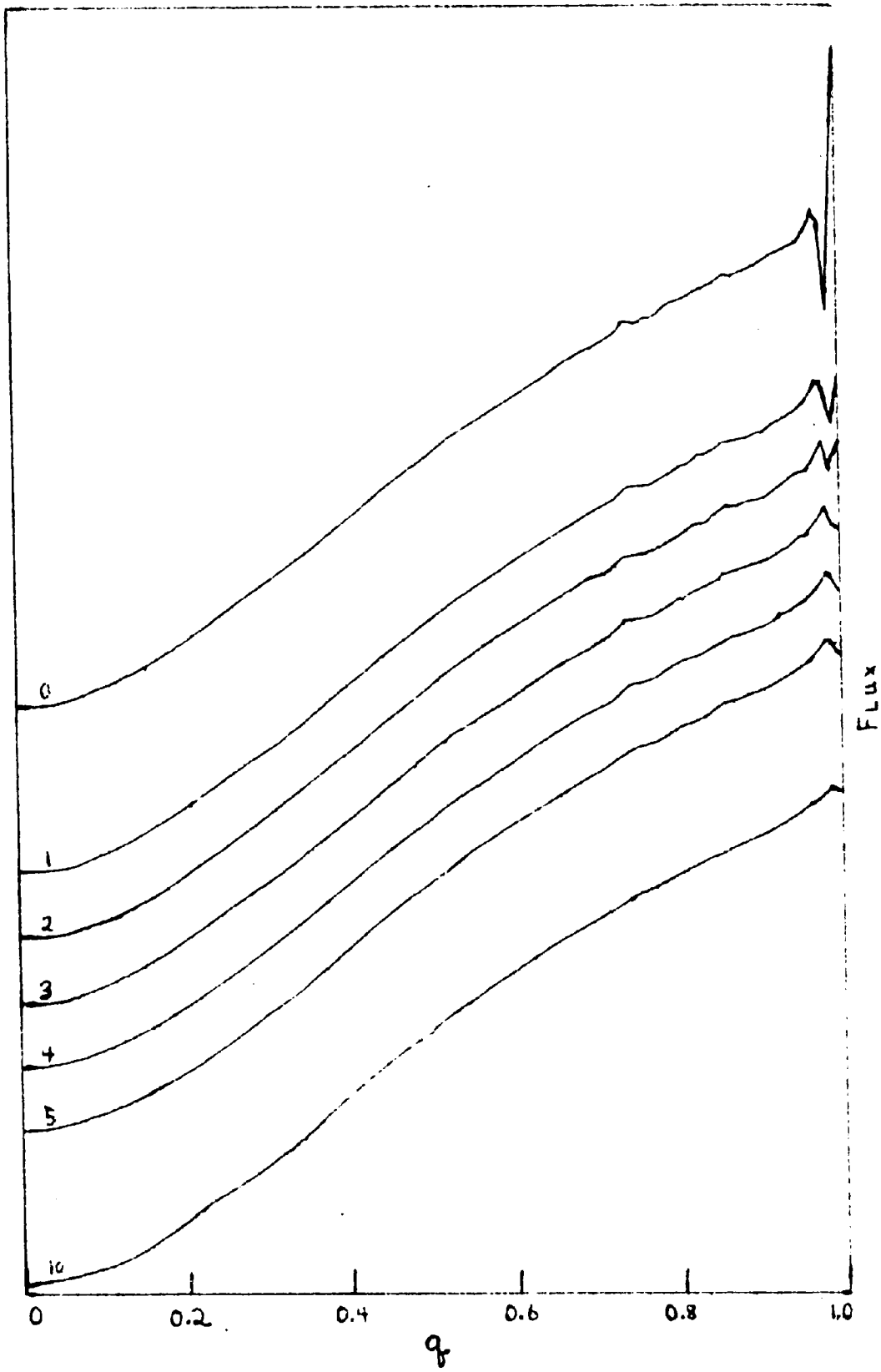


Figure 2. Outward energy flux as function of the mass variable q, in successive models of a typical sequence. The curves are displaced vertically from one another, each rising from zero flux at q = 0. The number over each curve is the serial number of the model in the sequence, 0 being the initial model.

have an adjustable coefficient so that a smooth transition can be made from the flux at q = .99 to the flux at q = 1.

13. It may be useful to review at this point what is being assumed about the thermal behavior of the star in different regions:

a) In the inner 99% of the mass, no special assumptions are made at all. The entropy distribution is entirely fixed by the past history. The treatment is rigorous insofar as the calculation of heat flow over finite space and time intervals permits.

b) From q = .99 to the photospheric level where $\tau = 2/3$ and q is indistinquishable from 1, it is assumed that any source of heat becomes smoothly distributed throughout this region in a time which is short compared to the interval between computational time steps. This means that the entropy is not remembered from one time step to the next, but instead it is instantaneously adjusted so as to smooth out the heat source. It is further assumed that the simple radiative transport equation (5.3) gives the correct energy flux all the way up to optical depth 2/3.

c) The region from $\tau = 2/3$ to $\tau = 0$ is assumed to be sufficiently like a grey atmosphere in radiative equilibrium for the Eddington atmosphere approximation to be used. The opacity formula in this region corresponds to pure electron scattering of a fully ionized gas.

It may be remarked here that our assumptions concerning regions b and c do not play a very important role in the overall problem, except that some scheme must be devised that approximates the correct

boundary conditions and allows the calculation to proceed smoothly. Any other physically plausible assumptions that meet these conditions should lead to the same results as ours.

CHAPTER III

SOME DETAILS OF THE CALCULATION

14. The basic physical and mathematical description of our approach to the contracting star problem has been presented in Chapter II. The present chapter will be concerned with certain details of procedure and of numerical methods which were used in the actual calculation of evolutionary sequences. These calculations were done on an IBM 7090 computer.

We recognize that there is no unique "best way" of doing a given numerical calculation. At every step of the way there are likely to be several acceptable procedures, of varying degrees of efficiency and simplicity, and each programmer will select or devise such methods according to his own taste. His main concern is to ensure that his methods are successful in obtaining solutions of the accuracy he requires within the computer time at his disposal.

The purpose in describing numerical methods, therefore, is not that those employed here are the best or the most elegant for this problem. The point is rather that certain procedural details have an important bearing on the nature and accuracy of the results one obtains, and on whether one obtains results at all. It is points of this kind which will be discussed in this chapter. The order of discussion does not correspond to the sequence of operations in an actual calculation, but is intended rather to correspond to the relative importance

of the various points in the overall problem. The relation of these separate parts to the complete program is shown in the form of flow diagrams in Appendix I.

In the discussion that follows it will be helpful to keep in mind the main theme of the calculation as it was set forth in section 4. It is primarily an explicit heat flow calculation in which changes of entropy are calculated in a sequence of finite time steps. After each step forward in time we have to calculate new distributions of pressure, density, and temperature that are consistent with the hydrostatic equations, with the new entropy values, with the equations of state, and with the boundary conditions. We have aimed for what may be roughly described as three place accuracy in the calculation, but we shall not attempt to define exactly what this means other than by illustrating how we attempt to attain it.

Method of calculating heat flow and entropy time derivative

15. The essential features of this calculation are most clearly seen if one forgets about changes of density and volume, and thinks in terms of heat flow in a rigid body. We first pick a fixed grid of reference points -- in our case, a set of spherical shells that move with the material -- spaced as closely as we wish to know the details of the temperature distribution. We shall call this the thermal grid and speak of its spacings as thermal grid intervals, in order to distinguish them from another kind of interval which is used for integrating the pressure equation. It is important to recognize that in a

numerical heat flow calculation we can attach no meaning to details of the temperature distribution which are finer than the thermal grid intervals. A thermodynamic function must be regarded as completely specified when its value is given at each of the thermal grid points. In most of the calculations we have used a grid with intervals $q = .02$ throughout most of the star, with a few extra points added near the center and near the surface, so that the star is essentially divided into 50 shells of equal mass.

Consider the entropy, temperature, opacity, etc. to be known at each of the thermal grid points at an initial time t_o. It is required to calculate changes of entropy at these same points in accordance with equations (5. 3) and (5. 4). For this purpose a scheme of calculation by finite differences is required that is equivalent to the differential equations. One possible approach is to eliminate all reference to derivatives and reformulate the problem entirely in terms of finite differences. This is the approach described by Henyey and his associates in a recent paper on their method of calculating evolutionary tracks (Henyey, Wilets, Bohm, LeLevier, and Levée, 1959). We have preferred, however, to keep the concept of the derivative in our calculations, in order that the programming for a computer may closely follow the original differential equations. This means that we must equate the derivatives to suitable differential quotients, a calculation that lends itself readily to incorporation as a subroutine in the programming. Although such a subroutine greatly simplifies the program-

ming wherever derivatives are required, we must still pay attention to the question of which differences are ultimately involved in the end product of the calculation. This is a matter of some importance in the treatment of equations (5. 3) and (5. 4).

For seeing how the pattern of differences affects the result, a preliminary discussion of an elementary heat flow problem will be useful. Let us take the problem of one-dimensional heat flow in a uniform solid bar with constant conductivity k and specific heat C (both per unit length). The equation of heat flow for this case can be expressed in the following classic form:

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2} \tag{15. 1}$$

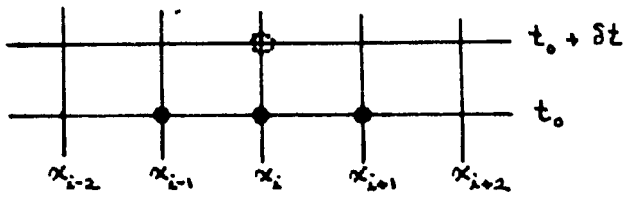
where

$$\alpha^2 = \frac{k}{C} \tag{15. 2}$$

To calculate the space derivative $\frac{\partial^2 T}{\partial x^2}$ we require at least three values of T. More than three points might conceivably be used, but the simplest reduction of (15. 1) to finite differences is the following:

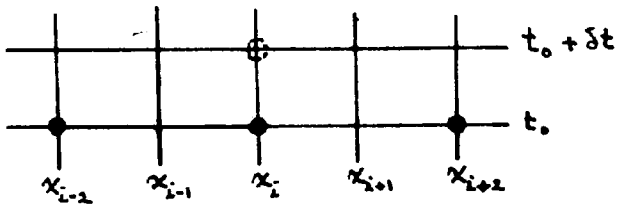
$$T(x_i, t_0 + \delta t) = T(x_i, t_0) + \alpha^2 \left[T(x_{i+1}, t_0) + T(x_{i-1}, t_0) - 2T(x_i, t_0) \right] \tag{15. 3}$$

This formula may be represented schematically by the following grid point pattern, which indicates that the new value of T at the point x_i is based on the old values at x_{i-1}, x_i, and x_{i+1} :



Pattern A

The same differential equation might be translated into other grid point patterns, but they would in general be less satisfactory than the above. A type of pattern whose shortcomings particularly concern us is the following:



Pattern B

It will be seen presently how this pattern may be achieved in a seemingly straightforward analysis. The point to be noticed is that Pattern B provides for no interaction between adjacent grid points, and hence it permits a wavy temperature distribution to develop.

Turning now to the stellar problem we find that a number of other quantities besides the temperature have space derivatives, and these derivatives also enter into the calculation. The principle still applies however that if ripples are to be avoided in the temperature distribution, the information from which $\frac{\partial S}{\partial t}$ is calculated should be based on the immediately adjacent grid points. This is especial-

ly true of the temperature information.

16. For the main body of the star we have adopted the following scheme of calculation, which is in accordance with the principle just discussed. It will be seen that the thermal grid point pattern is like pattern A.

a) Calculate the space derivatives $\frac{\partial P_r}{\partial m}$, $\frac{\partial^2 P_r}{\partial m^2}$, $\frac{\partial H}{\partial m}$, and $\frac{\partial r}{\partial m}$ by a method that makes use of the central point and the adjacent point on each side. The formulae used are for the general case of non-uniform spacing, viz:

$$f'(m_i) = \frac{f_{i+1} - f_i}{m_{i+1} - m_i} \cdot \frac{m_i - m_{i-1}}{m_{i+1} - m_{i-1}} + \frac{f_i - f_{i-1}}{m_i - m_{i-1}} \cdot \frac{m_{i+1} - m_i}{m_{i+1} - m_{i-1}} \tag{16.1}$$

$$f''(m_i) = \frac{2}{m_{i+1} - m_{i-1}} \left[\frac{f_{i+1} - f_i}{m_{i+1} - m_i} - \frac{f_i - f_{i-1}}{m_i - m_{i-1}} \right] \tag{16.2}$$

b) Calculate the heat current density H by equation (5.3).

c) Calculate L(m), the total energy flux through a spherical surface, by the formula:

$$L(m) = 4\pi r^2 H \tag{16.3}$$

d) Calculate the space derivative of L(m) by the formula:

$$\frac{\partial L(m)}{\partial m} = L(m) \left[\frac{4}{r} \frac{\partial r}{\partial m} - \frac{1}{\kappa} \frac{\partial \kappa}{\partial m} + \frac{1}{\frac{\partial P_r}{\partial m}} \frac{\partial^2 P_r}{\partial m^2} \right] \tag{16.4}$$

e) Calculate $\frac{\partial S}{\partial t}$ by equation (5.4), which may now be expressed as follows:

$$\frac{\partial S}{\partial t} = -\frac{1}{T}\frac{\partial L(m)}{\partial m} \qquad (16.5)$$

17. Near the center and the surface of the star the method just described cannot be employed, for the reason that $\frac{\partial r}{\partial m}$ varies too rapidly to be a useful derivative. An attempt to employ that method near the surface was found in some cases to give values of $\frac{\partial S}{\partial t}$ which were even of the wrong sign. In the regions q = .97 to q = 1 and q = .001 to q = .08 we rely therefore on a direct numerical differentiation of the flux itself. The procedure is as follows:

a) Calculate $\frac{\partial P_r}{\partial m}$ by formula (16.1), just as in the first method.

b) Calculate L(m) by equations (5.3) and (16.3), also as in the first method.

c) Calculate $\frac{\partial L(m)}{\partial m}$ by formula (16.1), thus differentiating the result of a previous differentiation.

d) Calculate $\frac{\partial S}{\partial t}$ by equation (16.5).

The shortcoming of this method is that it violates the principle of using adjacent points. If the thermal grid points are evenly spaced, as they are throughout most of the star, the double application of a three-point differentiation formula gives second derivatives that are based on alternate points as in pattern B, and no insurance is provided against the development of ripples. Fortunately it has proved possible to employ this method in the limited central and surface regions where the first method fails, without having any ripples actually develop. The criterion for its acceptability is the empiri-

cal one that it gives smooth results that are consistent with the trend of the flux.

Considerations governing size of time step

18. It is well known that in numerical calculations of non-stationary heat flow there is a critical time interval which depends on the size of the space intervals. (See for example W. E. Milne, 1953 pp. 119 - 121.) In the one-dimensional problem discussed in section 15 the relation is:

$$\Delta t_{crit} = \frac{1}{2a^2} (\Delta x)^2. \qquad (18.1)$$

If this critical time interval is exceeded, oscillations of increasing amplitude develop in the solutions. That is, in order to ensure that oscillations are damped we must not only use differentiation methods that are in accordance with the adjacent point principle described in section 15, but we must also select time intervals that satisfy the above condition. A rough physical statement of this condition is that the time intervals must not exceed the time it takes to transport heat from one grid point to the next, because in that time the temperature gradients may be completely changed. One may think of the critical time Δt as the thermal relaxation time for the element of length Δx. The proportionality of this relaxation time to $(\Delta x)^2$ is clearly of considerable importance in planning the numerical analysis. One factor Δx may be regarded as related to the heat capacity of the element, and the other as related to the heat current produced by a given temperature difference between the end points.

This same principle of the elementary relaxation time applies to the stellar case, but there the relaxation time is different for each element or shell. The relaxation time for a shell of stellar material can be obtained from (18. 1) and (15. 2), but the conductivity k must be replaced by the radiative conductivity

$$\frac{4acT^3}{K\rho}$$

and the specific heat per unit length C must be replaced by the specific heat per unit volume ρC_p, taken at constant pressure. This gives us the following expression:

$$\Delta t_{crit} = \frac{1}{2\rho C_p} \frac{K\rho}{4acT^3} (\Delta r)^2 \qquad (18.2)$$

where Δr is the distance between adjacent points in the thermal grid. To guard against oscillations we have chosen as our time step the smallest of these relaxation times for individual shells. The shells near the surface turn out to have the shortest relaxation times, but they are not so exceptionally short as to cause a serious problem.

As a consequence of this necessity to progress in time steps corresponding to the relaxation time of an individual shell, it takes a great many time steps to cover a significant evolution of the star. The division of the star into 50 elementary shells implies that each shell has a relaxation time about 2500 times smaller than that of the star as a whole. It follows that a significant evolution of the star requires several thousand time steps.

Method of finding total energy and entropy

19. The total energy and entropy of the star are not directly involved in the calculation of a time sequence of models, but they are very useful quantities for describing the evolution. The specific energy and entropy at each grid point are found in the course of integrating the hydrostatic equations for each model, and hence the calculation of total energy and entropy is a simple matter of integrating a tabulated function. This has been done by the trapezoidal formula, which was judged to be adequate in view of the large number of grid points used. The total energy in this calculation includes gas and radiation energy plus gravitational potential energy. Similarly, the total entropy includes both gas and radiation terms.

Finding the hydrostatic model for given entropy distribution

20. The basic principles on which this calculation is based were discussed in sections 4 and 5. Given an entropy distribution -- i. e., values at all the grid points -- we want to find the hydrostatic model in order to find the distributions of temperature, density, opacity, etc. This means we must integrate numerically the hydrostatic equations (5. 1) and (5. 2) in the range $q = 0$ to $q = .99$, using the equations of state and the known entropy function to relate density to pressure.

In the range $q = .99$ to $q = 1$ the method is different, for the reasons explained in section 12. In this last shell we employ the full set of four differential equations as they are used in steady state theory, assuming an energy source of one of the simple forms (12. 1),

(12.2), or (12.3). With mass as the independent variable these three alternative heat source equations appear as follows:

$$\frac{\partial}{\partial m}(4\pi r^2 H) = C \qquad \text{(uniform heat source)} \tag{20.1}$$

$$\frac{\partial}{\partial m}(4\pi r^2 H) = CT \qquad \left(\frac{\partial S}{\partial t} = \text{constant}\right) \tag{20.2}$$

$$\frac{\partial}{\partial m}(4\pi r^2 H) = CTS \qquad \left(\frac{1}{S}\frac{\partial S}{\partial t} = \text{constant}\right) \tag{20.3}$$

One of these ÷ it makes no difference which -- is used as the third differential equation for the outermost shell. The fourth equation is (5.3).

The details of the integration method will be discussed in the next section. Here we shall consider in broader terms how the hydrostatic model is found; that is, where the integration is started, how the starting values are chosen, and how they are adjusted so as to satisfy all the boundary conditions.

It is well known that solutions of (5.1) and (5.2) which satisfy the center boundary conditions diverge strongly as the surface is approached, and vice versa. For this reason it is the usual practice to run integrations from both ends and fit them at some mid-way point. This procedure has so much to recommend it that it has been adopted for these contracting stars, although they have no discontinuities of structure and hence have no naturally preferred fitting point. This absence

of a natural fitting point is a decided advantage, for it leaves us free to choose the fitting point in the most convenient way for avoiding the loss of accuracy that accompanies the divergency of the solutions. For outward integrations the divergence stems from the fact that the pressure is decreasing toward zero, while for the inward integrations it stems from the fact that the radius is decreasing toward zero. The most advantageous fitting point is therefore one which has a nearly as possible a pressure close to the central value and a radius close to the surface value. It turns out that in most of our contracting models these conditions are both satisfied within a factor 5 at a point near $q = 0.70$. We therefore adopt $q = 0.70$ as our standard fitting point.

When it comes to actually starting the outward (core) and inward (envelope) integrations, we find that there are four free parameters which must be specified as starting values -- one for the core and three for the envelope. For the core we take P_c, the central pressure, as the parameter. For the envelope we take R, the radius; S(R), the specific entropy at the photosphere; and C, the heat source coefficient for the outermost shell. We may contrast the situation here to that of a steady state star, where there are two free parameters for the core and two for the envelope. The fact that the entropy is specified in our problem eliminates one degree of freedom in the core and hence we have only one parameter there. But in the envelope, instead of removing a degree of freedom by specifying the entropy we have added a degree of freedom by introducing an arbitrary heat

source. This accounts for the three envelope parameters. We could if we liked have used the luminosity or the effective temperature instead of S(R).

By starting the integration from both ends we automatically satisfy all the boundary conditions, but we now have to consider certain continuity conditions instead. If the starting values are randomly selected there will be hydrodynamic discontinuities at q = .70 and thermodynamic discontinuities at q = .99. We refer to these points as the hydrodynamic and thermodynamic fitting points, respectively. The model desired is the one in which all discontinuities are removed at these points; that is, the one in which:

a) P is continuous at the hydrodynamic fitting point;

b) r is continuous at the hydrodynamic fitting point;

c) S is continuous at the thermodynamic fitting point;

d) H is continuous at the thermodynamic fitting point;

For our purposes we consider conditions (a), (b), and (d) to be satisfied when the discontinuity is less than .001 of the function itself. In the case of condition (c) we require that the discontinuity in S be less than .001 in the scaled units that we employ. (These are units in which S is the same as S/$\mathcal{R}$N.) The reason for this distinction is that the entropy is dominated by the gas term in all the situations we have to deal with, and this term is already a logarithm.

The procedure for adjusting the parameters to satisfy the fitting conditions is somewhat involved in detail, but in essence what we do is

to vary them one by one, redoing the integrations each time to see how each parameter affects the four discontinuities. This gives us a set of derivatives of the form

$$\frac{\partial (\text{discontinuity})_i}{\partial (\text{parameter})_j}$$

from which we calculate improved values of the parameters by linear interpolation or extrapolation. In the initial model of a sequence the eigenvalues have to be guessed, and if the guess is not a good one it may take many tries to reach the correct values. In later models however we can take advantage of the solutions already found, and after a few time steps it has generally proved possible to predict the eigenvalues closely enough to satisfy the fitting conditions on the first try. This prediction is made by fitting the best straight line to the ten preceding solutions of the sequence. The prediction and adjustment of parameters are incorporated as part of the overall program of calculation, so that the computing machine proceeds automatically from one model to the next. For the IBM 7090 computer the rate of progress is 10 to 12 models per minute.

Method of hydrostatic integration

21. In choosing a method for numerically integrating the two differential equations (5.1) and (5.2) which govern the pressure distribution, and for integrating the two additional thermodynamic equations in the case of the outermost layer, we have preferred simple and direct methods to those that achieve economy of integration steps at

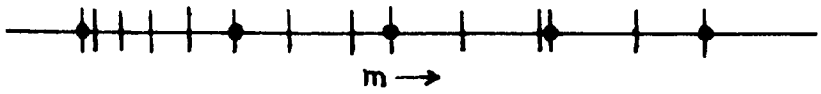
the price of complexity. The reason for this preference is to obtain programming simplicity and to arrange as far as possible that the finished program in a symbolic language (Fortran) resembles the original physical equations. It is easier to instruct a machine to go many times through the same relatively simple routine than to instruct it to go fewer times through a more complicated routine.

Besides simplicity, we require a knowledge of the accuracy of our solutions and a method for ensuring that any prescribed accuracy is attained. In accordance with these and other objectives we have devised a method whose essential features are as follows:

a) Direct physical variables are used, rather than logarithms or some other transformation. Units of pressure, temperature, etc. are chosen so as to eliminate constant factors from the equations and at the same time to avoid large powers of ten in the variables. This gives us something like Schwarzschild's scaled envelope variables, except that in the present case there is no intention of homologous transformations and all the scale factors are known in advance.

b) A distinction is made between the thermal grid interval, used in the calculation of heat flow, and the interval used in the hydrostatic integration. The latter we shall call a hydrostatic integration step. The hydrostatic integration steps are variable and are generally smaller than the thermal grid intervals. They have nothing to do with the the thermal grid, except that it is arranged that the fixed thermal grid points come at the ends of hydrostatic integration steps

rather than in the middle. This is to ensure that the physical variables are found at each thermal grid point in the course of the hydrostatic integration. A typical situation is shown schematically in the following sketch, where the filled circles represent thermal grid points and the vertical cuts represent the end points of hydrostatic integration steps.



c) The increments ΔP, Δr, etc. for each integration step are calculated by a Taylor series expansion of the two terms, in which the first term is the differential equation itself and the second term is obtained by analytic differentiation of the equation. This method is made possible by the fact that the differential equations are all explicit first order equations.

d) The errors inherent in this method are kept smaller than any prescribed fraction of the functions themselves by a suitable choice of the integration step. This step is always made to be as large as possible consistent with the accuracy requirements, but it is not permitted to overshoot the next thermal grid point in line. We shall illustrate in section 22 how this is done.

e) Since the hydrostatic integration steps are generally smaller than the thermal grid intervals, a knowledge of the entropy function between thermal grid points is required when we are doing the hydrostatic integration. For this purpose the entropy is approximated by an

interpolation scheme which puts a smooth curve through all the thermal grid points -- smooth in the sense that it has continuity of first and second derivatives and has little tendency to introduce spurious ripples.

It might be asked what is gained by subdividing the thermal grid interval into a number of smaller steps for hydrostatic integration, when the integration depends on an arbitrary interpolation of the entropy function. The answer is that the pressure changes much more drastically than the entropy between successive grid points, the latter being a very gradually increasing function of m. For example, an adiabatic model with S = constant is a well known possibility, but there is no possible model with constant pressure. Any arbitrary smooth interpolation method will therefore give considerably better values of S than of P, and the best way to find the pressure distribution is to integrate accurately on the basis of the interpolated entropy.

22. As an illustration of the detailed integration procedure, let us consider the pressure equation. We shall use the scaled form of the equation with Schwarzschild's notation, in which m becomes q, r becomes x, and P becomes p. Equation (5. 1) takes on the following appearance in these variables:

$$\frac{dp}{dq} = -\frac{q}{x^4} \qquad (22.\ 1)$$

This equation serves as a formula for finding the first term in the Taylor series for Δp. To find the second term we differentiate

(22. 1), obtaining:

$$\frac{d^2p}{dq^2} = \frac{dp}{dq}\left[1 - \frac{4}{x}\frac{dx}{dq}\right] \tag{22.2}$$

With these two formulae we calculate the increment Δp for any Δq, thus:

$$\Delta p_1 = \frac{dp}{dq}\Delta q = -\frac{q}{x^4}\Delta q \tag{22.3}$$

$$\Delta p_2 = \frac{1}{2}\frac{d^2p}{dq^2}(\Delta q)^2 = \frac{1}{2}\Delta p_1\left(1 - \frac{4}{x}\frac{dx}{dq}\right)\Delta q \tag{22.4}$$

$$\Delta p = \Delta p_1 + \Delta p_2 \tag{22.5}$$

The integration step Δq is to be chosen in such a way that the errors inherent in the truncation of the Taylor series do not exceed ϵp, where ϵ is a specified accuracy parameter. It is judged to be sufficient for this purpose if the two terms in the series satisfy the following conditions:

$$\left|\frac{\Delta p_2}{p}\right| \leqq \epsilon \tag{22.6}$$

$$\left|\frac{\Delta p_2}{\Delta p_1}\right| \leqq \frac{1}{2} \tag{22.7}$$

In the actual situations encountered the second condition is satisfied much more easily than the first, so that there can be no practical doubt that the series are strongly converging.

For each integration step we have chosen the largest Δq that satis-

fies these accuracy conditions for all of the equations being integrated; that is, the largest Δq that satisfies the accuracy conditions for two equations in most of the star and the largest that satisfies the accuracy conditions for four equations in the outermost shell. In most cases we have fixed the accuracy parameter ϵ at 0.002, so that the second term in the series is of this magnitude compared to the function itself. Since the series is strongly converging, the terms neglected should amount to considerably less than 1 part in 1000 of the function's running value.

In a typical model there are 75 to 120 hydrostatic integration steps required between q = .99 and q = 1, where the pressure changes by 6 or 7 powers of 10. Other intervals of the thermal grid are subdivided into 2 or 3 hydrostatic integration steps throughout most of the model, and as many as 10 to 15 near the boundaries.

Special procedures for the center

23. In all problems of hydrostatic integrations in stars one must devise special procedures for the central region because of the singularities in those equations which have m or r in the denominator. The usual procedure is to express everything as power series in r in this region and to work out analytically the coefficients that satisfy the differential equations.

A similar procedure is adopted here, but in keeping with the choice of mass as the independent variable we develop the series in terms of the scaled mass variable q. The general form of the expansion used,

for a function f, is:

$$f(q) = a_0 q^p \left[1 + a_2 q^{2/3} + a_2 a_4 q^{4/3}\right] \qquad (23.1)$$

The exponent p of the leading term depends of course on the function that is being expanded. A series in $q^{2/3}$ is equivalent to a series in r^2. The occurrence of these exponents reflects the fact that all functions of interest in a spherically symmetric star are either even or odd functions of r.

Corresponding to every quantity that is calculated in the main body of the star we find a series with three coefficients for the center. A feature which is perhaps novel in this work is that most of the series expansions have not been worked out analytically. Instead, a group of subroutines was developed that enable the computing machine to calculate the coefficients numerically whenever a new series is needed. For example, in the calculation of the heat flow near the center, we evaluate the quantity $4\pi r^2 H$ for use in equation (5.4). The series coefficients for r, κ, and P_r are already known, and our task is to find a series for $4\pi r^2 H$ in accordance with equation (5.3):

To do this we call four subroutines in turn:

a) a subroutine "SEXP" which operates on the series coefficients for r to produce the series coefficients for r^4;

b) a subroutine "SDERIV" which operates on the series coefficients

for P_r to produce the series coefficients for $\frac{\partial P_r}{\partial m}$;

c) a subroutine "SPROD" which calculates the series coefficients for the product of the previous two series;

d) a subroutine "SQUOT" which calculates the quotient of the last result by the series for K .

The region in which series have been used comprises .001 of the stellar mass; that is, they have been used in the region q = 0 to q = .001. Outside that region calculations have been done in the normal way; that is, by numerical methods based on finite differences. With one exception the use of the series is entirely straightforward and devoid of special points of interest. The one exception occurs in the example given above, where a numerical differentiation of P_r is required for the ultimate purpose of finding $\frac{\partial S}{\partial t}$. It is evident that the series expansions for P_r and for all other state variables such as T, ρ , S begin with a constant term, and from this it follows that when they are differentiated the number of terms in the series is diminished by one. In other words, when the series expansions for all variables are known to three terms, the calculated series for $\frac{\partial S}{\partial t}$ can have only two terms. The third term can be recovered only by taking into account values that have been calculated in the normal way at points outside the series region. This of course is not simply a mathematical inconvenience. It is an expression of the physical fact that the evolution of the central core cannot be calculated by itself without reference to the surrounding region.

Our procedure therefore is to calculate the first two terms of the $\frac{\partial S}{\partial t}$ series by means of the special subroutines mentioned above, this being equivalent to an analytical development. The third term we find by fitting the series to the calculated value of $\frac{\partial S}{\partial t}$ at q = .002.

When the "zero boundary conditions" (P = 0, T = 0) are used for the surface, a series development is required for that region as well as for the center. By avoiding these boundary conditions in our approach we eliminate the necessity for the surface series expansions.

CHAPTER IV

RESULTS AND DISCUSSION

24. Numerical calculations of evolutionary sequences were done for stars of ten solar masses and of the following composition by weight: hydrogen 90%, helium 9%, heavier elements in the Russell proportion 1%. The main sequence models for stars of this same mass and composition have been investigated by Kushwaha. The large mass was selected primarily in order to avoid the uncertainty with respect to outer convection zones that occurs at lower masses. A composition somewhat richer in helium and metals might be considered more interesting in connection with present-day stars, but on the other hand it is useful to have the main sequence models available for comparison. In this chapter we shall first present the results of the calculations and then discuss several aspects of stellar contraction which the calculations suggest or illustrate.

As the most fundamental measure of a star's evolutionary progress we take the total energy. This consists of kinetic energy of the ionized gas, energy of the radiation field inside the star, and gravitational potential energy. The latter has its zero point defined in the usual way so that the total energy is negative for a gravitationally bound system. As a star contracts its energy progresses to higher and higher negative values. We shall refer to a doubling of this negative total energy as one Kelvin time in the life of a star. The actual time required depends on how fast energy can escape from the

star, but this time rate of evolution is of secondary concern for us. The most meaningful comparison of different evolutionary paths, for the purpose of studying their convergence, will be one that disregards the time scale and focuses on the differences between models of the same total energy.

The initial models

25. Taking the mass and composition described above as standard, we consider four different initial configurations of the star and trace the subsequent evolution in each case. The overall properties of these four initial models are given in Table 1.

Table 1. Overall Properties of the Initial Models

Sequence	log(-E)	S/kN	E/E_{ms}	R/R_{ms}	L/L_{ms}
L	48.761	16.960	.0835	12.10	0.427
G	48.741	17.435	.0798	21.70	2.660
I	48.771	17.539	.0855	30.77	4.700
J	48.960	17.467	.1321	73.06	13.300

The successive columns in this table give: 1) a letter designation for the sequence; 2) logarithm of the total energy in ergs; 3) the total entropy per particle (N = number of nuclei plus number of electrons); 4) ratio of total energy to the main sequence value; 5) ratio of radius to the main sequence value; 6) ratio of luminosity to the main sequence value.

The first three models in the table all have practically the same energy, and they are therefore considered to be at the same stage of

evolution. There are considerable differences between them in entropy, radius and luminosity, due to the fact that they have different temperature distributions. The initial model of sequence J differs from the other three in a striking way that has nothing to do with the slightly different energy value. This may be seen in Figure 3, where we show the four temperature distributions all normalized to the same total energy by taking the mean temperature in each model as a scale factor. These normalized temperatures are shown as functions of the mass variable $q = m/M$, and hence the curves show directly the distribution of thermal energy in the stars.

Further details of the initial models are given in Table 2, where c. g. s. units are employed unless otherwise indicated. Logarithms in all our tables and diagrams are on the base 10. Most of the column headings in Table 2 are self-explanatory. κ is the mass absorption coefficient, including scattering. n is the polytropic index, defined below.

These initial models were chosen not for any individual significance which they are thought to possess, but for the variety which they represent with respect to temperature distribution and the energy-entropy relationship. None of them is a model which has previously been considered in stellar structure problems, except for model L which is close to the Emden polytropic model of index 2.5. Since we are investigating the general trend of thermal relaxation in stars it should not matter what models we start with, as long as they repre-

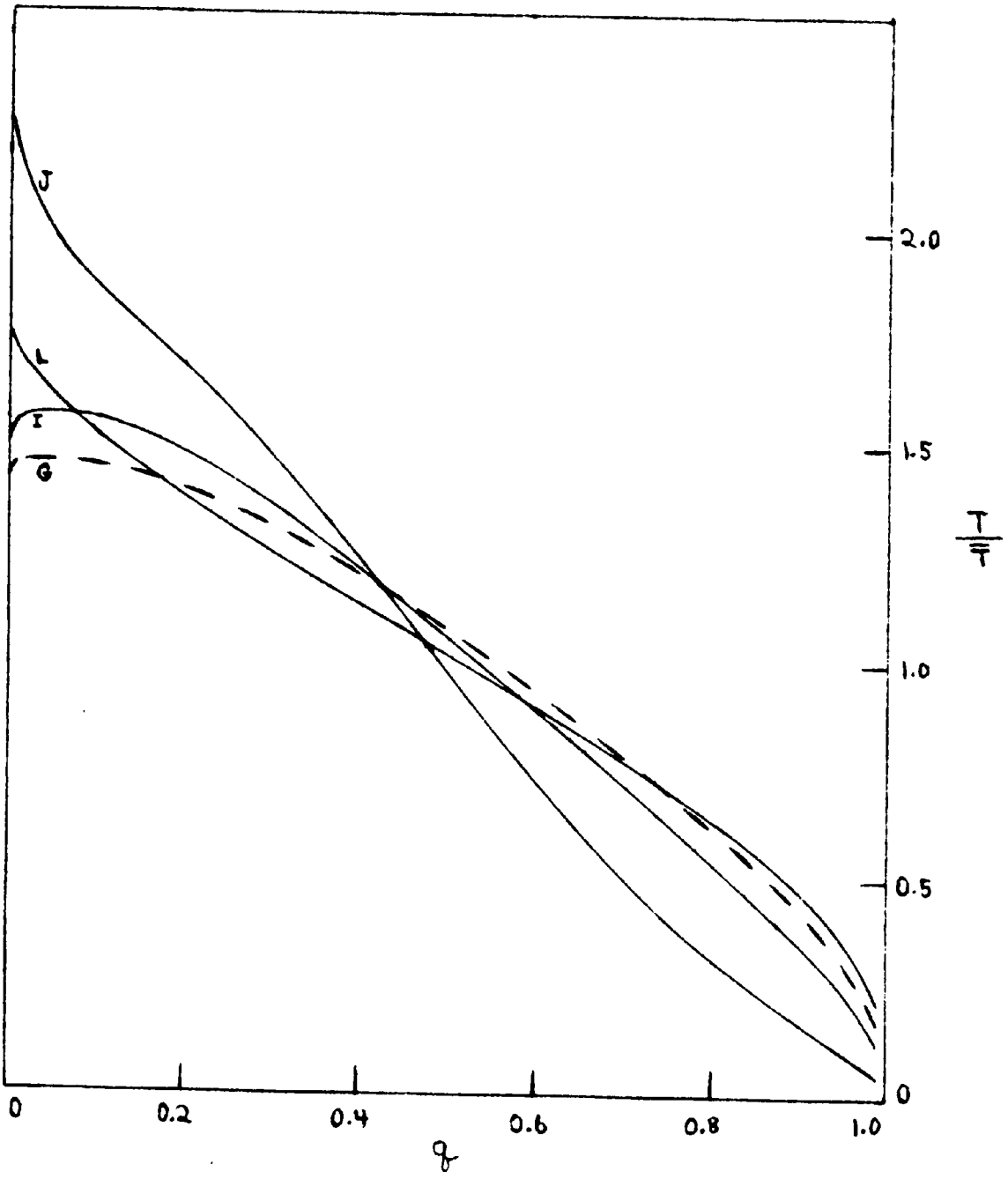


Figure 3. Temperature as function of the mass variable q in the four initial models. The ordinate is the ratio of local temperature to mean temperature.

Table 2. Detailed Properties of the Initial Models

Sequence L

q	$r/R_\odot$	S	log T	log ρ	log P	P_r/P	κ	$L/L_\odot$	n
0	0	16.328	6.329	-2.236	12.300	.0262	0.650	0	2.41
.10	6.70	16.477	6.269	-2.390	12.085	.0246	.691	521	2.56
.20	8.86	16.567	6.226	-2.500	11.932	.0236	.723	950	2.59
.30	10.64	16.668	6.184	-2.607	11.783	.0227	.755	1303	2.61
.40	12.29	16.776	6.141	-2.720	11.626	.0219	.791	1590	2.63
.50	13.95	16.895	6.094	-2.844	11.454	.0210	.833	1800	2.65
.60	15.71	17.034	6.040	-2.987	11.258	.0202	.884	1940	2.66
.70	17.71	17.203	5.976	-3.159	11.020	.0192	.950	1990	2.67
.80	20.16	17.430	5.891	-3.386	10.708	.0181	1.045	1920	2.71
.90	23.72	17.804	5.761	-3.747	10.217	.0169	1.198	1680	2.83
.99	32.20	19.027	5.411	-4.802	8.812	.0171	1.559	1280	3.19
1.00	43.93	23.927	3.722	-9.481	2.440	.0071	0.362	1280	7.00

Sequence G

q	$r/R_\odot$	S	log T	log ρ	log P	Pr/P	κ	$L/L_\odot$	n
0	0	14.387	6.224	-1.592	12.828	.0030	3.317	0	-2.19
.10	4.74	15.470	6.228	-2.047	12.380	.0086	1.367	9	33.30
.20	6.93	16.099	6.212	-2.333	12.081	.0148	.952	123	12.02
.30	9.08	16.616	6.185	-2.585	11.805	.0217	.772	521	7.39
.40	11.38	17.064	6.147	-2.834	11.532	.0288	.683	1370	5.41
.50	13.97	17.466	6.099	-3.059	11.251	.0352	.638	2690	4.34
.60	16.98	17.841	6.038	-3.303	10.949	.0404	.618	4330	3.71
.70	20.66	18.213	5.962	-3.573	10.604	.0441	.618	6030	3.33
.80	25.50	18.623	5.860	-3.899	10.178	.0462	.635	7360	3.12
.90	33.01	19.190	5.706	-4.374	9.549	.0475	.679	8110	3.06
.99	52.65	20.679	5.306	-5.612	7.913	.0518	.821	7750	3.20
1.00	78.77	25.722	3.789	-10.191	1.913	.0443	.361	7980	7.00

Sequence I

q	$r/R_\odot$	S	log T	log ρ	log P	Pr/P	κ	$L/L_\odot$	n
0	0	13.195	6.280	-0.994	13.482	.0011	7.931	0	-16.77
.10	3.31	14.744	6.290	-1.644	12.844	.0052	1.925	4	38.70
.20	5.22	15.652	6.269	-2.059	12.411	.0117	1.077	82	12.85
.30	7.36	16.408	6.232	-2.425	12.012	.0209	.774	472	7.99
.40	9.91	17.073	6.183	-2.768	11.624	.0322	.640	1530	5.88
.50	13.05	17.673	6.120	-3.100	11.236	.0443	.573	3530	4.70
.60	16.98	18.225	6.044	-3.432	10.832	.0556	.541	6450	3.96
.70	22.07	18.749	5.950	-3.783	10.392	.0646	.530	9860	3.46
.80	29.11	19.279	5.829	-4.184	9.872	.0700	.539	12900	3.16
.90	40.43	19.937	5.660	-4.734	9.144	.0721	.585	14600	3.03
.99	71.09	21.526	5.213	-6.071	7.372	.0762	.757	13500	3.15
1.00	111.70	26.528	3.775	-10.396	1.610	.0781	.361	14100	7.00

Sequence J

q	$r/R_\odot$	S	log T	log ρ	log P	Pr/P	κ	$L/L_\odot$	n
0	0	12.585	6.638	-0.191	14.643	.0020	3.024	0	3.67
.10	1.63	13.468	6.556	-0.693	14.060	.0037	1.974	10	9.02
.20	2.54	14.341	6.511	-1.135	13.575	.0074	1.207	61	9.44
.30	3.71	15.274	6.455	-1.609	13.049	.0150	.804	336	7.65
.40	5.44	16.258	6.380	-2.125	12.463	.0287	.610	1410	5.98
.50	8.16	17.282	6.277	-2.684	11.811	.0500	.518	4310	4.84
.60	12.62	18.336	6.145	-3.285	11.081	.0777	.475	10000	4.13
.70	20.18	19.406	5.983	-3.929	10.299	.1080	.456	18700	3.67
.80	33.59	20.481	5.784	-4.634	9.409	.1346	.454	29100	3.32
.90	60.32	21.610	5.518	-5.499	8.295	.1500	.472	37800	3.06
.99	146.20	23.419	4.955	-7.150	6.076	.1400	.502	39700	2.48
1.00	265.20	28.796	3.701	-11.146	0.859	.2214	.361	39900	7.00

sent a reasonable variety in terms of thermal structure.

26. To elucidate the variety which our models represent we shall compare them with the well-known polytropic models of Emden. These models were widely used as reference models in the early days of stellar structure analysis, because by taking different values of an index n one could construct a family of models whose properties varied in a systematic way. For stars composed of perfect gas with negligible radiation pressure, n has the significance that $\frac{\rho}{T^n}$ and $\frac{P}{T^{n+1}}$ are constant throughout the star. The values n = 1.5 and n = ∞ are cases of special interest, the former being an isentropic or adiabatic model (in the case of a monatomic gas) and the latter being isothermal. Values of n between -1 and +1.5 correspond to "superadiabatic" models; that is, models with entropy decreasing outward. Such models can not be considered as physically possible ones, for they would be unstable against redistribution of the material through convective motions. Values of n between -1 and - ∞ correspond to inverted temperature gradients (cooler at the center than at the surface) and hence would not be expected to occur except possibly as transient phenomena in limited regions. The physically interesting models are therefore the ones with n between 1.5 and ∞ . A complication arises with values n = 5 and higher, in that these models have to be terminated with a finite pressure rather than zero pressure at the boundary, but this should not be regarded as a physical restriction against the existence of stars built on these models.

It might be supposed that the Emden models with n between 1.5 and ∞ would represent the entire range of physically interesting temperature distributions, and that the n = 1.5 model would represent an upper limit as to how steep the temperature distribution can become without being convectively unstable. This is true enough if we confine our attention to temperature-pressure or temperature-density diagrams, but it is misleading as far as the distribution of temperature over mass is concerned. Figure 4 shows the situation from that point of view. In the range n = 1.5 to n = 4.5 the differences are disappointingly small in this diagram. We could obtain flatter curves by taking larger values of n, up to the limit n = ∞ which would be completely flat, but we could not get steeper curves with any Emden model that is stable against convection. In the Emden models the ratio of central temperature to mean temperature cannot exceed 1.89.

However, models with variable n can exceed this limit without introducing any superadiabatic region. Our initial model J is one of this type. This is an illustration of the fact that the class of hydrostatic models allows of a much greater variety than can be represented by a single parameter. Model J is a representative of a large class of models which have a greater central concentration of thermal energy than any of the stable Emden models. Later considerations will show that convectively stable models can in fact be arranged so as to have as high a central concentration of energy as we please. There is no upper limit to $T_c / \bar{T}$ for convectively stable models.

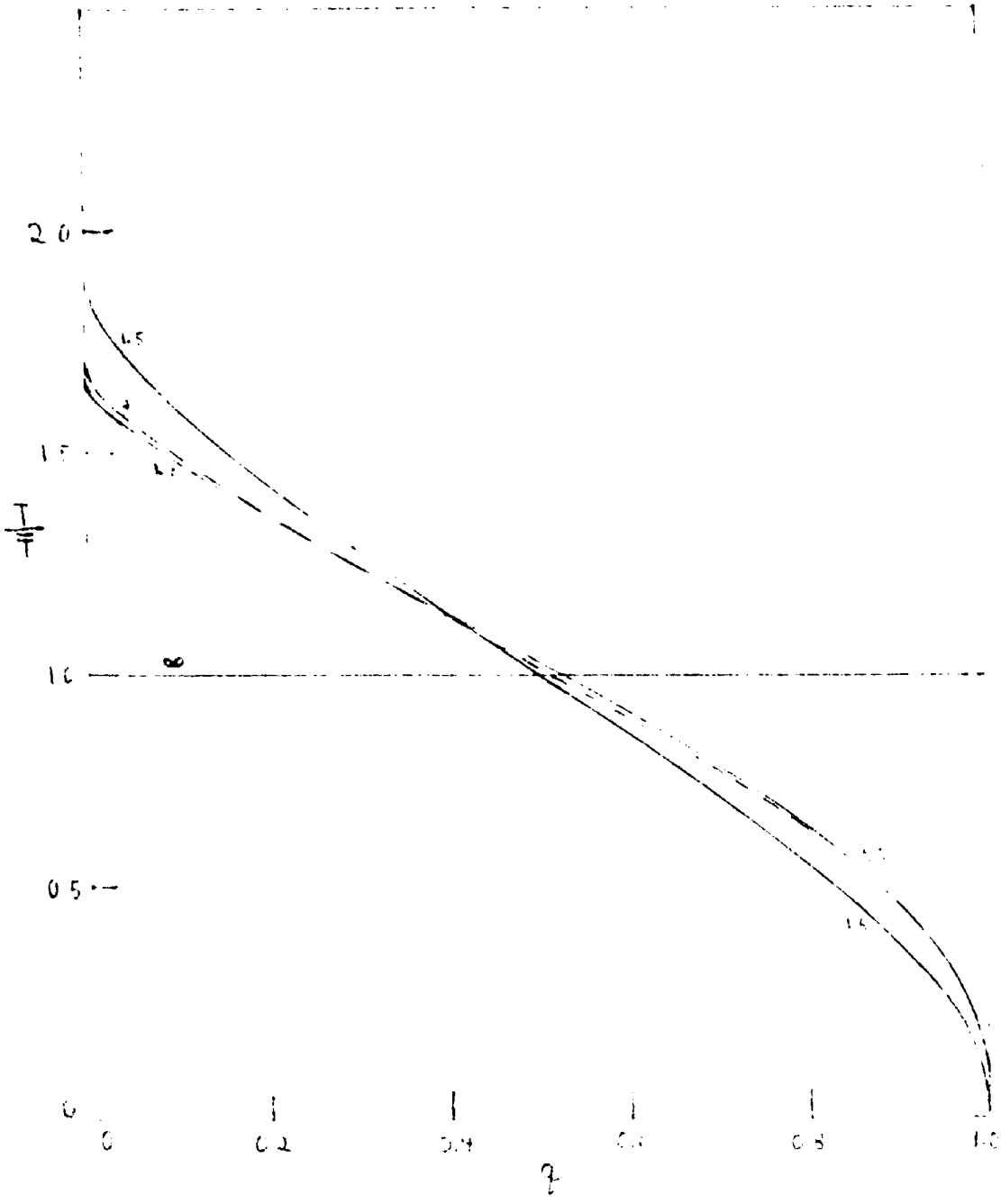


Figure 4. Temperature as function of the mass variable q in four Emden models. Each curve is labelled with the value of the polytropic index. The ordinate is the ratio of local temperature to mean temperature.

In Table 2 we list a local polytropic index in our initial models. Since these stars have significant radiation pressure n can be defined in a number of different ways, all of which agree with Emden's n in the limit of no radiation pressure. We have followed Schwarzschild in defining n as an indicator of the relative march of pressure and temperature in the star:

$$\frac{d \log P}{d \log T} = n + 1.$$

When n is defined in this way an adiabatic curve does not correspond exactly to n = 1.5. Any question as to convective stability can be settled more directly by looking at the run of entropy in the model than by looking at the n's. Our initial models were all made to have entropy increasing outward and therefore to be stable against convection.

27. The initial models G, I, and J were all generated by representing the entropy as a power series in the mass variable q. The values used were as follows:

Model G: $\frac{S}{kN} = 14.387 + 5.035q^{2/3}$

Model I: $\frac{S}{kN} = 13.195 + 7.195q^{2/3}$

Model J: $\frac{S}{kN} = 12.585 + 2.373q^{2/3} + 8.181q^{4/3}$.

A comparison between these series and the values given in Table 2 will reveal slight differences. The reason is that what we have called the initial model in Tables 1 and 2 and in Figure 3 is really a later

model in the evolutionary sequence, one that occurs 25 to 450 years later than the model that the series prescribes. This eliminates from our consideration the early and rapid readjustment of the outer layers that was described in section 12 and gives us a new initial model which has already made that readjustment. The readjustment was in general accompanied by a very large change of the star's position in the Hertzsprung-Russell diagram, but the evolution of the star as a whole in the same time was entirely negligible. In order to avoid confusion between this transient surface phenomenon and the thermal evolution of the star as a whole, we exclude the early models from our consideration altogether.

To generate the initial model L we did not use an analytic expression for the entropy as in the other three cases, but used instead a table of entropy values which corresponded approximately to an Emden polytropic model of index n = 2.5.

We consider that our four intial models represent a good variety of thermal structures. It is a variety that includes but exceeds that offered by the common Emden models n = 1.5 to n = 4.5. Our initial model J has an unusually high concentration of thermal energy toward the center compared to the Emden models or to main sequence models, but its ratio $\frac{T_c}{\bar{T}}$ = 2.24 is not so high that we should consider it a freak. Our models do not belong to any one-parameter or two-parameter family. This makes it the more likely that whatever convergent behavior they display in common should be a general

characteristic rather than a special one.

Before leaving our initial models we should notice how remarkably large are their differences in luminosity, as compared with the modest differences of temperature gradient in Figure 3. These luminosity differences must be attributed primarily to the differences in radius between the four models. The opacities, the temperature gradients, and the heat flow per cm^2 are not greatly different from one star to another, but the large differences in surface area lead to correspondingly large differences in the total energy output. Thus it is not the direct driving force of the temperature gradient that determines the luminosity, so much as the radius which the temperature distribution determines indirectly through its effect on the pressure-density relationship.

Evolutionary sequences

28. In Figures 5 through 15, we show various aspects of the contractive evolution of the four initial models. The most informative of these diagrams from a thermodynamical point of view is Figure 5, where we have plotted the total entropy of the stars (per particle) against the logarithm of their total energy. The evolutionary tracks run from upper left to lower right in this diagram, as the stars are losing both energy and entropy by radiation into space. Filled circles in this and other diagrams represent models that have certain standard energy values, evenly spaced on a logarithmic scale. The models with these energies are referred to as stages 1 through 6 of

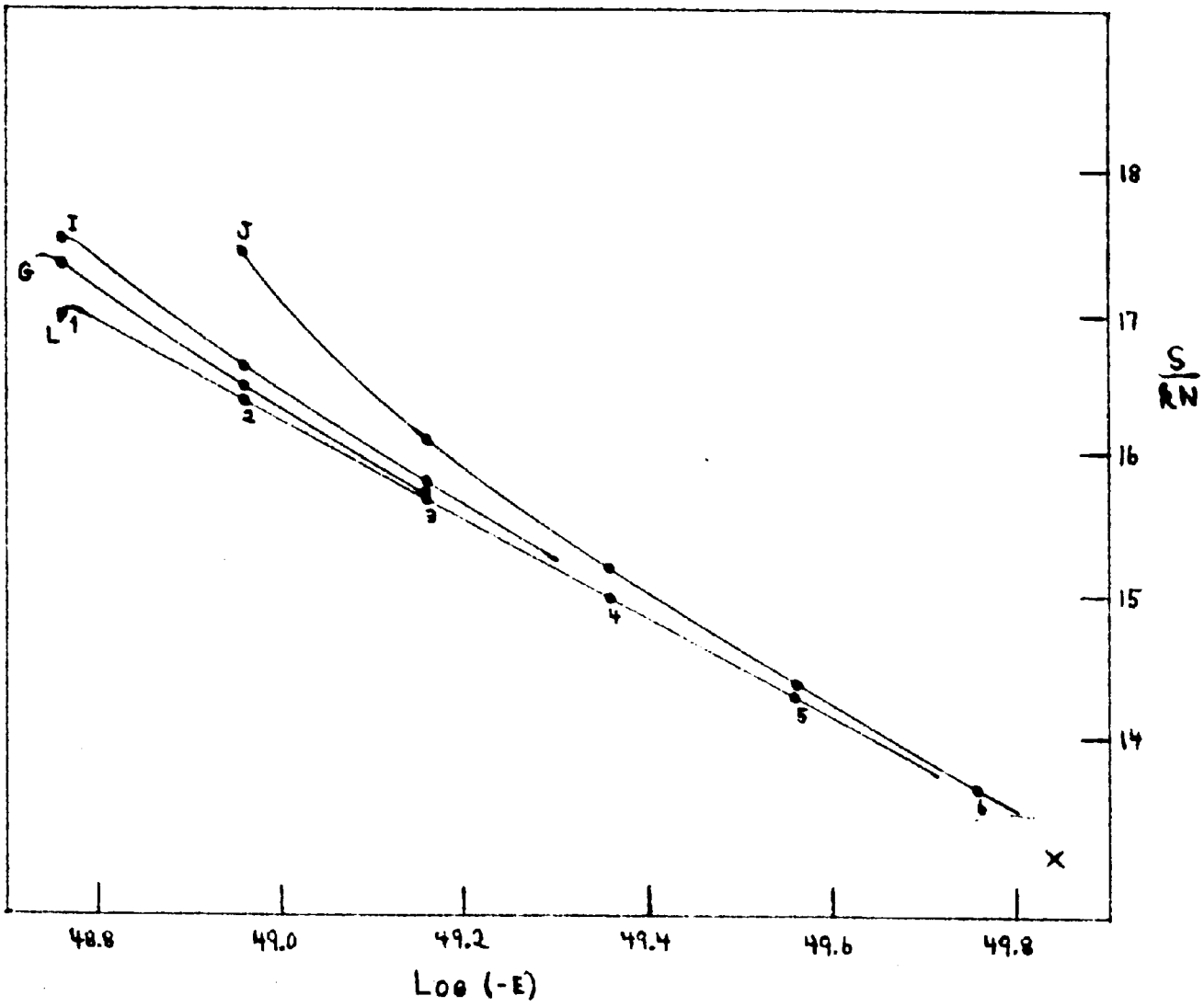


Figure 5. Evolutionary tracks in the energy–entropy diagram. E is the total energy and S is the total entropy of the star as defined in the text. The filled circles are the standard energy stages. The cross at lower right represents Kushwaha's main sequence model.

the evolution, according to the following scheme:

Table 3. Energies of Standard Evolutionary Stages

	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6
log(-E)	48.76	48.96	49.16	49.36	49.56	49.75
E/E_{ms}	0.0833	0.132	0.209	0.332	0.526	0.834

To go from one of these standard energy values to the next required about 1600 time steps in the calculation. In terms of the Kelvin time, which we have defined as a change by factor 2 in the total energy, the spacing between the standard stages is 2/3 Kelvin time. Detailed properties of these standard energy models will be presented in further diagrams, where we will have an opportunity to compare the structure of the different stars at the same total energy.

It will be seen that sequence J was started at a somewhat later evolutionary stage than the others, and that the four sequences were not all carried to the same stopping point. The general rule followed was to terminate a sequence either when it reached the main sequence energy (indicated by the cross in Figure 5) or when it became indistinguishable from some other sequence. Sequence L, the longest of the four, covers a factor 9.5 in the total energy.

Contrary to what was said in section 27 about excluding the first few time steps in each sequence, we have made an exception in Figure 5 and included these early readjustment phases in the diagram because the contrast between these phases and the main evolutionary trend illustrates nicely what is going on from the point of view of the second

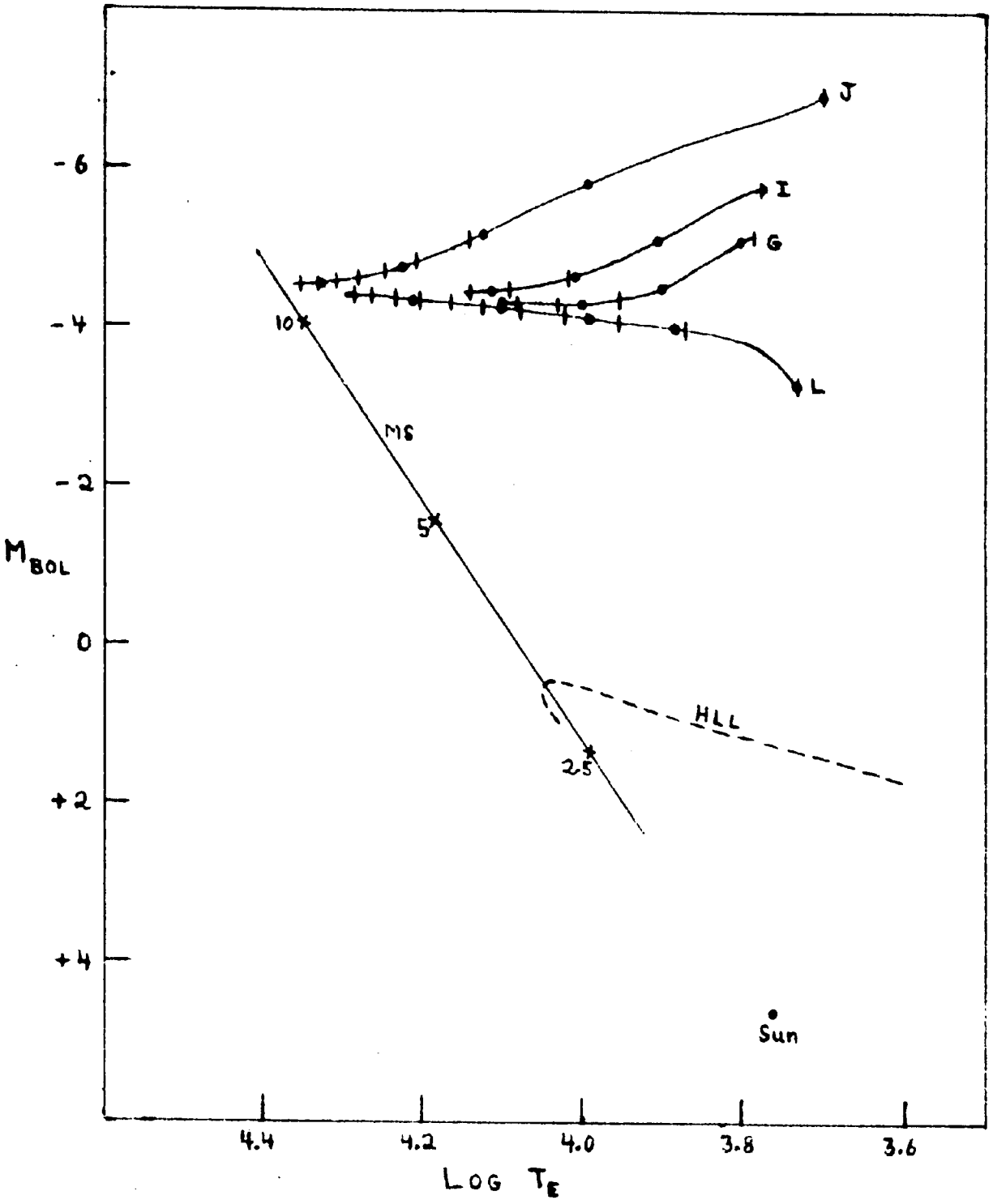


Figure 6. Evolutionary tracks in the Hertzsprung-Russell diagram. The tracks are divided into 10,000 year segments by the vertical lines. The line MS is Kushwaha's main sequence, with three masses indicated. The line HLL is one of the evolutionary tracks of Henyey, LeLevier, and Levée.

law of thermodynamics. We have stated that during most of the star's contraction its total entropy decreases. This is not in disagreement with the second law, for the star is not a closed system. It is possible to think in terms of a closed system comprising the star plus the rest of the universe, and from this point of view the contraction may be thought of as an attempt of the combined system to attain thermodynamic equilibrium. The main feature of this process is the transfer of energy from the hot star to the cold surrounding space. This exchange lowers the entropy of the star but raises the entropy of the surrounding space by a larger amount, so that the second law is satisfied for the combined system. At the beginning of each evolutionary track however the entropy of the star rises for a short while before the secular decrease sets in. This is the period where the outer layers are adjusting to constant flux conditions. This period may be regarded as an interlude in which the striving toward thermodynamic equilibrium is dominated by changes within the star instead of by interaction between the star and the rest of the universe.

For the present investigation the most interesting feature displayed by the tracks in Figure 5 is their convergence. There is no doubt that they converge in a systematic way. We shall postpone a discussion as to how this convergence ought to be described, however, until we have presented some other aspects of the evolution.

Figure 6 shows the evolutionary tracks in the theoretical

Hertzsprung-Russell diagram. The main sequence is the one calculated by Kushwaha (1957) for stars of the same composition, with the positions for three masses indicated by crosses. The position of the sun is also shown. These features are included primarily to illustrate the scale of the diagram. The filled circles on the evolutionary tracks represent the standard energy stages as in Figure 5. The time scale is indicated by the vertical cuts, which divide each track into 10,000 year segments. The track labelled HLL is one of the nearly homologous sequences calculated by Henyey, LeLevier and Levée (1955). It refers to a different mass and composition from our stars, and is included in the diagram for qualitative comparison only.

Figures 7 through 10 show the detailed development of the temperature distributions for each evolutionary sequence. The stages shown are the standard energy stages of Table 3, insofar as they occur in each sequence. In these diagrams the temperature scale is logarithmic, so that homologous contraction would be represented by a sequence of identical curves with equal vertical displacement between them. It will be seen that sequence L is not far from meeting this condition, while sequence J departs most drastically from it. Sequence L is therefore the most nearly homologous of the four, and sequence J is furthest from being homologous.

It so happens that sequences J and L also appear as the two extreme sequences in most of our other diagrams; for example, J

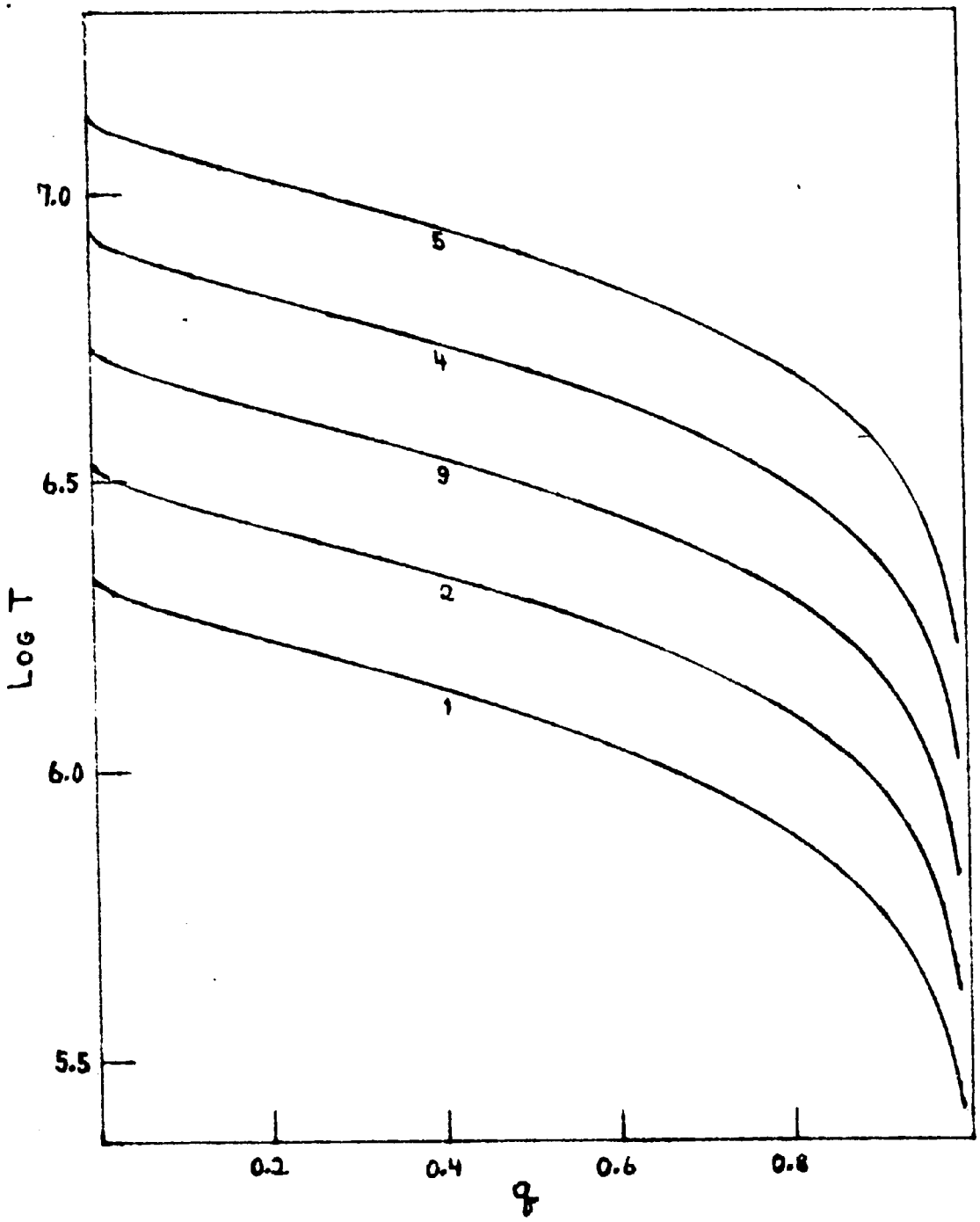


Figure 7. Successive temperature distributions in sequence L. The temperature is shown as function of the mass variable q, for five models in the sequence. These are the standard energy models of Table 3, identified by the number under each curve.

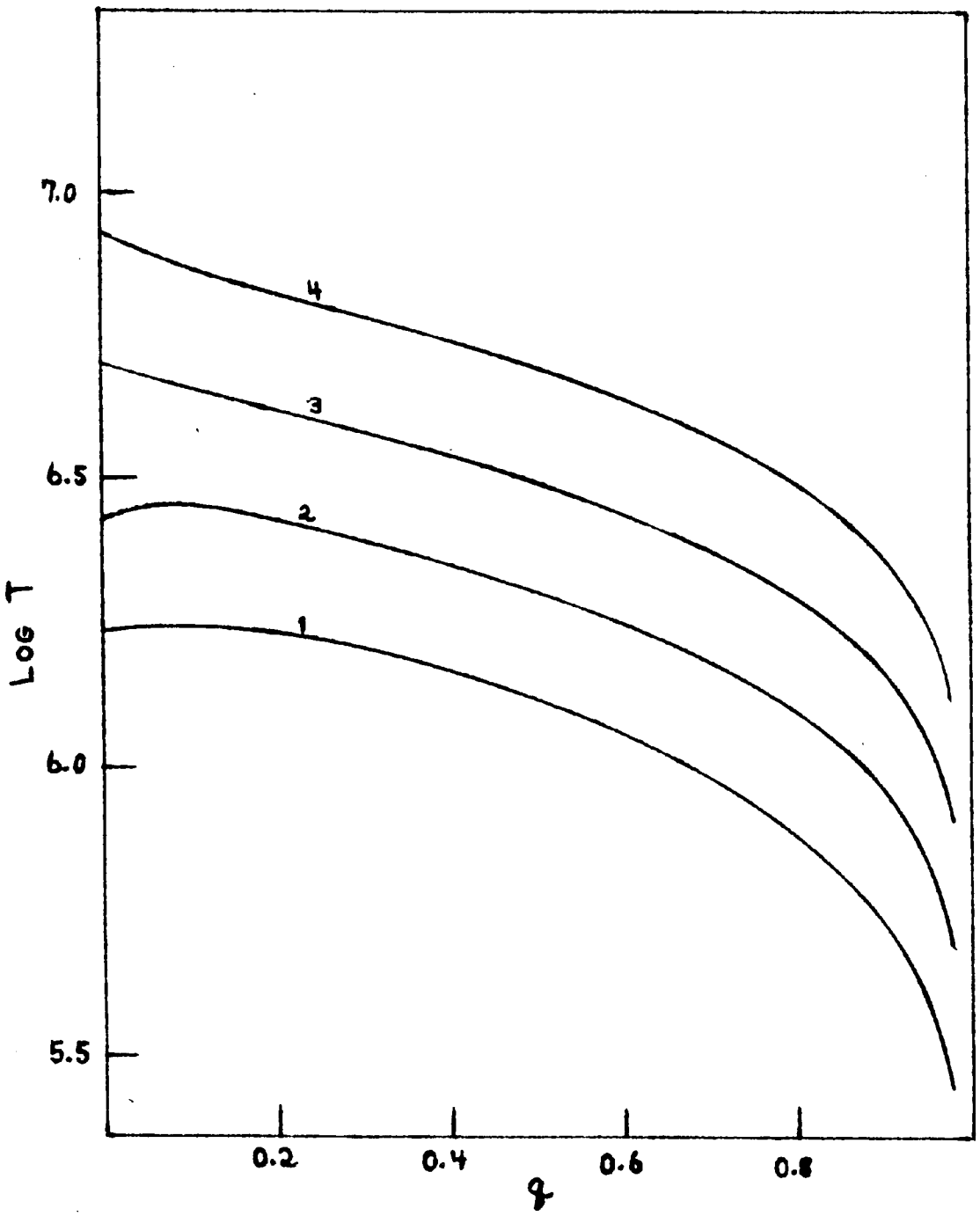


Figure 8. Successive temperature distributions in sequence C. The temperature is shown as function of the mass variable q, for four models in the sequence. These are the standard energy models of Table 3, identified by the number over each curve.

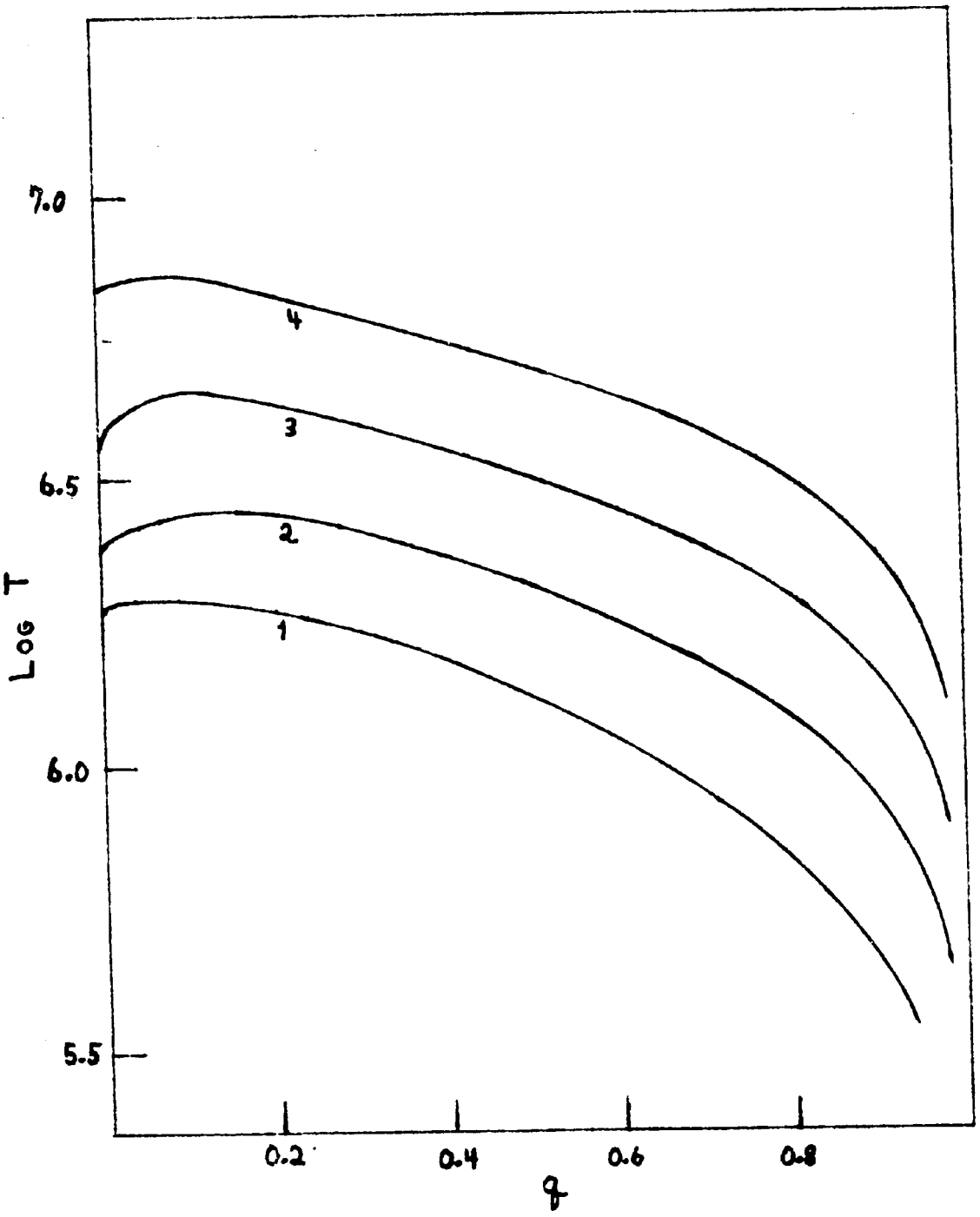


Figure 9. Successive temperature distributions in sequence I. The temperature is shown as function of the mass variable q, for four models in the sequence. These are the standard energy models of Table 3, identified by the number under each curve.

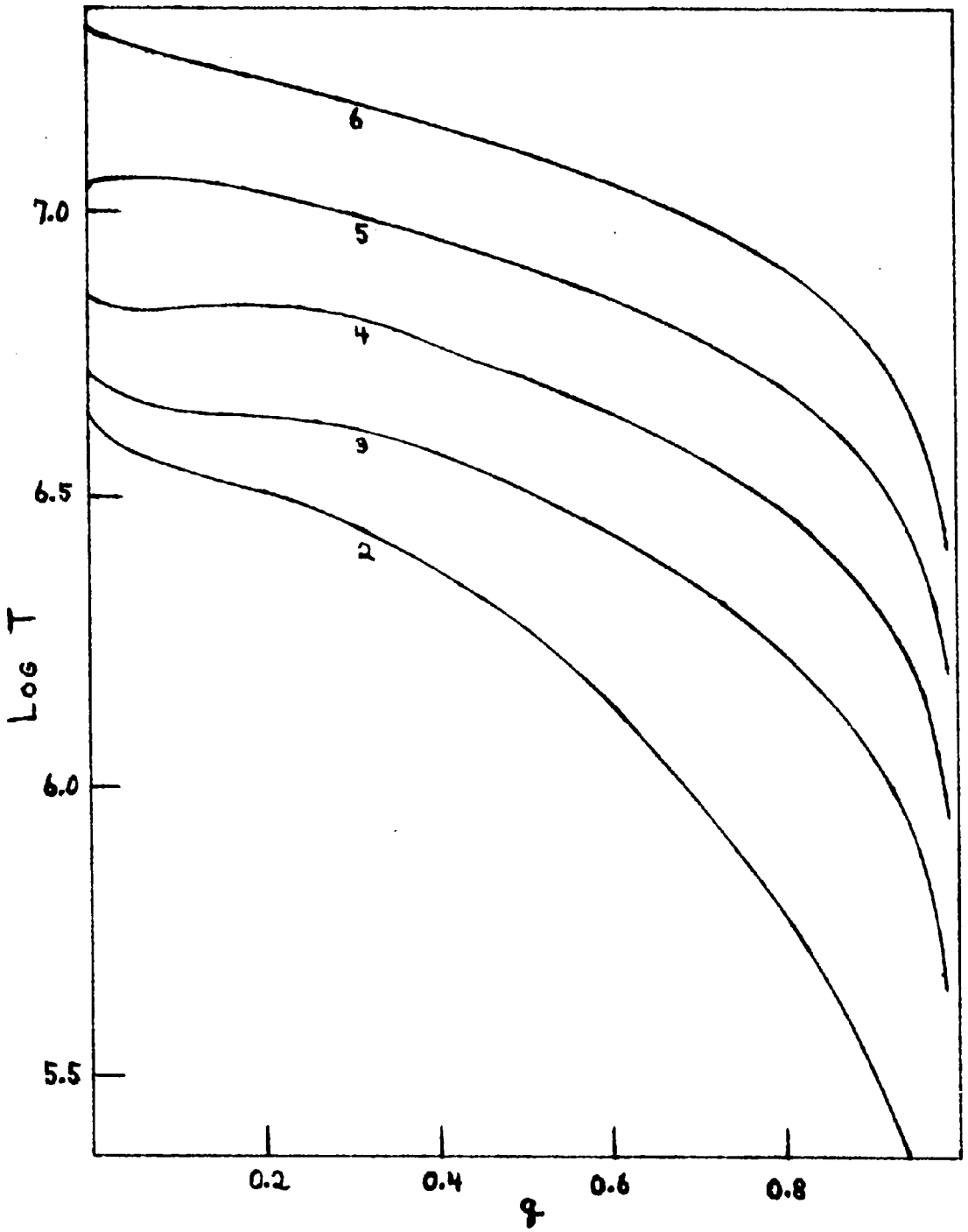


Figure 10. Successive temperature distributions in sequence J. The temperature is shown as function of the mass variable q, for five models in the sequence. These are the standard energy models of Table 3, identified by the number under each curve.

lies highest and L lies lowest in Figures 5 and 6. This should not be taken to imply that the most nearly homologous sequence is a limiting curve in these diagrams. Other sequences are possible which lie below L in Figures 5 and 6, and which depart further from the homology condition. In fact we have not been able to construct any diagram in which a homologous sequence is a limiting curve.

A curious phenomenon shown in Figures 7 through 10 is the tendency of some models to develop transient minima of temperature at the center. In sequences G and I our initial models had mild minima at the center, and these became accentuated in later stages. Sequence J went from a strong central peak to a pronounced dip before it attained the more or less constant shape of the top curve. The central temperature is not decreasing in any of these cases; it is merely increasing at a slower rate than the surrounding regions. This phenomenon seems to be connected with the fact that the stars in which it occurs have initially an opacity maximum at the center.

Changes of radius and of the relative mass distribution are illustrated in Figures 11 through 14. These diagrams show on a logarithmic scale the radii of ten shells of equal mass, labelled with their q values, as functions of the total energy of the star. An intercomparison of these figures again shows a detailed convergence of the different sequences.

Figure 15 shows the progress of central temperature and density for the four sequences. Again the models of standardized energy

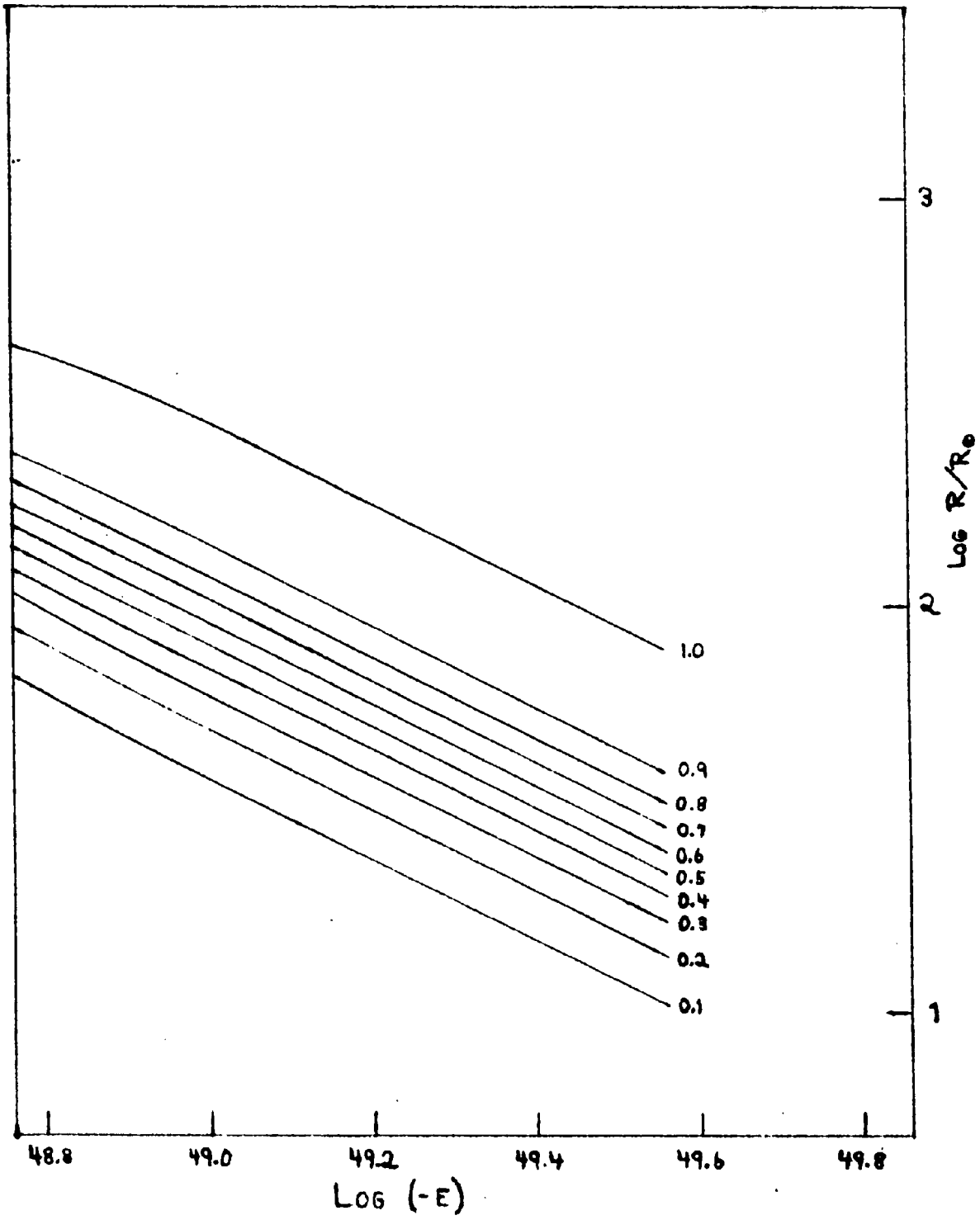


Figure 11. Changes in radius of ten shells of equal mass in sequence L, as function of the star's total energy. The ordinate is the logarithm of the radius in solar units. Each curve is labelled with its q value.

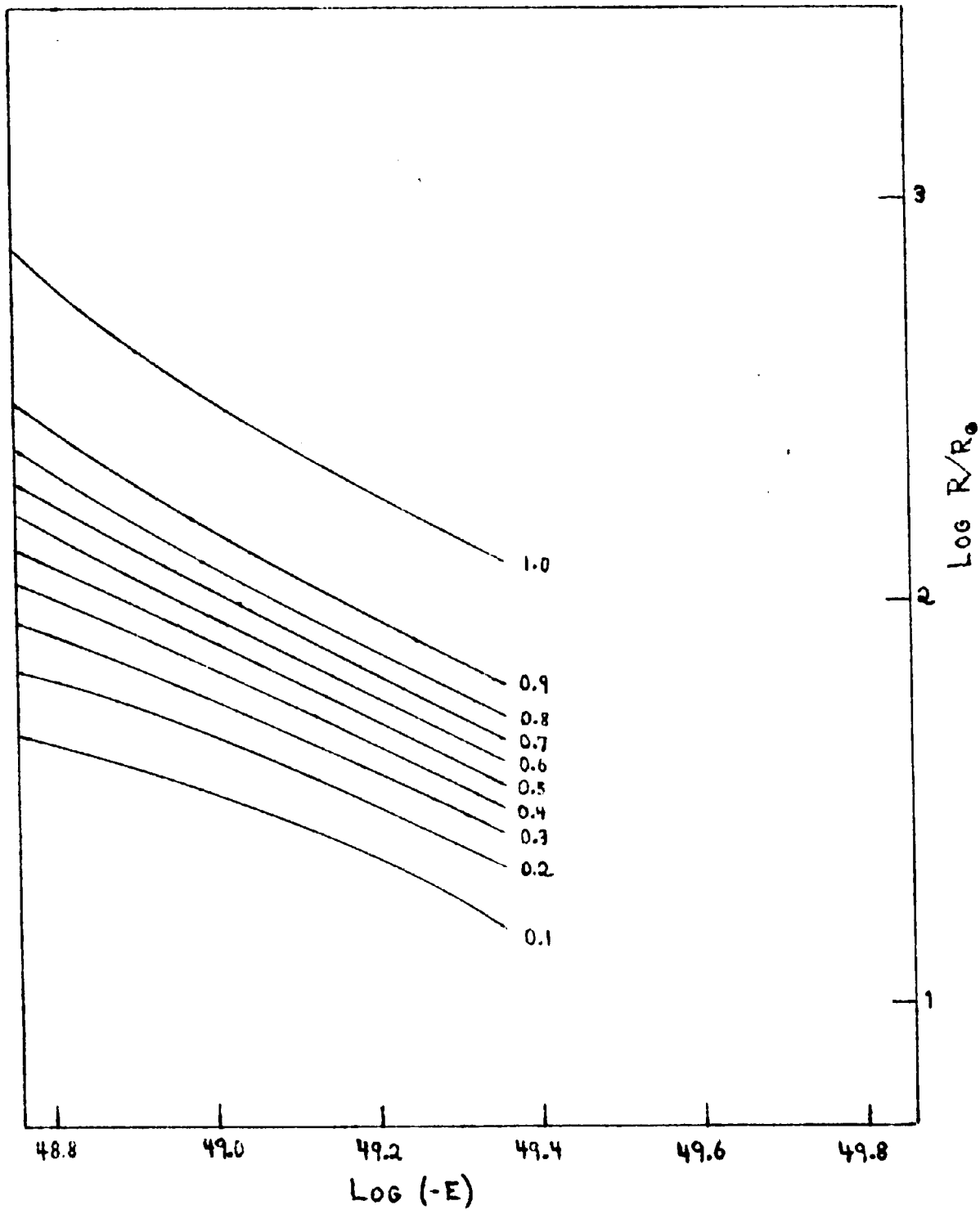


Figure 12. Changes in radius of ten shells of equal mass in sequence G, as function of the star's total energy. The ordinate is the logarithm of the radius in solar units. Each curve is labelled with its q value.

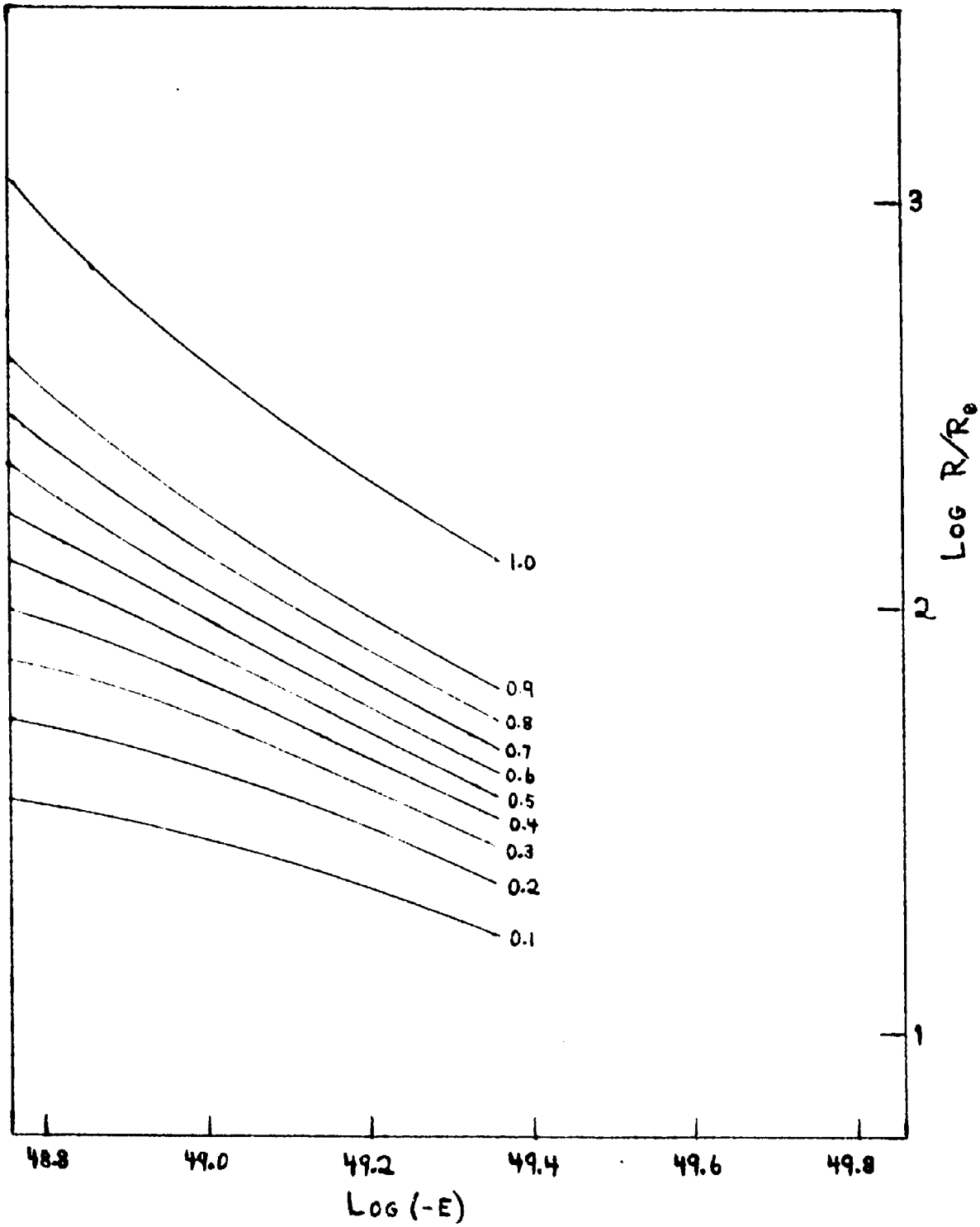


Figure 13. Changes in radius of ten shells of equal mass in sequence I, as function of the star's total energy. The ordinate is the logarithm of the radius in solar units. Each curve is labelled with its q value.

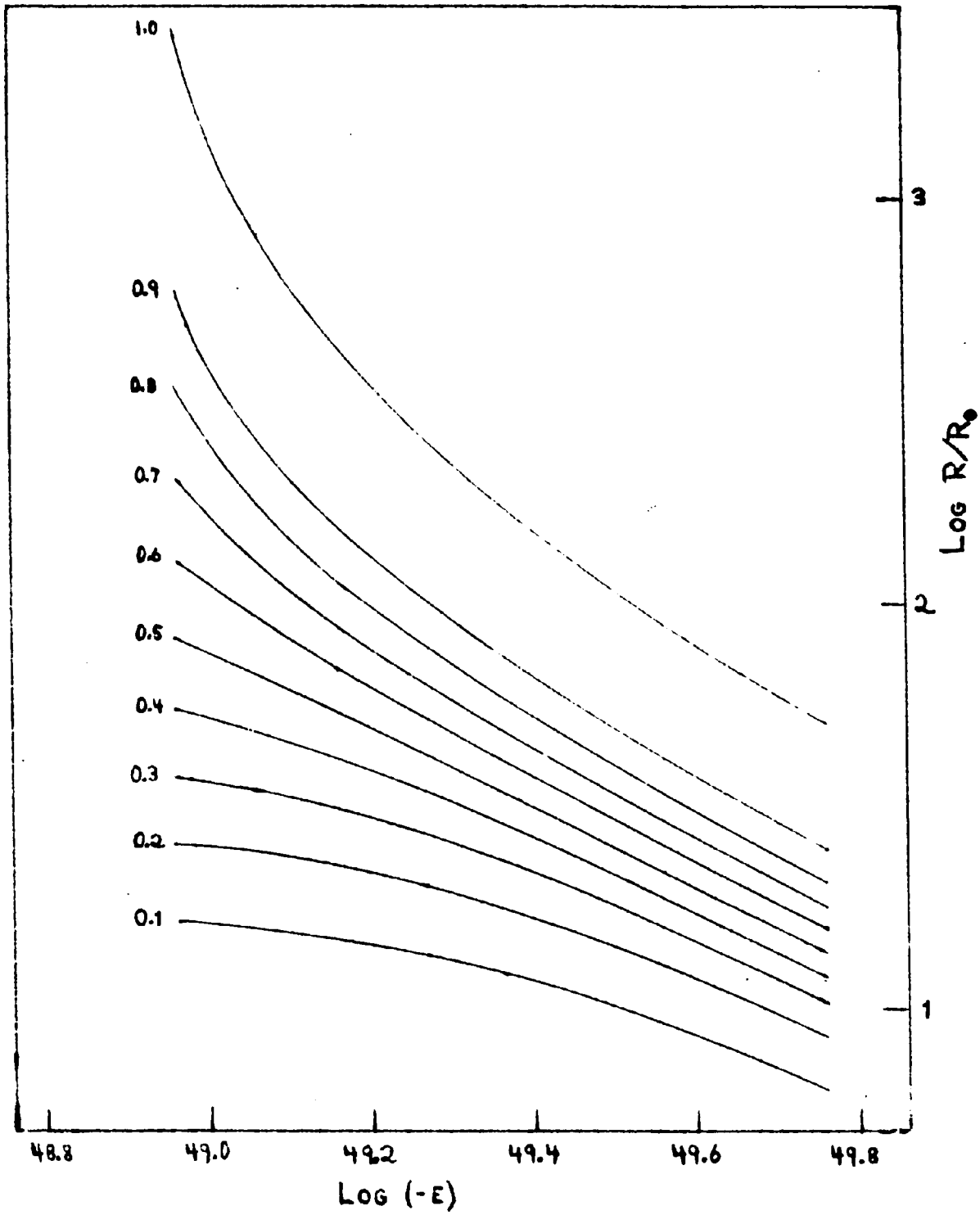


Figure 14. Changes in radius of ten shells of equal mass in sequence J, as function of the star's total energy. The ordinate is the logarithm of the radius in solar units. Each curve is labelled with its q value.

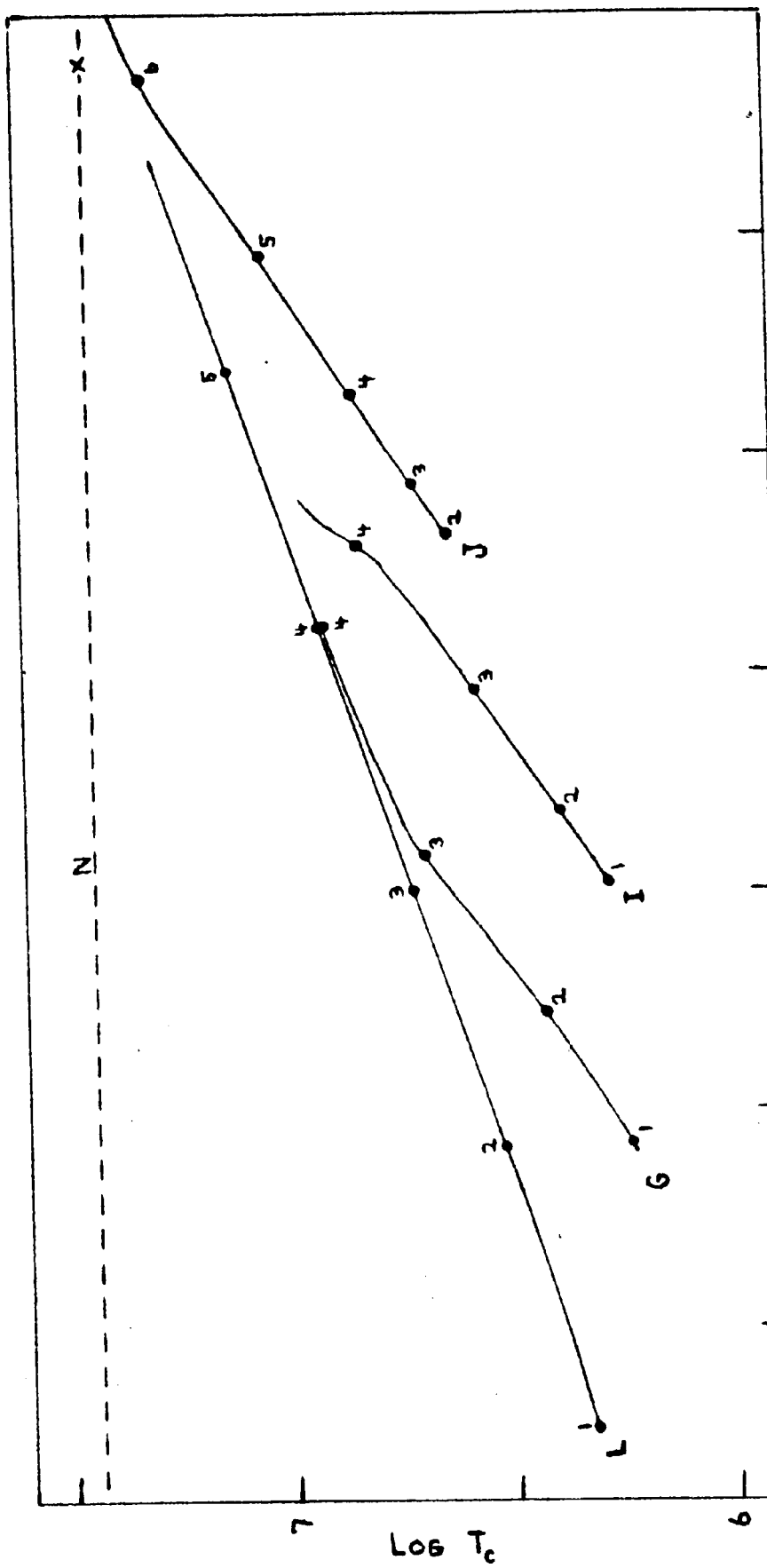


Figure 15. Evolutionary tracks in terms of central temperature and density. The filled circles are the standard energy models of Table 3. The cross at upper right represents Kushwaha's main sequence model. The line N, passing through this cross, is approximately the level the temperature must reach in order for nuclear reactions to become important.

are represented by filled circles on each track, while the main sequence model of Kushwaha is represented by a cross. Of course we do not expect that our tracks should head directly for the main sequence position in any of our diagrams, for until the central temperature rises close to the level of the cross in Figure 15 a star knows nothing of the nuclear reactions which will govern its main sequence structure. The main significance of Figure 15 is that it, rather than any of our other diagrams, shows at approximately what point the stars will cease to follow our tracks and will change over to their main sequence structure. When the temperature reaches the indicated level the stars discover a steady state, made possible by the nuclear energy now available to them, and they can be counted upon to find their way into that steady state. Their convergence or non-convergence in the contraction prior to that point cannot of course be influenced by the existence of a future steady state which depends on processes which have not yet come into play.

A second point of interest in Figure 15 is the abruptness with which some of the tracks change direction, indicating that the development of a local region -- in this case the center -- may proceed along rather unusual lines for several Kelvin times of the star taken as a whole.

Thermodynamics of non-radiating stars

29. We now proceed to a general discussion of stellar thermodynamics in the light of the results illustrated in Figures 5 through

15. We begin this discussion with some remarks concerning the hypothetical case of a star which is allowed to expand or contract as it pleases, but which is not able to radiate at the surface. In other words, its total energy is constant. We shall consider only the case of negative total energy. The consideration of this hypothetical case will bring to light a thermodynamical peculiarity which stars share in common with other unbounded gravitational systems. The thermodynamic peculiarity of these systems is that for any fixed energy they have no state of maximum probability, and hence they can attain no steady state.

The consequences of this principle for a star cluster are well known: the cluster must be continually losing members. Ambartsumian has recently pointed out (1961) that this process can be regarded as an instance of a system evolving to more and more probable states, which in this case are states of increasing radius, and that the complete dissolution of the cluster may be deduced as a consequence of the fact that no state of maximum probability exists. It is possible on this basis to predict in a very general way that a star cluster, however tightly bound, will lose members as long as it has enough of them for energy exchanges to take place. The final residue should be a close binary or multiple star system.

In this natural expansion of a cluster with negative energy -- a cluster which would seem to be gravitationally bound together -- there are two important thermodynamical factors. First, the expan-

sion is consistent with the conservation of energy, for the reason that there is no limit to the kinetic energy that may be gained by a sufficiently close approach of two stars (assuming them to be mass points). Hence it is possible to release all but a handful of stars, no matter what the negative energy of the cluster may be, by arranging that handful in such a way that they alone have the whole negative energy. Second, a transfer of energy in this direction is statistically to be expected because of the unlimited space available for the cluster to expand into. More precisely, the partition function for energy states of individual stars is completely dominated by states at infinity, as is well known in the case of electronic energy states for atoms. An assembly of atoms and a cluster of stars may be regarded as similar systems in this respect.

A star as a system of atoms is not analogous in all respects to a cluster as a system of stars, but if we stop the energy leak in a star we remove the main difference as far as statistical mechanics is concerned. There remain differences in the relation of mean free path of a particle to the dimensions of the system, and in the mechanism of energy interchanges between particles, but these differences do not affect conclusions which are based on probability arguments alone. A non-radiating star would have those same thermodynamical properties which cause a cluster to dissolve, and it should therefore follow an analogous course of evolution.

The general trend of this evolution can be understood in terms

of the analogy we have made with star clusters, but it can also be seen more directly by comparing a representative set of stellar models of the same energy. Consider for example our four models at stage 2, and compare their entropies (Fig. 5), their temperature distributions (Figs. 7-10), and their mass and radius distributions (Figs. 11-15). The natural evolution (at constant energy) must be from a low-entropy model like L toward a high-entropy model like J. The radius increases while the central regions become more condensed. The temperature rises in the central regions and falls in the outer layers, giving rise to a more and more uneven distribution. This is in sharp contrast to the evolution of a bounded system, which must inevitably approach a uniform temperature.

We may describe this evolution at constant energy in the following terms. Heat flows spontaneously down the temperature gradient from the central regions into the outer envelope. This transfer of heat causes the inner regions to contract and the outer layers to expand, with resultant heating in the former region and cooling in the latter. The end result of these contractions and expansions is that the original temperature difference is augmented rather than diminished. Thus the consequence of the virial theorem, that a loss of energy from a gravitational system leads to a rise in the temperature, may be thought of as applying in this case to limited regions as well as to the star as a whole. The increasing temperature difference between core and envelope does not lead to convective instability, because the transfer of heat causes the entropy gradient

to become increasingly positive outward.

The process we have been describing does not apply to real stars, because real stars have an efficient mechanism -- radiation -- for losing energy without losing mass. It is this radiation of energy which makes the essential difference between the evolution of a contracting star and that of a cluster. Our interest in the hypothetical case of a non-radiating star has been first to bring out clearly the effect of energy loss in real stars, and second to throw some further light on what was said in section 26 concerning the range of temperature distributions which are physically possible. In that section we stated that there is no upper limit to the ratio of central to mean temperature that can occur without convective instability. We have now seen that evolution at constant energy is a process which would -- if it occurred -- lead to ever increasing values of that ratio. A practical limit would be set by phenomena we have not considered in this discussion: the onset of nuclear reactions or of degeneracy.

Thermodynamics of radiating stars

30. For real stars our calculated tracks represent typical examples of evolution. It has been understood since Kelvin and Helmholtz that the stars must experience an overall contraction as a result of the energy loss at the surface. Our new result is that a kind of thermal relaxation occurs, whereby diverse initial structures converge as they evolve. It is not a convergence toward a common equilibrium that occurs, however, nor even convergence toward a

common steady state; for within the limitations of the theory neither equilibrium nor steady state is possible. In this respect a self-gravitating system of particlesdiffers fundamentally from the systems that are ordinarily considered in thermodynamics or statistical mechanics. We have already discussed in section 29 one thermodynamic peculiarity of stars which is due primarily to the fact that they are unbounded. A second peculiarity arises when the loss of energy is taken into account; and that is, that a star can radiate away many times the thermal energy which it initially contains. In other words, until incompressibility occurs or until nuclear reactions halt the contraction, the star has an unlimited supply of potential energy on which it can draw, and the nature of the process is such that for every erg lost at the surface a double replenishment automatically occurs. The contraction of a star may be described as a process of converting the potential energy to radiation at 50% efficiency and spilling it out into space. The important point for our present discussion is that as long as the gas is compressible the process is endless, and until nuclear reactions set up a temporary balance of energy it does not lead to any steady state.

Thus the kind of thermal relaxation which occurs in a contracting star is unlike what is normally considered in thermodynamics, and it is for this reason that it is not amenable to analysis by established thermodynamical principles. The status of that science with respect to relaxation problems may be summarized as follows.

The traditional and logically complete thermodynamics of Clausius, Kelvin, Helmholtz, Gibbs et al. is concerned only with equilibrium states. It predicts the equilibrium states which systems will attain if left to themselves, but it tells nothing of their approach to equilibrium. An extension of thermodynamics to non-equilibrium steady states has been made by Onsager, Prigogine, de Groot, and Meixner. (See for example I. Prigogine, 1955; S. de Groot, 1959). At the center of this development stands the principle of minimum entropy production. This principle, an inductive one based on a number of simple cases, states that systems which are prevented from reaching thermodynamic equilibrium by fixed constraints of a certain class may reach a steady state in which irreversible processes are taking place; and that this steady state is the one for which the rate of entropy production -- that is, the total rate of increase of entropy of the system and its surroundings -- is minimized subject to the given constraints. The role of the minimum entropy production principle for non-equilibrium steady states is much like that of the second law of thermodynamics for equilibrium states, except that the former principle rests upon a narrower base of experience than the latter, and its limitations have not been precisely established.

Since a contracting star is not in a steady state and cannot reach one, the principle of minimum entropy production does not apply to it, although it may possibly suggest a direction in which further theoretical progress can be made. A question of consider-

able interest would be whether a thermodynamical principle can be established for processes with no terminal state, such as the stellar contraction process or the dissolution of a star cluster, which would predict some property that an evolutionary track should assume after sufficient lapse of time. Our numerical calculations unfortunately do not suggest any general principle of this kind, nor do they provide a general answer to the question of what circumstances are necessary for the convergence of the sequences. What they do show is that convergence occurs for typical stellar conditions.

The rate of convergence

31. The nature of the convergence time scale can be seen best in Figure 5. It appears there that the time scale for convergence is qualitatively the same as the time scale for contraction; that is, every time the energy changes by factor 2 the difference between sequences diminishes by about 1/2. This is perhaps the behavior that one might expect, in view of the fact that the process by which energy is lost from the system is the same as the process by which it is transported from one part to another. A more precise appraisal of Figure 5 reveals that the difference between models decays in an approximately exponential fashion, and that the half-life for this thermal relaxation corresponds more nearly to a factor 1.6 in the total energy than a factor 2. No simple statement can be made about the convergence rate in time units, because it is tied to the rate of energy loss which varies greatly from one model to another.

The advantage of the energy-entropy diagram for the study of the convergence property is that these quantities are integrated properties of the star. Other integrated properties would presumably show a similar regular convergence, but in fact other integrated properties of direct physical interest are not very many in number. In a diagram which shows local properties, such as the Hertzsprung-Russell diagram (Fig. 6) or a temperature-density diagram for the center (Fig. 15), the tracks still converge but in a less regular fashion. Local regions may develop in a way that appears peculiar, and therefore it is better to avoid trying to make a general rule as to the way tracks in these diagrams should come together. We regard the energy-entropy diagram as the one with basic physical significance, and rely upon the convergence exhibited there to give a sensible convergence within a few Kelvin times in any other kind of diagram.

Although the Hertzsprung-Russell diagram is not the physically fundamental one it is the one in which observational properties of the stars appear. It is therefore important to form an estimate as to what the characteristic scale of convergence is likely to be in that diagram, even though we cannot express it in terms of a regular exponential decay. Let us say that convergence requires in general a tenfold change of total energy, on the basis of the energy-entropy tracks in Figure 5. Neglecting changes in the model and in luminosity, this corresponds to a tenfold change in radius and a change by $\sqrt{10}$ in surface temperature. This is about what our Figure 6

shows. The length of track required for convergence in that diagram is comparable with the length of track computed by Henyey, LeLevier and Levée (1959) in their contraction studies, and in fact it is comparable with the range of surface temperatures normally covered by the color-magnitude diagrams derived from photometry of clusters. This is a slower convergence than has sometimes been supposed.

32. It may be useful at this point to follow the careers of our four theoretical stars, taking into account the different rate of evolution for each. Let us suppose that in a region of active star formation these four stars have all arrived simultaneously at what we have called stage 2 of the evolution. All have the same mass and composition and energy, but their structural details are different. The most noticeable differences are in the radii, which range from 8.4 to 73.1 times the main sequence value, and in the luminosities, which range from 0.95 to 13.3 times the main sequence value. The luminosity differences are attributable mainly to the differences in radius, the surface temperatures being all about the same. The star of highest luminosity has the most uneven temperature distribution of the four and it also has the highest entropy.

With the passage of time the four stars lose energy and evolve along converging sequences. Since the rate of evolution is proportional to the luminosity, the luminous star J obtains a rapid lead in evolutionary progress which it never relinquishes. After sufficient lapse of time all four stars will be found going through nearly the same

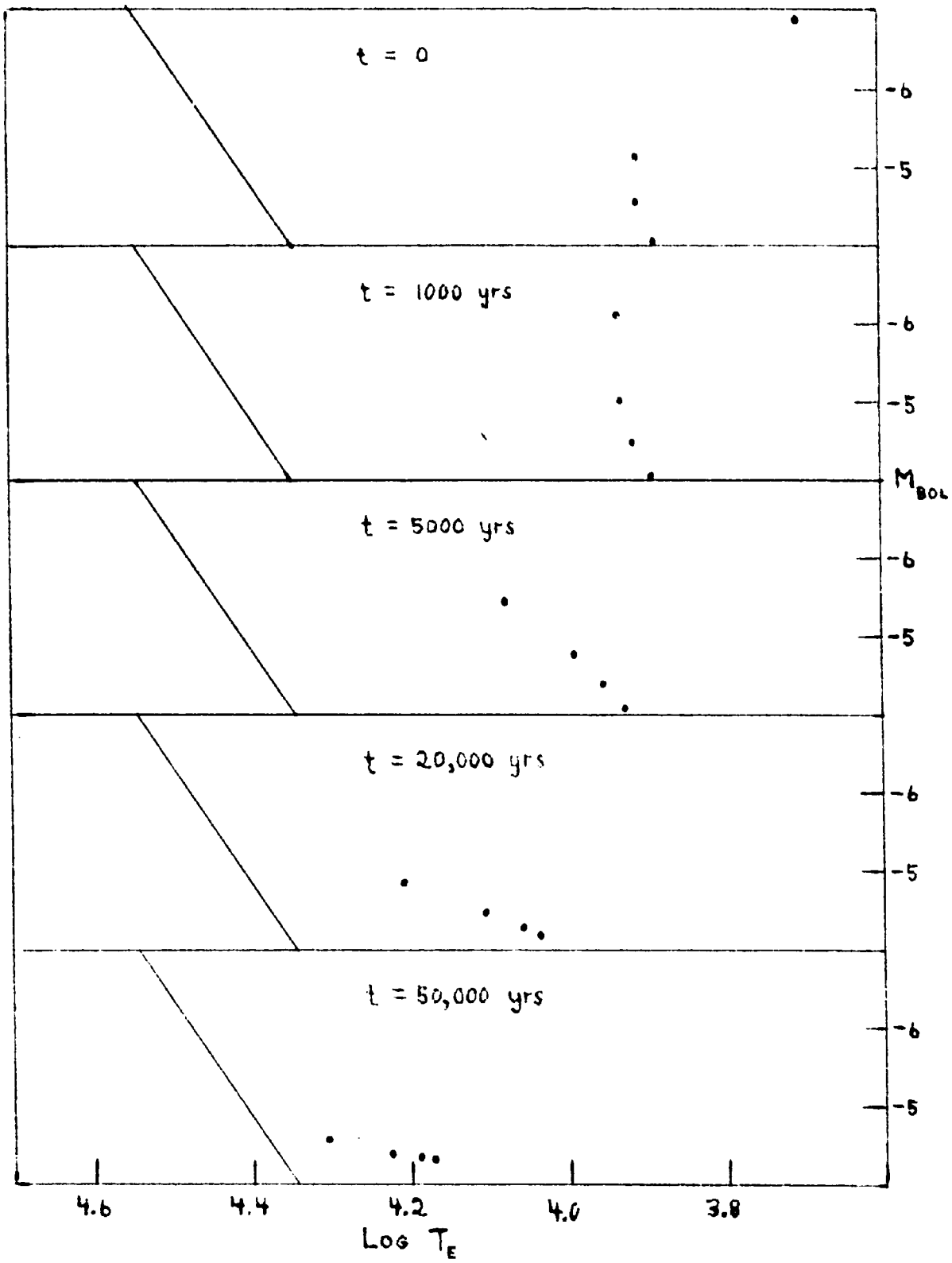


Figure 16. Five time snapshots of four stars crossing the Hertzsprung–Russell diagram. The diagonal line in each snapshot is a part of the main sequence of Kushwaha.

evolutionary stages, but they will be going through them at different times. This is illustrated in Figure 16, which shows a series of time snapshots (at unequal intervals) in the Hertzsprung-Russell diagram for our stars if they start at t = 0 with the energy corresponding to stage 2. Table 4 gives the number of years required for each of these stars to reach the other energy values which we have called stages 1 through 6.

Table 4. Time Scale of Evolution in Years

	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6
Star L	-11,250	0	13,940	33,740	62,470	104,000
Star G	-4,700	0	10,520	29,070	58,000	99,000
Star I	-2,440	0	6,880	21,390	47,000	88,000
Star J	---	0	1,960	8,160	24,390	58,670

There is a tendency for stars which are born at the same time to be strung out along a common track when their evolution is sufficiently advanced.

In the cases studied here star J reaches the main sequence in about half the time required by star L, and by a quite different route. It must be emphasized however that this is just an illustrative example, and the conclusions which may be drawn for the general case are not so specific as that. In order to make definite predictions as to the vertical and horizontal spread to be expected for stars of a given mass and composition as they cross the part of the Hertzsprung-Russell diagram which is normally observed, we would need two items of information which we do not have. First, at what stage in the formation of a star does the kind of contraction covered by this

theory become applicable? Second, what is the range of temperature distributions in stars when they begin this phase of their contraction? Both these questions lead to a consideration of the various processes that govern the earlier phases of contraction, and in particular to the question of how the primordial angular momentum is lost. Although this part of the evolution is still largely speculative it seems plausible that until the star solves its rotation problem its evolution is likely to be strongly influenced by the initial angular momentum and magnetic field strength, and perhaps by other accidental circumstances of its birth. If this is true, then the later the stars solve their rotation problem the more the final convergence of tracks will be shifted to the left in the Hertzsprung-Russell diagram; and the greater the scatter we are likely to find in the neighborhood of the main sequence in the color-magnitude diagrams of young clusters. The examples presented here indicate that a contraction through a factor 10 or so in the total energy is sufficient to wipe out the memory of considerable differences in the initial temperature distribution, and that if the observed stars have been relaxing longer than this they will probably exhibit different stages of progress for the same age but not differences in the tracks they are following. This would lead to a horizontal scatter in the Hertzsprung-Russell diagram for stars of the same mass, after their vertical scatter had become negligible.

Homologous contraction

33. Thus far very little has been said about the relation of our

results to homologous sequences. In view of the prominence which the homology idea has enjoyed in earlier treatments of stellar contraction, some readers might have preferred to see our results discussed from the beginning in terms of convergence toward a homologous sequence. Our idea however has been that it would be instructive to do without the homology concept in the initial discussion, particularly in view of the fact that exactly homologous sequences are possible neither for the idealized stars considered here nor for any real stars. One of our hopes in undertaking the calculations was to discover whether in these circumstances some sequence that is nearly homologous could be picked out as the one that satisfies some thermodynamical extremum condition. This hope has not been realized, however, and as far as we are able to tell all sequences are on an equal footing from the point of view of thermodynamics. In particular we have found nothing to suggest the view that one track should be singled out and considered as the one toward which all others converge. There seems to be no clear meaning in trying to give a special distinction of this kind to one track in the converging family. This is not to say that all tracks are equally probable for a contracting star. It means rather that if we want to pick out certain tracks as more probable than others, we will have to do it on the basis of our ideas as to how stars arrive at the state where simple thermodynamic contraction begins.

Having said this much in favor of the view that no one track

deserves a special distinction, let us now turn to an appraisal of the usefulness of the homology idea for contracting stars. In the first place we note that any kind of contraction -- in fact any kind of stellar evolution -- can be regarded as a combination of homologous changes with changes of the model. As a descriptive device for actual evolutionary processes this kind of separation will be useful to the extent that changes of scale are more pronounced than changes of the model. In view of the convergence property that we have found for contracting stars, it is clearly of interest to know whether a model can be found that changes very little while the energy is changing by factor 10 or more. Whenever such a model is physically possible it will be a model of special interest for the reasons that:

a) once the initial model is found, subsequent models in the sequence can be obtained by the simple application of scale factors in the Lane-Emden manner;

b) the convergence property assures us that an evolutionary sequence found in this way eventually takes on general validity for arbitrary initial models.

We have pointed out already that our sequence L is one of this nearly homologous type. This is evident in the fact that its track in Figure 5 is nearly a straight line, and in the fact that its successive temperature distributions in Figure 7 do not change noticeably in shape. Furthermore its track in the Hertzsprung-Russell diagram is nearly a straight line like the tracks of Henyey, LeLevier, and

Levée which are also nearly homologous. In accordance with what has been said above we could with some justification calculate the later evolution of all our four sequences by simply scaling one of the early models of sequence L. In this sense we might refer to the models of this sequence as the "relaxed models" for this range of energies; but since the implications of that term are not obvious we prefer to call them simply the homologous models, by which we mean that the deviations from homology are not conspicuous over the range of energies involved. It should be noted that the initial model L was not intended to be the best possible approximation to a homologously contracting star, and that other sequences could be found which approach this condition more closely. However, sequence L is very nearly homologous from stage 2 onward.

Table 5 gives the structure of star L at stage 5, as an example of the "homologous model" for our mass and composition and energy range. The arrangement of this table is the same as that of Table 2 in which the initial models are listed. It is interesting to compare this homologous model with Levée's homologous model (1953) for a star with Kramers absorption and no radiation pressure, and with the Emden model of index n = 3. Figure 17 shows the comparison in terms of Milne's dimensionless variables U and V, which are defined as follows:

$$U = \frac{d \log m}{d \log r}$$

Table 5. A Relaxed Model (Stage 5 of Sequence L)

q	$r/R_\odot$	S	log T	log ρ	log P	Pr/P	κ	$L/L_\odot$	n
0	0	13.466	7.125	0.197	15.529	.0237	.489	0	2.73
.10	1.04	13.628	7.065	0.034	15.305	.0228	.505	574	2.63
.20	1.38	13.750	7.023	-0.082	15.146	.0223	.515	1092	2.79
.30	1.66	13.871	6.982	-0.197	14.990	.0219	.526	1565	2.82
.40	1.93	14.001	6.940	-0.318	14.827	.0216	.536	1996	2.86
.50	2.20	14.146	6.893	-0.451	14.647	.0213	.549	2387	2.89
.60	2.49	14.315	6.841	-0.604	14.442	.0210	.563	2737	2.92
.70	2.82	14.524	6.778	-0.781	14.193	.0209	.580	3045	2.96
.80	3.24	14.803	6.696	-1.034	13.866	.0208	.602	3304	3.00
.90	3.86	15.254	6.568	-1.421	13.352	.0210	.638	3501	3.06
.99	5.41	16.577	6.215	-2.521	11.900	.0230	.738	3564	3.15
1.00	7.77	23.438	4.215	-8.503	3.917	.0221	.368	3572	7.00

$$V = -\frac{d\log P}{d\log r}$$

The three models are practically identical. We show the corresponding diagram for the initial models in Figure 18, in order to illustrate that this kind of diagram based on logarithmic derivatives is a sensitive one in exhibiting differences in the model.

The close resemblance between our nearly homologous model and Levée's strictly homologous one is the more striking when we consider that our stars have about equal contributions of Kramers absorption and electron scattering in the opacity, whereas Levée's stars had pure Kramers absorption. One may conclude therefore that a change from one of these absorbing mechanisms to the other -- i. e. from κ = constant to $\kappa = \kappa_0 \rho T^{-3.5}$ does not change the homologous model in any important way. Neither will a change in the ratio of radiation pressure to gas pressure change the model, for in the n = 3 model this ratio is constant throughout and any change is equivalent to a uniform change of the molecular weight. It appears therefore that the polytropic model of index 3, which was called the "standard model" by Eddington, has a rather general validity as the homologous model for contracting stars.

Although, as we have said, the homologous model does not appear to have any special thermodynamical significance, it does have two special distinctions from a practical point of view. One is that whenever a nearly homologous model is possible, it may be regarded

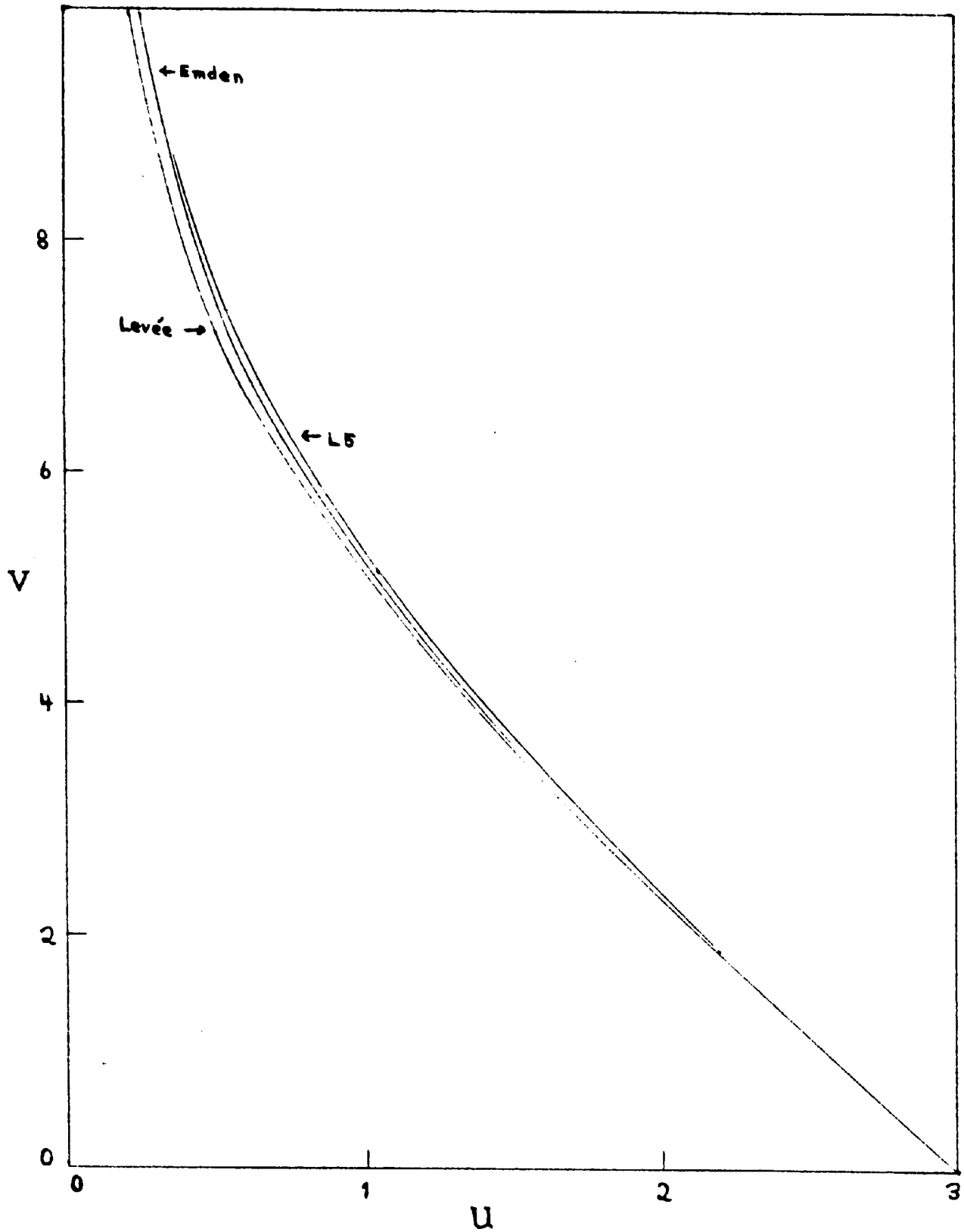


Figure 17. A comparison of three similar models in the U, V diagram. The close similarity is shown between model 5 of our sequence L, Levée's homologously contracting model, and the Emden model of polytropic index 3.

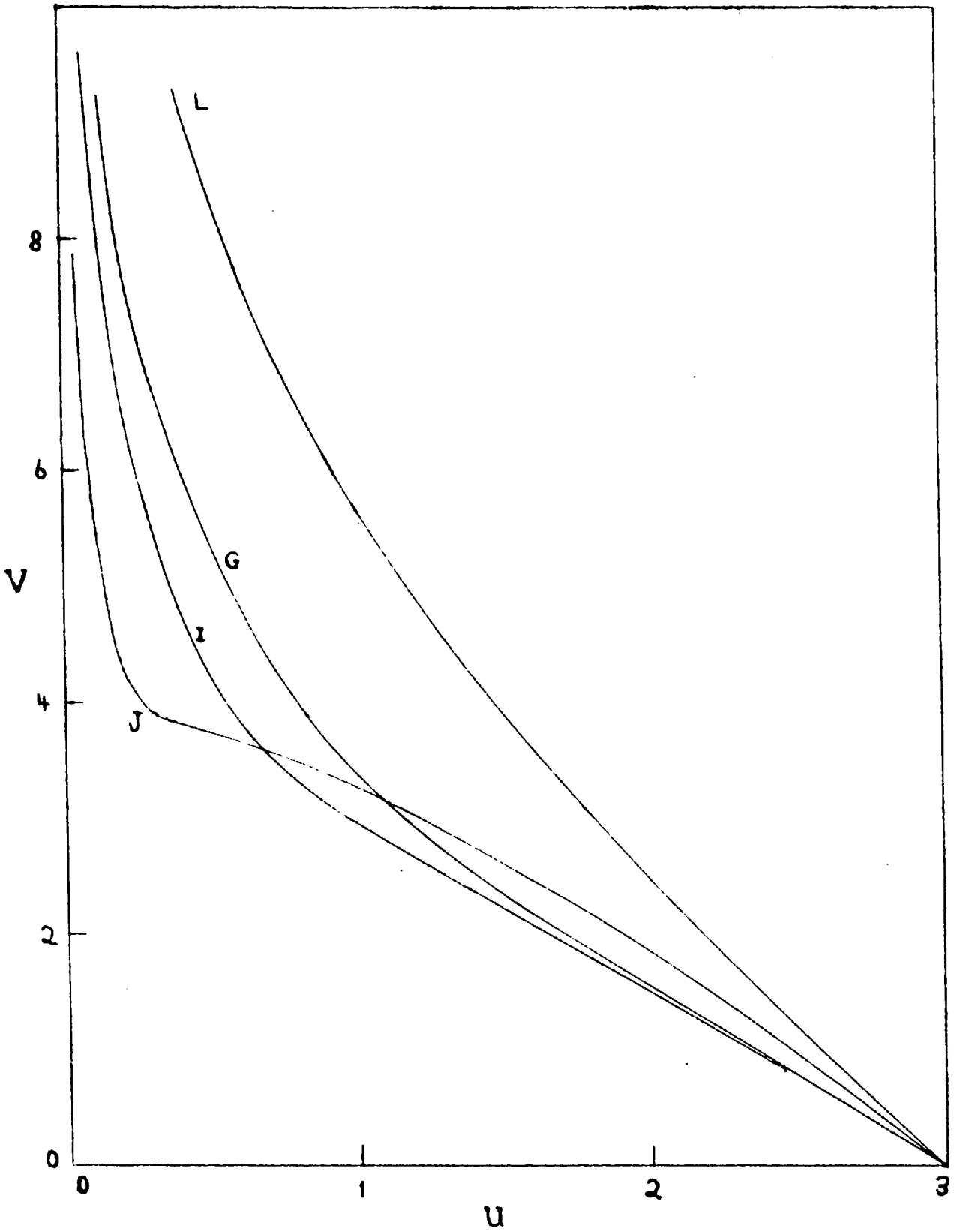


Figure 18. The four initial models in the U, V diagram.

as the form toward which other models evolve. This is of some interest in itself. The second distinction is that a homologous sequence is by far the easiest to calculate. Only one model has to be worked out in detail, and the whole evolution can then be found by a trivial calculation of scale factors. In a great many cases the homologous model does not even have to be calculated, for it is already known to us as the polytropic model of index 3.

Thus the case of homologous or nearly homologous contraction is a useful and informative case to study. We should bear in mind however, that the homologous model is not necessarily representative for stars at a given stage of their contractive evolution. Whether or not it is representative at a given stage depends on what has gone before.

34. The conditions under which homologous sequences are possible were first investigated by Jeans (1918, p. 195), and the investigation was later expanded by Thomas (1931a). They both showed that a homologous contraction is possible for stars which follow the equations of state for ideal gas plus radiation -- the equations we have used -- provided the opacity formula is of the right form. The right form, according to Thomas, is:

$$\kappa = T^{-n}\,\lambda\left(\frac{\rho}{T^3}\right) \tag{34.1}$$

where n is any constant and λ is any function of $\frac{\rho}{T^3}$.

For comparison with opacity formulae of the type generally used in modern work, and in particular for comparison with the formula we have used, we prefer to express the required form of the opacity formula as follows:

$$K = K_0 \rho^{\alpha} T^{-3.5-\beta} \left[1 + f\left(\frac{\rho}{T^3}\right)\right] \tag{34.2}$$

This is simply (34.1) rewritten, with $\lambda\left(\frac{\rho}{T^3}\right)$ replaced by

$$K_0 \left(\frac{\rho}{T^3}\right)^{\alpha} \left[1 + f\left(\frac{\rho}{T^3}\right)\right]$$

A constant opacity such as we would have in the case of pure electron scattering is of this required form, with $\alpha = 0$ and $\beta = -3.5$, and hence it permits a homologous sequence. This was the case considered by Jeans. Thomas pointed out that the Kramers formula also meets the requirement. The formula (10.2) which we have used and which has been found valuable in other recent work, i.e.

$$K = K_0 \rho^{\alpha} T^{-3.5-\beta} + \sigma$$

does not meet the homology condition unless

$$3\alpha = 3.5 + \beta .$$

For the values $\beta = 0$, $\alpha = 1$ which were used in our calculations the requirement is not met. This proves our earlier statement that there can be no homologous sequence for the idealized stars to which our calculations pertain.

When we consider the kinds of opacity formula that apply to real stars it appears that nearly homologous sequences will generally be possible, although strictly homologous ones will not. We have mentioned that formulae of the type (10.2) can be used to give reasonable representations of the opacity in individual stars and in the successive models of a short evolutionary sequence. For a long evolutionary sequence we would in general have to change the parameters κ_0, α, and β as we go along in order to maintain the closest approximation to the actual opacity values. It was not considered nedessary to do this in our present calculation, however, as the slow variation of these parameters would not affect the evolution of one star relative to another. Opacity formulae of the type (10.2) will allow a nearly homologous contraction when either of the two terms is much larger than the other. This occurs for example in stars of small mass and normal abundance of metals. In such stars the first term (photoelectric absorption) dominates and the constant term for electron scattering can be neglected. In very massive stars the situation is reversed, and the electron scattering term dominates. Therefore a homologous sequence is possible for those stars also. In cases where both terms are significant a nearly homologous contraction will still be possible if the first term is nearly a function of $\frac{\rho}{T^3}$. Since absorption in stars can usually be represented by taking something close to the Kramers formula for the first term,

and since this formula is not much different from a function of $\frac{\rho}{T^3}$, it follows that some kind of nearly homologous sequence such as our sequence L will usually be possible as long as our other approximations -- e. g. the equations of state -- are applicable.

Summary of this chapter

35. In this chapter we have presented the results of our calculations showing the evolution of four stars of the same mass and composition but different initial temperature distributions. From these results, together with other general physical considerations, we have drawn the following picture of pre-main-sequence contraction.

A nearly homologous sequence is possible for most masses and compositions. There is no reason to suppose, however, that stars are initially close to the nearly homologous sequences. A vast range of initial temperature distributions is possible, unlimited with respect to the degree of central heat concentration which is stable against convection. Present theory does not permit us to decide where in this unlimited range the initial models of real stars may lie. Accidental factors such as the primordial angular momentum of the protostar are likely to cause differences between one star and another in the evolutionary stages which precede the one we have studied, so that a spread of initial models for the present phase is likely.

Stars with different initial temperature distributions evolve

along converging sequences. The time scale for convergence is roughly the same as the time scale for contraction. That is, a star's thermal memory may be said to be half lost every time its energy changes by a factor 2. In the cases studied here, the central regions were slower in relaxing than the star as a whole. A few factors of 2 in the total energy are sufficient to bring together the sequences we have studied, but as in other decay phenomena the time required to attain a given condition -- for example, to reach a nearly homologous track -- depends on how far the initial condition is from the final one.

CHAPTER V

RANGE OF VALIDITY OF THE RESULTS

36. In this final chapter we shall consider briefly the extent to which the idealized stars we have been investigating may be representative of real stars. We shall ask whether limits can be set, for example in the Hertzsprung-Russell diagram, within which the evolutionary behavior we have considered is realistic.

As to whether our idealized stars are representative of real stars, it will be useful to distinguish between two different degrees of correspondence between them:

a) Detailed correspondence. By this we mean that all of our evolutionary tracks and diagrams are valid, and that we can obtain tracks for other masses and other stages of contraction by simply sliding the scales on our diagrams.

b) General correspondence. By this we mean that the basic convergence property, in terms of total energy and entropy, remains valid. That is, Figure 5 remains valid as it stands, and we can slide the scales to obtain energy-entropy tracks for other masses and other stages of contraction.

It is hardly necessary to state that we expect the general correspondence to hold good over a considerably wider range of conditions than the detailed correspondence. As was mentioned earlier, we regard Figure 5 as the most significant of our evolutionary dia-

grams, and we regard the convergence exhibited there as the principal result of our investigation. The evolutionary developments exhibited in our other diagrams must be regarded as details of a more restricted application, which will be altered whenever the opacity law, for example, is altered. It would go beyond our purpose to discuss all the possible alterations of detail which could occur for different masses and compositions and stages of contraction. We shall therefore confine our attention to the validity of the principal result, or the general correspondence as we have called it.

Some cases where the principal result is not valid

37. Our principal result is the quasi-exponential convergence of tracks in the energy-entropy diagram, as illustrated in Figure 5, with a half-life corresponding approximately to a factor 2 in the total energy. Our method of investigation does not allow us to determine necessary and sufficient conditions for the validity of this result, but we can at least see some of the essential factors by pointing out cases where the result must obviously break down.

One such case occurs when the opacity is much higher in some local region than it is in others, so as to make that region a thermal barrier in relation to the rest of the star. A white dwarf is generally considered to have such a thermal barrier at its surface, because while the surface opacity is normal the interior has an unusually high thermal conductivity. Under these conditions the thermal

relaxation of the interior will not have any connection with energy loss at the surface, and the situation will be more nearly one of thermal relaxation at constant energy.

It should be noted however that a local thermal barrier will upset the principal result only if the barrier is persistent. We saw in Chapter IV that in a compressible star a surface barrier leads to an expansion of the outer regions. This expansion may in many cases remove the barrier before the interior has changed very much, and the surface barrier will then cause only a temporary deviation from the main course of evolution. This is exactly the sort of thing that is shown by the slight distortions at the beginnings of the tracks in Figure 5. A surface barrier can lead to a serious breakdown of our main result only if it can be somehow maintained while the outer layers expand, or if the star departs so far from a perfect gas that the outer expansion does not occur. This last is the case of the white dwarf.

There is a second reason why the principal result must fail in the case of a white dwarf, regardless of what the opacity at the surface may be. It is that the contraction is halted by the onset of Fermi-Dirac degeneracy. In smaller bodies the contraction may be halted by the onset of the ordinary solid state, which may be regarded for our purpose as a related phenomenon (D. S. Kothari, 1936). When either of these final states is reached there can no longer be

any large changes in the total energy, and again Figure 5 loses its applicability.

A third breakdown of the principal result must occur when the contracting object is optically thin, as may be the case in the early stages of star formation. In this case the different parts of the object exchange heat primarily with the external radiation field rather than with each other, and there is no question of a thermal relaxation of the object as a whole. It is generally believed that the thermal independence of the separate parts deprives such an object of the means of maintaining pressure equilibrium, and that its contraction therefore becomes a hydrodynamic problem rather than a thermodynamic one.

On the basis of these examples it appears that at least three conditions are essential to the type of relaxation shown in Figure 5:

a) thermal interaction of the parts;

b) unlimited contraction possible;

c) no local regions of persistent high opacity.

A more precise listing of necessary and sufficient conditions is not possible within the scope of the present study.

Limiting processes for real stars

38. We shall now look specifically at the processes which limit the validity of our results for contracting stars. We consider first the case of pre-main-sequence contraction, and seek approxi-

mate limits in the Hertasprung-Russell diagram within which real stars will behave like our idealized ones.

Consider first the limitations on the right in the Hertzsprung-Russell diagram; that is, the limitations that apply at the beginning of the contraction. There are at least four independent factors which may cause real stars to behave in a fundamentally different way from our idealized stars:

a) outer convection zones;

b) partial ionization of the material;

c) rotation;

d) magnetic fields.

It is by no means certain which of these factors sets the practical limit; that is, which is the last to become unimportant as the contraction proceeds.

The uncertainty with respect to the outer convection zone has already been discussed in Chapter II. Hayashi's view is that this zone extends to the center or nearly to the center in the early stages of evolution, and that the star follows a path like the one shown in Figure 19, which is taken from his work. The vertical branch in that diagram is the convective phase of the contraction, and the horizontal branch is the radiative phase. The convective phase, if it really occurs, is sufficiently different from the evolution we have considered that we cannot expect any of our results to

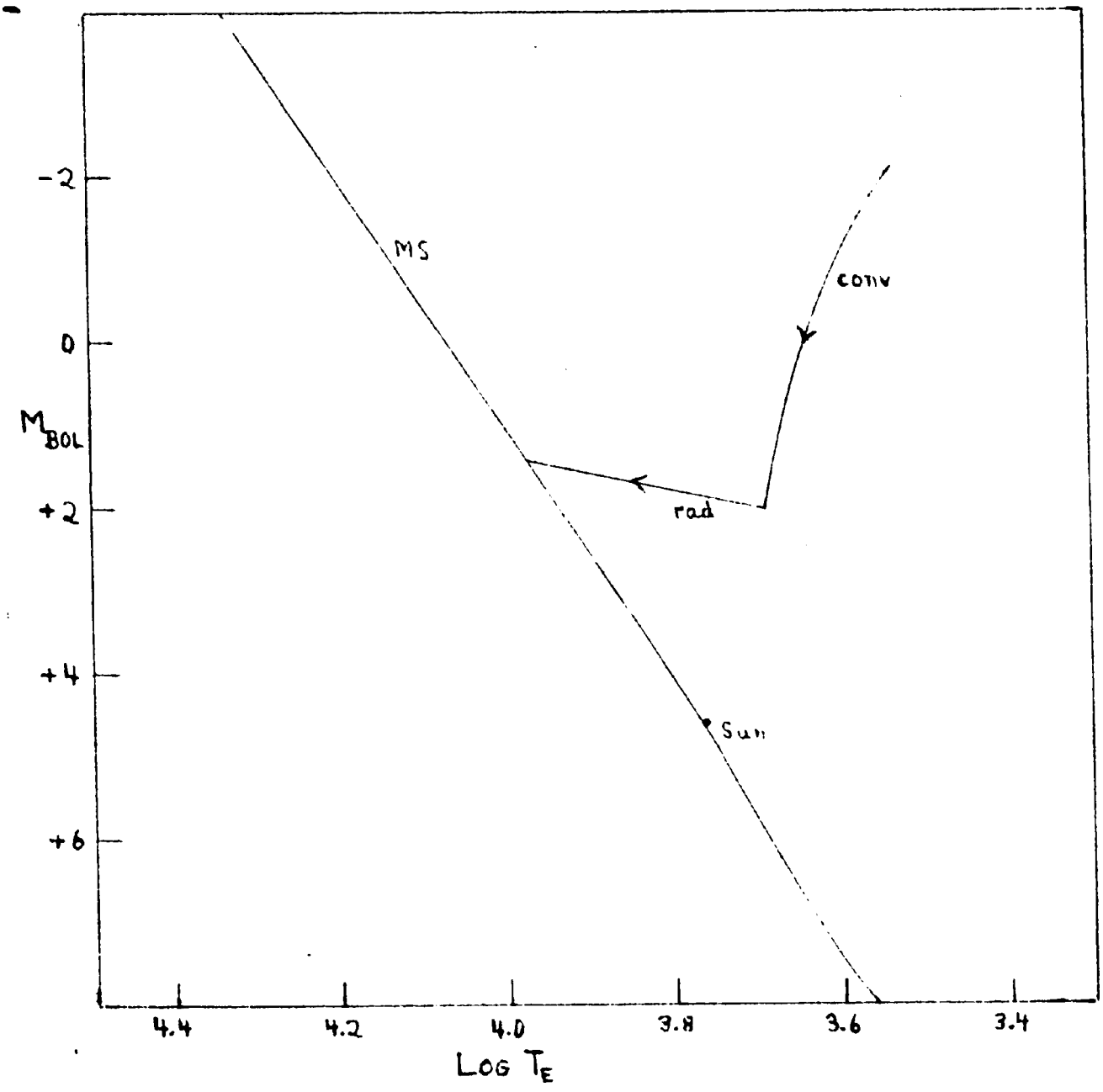


Figure 19. Pre-main-sequence contraction according to Hayashi. The track shown is for a Population I star of 1.9 solar masses. On the descending vertical branch the star has a predominantly convective structure, on the horizontal branch a radiative structure.

apply to it. There may be a quasi-exponential relaxation of temperature distributions in convective contraction, but this cannot be told from the present study.

In Figure 20 the line "1" marks roughly the limit set by the Hayashi convection theory. This line should vary somewhat with mass and composition if accurate limits were wanted. It corresponds in a general way to the convection criterion of Pecker and Bohm-Vitense that was illustrated in Figure 2, but it lies further to the right. The criterion in Figure 2 had to do with the onset of convection in the outer layers, whereas the criterion in Figure 20 is for the spread of convection through the entire star. Whether this line marks a real limit to the applicability of our results will be seen when the Hayashi theory is developed further. For the present we shall consider it as a convective limit whose consequences are not entirely settled.

At a certain stage of a star's contraction the hydrogen must be in a transition from the neutral to the ionized state. This condition favors a more rapid contraction than can occur in a fully ionized gas, since part of the energy which would ordinarily have to be radiated away now is taken up as energy of ionization. It has been suggested by A. G. W. Cameron (1962) that there is a possibility of a sudden collapse of the star at this stage. It is at least certain that the rate of contraction will be faster in relation to the

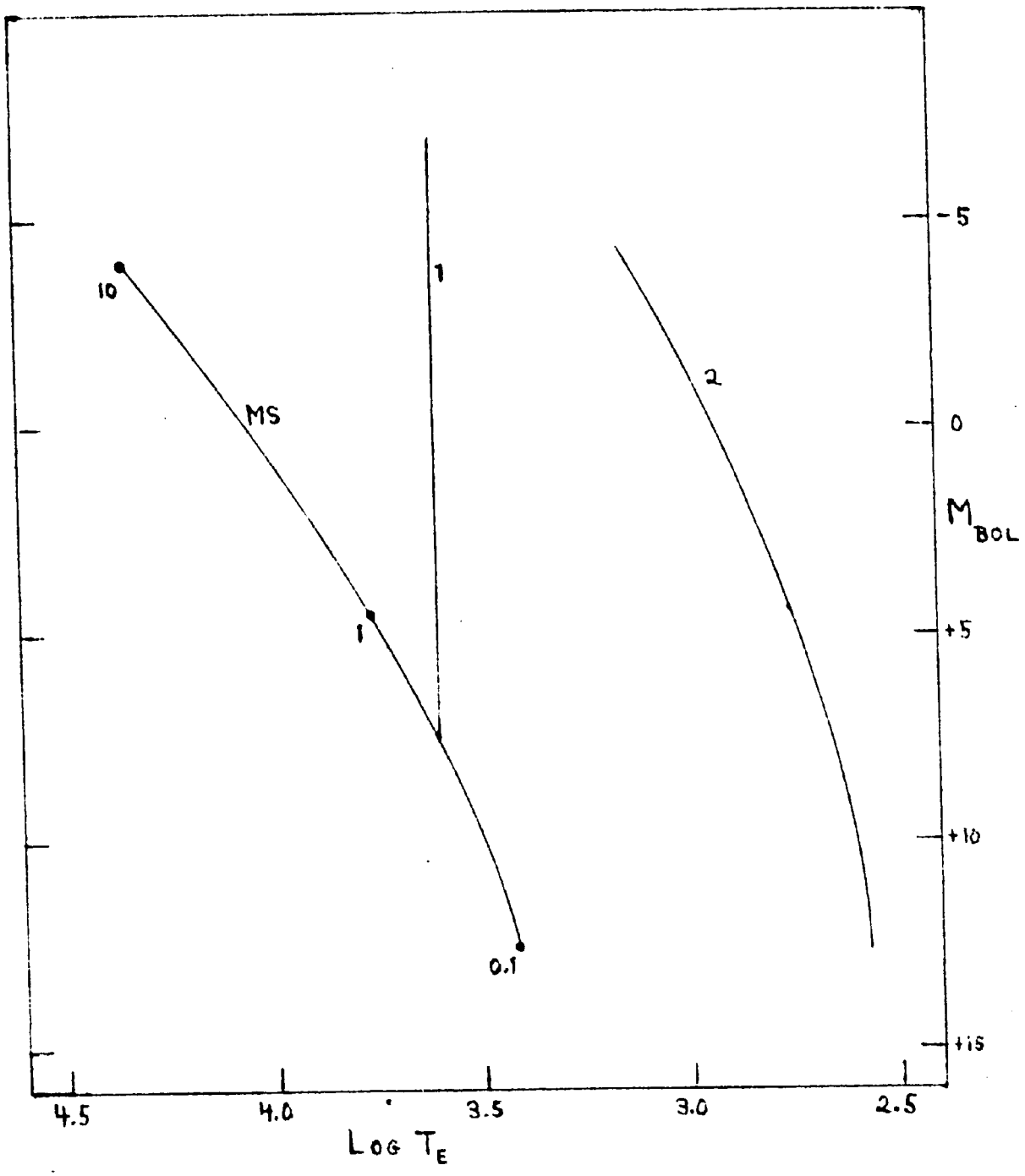


Figure 20. Approximate limits in the Hertzsprung-Russell diagram for application of the present theory. The line MS is an approximate main sequence with three masses indicated. This is the left-hand limit. The lines 1 and 2 are two possible right-hand limits. Line 1 is the convection limit of the Hayashi theory, and line 2 is the ionization limit discussed in the text.

changes of energy than it is in the fully ionized stages we have considered. We should therefore regard the condition of partial ionization as a cause of departure from our results.

The partial ionization should be at its most critical stage when the ionization energy of the star is about equal to its total energy. The energy of ionization for pure hydrogen is 1.3×10^{13} ergs/gram. Compare this with the total energies of three main sequence models of different masses:

Mass	Total energy	Reference
$10\ M_\odot$	-3.45×10^{15} ergs/gm	Kushwaha, 1957
1	-1.47	Schwarzschild, Howard, Härm, 1957
0.1	-0.68	Limber, 1958

This comparison shows that the peculiar behavior associated with partial ionization should occur when the energy is of the order of 1/50 to 1/250 the main sequence energy for stars of ordinary masses. This condition cannot be translated into a position in the Hertzsprung-Russell diagram in any unique way, but we can make a very rough translation by making homologous transformations of the main sequence models at constant luminosity. This gives us the line "2" in Figure 20 as a very rough ionization limit.

The rotational and magnetic effects associated with star formation are subjects that have been much discussed, but nothing like a definite evolutionary scheme has yet been established. The general arguments for the importance of these effects have been recently

reviewed by Hoyle (1960) in connection with a qualitative exposition of one possible evolutionary scheme. It seems certain that a pre-stellar cloud must lose most of its angular momentum before a star can be formed. It seems furthermore certain that the loss must begin fairly early in the contraction process, but it is not so easy to tell where it ends. While angular momentum is being lost, and while magnetic energy is being dissipated, we cannot say what the evolution of the star will be like. Our present view is that these processes terminate at some unknown point before the main sequence is reached, leaving a star whose rotational and magnetic energies are unimportant from that point onward. That star has some temperature distribution which depends on the processes just completed, and which may very well differ from one star to another. At any rate it is unknown to us at present. For our purpose, therefore, thermal relaxation begins at this point. It probably proceeds in the way we have described as our principal result, but this is not certain unless the star is already out of the convection and ionization regions of the Hertzsprung-Russell diagram. If it has those regions still to traverse, than the thermal relaxation process may be fundamentally different from the one described by our calculations. If the rotation and magnetic problems are solved as early, say, as when the radius is 10^4 times the main sequence value, then we should expect that all stars of the same mass and composition have already

converged to the same track before they become visible.

The left-hand limit in the Hertzsprung-Russell diagram is set by the onset of nuclear processes. It is the observed main sequence, which is represented by line "3" in Figure 20. We shall not distinguish here between main sequences for different compositions, as their positions in the theoretical Hertzsprung-Russell diagram do not seem to differ much from one another. It should be noted, however, that there can be some overshooting of the main sequence if thermal relaxation is not complete by the time that sequence is reached. The reason is that the halting of the contraction is determined by its surface conditions. If the models are not unique at this stage of the contraction, then there is not a one-to-one correspondence between the center and the surface properties. Thus it is possible that the left-hand limit for contracting stars may be a somewhat fuzzy one, the degree of fuzziness depending on the extent to which stars have converged to the same model at that point.

Other cases to which the results may apply

39. Our principal result may apply to a post-red-giant stage of evolution in some stars. This will be the case if the red giant stage is followed by a contractional stage in which neither nuclear energy nor degeneracy is an important factor. This stage of stellar evolution is still very much in the realm of speculation, however, and therefore it can only be mentioned as a possible instance of the

relaxation process which has been studied here.

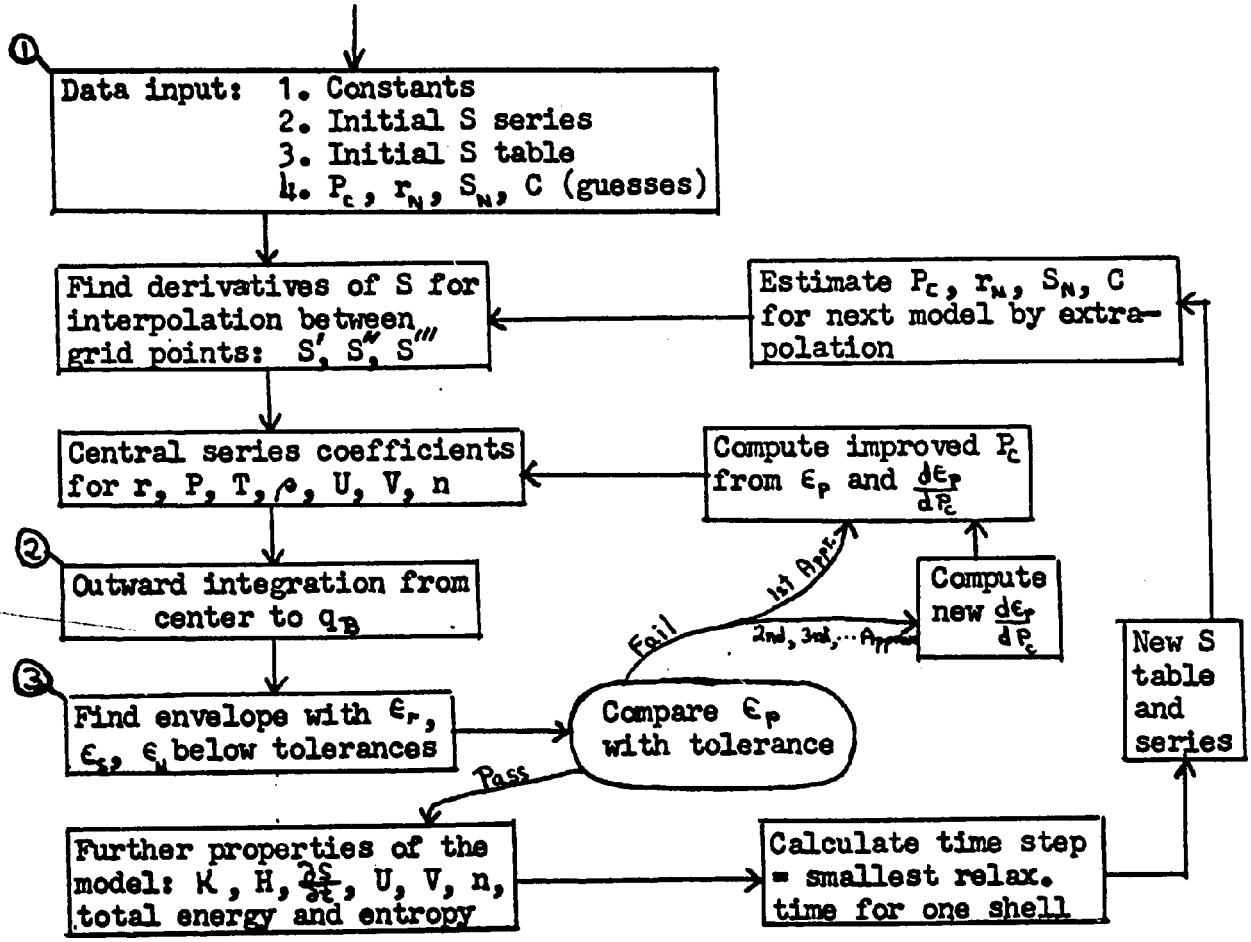
APPENDIX

FLOW DIAGRAMS ILLUSTRATING THE METHOD OF CALCULATION

These diagrams are intended to give a somewhat fuller view of the sequence and organization of operations than is provided by the text alone. The aim is to include just enough detail to enable the reader to form a clear picture of the order of operations on the large and intermediate scales, but not so much as to bewilder him.

The four discontinuities referred to in the text (page 53) are here called ϵ_P, ϵ_r, ϵ_S, and ϵ_H. The four free parameters are here called P_C, r_N, S_N, and C . The subscript N denotes the surface.

Overall view of the program



Block 1: Data input

The constants include the following items:

Physical constants.

Constants in the opacity formula.

The q values that constitute the thermal grid.

Special q values:

q_A = changeover point from series expansions to numerical integration.

q_B = hydrodynamic fitting point (value used = 0.70).

q_C = thermodynamic fitting point (value used = 0.99).

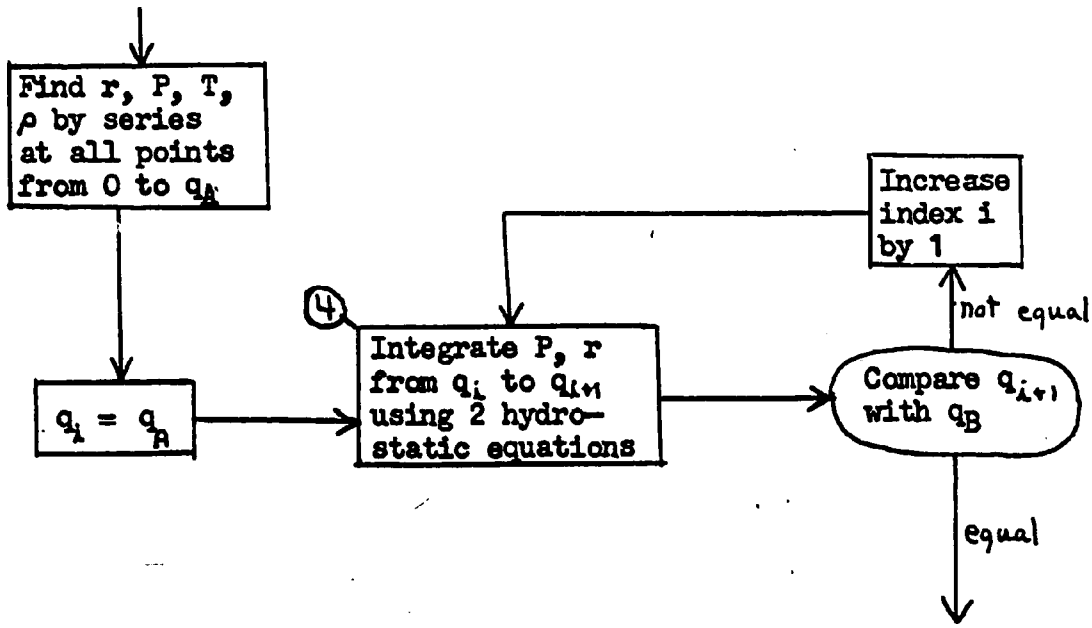
Molecular weight.

Tolerances for the four discontinuities.

Accuracy specified for hydrostatic integration.

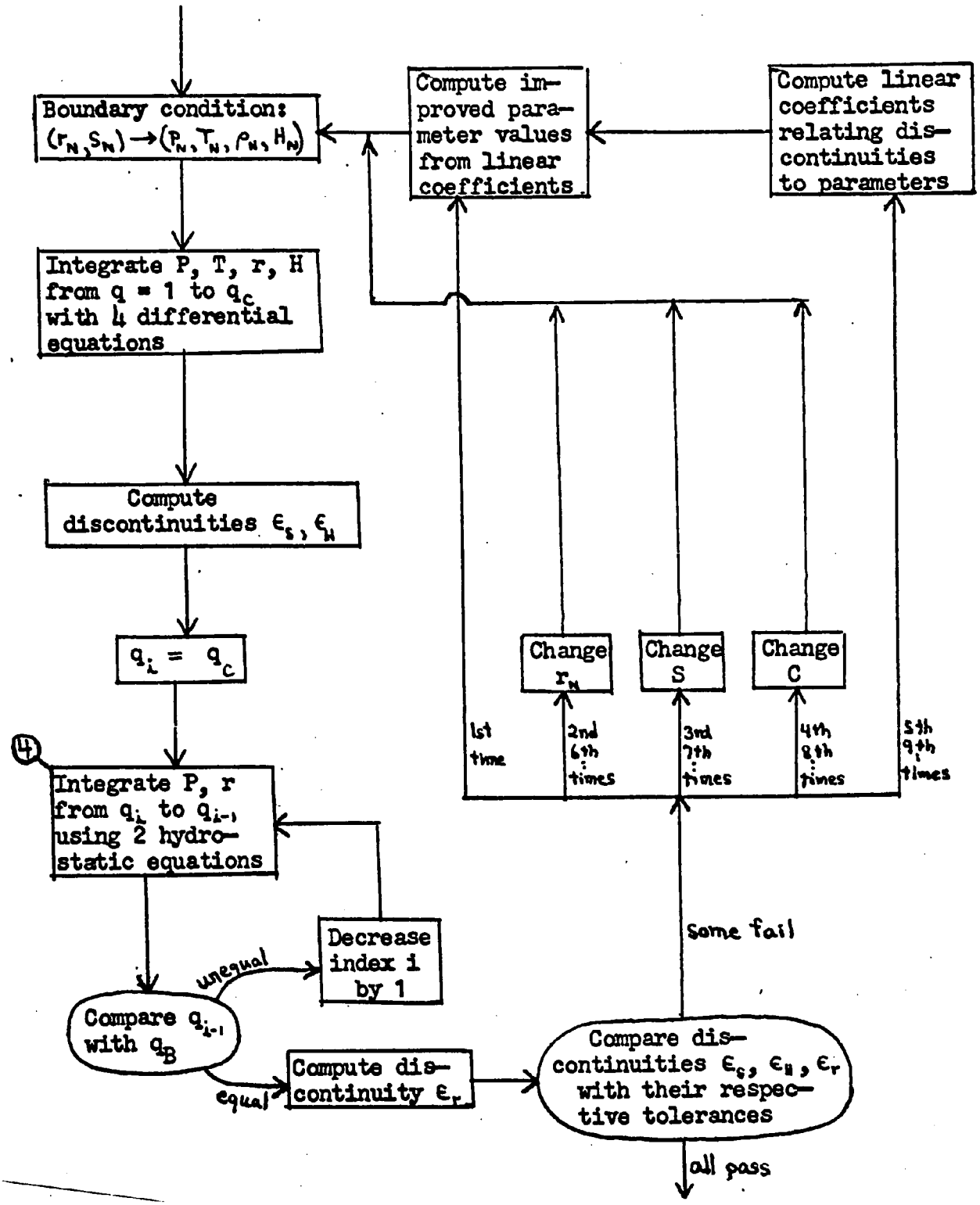
Number of models to be computed in the sequence.

Block 2: Outward integration from center to q_B



Block 3: Finding envelope for given core

The three envelope parameters r_N, S , and C are to be adjusted so that the three discontinuities ϵ_s, ϵ_H, and ϵ_r are reduced below their respective tolerances.

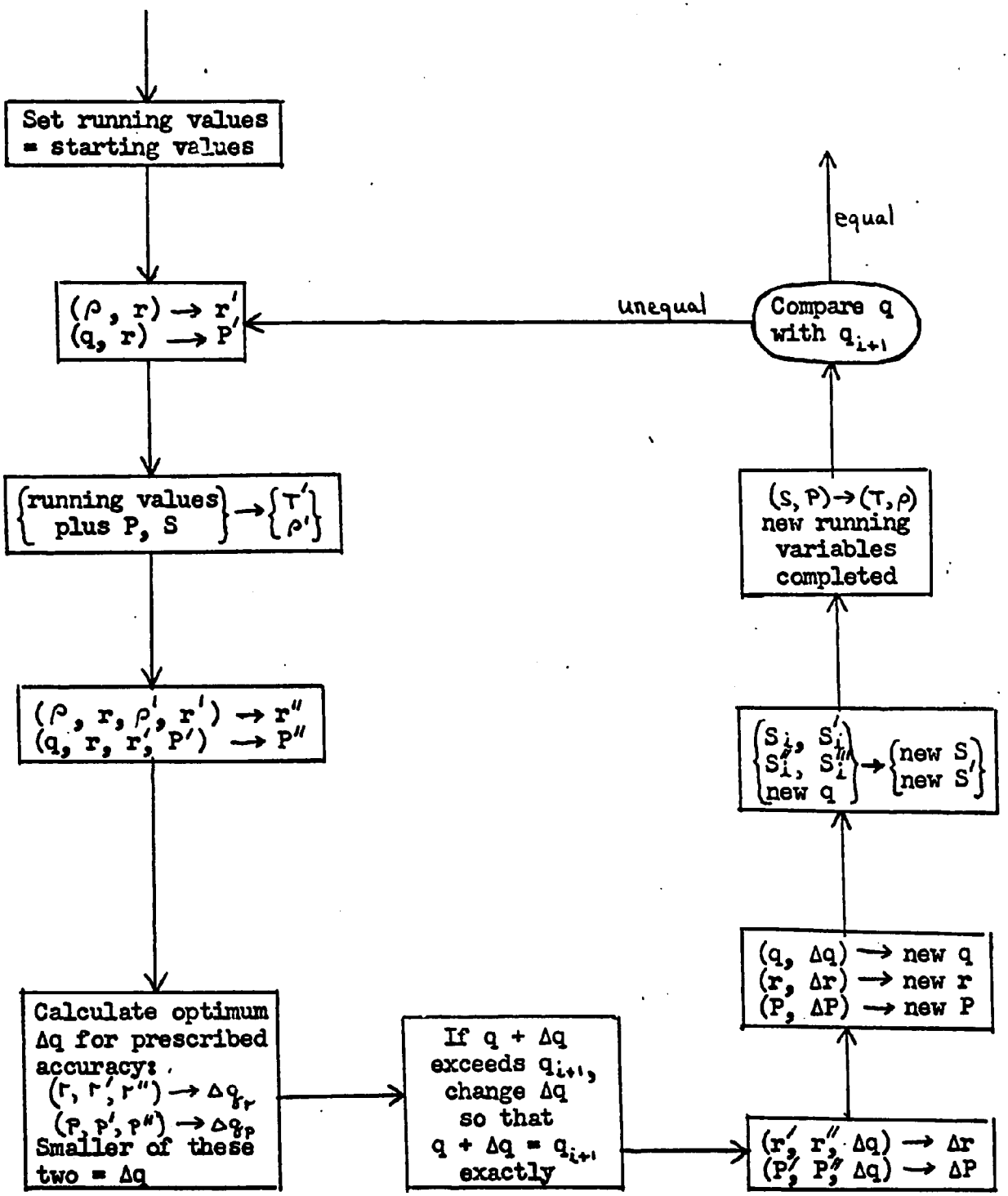


Block 4: Hydrostatic integration over one thermal grid interval

Running variables are called: q, r, P, T, ρ, S, S'.

Starting values at point i: q_i, r_i, P_i, T_i, ρ_i, S_i, S_i', S_i'', S_i'''.

Primes denote derivatives with respect to q. The object is to integrate the two hydrostatic equations from q_i to q_{i+1}.



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