Supporting Information for "Breaking Internal Waves and Ocean Diapycnal Diffusivity in a High-Resolution Regional Ocean Model: Evidence of a Wave-turbulence Cascade"

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Text ST1 The fundamental concept behind the Galerkin method is to expand all the unknown fields by using a set of orthogonal basis functions. This allows for the transformation of the boundary value problem into a matrix eigenvalue problem. In the main paper, this method was used to model the dynamical equations that describe the internal

wave modes, with both κ and ν assumed to be fixed and depth-independent values. Here, we derive the more general equations in which κ and ν can vary significantly with depth.

Similar to Eq. 8 of the main paper, we begin by writing the Reynolds-averaged Navier-Stokes equations for continuity, momentum, and energy conservation under the turbulent viscosity and gradient diffusion hypothesis:

$$\begin{array}{l} u_x + v_y + w_z = 0 \\ u_t + uu_x + vu_y + wu_z = -\frac{p_x}{\rho_0} + \nu\nabla^2 u + fv \\ v_t + uv_x + vv_y + wv_z = -\frac{p_y}{\rho_0} + \nu\nabla^2 v - fu \\ w_t + uw_x + vw_y + ww_z = -\frac{p_z}{\rho_0} + \nu\nabla^2 w - g\frac{\rho}{\rho_0} \\ \rho_t + u\rho_x + v\rho_y + w\rho_z = \kappa\nabla^2\rho \end{array} \tag{SE1}$$

This set of equations can be further simplified using the first-order perturbation theory. By assuming

$$\begin{array}{l} \{u, v, w\} = \{u'(x, y, z, t), v'(x, y, z, t), w'(x, y, z, t)\} \\ \{\rho, p\} = \{\bar{\rho}(z) + \rho'(x, y, z, t), \bar{p}(z) + p'(x, y, z, t)\} \end{array} \tag{SE2}$$

and neglecting higher-order terms, and also assuming hydrostatic balance, we get:

$$\begin{array}{l} u'_x + v'_y + w'_z = 0 \\ u'_t = -\frac{p'_x}{\rho_0} + \nu\nabla^2 u' + \frac{\partial\nu}{\partial z}\left(u'_z + w'_x\right) + fv' \\ v'_t = -\frac{p'_y}{\rho_0} + \nu\nabla^2 v' + \frac{\partial\nu}{\partial z}\left(v'_z + w'_y\right) - fu' \\ w'_t = -\frac{P'_z}{\rho_0} + \nu\nabla^2 w' + 2\frac{\partial\nu}{\partial z}w'_z - g\frac{\rho'}{\rho_0} \\ \rho'_t = -w'\bar{\rho}_z + \kappa\nabla^2\rho' + \frac{\partial\kappa}{\partial z}\rho'_z \end{array} \tag{SE3}$$

The perturbation terms are considered to have a plane waveform as follows:

$$\{u', v', w', p', \rho'\}(x, y, z, t) \ = \{\tilde{u}(z), \tilde{v}(z), \tilde{w}(z), \tilde{p}(z), \tilde{\rho}(z)\}e^{\left(\sigma t - i(kx+ly)\right)}, \tag{SE4}$$

where

$$\begin{array}{ll} \{\tilde{u}(z), \tilde{v}(z), \tilde{p}(z)\} & = \sum_{j=1}^{J}\{u_j, v_j, p_j\}\cos\left(m_j z\right), \\ \{\tilde{w}(z), \tilde{\rho}(z)\} & = \sum_{j=1}^{J}\{w_j, \rho_j\}\sin\left(m_j z\right). \end{array} \tag{SE5}$$

Now, by assuming Galerkin basis functions of Eq. SE4 and substituting the perturbation terms in Eq. SE3, the following equations set will be reached:

$$
\begin{aligned}
&iku' + ilv' - w'_z = 0\\
&\sigma u' = \tfrac{ikp'}{\rho_0} + \nu\nabla^2 u' + \tfrac{\partial\nu}{\partial z}\big(u'_z - ikw'\big) + fV'\\
&\sigma v' = \tfrac{ilp'}{\rho_0} + \nu\nabla^2 v' + \tfrac{\partial\nu}{\partial z}\big(v'_z - ilw'\big) - fu'\\
&\sigma w' = -\tfrac{p'_z}{\rho_0} + \nu\nabla^2 w' + 2\tfrac{\partial\nu}{\partial z}w'_z - g\tfrac{\rho'}{\rho_0}\\
&\sigma\rho' = -w'\bar{\rho}_z + \kappa\nabla^2\rho' + \tfrac{\partial\kappa}{\partial z}\rho'_z
\end{aligned}
\tag{SE6}
$$

Defining $G' \equiv iku' + ilv'$ and $\Omega' \equiv ikv' - ilu'$, the second and the third lines of Eq. SE6 can be rewritten as:

$$
\begin{aligned}
&iku' + ilv' - w'_z = 0\\
&\sigma G' = -\tfrac{k^2+l^2}{\rho_0}p' + \nu\nabla^2 G' + \tfrac{\partial\nu}{\partial z}G'_z + \tfrac{\partial\nu}{\partial z}(k^2+l^2)w' + f\Omega'\\
&\sigma\Omega' = \nu\nabla^2\Omega' + \tfrac{\partial\nu}{\partial z}\Omega'_z - fG'\\
&\sigma w' = -\tfrac{p'_z}{\rho_0} + \nu\nabla^2 w' + 2\tfrac{\partial\nu}{\partial z}w'_z - g\tfrac{\rho'}{\rho_0}\\
&\sigma\rho' = -w'\bar{\rho}_z + \kappa\nabla^2\rho' + \tfrac{\partial\kappa}{\partial z}\rho'_z
\end{aligned}
\tag{SE7}
$$

Now, by substituting $G' = iku' + ilv' = w'_z$ and eliminating p', Eq. SE7 can be simplified to the following equation set:

$$
\begin{aligned}
&\sigma\Omega' = \nu\nabla^2\Omega' + \nu_z\Omega'_z - fw'_z\\
&\sigma\Big[(k^2+l^2)w' - w'_{zz}\Big] = -\nu_z\nabla^2 w'_z - \nu\nabla^2 w'_{zz} - \nu_{zz}w'_{zz} - \nu_z w'zzz\\
&\qquad\qquad -\nu_z w'_z(k^2+l^2) - \nu_{zz}w'(k^2+l^2)\\
&\qquad\qquad +(k^2+l^2)\nu\nabla^2 w' + 2(k^2+l^2)\nu_z W'_z\\
&\qquad\qquad -(k^2+l^2)\tfrac{g}{\rho_0}\rho' - f\Omega'_z\\
&\sigma\rho' = -w'\bar{\rho}_z + \kappa\nabla^2\rho' + \tfrac{\partial\kappa}{\partial z}\rho'_z
\end{aligned}
\tag{SE8}
$$

Note that the perturbation terms are now:

$$
\begin{aligned}
&\Omega'(x,y,z,t) = \Big[\textstyle\sum_{j=1}^{J}\Omega_j cos(m_j z)\Big]e^{\left(\sigma t - i(kx+ly)\right)}\\
&\{w'(x,y,z,t),\rho'(x,y,z,t)\} = \Big[\textstyle\sum_{j=1}^{J}\{w_j,\rho_j\}sin(m_j z)\Big]e^{\left(\sigma t - i(kx+ly)\right)}
\end{aligned}
\tag{SE9}
$$

By substituting the above perturbation terms in Eq. SE8, the following equations set will be reached:

$$
\begin{aligned}
&\sigma\Omega_i = -\nu^{cc}_{ij}k_j^2\Omega_j - \nu^{cs}_{zij}m_j\Omega_j - fm_i w_i\\
&\sigma k_i^2 w_i = \nu^{sc}_{zij}k_j^2 m_j w_j - \nu^{ss}_{ij}k_j^2 m_j^2 w_j + \nu^{ss}_{zzij}m_j^2 w_j\\
&\qquad +\nu^{sc}_{zij}m_j^3 w_j - \nu^{sc}_{zij}(k^2+l^2)m_j w_j - \nu^{ss}_{zzij}(k^2+l^2)w_j\\
&\qquad -\nu^{ss}_{ij}k_j^2(k^2+l^2)w_j + 2\nu^{sc}_{zij}(k^2+l^2)m_j w_j - \tfrac{g}{\rho_0}(k^2+l^2)\rho_i\\
&\qquad +fm_i\\
&\sigma\rho_i = -\rho^{ss}_{zij}w_j - \kappa^{ss}_{ij}k_j^2\rho_j + \kappa^{sc}_{zij}m_j\rho_j
\end{aligned}
\tag{SE10}
$$

where the superscripts "cc", "cs", "sc", and "ss" for an arbitrary function h are defined as follows:

$$h_{ij}^{cc} = \frac{2}{L}\int_0^L cos(m_i z)cos(m_j z)h(z)dz \tag{SE11}$$

$$h_{ij}^{cs} = \frac{2}{L}\int_0^L cos(m_i z)sin(m_j z)h(z)dz \tag{SE12}$$

$$h_{ij}^{sc} = \frac{2}{L}\int_0^L sin(m_i z)cos(m_j z)h(z)dz \tag{SE13}$$

$$h_{ij}^{ss} = \frac{2}{L}\int_0^L sin(m_i z)sin(m_j z)h(z)dz \tag{SE14}$$

For instance:

$$\rho_{zij}^{ss} = \frac{2}{L}\int_0^L sin(m_i z)sin(m_j z)\frac{\partial\bar{\rho}(z)}{\partial z}dz \tag{SE15}$$

When the derivatives of κ and ν with respect to z are ignored in Eq. SE10, the same results as in Eq. 11 of the paper are reached. Therefore, Eq. SE10 above should be thought of as a more general form of Eq. 11 in the paper where full depth-dependence of κ and ν are taken into account.

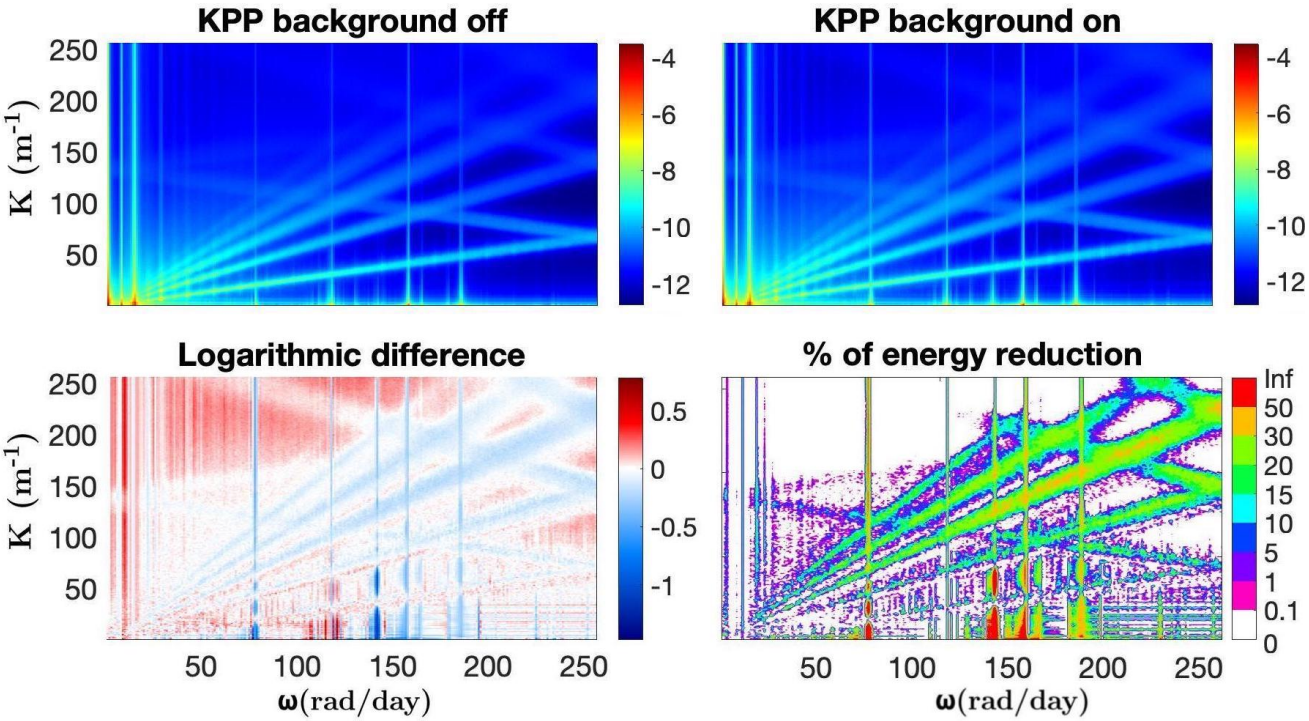


Figure SF1. The spectrum of sea surface height squared, SSH^2 ($\mathrm{m}^3\,\mathrm{day}\,\mathrm{rad}^{-1}$), in the horizontal wavenumber-temporal frequency ($K-\omega$) space for the dynamically-downscaled regional model. The two panels at the top row are identical to Figure 3c and Figure 3d in the main paper, showing two cases with the background component of KPP turned off and on. They are both shown in the logarithmic space. Although they appear similar, their difference, shown in the bottom-left panel, reveals the IW modal structure. Finally, the percentage of energy reduction (i.e., the red segments with positive energy accumulation filtered out, and the color code converted to the linear space), is shown in the bottom-right panel. The two panels in the bottom row are equivalent to Figure 8a of the main paper, albeit for the SSH data. This figure demonstrates that, after turning off the background component of KPP in the dynamically-downscaled regional domain, the energy of IW low-order modes at the surface is drained away and accumulated in the near-inertial wedge, although not as clearly captured as in the interior ocean.