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SYMBOLIC COMPUTER SOLUTION OF ELLIPTIC BOUNDARY VALUE PROBLEMS

by

Norman Loren Schryer

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ABSTRACT

SYMBOLIC COMPUTER SOLUTION OF ELLIPTIC BOUNDARY VALUE PROBLEMS

by

Norman Loren Schryer

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We study the elliptic boundary value problem:

$Lu = \Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0$ with $c(x,y) \leq 0$ on a simply connected domain D and $u = f$ given on ∂D.

Under the assumption that the coefficients a, b, and c are analytic functions of x and y, and following ideas of Stefan Bergman and of I. N. Vekua, a system of particular solutions, $p_k(x,y)$, of $Lu = 0$ is constructed. Each p_k is given by an infinite series whose elements are defined recursively in terms of a, b, and c. These solutions are independent of the domain D and represent generalizations of the harmonic polynomials $\mathrm{Re}\,(x+iy)^k$ and $\mathrm{Im}\,(x+iy)^k$ for the equation $\Delta u = 0$. Let $p_{k,T}$ be defined by truncating the series expansion of p_k after T terms. For any vector $s = (s_1, \ldots, s_N)$ let

$$u(z,s) = \sum_{k=1}^{N} s_k p_{k,T}(z) \quad .$$

Given boundary values f, a compact subset, S_N, of R^N is constructed that contains the necessary coefficients for convergence to the solution u.

To find these coefficients, we choose points $z_1, \ldots, z_M$ which are uniformly distributed around ∂D. Define $s^{N,M,T}$ to be a solution of the linear approximation problem

$$\min_{s \in S_N} \max_{j=1,\ldots,M} | f(z_j) - u(z_j, s)| \quad .$$

Then $u_{N,M,T}(z) = u(z, s^{N,M,T})$ is a best fit to u at $z_1, \ldots, z_M$. Our principal result is that

$$\lim_{N \to \infty} \left[\lim_{M,T \to \infty} \max_{D \cup \partial D} |u(z) - u_{N,M,T}(z)| \right] = 0 \quad .$$

The truncated solutions $p_{k,T}$ may be constructed exactly, but they are not exact solutions of $Lu = 0$. A computable a posteriori error bound is given for $|u - u_{N,M,T}|$ in terms of how well $u_{N,M,T}$ satisfies the boundary conditions and the differential equation $Lu = 0$.

An algorithm similar to the above is given for solving the inhomogeneous equation $Lu = g$ for analytic coefficients a, b, c, and g.

Several examples are given and numerical results presented for the special case where a, b, and c are polynomials in x and y.

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CHAPTER I

STATEMENT OF THE PROBLEM AND PRELIMINARY RESULTS

Introduction

We wish to solve the elliptic boundary value problem

$$Lu = \Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0$$

(I) $$u = f \quad \text{on} \quad \partial D \quad \text{with} \quad f \quad \text{given}$$

$$u \in C^{(0)}(D \cup \partial D) \cap C^{(2)}(D)$$

on a simply connected domain D in the plane. The coefficient $c(x,y)$ is assumed to be non-positive on D.

The algorithm we shall study for solving this problem consists of the following three steps.

Step 1: Generate a system of functions $p_k(x,y)$, $k = 0, 1, \ldots,$ which are formal solutions of $Lu = 0$. These functions are generalizations of the harmonic polynomials.

$$\left\{ \begin{matrix} \text{Re} \\ \text{Im} \end{matrix} \right\} (x + iy)^k \quad , \qquad k = 0, 1, \ldots,$$

for the equation $\Delta u = 0$. They are independent of the domain D.

Step 2: Determine coefficients in the linear combination $\sum_{k=0}^{N} a_k p_k(x,y)$ so that the sum approximately satisfies the boundary

conditions. The classical collocation method in which the boundary conditions are exactly satisfied at $N + 1$ points is studied briefly but abandoned in favor of a Chebyshev approximation.

Step 3: Given the coefficients a_k, produce an a posteriori error bound.

These three steps give an approximation to the true solution and a rigorous bound on the error. The approximation is defined on D in such a way that additional calculations are easily performed.

In the case where the coefficients a, b, and c are polynomials in x and y, the algorithm has been coded and run on a digital computer. The results of these runs are given in the last section of this paper.

We may, by a simple translation, assume that the origin is inside D. Then a classical result, see [5], states that if a, b, and c are real analytic functions of x and y on D and u is the solution of (I), then u can be extended to a complex solution, $U(z, z^*)$, of the complex hyperbolic problem $\mathcal{L}U = 0$, where

(II) $$\mathcal{L}U = U_{zz^*} + \mathcal{A}U_z + \mathcal{B}U_{z^*} + \mathcal{C}U = 0 \quad .$$

This is accomplished by setting $z = x + iy$, $z^* = x - iy$, and treating x and y as independent complex variables. That is, $z^* = \bar{z}$ iff x and y are real. The extension is valid on the domain $D \times D^*$ where D^* denotes the complex conjugate domain to D.

Then $\mathcal{A}(z, z^*) = (a(\frac{z + z^*}{2}, \frac{z - z^*}{2i}) + ib(\frac{z + z^*}{2}, \frac{z - z^*}{2i}))/4$,

$\mathcal{B} = (a - ib)/4$, and $\mathcal{C} = c/4$ as such complex extensions of the right-hand sides. We also have $U(z, \bar{z}) = u(x,y)$ if x and y are real. If we define $u(x,y) = U(x + iy, x - iy)$ for real x and y, then $Lu = 4\mathcal{L}U(z, \bar{z})$. Thus any complex solution U of (II) yields two real solutions of $Lu = 0$, namely, Re $U(z, \bar{z})$ and Im $U(z, \bar{z})$.

We will use the following notation throughout.

$$\| f(z) \| = \max_{D \cup \partial D} |f(z)|$$

$$\| f(z, z^*) \| = \max_{(D \cup \partial D) \times (D^* \cup \partial D^*)} |f(z, z^*)|$$

$$\| f(z) \|_\rho = \max_{|z| \leq \rho} |f(z)|$$

$$\| f(z, z^*) \|_\rho = \max_{|z|, |z^*| \leq \rho} |f(z, z^*)|$$

$$\| f(z) \|_{\partial D} = \max_{\partial D} |f(z)|$$

By combining some results of Stefan Bergman, R. S. Lehman, and I. N. Vekua, we will exhibit a bounded operator B which maps the set of functions analytic on D, with real values at the origin, in a 1-1 fashion onto the set of solutions of (I) with continuous boundary values, by the relation $u = \text{Re } B(g)$. This operator has the following form:

$$B(g) = \exp\Big(-\int_0^{z^*} \mathcal{A}(z,w)dw\Big[g(z) + \sum_{n=1}^{\infty} \frac{Q^{(n)}(z, z^*)}{2^{2n}\beta(n,n+1)} \int_0^z (z-\xi)^{n-1} g(\xi)d\xi\Big] \quad .$$

where β is the usual beta-function. The $Q^{(n)}$ depend only on the coefficients a, b, and c and can be recursively defined in terms of them. We then have for the solution u of (I) that $u(x,y) = \text{Re } B(g)(z, \bar{z})$ for some g analytic on D.

For the special case of $\mathcal{A} = \mathcal{B} = \mathcal{C} = 0$ we have $L = \Delta$ and we are studying $\Delta u = 0$. Then all the $Q^{(n)}$ are identically zero and $B(g) = g$ for all functions g. Thus for any solution u of $\Delta u = 0$ with $u \in C^{(0)}(D \cup \partial D) \cap C^{(2)}(D)$ we have an analytic g on D so that $u(x,y) = \text{Re } g(x + iy)$. This is a classical theorem about harmonic functions and represents the basis for investigating an operator of the above form. In this case, the function g is continuous on the closure of the domain D. By a theorem of Walsh [28] we can find coefficients $a_0^{(N)}, \ldots, a_N^{(N)}$ so that

$$\lim_{N\to\infty} \| g(z) - \sum_{k=0}^{N} a_k^{(N)} z^k \| = 0 \quad .$$

Letting $a_k^{(N)} = \alpha_k^{(N)} - i\beta_k^{(N)}$ we see that

$$\lim_{N\to\infty} \| u - \sum_{k=0}^{N} (\alpha_k^{(N)} \text{ Re } (z^k) + \beta_k^{(N)} \text{ Im } (z^k)) \| = 0 \quad .$$

Thus, in this case, the harmonic polynomials $\{\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\}(z^k)$ for $k = 0, 1, \ldots$ are complete in the space of solutions to $\Delta u = 0$ with continuous boundary values. We are led to suspect, in general, that the set of particular solutions of $Lu = 0$ given by

$$\{\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\} B(z^k) \quad \text{for} \quad k = 0, 1, \ldots,$$

will be complete in the space of solutions to (I) with continuous boundary values. This is indeed true. However, since $B(z^k)$ involves an infinite series in general, we cannot construct it practically. We define B_T to be the truncated operator

$$B_T(g) = \exp(-\int_0^{z^*} \text{Ⓡ}(z,w)dw)[g(z) + \sum_{n=1}^{T} \frac{Q^{(n)}(z,\, z^*)}{2^{2n}\beta(n,n+1)} \int_0^z (z-\xi)^{n-1} g(\xi)d\xi] \quad .$$

When $g(z) = z^k$ the integration may be carried out to obtain

$$B_T(z^k) = \exp(-\int_0^{z^*} \text{Ⓡ}(z,w)dw) z^k (1 + \sum_{n=1}^{T} \frac{(2n)!\, k!\, Q^{(n)}(z,\, z^*)}{2^{2n} n!\, (n+k)!}\, z^n)$$

in "closed" form.

We will show that for any solution u of (I) there are real coefficients $a_0^{(N)}, \ldots, a_N^{(N)}, b_0^{(N)}, \ldots, b_N^{(N)}$, so that

$$\lim_{N\to\infty} [\lim_{T\to\infty} \| u - \sum_{k=0}^{N} (a_k^{(N)} \mathrm{Re}\, B_T(z^k) + b_k^{(N)} \mathrm{Im}\, B_T(z^k)) \|] = 0 \quad .$$

Hence, we can use the closed form $B_T(z^k)$ to obtain arbitrarily accurate uniform approximations to u. The main aim of this paper is the construction of such coefficients in a practical manner, given only u on ∂D.

We will take the type of approach used by J. R. Cannon [9, 10] on the boundary value problem for $\Delta u = 0$. Choose M points on the boundary of the domain and determine coefficients in the linear combination

$$\sum_{k=0}^{N} (s_{2k} \operatorname{Re} B_T(z^k) + s_{2k+1} \operatorname{Im} B_T(z^k))$$

which produce a best fit to the given boundary values of u at these points. Call that linear combination $u_{N,M,T}$. It will be shown that $u_{N,M,T}$ approaches the true solution of the problem as the parameters become large. The details of the algorithm are given below.

Let u be the solution of (I). Then we can construct a compact coefficient space $S_N \subset R^{2(N+1)}$, with elements of the form $s = (a_0, \ldots, a_N, b_0, \ldots, b_N)$, with the property that if we define

$$u_{N,T}(z, s) = \sum_{k=0}^{N} (a_k \operatorname{Re} B_T(z^k) + b_k \operatorname{Im} B_T(z^k)) \quad ,$$

then

$$\lim_{N\to\infty}\left[\lim_{T\to\infty}\min_{s\in S_N}\|u - u_{N,T}(z,s)\|\right] = 0 \quad .$$

Thus the compact set S_N contains the necessary coefficients for convergence to the true solution. To find these coefficients, choose points $z_1^{(M)}, \ldots, z^{(M)}$ on ∂D with the property that as $M \to \infty$, the maximum distance between any two consecutive $z_j^{(M)}$'s goes to zero. Let $s^{N,M,T} \in S_N$ solve the linear approximation problem:

$$\min_{s\in S_N}\max_{j=1,\ldots,M}\left|u(z_j^{(M)}) - u_{N,T}(z_j^{(M)}, s)\right| \quad .$$

Then set $u_{N,M,T} = u_{N,T}(z, s^{N,M,T})$; it is the best fit to u at the $z_j^{(M)}$'s out of all possible $u_{N,T}$. It seems reasonable, and it will be shown that

$$\lim_{N\to\infty}\left[\lim_{M,T\to\infty}\|u - u_{N,M,T}\|\right] = 0 \quad .$$

The $u_{N,M,T}$ are explicitly computable using linear programming and a digital computer. Hence, we have an explicit method for constructing uniform approximations to solutions of (I).

Various examples have been run on a computer using the above technique, and the results have been very good, provided there are no corners in the domain. In the presence of a corner, or corners, in the domain, the convergence is very slow, too slow to be practical.

This problem arises because the partial derivatives of the solution near any corner may become infinite but the partial derivatives of $B_T(z^k)$ are bounded.

Let the domain D have a corner of interior angle π/α at a point $a \in \partial D$. Then R. S. Lehman [19] has shown that u and its partials near a behave like polynomials in $(z-a)^{\alpha}$. If there are corners $a_1, \ldots, a_L$ of angles $\pi/\alpha_1, \ldots, \pi/\alpha_L$ on ∂D, we take advantage of that fact and use $\{B_T((z-a_1)^{\alpha_1 k})\}_{k=0}^{\infty}, \ldots, \{B_T((z-a_L)^{\alpha_L k})\}_{k=0}^{\infty}$ as well as $\{B_T(z^k)\}_{k=0}^{\infty}$ in the above linear approximation scheme. This gives the approximating functions the same shape, or singularities in their partial derivatives, as the solution may have and the convergence rate is much faster.

The point to be stressed here is the fact that the functions $B_T((z-a)^{\alpha k})$ and $B_T(z^k)$ may be computed given only the coefficients a, b, and c. Thus, we may construct uniform approximations $u_{N,M,T}$ to solutions of (I).

Summary of Results

In Chapter 2 we show that $B_T(g) \to B(g)$ and $\mathcal{L}B_T(g) \to 0$ as $T \to \infty$, under suitable conditions. The main results of this chapter are in Theorems 2.2 and 2.3.

Chapter 3 extends the completeness properties of the operator V as defined by Vekua. The main results of this section are contained in Theorems 3.1 and 3.5.

A bound for u in terms of $\|u\|_{\partial D}$ and $\|Lu\|$ is obtained in Chapter 4. This gives an a posteriori error bound.

Chapter 5 shows how to construct the particular solutions ϕ_k and ψ_k. For the special case where the coefficients a, b, and c are polynomials in x and y, an algorithm is discussed for the closed form computation of the ϕ_k and ψ_k.

A collocation result is presented in Chapter 6 for the special case of the equation $\Delta u + c(r^2)u = 0$ on the unit disk.

Chapter 7 shows that the technique outlined in the last few pages actually must converge. The main result here is Theorem 7.13.

An algorithm is presented in Chapter 8 for the a posteriori calculation of error bounds. Theorem 8.1 is the basic error bounding inequality.

In Chapter 9, the requirement that $c(x,y)$ be analytic is relaxed to the restriction that $c(x,y)$ be merely continuous for an equation of the form $\Delta u + c(x,y)u = 0$.

In Chapter 10, the inhomogeneous equation $Lu = g$ is solved for analytic coefficients a, b, c, and g. This hypothesis may be weakened to the assumption that c and g are merely continuous, with a and b analytic as usual.

The computer technique we have used in practice is presented in Chapter 11. Two algorithms are discussed, one with and one without corners present.

The numerical results this algorithm produced for several examples are presented in Chapter 12.

Brief Review of the Literature

Many schemes have been developed to solve elliptic boundary value problems. Perhaps the foremost method, at the present time, is that of finite differences. It is relatively easy to program on a digital computer and has a wide range of applicability. The finite difference method has three distinct drawbacks, however. First, it is very expensive to run on any large scale problem, in terms both of time and storage. Second, even when the solution has been obtained, it is known only at a finite number of points. Third, there is no easy way to bound the error at these points. In spite of these

deficiencies, finite differences is the most successful known technique for solving such boundary value problems. There are, at present, very few ways to obtain useful solutions to these problems, but finite difference schemes work and are easy to program.

Recently, many mathematicians have begun using a "functional" approach to solve elliptic boundary value problems. They construct functions u^* which almost satisfy the boundary conditions and almost satisfy the differential equation. These u^*, intuitively, should then be close to the true solution of the problem.

For our problem, there is a constant M so that

(III) $$\|v\| \leq \|v\|_{\partial D} + M\|Lv\|$$

for all $v \in C^{(0)}(D \cup \partial D) \cap C^{(2)}(D)$. If one can generate functions u^* such that $\|u - u^*\|_{\partial D} \to 0$ and $\|Lu^*\| \to 0$, then by (III) we will have $\|u - u^*\| \to 0$. Thus, <u>any</u> process which yields such functions u^* will converge to the solution of the problem. This would give uniform convergence, over the closure of the domain, of known functions to the true solution. If one can find Lu^* explicitly, then (III) will provide an a posteriori error bound.

Since we may easily find $L(B_T(g))$ for any function g and the constant M is known, (III) provides an a posteriori error bound for our process. In fact, (III) states that

$$\|u - u^*\| \leq \|u - u^*\|_{\partial D} + M\|Lu^*\|$$

and the error over the closure of the domain may be bounded in terms of how well u^* satisfies the boundary conditions and the differential equation.

J. B. Rosen [24] has used this type of idea to generate polynomial approximations to solutions of mildly non-linear elliptic boundary value problems of the form $\Delta u = g(x,y,u)$. His method is conceptually quite simple:

Find $a \in C \subset R^{N^2}$, C compact, so that a minimizes the function

$$E(m,n,N,a) = \max_{j=1,\ldots,m} |u(z_j^{(m)}) - u(z_j^{(m)}, a)|$$

$$+ \max_{j=1,\ldots,n} |g(w_j^{(n)}, u(w_j^{(n)}, a)) - \Delta u(w_j^{(n)}, a)|$$

where $u(z,a) = \sum_{i,j=1}^{N} a_{ij} x^{i-1} y^{j-1}$, $z = x + iy$ and $z_j^{(m)} \in \partial D$, $w_j^{(n)} \in D$ are uniformly distributed around ∂D and D, respectively. This is a non-linear programming problem to be taken over larger and larger compact sets C. It is intuitively obvious that as $N \to \infty$ and $C \to R^{N^2}$, we will have $E(m,n,N,a) \to 0$. Let $a^* \in C$ be a solution of the above non-linear programming problem and set $u^* = u(z, a^*)$. Then $|u - u^*|$ and $|Lu - Lu^*|$ will be small at the $z_j^{(m)}$'s and $w_j^{(n)}$'s, hopefully. As more and more $z_j^{(m)}$'s and $w_j^{(n)}$'s are chosen, it is reasonable to expect

that $\|u - u^*\|$ will tend to zero. Although a convergence proof has not been given for this technique, Rosen uses an inequality like that in (III) to show that in practice it works quite well.

J. R. Cannon [9, 10] studied the equation $\Delta u = 0$. He uses linear combinations of the harmonic polynomials $\left\{ \begin{matrix} \text{Re} \\ \text{Im} \end{matrix} \right\} (z^k)$, $k = 0, 1, \ldots,$ to approximate the solutions of $\Delta u = 0$ which have continuous boundary values. Let

$$u_N(z, s) = a_0 + \sum_{k=1}^{N} (a_k \text{Re}\,(z^k) + b_k \text{Im}\,(z^k)), \quad \text{where}$$

$s = (a_0, \ldots, a_N, b_1, \ldots, b_N) \in R^{2N+1}$. He shows how to construct a compact set $S_N \subset R^{2N+1}$ with the property that

$$\lim_{N \to \infty} \left[\min_{s \in S_N} \|u - u_N(z, s)\| \right] = 0 \quad .$$

Thus, the compact set S_N contains the necessary coefficients for convergence to the true solution. To find these coefficients, choose $z_1^{(m)}, \ldots, z_m^{(m)}$ on ∂D so that as $m \to \infty$ the greatest distance between consecutive $z_j^{(m)}$'s goes to zero. Let $s_{N,m} \in S_N$ solve the linear programming problem:

$$\min_{s \in S_N} \max_{j=1,\ldots,m} \left| u(z_j^{(m)}) - u_N(z_j^{(m)}, s) \right| \quad ,$$

and set $u_{N,m} = u_N(z, s_{N,m})$. Cannon shows that this best fit to the boundary values of u at the $z_j^{(m)}$'s obeys

$$\lim_{N\to\infty} \left[\lim_{m\to\infty} \| u - u_{N,m} \| \right] = 0$$

Since $\Delta \left\{ \begin{matrix} \mathrm{Re} \\ \mathrm{Im} \end{matrix} \right\} (z^k) = 0$ for $k = 0, 1, \ldots,$ we have, by the Maximum Principle, $\| u - u_{N,m} \| = \| u - u_{N,m} \|_{\partial D}$ and we need only search ∂D for the maximum error. This makes the solution of the boundary value problem for $\Delta u = 0$ a well-understood process, with excellent error bounds. But there is one proviso: The method will probably not give good results if the domain has a corner in it. In that case we would be trying to approximate a function with singularities in its partial derivatives by functions which are entire. The method will converge, but the rate of convergence will be so slow that the process is not practical.

J. R. Whiteman and others [29, 30, 31] have studied, experimentally, the difficulties caused by corners in domains. The idea is to use approximating functions which have the same singularities in their partials as the solutions may have. It has been found experimentally that a great improvement in performance is obtained by using such functions.

Ciarlett, Schultz, Varga, Birkhoff and others [8, 11] have used polynomial splines in a variational procedure to approximately satisfy both the boundary conditions and the differential equation on domains which are unions of rectangles. They prove convergence of

their method to a solution of the problem, provided the solution is contained in a certain Sobolev Space. Thus, in practice, the method is subject to the restriction that there be no re-entrant corners in the domain. Since the only such domains without re-entrant corners are rectangles, this limits the applicability of their techniques.

George J. Fix [13] has modified these techniques by using local expansions, near corners in the domain, which have the proper singularities in their partial derivatives. These are based on an expansion, given by R. S. Lehman [19], for the solution of the problem near such corners. Away from the corners, he uses regular polynomial splines. However, since the number of spline coefficients to be manipulated goes up like $1/h^2$, where h is the mesh spacing, this results in very large storage and time requirements.

Vekua's Approach

Vekua [27] develops an operator we shall denote by V:

$$V(g(z)) = \mathrm{Re}(G(z,0,z,\bar{z})g(z) - \int_0^z g(t)H(t,0,z,\bar{z})dt) \quad ,$$

where G is the Riemann function for the complex hyperbolic problem $\mathcal{L}U = 0$ and

$$H(t,0,z,z^*) = \frac{\partial}{\partial t}G(t,0,z,z^*) - \mathcal{B}(t,0)G(t,0,z,z^*) \quad .$$

We shall call a solution of $Lu = 0$ <u>regular</u> if it is in $C^{(2)}(D)$. The operator V is a 1-1 map from the set of functions $g(z)$ which are analytic on D with real values at the origin onto the space of solutions of $Lu = 0$ which are regular. If $u = V(g)$ then $V^{-1}u$ is given by equation 12.11 on page 56 of [27]:

$$V^{-1}(u) = 2U(z,0) - U(0,0)G(0,0,z,0) \quad ,$$

where U is the complex extension of u. By the definition of the Riemann function G we have

$$(1) \qquad V^{-1}(u) = 2U(z,0) - U(0,0)\exp\left(-\int_0^z \mathcal{B}(z',0)dz'\right) \quad .$$

All of this is valid for simply connected domains with piecewise smooth boundaries. By restricting his attention to domains with smooth boundaries and solutions with smooth boundary values, Vekua is able to prove various smoothness and completeness theorems about the operator V.

Define the <u>class Ah</u> to be the collection of all simply connected domains which contain the origin and which have Hölder continuous tangents defined everywhere on ∂D. A domain of class Ah cannot have a corner in its boundary.

We now state the results of Vekua which we will use.

Vekua's Theorem 3 on page 128 states that

Theorem 1.1. If D is of class Ah and u is a regular solution of $Lu = 0$ with u Hölder continuous on ∂D, then the unique $g(z)$, analytic on D with $g(0)$ real, so that $u = V(g)$, is Hölder continuous on $D \cup \partial D$.

Vekua's Theorem 1 on page 156 states that

Theorem 1.2. For any regular solution of $Lu = 0$ on a domain D of class Ah with Hölder continuous boundary values, given any $\epsilon > 0$ there are $a_0^{(N)}, \ldots, a_N^{(N)}$ so that

$$\| u - V(\sum_{k=0}^{N} a_k^{(N)} z^k) \| \leq \epsilon \quad .$$

The above theorems provide us with certain smoothness and completeness properties of the operator V. However, the operator V requires the Riemann function for the equation $\mathcal{L}U = 0$; this is generally not known explicitly.

Bergman's Approach

Stefan Bergman [2, 3, 4, 5, 6, 7; especially 5] considers exactly the same problem as Vekua does, except that he requires the coefficients a, b, and c to be analytic on the larger set $\{ z \mid |z| \leq 3r \}$, $r = \max_{\partial D} |z|$. His aim is the construction of an operator of the same type as that of Vekua, but which can be constructed directly from the coefficients a, b, and c.

Define the functions

$$D(z, z^*) = -\int_0^{z^*} A_z dz^* + B, \quad F(z, z^*) = -A_z - A B + C \text{ and}$$

$$Q^{(1)} = -2 \int_0^{z^*} F dz^* .$$

Recursively define

$$Q^{(p)}(z, z^*) = -\frac{2}{2p-1}[Q_z^{(p-1)}(z, z^*) - Q_z^{(p-1)}(z, 0) + \int_0^{z^*} (DQ_{z^*}^{(p-1)} + FQ^{(p-1)})dz^*]$$

for $p = 2, 3, \ldots$. Then for any $g(z)$ analytic on D, define

$$B(g) = \exp(-\int_0^{z^*} A(z, w)dw)(g(z) + \sum_{n=1}^{\infty} \frac{Q^{(n)}(z, z^*)}{2^{2n}\beta(n, n+1)} \int_0^z (z-\xi)^{n-1} g(\xi)d\xi) .$$

In Bergman's notation, the above operator is denoted by $c_2(g)$.

Bergman shows that this operator B transforms $g(z)$ into a solution of $LU = 0$. Also, given a regular solution of $Lu = 0$ there is a unique analytic function $g(z)$ with $g(0)$ real so that $u(x,y) = \text{Re } B(g)(z, \bar{z})$. This function is given by equation 16 on page 24 of [5]:

$$(2) \qquad (\text{Re } B)^{-1}(u) = 2U(z, 0) - U(0, 0)\exp(-\int_0^z B(z', 0)dz') .$$

Notice that the operator B needs only the coefficients a, b, and c to be completely defined. The Riemann function for the equation $\mathcal{L}U = 0$ is not required.

We see from (1) and (2) that for any solution of $Lu = 0$ which is regular

$$(\text{Re } B)^{-1}(u) = V^{-1}(u) \quad .$$

Hence, $\text{Re } B = V$ and Vekua's smoothness and completeness results apply to the operator Bergman constructed directly from the coefficients.

Peter Henrici has published several papers which have interpreted the works of Vekua and Bergman [15, 16, 17]. The result that $V = \text{Re } B$ can be inferred indirectly by comparing the results of [15] and [17]. It is stated explicitly in [14], but not proved. Henrici's assertion that Vekua and Bergman had developed the same operator was the motivation for studying their work and also provided the impetus for our looking at "symbolic" solutions to elliptic boundary value problems.

Some Definitions

Anticipating the use of Vekua's and Lehman's results, we define the particular solutions of $\mathcal{L}U = 0$

$$\phi_k(z, z^*) = B(z^k) \quad \text{and} \quad \psi_k(z, z^*) = B((z - a)^{\alpha k}) \quad \text{for} \quad k = 0, 1, \ldots \quad ,$$

where ∂D has a corner of angle π/α at the point $a \in \partial D$. Also define the truncated particular solutions

$$\phi_{k,T}(z, z^*) = B_T(z^k) \quad \text{and} \quad \psi_{k,T}(z, z^*) = B_T((z-a)^{\alpha k}) \quad .$$

We will write $\phi_k(z) = \phi_k(z, \bar{z})$ and $\psi_k(z) = \psi_k(z, \bar{z})$; the the meaning will be clear from the context.

We now rigorously define the term "corner." A corner is a point $a \in \partial D$ at which the tangent to the boundary fails to be Hölder continuous and near which the boundary of D consists of two analytic arcs Γ_1 and Γ_2 so that $\Gamma_1 \cap \Gamma_2 = \{a\}$ and the tangents to Γ_1 and Γ_2 at the point a are non-zero, with an interior angle π/α where $\frac{1}{2} < \alpha < \infty$.

Let AD be the class of all domains which are simply connected, contain the origin, and whose boundaries have Hölder continuous tangents except at a finite number of corners.

A domain D of class AD is said to be nearly starlike if there are constants $K < 2$ and L such that: For any $z \in D$ there is a path, traversed by ξ, in the domain from 0 to z on which $|z - \xi| \leq K|z|$, and the length of which is $\leq L|z|$. These domains cannot differ too much from being starlike with respect to the origin.

CHAPTER 2

TRUNCATION EFFECTS

We will study the effect of truncating the series for the operator B at a finite number of terms. The following result is the basis for this section. It represents a minor, but practical, modification of Bergman's Theorem 2.1 on page 13 of [5].

Theorem 2.1. Let $r = \max_{\partial D} |z|$, then there exist constants $\overline{F}$ and A independent of n such that

$$\|Q^{(n)}\|_r \leq \frac{9}{2}\overline{F}\,(n + A - 1)\ldots(A + 1)/(r^{n-2}1.3\ldots(2n-1))$$

$$\|Q^{(n)}_{z^*}\|_r \leq \frac{9}{2}\overline{F}\,(n + A - 1)\ldots(A + 1)/(r^{n-1}1.3\ldots(2n-1)) .$$

Proof.

We will use the theory of dominants. We write $f << g$ if the coefficients in the MacLaurin series for g dominate those of of f.

Let $\overline{D}$ and $\overline{F}$ be constants chosen such that $D(z, z^*) << \overline{D}/((1 - z/3r)(1 - z^*/3r))$ and $F(z, z^*) << \overline{F}/((1 - z/3r)(1 - z^*/3r))$. This may be done since the functions a, b, and c are analytic for $|(x,y)| \leq 3r$.

Letting $Q^{(n)}_{z^*} = P^{(n)}$ we see that $P^{(1)} = -2F$ and

$$P^{(n+1)} = -2(P^{(n)}_z + DP^{(n)} + F\int_0^{z^*} P^{(n)}dz^*)/(2n+1) .$$

Define dominants $N^{(n+1)}$ by $N^{(1)} = 2\overline{F}/((1 - z/3r)(1 - z^*/3r))$ and

$$N^{(n+1)} = \frac{2}{2n+1}\left[N^{(n)}_z + \frac{\overline{D}}{(1-z/3r)(1-z^*/3r)} N^{(n)} + \frac{\overline{F}}{(1-z/3r)(1-z^*/3r)}\int_0^{z^*} N^{(n)}dz^*\right] \quad \text{for } n = 1, 2, \ldots \tag{2.1}$$

Write $N^{(1)} = R^{(1)}(z^*)/(1-z/3r)$ and for $n = 2, 3, \ldots$.

$$N^{(n)} = 2^{n-1}R^{(n)}(z^*)/((1 - z/3r)^n 1.3\ldots(2n-1)) \quad . \tag{2.2}$$

Thus $R^{(1)}(z^*) = 2\overline{F}/(1 - z^*/3r)$ and substitution of (2.2) into (2.1) yields the relation

$$R^{(n+1)}(z^*) = R^{(n)}(z^*)\left[\frac{n}{3r} + \frac{\overline{D}}{(1 - z^*/3r)}\right] + \frac{\overline{F}}{(1-z^*/3r)}\int_0^{z^*} R^{(n)}(z^*)dz^* \quad .$$

This last equation shows that $R^{(n)}$ depends only on z^* and

$$R^{(n+1)} << R^{(n)} [\frac{n}{3r} + \frac{\overline{D}}{(2/3)} + \frac{\overline{F}}{(2/3)}] \quad .$$

Thus by defining $A = 9r(\overline{D} + \overline{F}r)/2$, we see that

$$R^{(n+1)} << R^{(n)} [\frac{n+A}{3r}] \quad .$$

Now $R^{(1)} << 2\overline{F}/(1 - z^*/3r)$ and we have

$$R^{(n)} << 2\overline{F}(n + A - 1)\ldots(A + 1)/((3r)^{n-1}(1 - z^*/3r)) \quad .$$

This lets us dominate the dominants $N^{(n)}$

$$N^{(n)} << \frac{2^{n-1}2\overline{F}(n+A-1)\ldots(A+1)}{(3r)^{n-1}(1 - z^*/3r)(1 - z/3r)^n 1.3\ldots(2n-1)}$$

Using the relation $\| P^{(n)} \|_r \leq \| N^{(n)} \|_r$ we have the second inequality. The first inequality follows from the second by the relation

$$Q^{(n)} = \int_0^{z^*} Q^{(n)}_{z^*} dz^* \quad . \qquad \text{Q.E.D.}$$

We will later need numerical values for these bounds on $\| Q^{(n)} \|_r$. A typical case is an example given by Bergman and Herriot in [7]. They study the equation $\Delta u + (-4 + 2(x^2 + y^2))u = 0$ on the unit square with rounded corners; that is, each of the four corners are replaced by quadrants of circles. This domain will

henceforth be known as the "squirkle." For this example we have $\overline{F} = 9$, $r^2 = 2$, $\overline{D} = 0$. The dominance constant A from the above theorem is then 81. The A that Bergman and Herriot obtain is $54\sqrt{2}\ (1 + 3\sqrt{2})$ which is nearly 400. Since the bounds depend on $(n + A - 1)\ldots(A + 1)$, we see that the above theorem provides a much smaller bound than theirs.

We now study the operator B_T defined by truncating the series for B at T terms, i.e.,

$$B_T(g) = \exp\left(-\int_0^{z^*} R(z,w)dw\right)\left(g(z) + \sum_{n=1}^{T} \frac{Q^{(n)}(z,z^*)}{2^{2n}\beta(n,\,n+1)} \int_0^z (z-\xi)^{n-1} g(\xi)d\xi\right) .$$

We then expect that as $T \to \infty$, $\| \mathcal{L} B_T(g) \| \to 0$ and $\| (B - B_T)(g) \| \to 0$. This is indeed true as we will now show.

Theorem 2.2. Assume g is bounded and analytic on a nearly starlike domain $D_0 \supset D$, then

$$\lim_{T\to\infty} \| (B - B_T)(g) \| = 0 .$$

Proof.

Let $r = \max_{\partial D_0} |z|$. Then we have

$$\| (B - B_T)(g) \|$$

$$= \| \exp(-\int_0^{z^*} A(z, z^*)dz^*)[\sum_{n=T+1}^{\infty} \frac{Q^{(n)}(z, z^*)}{2^{2n}\beta(n, n+1)} \int_0^z (z-\xi)^{n-1} g(\xi)d\xi\,] \|$$

$$\leq \| \exp(-\int_0^{z^*} A(z, z^*)dz^*) \| \sum_{n=T+1}^{\infty} \frac{\| Q^{(n)} \|_r \| g \| (2n)!\, K^{n-1} L}{2^{2n}(n-1)!\, n!} r^n \quad ,$$

where K and L are the constants in the definition of nearly starlike, and the $\| g \|$ is taken over the domain D_0. Then by Theorem 2.1 there are constants M and A independent of T such that

$$\| (B - B_T)(g) \| \leq M \sum_{n=T+1}^{\infty} \frac{(n+A-1)\ldots(A+1)}{2^n (n-1)!} K^n \quad .$$

By the ratio test, this is the "tail" of a convergent series and thus goes to zero as $T \to \infty$. Q.E.D.

Theorem 2.3. Assume g is bounded and analytic on a nearly starlike domain $D_0 \supset D$, then

$$\lim_{T \to \infty} \| \mathcal{L} B_T(g) \| = 0 \quad .$$

Proof. A long, but straightforward, calculation shows

$$(2.3) \quad \mathcal{L}(B_T(g)) = \exp(-\int_0^{z^*} A(z, w)dw\,[-\frac{2T+1}{2} Q_{z^*}^{(T+1)} \frac{\int_0^z (z-\xi)^{T-1} g(\xi)d\xi}{2^{2T}\beta(T, T+1)}] \quad .$$

Let $U(z, z^*) = B_T(g)$, $r = \max_{\partial D_0} |z|$ and

$$M = \| \exp(-\int_0^{z^*} \mathcal{R}(z, w)dw) \| \quad .$$

Then

$$\| \mathcal{L}U \| \leq M^{\frac{2T+1}{2}} \| Q_{z^*}^{(T+1)} \|_r \frac{L(Kr)^{T-1} r \| g \|}{2^{2T} \beta(T, T+1)}$$

and by Theorem 2.1 we have

$$\| \mathcal{L}U \| \leq \frac{9\overline{F}}{4} L M \| g \| \frac{(T+A)\ldots(A+1)}{2^T (T-1)!} K^{T-1} \quad ,$$

which goes to zero as $T \to \infty$. Q.E.D.

The very simple equation for $\mathcal{L} B_T(g)$ given in (2.3) will be very useful later.

Corollary 2.4. For any integer k

$$\lim_{T \to \infty} \| \phi_k - \phi_{k,T} \| = 0 \quad \text{for any domain } D \quad ,$$

$$\lim_{T \to \infty} \| \psi_k - \psi_{k,T} \| = 0 \quad \text{if } D \text{ is nearly starlike} \quad .$$

Corollary 2.5. For any integer k

$$\lim_{T \to \infty} \| \mathcal{L}\phi_{k,T} \| = 0 \quad \text{for any domain } D \quad ,$$

$$\lim_{T \to \infty} \| \mathcal{L}\psi_{k,T} \| = 0 \quad \text{if } D \text{ is nearly starlike} \quad .$$

CHAPTER 3

COMPLETENESS THEOREMS

In this chapter we will study the operator V as defined by Vekua and extend the smoothness and completeness properties of V to more general domains and boundary conditions.

Theorem 3.1. Let D be a domain of class AD, u be any solution of (I) and $\{g_k(z)\}_{k=1}^{\infty}$ be any system of functions analytic on D, continuous on $D \cup \partial D$, which are dense in the space $\{g \mid g$ analytic on D, continuous on $D \cup \partial D\}$ in the maximum norm. Then for any $\epsilon > 0$ there exists an integer N and real coefficients $a_1^{(N)}, \ldots, a_N^{(N)}, b_1^{(N)}, \ldots, b_N^{(N)}$, so that

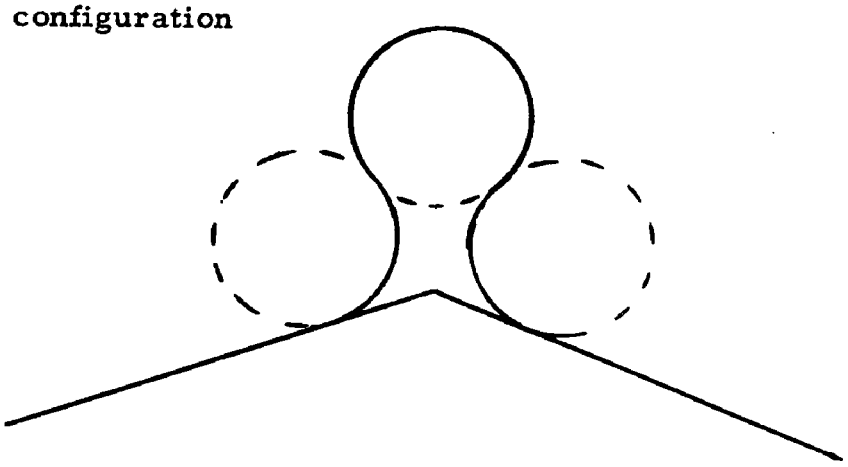
$$\| u - \sum_{k=1}^{N} (a_k^{(N)} \operatorname{Re} B(g_k) + b_k^{(N)} \operatorname{Im} B(g_k)) \| \leq \epsilon \quad .$$

This result extends Vekua's theory to more general domains and arbitrary continuous boundary values.

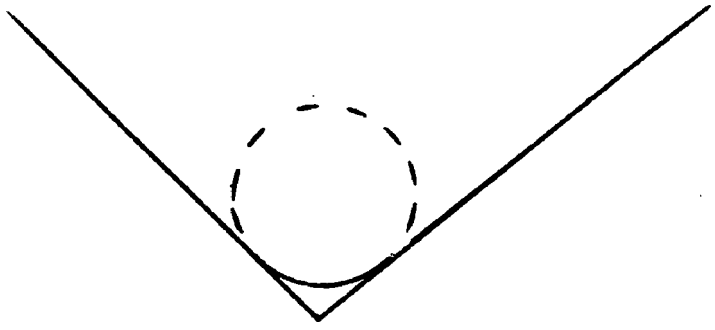
The first step in the proof of the above theorem is the following

Lemma. For any domain D of class AD there exists a sequence of domains D_n of class Ah so that $D_n \subset D$ for all n and any point in $D \cup \partial D$, except a corner, is contained in $D_n \cup \partial D_n$, for large n.

Proof. We give a "proof by picture." Using small circles, we round each corner in ∂D in the following way: If the corner is re-entrant, i.e., the angle π/α is greater than π, then we use the configuration



When the corner is not re-entrant we simply use



The resulting rounded boundary has Lipschitz continuous tangents. As the radii of the circles go to zero, the inscribed domains of class Ah approach the set $D \cup \partial D - \{\text{corners of } D\}$.

Q.E.D.

Define UD to be the class of all solutions of $Lu = 0$ on a domain D of class AD which satisfy the following smoothness conditions:

(1) u Hölder continuous on ∂D and analytic near any corners of ∂D, i.e., if the corner is at a point $a \in \partial D$ and the boundary of D near a consists of the union of two analytic arcs Γ_1 and Γ_2 then $u = f_1$ on Γ_1 and $u = f_2$ on Γ_2 with $f_1 = f_2$ at the corner a and f_1, f_2 analytic on Γ_1, Γ_2, respectively.

(2) $u \in C^{(2)}(D) \cap C^{(0)}(D \cup \partial D)$.

Theorem 3.2. Let D be a domain of class AD, u be of class UD, then $g = V^{-1}(u)$ is continuous on $D \cup \partial D$ and analytic on D.

Proof. Let D_n be as in the above lemma. Then by Theorem 1.1 and the uniqueness of $g(z)$ we have $g(z)$ analytic on D_n and continuous on $D_n \cup \partial D_n$. Letting $n \to \infty$ we see that $g(z)$ is analytic on D and continuous on $D \cup \partial D - \{\text{corners of } \partial D\}$.

Let a be any corner with angle π/α, $1/2 < \alpha < \infty$. We have

$$g(z) = 2U(z,0) - U(0,0)\exp(-\int_0^z \mathcal{B}(z',0)dz') \quad ,$$

and hence $g(z)$ is continuous at $z = a$ iff $U(z,0)$ is continuous at $z = a$. Let $w = z - a$, $w^* = z^* - \bar{a}$ and $V(w, w^*) = U(w + a, w^* + \bar{a})$,

$\mathcal{A}_a(w, w^*) = \mathcal{A}(w + a, w^* + \bar{a})$, $\mathcal{B}_a(w, w^*) = \mathcal{B}(w + a, w^* + \bar{a})$, and $\mathcal{C}_a(w, w^*) = \mathcal{C}(w + a, w^* + \bar{a})$. Define

$$\mathcal{L}_a W = W_{ww^*} + \mathcal{A}_a W_w + \mathcal{B}_a W_{w^*} + \mathcal{C}_a W$$

to find that $\mathcal{L}_a V = 0$ and $g(z)$ continuous at $z = a$ iff $V(w, -\bar{a})$ continuous at $w = 0$.

By (5.1) of R. S. Lehman's article [19] we have

$$V(w, w^*) = V(\bar{w}^*, w^*)R(w, w^*, \bar{w}^*, w^*)$$

$$(3.1) \qquad + \int_{\bar{w}^*}^{w} \{ (V_w(t, \bar{t})R(w, w^*, t, \bar{t}) + V(t, \bar{t})\mathcal{B}_a(t, \bar{t})R(w, w^*, t, \bar{t}))dt$$

$$+ V(t, \bar{t})(R_t - \mathcal{A}_a R)d\bar{t} \} \quad .$$

Also, by Theorem 3.1 of his article

$$(3.2) \qquad V(t, \bar{t}) = \begin{cases} P_1(t, t^{\alpha}, t, \bar{t}^{\alpha}) + o(t), & \text{if } \alpha \text{ irrational} \\ P_1(t, t^{\alpha}, t^p \log(t), t, \bar{t}^{\alpha}, \bar{t}^p \log(\bar{t})) + o(t), & \text{if } \alpha \text{ rational where } \alpha = p/q \text{ in reduced form} \end{cases} .$$

$P_1(\ldots)$ indicates a polynomial of "degree" 1 as defined by Lehman. $R(w, w^*, t, t^*)$ indicates the Riemann function for the complex hyperbolic problem $\mathcal{L}_a U = 0$.

By (3.1) we have

$$V(w, -\bar{a}) = V(-a, -\bar{a})R(w, -\bar{a}, -a, -\bar{a})$$

$$+ \int_{-a}^{w} \{(V_w(t,\bar{t})R(w,-\bar{a},t,\bar{t}) + V(t,\bar{t})\mathcal{B}_a(t,\bar{t})R(w,-\bar{a},t,\bar{t}))dt$$

$$+ V(t,\bar{t})(R_t - \mathcal{A}_a R)d\bar{t}\} \quad .$$

This shows that $\lim_{w\to 0} V(w,-\bar{a})$ exists iff $\int_{-a}^{0} V_w(t,\bar{t})dt$ exists.

Furthermore, by Lehman's article

$$V_w(t,\bar{t}) = \begin{cases} P_1^*(t, 1, t^{\alpha}, t^{\alpha-1}, \bar{t}, \bar{t}^{\alpha}) + o(1) & \text{if } \alpha \text{ irrational}, \\ P_1^*(t, 1, t^{\alpha}, t^{\alpha-1}, t^{p-1}(1 + p\log(t)), \bar{t}, \bar{t}^{\alpha}, \bar{t}^{p}\log(\bar{t})) + o(1) & \text{if } \alpha = p/q \text{ in reduced terms,} \end{cases}$$

where $P_1^*(\ldots)$ is a polynomial of "degree" 1. Then, as $t \to 0$, the worst possible singularity of $V_w(t,\bar{t})$ is of the form $t^{\alpha-1}$. Since this singularity is integrable through the origin, we have that

$$\lim_{w\to 0} \int_{-a}^{w} V_w(t,\bar{t})dt$$

exists and thus $V(w, -\bar{a})$ is continuous at $w = 0$. Q.E.D.

The above result extends Vekua's Theorem 1.1 to domains of class AD with solutions in UD.

Theorem 3.3. If D is any simply connected domain, then

$$\|V\| < \infty \quad .$$

Proof. We have

$$V(g) = \mathrm{Re}(G(z,0,z,\bar{z})g(z) - \int_0^z g(t)H(t,0,z,\bar{z})dt) \quad .$$

Let $M = \|G\|$ and $N = \|H\|$ as well as L = the maximum length of a path from 0 to z in D. Then

$$\|V(g)\| \leq \|M\| \, \|g\| + NL\|g\| \quad . \qquad \text{Q.E.D.}$$

Theorem 3.4. If D is of class AD, u of class UD, and $\{g_k(z)\}_{k=1}^{\infty}$ is any system of functions analytic on D, continuous on $D \cup \partial D$, which are dense in the space $\{g \mid g$ analytic on D and continuous on the closure of $D\}$, then for any $\epsilon > 0$ there exists an integer N and real coefficients $a_1^{(N)}, \ldots, a_N^{(N)}$, $b_1^{(N)}, \ldots, b_N^{(N)}$ so that

$$\| u - \sum_{k=1}^{N} (a_k^{(N)} \mathrm{Re}\, B(g_k) + b_k^{(N)} \mathrm{Im}\, B(g_k)) \| \leq \epsilon$$

Proof. By Theorem 3.2 we know $g(z) = V^{-1}(u)$ is analytic on D and continuous on $D \cup \partial D$. Then by the density of the g_k, there are coefficients $\alpha_1^{(N)}, \ldots, \alpha_N^{(N)}$, so that

$$\lim_{N\to\infty} \| \sum_{k=1}^{N} \alpha_k^{(N)} g_k - g \| = 0 \quad .$$

Setting $\alpha_k^{(N)} = a_k^{(N)} - ib_k^{(N)}$ we have

$$\| u - \mathrm{Re} \sum_{k=1}^{N} \alpha_k^{(N)} B(g_k) \| = \| V(g - \sum_{k=1}^{N} \alpha_k^{(N)} g_k) \|$$

$$\leq \| V \| \; \| g - \sum_{k=1}^{N} \alpha_k^{(N)} g \|$$

which goes to zero as $N \to \infty$, since $\| V \| < \infty$ by Theorem 3.3.

But

$$\mathrm{Re} \sum_{k=1}^{N} \alpha_k^{(N)} B(g_k) = \sum_{k=1}^{N} (a_k^{(N)} \mathrm{Re}\, B(g_k) + b_k^{(N)} \mathrm{Im}\, B(g_k))$$

and we are done. Q.E.D.

We are now in a position to prove Theorem 3.1.

Proof. Let s be arc-length, measured from some point on ∂D. Since u is continuous on ∂D there are polynomials $u_k(s)$ of degree k so that $\| u - u_k \|_{\partial D}$ approaches 0 as $k \to \infty$. These u_k may be chosen to satisfy $u_k(0) = u_k(\sigma)$ where σ is the length of the boundary of D; this follows because the function u is periodic on the interval $[0, \sigma]$.

A recent result of K. Miller [20] shows that if

(a) Each point of ∂D has an exterior cone touching it, and

(b) The coefficients a, b, and c are Hölder continuous, and

(c) The given boundary values are continuous,

then there is a solution to (I).

Now notice that any domain of class AD has the exterior cone property mentioned above. Then, since the boundary values $u_k(s)$ are Lipshitz continuous, there exists a solution of (I) with these prescribed boundary values. Call that solution $u_k(x,y) = u_k(z)$. Then, by the Maximum Principle, we have

$$\lim_{k\to\infty} \|u - u_k\| = \lim_{k\to\infty} \|u - u_k\|_{\partial D} = 0 \quad .$$

By Theorem 3.4, we can find u_{N_k}, a linear combination of the real and imaginary parts of $B(g_1), \ldots, B(g_{N_k})$, for some N_k, so that

$$\|u_k - u_{N_k}\| \leq 1/k \quad .$$

Then

$$\lim_{k\to\infty} \|u - u_{N_k}\| \leq \lim_{k\to\infty} \|u - u_k\| + \lim_{k\to\infty} \|u_{N_k} - u_k\| = 0 \ .$$

Q.E.D.

If z^k and $(z-a)^{\alpha k}$, for $k = 0, 1, \ldots,$ are dense in the space $W = \{ g(z) \mid g$ analytic on D and continuous on $D \cup \partial D \}$, then by Theorem 3.1 the ϕ_k and ψ_k will have the desired completeness properties.

Lemma (Walsh's Theorem). The functions z^k for $k = 0, 1, \ldots,$ are complete in the space W.

Proof. See [28]. Q.E.D.

Lemma. The functions $(z-a)^{\alpha k}$ for $k = 0, 1, \ldots,$ are complete in the space W.

Proof. Let $w = (z - a)^{\alpha}$. Then, under the map z to w, the domain D is mapped onto a simply connected domain E, with $0 \in \partial E$. Set $G(w) = g(z)$. By Walsh's Theorem there are coefficients $a_0^{(N)}, \ldots, a_N^{(N)}$ so that

$$\lim_{N \to \infty} \| G(w) - \sum_{k=0}^{N} a_k^{(N)} w^k \| = 0 \quad ,$$

i.e.,

$$\lim_{N \to \infty} \| g(z) - \sum_{k=0}^{N} a_k^{(N)} (z-a)^{\alpha k} \| = 0 \quad . \qquad \text{Q.E.D.}$$

Corollary to Theorem 3.1. The particular solutions of $Lu = 0$ $\left\{ \begin{matrix} \text{Re} \\ \text{Im} \end{matrix} \right\} B(z^k)$, $k = 0, 1, \ldots,$ are dense in the solution space of (I) for any domain of class AD. The particular solutions $\left\{ \begin{matrix} \text{Re} \\ \text{Im} \end{matrix} \right\} B((z-a)^{\alpha k})$, $k = 0, 1, \ldots,$ are dense in the solution space of (I) for any domain of class AD.

Proof. The above two lemmas. Q.E.D.

Theorem 3.5. Let D be a domain of class AD, u be any solution of (I) and $g_n(z)$, $n = 1, 2, \ldots,$ be any system of functions bounded and analytic on a nearly starlike domain $D_0 \supset D$, which are dense in the space $\{ g \mid g \text{ analytic on } D, \; g \in C^{(0)}(D \cup \partial D) \}$. Then for any $\epsilon > 0$ there exists an integer N and real coefficients

$a_1^{(N)}, \ldots, a_N^{(N)}, b_1^{(N)}, \ldots, b_N^{(N)}$, as well as an integer $T(\epsilon)$, so that $T_k \geq T(\epsilon)$ for $k = 1, \ldots, N$ implies

$$\| u - \sum_{k=1}^{N} (a_k^{(N)} \operatorname{Re} B_{T_k}(g_k) + b_k^{(N)} \operatorname{Im} B_{T_k}(g_k)) \| \leq \epsilon \ .$$

Proof. By Theorem 3.1 there is an N and real coefficients $a_1^{(N)}, \ldots, a_N^{(N)}, b_1^{(N)}, \ldots, b_N^{(N)}$, so that

$$u_N = \sum_{k=1}^{N} (a_k^{(N)} \operatorname{Re} B(g_k) + b_k^{(N)} \operatorname{Im} B(g_k))$$

satisfies

$$\| u - u_N \| \leq \epsilon/2 \ .$$

Now by Theorem 2.2 we can choose an integer $T(\epsilon)$ so that $T_k \geq T(\epsilon)$ implies

$$\| (B - B_{T_k})(g_k) \| \leq \begin{cases} \dfrac{\epsilon}{2^{k+1} |c_k^{(N)}|} & \text{if } c_k^{(N)} \neq 0 \\ 1 & \text{if } c_k^{(N)} = 0 \end{cases} ,$$

where $c_k^{(N)} = a_k^{(N)} - i b_k^{(N)}$, for $k = 1, \ldots, N$. We then have

$$\left\| u - \sum_{k=1}^{N} \left(a_k^{(N)} \operatorname{Re} B_{T_k}(g_k) + b_k^{(N)} \operatorname{Im} B_{T_k}(g_k)\right)\right\|$$

$$\leq \|u - u_N\| + \left\|u_N - \sum_{k=1}^{N} \left(a_k^{(N)} \operatorname{Re} B_{T_k}(g_k) + b_k^{(N)} \operatorname{Im} B_{T_k}(g_k)\right)\right\|$$

$$\leq \epsilon/2 + \sum_{k=1}^{N} \epsilon/2^{k+1} = (\epsilon/2)\left(1 + \sum_{k=1}^{N} 2^{-k}\right) < \epsilon \quad . \qquad \text{Q.E.D.}$$

Corollary to Theorem 3.5. The truncated particular solutions $\left\{ \begin{matrix} \operatorname{Re} \\ \operatorname{Im} \end{matrix} \right\} \phi_{k, T_k}$ for $k = 0, 1, \ldots,$ are complete in the solution space of (I) for any domain of class AD.

The truncated particular solutions $\left\{ \begin{matrix} \operatorname{Re} \\ \operatorname{Im} \end{matrix} \right\} \psi_{k, T_k}$ for $k = 0, 1, \ldots,$ are complete in the solution space of (I) for any nearly starlike domain.

The above analysis shows that the closed form truncated particular solutions we have defined can be used to generate arbitrarily accurate uniform approximations to solutions of (I) over the closure of the domain.

The next question we consider is: How do we obtain the coefficients to guarantee uniform convergence to a solution of (I), given only the boundary values?

CHAPTER 4

AN ERROR BOUND

The following theorem will be used to generate a posteriori error bounds and will be necessary in proving the convergence theorems in later chapters. K. Nickel [22] has used similar bounds in obtaining bounds on eigenvalues of elliptic equations. Protter and Weinberger [23] give a slightly modified version of this theorem.

Theorem 4.1. Let $w(x,y)$ be any function so that $Lw \geq 1$ on D; then for any $u \in C^{(2)}(D) \cap C^{(0)}(D \cup \partial D)$ we have

$$\|u\| \leq \|u\|_{\partial D} + 2\|w\| \, \|Lu\| \quad .$$

Proof. Let $\lambda = \|Lu\|$. Then $L(\lambda w \pm u) \geq 0$. This shows that the maximum of $\lambda w \pm u$ occurs on ∂D. Thus

$\lambda w \pm u \leq \|u\|_{\partial D} + \lambda \|w\|_{\partial D}$; i.e., $|u| \leq \|u\|_{\partial D} + 2\lambda \|w\|$. Q.E.D.

Note. If $w \leq 0$ or $w \geq 0$ on D, then we have

$$\|u\| \leq \|u\|_{\partial D} + \|w\| \, \|Lu\| \quad .$$

We now give explicit formulae for the functions w required in the above theorem.

CASE I. $Lu = \Delta u + c(x,y)u$ on a given domain D, with $c(x,y) \leq 0$ there. Let $w(x,y) = (x^2 - X^2)/2$, where $X = \max_{\partial D} |x|$. Then

$Lw = 1 + c(x,y)(x^2 - X^2)/2 \geq 1$. Similarly, $L((y^2 - Y^2)/2) \geq 1$, where $Y = \max_{\partial D} |y|$. This shows that the desired function w exists and actually gives a bound on its norm:

$$\|w\| \leq \min (X^2/2,\ Y^2/2) \quad .$$

CASE II. The general operator with the restriction that $q_1 = 1 + \min_{D \cup \partial D} (xa(x,y)) > 0$ or $q_2 = 1 + \min_{D \cup \partial D} (yb(x,y)) > 0$. Let $v = (x^2 - X^2)/2$. Then $Lv = 1 + xa + cv \geq q_1$. If $q_1 > 0$ we may define $w = v/q_1$ to see $Lw \geq 1$. Similarly, we may show $L((y^2 - Y^2)/2q_2 \geq 1$ if $q_2 > 0$. This yields the following bounds on w.

$$\|w\| \leq X^2/2q_1 \quad \text{if} \quad q_1 > 0 \quad .$$

$$\|w\| \leq Y^2/2q_2 \quad \text{if} \quad q_2 > 0 \quad .$$

CASE III. In the case where neither q_1 or q_2 are positive, we must resort to a more crude estimate. These are very much like those given on page 341 of [12].

Let $v(x,y) = \exp(\alpha x)$, $m = \min_{D \cup \partial D} c(x,y)$, and $n = \min_{D \cup \partial D} a(x,y)$. Then we have $Lv = (\alpha^2 + \alpha a + c)\exp(\alpha x) \geq (\alpha^2 + \alpha n + m)\exp(-\alpha X)$. This shows that $w = \exp(\alpha(x + X))/(\alpha^2 + \alpha n + m)$ has $Lw \geq 1$ whenever the quantity $\alpha^2 + \alpha n + m$ is positive. Let $\alpha^2 + \alpha n + m = \beta > 0$, i.e., $\alpha = (-n + (n^2 - 4(m - \beta))^{1/2}/2$.

Since we require that the α be positive, we let $\beta > 0$ to see that

$$\|w\| \leq \frac{1}{\beta} \exp((-n + (n^2 - 4(m - \beta))^{1/2})X) \ .$$

Similarly, for $p = \min_{D \cup \partial D} b(x,y)$ we may show

$$\|w\| \leq \frac{1}{\beta} \exp((-p + (p^2 - 4(m - \beta))^{1/2})Y) \ .$$

The above formulae explicitly give the functions w which are required by Theorem 4.1 and give bounds for their norms in terms of the operator and the domain.

In practice, Theorem 4.1 will be applied to the error $u - u^*$ of an approximate solution u^*.

CHAPTER 5

THE PRACTICAL COMPUTATION OF ϕ_k AND ψ_k

We have

$$\phi_k(z) = B(z^k)(z,\bar{z})$$

$$= \exp\Big(-\int_0^z A(z,w)dw\Big)\Big(z^k + \sum_{n=1}^{\infty} \frac{Q^{(n)}(z,\bar{z})}{2^{2n}\beta(n,n+1)} \int_0^z (z-\xi)^{n-1}\xi^k d\xi\Big) .$$

The integration in this expression can be eliminated and the entire sum simplified. Bergman in [5] shows

$$\phi_k(z) = \exp\Big(-\int_0^z A(z,w)dw\Big)(z^k)\Big(1 + \sum_{n=1}^{\infty} \frac{Q^{(n)}(z,\bar{z})z^n k!\,(2n)!}{2^{2n} n!\,(n+k)!}\Big), \tag{5.1}$$

as well as

$$\phi_{k,T}(z) = \exp\Big(-\int_0^z A(z,w)dw\Big)(z^k)\Big(1 + \sum_{n=1}^{T} \frac{Q^{(n)}(z,\bar{z})z^n k!\,(2n)!}{2^{2n} n!\,(n+k)!}\Big) . \tag{5.2}$$

We have

$$\psi_k(z) = B((z-a)^{\alpha k})(z,\bar{z})$$

$$= \exp(-\int_0^{\bar{z}} \mathcal{A}(z,w)dw)((z-a)^{\alpha k} + \sum_{n=1}^{\infty} \frac{Q^{(n)}(z,\bar{z})}{2^{2n}\beta(n,n+1)} \int_0^z (z-a)^{n-1}(\xi-a)^{\alpha k} d\xi) \quad .$$

Integrating by parts we may eliminate the integration. If we define $n!\,! = (2n-1)\ldots(3)(1)$ then we have

$$(5.3) \qquad \psi_k(z) = \sum_{n=1}^{\infty} Q^{(n)}(z,\bar{z}) \left[- \sum_{p=0}^{n-1} \frac{z^p (-a)^{\alpha k+n-p}\, n!\,!}{2^n(\alpha k+n-p)\ldots(\alpha k+1)p!} \right.$$

$$\left. + \frac{n!\,!\,(z-a)^{\alpha k}}{2^n(\alpha k+n)\ldots(\alpha k+1)} + (z-a)^{\alpha k} \right] \exp(-\int_0^{\bar{z}} \mathcal{A}(z,w)dw) \quad .$$

The formula for $\psi_{k,T} = B_T((z-a)^{\alpha k})$ is the same as the above with the "∞" replaced by T.

The above formulae give closed form expressions for the functions of interest. However, they presuppose that we have calculated the $Q^{(n)}$'s. Since these are defined by a recursion relation using integration and differentiation, the actual construction of the $Q^{(n)}$'s remains an impossible task in general. If the coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are polynomials in z and z^*, then all the $Q^{(n)}$ are polynomials and the necessary integration and differentiation may be done quite simply. Bergman and Herriot in [7] assume the coefficients are polynomials and give an algorithm

for computing the coefficients of as many $Q^{(n)}$ as are needed.

For the special case of the equation $\Delta u + c(r^2)u = 0$, where $r^2 = |z|^2$, the computation of the $Q^{(n)}$ may be carried out very economically. This algorithm is presented below.

Bergman in [5] gives the following result about the equation $\Delta u + c(r^2)u = 0$, where $c(r^2)$ is an analytic function of r^2. By defining $R^{(n)}(x) = z^n Q^{(n)}(z, z^*)$, where $x = r^2$, we see that the defining relations for the $Q^{(n)}$ become

$$R^{(1)}(x) = -2\int_0^x c(x)dx \quad \text{and}$$

(5.4)

$$R^{(n+1)}(x) = -\frac{2}{2n+1}\left[x\frac{\partial R^{(n)}}{\partial x} + \int_0^x cR^{(n)}dx - nR^{(n)}\right] .$$

This shows that if we define

(5.5) $$d_k(r^2) = 1 + \sum_{n=1}^{\infty} \frac{R^{(n)}(z\bar{z})k!\,(2n)!}{2^{2n}\,n!\,(n+k)!} \quad \text{then} \quad \phi_k(z) = z^k d_k(r^2).$$

This relation will be very useful later and is a most practical simplification.

The symmetry relation

(5.6) $$\phi_k(z) = e^{-i\theta k}\phi_k(z\,e^{i\theta})$$

follows directly from the above equation.

If $c(x) = \sum_{k=1}^{N} f_k x^{k-1}$ is a polynomial, then we can obtain the $R^{(n)}$ in the following way: Obviously, all the $R^{(n)}$ will be polynomials in x, so let

$$R^{(n)}(x) = \sum_{k=1}^{N_n} r_k^{(n)} x^{k-1} \quad .$$

It is sufficient to find the coefficients $r_k^{(n)}$.

We see that $R^{(1)}(x) = \sum_{k=1}^{N+1} (-2f_k/k)x^k$ and $r_1^{(1)} = 0$, $r_k^{(1)} = -2f_{k-1}/(k-1)$ for $k = 2, \ldots, N+1$. Thus, $N_1 = N + 1$ and the above recurrence relation(5.4) for the $R^{(n)}$ shows that $N_{n+1} = N_n + N$. Hence $N_n = nN + 1$.

Equation (5.4) shows that

$$R^{(n+1)} = \frac{-2}{2n+1}\left[\sum_{p=1}^{N_n} r_p^{(n)}(p-1-n)x^{p-1} + \sum_{k=1}^{N_n}\sum_{j=1}^{N} r_k^{(n)} f_j \frac{x^{k+j-1}}{k+j-1}\right] \quad .$$

Thus

$$r_1^{(n+1)} = [2n/(2n+1)]r_1^{(n)} \quad ,$$

$$r_p^{(n+1)} = -\frac{2}{2n+1}\left[r_p^{(n)}(p-1-n) + \frac{1}{p-1}\sum_{\substack{k+j=p \\ 1\le k\le N_n \\ 1\le j\le N}} r_k^{(n)} f_j\right]$$

for $p = 2, \ldots, N_n$, and

$$(5.7) \qquad r_p^{(n+1)} = -\frac{2}{2n+1}\,\frac{1}{p-1} \sum_{\substack{k+j=p \\ 1\le k\le N_n \\ 1\le j\le N}} r_k^{(n)} f_j$$

for $p = N_n + 1, \ldots, N_{n+1}$.

This completely determines the coefficients of all $R^{(n)}$ recursively. Note that the recursion relation (5.7) can very easily be programmed.

If we want $R^{(1)}, \ldots, R^{(T)}$, then to store the coefficients we will need

$$\sum_{n=1}^{T} (nN + 1) = T(T + 1)N/2 + T$$

storage locations. This shows that even for moderately large T, we will not need an excessive amount of storage. For example, if $c(r^2)$ is a polynomial of degree 5 and $T = 20$, then we only need 1070 words of storage. In the general case of polynomial coefficients, Bergman shows that the storage needed for T terms in the expansion of B_T grows like T^3. So special handling of the equation $\Delta u + c(r^2)u = 0$ results in important economies in both time and storage.

The only part of [7] we use is the algorithm presented for the computation of the $Q^{(n)}$; the orthonormalization process presented there is <u>not</u> used or studied in this paper.

CHAPTER 6

COLLOCATION

In this chapter we will study the process of collocation as applied to the equation $\Delta u + c(r^2)u = 0$ on the unit disk $D = \{ z \mid |z| < 1 \}$. We will assume that $c(r^2)$ is analytic for $r \leq 3$ and non-positive on $r \leq 1$. Then, if u is any solution of $Lu = \Delta u + c(r^2)u = 0$ which is in $C^{(0)}(D \cup \partial D) \cap C^{(2)}(D)$, we know by Theorem 3.1 that there are real coefficients $a_0^{(N)}, \ldots, a_{2N}^{(N)}$ such that

$$\lim_{N \to \infty} \| u - (a_0^{(N)} \operatorname{Re} \phi_0 + \sum_{k=1}^{N} (a_{2k-1}^{(N)} \operatorname{Re} \phi_k + a_{2k}^{(N)} \operatorname{Im} \phi_k)) \| = 0 .$$

Note that $\operatorname{Im} \phi_0 = 0$ by equation (5.5). We will show how to construct such coefficients given only the boundary values of u.

Collocation techniques have been used by many people to solve boundary value problems. However, very little is known about the convergence of such methods theoretically. E. B. Karpilovskaya [18] has used collocation to solve eigenvalue problems of the form $\Delta u + \lambda^2 q(x,y)u = 0$ on rectangles. He is able to give a bound on the rate at which the process converges. A. Shindler [26] has studied

the collocation techniques used in solving integral equations and was able to prove they converge in certain cases. The paper of Fox, Henrici, and Moler [14] used collocation to obtain approximations to eigenvalues of the equation $\Delta u + \lambda^2 u = 0$. The paper does not prove that the method used actually must converge, but shows that it works well in practice.

The basic idea of collocation is to choose points $z_0, \ldots, z_{2N}$ on ∂D and solve the system of linear equations

$$a_0 \operatorname{Re} \phi_0(z_j) + \sum_{k=1}^{N} (a_{2k-1} \operatorname{Re} \phi_k(z_j) + a_{2k} \operatorname{Im} \phi_k(z_j)) = u(z_j) \quad ,$$

$$j = 0, \ldots, 2N \quad ,$$

for $a_0^{(N)}, \ldots, a_{2N}^{(N)}$. Then set

$$u^{(N)}(z) = a_0^{(N)} \operatorname{Re} \phi_0(z) + \sum_{k=1}^{N} (a_{2k-1}^{(N)} \operatorname{Re} \phi_k(z) + a_{2k}^{(N)} \operatorname{Im} \phi_k(z)) \quad .$$

Since $u^{(N)}(z_j) = u(z_j)$ for $j = 0, \ldots, 2N$ we expect that $u^{(N)}$ will approach u as N tends to infinity.

We will show that if u is a regular solution of $Lu = 0$ with α-Hölder continuous boundary values and if the z_j are equally spaced around ∂D, then there is a constant G such that

$$\|u - u^{(N)}\| \leq G \begin{cases} \ln N/N^{\alpha}, & 0 < \alpha < 1 \\ (\ln N)^2/N \quad \text{if} \quad \alpha = 1 \end{cases} .$$

Thus collocation converges and the bounds given above give an indication of the rate at which the convergence takes place.

In order to prove the above result we will need to have an estimate of the rate of convergence that may be obtained using $\phi_0, \ldots, \phi_N$. The theorem below gives such an estimate. For $b = (b_0, \ldots, b_{2N})$ let

$$u_N(z, b) = b_0 \operatorname{Re} \phi_0 + \sum_{k=1}^{N} (b_{2k-1} \operatorname{Re} \phi_k + b_{2k} \operatorname{Im} \phi_k) .$$

Theorem 6.1. If u is a regular solution of $Lu = 0$ with α-Hölder continuous boundary values, then there exists a constant M and vectors $b^{(N)}$ so that

$$\|u - u_N(z, b^{(N)})\| \leq M \begin{cases} 1/N^{\alpha}, & 0 < \alpha < 1 \\ \ln N/N \quad \text{if} \quad \alpha = 1 \end{cases} .$$

Proof. First note that D, being a circle, is obviously of class Ah. Then $g = V^{-1}(u)$, by Theorem 1.1, is analytic on D, and Hölder continuous on $D \cup \partial D$.

A Maclaurin expansion of $g(z)$ about the origin gives the partial sum

$$S_N(z) = \frac{1}{2\pi i} \sum_{n=-N}^{N} z^n \int_{\partial D} \frac{g(t)dt}{t^{n+1}}$$

$$= \sum_{n=-N}^{N} e^{i\theta n} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\theta}) e^{-in\theta} \, d\theta \quad ,$$

where $z = e^{i\theta}$. But this is simply a partial sum of the Fourier series of $g(e^{i\theta})$ on $[-\pi, \pi]$.

Define the average Fourier sum

$$\bar{S}_N(z) = \frac{1}{N} \sum_{k=1}^{N} S_k(z) \quad .$$

Then, since g is α-Hölder continuous on ∂D, [21], page 161, shows that there exists a constant M such that

$$\| g(e^{i\theta}) - \bar{S}_N(e^{i\theta}) \|_{[-\pi, \pi]} \leq M \begin{cases} 1/N^{\alpha}, & 0 < \alpha < 1 \\ \ln N/N \text{ if } \alpha = 1 \end{cases} \quad . \tag{6.1}$$

Define $u_N(z, b^{(N)}) = V(\bar{S}_N)$. Then since $\bar{S}_N$ is a polynomial of degree N and $\operatorname{Im} \phi_0 = 0$ we see that $u_N(z, b^{(N)})$ has the desired form.

We then have

$$\| u - u_N \| = \| V(g - \bar{S}_N) \| \leq \| V \| \, \| g - \bar{S}_N \| = \| V \| \, \| g - \bar{S}_N \|_{[-\pi, \pi]}$$

by applying the Maximum Principle to the analytic function $g - \bar{S}_N$ on $D \cup \partial D$. Using (6.1) this shows that

$$\|u - u_N\| \le M\|V\| \begin{cases} 1/N^{\alpha}, & 0 < \alpha < 1 \\ \ln N/N \text{ if } \alpha = 1 \end{cases} \quad \text{Q.E.D.}$$

The following theorem is the main result of this chapter.

Theorem 6.2. For any regular solution of $\Delta u + c(r^2)u = 0$ with α-Hölder continuous boundary values, we can interpolate at $z_j = \exp(2\pi ij/(2N+1))$ for $j = 0, \ldots, 2N$, to obtain $a_0^{(N)}, \ldots, a_{2N}^{(N)}$ such that for some constant M we will have

$$\|u(z) - u_N(z, a^{(N)})\| \le M \begin{cases} \ln N/N^{\alpha}, & 0 < \alpha < 1 \\ (\ln N)^2/N \text{ if } \alpha = 1 \end{cases} .$$

Proof. For the equation $\Delta u + c(r^2)u = 0$ the ϕ_k have the form (see equation (5.5))

$$\phi_k(z) = z^k d_k(r^2) \quad \text{where} \quad r = |z| \ , \quad \text{for} \quad k = 0, 1, \ldots .$$

Let $z = re^{i\theta}$ and $D_k(r) = r^k d_k(r^2)$ to see that

$$\phi_k(z) = e^{ik\theta} D_k(r) \ , \tag{6.2}$$

where the D_k are real-valued functions.

Let $b^{(N)}$ be any element of R^{2N+1} and set

$$e^{(N)}(z) = u(z) - u_N(z, b^{(N)}) \ .$$

Solve the system of linear equations

(6.3) $$u(z_k) = u_N(z_k, a) \quad , \quad k = 0, \ldots, 2N \quad ,$$

for $a^{(N)}$ and set $u^{(N)}(z) = u_N(z, a^{(N)})$. Then, since $u^{(N)}(z_j) = u(z_j)$ for $j = 0, \ldots, 2N$, we expect that $\lim_{N\to\infty} \| u - u^{(N)} \| = 0$.

For fixed N let $\tau = 2\pi/(2N + 1)$. Then $z_j = e^{ij\tau}$, $j = 0, \ldots, 2N$ and we see that (6.3) is the system:

$$W\, Da^{(N)} = u \quad ,$$

$W_{00} = 1$, $W_{j0} = 1$ $W_{j,2k-1} = \cos(kj\tau)$, $W_{j,2k} = \sin(kj\tau)$ for $j = 1, \ldots, 2N$ and $k = 1, \ldots, N$. Also, $D = \text{diag}(D_0, D_1, D_1, \ldots, D_N, D_N)$ where $D_k = D_k(1)$ for $k = 0, 1, \ldots,$ and $u = u = (u(z_0), \ldots, u(z_{2N}))$.

Letting $e^{(N)} = (e^{(N)}(z_0), \ldots, e^{(N)}(z_{2N}))$, we have $WDa^{(N)} = u = W\,Db^{(N)} + e^{(N)}$ and thus

(6.4) $$a^{(N)} - b^{(N)} = D^{-1}W^{-1}e^{(N)} \quad .$$

This of course requires that no D_k be zero. If some D_k were zero, then we would have $\phi_k(z) = 0$ on ∂D. Now $\text{Re}\ \phi_k(z)$ satisfies $\Delta u + c(r^2)u = 0$ and by the Maximum Principle we then have $\| \text{Re}\ \phi_k \| = \| \text{Re}\ \phi_k \|_{\partial D} = 0$. But $V(g)$ is a 1-1 operator and thus the analytic function g such that $V(g) = \text{Re}\ \phi_k(z)$ must be $g(z) = 0$. This contradicts the fact that $V(z^k) = \text{Re}\ \phi_k$ and z^k is definitely not identically zero on any domain. This contradiction shows that no D_k may be zero and that D^{-1} exists.

Now if we set $u^{(N)}(z) = u_N(z, a^{(N)})$ we then have

$$u(z) - u^{(N)}(z) = u_N(z, b^{(N)}) - u_N(z, a^{(N)}) + e^{(N)}(z) \quad . \tag{6.5}$$

By (6.4) we have for $z = e^{i\theta}$ and $y = (1, \cos\theta, \sin\theta, \ldots, \cos(N\theta), \sin(N\theta))$ that

$$\begin{aligned} u_N(z, a^{(N)}) - u_N z, b^{(N)}) &= (a^{(N)} - b^{(N)}) \bullet (Dy) \\ &= (D^{-1}W^{-1}e^{(N)}) \bullet (Dy) \\ &= (W^{-1}e^{(N)}) \bullet (y) = e^{(N)} \bullet ((W^T)^{-1}y) \quad . \end{aligned}$$

Letting $x = (W^T)^{-1}y$ yields

$$u_N(z, a^{(N)}) - u_N(z, b^{(N)}) = e^{(N)} \bullet x \quad . \tag{6.6}$$

Since $z_j = \exp(2\pi ij/(2N+1))$ we can exhibit $(W^T)^{-1}$ explicitly. $W^{-T}_{i,0} = 1/(2N+1)$, $W^{-T}_{i,2j-1} = \cos(2\pi\, ij/(2N+1))/N$ and $W^{-T}_{i,2j} = \dfrac{\sin(2\pi\, ij/(2N+1))}{N}$ for $i = 0, \ldots, 2N$ and $j = 1, \ldots, N$.

This gives

$$x_k = \mathrm{Re}\left(\sum_{j=1}^{N} e^{ij(k\tau - \theta)} + \frac{N}{2N+1}\right)/N \quad . \tag{6.7}$$

Combining (6.6) and (6.7) gives for z on ∂D

$$(6.8)\quad |u_N(z, a^{(N)}) - u_N(z\, b^{(N)})|$$

$$\leq \frac{1}{N}\|e^{(N)}\| \int_{\partial D} \Big[\sum_{k=0}^{2N} \Big| \mathrm{Re} \sum_{j=1}^{N} e^{ij(k\tau-\theta)} \Big| + N \Big]$$

$$\leq \frac{1}{N}\|e^{(N)}\| \int_{\partial D} \Big[\sum_{k=0}^{2N} |(1 - e^{i(N+1)(k\tau-\theta)})/(1-e^{i(k\tau-\theta)})| + 3N + 1 \Big] \quad .$$

The function

$$H(\theta) = \sum_{k=0}^{2N} |(1 - e^{i(N+1)(k\tau-\theta)})/(1 - e^{i(k\tau-\theta)})| = \sum_{j=0}^{2N} f_j(\theta)$$

has period $2\pi/(2N+1)$. Using the inequalities

$$\sin(x) \geq \frac{2}{\pi} x \quad \text{on} \quad [0, \pi/2]$$

$$\sin(x) \geq \frac{2}{\pi} (\pi - x) \quad \text{on} \quad [\pi/2, \pi] \quad ,$$

one may show that

$$\|f_0(\theta)\|_{[0,\tau]} = 0(N) \quad ,$$

$$\|f_1(\theta)\|_{[0,\tau]} = 0(N) \quad ,$$

and

$$\|f_j(\theta)\|_{[0,\tau]} = 0(N/j) \quad \text{for} \quad j = 2, \ldots, N-1 \quad ,$$

$$\|f_j(\theta)\|_{[0,\tau]} = 0(N/(2N+1-j)) \quad \text{for} \quad j = N+2, \ldots, 2N \quad ,$$

$$\|f_j(\theta)\|_{[0,\tau]} = 0(1) \quad \text{for} \quad j = N, N+1 \quad .$$

Thus, we may bound $H(\theta)$ on the interval $[0, \tau]$, and obtain the estimate

$$\| H(\theta) \|_{[0,\tau]} = 0(N \ln N) \quad .$$

Thus

$$\| u_N(z, a^{(N)}) - u_N(z, b^{(N)}) \|_{\partial D} = \frac{\| e^{(N)} \|}{N} \partial D \; 0(N \ln N)$$

and

$$\| u_N(z, a^{(N)}) - u_N(z, b^{(N)}) \|_{\partial D} = 0(\| e^{(N)} \|_{\partial D} \ln N) \quad .$$

Referring to (6.5) we have

$$\| u(z) - u^{(N)}(z) \|_{\partial D} = 0(\| e^{(N)} \|_{\partial D} \ln N) \quad .$$

Then by the Maximum Principle,

$$\| u(z) - u^{(N)}(z) \| = 0(\| e^{(N)} \|_{\partial D} \ln N) \quad . \tag{6.9}$$

The equation (6.9) holds for all possible coefficients $b^{(N)}$ which define $e^{(N)}$. This shows that

$$\| u - u^{(N)} \| = 0(\ln N \inf_{b} \| u - u_N(z, b) \|) \quad .$$

Thus collocation converges at a rate which is at worst $\ln N$ times slower than the best possible rate of convergence.

To estimate this convergence rate we use Theorem 6.1. This shows that the $b^{(N)}$ may be chosen to satisfy

$$\| e^{(N)} \|_{\partial D} \leq G \begin{cases} 1/N^{\alpha}, & 0 < \alpha < 1 \\ \ln N / N \text{ if } \alpha = 1 \end{cases} \quad . \tag{6.10}$$

for all N, for some constant G. Combing (6.9) and (6.10) we see that there is a constant M such that

$$\|u - u^{(N)}\| \leq M \begin{cases} \ln N/N^{\alpha} & 0 < \alpha < 1 \\ (\ln N)^2/N \text{ if } \alpha = 1 \end{cases} . \qquad \text{Q.E.D.}$$

The above process of collocation is thus convergent. To use it in practice, just set $z_j = \exp(2\pi ij/(2N+1))$ for $j = 0, \ldots, 2N$ and solve the system of linear equations

$$a_0 \text{Re } \phi_0 + \sum_{k=1}^{N} (a_{2k-1} \text{Re } \phi_k(z_j) + a_{2k} \text{ Im } \phi_k(z_j)) = u(z_j) ,$$

$j = 0, \ldots, 2N$ for $a^{(N)}$. Then set $u^{(N)}(z) = u_N(z, a^{(N)})$ to have

$$\lim_{N \to \infty} \|u - u^{(N)}\| = 0 .$$

The above technique is very difficult, if not impossible, to extend to the general case of $Lu = 0$ on any domain of class Ah. For this reason a modification of the above method is needed. Such a modification is presented in the next two chapters.

CHAPTER 7

SOLUTION BY LINEAR CHEBYSHEV APPROXIMATION

This work was motivated in part by the work of J. R. Cannon in [9, 10]. Cannon applied linear programming techniques to solving the boundary value problem for $\Delta u = 0$. The following represents a generalization and extension of those ideas. The presentation will be in four stages involving progressively more complicated situations.

The basic idea is to use the particular solutions $\left\{ \begin{matrix} \text{Re} \\ \text{Im} \end{matrix} \right\} \phi_k$, $k = 0, \ldots, N$, to obtain a best fit, in the Chebyshev sense, to the given boundary data at M points distributed about the boundary of the domain. As both N and M approach infinity, we expect that the Chebyshev fit will approach the solution uniformly over the closure of the domain.

Solution of $\Delta u + c(r^2)u = 0$ using Particular Solutions

Let D be any domain of class AD and u be any regular solution of $\Delta u + c(r^2)u = 0$ with $u = f$ on ∂D, f continuous. We assume that $c(r^2)$ is an analytic function of r^2 on the interval $0 \leq r \leq 3R$ where $R = \max_{\partial D} |z|$ and is non-positive on $0 \leq r \leq R$.

Let the boundary data $f(z) = f(x,y)$ be known approximately only by $f^*(z)$ on ∂D, with $\|f - f^*\|_{\partial D} \leq \mu$. In practice, μ may represent experimental or numerical error in calculating f. The problem is to solve $Lu = 0$ on D, $u = f$ on ∂D as well as possible given only f^* on ∂D.

For this particular problem, the particular solutions $\left\{ \begin{matrix} \text{Re} \\ \text{Im} \end{matrix} \right\} \phi_k(z)$ take the form $\left\{ \begin{matrix} \text{Re} \\ \text{Im} \end{matrix} \right\} z^k d_k(r^2)$ where the functions $d_k(r^2)$ are real valued and never zero, as was seen in Chapter 6.

Let $r_0 = \min_{\partial D} |z|$. Define the compact sets

$$S_N = \left\{ s = (a_0, b_1, \ldots, b_N, c_1, \ldots, c_N) \mid |a_0| \leq (\|f\|_{\partial D} + 1)/|d_0(r_0^2)| \, , \left\{ \begin{matrix} |b_k| \\ |c_k| \end{matrix} \right\} \leq 2(\|f\|_{\partial D} + 1)/r_0^k \, |d_k(r_0^2)| \right\} .$$

For any s of the above form let

$$u_N(z, s) = a_0 \text{Re } \phi_0 + \sum_{k=1}^{N} (b_k \text{ Re } \phi_k + c_k \text{ Im } \phi_k) .$$

Theorem 7.1. For any $0 < \epsilon < 1$ there exists an N and $s^{(N)} \in S_N$ such that

$$\|u - u_N(z, s^{(N)})\| \leq \epsilon .$$

Proof. By the corollary to Theorem 3.1 there exists a coefficient vector $s^{(N)}$ such that

$$\| u - u_N(z, s^{(N)}) \| \leq \epsilon \quad .$$

We show that this $s^{(N)}$ is in S_N.

$$\int_0^{2\pi} u_N(r_0, \theta) d\theta = a_0^{(N)} 2\pi d_0(r_0^2) \quad ,$$

and thus

$$2\pi |a_0^{(N)}| \; |d_0(r_0^2)| \leq 2\pi \| u_N \|_{\partial D} \leq 2\pi (\| u \|_{\partial D} + 1) \quad .$$

This yields the desired bound on $|a_0^{(N)}|$. Also,

$$\int_0^{2\pi} u_N(r_0, \theta) \begin{Bmatrix} \cos(k\theta) \\ \sin(k\theta) \end{Bmatrix} d\theta \quad ,$$

$$= \begin{Bmatrix} |b_k^{(N)}| \\ |c_k^{(N)}| \end{Bmatrix} \pi r_0^k \, d_k(r_0^2) \quad ,$$

and as above we have

$$2\pi (\| f \|_{\partial D} + 1) \geq \begin{Bmatrix} |b_k^{(N)}| \\ |c_k^{(N)}| \end{Bmatrix} \pi r_0^k |d_k(r_0^2)| \qquad \text{Q.E.D.}$$

Let $z_j = r_j e^{i\theta_j}$ be chosen in order around ∂D for $j = 1, \ldots, M$. Then set η_M = the maximum length of ∂D between consecutive z_j's. We assume that $\eta_M \to 0$ as $M \to \infty$.

Now consider the continuous function

$$E_{N,M}(s) = \max_{1 \le j \le M} |f^*(z_j) - u_N(z_j, s)|$$

on the compact set S_N. Set

$$e_{N,M} = \min_{S_N} E_{N,M}(s) \quad .$$

Since $E_{N,M}$ is continuous on the compact set S_N we see that there exists an $s^{N,M} \in S_N$, not necessarily unique, so that

$$e_{N,M} = E_{N,M}(s^{N,M}) \quad .$$

Consider $u_{N,M}(z) = u_N(z, s^{N,M})$. It is a best fit, in the Chebyshev sense, to u of degree N in the ϕ_k at the points $z_1, \ldots, z_M$. Thus, out of the coefficient space S_N which is known to contain the proper coefficients for convergence to u, we pick the best coefficient vector at $z_1, \ldots, z_M$. This is simply a linear programming problem and can be solved in practice for some $s^{N,M}$. Define

$$w(f, \eta) = \sup_{\substack{z_1, z_2 \in \partial D \\ |z_2 - z_1| \le \eta}} |f(z_2) - f(z_1)| ,$$

the modulus of continuity of f on ∂D.

Theorem 7.2.

$$\|u - u_{N,M}\| \leq w(f, \eta_M) + \mu + e_{N,M} + w(u_{N,M}, \eta_M) \quad .$$

Proof. By the Maximum Principle we have that

$$\|u - u_{N,M}\| \leq \|u - u_{N,M}\|_{\partial D} \quad .$$

Let z be between z_j and z_{j+1} on ∂D. Then

$$|u(z) - u_{N,M}(z)| \leq |u(z) - u(z_j)| + |u(z_j) - f^*(z_j)|$$

$$+ |f^*(z_j) - u_{N,M}(z_j)| + |u_{N,M}(z_j) - u_{N,M}(z)| \quad .$$

Since $|z - z_j| \leq \eta_M$ we are done. Q.E.D.

For fixed N and varying M, we have $s^{N,M}$ bounded since they are in the compact set S_N. This means that the $u_{N,M}$ are equicontinuous on $D \cup \partial D$ for varying M and fixed N. Thus for fixed N, as $\eta_M \to 0$ we have $w(u_{N,M}, \eta_M) \to 0$ and also $w(f, \eta_M) \to 0$. This leads to

Theorem 7.3. For any $0 < \epsilon < 1$ there exist $N(\epsilon)$ and $M(\epsilon)$ so that $M \geq M(\epsilon)$ implies

$$\|u - u_{N(\epsilon),M}\| \leq 2\mu + \epsilon \quad .$$

Proof.

$$E_{N,M}(s) = \max_{1\le j\le M} |f^*(z_j) - u_N(z_j, s)|$$

$$\le \max_{1\le j\le M} |f^*(z_j) - f(z_j)| + \max_{1\le j\le M} |f(z_j) - u_N(z_j, s)|$$

$$\le \mu + \|f(z) - u_N(z, s)\|_{\partial D} .$$

This shows that $e_{N,M} \le \mu + \min_{S_N} \|f - u_N(z, s)\|_{\partial D}$, and hence by by Theorem 7.1 there is an integer $N(\epsilon)$ so that $e_{N(\epsilon),M} \le \mu + \epsilon/3$ for all M.

Using the remark just preceding this theorem we may choose an integer $M(\epsilon)$ so that $M \ge M(\epsilon)$ implies

$$w(f, \eta_M) \le \epsilon/3 \quad \text{and} \quad w(u_{N(\epsilon),M}, \eta_M) \le \epsilon/3 .$$

Then by Theorem 7.2 we see that for $M \ge M(\epsilon)$ we have

$$\|u - u_{N(\epsilon),M}\| \le \epsilon/3 + \mu + \mu + \epsilon/3 + \epsilon/3 . \quad \text{Q.E.D.}$$

This is an important result, since it shows that for N and M large, the error $\|u - u_{N,M}\|$ cannot be too much larger than twice the error made in simply computing the boundary values. Hence, ignoring the rounding error inherent in using linear programming in practice, this method is very stable with respect to perturbations in the given boundary values.

This method can be used in practice whenever very accurate expressions are known for the $d_k(r^2)$ and truncation effects are

negligible. For example, Bergman and Herriot [7] show that for the equation $\Delta u - 4u = 0$ the $d_k(r^2)$ have the form

$$d_k(r^2) = 1 + \sum_{n=1}^{\infty} \frac{k!\, r^{2n}}{2^{2n}\, n!\, (n+k)!} \quad .$$

This expression can be evaluated very accurately and makes the boundary value problem for $\Delta u - 4u = 0$ very well suited for the above type of analysis.

From a practical standpoint, there are two special points a about the equation $\Delta u + c(r^2)u = 0$. First, the above quite simple method can be used to solve it. Second, the functions $Q^{(n)}{}_z{}^n$ needed to calculate the ϕ_k can be obtained using only real arithmetic and real integration and differentiation, see equation (5.7). This results in a great saving in time and storage requirements.

In general, the $d_k(r^2)$ will not be known very accurately and the above process is not really practical. If we truncate the series for $d_k(r^2)$, or equivalently that of the operator B, then we can always evaluate $\phi_{k,T}$ in closed form.

Solution of $\Delta u + c(r^2)u = 0$ using Truncated Particular Solutions

Since the truncated particular solutions are not exact solutions of the differential equation, the above analysis needs to be modified.

If we define

$$d_{k,T}(r^2) = 1 + \sum_{n=1}^{T} \frac{R^{(n)}(r^2)\, k!\,(2n)!}{2^{2n}\, n!\,(n+k)!}$$

using the notation of (5.7), then we have $\phi_{k,T} = z^k d_{k,T}(r^2)$. We will follow the same procedure as we used above.

Define the compact set

$$S_N = \Big\{ s = (a_0, b_1, \ldots, b_N, c_1, \ldots, c_N) \mid |a_0| \leq 2(\|f\|_{\partial D} + 1)/|d_0(r_0^2)|, \begin{Bmatrix} |b_k| \\ |c_k| \end{Bmatrix} \leq 4(\|f\|_{\partial D} + 1)/r_0^k |d_k(r_0^2)| \Big\} .$$

For any integer vector $T = (T_0, \ldots, T_N)$ and vector s of the above form, define

$$u_{N,T}(z, s) = a_0 \operatorname{Re} \phi_0 + \sum_{k=1}^{N} (b_k \operatorname{Re} \phi_{k,T_k} + c_k \operatorname{Im} \phi_{k,T_k}) .$$

Theorem 7.4. For any $0 < \epsilon < 1$ there exist integers $T(\epsilon)$ and $N(\epsilon)$ so that for any T such that $T_k \geq T(\epsilon)$ there is an $s^{N(\epsilon)} \in S_{N(\epsilon)}$ which implies

$$\|u - u_{N(\epsilon),T}(z, s^{N(\epsilon)})\| \leq \epsilon .$$

Proof. By the Corollary to Theorem 3.5 there exists a coefficient vector $s^{N(\epsilon)}$ for some large integer $N(\epsilon)$ such that for all vectors T with large components $T_k \geq T'(\epsilon)$ we have

$$\| u - u_{N,T}(z, s^N) \| \leq \epsilon \quad .$$

We show that this vector s^N must be in S_N.

For fixed N, the $d_{k,T_k}(r_0^2)$ approach $d_k(r_0^2)$ as T_k goes to infinity. Pick $T(\epsilon) \geq T'(\epsilon)$ so that $T_k \geq T(\epsilon)$ implies that $|d_{k,T_k}(r_0^2)| \geq |d_k(r_0^2)|/2$ for $k = 0, \ldots, N$. This allows us to bound s^N. Q.E.D.

Choose $z_1, \ldots, z_M$ on ∂D and define η_M as before.

Consider the continuous function

$$E_{N,M,T}(s) = \max_{1 \leq j \leq M} |f^*(z_j) - u_{N,T}(z_j, s)|$$

on the compact set S_N. Set

$$e_{N,M,T} = \min_{S_N} E_{N,M,T}(s) \quad .$$

As before, there is an $s^{N,M,T} \in S_N$ so that

$$e_{N,M,T} = E_{N,M,T}(s^{N,M,T}) \quad .$$

Now consider $u_{N,M,T} = u_{N,T}(z\, s^{N,M,T})$. It is the best fit to u of degree N in the $\phi_{0,T_0}, \ldots, \phi_{N,T_N}$ at the z_j's.

Theorem 7.5. Let $Lw \geq 1$ on D; then

$$\|u - u_{N,M,T}\| \leq w(f, \eta_M) + \mu + e_{N,M,T} + w(u_{N,M,T}, \eta_M) + 2\|w\| \, \|Lu_{N,M,T}\| \quad .$$

Proof. By Theorem 4.1 we have

$$\|u - u_{N,M,T}\| \leq \|u - u_{N,M,T}\|_{\partial D} + 2\|w\| \, \|Lu_{N,M,T}\| \quad . \qquad \text{Q.E.D.}$$

Theorem 7.6. For any $0 < \epsilon < 1$ there exist integers $N(\epsilon)$, $M(\epsilon)$, and $T(\epsilon)$ so that $M \geq M(\epsilon)$, $T_k \geq T(\epsilon)$ for $k = 0, \ldots, N(\epsilon)$ implies

$$\|u - u_{N(\epsilon),M,T}\| \leq 2\mu + \epsilon \quad .$$

Proof. Let

$$u_{N,M}(z) = a_0^{N,M,T} \operatorname{Re} \phi_0 + \sum_{k=1}^{N} (b_k^{N,M,T} \operatorname{Re} \phi_k + c_k^{N,M,T} \operatorname{Im} \phi_k) \quad .$$

Then by Theorem 7.5 we have

$$\begin{aligned} \|u - u_{N,M,T}\| &\leq w(f, \eta_M) + \mu + e_{N,M,T} + w(u_{N,M,T}, \eta_M) \\ &\quad + 2\|w\| \, \|Lu_{N,M,T}\| \\ &\leq w(f, \eta_M) + \mu + e_{N,M,T} + 2\|u_{N,M,T} - u_{N,M}\|_{\partial D} \\ &\quad + w(u_{N,M}, \eta_M) + 2\|w\| \, \|Lu_{N,M,T}\| \quad . \end{aligned}$$

Precisely as before, we may choose $N(\epsilon)$, $M(\epsilon)$, and $T'(\epsilon)$ so that $M \geq M(\epsilon)$ and $T_k \geq T'(\epsilon)$ for $k = 0, \ldots, N(\epsilon)$ implies

$$w(f, \eta_M) + \mu + e_{N(\epsilon), M, T} + w(u_{N(\epsilon), M}, \eta_M) \leq \epsilon/3 + 2\mu \quad .$$

By Corollaries 2.4 and 2.5 and the fact that $s^{N(\epsilon), M, T}$ is bounded for fixed $N(\epsilon)$, we may choose $T(\epsilon) \geq T'(\epsilon)$ so that $T_k \geq T(\epsilon)$ for $k = 0, \ldots, N(\epsilon)$ implies

$$2\|w\| \quad \|Lu_{N(\epsilon), M, T}\| \leq \epsilon/3$$

and

$$2\|u_{N(\epsilon), M, T} - u_{N(\epsilon), M}\| \leq \epsilon/3 \quad .$$

Thus $M \geq M(\epsilon)$, $T_k \geq T(\epsilon)$ for $k = 0, \ldots, N(\epsilon)$ implies $\|u - u_{N(\epsilon), M, T}\| \leq 2\mu + \epsilon$. Q.E.D.

Theorem 7.6 shows that the method of truncated particular solutions converges and is very stable with respect to perturbations in the given boundary values of u.

Solution of $Lu = 0$ Using Particular and Singular Solutions

Let D be a domain of class AD. Let the coefficients a, b, and c be real analytic functions on the disk about the origin of radius $3R$ where $R = \max_{\partial D} |z|$. We assume that u is a regular solution of $Lu = 0$ with $u = f$ on ∂D continuous, and that $c \leq 0$ on D.

As in the previous sections, we assume that f is known only by f^* on ∂D with $\|f - f^*\|_{\partial D} \leq \mu$. The problem is to

solve $Lu = 0$ on D with $u = f$ on ∂D as well as possible given only f^* on ∂D.

Let V_k denote either ϕ_k or ψ_k, depending on which family we wish to use.

We shall follow the same outline used in the previous sections.

Lemma 1. If $g(z) = \sum_{k=0}^{N} c_k z^k$ then $|c_k| \leq \|g\|_\rho / \rho^k$.

Proof. We have

$$\int_0^{2\pi} g(\rho e^{i\theta}) e^{-ik\theta} d\theta = 2\pi c_k \rho^k \quad .$$

Thus, $2\pi |c_k| \rho^k \leq 2\pi \|g\|_\rho$. Q.E.D.

Lemma 2. If the point $a \in \partial D$ is a corner with angle π/α, $g(z) = \sum_{k=0}^{N} c_k (z-a)^{\alpha k}$ and $0 < \rho \leq r_0 = \min_{\partial D} |z|$, then

$$|c_k| \leq B_k^{(N)}(\rho) \|g\|_\rho \quad ,$$

where the $B_k^{(N)}(\rho)$ may be found recursively.

Proof. Let $w = (z-a)^\alpha - (-a)^\alpha$. Define $G(w) = g(z)$

Hence for $m = 0, \ldots, N$, we see by Lemma 1 and the maximum principle that if

$$R(\rho) = \min_{|z|=\rho} |(z-a)^{\alpha} - (-a)^{\alpha}| \quad ,$$

then

$$|d_m| \leq \|G\|_{R(\rho)}/R(\rho)^m \leq \|g\|_{\rho}/R(\rho)^m \quad .$$

Setting $m = N$ we have $|c_N| \leq \|g\|_{\rho}/R(\rho)^N$. Proceed inductively: Let $|c_j|$, $j = m+1, \ldots, N$, be bounded by $\overline{B}_j^{(N)}(\rho)\|g\|_{\rho}$ for $j = m+1, \ldots, N$. Then

$$|c_m| \leq |c_m + (-a)^{-m\alpha} \sum_{k=m+1}^{N} \binom{k-m}{k}(-a)^{\alpha k} c_k|$$

$$+ |(-a)^{-\alpha m} \sum_{k=m+1}^{N} \binom{k-m}{k}(-a)^{\alpha k} c_k|$$

$$\leq \|g\|_{\rho}/R(\rho)^m + |a^{\alpha}|^{-m} \sum_{k=m+1}^{N} \binom{k-m}{k}|a^{\alpha}|^k \overline{B}_k^{(N)}(\rho) \quad .$$

Thus we may set $\overline{B}_N^{(N)}(\rho) = 1/R(\rho)^N$ and for $m = N-1, \ldots, 0$ define

$$\overline{B}_m^{(N)}(\rho) = 1/R(\rho)^m + |a^{\alpha}|^{-m} \sum_{k=m+1}^{N} \binom{k-m}{k}|a^{\alpha}|^k \overline{B}_k^{(N)}(\rho) \quad .$$

Q. E. D.

For any vector $s = (s_0, \ldots, s_{2N+1})$ define

$$u_N(z,s) = \sum_{k=0}^{N} (s_{2k}\mathrm{Re}\, V_k + s_{2k}\mathrm{Re}\, V_k + s_{2k+1}\mathrm{Im}\, V_k) \quad .$$

For ρ sufficiently small, the operator B may be bounded below.

Theorem 7.7. There is a function $\delta(\rho)$ such that

$$\| V(g)\|_\rho \geq \delta(\rho)\| g\|_\rho$$

for all functions g, analytic on D with g(0) real. For ρ small we have $\delta(\rho) > 0$.

Proof. By definition

$$V(g) = \mathrm{Re}\{ \exp(-\int_0^{\bar{z}} A dz^*)(g(z) + \sum_{n=1} \frac{Q^{(n)}(z,\bar{z})}{2^{2n}} \int_0^z (z-\xi)^{n-1} g(\xi) d\xi \}$$

taking Real and Imaginary parts of the terms in the brackets, we see

$$V(g) = \mathrm{Re}\{ (e_1 + ie_2)(f + ih + P_1 + iP_2)\} \quad ,$$

where $g(z) = f(z) + ih(z)$ and f, h, e_1, e_2, P_1, P_2 are real functions of z. Thus

$$V(g) = e_1 f - e_2 h + e_1 P_1 - e_2 P_2$$

and

$$\| V(g)\|_\rho \geq \| e_1 f\|_\rho - \| e_2\|_\rho \| h\|_\rho - \| e_1\|_\rho \| P_1\|_\rho - \| e_2\|_\rho \| P_2\|_\rho \quad .$$

Now,

$$\| h\|_\rho \leq \| g\|_\rho \quad ,$$

$$\|P_1\|_\rho \le \|g\|_\rho \sum_{n=1}^{\infty} \frac{\|\Omega^{(n)}\|_\rho \rho^n}{2^{2n}\beta(n,n+1)n} \equiv \|g\|_\rho W(\rho) \quad ,$$

$$\|P_2\|_\rho \le \|g\|_\rho W(\rho) \quad ,$$

$$\|e_1\|_\rho \le \exp(\rho \|A\|_\rho) \quad ,$$

and

$$\|e_2\|_\rho \le \exp(\rho \|A\|_\rho) \quad .$$

Thus

$$\|V(g)\|_\rho \ge \|e_1 f\|_\rho - \|e_2\|_\rho \|g\|_\rho - 2\exp(\rho \|A\|_\rho)\|g\|_\rho W(\rho) \quad .$$

Now,

$$|e_1(z)| \ge \min_{|z|\le\rho} |\mathrm{Re}\exp(-\int_0^{\bar{z}} A\,dz^*)|$$

$$= \min_{|z|\le\rho} |\exp(-\mathrm{Re}\int_0^{\bar{z}} A\,dz^*)\cos(\mathrm{Im}\int_0^{\bar{z}} A\,dz^*)|$$

$$\ge \exp(-\rho\|A\|_\rho)(1 - \|A\|_\rho^2 \rho^2/2) \equiv m(\rho)$$

and

$$\|e_2\|_\rho = \max_{|z|\le\rho} |\operatorname{Im} \exp(-\int_0^{\bar z} \mathbb{A}\,dz^*)|$$

$$\le \max_{|z|\le\rho} \exp(-\operatorname{Re}\int_0^{\bar z} \mathbb{A}\,dz^*)\,|\sin(\operatorname{Im}\int_0^{\bar z} \mathbb{A}\,dz^*)|$$

$$\le \exp(\rho\|\mathbb{A}\|_\rho)\rho\|\mathbb{A}\|_\rho \equiv n(\rho) \quad .$$

These yield

$$\|V(g)\|_\rho \ge m(\rho)\|f\|_\rho - n(\rho)\|g\|_\rho - 2\exp(\rho\|\mathbb{A}\|_\rho)\|g\|_\rho W(\rho) \quad . \tag{7.1}$$

The function h is the conjugate harmonic function to f. Hence $h(z)$ may be written as

$$h(z) = \int_0^z (-f_t(s,t)ds + f_s(s,t)dt) + h(0) \quad .$$

But $h(0) = \operatorname{Im} g(0) = 0$ and thus

$$\|h\|_\rho \le 2\rho\|\operatorname{grad}(f)\|_\rho \quad . \tag{7.2}$$

We may bound $\|\operatorname{grad}(f)\|_\rho$ by the following lemma taken from [23], page 138.

Lemma. Let $0 < \rho < r_0 = \min_{\partial d}|z|$, then

$$\|\operatorname{grad}(f)\|_\rho \le 2\|f\|_{\partial D}/\pi d(\rho)$$

where $d(\rho) = \min_{|z|\le\rho} \operatorname{dist}(z, \partial D)$.

By (7.2) and the lemma we see

$$\|h\|_\rho \leq 4\|f\|_\rho \, \rho/(\pi d(\rho)) \quad .$$

Now $d(\rho) \geq r_0 - \rho$ and thus

$$\|h\|_\rho \leq 4\|f\|_\rho \, \rho/(\pi(r_0 - \rho)) \quad .$$

This shows

$$\|g\|_\rho = \|(f+ih)\|_\rho \leq \|f\|_\rho [1 + (4\rho/(\pi(r_0 - \rho)))^2]^{\frac{1}{2}} \quad .$$

This and (7.1) imply

$$\|V(g)\|_\rho \geq \delta(\rho)\|g\|_\rho$$

$$(7.3) \qquad \delta(\rho) = m(\rho)[1 + (4\rho/(\pi(r_0 - \rho)))^2]^{-\frac{1}{2}} - n(\rho)$$

$$- 2\exp(\rho\|\mathbb{A}\|_\rho)W(\rho) \quad .$$

We need a value of ρ for which $\delta(\rho) > 0$. We have $W(\rho) = \sum_{n=1}^{\infty} \frac{\|\Omega^{(n)}\|_\rho \, \rho^n}{2^{2n}\beta(n, n+1)n}$ and by Theorem 2.1 we have

$$W(\rho) \leq \sum_{n=1}^{\infty} \frac{9\bar{F}\rho^2}{2} \frac{(n+A(\rho) - 1)\ldots(A(\rho) + 1)}{2^n n!} \quad \text{where}$$

$$A(\rho) = \frac{9\rho}{2}(\bar{D} + \bar{F}\rho).$$

Let $0 < \rho_0 \leq r_0$ be such that $A(\rho_0) \leq 1$. That is, let

$$\rho_0 = \min(r_0, (-9\bar{D}/2 + (81\bar{D}^2/4 + 18\bar{F})^{\frac{1}{2}})/9\bar{F}) \quad .$$

Then, since $A(\rho)$ is monotone increasing as a function of ρ, we see that $0 < \rho \leq \rho_0$ implies

$$W(\rho) \leq \frac{9\bar{F}\rho^2}{2} \sum_{n=1}^{\infty} (1/2)^n = 9\bar{F}\rho^2/2 \quad .$$

This and (7.3) show that for $0 < \rho < \rho_0$ we have

$$(7.4) \quad \delta(\rho) \geq \exp(-\rho\|A\|_\rho)(1 - \rho^2\|A\|_\rho^2/2)[1 + (\frac{4\rho}{\pi(r_0 - \rho)})^2]^{-\frac{1}{2}}$$

$$- \exp(\rho\|A\|_\rho)\rho\|A\|_\rho - 9\bar{F}\rho^2\exp(\|A\|_\rho\ \rho) \ . \qquad \text{Q.E.D.}$$

Using the method of bisection on the interval $(0, \rho_0)$, we can find a value of ρ for which $\delta(\rho) > 0$. This furnishes us with a practical method for bounding the operator V below.

For a value of ρ such that $\delta(\rho) > 0$ define the compact sets

$$S_N = \left\{ s \mid \begin{Vmatrix} |s_{2k}| \\ |s_{2k+1}| \end{Vmatrix} \leq (\|f\|_{\partial D} + 1)B_k^{(N)}(\rho)/\delta(\rho) \right\}$$

where $B_k^{(N)}(\rho) = 1/\rho^k$ if $V_k = \phi_k$ and $B_k^{(N)}(\rho) = \bar{B}_k^{(N)}(\rho)$ if the V_k are the ψ_k.

Theorem 7.8. For $0 < \epsilon < 1$ there exists an integer N and $s^N \in S_N$ such that

$$\|u - u_N(z, s^N)\| \leq \epsilon \quad .$$

Proof. By Theorem 3.1 there is a coefficient vector s^N such that

$$\|u - u_N(z, s^N)\| \leq \epsilon \quad .$$

We show $s^N \in S_N$. There are two cases

Case (I): If V_k are the ϕ_k, then let $u_N(z, s^N) = V(g_N)$, where

$$g_N(z) = \sum_{k=0}^{N} c_k z^k \quad .$$

By Lemma 1, we have

$$|c_k| \leq \|g_N\|_\rho / \rho^k \quad .$$

But $\|g_N\|_\rho = \|V^{-1} u_N\|_\rho \leq \|u_N\|_\rho / \delta(\rho) \leq (\|f\|_{\partial D} + 1)/\delta(\rho)$.

Hence

$$|c_k| \leq (\|f\|_{\partial D} + 1)/(\rho^k \delta(\rho)) \quad \text{for} \quad k = 0, \ldots, N \quad .$$

Since $c_k = s_{2k}^{(N)} - i s_{2k+1}^{N}$, we are done.

Case (II): If the V_k are the ψ_k, then let $u_N(z, s^N) = V(g_N)$, where

$$g_N(z) = \sum_{k=0}^{N} c_k (z - a)^{\alpha k} \quad .$$

By Lemma 2 we have

$$|c_k| \leq \|g_N\|_\rho B_k^{(N)}(\rho) \quad .$$

But $\|g_N\|_\rho = \|V^{-1}u_N\|_\rho \le \|u_N\|_\rho/\delta(\rho) \le (\|f\|_{\partial D} + 1)/\delta(\rho)$.

Hence

$$|c_k| \le (\|f\|_{\partial D} + 1)B_k^{(N)}(\rho)/\delta(\rho) \quad \text{for} \quad k = 0, \ldots, N .$$

Since $c_k = s_{2k} - i\ s_{2k+1}$, we are done. Q.E.D.

Define the continuous function

$$E_{N,M}(s) = \max_{1\le j\le M} |f^*(z_j) - u_N(z_j, s)| ,$$

where the $z_1, \ldots, z_M$ are chosen as before. Set

$$e_{N,M} = \min_{S_N} E_{N,M}(s) .$$

Since $E_{N,M}$ is continuous on the compact set S_N there is an $s^{N,M} \in S_N$ so that $e_{N,M} = E_{N,M}(s^{N,M})$.

Define the solution $u_{N,M} = u_N(z, s^{N,M})$. This is the best Chebyshev approximation to u at the points $z_1, \ldots, z_M$ out of all possible u_N. Again, obtaining the $s^{N,M}$ is a linear programming problem and may be solved in practice.

Theorem 7.9.

$$\|u - u_{N,M}\| \le w(f, \eta_M) + \mu + e_{N,M} + w(u_{N,M}, \eta_M) .$$

Proof. Exactly as in the previous sections. Q.E.D.

Theorem 7.10. For any $0 < \epsilon < 1$ there are integers $N(\epsilon)$ and $T(\epsilon)$ so that $M \geq M(\epsilon)$ implies

$$\| u - u_{N(\epsilon),\, M} \| \leq 2\mu + \epsilon \quad .$$

Proof. Exactly as before. Q.E.D.

In practice, most problems do not have the V_k available in closed form and one must use the truncated particular solutions $V_{k,T}$ which can be evaluated in closed form. What happens if we use the truncated solutions instead of the V_k? The next section answers this question.

Solution of $Lu = 0$ Using Truncated Particular and Singular Solutions

Here the $V_{k,T}$ are used instead of the V_k. This is the problem of practical interest because generally V_k is not available in closed form, but $V_{k,T}$ is.

For any coefficient vector $s = (s_0, \ldots, s_{2N+1})$ and $T = (T_0, \ldots, T_N)$ define

$$u_{N,T} = \sum_{k=0}^{N} (s_{2k} \operatorname{Re} V_{k,T_k} + s_{2k+1} \operatorname{Im} V_{k,T_k}) \quad .$$

Define the compact sets S_N as before.

Theorem 7.11. For any $0 < \epsilon < 1$ there exist integers $N(\epsilon)$ and $T(\epsilon)$ so that for any T such that $T_k \geq T(\epsilon)$ there is an $s^{N(\epsilon)} \in S_{N(\epsilon)}$

which implies

$$\| u - u_{N(\epsilon),T}(z, s^{N(\epsilon)}) \| \leq \epsilon \quad .$$

Proof. By Theorem 7.8 of the previous section, we may choose $N(\epsilon)$ so that there are coefficients $s^{N(\epsilon)} \in S_{N(\epsilon)}$ such that

$$\| u - u_{N(\epsilon)}(z, s^{N(\epsilon)}) \| \leq \epsilon/2 \quad .$$

Now by Corollary 2.4 there is an integer $T(\epsilon)$ so that $T_k \geq T(\epsilon)$ for $k = 0, \ldots, N(\epsilon)$ implies

$$\| V_{k,T_k} - V_k \| \leq \begin{cases} \epsilon/(2^{k+2}|c_k|) & \text{if } c_k \neq 0 \\ 1 & \text{if } c_k = 0 \end{cases} ,$$

where $c_k = s_{2k}^{N(\epsilon)} - is_{2k+1}^{N(\epsilon)}$, for $k = 0, \ldots, N(\epsilon)$. Then we have

$$\| u - u_{N(\epsilon),T} \| \leq \| u - u_{N(\epsilon)} \| + \| u_{N(\epsilon)} - u_{N(\epsilon),T} \|$$

$$\leq \epsilon/2 + \sum_{k=0}^{N} \epsilon/2^{k+2} < \epsilon \quad . \qquad \text{Q. E. D.}$$

Choose $z_1, \ldots, z_M$ on ∂D as usual. Define the continuous functions

$$E_{N,M,T}(s) = \max_{1 \leq j \leq M} |f^*(z_j) - u_{N,T}(z_j, s)|$$

Since $E_{N,M,T}$ is continuous on the compact sets S_N, there is a vector $s^{N,M,T} \in S_N$ so that

$$e_{N,M,T} = \inf_{S_N} E_{N,M,T}(s) = E_{N,M,T}(s^{N,M,T}) \quad .$$

Consider $u_{N,M,T} = u_{N,T}(z, s^{N,M,T})$. It is a best fit to u at the z_j out of all possible $u_{N,T}$. It may be obtained using linear programming.

Theorem 7.12. Let $Lw \geq 1$ on D then

$$\|u - u_{N,M,T}\| \leq w(f, \eta_M) + \mu + e_{N,M,T} + w(u_{N,M,T}, \eta_M) + 2\|w\| \, \|Lu_{N,M,T}\| \quad .$$

Proof. Exactly as before. Q.E.D.

Theorem 7.13. If $\epsilon > 0$ then there are integers $N(\epsilon)$, $M(\epsilon)$, and $T(\epsilon)$ so that $M \geq M(\epsilon)$ and $T_k \geq T(\epsilon)$ for $k = 0, \ldots, N(\epsilon)$ implies

$$\|u - u_{N(\epsilon),M,T}\| \leq 2\mu + \epsilon \quad .$$

Proof. Exactly as before. Q.E.D.

Thus, even in this practical scheme, the process is convergent and stable with respect to perturbations in the given boundary values.

Theorem 7.13 has a very nice corollary which can be roughly stated as follows: For any fixed N and T, as M increases, $u_{N,M,T}$ approaches being a best fit to u from the space of solutions spanned by $V_{0,T_0}, \ldots, V_{N,T_N}$. To be more specific:

Let the span of $V_{0,T_0}, \ldots, V_{N,T_N}$ be denoted by $\langle V_{0,T_0}, \ldots, V_{N,T_N} \rangle$. Suppose that $f = f^*$, i.e., $\mu = 0$. Let

$$e_{N,T} = \min_{S_N} \| u_{N,T}(z,s) - u(z) \| \quad .$$

Then $e_{N,T}$ is the best possible error over the space $\langle V_{0,T_0}, \ldots, V_{N,T_N} \rangle$. Fix $\epsilon > 0$, then by Theorem 7.13 there exists an integer $M(\epsilon)$ such that $M \geq M(\epsilon)$ implies

$$e_{N,T} \leq \| u - u_{N,M,T} \| \leq e_{N,M,T} + \epsilon \quad ,$$

and thus

$$e_{N,T} \geq e_{N,M,T} \geq e_{N,T} - \epsilon \quad .$$

This shows that

$$\lim_{M \to \infty} e_{N,M,T} = e_{N,T} \quad ,$$

and thus $u_{N,M,T}$ is, in the limit, a best fit to u on the closure of the domain. Since the value $e_{N,M,T}$ is automatically produced by any linear programming scheme, and that value is very nearly the error $\| u - u_{N,M,T} \|$, there is no need to search the boundary of D to find the maximum error if M is very large.

CHAPTER 8

COMPUTABLE ERROR BOUNDS

Recall Theorem 4.1, if u is any solution of $Lu = 0$ with $u \in C^{(0)}(D \cup \partial D) \cap C^{(2)}(D)$, $u^* \in C^{(0)}(D \cup \partial D) \cap C^{(2)}(D)$ and w is any function such that $Lw \geq 1$ on D, then

$$\|u - u^*\| \leq \|u - u^*\|_{\partial D} + 2\|w\| \, \|Lu^*\| \quad . \tag{8.1}$$

If the above u^* is the $u_{N,M,T}$ as described in the previous sections, then since $L\phi_{k,T}$ and $L\psi_{k,T}$ are explicitly computable from equation (2.3) we can effectively bound the error $\|u - u^*\|$. To do this we will need bounds on $L\phi_{k,T}$ and $L\psi_{k,T}$, the following theorem provides such bounds.

Theorem 8.1. If the domain is starlike with respect to the origin, then

$$\|L\phi_{k,T}\| \leq \frac{2(T+1)!\,!\,E\,R^{T+k}\|Q^{(T+1)}_{z^*}\|}{2^T(T+k)\ldots(k+1)} \quad .$$

If the domain is also starlike with respect to the corner a, then

$$\|L\psi_{k,T}\| \leq \frac{2(T+1)!\,!\,E\,R^{T}\,R_a^{\alpha k}\|Q^{(T+1)}_{z^*}\|}{2^T(T-1)!} \quad ,$$

where $R = \max_{\partial D} |z|$, $R_a = \max_{\partial D} |z-a|$, and $E = \| \exp(-\int_0^{z^*} A(z,w)dw) \|$.

Proof. By Equation (2.3) we have

$$\mathcal{L}\phi_{k,T} = -\frac{2T+1}{2} \exp(-\int_0^{z^*} A dz^*) Q_{z^*}^{(T+1)} \frac{\int_0^z (z-\xi)^{T-1} \xi^k d\xi}{2^{2T}\beta(T,T+1)} .$$

But $\int_0^z (z-\xi)^{T-1}\xi^k d\xi = (T-1)!\, z^{T+k}/(T+k)\ldots(k+1)$ and the desired bound follows from the relation $L\phi_{k,T} = 4\mathcal{L}\phi_{k,T}$.

We also have that

$$\mathcal{L}\psi_{k,T} = -\frac{2T+1}{2} \exp(-\int_0^{z^*} A dz^*) Q_{z^*}^{(T+1)} \frac{\int_0^z (z-\xi)^{T-1} (\xi-a)^{\alpha k} d\xi}{2^{2T}\beta(T,T+1)}$$

Thus $\|\mathcal{L}\psi_{k,T}\| \le \frac{2T+1}{2} E \| Q_{z^*}^{(T+1)} \| R^T R_a^{\alpha k}/2^{2T}\beta(T,T+1)$,

and the result follows immediately. Q.E.D.

The above bounds are dominated by those given in Theorem 2.3 and thus approach zero as T goes to infinity.

An immediately obvious way to bound $Q_{z^*}^{(T+1)}$ required above is to use Theorem 2.1. This yields

$$\| L\phi_{k,T} \| \le 9E\bar{F}(2T+1) R^k \frac{(T+A)\ldots(A+1)}{2^T(T+k)\ldots(k+1)} .$$

A typical example is that of Bergman and Herriot in [7]. The equation is $\Delta u + (-4 + 2(x^2 + y^2))u = 0$ on the square with rounded corners, the "squirkle." In this case $\bar{F} = 9$, $R = \sqrt{2}$, $\bar{D} = 0$, and $E = 1$. Thus $A = 9\bar{F}R^2/2 = 81$. We then have the bound

$$\| L\phi_{0,T} \| \leq 81(2T+1)\frac{(T+81)\ldots(82)}{2^T T!} .$$

This bound for $T = 10$ is extremely large, greater than two billion. This shows two things. First, for an effective way to bound $\| L\phi_{k,T} \|$ we will need to use something other than Theorem 2.1. Second, it may be that for k "small" and T "large" $\| L\phi_{k,T} \|$ will be excessively large. It is easily seen that $\| L\phi_{0,4} \| \geq 4.0$ if the equation is $\Delta u - 4(1 + x^2 + y^2)u = 0$ on the "squirkle." This is rather disquieting and seems to indicate that for "small" k we will actually <u>need</u> "large" T.

Another approach is to calculate $Q^{(T+1)}$. We will have $Q^{(T+1)}_{z^*}$ explicitly available. This would give us a smaller bound for $Q^{(T+1)}_{z^*}$ and hence for $L\phi_{k,T}$. The cost of this procedure is additional computer storage requirements.

In the case where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are polynomials in z and z^*, the $Q^{(T+1)}_{z^*}$ is also a polynomial. Let

$$Q^{(T+1)} = \sum_{j=0}^{J} \sum_{k=0}^{K} q_{jk}^{(T+1)} z^j z^{*k}$$

$$\|Q_{z^*}^{(T+1)}\| \leq \sum_{j=0}^{J} \sum_{k=1}^{K} k|q_{jk}^{(T+1)}| R^{j+k-1} \quad . \tag{8.2}$$

The dominance argument used in Theorem 2.1 shows that the bound in (8.2) is dominated by the bound given in Theorem 2.1. Thus, (8.2) provides a "tighter" bound on the $Q^{(T+1)}$. That this bound is actually "tighter" than those given by Theorem 2.1 may be seen by the following example: Let the equation be $\Delta u + 4(-1 + \frac{1}{2}(x^2 + y^2))u = 0$ on the squirkle. Then the bound given in (8.2) yields

$$\|L\phi_{0,6}\| \leq 0.48 \quad .$$

The bound which Theorem 2.1 gave was greater than two <u>billion</u>.

Given

$$u^* = \sum_{k=0}^{N} \left(a_k \mathrm{Re}\begin{Bmatrix}\phi_{k,T_k}\\ \psi_{k,T_k}\end{Bmatrix} + b_k \mathrm{Im}\begin{Bmatrix}\phi_{k,T_k}\\ \psi_{k,T_k}\end{Bmatrix}\right) \quad ,$$

we can write

$$\|Lu^*\| \leq \sum_{k=0}^{N} (|a_k| + |b_k|)\|L\begin{Bmatrix}\phi_{k,T_k}\\ \psi_{k,T_k}\end{Bmatrix}\| \quad .$$

Using the inequality in (8.2) we can then bound Lu^* on D. This technique has been found to be quite practical.

In practice all we need to is generate an approximation u^* to a solution u of $Lu = 0$ and then use the inequality in (8.1) with those of Theorem 8.1 to bound the error $\|u - u^*\|$ in an a posteriori fashion. The norm of the function w may be bounded using the formulae given in Chapter 4 for w.

Since we will require "symbolic" solutions, the T_k cannot be too large, otherwise the solution will be too unwieldy to be truly symbolic or usable. Some rule of thumb is needed to give a good value for the T_k. The following argument is a step in that direction. The problems we will study will have a, b, and c polynomials in x and y, and thus $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ will be polynomials in z and z^*. Let M' be the maximum degree with respect to z of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Let M'' be the maximum degree with respect to z^* of the same coefficients. Then Bergman and Harriot in [7] show that

$$Q^{(p)} = \sum_{j=0}^{M_p} \sum_{k=0}^{N_p} q_{jk}^{(p)} z^j z^{*k} \quad ,$$

where $M_p = 2pM'$ and $N_p = 2pM'' + p$. Let $d(x^p y^q) = p + q$ denote the degree of a polynomial in x and y. Then

$$d(\phi_{k, T_k}) = k + \max_{1 \le n \le T_k} (n + 2nM' + 2nM'' + n) \quad ,$$

and letting $m = M' + M'' + 1$ we see

$$d\Big(\sum_{k=0}^{N} (a_k \operatorname{Re} \phi_{k,T_k} + b_k \operatorname{Im} \phi_{k,T_k})\Big) = \max_{1 \le k \le N} (k + 2T_k m) .$$

This shows that given an acceptable degree $d \ge N + 2m$ of our approximation by the $\phi_{0,T_0}, \ldots, \phi_{N,T_N}$, any T_k may be as large as $[(d-k)/2m]$ where $[\cdot]$ denotes the greatest integer function.

For example, $\Delta u + (-4 + 2(x^2 + y^2))u = 0$ on the squirkle has $M' = M'' = 1$ and $m = 3$. Bergman and Herriot choose $N = 24$ and $T_{24} = 4$. This shows that

$$d\Big(\sum_{k=0}^{24} (a_k \operatorname{Re} \phi_{k,T_k} + b_k \operatorname{Im} \phi_{k,T_k})\Big) \ge 24 + 8 \cdot 3 = 48 .$$

Hence $T_k = [(48 - k)/6]$ is a perfectly acceptable choice for the T_k's.

The relation $T_k = [(d - k)/2m]$ allows T_k to be "large" for "small" k. As we have already seen, this is a good thing to have. The relation also lets T_k get smaller as k increases. This is also reasonable because the error bounds on $L\phi_{k,T_k}$ given in Theorem 8.1 decrease very rapidly as k increases.

The above error bounding techniques have been used extensively and found to be quite practical.

CHAPTER 9

ELIMINATION OF ANALYTICITY REQUIREMENT FOR $\Delta u + c(x,y)u = 0$

Let u be a solution of $\Delta u + c(x,y)u = 0$ on a domain D of class AD, with $u \in C^{(0)}(D \cup \partial D) \cap C^{(2)}(D)$ and $c(x,y)$ merely continuous and non-positive on D. Let the boundary values f be known only by f^* where $\|u - f^*\|_{\partial D} \leq \mu$. Using the techniques of the previous sections, we can solve the boundary value problem for this equation.

Since $c(x,y)$ is continuous on the closure of the domain D, there exist functions c_p possessing analytic extensions to the domain $\{(z, z^*) \mid |z|, |z^*| \leq 3\ \mathrm{rad}(D)\}$, so that

$$\lim_{p \to \infty} \|c - c_p\| = 0 \quad .$$

An example of such c_p: Let c_p be a polynomial of degree p in x and y which fits c most closely in the maximum norm. Then, since the polynomials in x and y are dense in $C^{(0)}(D \cup \partial D)$, we must have $\lim_{p \to \infty} \|c - c_p\| = 0$.

We may assume that the approximating functions c_p are non-positive on D. This follows since we may subtract $\|c - c_p\|$ from

c_p to obtain a function $c_p - \|c - c_p\|$ which is non-positive on D and obeys

$$\|c - (c_p - \|c - c_p\|)\| \leq 2\|c - c_p\| \quad ,$$

which goes to zero as p goes to infinity.

Now set $L_p u = \Delta u + c_p u$ and $w(x,y) = (x^2 + y^2 - R^2)/4$ where R is the radius of the domain. Then $L_p w \geq 1$ on D for all p.

The previous theory may be applied to solve the boundary value problem:

$$L_p u^* = 0 \quad \text{on} \quad D$$

and

$$u^* = f \quad \text{on} \quad \partial D \quad ,$$

knowing only f^* on ∂D, for an approximate solution $u^{(p)}_{N,M,T}$.

Theorem. $\overline{\lim_{p\to\infty}} \{\overline{\lim_{N\to\infty}} [\overline{\lim_{M,T\to\infty}} \|u - u^{(p)}_{N,M,T}\|]\} \leq 2\mu$.

Proof. We have by Theorem 4.1 that

$$\|u - u^{(p)}_{N,M,T}\| \leq \|u - u^{(p)}_{N,M,T}\|_{\partial D} + 2\|w\| \, \|L_p(u - u^{(p)}_{N,M,T})\| .$$

But $L_p u = Lu + (c_p - c)u = (c_p - c)u$ and hence

$$\|u - u^{(p)}_{N,M,T}\| \leq \|u - u^{(p)}_{N,M,T}\|_{\partial D} + 2\|w\|[\|c - c_p\| \, \|u\| + \|L_p u^{(p)}_{N,M,T}\|] \quad . \tag{9.1}$$

Fix $\epsilon>0$ and choose p' so that

$$2\|w\| \, \|c - c_{p'}\| \, \|u\| \leq \epsilon/3 \quad .$$

Then there exist integers N', M', and T' so that $M \geq M'$, $T_k \geq T'$, for $k = 0, \ldots, N'$ implies

$$\|u - u^{(p')}_{N', M, T}\|_{\partial D} \leq \epsilon/3 + 2\mu \quad \text{and} \quad 2\|w\| \, \|L_{p'} u^{(p')}_{N', M, T}\| \leq \epsilon/3.$$

Using (9.1) this yields

$$\|u - u^{(p')}_{N', M, T}\| \leq 2\mu + \epsilon \quad ,$$

whenever $M \geq M'$ and $T_k \geq T'$ for $k = 0, \ldots, N'$. Q.E.D.

The inequality (9.1) should provide a satisfactory a posteriori error bound.

CHAPTER 10

SOLUTION OF THE INHOMOGENEOUS EQUATION, $Lu = g$

For a domain D of class AD, let $u \in C^{(0)}(D \cup \partial D) \cap C^{(2)}(D)$ be a solution of the inhomogeneous equation $Lu = g$. We assume that the coefficients a, b, and c and the right hand side g are analytic functions of x and y on the disk about the origin of radius $3R$, where $R = \max_{\partial D} |z|$, and that $c(x,y) \leq 0$ on the domain D.

Given boundary values f of u, we wish to find u on $D \cup \partial D$.

Let $A(z, z^*)$, $B(z, z^*)$, $C(z, z^*)$, and $G(z, z^*)$ be the usual complex extensions of $(a + ib)/4$, $(a - ib)/4$, $c/4$, and $g/4$. The complex form of $Lu = g$ is then $\mathcal{L}U = G$.

For the special case where $a(x,y)$ and $b(x,y)$ are identically zero, Bergman and Herriot in [7] give a particular solution of $\mathcal{L}U = G$. We present a generalization of that argument to the full operator $Lu = g$.

Let

$$U^{(0)} = \sum_{k=0}^{\infty} F_k$$

represent a solution of $\mathcal{L} U = G$. Then, formally, we need

$$G = \mathcal{L} U^{(0)} = \sum_{k=0}^{\infty} \mathcal{L} F_k = \sum_{k=0}^{\infty} [(F_k)_{zz^*} + A(F_k)_z + B(F_k)_{z^*} + C(F_k)]$$

$$= (F_0)_{zz^*} + \sum_{k=0}^{\infty} [A(F_k)_z + B(F_k)_{z^*} + C(F_k) + (F_{k+1})_{zz^*}] \quad .$$

Thus, if we let $(F_0)_{zz^*} = G$ and for $k = 0, 1, \ldots,$

$$(F_{k+1})_{zz^*} = -[A(F_k)_z + B(F_k)_{z^*} + C(F_k)] \quad ,$$

then we have $\mathcal{L} U^{(0)} = G$. The F_k are thus recursively defined by the relations

$$F_0 = \int_0^z \int_0^{z^*} G \, dz dz^*$$

and

$$F_{k+1} = -\int_0^z \int_0^{z^*} [A(F_k)_z + B(F_k)_{z^*} + C(F_k)] dz dz^* \quad ,$$

$$k = 0, 1, \ldots \quad .$$

Integration by parts yields the recursion relation

$$F_0 = \int_0^z \int_0^{z^*} G\, dzdz^* \quad , \qquad \text{and for } k = 0, 1, \ldots,$$

(10.1)

$$F_{k+1} = -\left(\int_0^{z^*} AF_k dz^* + \int_0^z BF_k dz + \int_0^z \int_0^{z^*} (-A_z - B_{z^*} + C)F_k\, dzdz^*\right)$$

The equations above define a formal solution $U^{(0)}$ of $\mathcal{L}U = G$.

To study the validity of this formal argument, truncate the series of $U^{(0)}$, i.e., let

$$U_t^{(0)} = \sum_{k=0}^{t} F_k \quad .$$

Lemma. $\lim_{t\to\infty} \| \mathcal{L}U_t^{(0)} - G \| = 0$.

Proof. It is easy to see that

(10.2) $$\mathcal{L}\, U_t^{(0)} = G + A(F_t)_z + B(F_t)_{z^*} + C(F_t) \quad .$$

Now

$$-(F_{t+1})_{zz^*} = A(F_t)_z + B(F_t)_{z^*} + C(F_t) = \left(A\int_0^{z^*} + B\int_0^z + C\int_0^z \int_0^{z^*}\right)(F_t)_{zz^*}$$

$$= \ldots = (-1)^t \left(A\int_0^{z^*} + B\int_0^z + C\int_0^z \int_0^{z^*}\right)^{t+1} (F_0)_{zz^*} \quad ,$$

and thus, by (10.2)

$$\|\mathcal{L}U_t^{(0)} - G\| \le \|(A\int_0^{z^*} + B\int_0^{z} + C\int_0^{z}\int_0^{z^*})^{t+1}\| \, \|G\| \quad . \tag{10.3}$$

We have

$$\|(A\int_0^{z^*} + B\int_0^{z} + C\int_0^{z}\int_0^{z^*})^{t+1}\| \le (\|A\|\int_0^{R} + \|B\|\int_0^{R} + \|C\|\int_0^{R}\int_0^{R})^{t+1}$$

$$\le \sum_{i=0}^{t+1}\binom{t+1}{i}\|C\|^{t+1-i}(\int_0^{R}\int_0^{R})^{t+1-i}\sum_{j=0}^{i}\binom{i}{j}\|A\|^{j}(\int_0^{R})^{j}\|B\|^{i-j}(\int_0^{R})^{i-j} \quad .$$

Thus, we have

$$\|\mathcal{L}U_t^{(0)} - G\| \tag{10.4}$$

$$\le \|G\| \sum_{i=0}^{t+1}\binom{t+1}{i}\|C\|^{t+1-i}\frac{R^{2(t+1)-i}}{[(t+1-i)!]^2}\sum_{j=0}^{i}\binom{i}{j}\frac{\|A\|^{j}\|B\|^{i-j}}{j!\,(i-j)!} \quad .$$

From Equation (10.4) we see that there is a constant P such that

$$\|\mathcal{L}U_t^{(0)} - G\| \le P^t \sum_{i=0}^{t+1}\frac{\binom{t+1}{i}}{[(t+1-i)!]^2}\sum_{j=0}^{i}\binom{i}{j}\frac{1}{j!\,(i-j)!} \quad .$$

Thus, since $\sum_{j=0}^{i}\binom{i}{j} = 2^i$, we have

$$\|\mathcal{L}U_t^{(0)} - G\| \le P^t \sum_{i=0}^{t+1}\binom{t+1}{i}\frac{2^i}{[(t+1-i)!]^2[i/2]!} \quad ,$$

and since $\sum_{i=0}^{t+1} \binom{t+1}{i} 2^i = 3^{t+1}$ it follows that

$$\| \mathcal{P}_\alpha U_t^{(0)} - G \| \leq P^t 3^{t+1} / [t/4]! \quad ,$$

where $[\cdot]$ denotes the greatest integer function. Q.E.D.

Now set $u_t = \mathrm{Re}\, U_t^{(0)}(z, \bar{z})$. The above lemma then shows that u_t is nearly a particular solution of $Lu = g$ for large t.

By the results of the previous sections, we may solve

$$\begin{aligned} &Lu = 0 \quad \text{on} \quad D \\ &u = f - u_t \quad \text{on} \quad \partial D \end{aligned} \tag{10.5}$$

for an approximate solution $u_{N,M,T,t}$.

Theorem. Let u solve $Lu = g$ on D, $u = f$ given on ∂D, then

$$\lim_{t \to \infty} \left\{ \lim_{N \to \infty} \left[\lim_{M,T \to \infty} \| u - (u_t + u_{N,M,T,t}) \| \right] \right\} = 0 \quad .$$

Proof. By Theorem 4.1, there is a function $w(x,y)$ such that

$$\begin{aligned} \| u - (u_t + u_{N,M,T,t}) \| &\leq \| (u - u_t) - u_{N,M,T,t} \|_{\partial D} \\ &\quad + 2 \| w \| \, \| L(u - u_t - u_{N,M,T,t}) \| \\ &\leq \| (f - u_t) - u_{N,M,T,t} \|_{\partial D} \\ &\quad + 2 \| w \| [\| L(u - u_t) \| + \| L u_{N,M,T,t} \|] \quad . \end{aligned} \tag{10.6}$$

Fix $\epsilon > 0$. Then by the above lemma we may choose t' such that $t \geq t'$ implies

$$2\|w\| \; \|L(u - u_t)\| \leq 2\|w\| \; \|4 \operatorname{Re} \mathcal{L}(U - U_t^{(0)})\| \leq \epsilon/3 \quad .$$

Let $t \geq t'$ be fixed. Then using the results of the previous sections, we may choose N', T', and M' so that $T_k \geq T'$ for $k = 0, \ldots, N'$ and $M \geq M'$ implies

$$\|(f - u_t) - u_{N', M, T, t}\|_{\partial D} \leq \epsilon/3$$

and

$$2\|w\| \; \|Lu_{N', M, T, t}\| \leq \epsilon/3 \quad .$$

Then by Equation (10.6), for $M \geq M'$ and $T_k \geq T'$ we have

$$\|u - (u_t + u_{N', M, T, t})\| \leq \epsilon \quad . \qquad \text{Q.E.D.}$$

Given the boundary values of the solution u, all one need do is construct u_t using (10.1) and solve (10.5) for $u_{N, M, T, t}$. Then $u_t + u_{N, M, T, t}$ should be close to the solution u over the closure of the domain.

The bounds given in (10.4) and (10.6) provide an a posteriori error bound:

$$\begin{aligned} (10.7) \qquad \|u - (u_t + u_{N, M, T, t})\| \leq \; & \|f - (u_t + u_{N, M, T, t})\|_{\partial D} \\ & + 2\|w\| \; \|Lu_{N, M, T, t}\| \\ & + 8\|G\| \; \|w\| \sum_{i=0}^{t+1} \binom{t+1}{i} \|C\|^{t+1-i} \frac{R^{2(t+1)-i}}{[(t+1-i)!]^2} \sum_{j=0}^{i} \binom{i}{j} \frac{\|A\|^j \|B\|^{i-j}}{j!\,(i-j)!} \quad . \end{aligned}$$

The inequality (10.7) gives an explicit a posteriori error bound for the approximate solution $u_t + u_{N,M,T,t}$.

The assumption that the coefficient c and the right hand side g are analytic is not needed. Using a technique similar to that of Chapter 9, one can solve the problem under the hypothesis that both c and g are merely continuous on $D \cup \partial D$ and that the coefficients a and b are analytic as usual. In that case, the error bound (10.7) has three additional terms in it.

CHAPTER 11

COMPUTER IMPLEMENTATION OF THE METHOD

We wish to solve

$$Lu = \Delta u + a u_x + b u_y + c u = 0$$

on a domain D, with $u = f$ given on ∂D. The coefficients are assumed to be polynomials in x and y. These polynomials may be regarded as truncated power series for coefficients analytic on $|(x,y)| \leq 3\ \mathrm{rad}(D)$.

The input to the computer program consists of four data blocks:

(1) N, M, $T = (T_0, \ldots, T_N)$ and points $z_1, \ldots, z_M$ on ∂D;

(2) The coefficients of the polynomials a, b, and c;

(3) A subroutine that computes $f(x,y)$ for $(x,y) \in \partial D$;

(4) M', the number of points to be used in a search of ∂D for $\| u - u_{N,M,T} \|_{\partial D}$.

The steps by which the problem is solved are:

STEP (I): Given the polynomials a, b, and c in x and y, calculate the polynomials $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in z and z^* by the relations

$$\mathcal{A}(z, z^*) = \frac{1}{4}\left(a\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right) + ib\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right)\right) ,$$

$$\mathcal{B}(z, z^*) = \frac{1}{4}\left(a\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right) - ib\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right)\right) ,$$

and

$$\mathcal{C}(z, z^*) = \frac{1}{4}c\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right) .$$

We use the term "calculate" a polynomial $P(z, z^*)$ to mean the computation of the array of coefficients p_{jk} such that

$$P(z, z^*) = \sum_{j=0}^{J} \sum_{k=0}^{K} p_{jk} z^j z^{*k} .$$

Then, using the algorithm described by Bergman and Herriot in [7], calculate $Q^{(1)}, \ldots, Q^{(|T|)}$, where $|T|$ is the greatest level of truncation we wish to use, i.e., $|T| = \max_{1 \le k \le N} T_k$. If the equation is of the form $\Delta u + c(r^2)u = 0$, the calculation of the $Q^{(n)}$ may be carried out very efficiently by use of the algorithm given in Chapter 5, see (5.7).

When the $Q^{(1)}, \ldots, Q^{(|T|)}$ have been calculated, the functions ϕ_{k, T_k} and ψ_{k, T_k} for $T_k \le |T|$ can be evaluated at any point by the expressions given in Chapter 5, see (5.2) and (5.3).

The results of Step (I) are the coefficients of the truncated series for the particular solutions ϕ_{k,T_k} and ψ_{k,T_k}.

STEP (II): Given points $z_1, \ldots, z_M$ on ∂D with $M \geq 2(N+1)$, let A be the matrix (a_{jk}) given by $a_{j,2k} = \mathrm{Re}\, \phi_{k,T_k}(z_j)$ and $a_{j,2k+1} = \mathrm{Im}\, \phi_{k,T_k}(z_j)$ for $j = 1, \ldots, M$ and $k = 0, \ldots, N$. Let $b = (u(z_1), \ldots, u(z_M))^T$.

Solve the overdetermined system of linear equations

$$\min_x \|AX-b\|_\infty \tag{11.1}$$

for the minimum $e_{N,M,T}$ and a minimizing vector x.

Then define

$$u_{N,M,T} = \sum_{k=0}^{N} (x_{2k}\, \mathrm{Re}\, \phi_{k,T_k} + x_{2k+1}\, \mathrm{Im}\, \phi_{k,T_k}) .$$

The result of Step (II) is the approximate solution of the problem.

STEP (III): Using Theorem 8.1 and the remarks following it, bound $\|Lu_{N,M,T}\|$.

Search ∂D for $\|u - u_{N,M,T}\|_{\partial D}$ using M' points on ∂D.

Using the results of the above calculations, we may bound

$$\|u - u_{N,M,T}\| \leq \|u - u_{N,M,T}\|_{\partial D} + 2\|w\|\, \|Lu_{N,M,T}\| .$$

This step gives an a posteriori error bound.

The above three steps compute the approximate solution $u_{N,M,T}$ and give an indication of how accurate this approximation is.

Step (I) may be carried out symbolically using FORMAC, SYMBAL or some other symbolic processor. This has been done on a small scale and may be a useful technique as such symbolic processors become more economical to use. The method we have employed can be called at best "quasi-symbolic." It consists of simply manipulating arrays of coefficients which represent polynomials using FORTRAN.

Obviously, Step (II) is not the same algorithm as was described in Chapter 7. The compact coefficient space S_N is never mentioned in Step (II) above. According to the theory of the method, we should solve

$$\min_{x \in S_N} \|Ax-b\|_\infty \quad . \tag{11.2}$$

This problem is equivalent to the linear programming problem

(11.3) minimize $g(e) = e$ subject to the linear inequalities

$$-(\|f\|_{\partial D} + 1)/(\rho^{[k/2]}\delta(\rho)) \le s_k \le (\|f\|_{\partial D} + 1)/(\rho^{[k/2]}\delta(\rho)) \qquad \text{for } k = 0, \ldots, 2N+1$$

$$-e \le u(z_j) - u_{N,M,T}(z_j) \le e, \qquad j = 1, \ldots, M \quad ,$$

where $\rho < \min_{\partial D} |z|$ and [ ] denotes the greatest integer function. Cannon in [8] arbitrarily weakens the inequalities in (11.3) to the one-sided inequalities in

$$-e \leq u(z_j) - u_{N,M,T}(z_j) \quad , \qquad j = 1, \ldots, M$$

$$-(\|u\|_{\partial D} + 1)/(\rho^{\lceil k/2 \rceil}\delta(\rho)) \leq s_k \quad \text{for} \quad k = 0, \ldots, 2N+1 \tag{11.4}$$

$$e \geq 0 \quad .$$

He notes that <u>in practice</u> a solution of (11.4) seems to be a solution of (11.3).

There is, at present, no fast numerical method for solving linear programming problems such as (11.3) and (11.4). For this reason, it was decided that the linear programming techniques of (11.3), or even of (11.4), may not be necessary, or desirable, in practice.

R. H. Bartels and G. H. Golub [1] have given a fast and very accurate algorithm for obtaining the Chebyshev solution to an overdetermined system. Thus (11.1) may be solved practically using their algorithm. A FORTRAN translation of their ALGOL code was made and used to solve all examples given in the next chapter.

Step (II) does not take into account any corners, if there are any, in the ∂D. When there are corners $a_1, \ldots, a_C$ on ∂D with angles $\pi/\alpha_1, \ldots, \pi/\alpha_C$ one should replace the input data

block (1) by:

(1') $N = (N_0, \ldots, N_C)$, M, a truncation array T, points $z_1, \ldots, z_M$ on ∂D, $a = (a_1, \ldots, a_C)$, and $\alpha = (\alpha_1, \ldots, \alpha_C)$.

This means that for $k = 0, \ldots, N_0$ we will use ϕ_{k, T_k}. We will also use $\psi_{k, T_{k,p}}(z, a_p)$ for $k = 1, \ldots, N_p$ and $p = 1, \ldots, C$. Note that $\phi_0 = \psi_0 = B(1)$ and thus there is no need to use any of the $\psi_0(z, a_p)$.

Then replace Step (II) by:

STEP (II'): Let

$$M_p = 2 \sum_{q=1}^{p-1} N_q + 2N_0 - 1$$

for $p = 1, \ldots, C$. Then let $A = (a(j,k))$ be the matrix given by $a(j, 2k) = \text{Re } \phi_{k, T_k}(z_j)$, $a(j, 2k+1) = \text{Im } \phi_{k, T_k}(z_j)$ for $j = 1, \ldots, M$ and $k = 0, \ldots, N_0$, $a(j, 2k+M_p) = \text{Re } \psi_{k, T_{k,p}}(z_j, a_p)$ and $a(j, 2k+1+M_p) = \text{Im } \psi_{k, T_{k,p}}(z_j, a_p)$ for $j = 1, \ldots, M$ and $k = 1, \ldots, N_p$ and $p = 1, \ldots, C$. Set $b = (u(z_1), \ldots u(z_M))^t$.

Solve the overdetermined system of linear equations

$$\min_x \| Ax - b \|_\infty$$

for the minimum $e_{N, M, T}$ and a minimizing vector x.

Define

$$u_{N,M,T} = \sum_{k=0}^{N_0} (x_{2k} \operatorname{Re} \phi_{k,T_k} + x_{2k+1} \operatorname{Im} \phi_{k,T_k})$$

$$+ \sum_{p=1}^{C} \sum_{k=1}^{N_p} (x_{2k+M_p} \operatorname{Re} \psi_{k,T_{k,p}}(z,a_p) + x_{2k+1+M_p} \operatorname{Im} \psi_{k,T_{k,p}}(z,a_p)) .$$

This is the desired approximate solution.

Steps (I), (II'), and (III) provide an effective method for solving $Lu = 0$, $u = f$ given on ∂D, in the presence of any finite number of corners on ∂D.

Some discretion must be used when applying Steps (II) or (II'). If the equation being solved is of the form $\Delta u + c(r^2)u = 0$, then $\phi_{k,T_k}(z) = z^k d_{k,T_k}(r^2)$ where the d_{k,T_k} are real valued functions. Thus, $\operatorname{Im} \phi_{0,T_0}(z) = 0$. This is a very uninteresting particular solution and should not be included in the matrix A. If it is included in the matrix A, then the overdetermined linear system

$$\min_x \| Ax - b \|_\infty$$

is not well posed. That is, an infinite number of solutions exist and they may be unbounded. As long as the above care is taken, the overdetermined system approach has been, in practice, well posed.

We know that the value $e_{N,M,T}$ will be, for large M, very nearly equal to $\| u - u_{N,M,T} \|_{\partial D}$. So if M were very large

we could skip the search of ∂D in Step (III) and write

$$\|u - u_{N,M,T}\| \leq e_{N,M,T} + 2\|w\| \, \|Lu_{N,M,T}\| \quad .$$

For numeric and economic reasons, it is not desirable to have M very large. However, the value of $e_{N,M,T}$ can be useful. If when the ∂D is searched for $\|u - u_{N,M,T}\|_{\partial D}$ that value is <u>much</u> larger than the value of $e_{N,M,T}$ returned by the Bartels and Golub routine, then it would be wise to increase M and try Step (II) or (II') again.

CHAPTER 12

NUMERICAL RESULTS

The algorithm of Bartels and Golub [1] was translated into FORTRAN IV making the necessary corrections as noted in [25]. The algorithms of Chapter 11 for the construction of the ϕ_k and ψ_k, calculation of the coefficients and bounding of Lu^* were coded in FORTRAN IV using REAL*8 arithmetic. All computations were carried out on an IBM 360 Model 67. The iterative improvement in [1] was accomplished using extended precision arithmetic.

To test the routines we used the following problem. The equation is $\Delta u - 4(1 + x^2 + y^2)u = 0$ on the "squirkle" with $u(x,y) = \exp(x^2 + y^2)$ on ∂D. The "squirkle" is the square with sides given by $x = \pm 1$, $y = \pm 1$ and its corners rounded by quadrants of circles. See Fig. 1.a. The northeast corner of the square is replaced by the northeast quadrant of the circle

$$(x - x_0)^2 + (y - y_0)^2 = (1 - y_0)^2 \quad ,$$

where $x_0 = y_0 = \tan(39.0 * 0.017453293)$, this is a "magic" number given by Bergman and Herriot in [7]. The other corners are rounded similarly.

The squirkle is a domain of class Ah and has no corners. Thus we need only use the ϕ_k in solving the problem.

The exact solution of the problem is $u = \exp(x^2 + y^2)$ in the entire xy-plane and provides a check case for our routines.

Table 1 lists the results for this example. The "METHOD" column indicates whether collocation or the Chebyshev method was used. The "ERROR 1" column gives the error found by searching the curves $L_1 = \frac{1}{4}\partial D$, $L_2 = \frac{1}{2}\partial D$, $L_3 = \frac{3}{4}\partial D$ with 48 points on each. "E" is the best fit tolerance returned by the Bartels and Golub routine. "ERROR 2" gives an estimate of the size of $\|w\| \, \|Lu^*\|$. If the entry is "$\geq$?" then $\|w\| \, \|Lu^*\| \geq$? as determined by a hand calculation. If the entry is "$\leq$?" then $\|w\| \, \|Lu^*\| \leq$? as determined by the bounding algorithm given in Chapter 8. The w used here was $w(x,y) = \frac{1}{2}(-1 + x^2)$, i.e., $\|w\| = \frac{1}{2}$. The "TIME" column gives the total running time in seconds for all calculations except compilations. The compilation time for the Chebyshev routine of Bartels and Golub is 18 seconds. The compilation time of ϕ_k version of the algorithm takes 18 seconds while the version which uses the ϕ_k and ψ_k takes 22 seconds to compile. Once compiled, the object modules for the routines were placed on disk packs and read off them for each run. The "N,M,T" column gives the number of ϕ_k used, the number of boundary points used, and the truncation vector.

The two very long runs were made using a FORTRAN translation of the ALGOL code given by Bergman and Herriot in [7] for the computation of the $\phi_{k,4}$. In these two cases, the compilation time is included in the time estimate. These routines had two distinct drawbacks as Table 1 shows. First, the fixed truncation level of 4 is not sufficient to have $\|Lu^*\|$ small. Second, they are very time consuming. After these two runs had been made, we decided to write our own routines for the construction of ϕ_{k,T_k} for any T_k. Both a general algorithm like that of Bergman and Herriot and the special algorithm of Chapter 5 (see Equation 5.7) for $\Delta u + c(r^2)u = 0$ were coded and used on this example. The first run in the table was made using the general purpose routines for the equation $\Delta u + au_x + bu_y + cu = 0$, and the time estimate there includes the compilation time. It may easily be seen that the running time is greatly reduced by recoding the algorithm of [7]. The rest of the runs were made using the special algorithm for $\Delta u + c(r^2)u = 0$ which is even more efficient.

One indication given by Table 1 is that collocation should not be used. It gives less accurate results than the Chebyshev method. The same behavior will be exhibited in other examples.

After the runs for Table 1 had been made, it was noticed that all the coefficients of the ϕ_k were nearly zero, except the

first one which was very close to one. This led to the conjecture that $\exp(x^2 + y^2) = \text{Re } \phi_0 = \phi_0$, which, in turn, led to the following characterization of the functions ϕ_k for the equation $\Delta u + c(r^2)u = 0$.

Theorem. All solutions of

$$U_{zz^*} + \frac{1}{4}c(r^2) = 0 \quad , \quad r^2 = zz^*$$

which are of the form $U(z, z^*) = z^k g(r^2)$ are constant multiples of particular solutions ϕ_k; specifically we have

$$U(z, \bar{z}) = 2g(0)\, \phi_k(z, \bar{z}) \quad .$$

Proof. We have

$$V^{-1}(\text{Re } U(z, \bar{z})) = 2z^k g(0) - 0 * g(0) = 2z^k g(0) \quad ,$$

and hence

$$\begin{aligned} \text{Re } U(z, \bar{z}) &= V(2z^k g(0)) = \text{Re } B(2z^k g(0))(z, \bar{z}) \\ &= 2\,\text{Re}(g(0)\phi_k(z, \bar{z})) \quad . \end{aligned}$$

We also have

$$V^{-1}(\text{Im } U(z, \bar{z})) = 2iz^k g(0) - 0 * i\, g(0) = 2iz^k g(0) \quad .$$

As above, we then have $\text{Im } U(z, \bar{z}) = -\text{Re}(i\, U(z, \bar{z}))$, i.e.,

$$\begin{aligned} \text{Im } U(z, \bar{z}) &= 2\, \text{Re}(-ig(0)\phi_k(z, \bar{z})) \\ &= 2\, \text{Im}(g(0)\phi_k(z, \bar{z})) \quad . \end{aligned}$$

Q.E.D.

The above theorem proves that $\phi_0 = \exp(x^2 + y^2)$ is the correct first particular solution of Bergman.

In [7] Bergman and Herriot assume that for $T = 4$ the approximations will satisfy the differential equation, and hence the Maximum Principle, fairly well. A glance at Table 1 shows that $T = 4$ does not produce such results. We have $\| L(\mathrm{Re}\ \phi_{0,4}) \| > 4$ and $\| Lu_{12,48,4} \| > 4$ for this example. So although this approximation nearly satisfies the boundary conditions, it does not satisfy the differential at all well. We searched the ∂D for $\| u - u_{12,4} \|_{\partial D}$ in the collocation result and also searched inside D. We found that $\| u - u^* \|_{\partial D} = 6.7 * 10^{-4}$ while $| (u - u^*)(x', y') | = 4.5 * 10^{-3}$ for a point (x', y') inside D. Thus, in this case, we have

$$\| u - u^* \| > 6 \| u - u^* \|_{\partial D}$$

and the Maximum Principle does not hold even approximately. This shows that the error bound given in Theorem 4.1 must be used, for in general Lu^* will not be zero and the Maximum Principle will not hold.

Example 1: The equation is

$$\Delta u + (-4 + 2(x^2 + y^2))u = 0$$

on the squirkle with $u(x,y) = x^2 + y^2$ on ∂D. This example is given by Bergman and Herriot in [7]. The exact solution of this

problem is unknown.

Table 2 gives the results for this example. The "ERROR 1" column gives $\|u - u^*\|_{\partial D}$ as determined by a search of ∂D using the number of points given in the "SEARCH" column. "ERROR 2" gives a bound on the size of $\|w\| \, \|Lu^*\|$. Thus, by Theorem 4.1 we may write

$$\text{ERROR 1} + \text{ERROR 2} \geq \|u - u^*\| \quad ,$$

and we have an a posteriori error bound.

The symmetry relation (5.6) may be used to reduce the amount of computation needed to solve a given problem. If the boundary and boundary values are invariant with respect to rotation about the origin by an angle of $2\pi/s$ where s is an integer, then obviously the only ϕ_k that will be needed are the ϕ_{ks, T_k} for $k = 0, 1, \ldots$. The routine takes advantage of this fact and s is passed as a parameter to it. In the present example we have $s = 4$. The s used for each run is listed under the "s" in the "N, M, T, s" column of Table 2.

The two very long collocation runs were made with the FORTRAN version of Bergman and Herriot's ALGOL routines. The rest of the runs were made with our own FORTRAN variable truncation routines for $\Delta u + c(r^2)u = 0$.

Example 2: The equation is

$$\Delta u = 0$$

on the ellipse $x^2 + 4y^2 < 1$, with $u(x,y) = \sin(x)$ on ∂D.

The results for this example are given in Table 3. No symmetry properties were used for this example; i.e., $s = 1$ was passed to the routines. No compilations were made and the total running time for all runs was 69 seconds.

In this case, all the $Q^{(n)}$ are identically zero and hence $\phi_{k,1}$ is exactly equal to ϕ_k for all k. This means that we have $\|Lu^*\| = 0$ for all our approximations. We then have

$$\|u - u^*\| \leq \|u - u^*\|_{\partial D}$$

which may be estimated by the value "ERROR" in the table.

One collocation run was made with the result that $\|u - u_4\| \leq 2.1 * 10^{-2}$ using a 25 point search of ∂D. The Chebyshev result gave $\|u - u_{4,10,1}\| \leq 5.0 * 10^{-4}$ with a 25 point search of ∂D. This is more evidence that collocation is not the best method for solving elliptic boundary value problems.

The first seven runs in Table 3 are especially interesting. There the number of terms is kept constant while the number of boundary points is increased. It is seen that there is very little to be gained by taking M very large. It should also be noted that

for M nearly equal to $4N$; namely, for $N = 6$ and $M = 28$ we have the best results observed. The relation $M = 4N$ has been used as a rule of thumb throughout our applications. As the results show, it seems to work fairly well. For more oscillatory boundary values we would have to use more boundary points, but for moderately smooth boundary values, $M = 4N$ works well.

Example 3: The equation is

$$\Delta u - 4u = 0$$

with $u = 1$ on ∂D. The domain is the rhombus with corners $(\pm 1, 0)$ and $(0, \pm \tan(\pi/12.5))$. See Fig. 1.b. This domain has four corners in it, and we should use the ψ_k at these corners. The corners at $(0, \pm \tan(\pi/12.5))$ have an angle of $\pi/(6.25/5.25)$ while the corners at $(\pm 1, 0)$ have an angle of $\pi/6.25$. Since the solution at a corner a of angle π/α behaves like a polynomial in $(z-a)^{\alpha}$, we see that the corners at $(0, \pm \tan(\pi/12.5))$ will probably be more "important" in the convergence process than the corners at $(\pm 1, 0)$.

Table 4 gives the results for this example. The parameters "PAR" are described in the footnotes. The last two runs in the table were made using no corner terms at all, i.e., just the ϕ_k were used.

From Table 4 it is seen that by using 4 of the ψ_k at each of the corners $(0, \pm\tan(\pi/12.5))$ and 10 of the ϕ_k, one may obtain an approximate solution u^* such that

$$\|u - u^*\| \leq 1.0 * 10^{-5} \quad .$$

The last two runs in Table 4 show that the process converges quite slowly if we use only the ϕ_k. By taking $\phi_0, \ldots, \phi_{12}$, we can only obtain $\|u - u_{13,52,10}\| \leq 2.3 * 10^{-2}$. However, by using $\phi_0, \ldots, \phi_6$ and 3 of the ψ_k at each of the corners $(0, \pm\tan(\pi/12.5))$ we may obtain u^* such that $\|u - u^*\| \leq 5.7 * 10^{-5}$. Both of these runs use a total of 13 particular solutions from the set of all ϕ_k and ψ_k. Thus the necessity of using the ψ_k for obtaining fast convergence, in the presence of corners, is quite apparent. By simply exchanging 6 of the ϕ_k for 6 of the ψ_k the approximation is made nearly 300 times more accurate.

The relative "importance" of the corners $(\pm 1, 0)$ and $(0, \pm\tan(\pi/12.5))$ may be weighed by comparing the second and third runs in Table 4. Exchanging the two ψ_k which were used at each of the corners $(\pm 1, 0)$ in the second run for one more ψ_k at each of the corners $(0, \pm\tan(\pi/12.5))$ and two more ϕ_k in the third run, the error bound dropped slightly from $7.3 * 10^{-5}$ to $5.7 * 10^{-5}$. Thus, the singularities in the partial derivatives of u at $(\pm 1, 0)$ are, experimentally, much less important than those at

the other corners. As mentioned above, this should be the case due to the size of the angles at the respective corners.

Example 4: The equation is

$$\Delta u - 4u = 0$$

with $u = 1$ on ∂D, where D is the L-shaped domain in Fig. 1.c.

The results for this example are given in Table 5.

This domain is the classical example of a domain with a re-entrant corner. The corner at $(\frac{1}{2}, -\frac{1}{2})$ has an angle of $3\pi/2$ and is thus re-entrant. The first order partial derivatives of u may be unbounded near this corner.

The fourth run in Table 5 shows that using only the ϕ_k does not give good results. Using 25 of the ϕ_k we obtain the error estimate $\|u - u_{25,88,10}\| \leq 0.23$. This shows the very slow rate of convergence that a re-entrant corner produces when only the ϕ_k are used. The last two runs in the table show that using the ϕ_k or the ψ_k still results in very slow convergence rates. Thus, the ϕ_k or the ψ_k alone are not enough to give practical rates of convergence. A mixture of the two must be used. The proper mix must be found experimentally. The first two runs illustrate this quite well. By exchanging five of the ψ_k for five of the ϕ_k the error is nearly halved. Just what the optimal combination of ϕ_k and ψ_k should be is not at all obvious.

Example 5: The equation is

$$\Delta u + 4(-1 + \frac{1}{2}(x^2 + y^2))u = 4x$$

on the ellipse $x^2 + 4y^2 < 1$, with $u(x,y) = x$ on ∂D.

The technique of Chapter 10 was coded for polynomial coefficients. Using the notation of that chapter and the a posteriori error bound (10.7), we obtained:

$$\|u - (u_{8,32,15,10} + u_{10})\| \leq 1.2*10^{-6} + 2.5*10^{-15} + 5.3*10^{-15} ,$$

using a 300 point search of the boundary of the ellipse. There is no need to increase the number of points M because the $e_{8,32,15,10}$ returned by the Bartels and Golub routine was $8.9*10^{-7}$ which is only slightly smaller than the searched-for bound of $1.2*10^{-6}$. The total running time for this run was 40 seconds, including the compilation of the routine for computing the particular solution u_t.

Example 6: The equation is

$$\Delta u + xu_x + yu_y + (x+y)u = 0$$

on the ellipse $x^2 + 4y^2 < 1$, with $u(x,y) = \sin(x)$ on ∂D.

The general purpose algorithm for the construction of the ϕ_k and the ψ_k was used to obtain:

$$\|u - u_{8,32,15}\| \leq 9.1*10^{-7} + 1.0*10^{-8} ,$$

using a boundary search of 200 points. The $e_{8,32,15}$ returned by the Bartels and Golub routine was $6.3*10^{-7}$, which indicates that 32 points is sufficient in the fitting procedure.

The total running time, including compilation of the general purpose algorithm, was 233 seconds.

Example 7: The equation is

$$\Delta u + xu_x + yu_y + (x+y)u = 1 + x^2 + y^2$$

on the ellipse $x^2 + 4y^2 < 1$, with $u(x,y) = \sin(x)$ on ∂D.

Using the notation of Chapter 10 and inequality (10.7), we obtained

$$\| u - (u_{8,32,15,10} + u_{10}) \| \leq 1.3*10^{-6} + 1.5*10^{-7} + 3.3*10^{-6}$$

with a 200-point search of the ∂D. There is no need to increase the number of points M because the $e_{8,32,15,10}$ returned by the Bartels and Golub routine was $9.3*10^{-7}$ which is only slightly smaller than the searched-for bound of $1.3*10^{-6}$. The total running time for this run was 266 seconds, including the compilation of the general purpose routine for the computation of the ϕ_k and the routine for the construction of u_t.

BIBLIOGRAPHY

1. R. H. Bartels and G. H. Golub, "Chebyshev Solution to an Overdetermined Linear System," Comm. ACM., 11, (1968), 428-430.

2. Stefan Bergman, "Zur Theorie der Functionen, die eine Lineare Partielle Differentialgleichung Befreiedigen," Recueil Mathematique, New Series, 2, (1937), 1169-1198.

3. _____, "The Approximation of Functions Satisfying a Linear Partial Differential Equation," Duke Math. J., 6, (1940), 537-561.

4. _____, "Construction of Complete Solution Sets," Quart. Appl. Math., 4, (1946), 233-245.

5. _____, "Integral Operators and Partial Differential Equations," Ergibnisse der Mathematik und Ihrer Grenzgebiete, Neue Folge, Heft 23, 1961.

6. Stefan Bergman and John G. Herriot, "Application of the Method of the Kernel Function for Solving Boundary Value Problems," Numer. Math., 3 (1961), 209-225.

7. _____, "Numerical Solution of Boundary Value Problems by the Method of Integral Operators," Numer. Math., 7, (1965), 42-65.

8. G. Birkhoff, M. H. Schultz, and R. S. Varga, "Piecewise Hermite Interpolation in One and Two Variables with Applications to Partial Differential Equations," Numer. Math., 11, (1968), 232-256.

9. J. R. Cannon, "The Numerical Solution of the Dirichlet Problem for Laplace's Equation by Linear Programming," SIAM J. Appl. Math., 12, (1964), 233-237.

10. J. R. Cannon and Maria M. Cecchi, "The Numerical Solution of Some Biharmonic Problems by Mathematical Programming Techniques," SIAM J. Numer. Anal., 3, (1966), 451-466.

11. P. G. Ciarlet, M. H. Schultz, and R. S. Varga, "Numerical Methods of High-Order Accuracy for Non-Linear Boundary Value Problems";
One-Dimensional Problems, Numer. Math., 9, (1967), 394-430;
Nonlinear Boundary Conditions, Numer. Math., 11, (1968), 331-345.
Eigenvalue Problems, Numer. Math., 12, (1968), 120-133.

12. R. Courant and D. Hilbert, Methods of Mathematical Physics, Volume II, Interscience, New York, 1962.

13. George J. Fix, "Bounds and Approximations for Eigenvalues of Self-Adjoint Boundary Value Problems," Doctoral Thesis, Harvard University, Cambridge, Mass., 1968.

14. L. Fox, P. Henrici, and C. Moler, "Approximations and Bounds for Eigenvalues of Elliptic Operators," SIAM J. Numer. Anal., 4, (1967), 89-102.

15. Peter Henrici, "Bergman's Intergraloperator erster Art und Riemannsche Funktion," ZAMP, 3, 1952, 228-232.

16. _____, "Zur Funktiontheorie der Wellengleichung," Comm. Math. Helv., 27, (1953), 235-293.

17. _____, "A Survey of I. N. Vekua's Theory of Elliptic Partial Differential Equations with Analytic Coefficients," ZAMP, 8, (1957), 169-203,

18. E. B. Karpilovskaya, "Convergence of the Collocation Method for Certain Boundary Value Problems of Mathematical Physics," Sibirsk. Mat. Z. 4, 632-640.

19. R. Sherman Lehman, "Developments at an Analytic Corner of Solutions of Elliptic Partial Differential Equations," J. Math. Mech., 8, (1959), 729-760.

20. Keith Miller, "Barriers on Cones for Uniformly Elliptic Operators," Ann. Mat. Pura Appl., 4, (1967), 93-105.

21. I. P. Natanson, Constructive Function Theory, Volume I, Ungar, New York, 1964.

22. Karl Nickel, "Extension of Fox, Henrici, and Moler on Eigenvalues of Elliptic Operators," SIAM J. Numer. Anal., 4, (1967), 483-488.

23. M. Protter and H. Weinberger, Maximum Principles in Differential Equations, Prentice Hall, Englewood Cliffs, N. J., 1967.

24. J. B. Rosen, "Approximate Solution and Error Bounds for Quasilinear Elliptic Boundary Value Problems," Technical Report No. 30, 1968, Computer Science Department, University of Wisconsin.

25. N. L. Schryer, "Certification of Algorithm 328," Comm. ACM., 12, (1969), 326.

26. A. Shindler, "Some Theorems of the General Theory of Approximate Methods of Analysis and their Applications to Collocation, Moment, and Galerkin Methods," Siberian Math. J., 8, (1967), 302-314.

27. I. N. Vekua, New Methods for Solving Elliptic Equations, North-Holland Publishing Company, Amsterdam, 1967.

28. J. L. Walsh, Interpolation and Approximation, American Mathematical Association, Colloquium Publications, Volume XX.

29. J. R. Whiteman, "Singularities due to Re-entrant Corners in Harmonic Boundary Value Problems," MRC Technical Summary Report No. 829, (1967), University of Wisconsin.

30. _____, "Numerical Solution of Harmonic Mixed Boundary Value Problems by Linear Programming," MRC Technical Summary Report No. 857, (1968), University of Wisconsin.

31. J. R. Whiteman and M. J. M. Bernal, "Numerical Treatment of Biharmonic Boundary Value Problems with Re-entrant Boundaries," MRC Technical Summary Report No. 914, University of Wisconsin.

APPENDIX I

TABLES AND FIGURE

TABLE 1

METHOD	N,M,T	E	ERROR 1	ERROR 2	TIME
Cheby	12,48,(1)	$1.3*10^{-5}$	$1.9*10^{-4}$	$1.1 \leq \quad \leq 1.3$	46
Cheby	12,48,4	$2.8*10^{-4}$	$4.0*10^{-3}$	$2.0 \leq$	111
Colloc	12,23,4	---	$4.5*10^{-3}$	$2.0 \leq$	152
Cheby	12,48,(2)	$2.5*10^{-7}$	$7.2*10^{-7}$	$\leq 1.9*10^{-9}$	22
Cheby	12,48,6	$1.3*10^{-5}$	$1.9*10^{-4}$	≤ 0.24	16

The truncation vectors are:

(1): $T_k = (36-k)/6$ for $k = 0, \ldots, 11$.

(2): $T_k = \begin{cases} 6 & \text{if } k = 0, 4, 8, 12 \\ 20 & \text{if } k = 0, 4, 8, 12 \end{cases}$

TABLE 2

METHOD	N,M,T,s	E	SEARCH	ERROR 1	ERROR 2	TIME
Colloc	32,63,4,1	---	120	$4.2*10^{-4}$	$1/8 \leq$	241
Colloc	28,55,4,1	---	120	$1.3*10^{-3}$	$1/8 \leq$	211
Cheby	12,48,(1),4	$7.2*10^{-5}$	320	$1.5*10^{-4}$	$\leq 3.3*10^{-15}$	32
Cheby	8,48,(2),4	$2.2*10^{-4}$	320	$3.1*10^{-4}$	$\leq 2.1*10^{-14}$	28
Cheby	8,48,10,4	$2.2*10^{-4}$	300	$3.1*10^{-4}$	$\leq 1.6*10^{-5}$	18
Cheby	12,48,(3),1	$2.9*10^{-3}$	320	$4.4*10^{-3}$	$\leq 5.8*10^{-11}$	34
Cheby	24,80,(4),4	$9.0*10^{-6}$	300	$4.9*10^{-5}$	$\leq 9.4*10^{-11}$	113

The truncation vectors are:

(1): $T_k = \begin{cases} 20 & \text{if } k \leq 6 \\ 10 & \text{if } k \geq 7 \end{cases}$

(2): $T_k = \begin{cases} 20 & \text{if } k \leq 4 \\ 10 & \text{if } k \geq 5 \end{cases}$

(3): $T_k = \begin{cases} 20 & \text{if } k = 0,4,8,12 \\ 6 & \text{if } k \neq 0,4,8,12 \end{cases}$

(4): $T_k = \begin{cases} 15 & \text{if } k \leq 12 \\ 10 & \text{if } k \geq 13 \end{cases}$

TABLE 3

METHOD	N,M,T	E	SEARCH	ERROR
Colloc	6,11,1	---	100	$5.8*10^{-6}$
Cheby	6,20,1	$2.5*10^{-6}$	100	$3.8*10^{-6}$
Cheby	6,28,1	$3.0*10^{-6}$	100	$3.0*10^{-6}$
Cheby	6,36,1	$2.8*10^{-6}$	100	$3.2*10^{-6}$
Cheby	6,44,1	$2.9*10^{-6}$	100	$3.1*10^{-6}$
Cheby	6,52,1	$2.9*10^{-6}$	100	$3.1*10^{-6}$
Cheby	6,60,1	$2.9*10^{-6}$	100	$3.1*10^{-6}$
Cheby	4,10,1	$5.0*10^{-4}$	25	$5.0*10^{-4}$
Cheby	5,12,1	$4.0*10^{-4}$	30	$6.2*10^{-4}$
Cheby	6,20,1	$2.5*10^{-6}$	40	$3.7*10^{-6}$
Cheby	7,30,1	$2.8*10^{-6}$	100	$3.3*10^{-6}$
Colloc	4,7,1	---	25	$2.1*10^{-2}$

TABLE 4

METHOD	PAR	E	SEARCH	ERROR 1	ERROR 2	TIME
Cheby	(1)	$2.1*10^{-7}$	200	$1.0*10^{-5}$	$4.1*10^{-10}$	82
Cheby	(2)	$2.2*10^{-5}$	72	$7.3*10^{-5}$	$1.7*10^{-6}$	39
Cheby	(3)	$1.7*10^{-5}$	72	$5.7*10^{-5}$	$9.2*10^{-6}$	34
Cheby	(4)	$1.3*10^{-3}$	300	$6.0*10^{-3}$	$1.4*10^{-13}$	16
Cheby	(5)	$6.1*10^{-4}$	300	$2.3*10^{-2}$	$1.4*10^{-13}$	45

The parameters PAR are:

(1): Use 10 of the $\phi_{k,8}$ and 4 of the $\psi_{k,9}$ at each of the corners $(0, \pm\tan(\pi/12.5))$ and fit the boundary conditions at 60 points.

(2): Use 5 of the $\phi_{k,10}$ and 2 of the $\psi_{k,7}$ at each of the four corners of the domain and fit the boundary conditions at 52 points.

(3): Use 7 of the $\phi_{k,10}$ and 3 of the $\psi_{k,6}$ at each of the corners $(0, \pm\tan(\pi/12.5))$ and fit the boundary conditions at 52 points.

(4): Use 8 of the $\phi_{k,10}$ and fit the boundary conditions at 92 points. Use the two-fold symmetry of the domain.

(5): Use 13 of the $\phi_{k,10}$ and fit the boundary conditions at 52 points.

TABLE 5

METHOD	PAR	E	SEARCH	ERROR 1	ERROR 2	TIME
Cheby	(1)	$2.7*10^{-3}$	200	$8.5*10^{-3}$	$1.1*10^{-7}$	148
Cheby	(2)	$2.2*10^{-3}$	200	$1.5*10^{-2}$	$5.5*10^{-7}$	113
Cheby	(3)	$2.1*10^{-2}$	200	$6.5*10^{-2}$	$1.8*10^{-7}$	46
Cheby	(4)	0.13	240	0.23	$1.4*10^{-8}$	138
Cheby	(5)	0.031	160	0.13	$6.9*10^{-11}$	36
Cheby	(6)	0.015	160	0.20	$5.6*10^{-9}$	60

The parameters are:

(1): Use 16 of the $\phi_{k,10}$ and 5 of the $\psi_{k,10}$. Fit the boundary conditions at 80 points.

(2): Use 11 of the $\phi_{k,10}$ and 10 of the $\psi_{k,10}$. Fit the boundary conditions at 80 points.

(3): Use 8 of the $\phi_{k,10}$ and 5 of the $\psi_{k,10}$. Fit the boundary conditions at 48 points.

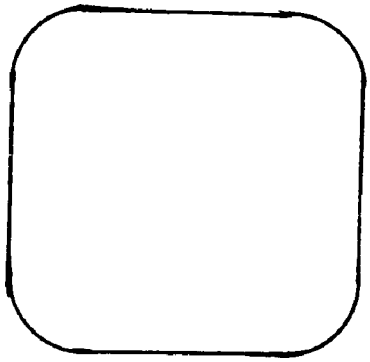
(4): Use 25 of the $\phi_{k,10}$ and fit the boundary conditions at 88 points.

(5): Use 12 of the $\phi_{k,10}$ and just one of the $\psi_{k,10}$. Fit the boundary conditions at 48 points.

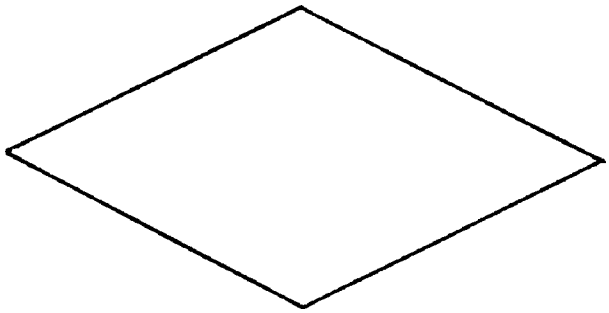
(6): Use $\psi_{0,10}, \cdots, \psi_{12,10}$ and fit the boundary conditions at 48 points.

Figure 1

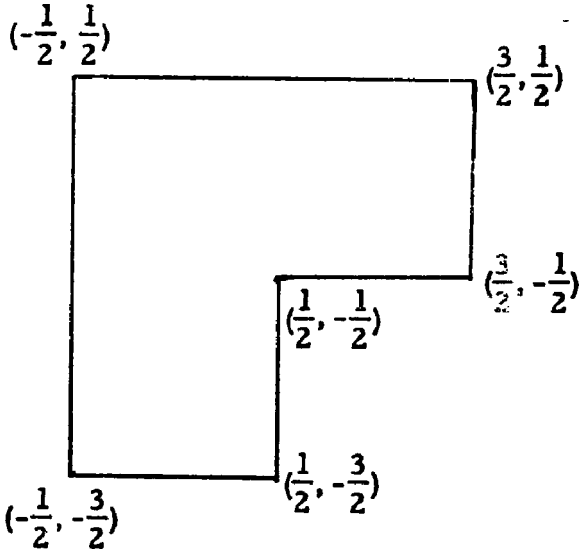
(a)



(b)



(c)


$(-\frac{1}{2}, \frac{1}{2})$
$(\frac{3}{2}, \frac{1}{2})$
$(\frac{3}{2}, -\frac{1}{2})$
$(\frac{1}{2}, -\frac{1}{2})$
$(\frac{1}{2}, -\frac{3}{2})$
$(-\frac{1}{2}, -\frac{3}{2})$