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VECTOR SEQUENCE SPACES

by

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VECTOR SEQUENCE SPACES

By David A. Gregory

A general study of scalar sequence spaces (s.s.s.) was begun in 1934 by Köthe and Toeplitz. In this dissertation, vector sequence spaces (v.s.s.) are studied. A v.s.s. is a linear space $S(E)$ of sequences of vectors chosen from a vector space E. The operations on $S(E)$ are the usual componentwise ones and $S(E)$ is assumed to contain all those sequences in E that have only a finite number of entries not equal to the zero vector.

In Chapter 1, $S(E)$ is provided with a topology T so that $(S(E),T)$ is an l.c.t.v.s. The notions of solid subset and solid topology that occur in s.s.s. are then extended to v.s.s. For each $j = 1,2,\ldots,$ an isomorphic copy of E is obtained in $S(E)$ by restricting attention to those sequences in $S(E)$ with all terms but the j-th term equal to the zero vector. The relative topology on E is denoted by T_j and the topological properties of the natural imbedding and projection maps that arise between (E,T_j) and $(S(E),T)$ are studied. Results on compact, countably compact, and sequentially compact subsets of $(S(E),T)$ are obtained. In particular, it is shown that any two of these classes of subsets of $(S(E),T)$ are the same if and only if the corresponding class of subsets of (E,T_j) are the same for each j. Compact subsets and sequentially compact subsets of $(S(E),T)$ are characterized.

In Chapter 2, a dual pair $\langle E,E' \rangle$ of vector spaces is given and $S(E)$ is assumed to be a member of a dual pair $\langle S(E),S'(E') \rangle$ of v.s.s. where the bilinear form on the latter dual pair is induced by the bilinear form on the former dual pair by summation. The l.c.t.v.s. $(S(E),T)$ is then considered where T is a polar topology with respect to the dual pair $\langle S(E),S'(E') \rangle$. Such an l.c.t.v.s. $(S(E),T)$ is said to be of Köthe type. Much of the terminology of this chapter is borrowed from s.s.s. and many of the results are easily reduced to known properties of s.s.s. In particular, normal topology, α-dual, and α-perfect are defined as expected and weakly convergent sequences in $S(E)$ are shown to be normally convergent if $S'(E')$ is solid. Under certain additional assumptions, the Mackey, the strong, and the strong dual topologies on $S(E)$ are shown to be solid. A theorem on matrices of linear maps is given. The chapter is concluded with seminorm conditions which determine when certain l.c.t.v.s. $(S(E),T)$ are of Köthe type.

In Chapter 3, necessary and sufficient conditions are given for spaces of Köthe type to be complete, quasicomplete, sequentially complete, nuclear, barrelled, quasibarrelled (= evaluable), reflexive, or semireflexive. The completeness results require T to be solid. The result on nuclearity is given for the normal topology only. The remaining results are stated for the case when T is the Mackey topology.

In Chapter 4, we study a v.s.s. that is formed in a natural way from a s.s.s. and an l.c.t.v.s. (E,T) . The "generalized

perfect spaces" of Pietsch are examples of spaces of this type and are shown to be α-perfect. Additional results are obtained for the case when (E,T) is a Banach space.

The theory of s.s.s. has been a fundamental tool in summability theory and has enjoyed occasional applications to the theory of bases, analytic function theory, and distribution theory. The theory of v.s.s. therefore shows promise as a useful tool in the study of the vector valued generalizations of these theories.

Preface

A theory of scalar sequence spaces (s.s.s.), i.e. of spaces such as ℓ_1, ℓ_2, c or c_o, was given in 1934 by Köthe and Toeplitz [11]. It was in this paper that the "vollkommene" or "perfect" spaces were first introduced. The theory was later reformulated within the framework of locally convex spaces in 1951 by Köthe [9]. The purpose of the present study is to develop a theory of vector sequence spaces (v.s.s.), i.e. of linear spaces of sequences of vectors chosen from a fixed vector space E. Examples of such spaces are the space of all convergent sequences and the space of all bounded sequences in a locally convex topological vector space (l.c.t.v.s.). The results we obtain are motivated by well-known properties of scalar sequence spaces, but the proofs we give are usually quite different. The key to many of the results in the case of scalars is a theorem on compactness due to Köthe ([9], p. 418) which need not hold in the vector case.

Just as Köthe considered a dual pair $\langle\lambda,\lambda^*\rangle$ of s.s.s. with a bilinear form induced by summation of componentwise products, we consider the v.s.s. S(E) as a member of a dual pair $\langle S(E),S'(E')\rangle$ of v.s.s. with a bilinear form induced by summation of a bilinear form on a dual pair $\langle E,E'\rangle$ of vector spaces. Except for a more general viewpoint adopted in Chapter 1, we study the v.s.s. S(E) within the framework of this duality.

In Chapter 1, we provide S(E) with a topology T to form an l.c.t.v.s. (S(E),T). No use is made of the notion of dual pair at this point. The main theorems deal with compact subsets of (S(E),T) and are motivated by the useful compactness theorem of Köthe already mentioned. The proof of Theorem 4.1 is a simple modification of a type of proof used by Pietsch ([13], p. 34).

An analogue of Köthe's compactness theorem is given for metrisable spaces. Even in the scalar case, it is more general than the original theorem since $S(E)$ is not assumed to be perfect. In the succeeding chapters, the topologies T we put on a v.s.s. $S(E)$ arise from a dual pair $\langle S(E), S'(E') \rangle$. Such an l.c.t.v.s. $(S(E), T)$ is said to be of Köthe type. The results we obtain, even when reduced to the scalar case, are better than those generally known since we do not assume that $S'(E') = S(E)^{\times}$ (see Chapter 2, Section 2).

In Chapter 2, we develop some basic properties of spaces of Köthe type. The results on convergence which we give are familiar in the scalar case and actually follow quickly from the scalar analogues with a little manipulation. We also show in this chapter that under certain additional assumptions the Mackey, strong, and strong dual topologies are all solid and thus eligible for the completeness criteria given in Chapter 3. A theorem on matrix maps which generalizes a result due to Köthe and Toeplitz ([11], p. 206) is also given. The chapter is concluded with a characterization of certain spaces of Köthe type. Such characterizations are essential since v.s.s. do not in practice generally occur as members of dual pairs $\langle S(E), S'(E') \rangle$. The characterizations given apply in most cases that arise; the author was unable to obtain a complete characterization.

In Chapter 3, we investigate v.s.s. of Köthe type that are complete, quasicomplete, sequentially complete, barrelled, quasi-barrelled, reflexive, semireflexive, Montel, or nuclear. These properties are hereditary, i.e. E with a relative topology T_j inherits each of the above properties that $(S(E), T)$ has. The author was unable to determine which spaces of Köthe type are

bound (=bornological). The problem is discussed in the scalar case by Komura and Komura [8].

In Chapter 4, we consider a v.s.s. $\lambda(E,T)$ that is formed in a natural way from a s.s.s. λ and an l.c.t.v.s. (E,T). Spaces of this type occur frequently in practice and include the spaces $\lambda(E)$ of Pietsch [14] and the examples $\mathcal{F}(E)$ of Cac [1] as special cases.

Gribanov ([6], [7]) and Cac ([1], [2]) have also considered v.s.s. from the viewpoint of dual pairs. The few results that their papers have in common with this thesis are given here in greater generality.

The theory is not presented here in the most general form. Out of convenience, we have taken sequences from a fixed vector space E. All of the results can be extended without difficulty to spaces of vector sequences chosen from a sequence E_n, $n = 1,2,\cdots$, of vector spaces. Some of the results can be extended to spaces of directed vector sequences chosen from a directed family E_α, $\alpha \in D$ of vector spaces. Examples of such spaces can be found in Pietsch [13]. Dieudonné [4] has generalized the results of scalar sequence spaces, i.e. of spaces of scalar valued functions on the integers, to theorems on spaces of scalar valued functions on a measure space. The author was unable to develop a theory of vector valued functions on a measure space which embraced both the paper of Dieudonné and the present work as special cases.

Aside from its intrinsic interest, the scalar theory of Köthe and Toeplitz has proved valuable as a rich source of examples and counterexamples in the theory of locally convex spaces. Moreover, the theory has been a fundamental tool in the study of summability and has enjoyed occasional application to the theory of bases,

distribution theory, and analytic function theory. Vector sequence spaces therefore show promise as useful tools in studying the vector valued generalizations of these theories.

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CHAPTER 1

VECTOR SEQUENCE SPACES

In this chapter, we study a vector sequence space $S(E)$ with a locally convex topology T. In practice, $S(E)$ is given beforehand and T is usually determined by a class of seminorms. The treatment we give here is more general than that in Chapter 2, and the results we obtain can be applied there.

1. Definitions and Notation

Let E be a vector space over the scalar field K of real or complex numbers. By a vector sequence space (v.s.s.) $S(E)$ over E, we mean a set $S(E)$ of sequences $(x_n) = (x_1, x_2, \cdots)$ of vectors x_n from E that forms a linear space over K under the usual componentwise operations. We also assume that $S(E)$ contains $\phi(E)$, the space of all sequences of vectors from E with only a finite number of entries not equal to the zero vector. The zero vectors in E and $S(E)$ will be denoted by θ and (θ) respectively. We let $\omega(E)$ denote the v.s.s. of all sequences of vectors from E. Thus $\phi(E) \subset S(E) \subset \omega(E)$ always holds.

The spaces λ of scalar sequences considered by Köthe ([10], pp. 408-427) are examples of v.s.s. over $E = K$. In the case of scalar sequence spaces (s.s.s.), we employ the customary notation and usually do not carry the K. In particular:

$\phi (= \phi(K))$ denotes the space of all scalar sequences with only a finite number of non zero terms;

$\omega (= \omega(K))$ denotes the space of all scalar sequences;

c denotes the space of all convergent scalar sequences;

c_o denotes the space of all scalar sequences converging to zero;

ℓ_p, $1 \leq p < \infty$, denotes the space of all scalar sequences (a_n) such that $\Sigma |a_n|^p < \infty$; and,

ℓ_∞ denotes the space of all bounded scalar sequences.

A subset X of $\omega(E)$ is called <u>solid</u> if $(x_n) \in X$ implies that $(c_n x_n) \in X$ for all scalar sequences (c_n) such that $|c_n| \leq 1$ for all n. $|X| = \{(c_n x_n) : (x_n) \in X, c_n \in K, |c_n| \leq 1\}$ is the smallest solid set containing X and is called the <u>solid hull</u> of X. The solid hull of a singleton, $\{(x_n)\}$, is denoted by $|(x_n)|$. An absolutely convex subset of $\omega(E)$ need not be solid, and a solid subset of $\omega(E)$ need not be absolutely convex, even if $E = K$ (Cf. [10], p. 411). The solid hull of an absolutely convex set is not necessarily absolutely convex. However, the absolutely convex hull of a solid set is solid.

Most known results on s.s.s. require the s.s.s. considered to be solid. The s.s.s. ϕ, ω, ℓ_p, and c_o are fortunately solid but one of the most interesting s.s.s., namely c, is not. It is to deal with such v.s.s. that we introduce the term "solid in $S(E)$".

A subset X of $S(E)$ is called <u>solid in $S(E)$</u> if $|X| \cap S(E) = X$. $|X| \cap S(E)$ is called the <u>solid hull of X in $S(E)$</u>.

When we wish to state or quote results in the general theory

of locally convex spaces, we let (L,T) denote an l.c.t.v.s.; i.e., we let (L,T) denote a vector space L over K with a locally convex Hausdorff topology T compatible with the linear structure of L. In the particular case that $L = S(E)$ is a v.s.s., we call $(S(E),T)$ an l.c.t.v.s.s. (locally convex topological vector sequence space).

If T is determined by a norm, $||\ ||$, we often write $T = ||\ ||$ for convenience. Thus $(\ell_p, ||\ ||_p)$, $1 \leq p < \infty$, and $(\ell_\infty, ||\ ||_\infty)$ are examples of l.c.t.v.s.s. over $E = K$, where $||(a_n)||_p = (\Sigma\ |a_n|^p)^{1/p}$, $1 \leq p < \infty$, and $||(a_n)||_\infty = \sup_n |a_n|$. We let B_p denote the unit ball in $(\ell_p, ||\ ||_p)$, $1 \leq p \leq \infty$.

The topology T on $S(E)$ is called solid if $(S(E),T)$ has a base of neighborhoods of (θ) that are solid in $S(E)$.

We generalize the terminology of Garling ([5], p. 86) and call a seminorm p on $S(E)$ absolutely monotone if $p[(z_n)] \leq p[(x_n)]$ whenever (z_n) and (x_n) are in $S(E)$ and $|(z_n)| \subset |(x_n)|$. The topology T on $S(E)$ is then solid if and only if T can be determined by a set of absolutely monotone seminorms on E.

We note that the norms, $||\ ||_p$, $1 \leq p \leq \infty$ are absolutely monotone and determine solid topologies on the s.s.s. ℓ_p, $1 \leq p \leq \infty$, respectively. Note also that $||\ ||_\infty$ determines a solid topology on the non-solid s.s.s. c.

2. Projection Maps

The maps regarded as projections here will not in general map a vector space into itself, but for convenience, will be assumed to map onto another vector space. We will be interested mainly in the maps:

$P_j : S(E) \to E$ defined by $P_j[(x_n)] = x_j$; and,

$I_j : E \to S(E)$ defined by $I_j(x) = (\theta,\theta,\cdots,\theta,\underset{j}{x},\theta,\cdots)$.

However, we shall first consider the following more general situation.

Let L and M be vector spaces and let $P : L \to M$ and $I : M \to L$ be linear maps such that PI is the identity map on M. Then I is injective, P is onto, and IP is a projection of L onto $I(M)$ in the usual sense. If L is a Hausdorff topological vector space with topology T, we give M the relative topology $T_I = \{I^{-1}(U) : U \in T\}$. Equivalently, if T is determined by the seminorms p_α, $\alpha \in D$, we let T_I be the topology determined by the seminorms $p_\alpha \circ I$, $\alpha \in \mathbf{D}$.

Note that I is T_I, T continuous and P is T, T_I open ([15], p. 19).

Also, if P is T, T_I continuous then I is T_I, T closed. For if P is continuous, $I(M)$ has a topological supplement ([15], p. 20). Thus $I(M)$ is closed since it is the kernel of a continuous projection. Consequently, I is a closed map.

Note also that P is T, T_I continuous if and only if IP is T, T continuous.

In the particular case when $(L,T) = (S(E),T)$ is an l.c.t.v.s.s., $M = E$, and $I = I_j$, $P = P_j$, then PI is the identity map on M. We let $T_{I_j} = T_j$ be the relative topology on E.

<u>Proposition 2.1</u> Let $(S(E),T)$ be an l.c.t.v.s.s. Then:

(1) I_j is T_j, T continuous and P_j is T, T_j open.

(2) If P_j is T, T_j continuous then I_j is T_j, T closed.

(3) If T is solid, then P_j is T, T_j continuous.

Proof. (1) and (2) are particular cases of the results just discussed. (3) holds since $T_j = \{I_j^{-1}(U) : U \in T\}$, and $P_j^{-1}[I_j^{-1}(U)] \supset U$ if U is solid.

Let $E^{(j)} = \overbrace{E \times E \times \cdots \times E}^{j}$ be the vector space formed by taking the Cartesian product of the vector space E j times. We shall be using the maps: $P^{(j)} : S(E) \to E^{(j)}$ defined by $P^{(j)}[(x_n)] = (x_1, x_2, \cdots, x_j)$; and, $I^{(j)} : E^{(j)} \to S(E)$ defined by $I^{(j)}[(x_1, x_2, \cdots, x_j)] = (x_1, x_2, \cdots, x_j, \theta, \theta, \cdots)$. If we take $(L,T) = (S(E),T)$, $M = E$, and $I = I^{(j)}$, $P = P^{(j)}$, then PI is the identity map on M , and we have the following analogue of Proposition 2.1 for the maps $P^{(j)}$, $I^{(j)}$. Let $T^{(j)} = T_{I^{(j)}}$ denote the relative topology on $E^{(j)}$.

Proposition 2.2 Let $(S(E),T)$ be an l.c.t.v.s.s. Then for each fixed i , the following are equivalent:

(1) P_j is T, T_j continuous for $j = 1,2,\cdots,i$.

(2) $P^{(j)}$ is $T, T^{(j)}$ continuous for $j = 1,2,\cdots,i$.

(3) $(E^{(j)}, T^{(j)})$ is the product of (E, T_j) , $j = 1,2,\cdots,i$.

Proof. Recalling that the general projection mapping P is T, T_I continuous if and only if IP is T, T continuous, the equivalence of (1) and (2) follows from the relations $I_1P_1 = I^{(1)}P^{(1)}$, $I_jP_j = I^{(j)}P^{(j)} - I^{(j-1)}P^{(j-1)}$ for $j = 2,3,\cdots,i$, and $I^{(j)}P^{(j)} = I_1P_1 + I_2P_2 + \cdots + I_jP_j$ for $j = 1,2,\cdots,i$.

The equivalence of (1) and (3) follows from the fact that $I^{(i)}E^{(i)}$ is the topological direct sum of I_jE_j, $j = 1,2,\cdots,i$, if and only if I_jP_j is continuous for each j ([15], p. 20).

3. Some Properties of Solid Sets and Solid Topologies

In this section, we collect a few simple properties of the l.c.t.v.s.s. $(S(E),T)$ with T solid.

For each $i = 1,2,\cdots,$ and each (x_n) in $\omega(E)$, we call $(x_n)(\leq i) = (x_1,x_2,\cdots,x_i,\theta,\theta,\cdots)$ the <u>i-th slice</u> of (x_n). We let $(x_n)(> i) = (x_n) - (x_n)(\leq i)$. Note that for each $i = 1,2,\cdots,$ and each (x_n) in $\omega(E)$, $(x_n)(\leq i)$ is in every v.s.s. $S(E)$, while $(x_n)(> i)$ is in a v.s.s. $S(E)$ if and only if (x_n) is in $S(E)$.

If $(S(E),T)$ is an l.c.t.v.s.s., we let $(S(E),T)_a$ be the set of all (x_n) in $(S(E),T)$ for which $(x_n)(\leq i) \to (x_n)$ as $i \to \infty$.

In particular, $(\ell_\infty, ||\ ||_\infty)_a = c_o$.

<u>Proposition 3.1</u> If $(S(E),T)$ is an l.c.t.v.s.s. with T solid, then $(S(E),T)_a$ is the closure of $\phi(E)$ in $(S(E),T)$. In particular, $(S(E),T)_a$ is a closed subspace.

<u>Proof.</u> Clearly $(S(E),T)_a$ is contained in the closure of $\phi(E)$. Suppose then that (x_n) is in the closure of $\phi(E)$ and U is a solid neighborhood of (θ) in $(S(E),T)$. We wish to show that $(x_n)(> i) = (x_n) - (x_n)(\leq i)$ is in U for all i sufficiently large. Choose (z_n) in $\phi(E)$ such that $(x_n) - (z_n)$ is in U. Since U is solid, $(x_n)(> i) - (z_n)(> i)$ is in U for all i. Thus,

$(x_n)(> i)$ is in U for all i sufficiently large since (z_n) is in $\phi(E)$.

Proposition 3.2 If T is a solid topology and $|X| \subset S(E)$, then the closure of $|X|$ in $(S(E),T)$ is solid in $S(E)$.

Proof. Since $|X|$ is solid, $|X| + U$ is solid in $S(E)$ for every U in $\mathcal{U}$, where $\mathcal{U}$ is the set of all neighborhoods of (θ) that are solid in $S(E)$. Thus, $\bigcap_{U\in\mathcal{U}}(|X| + U)$ is solid in $S(E)$; that is, the closure of $|X|$ is solid in $S(E)$.

4. Compactness

In this section, we consider compact, countably compact, and sequentially compact subsets of an l.c.t.v.s.s. $(S(E),T)$.

Assumption Throughout this section, it is assumed that $(S(E),T)$ is an l.c.t.v.s.s. such that P_j is T, T_j continuous for each j .

We first list a few facts associated with the notion of compactness (see [10], pp. 9-20). A net $x_\alpha,\alpha \in D$, in a Hausdorff space X is said to have x as a cluster point if for each neighborhood U of x and β_o in D , there is some $\beta > \beta_o$ such that x_β is in U . Convergent nets have only one cluster point. If X is compact, and x is the only cluster point of the net $x_\alpha,\alpha \in D$, then x_α converges to x . If $P : X \to Y$ is continuous, where Y is a Hausdorff space, and if $x_\alpha,\alpha \in D$ has a cluster point x in X , then $P(x)$ is a cluster point of $P(x_\alpha),\alpha \in D$, in Y . X is compact (respectively, countably compact) if and only if every net $x_\alpha,\alpha \in D$, (respectively, every sequence x_i, $i = 1,2,\cdots,$) in X has a cluster point in X ; equivalently, if and only if every family (respectively, countable family),

$F_\alpha, \alpha \in D$, of closed subsets of X satisfying the finite intersection property has a non-empty intersection. In terms of filters, X is compact if and only if every ultrafilter in X converges. If $P : X \to Y$ is onto, and $\mathcal{F}$ is an ultrafilter, then $P(\mathcal{F}) = \{P(F) : F \in \mathcal{F}\}$ is an ultrafilter.

Given a filter $\mathcal{F}$ in X, choose an element $x^{(F)}$ in F for each F in $\mathcal{F}$. Then $x^{(F)}$, $F \in \mathcal{F}$, is a net if $\mathcal{F}$ is directed by inclusion. If $\mathcal{F}$ converges to x, then $x^{(F)}$, $F \in \mathcal{F}$, converges to x; the converse holds when $\mathcal{F}$ is an ultrafilter.

X is called sequentially compact if every sequence in X has a subsequence converging to an element of X. If X is a subspace of a Hausdorff space Y, we shall call X relatively sequentially compact if every sequence in X has a subsequence converging to an element of Y. X will be called relatively compact in Y if its closure $\overline{X}$ in Y is compact.

If X is a compact (respectively, relatively compact) subset of $(S(E),T)$, then $P_j(X)$ is a compact (respectively, relatively compact) subset of (E,T_j) for each j. The converse is of course not true; some additional assumption extending over all of the $P_j(X)$, $j = 1,2,\cdots$, is needed to ensure the compactness (respectively, relative compactness) of X. The assumption we need is stated in terms of the following definition.

A net $(x_n^{(\alpha)})$, $\alpha \in D$, in an l.c.t.v.s.s. $(S(E),T)$ is called <u>TK-convergent</u> (<u>respectively, TK-Cauchy</u>) if for each j, $x_j^{(\alpha)}$, $\alpha \in D$, is convergent (respectively, Cauchy) in (E,T_j). Note that if P_j is T, T_j continuous for each j, then convergent (respectively, Cauchy) nets in $(S(E),T)$ are TK-convergent (respectively, TK-Cauchy).

Theorem 4.1 A subset X of $(S(E),T)$ is relatively compact if and only if:

(1) For each j, $P_j(X)$ is a relatively compact subset of (E,T_j); and,

(2) Every TK-convergent net in X converges in $(S(E),T)$.

Proof.

Necessity. Since P_j is T, T_j continuous for each j, $P_j(X)$ is relatively compact in (E,T_j) if X is relatively compact in $(S(E),T)$.

Let $(x_n^{(\alpha)})$, $\alpha \in D$, be a TK-convergent net in X. Then for each j, there is an x_j such that $x_j^{(\alpha)}$, $\alpha \in D$, converges to x_j in (E,T_j). Since $\overline{X}$ is compact, $(x_n^{(\alpha)})$, $\alpha \in D$, has a cluster point (z_n) in $(S(E),T)$. Since P_j is T, T_j continuous, z_j is a cluster point of $x_j^{(\alpha)}$, $\alpha \in D$. Because $x_j^{(\alpha)}$, $\alpha \in D$, converges to x_j, we must have $z_j = x_j$. Thus, (x_n) is the only cluster point of $(x_n^{(\alpha)})$, $\alpha \in D$. Since $\overline{X}$ is compact, it follows that $(x_n^{(\alpha)})$, $\alpha \in D$, converges to (x_n) in $\overline{X}$.

Sufficiency. Let $\mathcal{F}$ be an ultrafilter in $\overline{X}$. Then $P_j(\mathcal{F}) = \{P_j(F) : F \in \mathcal{F}\}$ is an ultrafilter in $\overline{P_j(X)}$. For each F in $\mathcal{F}$, choose some member $(x_n^{(F)})$ of F. Since $\overline{P_j(X)}$ is compact, $P_j(\mathcal{F})$ converges to some x_j in (E,T_j). Thus, the net $x_j^{(F)}$, $F \in \mathcal{F}$, converges to x_j in (E,T_j); i.e., $(x_j^{(F)})$, $F \in \mathcal{F}$, is TK-convergent. By (2), it is therefore convergent in $(S(E),T)$. Since $\mathcal{F}$ is an ultrafilter, $\mathcal{F}$ is also convergent and the compactness follows.

The following corollary is an immediate consequence of Theorem 4.1.

Corollary 4.2 A subset X of $(S(E),T)$ is compact if and only if:

(1) For each j, $P_j(X)$ is a compact subset of (E,T_j); and,

(2) Every TK-convergent net in X converges in $(S(E),T)$ to a member of X.

Theorem 4.3 A subset X of $(S(E),T)$ is sequentially compact if and only if:

(1) For each j, $P_j(X)$ is a (relatively) sequentially compact subset of (E,T_j); and,

(2) Every TK-convergent sequence in X converges in $(S(E),T)$ to a member of X.

Proof. The necessity is proved as in Theorem 4.1 except that we use sequences in place of nets, and limits of subsequences in place of cluster points.

To prove the sufficiency, suppose that $(x_n^{(i)})$, $i = 1,2,\cdots$, is a sequence in X. Since for each j, P_jX is a relatively sequentially compact subset of (E,T_j), by using a diagonal process, we can obtain a TK-convergent subsequence of $(x_n^{(i)})$, $i = 1,2,\cdots$. By (2), this subsequence is convergent to a member of X. Thus X is sequentially compact.

Proposition 4.4 The following conditions are necessary for a subset X of $(S(E),T)$ to be countably compact:

(1) For each j, $P_j(X)$ is a countably compact subset of (E,T_j); and,

(2) Every TK-convergent sequence in X converges in $(S(E),T)$ to a member of X.

The proof is similar to the proof of the necessity in Theorem 4.1 and is omitted here.

It is always true that sequentially compact sets are countably compact and compact sets are countably compact. Moreover, in several familiar spaces, some of these types of compactness are the same. In particular, by the theorem of Eberlein ([10], p. 316), in a complete l.c.t.v.s., countably compact sets are compact. The following theorem shows that if two types of compactness are the same in (E,T_j) for each j, then the two types of compactness are the same in $(S(E),T)$.

Theorem 4.5 Let "A", "B" stand for any two of the three expressions "compact", "countably compact", "sequentially compact". Then the A subsets of $(S(E),T)$ are B subsets if and only if for each j, the A subsets of (E,T_j) are B subsets.

Proof. The statements relating countable compactness and sequential compactness follow from Theorem 4.3, Proposition 4.4 and the fact that sequentially compact sets are always countably compact. Moreover, by Corollary 4.2 and Theorem 4.3, it is clear that compact subsets of $(S(E),T)$ are sequentially compact if for each j compact subsets of (E,T_j) are sequentially compact.

We now deal with the two remaining possibilities. Suppose that X is a sequentially compact (respectively, countably compact) subset of $(S(E),T)$, and that for each j, the sequentially compact (respectively, countably compact) subsets of (E,T_j) are compact. Since P_j is T, T_j continuous for each j, $T^{(j)}$ is the product topology on $E^{(j)}$, and thus the sequentially compact (respectively,

countably compact) subsets of $(E^{(j)},T^{(j)})$ are compact. To show that X is compact, we give X the induced topology and consider a family F_α, $\alpha \in D$, of closed subsets of X that satisfies the finite intersection property. Since $P^{(j)}$ is $T, T^{(j)}$ continuous, $P^{(j)}(X)$ and the sets $P^{(j)}(F_\alpha)$, $\alpha \in D$, are sequentially compact (respectively countably compact) and thus compact in $(E^{(j)}, T^{(j)})$. Since the $P^{(j)}(F_\alpha)$, $\alpha \in D$, are closed subsets of the compact set $P^{(j)}(X)$ and satisfy the finite intersection property, there is an element $(x_n^{(j)})$ of X such that $P^{(j)}[(x_n^{(j)})]$ is in $\bigcap_{\alpha\in D} P^{(j)}(F_\alpha)$. Since X is sequentially compact (respectively, countably compact), $(x_n^{(j)})$, $j = 1,2,\cdots$, has a cluster point (x_n) in X. We shall show that (x_n) is in $\bigcap_\alpha F_\alpha$, and thus obtain the compactness of X. Since $P^{(j)}[(x_n^{(j)})] \in P^{(j)}(F_\alpha)$ for each j, we have $P^{(i)}[(x_n^{(j)})] \in P^{(i)}(F_\alpha)$ whenever $j \geq i$. Thus, $P^{(i)}[(x_n)] \in P^{(i)}(F_\alpha)$ since $P^{(i)}$ is continuous and $P^{(i)}(F_\alpha)$ is closed. Therefore, for each i, there is a $(z_n^{(i)})$ in F_α such that $P^{(i)}[(z_n^{(i)})] = P^{(i)}[(x_n)]$; that is, there is a $(z_n^{(i)})$ such that $x_n = z_n^{(i)}$ for $n = 1,2,\cdots,i$. Since F_α is sequentially compact (respectively, countably compact), the TK-convergent sequence $(z_n^{(i)})$, $i = 1,2,\cdots$, must converge in $(S(E),T)$ to an element of F_α by Theorem 4.3 (respectively, Proposition 4.4). Thus, (x_n) is in F_α. Since this holds for each $\alpha \in D$, we have $(x_n) \in \bigcap_{\alpha\in D} F_\alpha$ as required.

From Theorems 4.3 and 4.5, we obtain the following proposition.

Proposition 4.6 Let (E,T_j) be metrisable for each j. Then the following conditions on a subset X of $(S(E),T)$ are equivalent:

(1) X is compact.

(2) X is countably compact.

(3) X is sequentially compact.

(4) For each j, $P_j(X)$ is a compact subset of (E,T_j); and, every TK-convergent sequence in X converges in $(S(E),T)$ to a member of X.

Remark. If for each j the closed bounded subsets of (E,T_j) are compact, then condition (4) in the proposition above can be replaced by:

(4') X is bounded and every TK-convergent sequence in X converges in $(S(E),T)$ to a member of X.

It is clear that (4) implies (4'). Moreover, since $P_j(X)$ is relatively sequentially compact in (E,T_j) whenever X is bounded in $(S(E),T)$, (4') implies (3) by Theorem 4.3. Thus, the asserted equivalence holds.

In the following theorem, we characterize those relatively compact subsets of $(S(E),T)$ that are solid in $S(E)$. This characterization turns out in practice to be more convenient to apply than the one given in Theorem 4.1.

Let X be a subset of an l.c.t.v.s.s. $(S(E),T)$ and suppose that (x_n) is an element of $\omega(E)$ such that $(x_n)(\leq i)$, $i = 1,2,\cdots$, is a Cauchy sequence in X. If we can always conclude for such sequences that (x_n) is in X and $(x_n)(\leq i) \to (x_n)$ as $i \to \infty$, then we say that X is an AK-complete subset of $(S(E),T)$. X is called relatively AK-complete if its closure $\overline{X}$ in $(S(E),T)$ is AK-complete.

It is clear that if X is a complete subset of $(S(E),T)$, then X is AK-complete. Also, if X is a relatively complete subset of $(S(E),T)$ (i.e., if $\overline{X}$ is complete), then X is relatively AK-complete.

Theorem 4.7 Let X be a subset of $(S(E),T)$ that is solid in $S(E)$. Then X is relatively compact if and only if the following conditions hold:

(1) For each j, $P_j(X)$ is relatively compact in (E,T_j);

(2) $(x_n)(> i)$, $i = 1,2,\cdots$, converges to (θ) in $(S(E),T)$ uniformly for all (x_n) in X; and,

(3) X is relatively AK-complete in $(S(E),T)$.

Proof.

Necessity. (1) follows from the continuity of P_j for each j.

If X is relatively compact, then it is relatively complete and thus relatively AK-complete. Therefore (3) holds.

Suppose (2) did not hold. Then for some neighborhood U of (θ) in $(S(E),T)$, there would be a sequence $(x_n^{(i)})$, $i = 1,2,\cdots$, in X and integers $j_1 < j_2 < \cdots < j_i \cdots$, such that $(x_n^{(i)})(> j_i) \not\in U$ for each $i = 1,2,\cdots$. Now the sequence $(x_n^{(i)})(> j_i)$, $i = 1,2,\cdots$, is TK-convergent to (θ) in $(S(E),T)$. Therefore, by Theorem 4.1, it converges to (θ) in $(S(E),T)$. Thus, we obtain a contradiction.

Sufficiency. We use Theorem 4.1 to show that X is relatively compact. Let $(x_n^{(\alpha)})$, $\alpha \in D$, be a net in X such that for each j, $x_j^{(\alpha)} \to x_j$ in (E,T_j) and let U be an absolutely convex neighborhood of (θ) in $(S(E),T)$. By the TK-convergence, we have for each fixed i that

$$(x_n^{(\alpha)})(\leq i) - (x_n)(\leq i) = \sum_{n=1}^{i} I_n(x_n^{(\alpha)} - x_n)$$

$$\in U/i + U/i + \cdots + U/i = U$$

for α sufficiently large. Thus, for each i, $(x_n^{(\alpha)})(\leq i) \to (x_n)(\leq i)$ in $(S(E),T)$. In particular, $(x_n)(\leq i)$ is in $\overline{X}$. Moreover, using this convergence and (2), we can choose α such that

$$(x_n)(\leq i) - (x_n)(\leq j) = [(x_n)(\leq i) - (x_n^{(\alpha)})(\leq i)] + [(x_n^{(\alpha)})(\leq i) - (x_n^{(\alpha)})]$$

$$+[(x_n^{(\alpha)}) - (x_n^{(\alpha)})(\leq j)] + [(x_n^{(\alpha)})(\leq j) - (x_n)(\leq j)]$$

$$\varepsilon\ U/4 + U/4 + U/4 + U/4 = U$$

for each i, j sufficiently large. That is, $(x_n)(\leq i)$, $i = 1,2,\cdots$, is a Cauchy sequence in $\overline{X}$. By (3), we thus have $(x_n)(\leq i) \to (x_n)$ in $(S(E),T)$ as $i \to \infty$. From the above convergences and (2), we can choose i such that

$$(x_n^{(\alpha)}) - (x_n) = [(x_n^{(\alpha)}) - (x_n^{(\alpha)})(\leq i)] + [(x_n^{(\alpha)})(\leq i) - (x_n)(\leq i)]$$

$$+[(x_n)(\leq i) - (x_n)]$$

$$\varepsilon\ U/3 + U/3 + U/3 = U$$

for α sufficiently large. Thus $(x_n^{(\alpha)}) \to (x_n)$ in $(S(E),T)$ and the theorem is proved.

Remarks. 1. From (2) of Theorem 4.7, we see that if X is a relatively compact subset of $(S(E),T)$ that is solid in $S(E)$, then $X \subset (S(E),T)_a$. In particular, $(S(E),T)_a$ contains all those (x_n) in $S(E)$ whose solid hulls in $S(E)$ are relatively compact. Thus, the solid hull in $S(E)$ of a relatively compact subset of $(S(E),T)$

need not in general be relatively compact.

2. If $(S(E),T)$ is AK-complete (in particular, if it is sequentially complete), then every subset X of $(S(E),T)$ is relatively AK-complete and condition (3) in Theorem 4.7 is always satisfied.

3. If T is solid and $X = |Z| \cap S(E)$ for some Z contained in $S(E)$, then condition (2) of Theorem 4.7 need only be satisfied for (x_n) in Z.

Proposition 4.8 Let $(S(E),T)$ be an l.c.t.v.s.s. with T solid and let $|(x_n)| \subset S(E)$. Then $|(x_n)|$ is compact if and only if (x_n) is in $(S(E),T)_a$.

Proof. The necessity follows from condition (2) of Theorem 4.7.

Since P_j is T, T_j continuous for each j, any net in $|(x_n)|$ that converges in $(S(E),T)$ is TK-convergent, and thus has its limit in $|(x_n)|$. Therefore, $|(x_n)|$ is closed. By the same type of reasoning, $|(x_n)|$ is (relatively) AK-complete. Thus, condition (3) of Theorem 4.7 holds. Condition (1) certainly holds. Condition (2) holds by Remark 3, since T is solid and $(x_n)(\leq i) \to (x_n)$ in $(S(E),T)$ as $i \to \infty$.

CHAPTER 2

VECTOR SEQUENCE SPACES OF KÖTHE TYPE

In this chapter, we study v.s.s. from a dual pair viewpoint. If $\langle E,E'\rangle$ is a dual pair of vector spaces, we consider a dual pair $\langle S(E),S'(E')\rangle$ of v.s.s., where the bilinear form on $\langle S(E),S'(E')\rangle$ is induced by the bilinear form on $\langle E,E'\rangle$ by summation. $S(E)$ is then assigned a polar topology T of convergence on some class of subsets of $S'(E')$. Such an l.c.t.v.s.s. $(S(E),T)$ is said to be of Köthe type. The viewpoint we adopt in this chapter does not cover the l.c.t.v.s.s. considered in Chapter 1. The reason for this is that we require the second member of the dual pair to be a v.s.s. A characterization of a certain l.c.t.v.s.s. $(S(E),T)$ of Köthe type is given in Section 9.

1. Preliminaries

The results stated in this section are basic to the theory of l.c.t.v.s. and can all be found in Köthe [10].

We assume throughout this chapter that $\langle E,E'\rangle$ is a dual pair of vector spaces over the field K of real or complex scalars. We also assume that E and E' distinguish one another by the bilinear form $\langle\ ,\ \rangle$. Using this bilinear form, each member of E' can be regarded as a linear functional on E, and vice-versa. If $\mathcal{A}$ is the class of all finite subsets of E', by the weak topology $T_s(E',E)$ on E we mean that topology on E having as neighborhood basis at the origin the class of all sets of the form $A^\circ = \{x \in E : |\langle x,y\rangle| \leq 1 \text{ for all } y \in A\}$, $A \in \mathcal{A}$. $T_s(E',E)$ is thus the coarsest topology on E for which each member of E' is continuous. Similarly, we can define the weak topology $T_s(E,E')$ on E'.

More generally, we let $\mathscr{A}$ be a class of weakly bounded subsets of E' such that the linear hull of $\bigcup_{A \in \mathscr{A}} A$ in E' is equal to E'. By the polar topology $T(E',E)$ on E of $\mathscr{A}$-convergence, we mean that topology on E having as neighborhood subbasis at the origin the class of all sets of the form $A^{\circ} = \{x \in E : |\langle x,y\rangle| \leq 1 \text{ for all } y \in A\}$, where each A is in $\mathscr{A}$. In particular, if $\mathscr{A}$ is the set of all:

1. finite subsets of E' (respectively, of E), then we get the weak topology $T_s(E',E)$ on E (respectively, $T_s(E,E')$ on E')
2. absolutely convex weakly compact subsets of E' (respectively, of E), then we get the Mackey topology $T_k(E',E)$ on E (respectively, $T_k(E,E')$ on E')
3. weakly bounded subsets of E' (respectively, of E), then we get the strong topology $T_b(E',E)$ on E (respectively, $T_b(E,E')$ on E')
4. strongly bounded subsets of E' (respectively, of E), then we get the strong dual topology $T_{b^*}(E',E)$ on E (respectively, $T_{b^*}(E,E')$ on E').

It turns out that $T_s(E',E) \subset T_k(E',E) \subset T_{b^*}(E',E) \subset T_b(E',E)$. $T_b(E',E)$ is the finest polar topology that can be defined on E using the dual pair $\langle E,E'\rangle$; $T_s(E',E)$ is the coarsest. If (E,T) is an l.c.t.v.s., we let $(E,T)'$ be the vector space of all continuous linear functionals on E. It turns out that $T_k(E',E)$ is the finest polar topology T on E with respect to the dual pair $\langle E,E'\rangle$ for which $(E,T)' = E'$.

If (E,T) is an l.c.t.v.s., then E is a member of the dual pair $\langle E,(E,T)'\rangle$. Moreover, T is the polar topology of $\mathscr{A}$-convergence where $\mathscr{A} = \{U^{\circ} : U \text{ is a neighborhood of } (\theta) \text{ in } (E,T)\}$. Thus, the viewpoint of dual pairs can be used on any l.c.t.v.s.

2. The Dual Pair $\langle S(E), S'(E') \rangle$

Recall that $\langle E, E' \rangle$ is a dual pair of vector spaces over the field K of real or complex scalars.

The β-dual X^* of a subset X of $\omega(E)$ is defined by $X^* = \{(y_n) \in \omega(E') : |\sum \langle x_n, y_n \rangle| < \infty \text{ for all } (x_n) \text{ in } X\}$. Y^* is defined similarly for a subset Y of $\omega(E')$. Note that $\omega(E)^* = \phi(E')$ and $\phi(E)^* = \omega(E')$. Note also that * is an inclusion reversing map and thus $X^* = X^{***}$ for all X contained in $\omega(E)$. Also, X^* is a v.s.s. over E'.

A subset X of $\omega(E)$ (respectively, Y of $\omega(E')$) is called β-perfect if $X = X^{**}$ (respectively, $Y = Y^{**}$). From the previous paragraph, X^* and X^{**} are always β-perfect and X^{**} is the smallest β-perfect subset of $\omega(E)$ containing X.

If $S(E)$, $S'(E')$ are v.s.s. over E, E' respectively, and if $S'(E') \subset S(E)^*$, then $\langle S(E), S'(E') \rangle$ is a dual pair of vector spaces over the field K with bilinear form $\langle (x_n), (y_n) \rangle = \sum \langle x_n, y_n \rangle$. $S(E)$ and $S'(E')$ distinguish one another since $S(E) \supset \phi(E)$ and $S'(E') \supset \phi(E')$. We shall assume throughout that $\langle S(E), S'(E') \rangle$ denotes a dual pair of v.s.s. with $S'(E') \subset S(E)^*$ and with the bilinear form just defined.

We note that if $T = T(S'(E'), S(E))$ is the polar topology on $S(E)$ of $\mathscr{A}$-convergence, then T is determined by the seminorms:

$$(x_n) \longrightarrow \sup_{(y_n) \in A} |\sum \langle x_n, y_n \rangle| , \quad A \in \mathscr{A}.$$

If X is solid in $S(E)$, then it is easy to show that $X^\circ = (X \cap \phi(E))^\circ$, where $^\circ$ is taken in $S'(E')$. Thus, X° is

solid in $S^{\beta}(E^{\beta})$ if X is solid in $S(E)$. Using this fact and a related one for subsets of $S^{\beta}(E^{\beta})$, we see that the polar topology $T = T(S^{\beta}(E^{\beta}), S(E))$ of $\mathscr{A}$-convergence is solid if and only if each A in $\mathscr{A}$ can be chosen to be solid in $S^{\beta}(E^{\beta})$. Because $S^{\beta}(E^{\beta})$ contains $\phi(E^{\beta})$, it follows that such a topology T is determined by the seminorms:

$$(x_n) \longrightarrow \sup_{(y_n)\in A} \sum |\langle x_n, y_n \rangle| , \quad A \in \mathscr{A} .$$

We have defined $S(E)^*$ to be the largest v.s.s. $S^{\beta}(E^{\beta})$ such that $\langle S(E), S^{\beta}(E^{\beta})\rangle$ is a dual pair with the bilinear form defined above. To ensure that the normal topology (defined below) can be put on $S(E)$, it is necessary to introduce another type of dual.

The <u>α-dual</u> $X^{\times}$ of a subset X of $\omega(E)$ is defined by

$$X^{\times} = \{(y_n) \in \omega(E^{\beta}) : \sum |\langle x_n, y_n \rangle| < \infty \text{ for all } (x_n) \text{ in } X\} .$$

$Y^{\times}$ is defined similarly for a subset Y of $\omega(E^{\beta})$. The map $^{\times}$, like the map *, is inclusion reversing and $X^{\times} = X^{\times\times\times}$ for all X contained in $\omega(E)$. We call X <u>α-perfect</u> if $X = X^{\times\times}$. Similarly, we call Y α-perfect if $Y = Y^{\times\times}$, where Y is a subset of $\omega(E^{\beta})$. Note that $X^{\times}$ is a solid v.s.s. over E^{β}, and that $X^{\times} = X^*$ if X is solid.

Given the dual pair $\langle S(E), S^{\beta}(E^{\beta})\rangle$ of v.s.s., take $\mathscr{A}$ to be the set of all solid hulls in $S^{\beta}(E^{\beta})$ of elements of $S^{\beta}(E^{\beta})$. $\mathscr{A}$ determines a polar topology if and only if each member of $\mathscr{A}$ is weakly bounded or, equivalently, if and only if $S^{\beta}(E^{\beta}) \subset S(E)^{\times}$.

When it is defined, we call this topology the normal topology and denote it by $T_o(S'(E'), S(E))$. We note that whenever $S(E)$ or $S'(E')$ is solid, then $S'(E') \subset S(E)^\times$ and $T_o(S'(E'), S(E))$ is defined. We also note that if $T(S'(E'), S(E))$ is solid, then $T(S'(E'), S(E)) \supset T_o(S'(E'), S(E))$ and $S'(E') \subset S(E)^\times$.

The dual pair $\langle\lambda,\lambda^\times\rangle$ of s.s.s. was originally studied by Köthe and Toeplitz [11], and later, using the theory of l.c.t.v.s., by Köthe [9]. The dual pair $\langle\lambda,\lambda^*\rangle$ has been considered by Chillingworth [3], Matthews [12], and Komura and Komura [8]. We list below the α-duals and β-duals of some s.s.s. Recall that $\lambda^* = \lambda^\times$ if λ is solid.

$$\ell_1^* = \ell_\infty \ , \quad \ell_\infty^* = \ell_1$$

$$\ell_p^* = \ell_q \ , \quad \text{where} \quad 1 < p,q < \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$c_o^* = c^* = c^\times = \ell_1$$

$$\alpha^\times = \ell_1 \ , \quad \text{where } \alpha \text{ is the s.s.s. spanned by } \phi \text{ and the alternating sequences } (c,-c,\ c,-c,\cdots) \text{ where } c \in K \ .$$

$$\alpha^* = \{(a_n) \in \omega : \sum(-1)^n a_n \text{ converges}\}$$

Since the results that we will be obtaining in the succeeding chapters on the v.s.s. $S(E)$ with polar topology $T(S'(E'), S(E))$ generalize the scalar theory developed mostly by Köthe, we make the following definition.

An l.c.t.v.s.s. $(S(E),T)$ is said to be of Köthe type if:

1. There is a vector space E' and a v.s.s. $S'(E')$ such that $\langle E,E'\rangle$ and $\langle S(E), S'(E')\rangle$ are dual pairs with the bilinear form on the latter induced

by that on the former by summation; and,

2. There is a class $\mathscr{A}$ of subsets of $S'(E')$ such that T is the polar topology on $S(E)$ of $\mathscr{A}$-convergence.

3. Projection Maps

In this chapter, we have been considering a v.s.s. $S(E)$ which is a member of a dual pair $\langle S(E), S'(E')\rangle$ of v.s.s. In addition to the projection and injection maps P_j and I_j between $S(E)$ and E (defined in Section 2 of Chapter 1), we have corresponding maps P_j' and I_j' between $S'(E')$ and E'. It is easy to verify that:

$$P_j^* = I_j' \quad \text{and} \quad I_j^* = P_j'$$

where $*$ denotes the adjoint map. Since the adjoints are defined, P_j and I_j are weakly continuous ([15], p. 128). Similarly P_j' and I_j' are weakly continuous. As in Section 2 of Chapter 1, we first obtain our results for projection maps in a more general situation.

Suppose now that L and M are members of the dual pairs $\langle L,L'\rangle$ and $\langle M,M'\rangle$ respectively, that PI is the identity map on M and that P and I are weakly continuous with adjoints P^* and I^*. If we let $I' = P^*$ and $P' = I^*$, then $P'I' = (PI)^*$ is the identity map on M'. Thus P' and I' act on L' and M' as P and I act on L and M.

Lemma 3.1 Let P and I be weakly continuous and let T be the polar topology on L of $\mathscr{A}$-convergence. Then:

(1) T_I is the polar topology on M of $P'(\mathscr{A})$-convergence, where $P'(\mathscr{A}) = \{P'(A) : A \in \mathscr{A}\}$.

(2) P is T, T_I continuous if and only if the class $\mathscr{A}$

determining T can be chosen so that $I'P'(A) \in \mathscr{A}$ whenever $A \in \mathscr{A}$.

Proof. Since P' is weakly continuous and onto, $P'(\mathscr{A})$ is a class of weakly bounded sets with linear hull equal to M'. Also, for each A in $\mathscr{A}$, $I^{-1}(A^\circ) = [I^*(A)]^\circ = [P'(A)]^\circ$. The result (1) follows from this equation if we recall that the $I^{-1}(A^\circ)$ generate T_I and $P'(\mathscr{A}) = \{P'(A) : A \in \mathscr{A}\}$.

For each A in $\mathscr{A}$, we have $P^{-1}[P'(A)^\circ] = [P^*P'(A)]^\circ = [I'P'(A)]^\circ$. Suppose that T is continuous. Then $[I'P'(A)]^\circ$ is in T since $P'(A)^\circ$ is in T_I by (1). Thus, $I'P'(A)$ can be assumed to be in $\mathscr{A}$. Conversely, if $I'P'(A)$ is in $\mathscr{A}$, then $P^{-1}[P'(A)^\circ]$ is open and P is continuous since T_I is the topology of $P'(\mathscr{A})$-convergence.

Remark 3.2 If $\mathscr{A}$ is the class of all subsets of L' that are bounded or compact or absolutely convex and compact or precompact with respect to any one of the weak, Mackey, or strong topologies, then $P'(\mathscr{A})$ is the class of all subsets of M' of the same type. Thus, in particular, if T is the weak, Mackey, or strong topology respectively, then T_I is the weak, Mackey, or strong topology. These results follow from the weak (and thus strong and Mackey) continuity of the maps I' and P' ([15], p. 158) and from the fact that $P'I'$ is the identity on M'.

Moreover, each of the above classes satisfies the condition in (2) of Lemma 3.1. Thus, P is T, T_I continuous for the corresponding topologies.

If we take $\langle L,L'\rangle = \langle S(E), S'(E')\rangle$, $\langle M,M'\rangle = \langle E,E'\rangle$, $P = P_j$, $I = I_j$, then P_j and I_j are weakly continuous, P_jI_j

is the identity map on E and $P' = P_j'$, $I' = I_j'$.

From the previous lemma and remark, we get the following proposition and corollary.

Proposition 3.3 Let $\langle S(E), S'(E') \rangle$ be a dual pair of v.s.s. and let T be the polar topology on $S(E)$ of $\mathscr{A}$-convergence. Then:

(1) T_j is the polar topology on E of $P_j'(\mathscr{A})$-convergence

(2) Γ is T, T_j continuous if and only if the class $\mathscr{A}$ determining T can be chosen so that $I_j' P_j'(A) \in \mathscr{A}$ whenever $A \in \mathscr{A}$.

Corollary 3.4 Let $\langle S(E), S'(E') \rangle$ be a dual pair of v.s.s. Then for each j, P_j is continuous and open and I_j is continuous and closed if $S(E)$ and E both have the weak, the Mackey, the strong, or the strong dual topologies, or if $S(E)$ has the normal topology and E has the weak topology.

4. Convergence Properties

In this section, we shall develop properties of the dual pair $\langle S(E), S'(E') \rangle$ that are well-known for the scalar case where $E = E' = K$ and $S'(E') = S(E)^{\times}$. The key results are obtained from corresponding properties of the particular case $\langle \ell_1, \ell_\infty \rangle$.

Proposition 4.1 Let $\langle S(E), S'(E') \rangle$ be a dual pair of v.s.s. with $S'(E')$ solid. A sequence of elements in $S(E)$ is weakly convergent (respectively, weakly Cauchy) if and only if it is normally convergent (respectively, normally Cauchy).

Proof. Since $S'(E')$ is solid, the normal topology is defined. A normally convergent (respectively, normally Cauchy) sequence is weakly

convergent (respectively, weakly Cauchy) since the normal topology is finer than the weak topology.

To prove the reverse implication for convergent sequences, suppose that $(x_n^{(i)}) \to (x_n)$ weakly in $S(E)$ as $i \to \infty$. Since $S'(E')$ is solid, we have for all (c_n) in ℓ_∞ and (y_n) in $S'(E')$ that $(c_n y_n)$ is in $S'(E')$ and thus that

$$| \langle (x_n^{(i)}) - (x_n), (c_n y_n) \rangle | = | \sum_{n=1}^{\infty} \langle x_n^{(i)} - x_n, y_n \rangle \; c_n | \to 0$$

as $i \to \infty$. This implies that $(\langle x_n^{(i)} - x_n, y_n \rangle)$ converges to (0) weakly in ℓ_1 as $i \to \infty$, where the weak topology on ℓ_1 is defined by the dual pair $\langle \ell_1, \ell_\infty \rangle$. Since weak convergence implies normal convergence in ℓ_1 ([10], p. 198), the sequence also converges normally. Consequently,

$$\sum_{n=1}^{\infty} | \langle x_n^{(i)} - x_n, y_n \rangle | \to 0 \quad \text{as} \quad i \to \infty$$

for each (y_n) in $S'(E')$. That is, $(x_n^{(i)}) \to (x_n)$ normally as $i \to \infty$.

If there were a weakly Cauchy sequence $(x_n^{(i)})$, $i = 1,2,\cdots$, in $S(E)$ which was not normally Cauchy, then for some $\varepsilon > 0$ and (y_n) in $S'(E')$, there would be a sequence of pairs i_k, j_k where $k = 0,1,2,\cdots$, such that

$$\sum_{n=1}^{\infty} | \langle x_n^{(i_k)} - x_n^{(j_k)}, y_n \rangle | \geq \varepsilon$$

contradicting the fact that $(x_n^{(i_k)}) - (x_n^{(j_k)})$ converges to (θ)

weakly and normally as $k \to \infty$.

Remark. The proposition need not hold if $S'(E')$ is not solid, even in the case when $E = E' = K$. For if we take $S(E) = \alpha^{\times} = \ell_1$, $S'(E') = \alpha$, and $(x_n^{(i)}) = (0,0,\ldots,\overset{2i}{0},1,1,0,0,\ldots)$ in $\alpha^{\times}$, then $(x_n^{(i)})$ converges weakly to (0) , but not normally.

In the remainder of this section, we investigate "slice convergence"; i.e., we examine the convergence of the "slices" $(x_n)(\leq i)$, $i = 1,2,\ldots$, for (x_n) in $S(E)$.

Given the dual pair $\langle S(E), S'(E')\rangle$, it is clear that for each (x_n) in $S(E)$, $(x_n)(\leq i) \to (x_n)$ weakly as $i \to \infty$. Moreover, if the normal topology on $S(E)$ is defined, then $S'(E') \subset S(E)^{\times}$ and thus, $(x_n)(\leq i) \to (x_n)$ normally as $i \to \infty$ for each (x_n) in $S(E)$. We now show that if $S(E)$ is solid, the convergence also holds in the Mackey topology.

Lemma 4.2 Let $\langle S(E), S'(E')\rangle$ be a dual pair of v.s.s. with $S(E)$ solid. Then the map $(x_n)[\ ] : S'(E') \to \ell_1$ defined by $(x_n)[(y_n)] = (\langle x_n, y_n\rangle)$ is $T_s(S(E), S'(E'))$, $T_s(\ell_\infty, \ell_1)$ continuous.

Proof. Since $S(E)$ is solid, the image of $(x_n)[\ ]$ lies in ℓ_1 . Let (a_n) be in ℓ_∞ . Then $\{(a_n)\}^{\circ}$ is a typical subbasic weak neighborhood of (0) in ℓ_1 . For (y_n) in the weak neighborhood $\{(a_n x_n)\}^{\circ}$ of (θ') in $S'(E')$ we have

$$|\sum \langle x_n, y_n\rangle a_n| = |\sum \langle a_n x_n, y_n\rangle| \leq 1 .$$

Thus, $(x_n)[\ ]$ takes $\{(a_n x_n)\}^{\circ}$ into $\{(a_n)\}^{\circ}$, and continuity follows.

Proposition 4.3 Let $\langle S(E), S^{\circ}(E^{\circ})\rangle$ be a dual pair of v.s.s. with $S(E)$ solid. Then for each (x_n) in $S(E)$, $(x_n)(\leq i) \to (x_n)$ in the topology $T_k(S^{\circ}(E^{\circ}), S(E))$ as $i \to \infty$.

Proof. Let C be an absolutely convex weakly compact subset of $S^{\circ}(E^{\circ})$. By Lemma 4.2, $(x_n)[C]$ is an absolutely convex weakly compact subset of ℓ_1. Thus ([10], p. 284),

$$\sup_{(y_n)\in C} \sum_{n=i}^{\infty} |\langle x_n, y_n\rangle| \to 0 \quad \text{as} \quad i \to \infty .$$

In particular,

$$\sup_{(y_n)\in C} |\langle (x_n) - (x_n)(\leq i), (y_n)\rangle| \to 0 \quad \text{as} \quad i \to \infty ,$$

and the proposition follows.

Remark. Proposition 4.3 need not hold if $S(E)$ is not solid. Let $S(E) = \alpha$, $S^{\circ}(E^{\circ}) = \alpha^{\times}$, and $(x_n) = (1,-1,1,-1,\cdots)$. Because it is orthogonal to the alternating sequences, it turns out that $C = \{(c_1,c_1,c_2,c_2,\cdots) : \sum |c_n| \leq 1\}$ is an absolutely convex, weakly bounded and weakly complete (and therefore weakly compact) subset of $\alpha^{\times}$. However, $(x_n)(\leq i)$ does not converge to (x_n) uniformly on C as $i \to \infty$.

The following theorem asserts that "slice convergence" is a property characteristic of spaces of Köthe type.

Theorem 4.4 Let $S(E)$ be a solid v.s.s. and let $E' = E^*$ be the algebraic dual of E. Then there is a finest l.c.t.v.s.s. topology T_a on $S(E)$ for which $(S(E),T_a)_a = S(E)$. Moreover,

$T_a = T_k(S(E)^*, S(E))$.

Proof. If $S(E)$ is solid, then $(S(E),T_k)_a = S(E)$ by Proposition 4.3, where $T_k = T_k(S(E)^*, S(E))$.

Suppose $(S(E),T)_a = S(E)$. Let f be in $(S(E),T)'$. Since $(x_n)(\leq i) \to (x_n)$ in $(S(E),T)$ as $i \to \infty$, we have for each (x_n) in $S(E)$ that

$$f[(x_n)] = \lim_{i \to \infty} f[(x_n)(\leq i)] = \lim_{i \to \infty} \sum_{n=1}^{i} \langle x_n, y_n \rangle = \langle (x_n), (y_n) \rangle$$

where $y_n = f \circ I_n$ is in E^* . Thus, $(S(E),T)' \subset S(E)^*$. Therefore, $T \subset T_k$. Thus, T_a is defined and equals $T_k(S(E)^*, S(E))$.

5. Solid Topologies

In this section, we investigate the solid polar topologies on $S(E)$ induced by the dual pair $\langle S(E), S'(E') \rangle$.

In the proof of the following proposition, we use the Banach Boundedness Principle ([15], p.26) which asserts: "A subset X of an l.c.t.v.s. (L,T) is bounded if and only if for every null sequence c_n , $n = 1,2,\cdots$, in K and every sequence x_n , $n = 1,2,\cdots$, in X , the sequence $c_n x_n$, $n = 1,2,\cdots$, converges to the zero vector in L as $n \to \infty$."

Proposition 5.1 Let $\langle S(E), S'(E') \rangle$ be a dual pair of v.s.s. If $S(E)$ is solid, then the solid hull in $S'(E')$ of a weakly bounded subset of $S'(E')$ is weakly bounded. Thus, $T_b(S'(E'), S(E))$ is solid if $S(E)$ is solid.

Proof. Let Y be a weakly bounded subset of $S'(E')$. Let

$(a_n^{(i)} y_n^{(i)})$, $i = 1,2,\cdots$, be any sequence in $|Y| \cap S'(E')$, where $(y_n^{(i)})$ is in Y and $|a_n^{(i)}| \leq 1$ for all i , $n = 1,2,\cdots$. Let $c^{(i)}$, $i = 1,2,\cdots$, be a null sequence in K . Since Y is weakly bounded, $c^{(i)}(y_n^{(i)})$ converges weakly to (θ') as $i \to \infty$. By Proposition 4.1, it converges normally as well. Thus, $c^{(i)}(a_n^{(i)} y_n^{(i)})$ converges normally and weakly to (θ') . Since this convergence holds for all null sequences $c^{(i)}$ and sequences $(a_n^{(i)} y_n^{(i)})$ in $|Y| \cap S'(E')$, it follows that $|Y| \cap S'(E')$ is weakly bounded.

Corollary 5.2 If $S'(E')$ is solid, then the solid hull of a strongly bounded subset of $S'(E')$ is strongly bounded. Thus, $T_{b^*}(S'(E'), S(E))$ is solid if $S'(E')$ is solid.

Proof. Let Y be a strongly bounded subset of $S'(E')$ and let X be a weakly bounded subset of $S(E)$. Since $|X| \cap S(E)$ is weakly bounded, there is a positive constant M such that $| \sum \langle x_n, y_n \rangle | \leq M$ for all (x_n) in $|X| \cap S(E)$, (y_n) in Y . Because $S(E) \supset \phi(E)$ and $|X| \cap S(E)$ is solid in $S(E)$, this implies that $\sum | \langle x_n, y_n \rangle | \leq M$ for all (x_n) in X , (y_n) in Y . Since such an inequality holds for each weakly bounded X , $|Y|$ is strongly bounded.

Remark. Proposition 5.1 need not hold if $S(E)$ is not solid. For the sequence $(x_n^{(i)}) = (0,0,\cdots,0,\overset{2^i}{1},1,\cdots,1,\overset{2^{i+1}}{0},0,\cdots)$, $i = 1,2,\cdots$, is $T_s(\alpha,\alpha^\times)$ bounded in $\alpha^\times$ but its solid hull is not.

Given a polar topology $T(S'(E'), S(E))$ on $S(E)$ of $\mathscr{A}$-convergence, we note from Section 2 that there will be a solid polar topology on $S(E)$ with respect to the dual pair $\langle S(E), S'(E') \rangle$ that is finer than T if each

member of $|\mathcal{A}| \cap S'(E') = \{|A| \cap S'(E') : A \in \mathcal{A}\}$ is weakly bounded. Moreover, the topology $|T|$ of $|\mathcal{A}| \cap S'(E')$-convergence would then be the coarsest solid polar topology of the dual pair $\langle S(E), S'(E')\rangle$ finer than T. From Proposition 5.1, we see that $|T|$ is defined whenever $S(E)$ is solid. Also, $S'(E') \subset S(E)^{\times}$ whenever $|T|$ is defined.

Remark. In particular, given the dual pair $\langle S(E), S'(E')\rangle$ of v.s.s., we can form the polar topology $|T_k|(S'(E'), S(E))$ on $S(E)$. Since the series occuring in the proof of Proposition 4.3 is absolutely convergent, for each (x_n) in the solid v.s.s. $S(E)$, we can conclude that $(x_n)(\leq i) \to (x_n)$ in the topology $|T_k|(S'(E'), S(E))$ as $i \to \infty$. Using this fact, we will be able to show that $T_k(S(E)^*, S(E))$ is solid if $S(E)$ is solid. We first require a result on duals of spaces of Köthe type.

Theorem 5.3 Let $\langle S(E), S'(E')\rangle$ be a dual pair of v.s.s. with $S(E)$ solid and $S'(E') = S(E)^*$, and let T be a polar topology on $S(E)$. Then $(S(E),T)' = S(E)^*$ if and only if $(S(E),T)_a = S(E)$ and $(E,T_j)' = E'$ for each j.

Sufficiency. Since T is a polar topology, it is enough to show that $(S(E),T)' \subset S(E)^*$. If f is in $(S(E),T)'$, then since $(x_n)(\leq i) \to (x_n)$ in $(S(E),T)$ as $i \to \infty$, we have

$$f[(x_n)] = \lim_{i \to \infty} f[(x_n)(\leq i)] = \lim_{i \to \infty} \sum_{n=1}^{i} \langle x_n, y_n\rangle = \langle (x_n), (y_n)\rangle ,$$

where $y_n = f \circ I_n$ is in E' since $f \circ I_n$ is in $(E,T_j)'$. (y_n) is in $S(E)^*$ since the series converges for all (x_n) in $S(E)$.

Necessity. Since $T \subset T_k(S(E)^*, S(E))$, the weakly bounded subsets of $S(E)^*$ determining T can be assumed to be absolutely convex and weakly compact. Since P_j' is weakly continuous, it follows that the weakly bounded subsets of E' determining T_j can also be assumed to be absolutely convex and weakly compact. Thus, $T_j \subset T_k(E',E)$; equivalently, $(E,T_j)' = E'$. $(S(E),T)_a = S(E)$ by Proposition 4.3.

Proposition 5.4 $T_k(S(E)^*, S(E))$ is solid if $S(E)$ is solid.

Proof. By the remark above, $(S(E),|T_k|)_a = S(E)$. Clearly $T_k(E',E) = |T_k|_j$ for each j . Thus, $|T_k| \subset T_k$ by Theorem 5.3. Since $|T_k| \supset T_k$ always holds, we have equality and the proposition follows.

Corollary 5.5 If $S(E)$ is a solid v.s.s., then the topology T_a is solid.

Proof. Recall (Theorem 4.4) that $T_a = T_k(S(E)^*, S(E))$.

From the next proposition, we obtain the interesting fact that the strong topology $T_b(S'(E'), S(E))$ is independent of the v.s.s. $S'(E')$ if $S(E)$ is solid.

Proposition 5.6 Let $\langle S(E), S'(E') \rangle$ be a dual pair of v.s.s. with $S(E)$ solid. Then $T_b(S'(E'), S(E)) = T_b(S(E)^*, S(E))$.

Proof. Since $\phi(E') \subset S'(E') \subset S(E)^*$, it is clear that the inclusions $T_b(\phi(E'), S(E)) \subset T_b(S'(E'), S(E)) \subset T_b(S(E)^*, S(E))$ always hold. In proving the other inclusions, we assume that polars are always taken in $S(E)$. Suppose U is a neighborhood of (θ) in

$T_b(S(E)^*, S(E))$. By Proposition 5.1, $U \supset A^\circ$ for some $T_s(S(E), S(E)^*)$ bounded set A that is solid in $S(E)^*$. Since A is solid in $S(E)^*$, $A^\circ = [A \cap \phi(E')]^\circ$. Therefore, $U \supset [A \cap \phi(E')]^\circ$, a neighborhood of (θ) in $T_b(\phi(E'), S(E))$. Thus, $T_b(S(E)^*, S(E)) \subset T_b(\phi(E'), S(E))$ and the desired equality follows.

6. AK-Completeness and the Space $S_{\mathscr{A}}(E)$

In this section, we study the notion of AK-completeness in spaces of Köthe type. The results we obtain will be particularly useful in Section 1 of Chapter 3 where complete spaces of Köthe type are considered. In the course of our investigation, we introduce a v.s.s. $S_{\mathscr{A}}(E)$ which has been studied in the scalar case by Garling [5].

Proposition 6.1 Let $(S(E),T)$ be of Köthe type. If $(x_n)(\leq i)$, $i = 1,2,\cdots,$ is Cauchy in $(S(E),T)$ and if (x_n) is in $S(E)$, then $(x_n)(\leq i) \to (x_n)$ in $(S(E),T)$ as $i \to \infty$.

Proof. Let T be the polar topology of $\mathscr{A}$-convergence for some class $\mathscr{A}$ of weakly bounded subsets of a v.s.s. $S'(E')$. Since $(x_n)(\leq i)$, $i = 1,2,\cdots,$ is Cauchy, we have for each $\varepsilon > 0$ and A in $\mathscr{A}$ that

$$\left| \sum_{n=i+1}^{j} \langle x_n, y_n \rangle \right| = \left| \langle (x_n)(\leq j) - (x_n)(\leq i), (y_n) \rangle \right| \leq \varepsilon$$

uniformly for (y_n) in A when i,j are sufficiently large. Thus,

$$\left| \sum_{n=i+1}^{\infty} \langle x_n, y_n \rangle \right| = \left| \langle (x_n) - (x_n)(\leq i), (y_n) \rangle \right| \leq \varepsilon$$

uniformly for (y_n) in A when i is sufficiently large. Therefore, $(x_n)(\leq i) \to (x_n)$ in $(S(E),T)$ as $i \to \infty$.

The next two corollaries follow directly from Proposition 6.1 and the definitions of AK-completeness and relative AK-completeness.

Corollary 6.2 If $(S(E),T)$ is of Köthe type, then a subset X of $(S(E),T)$ is AK-complete (respectively, relatively AK-complete) if and only if X (respectively, $S(E)$) contains all those (x_n) in $\omega(E)$ for which $(x_n)(\leq i)$, $i = 1,2,\ldots,$ is a Cauchy sequence in X (respectively, in $\overline{X}$).

Corollary 6.3 Let $(S(E),T)$ be an l.c.t.v.s.s. and let $\hat{T}$ be a topology on $S(E)$ finer than T such that $(S(E),\hat{T})$ is of Köthe type. Then the AK-complete (respectively, the relatively AK-complete) subsets of $(S(E),T)$ are AK-complete (respectively, relatively AK-complete) in $(S(E),\hat{T})$.

Given a dual pair $\langle E,E'\rangle$ of vector spaces, let $\mathscr{A}$ be a class of subsets of $\omega(E')$ such that:

(1) For each $j = 1,2,\ldots$, and each A in $\mathscr{A}$, the set $P_j'(A)$ is weakly bounded in E'; and,

(2) The linear hull of $\bigcup_{A\in\mathscr{A}} A$ in $\omega(E')$ contains $\phi(E')$.

For such a class $\mathscr{A}$, we let

$$S_{\mathscr{A}}(E) = \{(x_n) \in \omega(E) : \sup_{(y_n)\in A} | \sum \langle x_n,y_n\rangle | < \infty \text{ for all } A \in \mathscr{A}\}.$$

From condition (1), we see that $S_{\mathscr{A}}(E) \supset \phi(E)$. If we let $S_{\mathscr{A}}'(E')$ be the linear hull of $\bigcup_{A\in\mathscr{A}} A$ in $\omega(E')$, then from condition (2), we see that $S_{\mathscr{A}}'(E') \supset \phi(E')$. Moreover, $\langle S_{\mathscr{A}}(E),S_{\mathscr{A}}'(E')\rangle$ is a dual pair of v.s.s., and $\mathscr{A}$ is a class of weakly bounded subsets of

$S_{\mathscr{A}}^{\circ}(E')$ determining a polar topology $T_{\mathscr{A}}$ on $S_{\mathscr{A}}(E)$. $T_{\mathscr{A}}$ is determined by the seminorms:

$$(x_n) \longrightarrow \sup_{(y_n)\in A} \left| \Sigma < x_n, y_n > \right| , \quad A \in \mathscr{A} .$$

$S_{\mathscr{A}}(E)$ can be thought of as the largest v.s.s. in $\omega(E)$ on which these seminorms determine a topology.

If $E = E^{\circ} = K$ and if the members of $\mathscr{A}$ are solid, then $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$ turns out to be the l.c.t.v.s.s. considered by Garling [5].

If $\mathscr{A}$ is the class of all finite subsets of a v.s.s. $S^{\circ}(E')$, then $S_{\mathscr{A}}(E) = S^{\circ}(E')^{*}$. If $\mathscr{A}$ is the class of all solid hulls in $S^{\circ}(E')$ of elements of $S^{\circ}(E')$, then $S_{\mathscr{A}}(E) = S^{\circ}(E')^{\times}$. Thus, $S_{\mathscr{A}}(E)$ appears to be a general kind of dual which includes the α-dual and β-dual as special cases. We see in Section 3 of Chapter 3 that for some dual pairs $<E, E'>$ this new dual is α-perfect when the sets in $\mathscr{A}$ are solid.

If $<S(E), S^{\circ}(E')>$ is a dual pair of v.s.s. and T is the polar topology on $S(E)$ of $\mathscr{A}$-convergence, then $\mathscr{A}$ is an example of a class of subsets of $\omega(E')$ that satisfies conditions (1) and (2). Thus, $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$ is defined. We note that $S(E) \subset S_{\mathscr{A}}(E) \subset S^{\circ}(E')^{*}$ and that T is the restriction of $T_{\mathscr{A}}$ to $S(E)$.

In the following proposition, we use $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$ to characterize those $(S(E), T)$ of Köthe type that are AK-complete.

<u>Proposition 6.4</u> Let $T = T(S^{\circ}(E'), S(E))$ be the polar topology on $S(E)$ of $\mathscr{A}$-convergence. Then $(S(E), T)$ is AK-complete if and only if $S(E) \supset (S_{\mathscr{A}}(E), T_{\mathscr{A}})_a$.

Proof.

Necessity. If (x_n) is in $(S_{\mathscr{A}}(E),T_{\mathscr{A}})_a$ then $(x_n)(\leq i)$, $i = 1,2,\cdots$, is Cauchy in $(S_{\mathscr{A}}(E),T_{\mathscr{A}})$, and thus Cauchy in $(S(E),T)$. If $(S(E),T)$ is AK-complete, it then follows that (x_n) is in $S(E)$.

Sufficiency. Note that $(S_{\mathscr{A}}(E),T_{\mathscr{A}})$ is AK-complete. For if $(x_n)(\leq i)$, $i = 1,2,\cdots$, is Cauchy in $(S_{\mathscr{A}}(E),T_{\mathscr{A}})$, then for each A in $\mathscr{A}$, $|\sum_{n=1}^{i} \langle x_n,y_n\rangle| = |\langle (x_n)(\leq i), (y_n)\rangle|$ is bounded uniformly for (y_n) in A and $i = 1,2,\cdots$. Thus, since $\sum \langle x_n,y_n\rangle$ converges,

$$\sup_{(y_n)\in A} |\sum \langle x_n,y_n\rangle| < \infty ;$$

that is, (x_n) is in $S_{\mathscr{A}}(E)$. Now suppose that $S(E) \supset (S_{\mathscr{A}}(E),T_{\mathscr{A}})_a$ and $(x_n)(\leq i)$, $i = 1,2,\cdots$, is Cauchy in $(S(E),T)$. Then it is also Cauchy in $(S_{\mathscr{A}}(E),T_{\mathscr{A}})$. Thus, $(x_n)(\leq i) \to (x_n)$ in $(S_{\mathscr{A}}(E),T_{\mathscr{A}})$ as $i \to \infty$, since $(S_{\mathscr{A}}(E),T_{\mathscr{A}})$ is AK-complete. Therefore, (x_n) is in $(S_{\mathscr{A}}(E),T_{\mathscr{A}})_a$, and thus in $S(E)$.

Corollary 6.5 If $T = T(S'(E'),S(E))$ is a polar topology on $S(E)$ and $S(E) = S^{\circ}(E')^*$, then $(S(E),T)$ is AK-complete.

7. Compactness

The results in Section 4 of Chapter 1 can of course all be applied to any l.c.t.v.s.s. $(S(E),T)$ of Köthe type.

Because a v.s.s. $S^{\circ}(E')$ always contains $\phi(E')$ and because the balanced hull of relatively compact set is relatively compact, Theorem 4.7 of Chapter 1 can be restated in the following way.

Theorem 7.1 Let $\langle S(E), S'(E') \rangle$ be a dual pair of v.s.s. and let T be the polar topology on S(E) of $\mathscr{A}$-convergence. Then the solid hull in S(E) of a subset X of S(E) is relatively compact if and only if:

(1) For each j, $P_j(X)$ is relatively compact in (E,T_j);

(2) For each A in $\mathscr{A}$,

$$\sup_{\substack{(x_n)\in X \\ (y_n)\in A}} \sum_{n=i}^{\infty} | \langle x_n, y_n \rangle | \to 0$$

as $i \to \infty$; and,

(3) $|X| \cap S(E)$ is relatively AK-complete in $(S(E),T)$.

Remark. If $S(E) = S'(E')^*$, then by Corollary 6.5 $(S(E),T)$ is AK-complete and condition (3) is automatically satisfied.

Corollary 7.2 Let $(S(E),T)$ be of Köthe type and let $|T|$ be defined. Then the solid hull in S(E) of a subset X of S(E) is relatively compact in $(S(E),T)$ if and only if it is relatively compact in $(S(E),|T|)$.

This corollary follows directly from Theorem 7.1, Corollary 6.3, and the fact that $|T|_j = T_j$ for each j.

Proposition 7.3 If $\langle S(E), S'(E') \rangle$ is a dual pair of v.s.s. with S'(E') solid, then $|(y_n)|$ is a weakly compact subset of S'(E') for each (y_n) in S'(E'). Thus, $T_o(S'(E'), S(E)) \subset T_k(S'(E'),S(E))$ when S'(E') is solid.

Proof. Since S'(E') is solid, the normal topology is defined on S'(E'). Also, for each (y_n) in S'(E'), $(y_n)(\leq i) \to (y_n)$ in

the normal topology on $S^{\circ}(E^{\circ})$ as $i \to \infty$. By Proposition 4.8 in Chapter 1, $\{y_n\}$ is normally compact, in particular, it is weakly compact. Thus, the solid hull of an element of $S^{\circ}(E^{\circ})$ is absolutely convex and weakly compact; i.e., $T_o(S^{\circ}(E^{\circ}), S(E)) \subset T_k(S^{\circ}(E^{\circ}), S(E))$.

8. Matrix Representation of Linear Maps

In this section, we consider two dual pairs $\langle E, E^{\circ}\rangle$ and $\langle F, F^{\circ}\rangle$ of vector spaces and two dual pairs $\langle S(E), S^{\circ}(E^{\circ})\rangle$ and $\langle S(F), S^{\circ}(F^{\circ})\rangle$ of v.s.s. The projection mappings and the bilinear forms for these spaces will be distinguished by subscripts.

A matrix (Z_{ij}), $i,j = 1,2,\cdots$, of linear maps $Z_{ij} : E \to F$ is said to represent the linear map $Z : S(E) \to S(F)$ if for each (x_n) in $S(E)$, $\sum_{j=1}^{\infty} Z_{ij}x_j$ is a weakly convergent sum in F for each i and $P_{i,F}(Z(x_n)) = \sum_{j=1}^{\infty} Z_{ij}x_j$ for all i.

Theorem 8.1 Let $S^{\circ}(E^{\circ})$ be weakly sequentially complete and let $S^{\circ}(E^{\circ}) = S(E)^*$. Then a linear map Z from $S(E)$ to $S(F)$ is weakly continuous if and only if it is represented by a matrix of weakly continuous linear maps from E to F.

Proof. The necessity follows by a straightforward application of the weak continuity of the maps $P_{i,F}$ and $I_{j,E}$ and the weak convergence of $(x_n)(\leq i)$ to (x_n) for each (x_n) in $S(E)$.

To obtain the sufficiency, it is enough to show that if Z is represented by a matrix (Z_{ij}), where the $Z_{ij} : E \to F$ are weakly continuous, then Z has an adjoint. For all (x_n) in $S(E)$ and (y_n) in $S^{\circ}(F^{\circ})$ we have

$$\langle Z(x_n), (y_n) \rangle_{\langle S(F), S'(F') \rangle} = \sum_{i=1}^{\infty} \langle \sum_{j=1}^{\infty} Z_{ij} x_j, y_i \rangle_{\langle F, F' \rangle}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle Z_{ij} x_j, y_i \rangle_{\langle F, F' \rangle}$$

(since the sum is weakly convergent)

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle x_j, Z_{ij}^* y_i \rangle_{\langle E, E' \rangle}$$

where Z_{ij}^*, the adjoint of Z_{ij}, is defined since Z_{ij} is assumed weakly continuous. In particular, this implies that $(Z_{ij}^* y_i)$ is in $S(E)^*$ for each i. Therefore, it is in $S'(E') = S(E)^*$. Thus,

$$\langle Z(x_n), (y_n) \rangle_{\langle S(F), S'(F') \rangle} = \sum_{i=1}^{\infty} \langle (x_n), (Z_{in}^* y_i) \rangle_{\langle S(E), S'(E') \rangle}.$$

Since this sum converges for all (x_n) in $S(E)$, we get that $\sum_{i=1}^{k} (Z_{in}^* y_i)$ is a weak Cauchy sequence of partial sums of elements of $S'(E')$. By the assumption of weak sequential completeness, it converges to an element $\sum_{i=1}^{\infty} (Z_{in}^* y_i)$ say, of $S'(E')$. Since the convergence is weak, from the previous equality we have

$$\langle Z(x_n), (y_n) \rangle_{\langle S(F), S'(F') \rangle} = \langle (x_n), \sum_{i=1}^{\infty} (Z_{in}^* y_i) \rangle_{\langle S(E), S'(E') \rangle}$$

Thus the adjoint of Z is defined and Z is weakly continuous.

Remark. From (2) of Theorem 1.3 in Chapter 3, we see that the conditions of Theorem 8.1 are satisfied if E' is weakly sequentially complete, $S(E)$ is solid, and $S'(E') = S(E)^*$.

9. Characterizations of Certain Spaces of Köthe Type

In this section, we obtain a necessary and sufficient condition for a special class of l.c.t.v.s.s. $(S(E),T)$ to be of Köthe type.

Theorem 9.1 Let $(S(E),T)$ be an l.c.t.v.s.s. such that $(E,T_i)' = (E,T_j)'$ for all $i,j = 1,2,\ldots$. Then $(S(E),T)$ is of Köthe type with T solid if and only if T is determined by a family of absolutely monotone seminorms $p_\alpha, \alpha \in D$, satisfying the following condition:

$$p_\alpha[(x_n)] = \lim_{i \to \infty} p[(x_n)(\leq i)] \quad \text{for each } \alpha \in D \text{ and}$$

$$(x_n) \in S(E) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (S).$$

Proof.

Necessity. If $(S(E),T)$ is of Köthe type, then T is determined by a class $\mathscr{A}$ of subsets of a v.s.s. $S'(E')$. If in addition T is solid, then the sets in $\mathscr{A}$ can be chosen to be solid in $S'(E')$. The seminorms determined by the sets in $\mathscr{A}$ then satisfy the seminorm condition (S).

Sufficiency. Let $E' = (E,T_j)'$, $j = 1,2,\ldots$. Since the seminorms are absolutely monotone, T is solid and P_j is T, T_j continuous for each j. Thus for each y in E' and each $j = 1,2,\ldots$, the linear functional $(x_n) \to \langle P_j[(x_n)], y\rangle = \langle (x_n), I_j'(y)\rangle$ (where $I_j' : S'(E') \to \phi(E')$), is in $(S(E),T)'$. Thus, $I_j'(y)$ is in $(S(E),T)'$ for each y in E' and $j = 1,2,\ldots$. Therefore $(S(E),T)' \supset \phi(E')$. Thus $T \supset T_s(\phi(E'), S(E))$. We now show that T is determined by a class of weakly bounded subsets of $\phi(E')$.

We first note that since $T_s(E',E) \subset T_j \subset T_k(E',E)$ for each j, we have $T_s(E'^{(i)}, E^{(i)}) \subset T^{(i)} \subset T_k(E'^{(i)},E^{(i)})$ for each i. The polars we use in the proof are taken either with respect to the dual pair $\langle E^{(i)},E'^{(i)}\rangle$ or with respect to the dual pair $\langle S(E),\phi(E')\rangle$.

For each $\alpha \in D$, let $U_\alpha = \{(x_n) : p_\alpha[(x_n)] \leq 1\}$ and let $U_{\alpha,i} = \{(x_n) : p_\alpha[(x_n)(\leq i)] \leq 1\} = (I^{(i)}P^{(i)})^{-1}(U_\alpha)$. Since U_α is a closed absolutely convex neighborhood in $(S(E),T)$ and since $I^{(i)}$ is $T^{(i)}, T$ continuous, $(I^{(i)})^{-1}(U_\alpha)$ is a closed absolutely convex neighborhood in $(E^{(i)},T^{(i)})$. Since $T^{(i)} \subset T_k(E'^{(i)},E^{(i)})$, $(I^{(i)})^{-1}(U_\alpha)$ is weakly closed and it follows that there is a weakly bounded subset $A_{\alpha,i}$ of $E'^{(i)}$ such that:

$$A_{\alpha,i}^{o} = (I^{(i)})^{-1}(U_\alpha) .$$

Thus, $(P^{(i)})^{-1}(A_{\alpha,i}^{o}) = (P^{(i)})^{-1}(I^{(i)})^{-1}(U_\alpha)$;

$$\text{i.e., } B_{\alpha,i}^{o} = U_{\alpha,i},$$

where $B_{\alpha,i} = I'^{(i)}(A_{\alpha,i})$ is a subset of $\phi(E')$. Therefore,

$$\begin{aligned} U_\alpha &= \bigcap_i U_{\alpha,i} \quad \text{(by condition (S))} \\ &= \bigcap_i B_{\alpha,i}^{o} \\ &= (\bigcup_i B_{\alpha,i})^{o} \end{aligned}$$

Thus the sets $\bigcup_i B_{\alpha,i}, \alpha \in D$, are weakly bounded in $\phi(E')$ and determine T. Since we have already shown that $T \supset T_s(\phi(E'),S(E))$, it follows that T is a polar topology on $S(E)$ with respect to the dual pair $\langle S(E),\phi(E')\rangle$. Thus $(S(E),T)$ is of Köthe type and the theorem is proved.

Remarks. 1. If $(S(E),T)$ is of Köthe type with $T = T(S^{\circ}(E^{\circ}),S(E))$ solid then the seminorm condition (S) of Theorem 9.1 holds for any family of absolutely monotone seminorms $p_{\alpha},\alpha \in D$, determining T. For if we let $A_{\alpha} = \{(x_n) \in S(E) : p_{\alpha}[(x_n)] \leq 1\}$ for each $\alpha \in D$, then A_{α}° is a subset of $S^{\circ}(E^{\circ})$ that is solid in $S'(E')$ and p_{α} is the seminorm associated with A_{α}°. Thus, the seminorms $p_{\alpha},\alpha \in D$, satisfy condition (S).

2. Note that the condition "$(E,T_i)^{\circ} = (E,T_j)^{\circ}$ for all $i,j = 1,2,\circ\circ\circ,$" will always hold when $E = K$, i.e. when $S(E) = S(K)$ is a s.s.s. Note also that since $\langle K,K\rangle$ is the only dual pair of vector spaces in which K can occur, it follows that if $(S(K),T)$ is of Köthe type then T is a polar topology with respect to a dual pair $\langle S(K), S^{\circ}(K)\rangle$ of scalar sequence spaces.

Let $\langle E,E^{\circ}\rangle$ be a dual pair of vector spaces and let $(S(E),T)$ be an l.c.t.v.s.s. over E. If $(S(E),T)^{\circ}$ is a v.s.s. over E° (i.e., if the elements f of $(S(E),T)^{\circ}$ can be identified with the elements (y_n) of a v.s.s. over E' so that $f[(x_n)] = \Sigma \langle x_n,y_n\rangle$ for all (x_n) in $S(E)$), then it is clear that $(S(E),T)^{\circ} \subset S(E)^{*}$. Thus a sufficient condition for $(S(E),T)$ to be of Köthe type is that $(S(E),T)'$ be a v.s.s. over E'. The following theorem determines those solid l.c.t.v.s.s. $(S(E),T)$ for which $(S(E),T)^{\circ}$ is a v.s.s. over E'. The proof is similar to the proof of Theorem 5.3 and is omitted.

Theorem 9.2 Let $\langle E,E^{\circ}\rangle$ be a dual pair of vector spaces and let $(S(E),T)$ be a solid l.c.t.v.s.s. over E. Then $(S(E),T)^{\circ}$ is a v.s.s. over E° if and only if $(S(E),T)' \supset \phi(E^{\circ})$, $(S(E),T)_a = S(E)$, and $(E,T_j)' = E^{\circ}$ for each $j = 1,2,\circ\circ\circ$.

Remark. From Theorem 9.2, we see that $(\ell_{\infty},||\ ||_{\infty})^{\circ}$ is not a

s.s.s. since $(\ell_\infty, \|\ \|_\infty)_a = c_o \neq \ell_\infty$.

If $(S(E),T)$ is an l.c.t.v.s.s., and if T is determined by a family of seminorms p_α, $\alpha \in D$, we let

$$|p_\alpha|[(x_n)] = \sup_{(c_n)\in\phi\cap B_\infty} p_\alpha(c_n x_n)$$

for each (x_n) in $S(E)$ and $\alpha \in D$. If $|p_\alpha|[(x_n)] < \infty$ for each (x_n) in $S(E)$ and $\alpha \in D$, then $|p_\alpha|, \alpha \in D$, is a family of absolutely monotone seminorms determining a solid topology on $S(E)$ which we denote by $|T|$. Moreover, the seminorms $|p_\alpha|, \alpha \in D$, satisfy the seminorm condition (S) of Theorem 9.1.

Theorem 9.3 Let $(S(E),T)$ be a solid l.c.t.v.s.s. such that $E^0 = (E,T_j)^0$ for each $j = 1,2,\ldots,$ and suppose that $(S(E),T)' \supset \phi(E^0)$. Then $(S(E),T)$ is of Köthe type if and only if T is determined by a family of seminorms $p_\alpha, \alpha \in D$ such that $|p_\alpha|[(x_n)] < \infty$ for each $(x_n) \in S(E)$ and $\alpha \in D$.

Proof.

Necessity. If $(S(E),T)$ is of Köthe type then T is a polar topology on $S(E)$ of $\mathscr{A}$-convergence for a class $\mathscr{A}$ of subsets of a v.s.s. $S^0(E^0)$. If p_A is the seminorm associated with A in $\mathscr{A}$, then $|p_A|[(x_n)] < \infty$ for each (x_n) in $S(E)$.

Sufficiency. From the proof of Theorem 9.1, $|T|$ is a polar topology on $S(E)$ with respect to the dual pair $\langle S(E),\phi(E')\rangle$. Since $(S(E),T)^0 \supset \phi(E^0)$, $T \supset T_s(\phi(E^0),S(E))$. Since $T \subset |T|$, T is also a polar topology of the dual pair $\langle S(E),\phi(E')\rangle$, i.e. $(S(E),T)$ is of Köthe type.

CHAPTER 3

HEREDITARY PROPERTIES OF VECTOR SEQUENCE SPACES OF KÖTHE TYPE

If $(S(E),T)$ is an l.c.t.v.s.s., we might ask how the properties of $(S(E),T)$ are related to those of the (E,T_j), $j = 1,2,\cdots$. In particular, is $(S(E),T)$ complete or barrelled or reflexive when (E,T_j) is complete or barrelled or reflexive for each j? If P_j is T,T_j continuous for each j, an immediate partial answer to the above question is that (E,T_j) inherits those properties of $(S(E),T)$ that are preserved under continuous open onto maps. The problem is in going the other way. Most of this chapter is therefore devoted to determining when $(S(E),T)$ inherits a property that the (E,T_j), $j = 1,2,\cdots,$ all have.

We assume throughout this chapter that $(S(E),T)$ is of Köthe type.

1. Completeness

Recall that an l.c.t.v.s. (L,T) is <u>complete</u> if every Cauchy net in (L,T) converges. (L,T) is <u>quasicomplete</u> if every closed bounded subset of L is complete; equivalently, (L,T) is quasicomplete if every bounded Cauchy net in (L,T) converges. Finally, (L,T) is <u>sequentially complete</u> if every Cauchy sequence in (L,T) converges.

In this section, we determine when an l.c.t.v.s.s. $(S(E),T)$ of Köthe type with T solid is complete, quasicomplete, or sequentially complete. The results we obtain on completeness are stated in terms of the l.c.t.v.s.s. $(S_{\mathscr{A}}(E),T_{\mathscr{A}})$. The dependence on the class $\mathscr{A}$ is only apparent; i.e., if $T(S^{\times}(E^{\times}),S(E))$ is a solid polar topology on $S(E)$, then $S_{\mathscr{A}}(E)$ is independent of the class $\mathscr{A}$ determining T.

The following proposition is the key to most of the results in this section.

Proposition 1.1 Let $(S(E),T)$ be of Köthe type, where T is a solid polar topology of $\mathscr{A}$-convergence. Then each of the following statements implies the one below it:

(1) $S(E) = S^{\circ}(E^{\circ})^{*}$.

(1') $S(E)$ is a closed subspace of $(S_{\mathscr{A}}(E),T_{\mathscr{A}})$.

(2) Every TK-convergent Cauchy net in $(S(E),T)$ converges.

(2') Every TK-convergent bounded Cauchy net in $(S(E),T)$ converges.

(2") Every TK-convergent Cauchy sequence in $(S(E),T)$ converges.

(3) $(S(E),T)$ is AK-complete.

Moreover, (1') and (2) are equivalent. Also, if T is the normal topology, then all the above statements are equivalent.

Proof. If (1) holds, then $S(E) = S^{\circ}(E^{\circ})^{*} = S_{\mathscr{A}}(E)$ and thus (1') holds.

Suppose (1') holds and $(x_n^{(\alpha)})$ is a Cauchy net in $(S(E),T)$ such that for each j, $x_j^{(\alpha)} \to x_j$ in (E,T_j). Let $\varepsilon > 0$ and A in $\mathscr{A}$ be given. Then

$$\sum_{n=1}^{k} |\langle x_n^{(\alpha)} - x_n, y_n \rangle| \leq \sum_{n=1}^{\infty} |\langle x_n^{(\alpha)} - x_n^{(\beta)}, y_n \rangle| + \sum_{n=1}^{k} |\langle x_n^{(\beta)} - x_n, y_n \rangle| .$$

Since $(x_n^{(\alpha)})$ is Cauchy in $(S(E),T)$, there is some γ such that $\alpha,\beta \geq \gamma$ implies that $\sum_{n=1}^{\infty} |\langle x_n^{(\alpha)} - x_n^{(\beta)}, y_n \rangle| \leq \varepsilon/2$ for all (y_n) in A. Since for each j, $x_j^{(\alpha)} \to x_j$ in the topology T_j of $\mathscr{A}_j$-convergence, for sufficiently large β, $\sum_{n=1}^{k} |\langle x_n^{(\beta)} - x_n, y_n \rangle| \leq \varepsilon/2$ for all (y_n) in A. Since the choice of γ was independent of k, we have that

$\alpha \geq \gamma$ implies that $\sum_{n=1}^{\infty} |\langle x_n^{(\alpha)} - x_n, y_n \rangle| \leq \varepsilon$ for all (y_n) in A. In particular, it follows that (x_n) is in $S_{\mathscr{A}}(E)$ since $(x_n^{(\alpha)}) - (x_n)$ and $(x_n^{(\alpha)})$ are in $S_{\mathscr{A}}(E)$. Also, $(x_n^{(\alpha)}) \to (x_n)$ in $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$. Since we are assuming (1'), this implies that $(x_n^{(\alpha)}) \to (x_n)$ in $(S(E), T)$. Thus (1') implies (2).

The remaining implications follow immediately.

If T is the normal topology, then $(x_n)(\leq i)$ is normally Cauchy for each (x_n) in $S^{\circ}(E^{\circ})^*$. Thus, (3) implies (1) if T is the normal topology.

Remark. Recall that if $S^{\circ}(E^{\circ})$ is solid, then weakly convergent (respectively, weakly Cauchy) sequences are normally convergent (respectively, normally Cauchy). Thus if $S^{\circ}(E^{\circ})$ is solid, (2") and (3) are equivalent to the condition $S(E) = S^{\circ}(E^{\circ})^*$ when T is the weak topology.

Suppose that I_j is T_j, T closed. (This condition is satisfied when T is solid). Then $I_j(E)$ is a closed subspace of $(S(E), T)$. Consequently, (E, T_j) is complete (respectively, quasicomplete, sequentially complete) if $(S(E), T)$ is complete (respectively, quasi-complete, sequentially complete). From these facts and the preceding proposition and remark, we obtain the following theorems.

Theorem 1.2 Let $(S(E), T)$ be of Köthe type, and let T be solid. Then

(1) $(S(E), T)$ is complete if and only if (E, T_j) is complete for all j and $S(E)$ is a closed subspace of $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$.

(2) If $S(E)$ is a closed subspace of $(S_{\mathscr{A}}(E), T_{\mathscr{A}})$ then

$(S(E),T)$ is quasicomplete (respectively, sequentially complete) if and only if (E,T_j) is quasicomplete (respectively, sequentially complete) for all j.

Theorem 1.3 (1) $S(E)$ is normally complete (respectively, quasicomplete, sequentially complete) if and only if E is weakly complete (respectively, quasicomplete, sequentially complete) and $S(E) = S'(E')^*$.

(2) If $S'(E')$ is solid, then $S(E)$ is weakly sequentially complete if and only if E is weakly sequentially complete and $S(E) = S'(E')^*$.

Remarks. 1. If T is solid and $S(E) = S'(E')^*$, then $S(E) = S_{\mathcal{A}}(E)$ is a closed subspace of $(S_{\mathcal{A}}(E),T_{\mathcal{A}})$ and (1) and (2) of Theorem 1.2 apply. In particular, the s.s.s. $(\ell_p,|| \ ||_p)$, $1 \leq p \leq \infty$ are all complete.

2. By Proposition 3.1 of Chapter 1, we see that $(S_{\mathcal{A}}(E),T_{\mathcal{A}})_a$ is another example of a closed subspace of $(S_{\mathcal{A}}(E),T_{\mathcal{A}})$. In particular, $c_o = (\ell_\infty,|| \ ||_\infty)_a$ is a closed subspace of $(\ell_\infty,|| \ ||_\infty)$ and thus $(c_o,|| \ ||_\infty)$ is complete. This example shows that a s.s.s. need not be an α-dual to be complete, even if the topology is coarser than the Mackey topology $(|| \ ||_\infty \subset T_k(\ell_1,c_o)$ since $(c_o,|| \ ||_\infty)' = \ell_1$ by Theorem 5.3 in Chapter 2.)

3. There may be closed vector sequence subspaces of $(S_{\mathcal{A}}(E),T_{\mathcal{A}})$ besides those mentioned above. In particular, c is a closed subspace of $(\ell_\infty,|| \ ||_\infty)$ which isn't even solid. $(c,|| \ ||_\infty)$ is also complete.

Theorem 1.4 Let $(S(E),T)$ be of Köthe type with T solid and $(S(E),T)_a = S(E)$. Then $(S(E),T)$ is complete if and only if (E,T_j)

is complete for all j and $(S(E),T)$ is AK-complete.

Proof. The necessity is immediate.

To prove the sufficiency, suppose that T is the polar topology of $\mathscr{A}$-convergence. If $(S(E),T)$ is AK-complete and $(S(E),T)_a = S(E)$, then $S(E) = (S_{\mathscr{A}}(E),T_{\mathscr{A}})_a$ by Proposition 6.4 in Chapter 2. Thus, $S(E)$ is a closed subspace of $(S_{\mathscr{A}}(E),T_{\mathscr{A}})$. Therefore, $(S(E),T)$ is complete.

Remark. By Proposition 4.3 in Chapter 2, if $S(E)$ is solid, the condition $(S(E),T)_a = S(E)$ of Theorem 1.4 holds for any polar topology T coarser than $T_k(S'(E'),S(E))$.

Except for (2) of Theorem 1.3, the results of this section have all been stated for an l.c.t.v.s.s. $(S(E),T)$ of Kothe type with T solid. If T is not necessarily solid, the condition that $(S(E),T)$ be complete can turn out to be a very strong one. The following proposition on weak completeness illustrates this fact. Weak sequential completeness has already been considered; weak quasicompleteness is characterized in the next section.

Proposition 1.5 Let $\langle S(E),S'(E')\rangle$ be a dual pair of v.s.s. Then $S(E)$ is weakly complete if and only if E is weakly complete, $S(E) = \omega(E)$, and $S'(E') = \phi(E')$.

Proof. The sufficiency follows by noting that a net $(x_n^{(\alpha)})$, $\alpha \in D$ is weakly convergent (respectively, weakly Cauchy) in $S(E) = \omega(E)$ if and only if for each j, $x_j^{(\alpha)}$, $\alpha \in D$, is weakly convergent (respectively, weakly Cauchy) in E.

To prove the necessity, we first note that if $S(E)$ is weakly complete, then $S(E) = L$, the algebraic dual of $S'(E')$. This

follows because $S(E)$ is $T_s(S^*(E^*),L)$ dense in L ([15], p. 125) and because $T_s(S^0(E^0),S(E))$ is the restriction of $T_s(S^0(E^0),L)$ to $S(E)$. We note also that (θ) is the only member of $S(E)$ which, when regarded as a linear functional on $S^*(E^0)$, is equal to zero on $\phi(E^0)$. From this we conclude that $S^0(E^0) = \phi(E^0)$; otherwise, we could construct a non-zero linear functional in $L = S(E)$ that is zero on $\phi(E^*)$. Since $S^0(E^0) = \phi(E^0)$, the weak and normal topologies on $S(E)$ are the same. Thus, $S(E)$ is normally complete. Therefore, by (1) of Theorem 1.3, $S(E) = \phi(E^0)^* = \omega(E)$ and E is weakly complete.

2. Spaces of Köthe Type that are Quasibarrelled, Barrelled, Semireflexive, Reflexive, or Montel.

Let (L,T) be an l.c.t.v.s. If we let $L' = (L,T)'$, and $L'' = (L^0, T_b(L,L^0))'$, then L is a subset of L'' when considered as a set of linear functionals on L^0.

We say that (L,T) is quasibarrelled if the injection map of (L,T) into $(L'', T_b(L',L''))$ is continuous or, equivalently ([10], p. 304), if $T = T_{b^*}(L^0,L)$.

We say that (L,T) is barrelled if every set of pointwise bounded continuous linear functionals on (L,T) is equicontinuous. Since T is the polar topology determined by the class of equicontinuous subsets of L^0, (L,T) is barrelled if and only if $T = T_b(L',L)$.

We note that if $\langle L,L^0\rangle$ is a dual pair of vector spaces and if (L,T) is barrelled (respectively, quasibarrelled) for some polar topology T coarser than $T_k(L',L)$, then $(L, T_k(L',L))$ is barrelled (respectively, quasibarrelled). In this section, the barrelled (respectively, quasibarrelled) spaces $(S(E),T)$ of Köthe type that we investigate are assumed to have the Mackey topology. The following

proposition is a simple restatement of Theorem 5.3 in Chapter 2.

Proposition 2.1 Let $\langle S(E), S^{\circ}(E^{\circ})\rangle$ be a dual pair of solid v.s.s. with $S^{\circ}(E^{\circ}) = S(E)^{*}$, and let $T = T(S(E)^{*}, S(E))$ be a polar topology on $S(E)$. Then $T \subset T_k(S(E)^{*}, S(E))$ if and only if $(S(E), T)_a = S(E)$ and $T_j \subset T_k(E^{\circ}, E)$ for each $j = 1, 2, \cdots$.

Theorem 2.2 Let $\langle S(E), S^{\circ}(E^{\circ})\rangle$ be a dual pair of solid v.s.s. with $S^{\circ}(E^{\circ}) = S(E)^{*}$. Then $(S(E), T_k(S(E)^{*}, S(E)))$ is quasibarrelled (respectively, barrelled) if and only if $(E, T_k(E^{\circ}, E))$ is quasibarrelled (respectively, barrelled) and $(S(E), T_{b^{*}}(S(E)^{*}, S(E)))_a = S(E)$ (respectively, $(S(E), T_b(S(E)^{*}, S(E)))_a = S(E)$).

Proof. Since $(S(E), T_k(S(E)^{*}, S(E)))$ is quasibarrelled (respectively, barrelled) if and only if $T_k(S(E)^{*}, S(E)) \supset T_{b^{*}}(S(E)^{*}, S(E))$ (respectively, $T_k(S(E)^{*}, S(E)) \supset T_b(S(E)^{*}, S(E))$), the theorem follows from Proposition 2.1.

We recall from Proposition 5.6 in Chapter 2 that if $S(E)$ is solid, the strong topology $T_b(S(E)) = T_b(S^{\circ}(E^{\circ}), S(E))$ on $S(E)$ is independent of the v.s.s. $S^{\circ}(E^{\circ})$, where $\phi(E^{\circ}) \subset S^{\circ}(E^{\circ}) \subset S(E)^{*}$. This independence allows us to give the following theorem.

Theorem 2.3 Let $\langle S(E), S^{\circ}(E^{\circ})\rangle$ be a dual pair of v.s.s. with $S(E)$ solid. Then $(S(E), T_k(S^{\circ}(E^{\circ}), S(E)))$ is barrelled if and only if $S^{\circ}(E^{\circ}) = S(E)^{*}$, $(E, T_k(E^{\circ}, E))$ is barrelled, and $(S(E), T_b(S(E)))_a = S(E)$.

Proof. The sufficiency follows from Theorem 2.2.

To prove the necessity, we recall that if $(S(E), T_k(S^{\circ}(E^{\circ}), S(E)))$ is barrelled, then $T_b(S(E)) = T_k(S^{\circ}(E^{\circ}), S(E))$. Thus,

$S^0(E^0) = (S(E), T_b(S(E)))' \supseteq (S(E), T_k(S(E)^*, S(E)))' = S(E)^*$. Therefore, $S^0(E^0) = S(E)^*$. The necessity of the remaining conditions now follows from Theorem 2.2.

Given a dual pair $\langle L, L' \rangle$ of vector spaces, we say that L is semireflexive in the dual pair $\langle L, L' \rangle$ if and only if $L = L''$. Since L is semireflexive in the dual pair $\langle L, L^0 \rangle$ if and only if $(L, T_k(L, L^0))$ is barrelled ([10], p. 308), the following proposition is an immediate consequence of Theorem 2.3.

Proposition 2.4 Let $\langle S(E), S^0(E^0) \rangle$ be a dual pair of v.s.s. with $S^0(E^0)$ solid. Then $S(E)$ is semireflexive in the dual pair $\langle S(E), S^0(E^0) \rangle$ if and only if E is semireflexive in the dual pair $\langle E, E^0 \rangle$, $S(E) = S^0(E^0)^*$, and $(S^0(E^0), T_b(S^0(E^0)))_a = S^0(E^0)$.

Since L is semireflexive in the dual pair $\langle L, L^0 \rangle$ if and only if L is weakly quasicomplete ([10], p. 301), from Proposition 2.4 we obtain the following corollary.

Corollary 2.5 Let $\langle S(E), S^0(E^0) \rangle$ be a dual pair of v.s.s. with $S^0(E^0)$ solid. Then $S(E)$ is weakly quasicomplete if and only if E is weakly quasicomplete, $S(E) = S^0(E^0)^*$, and $(S^0(E^0), T_b(S^0(E^0)))_a = S^0(E^0)$.

Remark. From Corollary 2.5, we see that a perfect s.s.s. need not be weakly quasicomplete. For in the dual pair $\langle \ell_1, \ell_\infty \rangle$, ℓ_1 is not $T_s(\ell_\infty, \ell_1)$ quasicomplete since $c_o = (\ell_\infty, \| \|_\infty)_a \neq \ell_\infty$.

A dual pair $\langle L, L^0 \rangle$ of vector spaces is called reflexive if both L and L^0 are semireflexive in the dual pair. It turns out ([10], p. 307) that $\langle L, L^0 \rangle$ is reflexive if and only if $(L, T_k(L^0, L))$

is reflexive in the usual sense, i.e., if and only if $L = L''$ and $T_k(L', L) = T_{b^*}(L'', L)$. From this fact and Proposition 2.4, we obtain the following proposition

Proposition 2.6 Let $\langle S(E), S'(E') \rangle$ be a dual pair of solid v.s.s. Then $(S(E), T_k(S'(E'), S(E)))$ is reflexive if and only if $(E, T_k(E', E))$ is reflexive, $S(E) = S'(E')^*$, $S'(E') = S(E)^*$, $(S(E), T_b(S(E)))_a = S(E)$, and $(S'(E'), T_b(S'(E')))_a = S'(E')$.

A barrelled l.c.t.v.s. (L, T) is said to be Montel if every closed bounded subset of (L, T) is compact or, equivalently, if every bounded subset of (L, T) is relatively compact.

Proposition 2.7 Let $\langle S(E), S'(E') \rangle$ be a dual pair of v.s.s. with $S(E)$ solid. Then $(S(E), T_k(S'(E'), S(E)))$ is Montel if and only if the following conditions hold:

(1) $(E, T_k(E', E))$ is Montel;

(2) For each bounded set X in $(S(E), T_b(S(E)))$, (x_n) $(\in X)$, $i = 1, 2, \cdots$, converges to (θ) in $(S(E), T_b(S(E)))$ uniformly for (x_n) in X; and,

(3) $S(E) = S'(E')^*$ and $S'(E') = S(E)^*$.

Proof.

Necessity. Let $T = T_k(S'(E'), S(E))$. If $(S(E), T)$ is Montel, then it is barrelled. Thus, $(E, T_k(E', E))$ is barrelled by Theorem 2.3. Moreover, since P_1 is T, $T_k(E', E)$ continuous and I_1 is $T_k(E', E)$, T continuous and closed, it follows that the closed bounded subsets of $(E, T_k(E', E))$ are compact. Thus, condition (1) holds.

If $(S(E), T)$ is Montel, then $T = T_b(S(E))$ and the bounded

subsets of $S(E)$ are the same for every polar topology on $S(E)$ with respect to the dual pair $\langle S(E), S^{\sim}(E^{\sim})\rangle$. Thus, solid hulls of bounded subsets of $S(E)$ are bounded by Corollary 5.2 of Chapter 2. Therefore, if $(S(E),T)$ is Montel, then the solid hulls of bounded subsets of $S(E)$ are relatively compact with respect to $T = T_b(S(E))$. Condition (2) then follows from condition (2) of Theorem 4.7 in Chapter 1.

Since Montel spaces are reflexive ([10], p. 372), condition (3) follows from Proposition 2.6.

Sufficiency. If conditions (1), (2), and (3) hold, then $(S(E),T)$ is barrelled by Theorem 2.3. Thus, $T = T_b(S(E))$. We note that $(S(E),T)$ is AK-complete since $S(E) = S^{\sim}(E')^*$, and that $P_j(X)$ is relatively compact in the Montel space $(E, T_k(E^{\sim},E))$ if X is bounded in $(S(E),T)$. From Theorem 4.7 in Chapter 1, it follows that the solid bounded subsets of $(S(E),T)$ are relatively compact. By Corollary 5.2 in Chapter 2, the solid hull of a bounded subset of $(S(E),T)$ is bounded. Thus, bounded subsets of $(S(E),T)$ are relatively compact. Therefore $(S(E),T)$ is Montel.

3. Weakly Bounded and Strongly Bounded Subsets of Spaces of Köthe Type.

If $\langle L,L^{\sim}\rangle$ is a dual pair of vector spaces, and if

$\mathcal{C}$ denotes the class of all absolutely convex weakly compact subsets of $L^{\sim}$;

$\mathcal{S}$ denotes the class of all absolutely convex, weakly closed, strongly bounded subsets of $L^{\sim}$; and,

$\mathcal{W}$ denotes the class of all absolutely convex, weakly closed, weakly bounded subsets of $L^{\sim}$;

then $\mathcal{C} \subset \mathcal{S} \subset \mathcal{W}$. Moreover, the equality $\mathcal{C} = \mathcal{W}$ is equivalent to the condition that $(L, T_k(L',L))$ be barrelled; and, the equality $\mathcal{C} = \mathcal{S}$ is equivalent to the condition that $(L, T_k(L',L))$ be quasibarrelled. We have studied the conditions $\mathcal{C} = \mathcal{W}$ and $\mathcal{C} = \mathcal{S}$ in Section 2. In this section, we consider the condition $\mathcal{S} = \mathcal{W}$. It follows that $\mathcal{S} = \mathcal{W}$ if and only if every weakly bounded subset of L' is strongly bounded. Moreover, the weakly bounded subsets of L' are strongly bounded if and only if the weakly bounded subsets of L are strongly bounded. If L (or L') is weakly sequentially complete, then $\mathcal{S} = \mathcal{W}$ by the Banach-Mackey Theorem ([10], p. 256). Thus, in particular, weakly bounded subsets are strongly bounded in each member of the dual pair $\langle \lambda, \lambda^* \rangle$ of solid s.s.s. This result can be generalized to the dual pair $\langle S(E), S(E)^* \rangle$ of solid v.s.s. without requiring $S(E)^*$ to be weakly sequentially complete.

Theorem 3.1 Let $\langle S(E), S'(E') \rangle$ be a dual pair of v.s.s. with $S(E)$ solid and $S'(E') = S(E)^*$. Then the weakly bounded subsets of $S(E)$ are strongly bounded if and only if the weakly bounded subsets of E are strongly bounded.

Proof. The necessity follows from the continuity of the maps P_j and I_j.

To prove the sufficiency, we argue by contradiction. Suppose then that the weakly bounded subsets of E are strongly bounded, that $S(E)$ is solid, that $S'(E') = S(E)^*$, and suppose that there is a weakly bounded subset of $S(E)$ which is not strongly bounded. Then there is also a weakly bounded subset of $S(E)^*$ which is not strongly bounded. Thus by the Banach Boundedness Principle, there

is a weakly bounded sequence $(y_n^{(i)})$, $i = 1,2,\cdots$, in $S(E)^*$ and a scalar null sequence $c^{(i)}$, $i = 1,2,\cdots$, such that $c^{(i)}(y_n^{(i)})$, $i = 1,2,\cdots$, does not converge to (θ') strongly. Consequently, there is an $\varepsilon > 0$ and a weakly bounded sequence $(x_n^{(i)})$, $i = 1,2,\cdots$, in $S(E)$ such that $\sum_{n=1}^{\infty} |\langle x_n^{(i)}, c^{(i)} y_n^{(i)}\rangle| \geq \varepsilon$ for infinitely many $i = 1,2,\cdots$. By choosing subsequences if necessary, we assume that this inequality holds for all $i = 1,2,\cdots$, and that $(\sqrt{|c^{(i)}|})$ is in ℓ_1. Let $a_n^{(i)} = |\langle x_n^{(i)}, c^{(i)} y_n^{(i)}\rangle|$. Then

$$\sum_{n=1}^{\infty} a_n^{(i)} \geq \varepsilon \quad \text{for all} \quad i \quad \cdots\cdots \quad (*)$$

Since P_j is weakly continuous, $y_j^{(i)}$, $i = 1,2,\cdots$, is a weakly bounded (and therefore a strongly bounded) sequence in E' for each j. Thus for each j, $c^{(i)} y_j^{(i)} \to \theta'$ strongly as $i \to \infty$. Since for each j, $x_j^{(i)}$, $i = 1,2,\cdots$, is a weakly bounded sequence in E, we have for each fixed m that

$$\sum_{n=1}^{m} a_n^{(i)} \longrightarrow 0 \quad \text{as} \quad i \to \infty \quad \cdots\cdots \quad (**)$$

Thus, we can choose a subsequence $(a_n^{(i_k)})$, $k = 1,2,\cdots$, of $(a_n^{(i)})$, $i = 1,2,\cdots$, and integers $1 = n_1 < m_1 < n_2 < m_2 < \cdots < n_k < m_k < \cdots$, such that

$$\sum_{n=1}^{n_k - 1} a_n^{(i_k)} \leq \varepsilon/4 \ ,$$

$$\sum_{n=n_k}^{m_k} a_n^{(i_k)} \geq \varepsilon/2 \ ; \text{ and,}$$

$$\sum_{n=m_k+1}^{\infty} a_n^{(i_k)} \leq \varepsilon/4 \quad \text{for all} \quad k = 1,2,\cdots.$$

Loosely speaking, for each n_k we concentrate most of (*) in $\sum_{n=n_k}^{m_k}$ by choosing i_k so large that $\sum_{n=1}^{n_k-1}$ is small by (**), and by taking m_k so large that $\sum_{n=m_k+1}^{\infty}$ is small.

Let $b^{(i)} = \sqrt{|c^{(i)}|}$ and define (z_n) by $z_n = b^{(i_k)} y_n^{(i_k)}$ if $n_k \leq n \leq m_k$, and $z_n = \theta'$ otherwise. Since the sequence $(y_n^{(i_k)})$, $k = 1,2,\cdots,$ is weakly bounded and since $S(E)$ is solid and $(b^{(i)})$ is in ℓ_1, it follows that (z_n) is in $S(E)^*$. Since the sequence $(x_n^{(i_k)})$, $k = 1,2,\cdots,$ is weakly bounded and $b^{(i)}$, $i = 1,2,\cdots,$ is a scalar null sequence, $b^{(i_k)}(x_n^{(i_k)})$, $k = 1,2,\cdots,$ converges to (θ) weakly (and thus normally). But

$$\sum_{n=1}^{\infty} |\langle b^{(i_k)} x_n^{(i_k)}, z_n \rangle| \geq \sum_{n=n_k}^{m_k} |\langle b^{(i_k)} x_n^{(i_k)}, b^{(i_k)} y_n^{(i_k)} \rangle| = \sum_{n=n_k}^{m_k} a_n^{(i_k)} \geq \varepsilon/2$$

for all k. This contradicts the fact that $b^{(i_k)}(x_n^{(i_k)})$, $k = 1,2,\cdots,$ converges to (θ) normally.

Remark. By considering the dual pair $\langle \phi, \phi \rangle$, we see that Theorem 3.1 need not hold if $S'(E') \neq S(E)^*$. Theorem 3.1 need not hold if $S(E)$ is not solid. If it did, then in the remark following Corollary 5.2 in Chapter 2, the solid hull of the sequence $(x_n^{(i)})$, $i = 1,2,\cdots,$ would be weakly bounded by Corollary 5.2.

The next proposition does not involve a hereditary property of v.s.s., but does follow naturally from Theorem 3.1. We assume that $\langle E, E' \rangle$ is a dual pair of vector spaces and that $\mathscr{A}$ is a class of

subsets of $\phi(E^0)$ satisfying conditions (1) and (2) in the definition of $S_{\mathscr{A}}(E)$ in Section 6 of Chapter 2.

Proposition 3.2 If the weakly bounded subsets of E are strongly bounded, and if the members of $\mathscr{A}$ are solid, then $S_{\mathscr{A}}(E)$ is α-perfect.

Proof. Since $S_{\mathscr{A}}(E)^*$ is solid, it follows from Proposition 5.6 in Chapter 2 that the strongly bounded subsets of $S_{\mathscr{A}}(E)^*$ are the same for the dual pairs $\langle S_{\mathscr{A}}(E), S_{\mathscr{A}}(E)^* \rangle$ and $\langle S_{\mathscr{A}}(E)^{**}, S_{\mathscr{A}}(E)^* \rangle$. By Theorem 3.1, the weakly bounded subsets of $S_{\mathscr{A}}(E)^*$ are strongly bounded for each of these dual pairs. Thus, the class $\mathscr{W}$ of all weakly bounded subsets of $S_{\mathscr{A}}(E)^*$ is the same for each of the dual pairs. Therefore, $S_{\mathscr{A}}(E)^{**} \subset S_{\mathscr{W}}(E) \subset S_{\mathscr{A}}(E)$ since $\mathscr{A} \subset \mathscr{W}$. Thus, $S_{\mathscr{A}}(E) = S_{\mathscr{A}}(E)^{**}$ and $S_{\mathscr{A}}(E)$ is α-perfect since it is solid.

4. Nuclearity

In this section, we obtain a necessary and sufficient condition for a v.s.s. to be nuclear with respect to the normal topology.

If (L,T) is an l.c.t.v.s. and B is an absolutely convex non-empty bounded subset of L, we let $L_B = \bigcup_{n=1}^{\infty} nB$ and give L_B the norm topology induced by p_B, the seminorm associated with B. The following definitions of nuclear map and nuclear space are equivalent to the usual definitions (see Schaeffer [15], p. 97-101).

A linear map Z from an l.c.t.v.s. (L,T) to an l.c.t.v.s. $(M,\hat{T})$ is nuclear if it is of the form

$$Z(x) = \sum_{i=1}^{\infty} a^{(i)} \langle x, y^{(i)} \rangle z^{(i)}$$

where $\Sigma_{i=1}^{\infty} |a^{(i)}| < \infty$, $y^{(i)}$, $i = 1,2,\cdots$, is an equicontinuous sequence in $(L,T)'$, and $z^{(i)}$, $i = 1,2,\cdots$, is contained in an absolutely convex bounded subset B of L for which M_B is complete.

An l.c.t.v.s. (L,T) is nuclear if every continuous linear map of E into any Banach space is nuclear.

It turns out ([13], p. 64) that an l.c.t.v.s. (L,T) is nuclear if and only if for some (every) neighborhood base $\mathcal{U}$ of absolutely convex neighborhoods of zero the following property holds:

(P) "For each neighborhood U in $\mathcal{U}$ there is a neighborhood V in $\mathcal{U}$ and a sequence $y^{(i)}$, $i = 1,2,\cdots$, in $(L,T)'$ with $\Sigma_{i=1}^{\infty} p_V^{\circ}(y^{(i)}) < \infty$ so that the inequality $p_U(x) \leq \Sigma_{i=1}^{\infty} |\langle x, y^{(i)} \rangle|$ holds for all x in E".

It is actually sufficient to obtain this inequality for neighborhoods U in a subbase generating $\mathcal{U}$ by finite intersections and for V in $\mathcal{U}$.

Lemma 4.1 A subset X of an l.c.t.v.s. (L,T) is bounded if and only if the sequence of partial sums $\Sigma_{i=1}^{n} c^{(i)}x^{(i)}$, $n = 1,2,\cdots$, is Cauchy for each sequence $x^{(i)}$, $i = 1,2,\cdots$, in X and each sequence $c^{(i)}$, $i = 1,2,\cdots$, in ℓ_1.

Proof.

Necessity. Suppose X is bounded in (L,T), $x^{(i)}$, $i = 1,2,\cdots$, is a sequence in X and $c^{(i)}$, $i = 1,2,\cdots$, is in ℓ_1. Given an absolutely convex neighborhood U of zero, choose r so large that $X \subset rU$ and n_o so large that $\Sigma_{i=n}^{i=m} |c^{(i)}| < 1/r$ when $m \geq n \geq n_o$. Then, since U is absolutely convex,

$\Sigma_{i=n}^{m} c^{(i)}x^{(i)} \in c^{(n)}rU + \cdots + c^{(m)}rU = (c^{(n)} + \cdots + c^{(m)})rU \subset U$
whenever $m \geq n \geq n_o$.

Sufficiency. Suppose that the condition stated in the lemma holds for some subset X of L. If X is not bounded in (L,T), then for some neighborhood U of zero we have $X \not\subset i^2U$ for all $i = 1,2,\cdots$. Thus, there is a sequence $x^{(i)}$, $i = 1,2,\cdots$, in X such that $x^{(i)} \notin i^2U$ for $i = 1,2,\cdots$. Since $1/i^2$, $i = 1,2,\cdots$, is in ℓ_1, $\Sigma_{i=1}^{n}(1/i^2)x^{(i)}$, $n = 1,2,\cdots$, is Cauchy. Thus, $(1/i^2)x^{(i)}$, $i = 1,2,\cdots$, converges to zero, contradicting $x^{(i)} \notin i^2U$ for all $i = 1,2,\cdots$.

If $S(E)$ is a v.s.s. and λ is a s.s.s., we let
$\lambda \circ S(E) = \{(a_n x_n) : (a_n) \in \lambda , (x_n) \in S(E)\}$.

Theorem 4.2 Let $\langle S(E), S'(E')\rangle$ be a dual pair of solid v.s.s. Then $(S(E), T_o(S'(E'), S(E)))$ is nuclear if and only if $S'(E') \subset \ell_1 \circ S'(E')$.

Proof.

Sufficiency. A neighborhood subbasis for the normal topology on $S(E)$ is given by the sets $|(y_n)|^o$ where (y_n) is in $S'(E')$. Given a subbasic neighborhood $U = |(y_n)|^o$, let $(y_n) = (c_n z_n)$ where (c_n) is in ℓ_1 and (z_n) is in $S'(E')$. Take $V = |(z_n)|^o$ and $(y_n^{(i)}) = I_i'(c_i z_i)$. Since $|(z_n)|$ is absolutely convex and weakly closed, we have $V^o = |(z_n)|$. Thus,

$\Sigma_{i=1}^{\infty} p_{V^o}[(y_n^{(i)})] = \Sigma_{i=1}^{\infty} |c_i| \, p_{|(z_n)|}(I_i z_i) = \Sigma_{i=1}^{\infty} |c_i| < \infty$ so that V and $(y_n^{(i)})$ satisfy the condition in (P) . Also,

$p_U[(x_n)] = \Sigma_{n=1}^{\infty} |\langle x_n, y_n\rangle| = \Sigma_{i=1}^{\infty} |\langle x_i, c_i z_i\rangle| = \Sigma_{i=1}^{\infty} |\langle (x_n), (y_n^{(i)})\rangle|$

for all (x_n) in $S(E)$. The nuclearity now follows from property (P).

Necessity. Let (y_n) be in $S'(E')$. Then as in Lemma 4.2 one can show that the map $(y_n)[(x_n)] = (\langle x_n, y_n \rangle)$ is a continuous map from $(S(E), T_o(S'(E'),S(E)))$ to $(\ell_1, \| \|_1)$. Since $S(E)$ is nuclear and ℓ_1 is a Banach space, this map is nuclear. Thus there is a sequence $a^{(i)}$, $i = 1,2,\cdots$, in ℓ_1, an equicontinuous sequence $(y_n^{(i)})$ in $S'(E') = (S(E), T_o(S'(E'),S(E)))'$, and a sequence $(b_n^{(i)})$, $i = 1,2,\cdots$, from a bounded set B in ℓ_1, with $(\ell_1)_B$ complete such that $(y_n)[(x_n)] = (\langle x_n, y_n \rangle) = \Sigma_{i=1}^{\infty} a^{(i)} \langle (x_n), (y_n^{(i)}) \rangle (b_n^{(i)})$, where the series converges in the p_B topology on $(\ell_1)_B$ and thus in the norm topology on ℓ_1. Taking $(x_n) = I_n(x)$ in the last equation gives $\langle x, y_n \rangle = \Sigma_{i=1}^{\infty} a^{(i)} \langle x, y_n^{(i)} \rangle b_n^{(i)}$ for all x in E. Thus $y_n = \Sigma_{i=1}^{\infty} a^{(i)} b_n^{(i)} y_n^{(i)}$ for each $n = 1,2,\cdots$. Note that the $(y_n^{(i)})$, being equicontinuous, are members of the absolutely convex hull Γ of the solid hulls of a finite number of points $(z_n^{(1)}), \cdots (z_n^{(k)})$ in $S'(E')$. Γ is weakly complete since each of the $(z_n^{(i)})$ are. We now have $(y_n) = \Sigma_{i=1}^{\infty} a^{(i)} (b_n^{(i)} y_n^{(i)})$. Indeed, the partial sums are weakly Cauchy by Lemma 4.1 since the sequence $(b_n^{(i)} y_n^{(i)})$, $i = 1,2,\cdots$, is weakly bounded and $a^{(i)}$, $i = 1,2,\cdots$, is in ℓ_1. Since the partial sums all lie in $r\Gamma$ for some fixed $r > 0$, the series converges since Γ is weakly complete.

Because $(y_n^{(i)})$ is in Γ, we can write $(y_n^{(i)}) = \Sigma_{j=1}^{k} a^{(i,j)} (c_n^{(i,j)} z_n^{(j)})$ where $\Sigma_{j=1}^{k} |a^{(i,j)}| \leq 1$ for each $i = 1,2,\cdots$, and $(c_n^{(i,j)}) \in rB_\infty$ for each $i = 1,2,\cdots$, and for each $j = 1,2,\cdots,k$. Thus,

$$(y_n) = \sum_{i=1}^{\infty} a^{(i)} (b_n^{(i)} y_n^{(i)})$$

$$= \sum_{i=1}^{\infty} a^{(i)} \sum_{j=1}^{k} a^{(i,j)} (c_n^{(i,j)} b_n^{(i)} y_n^{(i)})$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{\infty} a^{(i)} a^{(i,j)} (c_n^{(i,j)} b_n^{(i)} y_n^{(i)})$$

$$= \sum_{j=1}^{k} (\alpha_n^{(j)} z_n^{(j)})$$

where $(\alpha_n^{(j)}) = (\sum_{i=1}^{\infty} a^{(i)} a^{(i,j)} c_n^{(i,j)} b_n^{(i)})$ is in ℓ_1 for each $j = 1,2,\cdots,k$, since $a^{(i)}$, $i = 1,2,\cdots$, is in ℓ_1, $(b_n^{(i)})$, $i = 1,2,\cdots$, is a bounded sequence in ℓ_1, and $|a^{(i,j)} c_n^{(i,j)}| \leq 1$ for all i,j,n. Let $\alpha_n = \max_{j=1,2,\cdots,k} |\alpha_n^{(j)}|$. Then (α_n) is in ℓ_1 and from the last equation

$$(y_n) = \sum_{j=1}^{k} (\alpha_n \circ \frac{\alpha_n^{(j)}}{\alpha_n} \circ z_n^{(j)}) \quad \text{(where we set } \frac{\alpha_n^{(j)}}{\alpha_n} = 0 \text{ if } \alpha_n = 0\text{)}$$

$$= (\alpha_n \sum_{j=1}^{k} \frac{\alpha_n^{(j)}}{\alpha_n} z_n^{(j)})$$

$$= (\alpha_n z_n) \quad \text{for some } (z_n) \text{ in } S^0(E^0).$$

This completes the proof.

Remark. Since a subspace of a nuclear space is nuclear ([15], p. 103), $S(E)$ is nuclear with the normal topology when $S^0(E^0) \subset \ell_1 \circ S^0(E^0)$ and $S'(E')$ is solid even if we do not assume that $S(E)$ is solid.

CHAPTER 4

THE VECTOR SEQUENCE SPACE $\lambda(E,T)$

In this chapter, we study a v.s.s. $\lambda(E,T)$ that is formed in a natural way from a solid s.s.s. λ and an l.c.t.v.s. (E,T). These spaces include as special cases the spaces $\lambda(E)$ of Pietsch [14], and the spaces $\mathcal{F}(E)$ of Cac [1].

If λ is provided with a solid topology T_λ, then a solid topology $T_\lambda \circ T$ is defined on $\lambda(E,T)$ in a natural way. In Section 2, we consider the l.c.t.v.s.s. $(\lambda(E,T), T_\lambda \circ T)$. In Section 3, we examine this space in the particular case when E is normed.

1. The Space $\lambda(E,T)$

If (E,T) is an l.c.t.v.s. with topology T determined by seminorms p_α, $\alpha \in D$, and if λ is a solid s.s.s., we let

$$\lambda(E,T) = \{(x_n) \in \omega(E) : (p_\alpha(x_n)) \in \lambda \text{ for each } \alpha \in D\} .$$

Since λ is solid, it follows that $\lambda(E,T)$ is a v.s.s. which is independent of the family of seminorms chosen to determine T.

The following spaces are all examples of the v.s.s. $\lambda(E,T)$:

$c_o(E,T)$, the space of all sequences in (E,T) that converge to θ ;

$\ell_\infty(E,T)$, the space of all bounded sequences in (E,T) ;

$\ell_p(E,||\ ||) = \{(x_n) \in \omega(E) : \Sigma\ ||x_n||^p < \infty$, where $(E,||\ ||)$ is a normed linear space and $1 \leq p < \infty$;

$\lambda(K,T) = \lambda$;

$\omega(E,T) = \omega(E)$; and,

$\phi(E,T) = \phi(E)$.

$\lambda(E,T)$ has only been defined for those s.s.s. λ which are solid. We can therefore define $c(E,T)$ to be the space of all convergent sequences in (E,T).

If E is a member of a dual pair $\langle E,E' \rangle$ of vector spaces and if T is the polar topology on E of $\mathscr{A}$-convergence, it is convenient to express $\lambda(E,T)$ by the equality

$$\lambda(E,T) = \{(x_n) \in \omega(E) : (|\langle x_n, A \rangle|) \in \lambda \text{ for each } A \in \mathscr{A}\} .$$

where $|\langle x,A \rangle| = \sup_{y \in A} |\langle x,y \rangle|$ for each x in E and A in $\mathscr{A}$. If we take $\mathscr{A}$ to be the class of all singleton subsets of E' and if we assume that λ is α-perfect, then $\lambda(E,T) = \lambda(E,T_s(E',E))$ is the "generalized perfect space $\lambda(E)$" introduced by Pietsch ([14], p. 30). We observe that the notation $\lambda(E)$ is consistent in the case when $\lambda = \phi$ or $\lambda = \omega$ with the notation $\phi(E)$, $\omega(E)$ introduced in Chapter 1.

If T is the polar topology on E of $\mathscr{A}$-convergence with respect to the dual pair $\langle E,E' \rangle$, it is not in general true that $\lambda(E,T)^* = \lambda^*(E',T')$ for some polar topology $T' = T'(E,E')$ on E'. However, if we let

$$\xi(E',\mathscr{A}) = \bigcup_{A \in \mathscr{A}} \{(y_n) \in \omega(E') : y_n \in A \text{ for each } n = 1,2,\ldots\} ; \text{ and,}$$

$$\lambda^* \cdot \xi(E',\mathscr{A}) = \{(c_n y_n) : (c_n) \in \lambda^* , (y_n) \in \xi(E',\mathscr{A})\}$$

then the following proposition holds.

Proposition 1.1 Let $\langle E,E' \rangle$ be a dual pair of vector spaces, let T be the polar topology on E of $\mathscr{A}$-convergence, and let T^* be the polar topology on E' of convergence on the class of all

bounded subsets of (E,T) . If λ is a solid s.s.s. then:

(1) $\lambda(E,T)^* \subset \lambda^*(E',T^*)$; and,

(2) $\lambda^{**}(E,T) = [\lambda^* \circ \xi(E',\mathscr{A})]^*$

Proof. (1) If (y_n) is in $\lambda(E,T)^*$, then $\Sigma\, |\langle x_n,y_n\rangle| < \infty$ for all (x_n) in $\lambda(E,T)$. In particular, $\Sigma\, |\langle c_n z_n,y_n\rangle| < \infty$ for all (c_n) in λ and (z_n) in $\ell_\infty(E,T)$. Thus, $\Sigma\, |c_n|\, |\langle B,y_n\rangle| < \infty$ for each bounded set B in (E,T) and each sequence (c_n) in λ ; otherwise, there would be a sequence (z_n) in $\ell_\infty(E,T)$ such that $\Sigma\, |c_n|\, |\langle z_n,y_n\rangle|$ diverged. Thus, $(|\langle B,y_n\rangle|)$ is in λ^* for each bounded set B in (E,T) . Therefore, (y_n) is in $\lambda^*(E',T^*)$.

(2) Recalling the appropriate definitions, we have:

$(x_n) \in \lambda^{**}(E,T) \iff (|\langle x_n,A\rangle|) \in \lambda^{**}$ for all $A \in \mathscr{A}$

$\iff \Sigma\, |c_n|\, |\langle x_n,A\rangle| < \infty$ for all $A \in \mathscr{A}$ and $(c_n) \in \lambda^*$

$\iff \Sigma\, |c_n|\, |\langle x_n,y_n\rangle| < \infty$ for all $(y_n) \in \xi(E',\mathscr{A})$ and $(c_n) \in \lambda^*$

$\iff \Sigma\, |\langle x_n,c_n y_n\rangle| < \infty$ for all $(y_n) \in \xi(E',\mathscr{A})$ and $(c_n) \in \lambda^*$

$\iff (x_n) \in [\lambda^* \circ \xi(E',\mathscr{A})]^*$

Corollary 1.2 If λ is a solid s.s.s. such that $\lambda^* = \ell_1$, then $\lambda(E,T)^* = \ell_1(E',T^*)$. Consequently, $c_o(E,T)^* = c(E,T)^* = \ell_\infty(E,T)^* = \ell_1(E',T^*)$.

Proof. Since $\lambda^* = \ell_1$, $\lambda \subset \lambda^{**} = \ell_\infty$. Thus, $\lambda(E,T) \subset \ell_\infty(E,T)$. Therefore, $\lambda(E,T)^* \subset \ell_\infty(E,T)^* \subset \ell_1(E',T^*)$, where the last inclusion

follows immediately from the definitions. The reverse inclusion holds by (1) of Proposition 1.1. The final equality follows from the inclusions $c_o(E,T) \subset c(E,T) \subset \ell_\infty(E,T)$ and the fact that $c_o^* = \ell_\infty^* = \ell_1$.

Remarks. 1. If λ is α-perfect, then $\lambda(E,T)$ is the β-dual of a solid v.s.s. by (2) of Proposition 1.1. Thus, $\lambda(E,T)$ is α-perfect if λ is α-perfect. In particular, the "generalized perfect spaces $\lambda(E)$" of Pietsch are α-perfect. Moreover, if we take $S(E) = \lambda(E)$ and $S'(E')$ to be the linear hull of $\lambda^* \circ \xi(E', \mathscr{A})$ where $\mathscr{A}$ is the class of all singletons in E', then the topologies considered by Pietsch ([12], p. 21) are polar topologies of the dual pair $\langle S(E), S'(E') \rangle$. Note, in this case, that $S(E) = S'(E')^*$ by (2) of Proposition 1.1, and that $S(E)$ and $S'(E')$ are solid. Also, $S(E) = \lambda(E)$ is of Köthe type when provided with one of the topologies of Pietsch. Thus the results of Chapters 2 and 3 can be applied here.

2. In Proposition 1.1, we see that $T^* \subset T_b(E',E)$ always holds, while if $T \subset T_k(E',E)$, then $T^* = T_b(E',E)$.

2. The Space $(\lambda(E,T), T_\lambda \circ T)$

If (E,T) is an l.c.t.v.s. with topology T determined by the seminorms p_α, $\alpha \in D$, and if (λ, T_λ) is an l.c.t.s.s.s. with solid topology T_λ determined by the absolutely monotone seminorms q_β, $\beta \in G$, we let $T_\lambda \circ T$ be the topology on $\lambda(E,T)$ determined by the seminorms $q_\beta \circ p_\alpha$, $\alpha \in D$, $\beta \in G$, where $(q_\beta \circ p_\alpha)[(x_n)]$ is the seminorm q_β applied to the sequence $(p_\alpha(x_n))$ in λ. Because T_λ is solid, it turns out that $T_\lambda \circ T$ is independent of the seminorms chosen to represent T and of the absolutely monotone seminorms chosen to represent T_λ. Also, $(\lambda(E,T), T_\lambda \circ T)$ is a

solid l.c.t.v.s.s. with $T_\lambda \circ T$ solid.

$\lambda(E,T)$ can of course be provided with l.c.t.v.s.s. topologies other than $T_\lambda \circ T$. In particular, the topologies studies by Pietsch on $\lambda(E)$ (see Remark 1 at the end of the previous section) are not of this type.

Proposition 2.1 $(\lambda(E,T), T_\lambda \circ T)$ is of Köthe type if and only if (λ, T_λ) is of Köthe type.

Proof.

Necessity. If $(\lambda(E,T), T_\lambda \circ T)$ is of Köthe type then by the remark following Theorem 9.1 in Chapter 2, the absolutely monotone seminorms $q_\beta \circ p_\alpha$, $\alpha \in D, \beta \in G$, must satisfy the seminorm condition (S) of Theorem 9.1. Choose a non-zero vector x in E and let p_α be such that $p_\alpha(x) \neq 0$. Then for each (a_n) in λ and $\beta \in G$ we have

$$q_\beta \circ p_\alpha[(a_n x)] = \lim_{i \to \infty} q_\beta \circ p_\alpha[(a_n x)(\leq i)] \text{ ; i.e.,}$$

$$q_\beta[(p_\alpha(a_n x))] = \lim_{i \to \infty} q_\beta[(p_\alpha(a_n x))(\leq i)]. \text{ Thus}$$

$$p_\alpha(x)\, q_\beta[(a_n)] = p_\alpha(x) \lim_{i \to \infty} q_\beta[(a_n)(\leq i)].$$

Dividing by $p_\alpha(x)$, we see that the seminorms $q_\beta, \beta \in G$ satisfy the seminorm condition (S) of Theorem 9.1. Since $(K, (T_\lambda)_j)' = K$ for each $j = 1, 2, \ldots,$ it follows from Theorem 9.1 that (λ, T_λ) is of Köthe type.

Sufficiency. Suppose (λ, T_λ) is of Köthe type. Then T_λ is the topology of $\mathcal{M}$-convergence with respect to a dual pair $\langle \lambda, \lambda' \rangle$ of s.s.s., where λ' can be assumed solid since T_λ is solid and $\mathcal{M}$ is a class of solid subsets of λ'. Let $E' = (E,T)'$ and let T

be the polar topology of $\mathscr{A}$-convergence with respect to the dual pair $\langle E, E^0 \rangle$, where $\mathscr{A}$ is a class of subsets of E^0 such that $\bigcup_{A \in \mathscr{A}} A = E^0$ and such that for each pair A_1, A_2 of sets in $\mathscr{A}$, there is a set A_3 in $\mathscr{A}$ such that $A_1 + A_2 \subset A_3$. Then $\xi(E^0, \mathscr{A})$ is a v.s.s. over E^0. We let $\mathscr{M} \circ \mathscr{A} = \{M \circ A : M \in \mathscr{M}, A \in \mathscr{A}\}$ where $M \circ A = \{(m_n y_n) : (m_n) \in M \text{ and } y_n \in A \text{ for each } n = 1,2,\ldots\}$. Then it follows that $T_\lambda \circ T$ is the topology of $\mathscr{M} \circ \mathscr{A}$-convergence with respect to the dual pair $\langle \lambda(E,T), \lambda^\times \circ \xi(E^0, \mathscr{A}) \rangle$. Thus $(\lambda(E,T), T_\lambda \circ T)$ is of Köthe type.

Remark. We could have proved the sufficiency in Proposition 2.1 more easily by applying Theorem 9.1. The purpose of the proof we have given is to introduce notation that will be of use later on.

Proposition 2.2 Let $\lambda_a = (\lambda, T_\lambda)_a$. Then $(\lambda(E,T), T_\lambda \circ T)_a = \lambda_a(E,T)$.

Proof. Note that λ_a is solid since T_λ is solid and thus $\lambda_a(E,T)$ is defined. Let $(x_n) \in (\lambda(E,T), T_\lambda \circ T)_a$. Then $(p_\alpha(x_n)) \in \lambda$ and $q_\beta \circ p_\alpha[(x_n)(> i)] \to 0$ as $i \to \infty$ for each $\alpha \in D, \beta \in G$. Thus, $q_\beta[(p_\alpha(x_n))(> i)] \to 0$ as $i \to \infty$ for each $\alpha \in D, \beta \in G$; i.e. $(p_\alpha(x_n)) \in \lambda_a$ for each $\alpha \in D$. Thus $(x_n) \in \lambda_a(E,T)$. The reverse inclusion follows directly by the same type of seminorm argument.

Corollary 2.3 $(\lambda(E,T), T_\lambda \circ T)_a = \lambda(E,T)$ if and only if $(\lambda, T_\lambda)_a = \lambda$.

Since $T_\lambda \circ T$ is solid, P_j is $T_\lambda \circ T$, $(T_\lambda \circ T)_j$ continuous for each j. From this, we can conclude (as in the proof of

Theorem 9.1 in Chapter 2) that $(\lambda(E,T), T_\lambda \circ T)' \supset \phi(E')$ where $E' = (E,T)^0$. Using this fact, the next corollary follows from Corollary 2.3 and from Theorem 9.2 in Chapter 2.

Corollary 2.4 $(\lambda(E,T), T_\lambda \circ T)^0$ is a v.s.s. over E' if and only if $(\lambda, T_\lambda)^0$ is a s.s.s. and $E' = (E,T)'$.

Remark. If $(\lambda(E,T), T_\lambda \circ T)^0$ is a v.s.s. over E', then $(\lambda(E,T), T_\lambda \circ T)$ is of Köthe type. Using the notation in the proof of Proposition 2.1, we thus have

$$\lambda' \circ \xi(E', \mathscr{A}) \subset (\lambda(E,T), T_\lambda \circ T)^0 \subset \lambda(E,T)^* .$$

3. The Spaces $(\lambda(E, \|\ \|), T_\lambda \circ \|\ \|)$

Let E be a normed linear space with norm topology denoted by $\|\ \|$ and with unit ball B. Let $(E', \|\ \|')$ be the norm dual of E and let B^0 denote the unit ball in E'. In this section, we study the class $\lambda(E, \|\ \|)$ of v.s.s. where λ is solid.

Proposition 3.1

(1) $\lambda(E, \|\ \|) = \lambda \circ B = \{(a_n z_n) \mid (a_n) \in \lambda\ ,\ z_n \in B\ ,$ for $n = 1,2,\cdots\}$,

(2) $\lambda(E, \|\ \|)^* = \lambda^*(E', \|\ \|')$.

Proof. (1) From the definition in Section 2, we see that $\lambda(E, \|\ \|) = \{(x_n) : (\|x_n\|) \in \lambda\ ,\ x_n \in E\}$. If (a_n) is in λ and z_n, $n = 1,2,\cdots$, is a sequence in B, we have $\|a_n z_n\| \leq |a_n|$ for all n. Thus, $(\|a_n z_n\|)$ is in λ, and the inclusion $\lambda(E, \|\ \|) \supset \lambda \circ B$ follows. On the other hand, if (x_n) is in $\lambda(E, \|\ \|)$, let $c_n = \|x_n\|$ if $\|x_n\| \neq 0$ and $c_n = 1$ if

$||x_n|| = 0$. Let $z_n = x_n/c_n$. Then, $(x_n) = (||x_n|| \cdot z_n)$ is in $\lambda \cdot B$ since $||x_n||$ is in λ .

(2) Note that $\lambda^*(E',|| \ ||') = \lambda^* \cdot B'$ by part (1). For all $(a_n z_n)$ in $\lambda \cdot B$ and all $(b_n w_n)$ in $\lambda^* \cdot B'$ we have $\Sigma \ | <a_n z_n, b_n w_n> | \leq \Sigma \ |a_n b_n| < \infty$. Thus, $\lambda(E,|| \ ||)^* \supset \lambda^*(E',|| \ ||')$. The reverse inclusion holds by (2) of Proposition 1.1.

Proposition 3.2 Let $T_k = T_k(\lambda^*,\lambda)$ and let $(E,|| \ ||)$ be barrelled. Then $(\lambda(E,|| \ ||), T_k \circ || \ ||)' = \lambda^*(E',|| \ ||')$.

Proof. Since $(\lambda,T_k)' = \lambda^*$ is a s.s.s., $(\lambda(E,|| \ ||), T_k \circ || \ ||)'$ is a v.s.s. over E' by Corollary 2.4. Since $(E,|| \ ||)$ is barrelled, $|| \ ||$ is determined by the subset B' of E' . Letting $\mathcal{A} = \{nB' : n = 1,2,\ldots\}$, we have by the remark following Corollary 2.4 that $\lambda^* \cdot \xi(E',\mathcal{A}) = \lambda^* \cdot B' \subset (\lambda(E,|| \ ||) , T_k \circ || \ ||)' \subset \lambda(E,|| \ ||)^*$. Since $\lambda^* \cdot B' = \lambda(E,|| \ ||)^*$, equality follows.

Remark. From the previous proposition, we see that if $(E,|| \ ||)$ is barrelled, $(\ell_p(E,|| \ ||), || \ ||_p \circ || \ ||)' = \ell_q(E',|| \ ||')$ for $1 \leq p < \infty$ where $(|| \ ||_p \circ || \ ||)[(x_n)] = (\Sigma \ ||x_n||^p)^{1/p}$. Also, $(c_o(E,|| \ ||), || \ ||_\infty \circ || \ ||)' = \ell_1(E',|| \ ||')$, where $(|| \ ||_\infty \circ || \ ||)[(x_n)] = \sup_n ||x_n||$.

We conclude this section with a few results on the dual pair $<\lambda(E,|| \ ||), \lambda^*(E',|| \ ||')>$.

Proposition 3.3 If $(E,|| \ ||)$ is barrelled and $T_b = T_b(\lambda^*,\lambda)$, then $T_b \circ || \ ||$ is the strong topology on $\lambda(E,|| \ ||)$ with respect to the dual pair $<\lambda(E,|| \ ||), \lambda^*(E',|| \ ||')>$.

Proof. Since $(E,||\ ||)$ is barrelled, the weakly bounded subsets of E' are strongly bounded, i.e. are $||\ ||'$ bounded. Thus we wish to show that each weakly bounded subset of $\lambda^*(E',||\ ||')$ is contained in $M \circ B'$ for some weakly bounded M in λ^*.

Let X be a weakly bounded subset of $\lambda^*(E',||\ ||')$. Each member of X can be written as $(b_n y_n)$ where (b_n) is in λ^* and $||y_n|| = 1$ for all n. Since X is weakly bounded, it is strongly bounded by Theorem 3.1 in Chapter 3. In particular, it is bounded on all sets of the form $(a_n) \circ B$ where (a_n) is in λ. Thus, if M denotes the set of all (b_n) in λ^* obtained from X in the above manner, and Y the corresponding set of all (y_n), we have

$$\sup_{\substack{(b_n)\in M \\ (y_n)\in Y \\ x_n \in B}} \left| \sum_{n=1}^{\infty} \langle a_n x_n, b_n y_n \rangle \right| < \infty$$

The absolute value can clearly be taken inside the sum. Also, for each y_n there is an x_n in B such that $|\langle x_n, y_n \rangle| \geq \frac{1}{2}||y_n|| = \frac{1}{2}$. Thus

$$\sup_{(b_n)\in M} \sum_{n=1}^{\infty} |a_n b_n| < \infty .$$

Thus, M is weakly bounded and we have $X \subset M \circ B'$ as required.

Choose x in E and y in E' such that $\langle x,y \rangle = 1$. We let:

$P_y : \lambda(E,||\ ||) \to \lambda$ be defined by $P_y[(x_n)] = (\langle x_n, y \rangle)$; and,

$I_x : \lambda \to \lambda(E,||\ ||)$ be defined by $I_x[(a_n)] = (a_n x)$.

Note that $P_y \circ I_x$ is the identity mapping on λ. Thus, the

results on projection maps in Section 2 of Chapter 2 can be applied. Maps $P_x^0 : \lambda^*(E^0, ||\ ||^0) \to \lambda^*$ and $I_y^0 : \lambda^* \to \lambda^*(E', ||\ ||')$ can be defined in a similar way. It turns out that $I_x^* = P_x^0$ and $P_y^* = I_y^0$ where $*$ denotes the adjoint map. In particular, all these maps are weakly continuous.

The maps I_x, P_y can be used to relate the properties of $\lambda(E, ||\ ||)$ and λ in the same manner that the maps I_j, P_j are used to relate the properties of $S(E)$ and E. The following theorem illustrates this technique.

Theorem 3.4 Let $T_k = T_k(\lambda^*(E', ||\ ||'), \lambda(E, ||\ ||))$. Then $(\lambda(E, ||\ ||), T_k)$ is barrelled if and only if $(\lambda, T_k(\lambda^*, \lambda))$ is barrelled and $(E, T_k(E', E))$ is barrelled.

Proof.

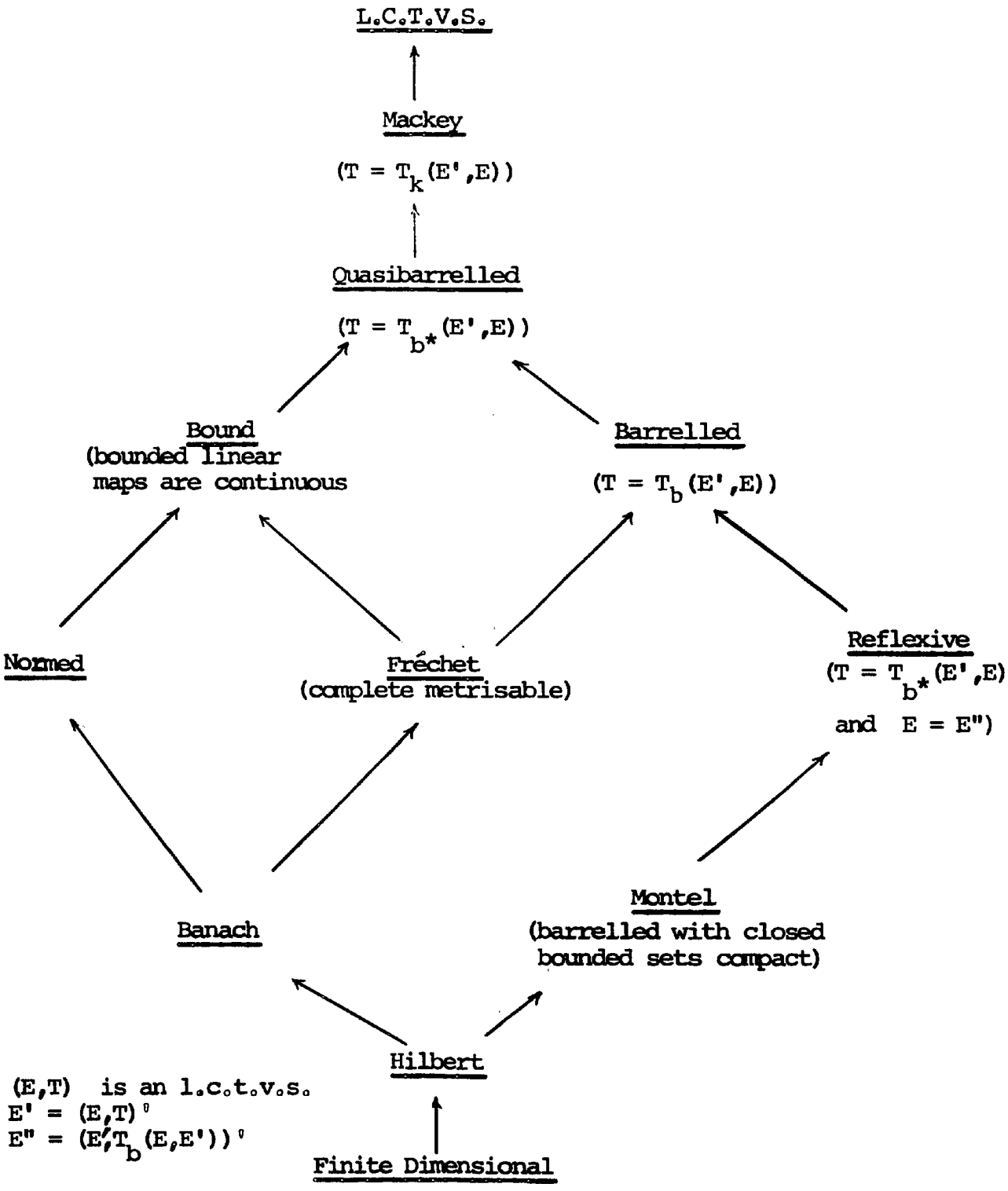
Necessity. Since P_1 is $T_k, ||\ ||$ continuous and open, $(E, T_k(E^0, E))$ is barrelled. To prove that $(\lambda, T_k(\lambda^*, \lambda))$ is barrelled, we show that every absolutely convex weakly bounded subset C of λ^* is weakly relatively compact, i.e. has weakly compact closure. Since I_y^0 is weakly continuous, $I_y^0(C)$ is an absolutely convex weakly bounded subset of $\lambda^*(E^0, ||\ ||')$ and hence is weakly relatively compact. Since P_x^0 is weakly continuous, we have that $C = P_x^0 I_y^0(C)$ is weakly relatively compact, as required.

Sufficiency. Since $(\lambda, T_k(\lambda^*, \lambda))$ is barrelled, $T_k(\lambda^*, \lambda) = T_b(\lambda^*, \lambda)$. Since $(E, T_k(E', E))$ is barrelled, by Proposition 3.3 we have $T_k(\lambda^*, \lambda) \circ ||\ || = T_b(\lambda^*, \lambda) \circ ||\ ||$ is the strong topology on $\lambda(E, ||\ ||)$. But by Proposition 3.2, the dual of $\lambda(E, ||\ ||)$ with respect to this topology is $\lambda^*(E', ||\ ||')$.

Thus, $(\lambda(E,||\ ||), T_k)$ is barrelled.

Corollary 3.5 The dual pair $<\lambda(E,||\ ||), \lambda^*(E',||\ ||^0)>$ is reflexive if and only if the dual pairs $<\lambda,\lambda^*>$ and $<E,E'>$ are reflexive.

TABLE OF LOCALLY CONVEX HAUSDORFF TOPOLOGICAL VECTOR SPACES (L.C.T.V.S.)



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