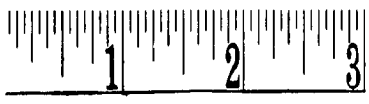
DOCTORAL DISSERTATION SERIES

TITLE Some Geometric Aspects of Linear Transformations

AUTHOR Theodore Ernest Raiford DATE 1932

UNIVERSITY U of Michigan

DEGREE Ph.D. PUBLICATION NO. 382



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Some Geometric Aspects of Linear Transformations

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A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the University of Michigan

Ann Arbor 1932

Table of Contents

Part 1.
Geometry as a basis for the Classical Canonical Form.

Section		Page
1.	Introduction	1.
2.	Sequences and Related Zero Spaces.	2.
3.	Choice of Axes for $A^{\kappa}=o$.	4.
4.	Vectors of an n-space Containing a Zero Space.	5.
5.	Non-singular Transformations.	7.
6.	Vector Coordinate System for any A.	8.
7.	Conclusion.	9.

Part 2.
Invariant Configurations and a Real Canonical Form.

1.	Introduction.	10.
2.	Some Theorems on Invariant Configurations.	12.
3.	Transformations in which the Roots of $(A-\lambda I)=o$ are Complex.	17.
4.	Two Special Problems in 4-space.	22.
5.	Real Canonical Form.	26.

Some Geometric Aspects of Linear Transformations.

For the purposes of this paper a discussion of the geometry of linear transformations is most conveniently presented in two parts. Although some of the underlying ideas have appeared in other publications, Part 1, by a new method of approach, arrives at the well known "classical canonical form" through purely geometric considerations. Theorems established by this method of approach make possible the study in Part 2 of certain invariant configurations and the establishment of a canonical form designated as "the real canonical form".

Part 1.
Geometry as a Basis for the Classical Canonical Form.

1. Introduction. We consider transformations of the type

$$\alpha_i' = \sum_{j=1}^{n} a_{ij}\alpha_j , \qquad i = 1, 2, \cdots n.$$

in which the α's may be considered as the coordinates of a point in a Cartesian system in a space of n dimensions. Such transformations, written for brevity $\alpha' = A(\alpha)$, always carry a straight line into a straight line, parallel lines into parallel lines, and leave unchanged the ratio of two directed distances on two parallel lines.[1] These properties are invaluable in studying certain relations existing between vector quantities and linear transformations.

The variables $(\alpha_1, \alpha_2, \ldots \alpha_n)$ collectively are called a vector and will be designated by the single letter χ.

The n vectors $\chi_1, \chi_2, \ldots \chi_n$, are said to be linearly dependent[2] if there exist n constants $c_1, c_2, \ldots c_n$, not all zero, such that $c_1\chi_1 + c_2\chi_2 + \cdots c_n\chi_n = 0$.

Given then n independent vectors C_i every vector χ may be represented as a linear combination of them, $\chi = \sum_{i=1}^{n} \lambda_i C_i$, and the num-

bers λ_i are called the components of x with respect to the vectors C_i.

A linear transformation may be considered as an operation assigning to a vector x the vector x' and is a function of A which possesses the property $\mathbf{A}(\nu x + \eta y) = \nu\mathbf{A}(x) + \eta\mathbf{A}(y)$. Conversely, if to every vector x a vector $x' = \mathbf{A}(x)$ is assigned in such a way that $\mathbf{A}(\nu x + \eta y) = \nu\mathbf{A}(x) + \eta\mathbf{A}(y)$, then the components of x' are obtained from those of x by a linear transformation of this same type.[3] In fact if $x = \sum_{i=1}^{n} \xi_i C_i$, we have $x' = A\sum_{i=1}^{n} \xi_i C_i = \sum_{i=1}^{n} \xi_i A(C_i)$. If now we have another set of linearly independent vectors D_i there is a non-singular transformation $P = (P_{ij})$ such that $D_i = \sum_{j=1}^{n} p_{ji} C_j$. If $y = \sum_{i=1}^{n} \xi_i D_i = \sum_{i=1}^{n} \eta_i C_i$ we have $\eta_i = \sum_{i=1}^{n} \xi_i p_{ij}$ hence $y = P(x)$. From $A(y) = y'$ and $\mathbf{A}(x) = x'$ we have $PAP^{-1}y = y'$. Thus, a change in the coordinate system results in replacing $\mathbf{A}$ by a so-called "similar" matrix[4] PAP^{-1}.

We shall need the following definitions:

A space is defined as the totality of vectors such that given any two vectors x and y of that totality every linear combination $\theta x + \phi y$, θ and ϕ being any two constants, is again a vector of that totality.

A zero vector of a transformation is defined as any vector which satisfies the equation $\mathbf{A}(x) = 0$.

The number of dimensions of a space is defined as the maximum number of independent vectors of that space.

We shall denote $\mathbf{A}[\mathbf{A}(x)]$ by $\mathbf{A}^2(x)$, and similarly refer to $\mathbf{A}^n$ as the nth power of the matrix A. Such a matrix is said to be singular if the determinant formed by its elements is zero.

2. Sequences and related zero spaces. From the foregoing definitions the totality of all vectors satisfying the equation $\mathbf{A}(x)=0$ constitutes a zero space. There are zero spaces associated with

powers of A other than those defined by this equation. Consider

THEOREM 1. A necessary condition that a sequence of positive integers $a_1, a_2, \ldots a_n$, represent the number of dimensions of the zero spaces S_K, of the successive powers of a singular matrix A of a transformation is that the differences $\Delta a_i = a_{i+1} - a_i$ should be non-negative and form a non-increasing sequence, and that $a_2 \leqq 2a_1$.

Every solution of the equation $A(x) = 0$ is also a solution of the equation $A^2(x) = 0$ since $A^2(x) = A[A(x)] = A(0) = 0$. Similarly every solution of $A^3(x) = 0$ is a solution of $A^2(x) = 0$ and so on for higher powers of A. This imposes the condition $a_1 \leq a_2 \leq \cdots \leq a_n$ and consequently the first differences are either positive or zero.

The remainder of the theorem may be shown by establishing the

LEMMA: If p is the maximum number of new independent zero vectors of A^{K+1} no linear combination of which, say $y = \sum a_i x_i$, is a zero vector of A^K, then the p vectors obtained from applying A to each are zero vectors of A^K and no linear combination of them is a zero vector of A^{K-1}.

The proof of this lemma follows readily from the definition of the terms used in its statement. The maximum number of independent zero vectors of A^{K+1} is a_{K+1} and the maximum number of independent zero vectors of A^K is a_K, and consequently the maximum number , p, of new zero vectors of A^{K+1} such that no linear combination of them is a zero vector of A^K is given by $p = a_{K+1} - a_K$. Similarly the maximum number of zero vectors, p', which belong to A^K such that no linear combination of them is a zero vector of A^{K-1} is $p' = a_K - a_{K-1}$. However if A is applied to any y the resulting vector is a zero vector of A^K since $A^K[A(y)] = A^{K+1}(y) = 0$; but $A^{K+1}[A(y)] = A^K(y) \neq 0$, and consequently none of the zero vectors gotten in this way are zero vectors of A^{K-1}, establishing the truth of the lemma.

The fact that $a_2 \leqq 2a_1$ follows directly from this lemma since the number of zero vectors p gotten by applying A to each of the p zero vectors cannot exceed the maximum number of vectors p' and consequently p is at most equal to p', that is $a_{K+1} - a_K \leqq a_K - a_{K-1}$.

It will be shown later that the conditions of Theorem 1 are also sufficient.

3. Choice of axes for $A^K = 0$. In studying the zero spaces associated with the singular matrix A^K, $K = 1, 2, \ldots n$, we wish to make the simplest choice of coordinate axes that will exhibit its geometric properties. To do this we shall use the

DEFINITION: By a set of "coordinate vectors" is meant any set of n independent vectors, say $i^{(1)}, i^{(2)}, \ldots i^{(n)}$.

Consider the case in which for a sufficiently high power k of A the value of A^k is zero. Let the number of dimensions a_i of the zero spaces S_i corresponding to A^i, be given by the terms of the sequence

i	a_i	Δa_i	$\Delta^2 a_i$
1	2	2	1
2	4	1	
3	5		

Here there are five independent zero vectors of S_3 but only one (in this case) new independent zero vector. Choose then as $i^{(1)}$ the new zero vector of S_3. This means that $i^{(1)}$ is a vector to which A must be applied three times in order to produce a zero vector and consequently all lower powers of A when applied to $i^{(1)}$ produce vectors which are not zero vectors. We have $A^3(i^{(1)}) = 0$ and may set

$$A(i^{(1)}) = i^{(2)}, \qquad A^2(i^{(1)}) = i^{(3)}.$$

These relations also determine one of the zero vectors of S_2 and one of S_1 as is evident from writing

$$A(i^{(2)}) = i^{(3)}, \quad A^2(i^{(2)}) = 0, \quad \text{and} \quad A(i^{(3)}) = 0$$

There remains one new zero vector of S_2. Choose $i^{(4)}$ as this vector

and in a similar way

$$A(i^{(4)}) = i^{(5)}, \qquad A^2(i^{(4)}) = o, \quad \text{and} \qquad A(i^{(5)}) = o.$$

All of the five independent zero vectors of the 5-space are now accounted for and from them we can determine the form of A for which the number of dimensions of the zero spaces of the successive powers of A are the numbers of the given sequence. On choosing any vector

(1) $$x = \alpha_1 i^{(1)} + \alpha_2 i^{(2)} + \alpha_3 i^{(3)} + \alpha_4 i^{(4)} + \alpha_5 i^{(5)}$$

and applying A to x the transformed vector becomes

(2) $$A(x) = x' = \alpha_1 i^{(2)} + \alpha_2 i^{(3)} + \alpha_4 i^{(5)}.$$

Since by α'_k is meant the coefficient of $i^{(k)}$ in the expression for x' one has from the last two equations

(3) $$\alpha'_1 = o, \quad \alpha'_2 = \alpha_1, \quad \alpha'_3 = \alpha_2, \quad \alpha'_4 = o, \quad \alpha'_5 = \alpha_4$$

The singular transformation given by (3) is

(4) $$A = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{Vmatrix}$$

and is indeed a very simple form for this special case. Moreover the method is easily generalized; and since it is always possible to choose the coordinate axes by the foregoing method with the resulting simple form for A the necessary conditions on the sequence as stated in Theorem 1 are also sufficient. Note that the above type of choice always yields an A whose every element a_{ij} is zero except the elements $a_{i(i-1)}\ (i = 2, 3, \cdots n)$ which are either 0 or 1, and that the non-zero rows of A are the coordinate vectors of $A(x)$.

4. Vectors of an n-space containing a zero space. A vector will be defined as a "value vector" if there exists a vector y such that $A(y) = v$. A value vector of A may be a zero vector of A e.g., $v = (0,0,1,0,0)$ and $y = (0,1,0,0,0,)$ for the example above.

Given any two vectors x and y with μ and ρ any two arbitrary constants then $A(\mu x + \rho y) = \mu A(x) + \rho A(y)$ and hence any linear combination of two or more value vectors is also a value vector. The totality of all such value vectors constitutes a space which will be called a "value space".

If a singular transformation A be applied to a vector x then the equation $A(x) = 0$ defines a zero space of a_1 dimensions; if A^2 be applied to x the equation $A^2(x) = 0$ defines a zero space of a_2 dimensions, and so on. Suppose the number of dimensions corresponding to the successive powers of A increases to a_K corresponding to the equation $A^K(x) = 0$ but the number of dimensions corresponding to the equation $A^{K+l}(x) = 0$ for all $l > 0$ remains a_K, then the zero space of a_K dimensions is defined as the "ultimate zero space" of A.

The characterization of a value space is facilitated by

THEOREM 2. Any vector x may be expressed as the sum of a zero vector z of the ultimate zero space of A and a vector v of the value space.

Proof: Given a vector x, form $A(x), A^2(x), \ldots A^n(x)$. For $n+1$ vectors of an n-space there must be at least one linear combination that is zero, and it is therefore possible to write

$$(5) \qquad \alpha_0 x + \alpha_1 A(x) + \alpha_2 A^2(x) + \cdots \alpha_m A^n(x) = 0$$

where not all the α's are zero. If the constant α_0 is different from zero then it follows that x can be expressed as

$$(6) \qquad x = A[\beta_0 x + \beta_1 A(x) + \beta_2 A^2(x) - - - \beta_{n-1} A^{n-1}(x)]$$

which from the definition above says that x is a value vector. If the first term appearing in (5) different from zero is the term A^n, then (5) can be written

$$(7) \qquad A^n[x + \nu_1 A(x) + \nu_2 A^2(x) \cdots \nu_{n-k} A^{n-k}(x)] = 0$$

and consequently

$$(8) \qquad x + \nu_1 A(x) + \nu_2 A^2(x) \cdots \nu_{n-k} A^{n-k}(x)$$

is a zero vector of A^κ. Denote this vector by z. Then $x = z + v$ where v is a value vector given by a quantity, with the signs changed, in parenthesis. The theorem is thus proved.

5. Non-singular transformations. If there is no power κ such that $A^\kappa = 0$ then it becomes necessary to make a choice of coordinate axes for the value space.

Let A be the matrix of the non-singular transformation. We seek a number λ such that when subtracted from each element of the principal diagonal of A (usually described as subtracting the product, λ times the unit matrix I, from the matrix A) there results a matrix A_1 which is singular. Since a singular matrix is one whose determinant is zero such a value can be found by solving the n-th degree equation in λ gotten by equating this determinant to zero, thus, $|A - \lambda I| = 0$. Let λ_1 be the root of this equation. The matrix $A_1 = A - \lambda_1 I$ is then a singular matrix. If it happens that there is some value κ such that $A_1^\kappa = 0$ then the desired choice of coordinate axes for A_1 is known from the previous discussion. If there is no such value κ then there is the question of the simplest choice of axes for the value space still unanswered. We consider then the zero space of A_1 and the value space of A_1 separately as given by

THEOREM 3. The zero space of A_1 and the value space of A_1 are invariant spaces of A.

By applying A to each side of the equation $A_1 = A - \lambda I$ it is seen that A and A_1 are commutative since

$$AA_1 = AA - \lambda AI = (A - \lambda I)A = A_1 A \tag{9}$$

If therefore v is a value vector of A_1, that is $v = A_1(y)$, the result of applying A to this relation gives

$$Av = A[A_1(y)] = A_1[A(y)] \tag{10}$$

showing tha value space of A_1 to be invariant under the transfor-

mation A and similarly for the zero space of A_1.

6. Vector coordinate system for any A. Given then a non-singular transformation A, reduce as indicated by equation $A - \lambda_1 I = A_1$, to a singular transformation having an ultimate zero space of dimensions s_1 and a value space of dimensions $n - s_1$. By Theorem 3 it is possible to treat the value space separately and in exactly the same way as was the matrix of the original transformation. Therefore let A_1 be written as

$$A_1 = \left\| \begin{array}{cccc} 0 & 0 & 0 \cdots & 0 \\ a & 0 & 0 \cdots & 0 \\ 0 & a & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots & 0 \end{array} \right\| = \left\| \begin{array}{c} \\ \\ A_{01} \\ \\ \\ \\ \end{array} \right\|$$

$$\left\| \begin{array}{cccc} b_{s_1+1,s_1+1} & b_{s_1+1,s_1+2} & \cdots & b_{s_1+1,n} \\ b_{s_1+2,s_1+1} & b_{s_1+2,s_1+2} & \cdots & b_{s_1+2,n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n,s_1+1} & b_{n,s_1+2} & \cdots & b_{n,n} \end{array} \right\| = \left\| \begin{array}{c} \\ \\ \\ \\ A_{v1} \end{array} \right\|$$

where a is either zero or 1 as given by §3 and where A_{01} and A_{v1} denote the singular matrix of the ultimate zero space of A_1 and the non-singular matrix of the value space of A_1 respectively. Similarly in the $(n - s_1)$th degree equation $|A_{v1} - \lambda I| = 0$ a value λ'_2 can be found which on being substituted for λ will produce a singular matrix, say A_2. Every vector of this $(n - s_1)$-space can be expressed as the sum of a zero vector of the ultimate zero space of A_2 of dimensions s_2 and a value vector of the value space of A_2 of dimensions $n - s_1 - s_2$. Again in a similar way we designate the ultimate zero space and that of the value space of A_2 by A_{02} and A_{v2} respectively. Since n is finite this process can be repeated until A has been completely expressed by a sequence of ultimate zero spaces and for

each of these a very simple choice of coordinate axes is given by the method of §3. The transformation can therefore be written

(11) $$A = \begin{Vmatrix} \lambda_1 & & & & \\ a & \lambda_1 & & & \\ & a & & & \\ & & \lambda_2 & & \\ & & a & \lambda_2 & \\ & & & & \lambda_\omega \\ & & & & a \quad a\lambda_\omega \end{Vmatrix}$$

where $\lambda_2 = \lambda_1 + \lambda'_2$, $\lambda_3 = \lambda_1 + \lambda'_2 + \lambda'_3$, etc.

From (11) the numbers $\lambda_1, \lambda_2, \ldots, \lambda_\omega$ form the principal diagonal of A and are consequently the roots of $|A - \lambda I| = 0$. This equation is called the characteristic equation[6] which we write

(12) $$D(\lambda) = |A - \lambda I| = (\lambda - \lambda_1)^{s_1}(\lambda - \lambda_2)^{s_2} \cdots (\lambda - \lambda_\omega)^{s_\omega}.$$

Moreover the roots of such an equation of a transformation are independent of the form of the transformation[7] and therefore the values of λ thus obtained are also the roots of the characteristic equation $|A - \lambda I| = 0$, of the original matrix A.

3. Conclusion. The very simple form given by (11) in which any transformation can be written by choosing a coordinate system as here developed is readily recognized as the well known "classical canonical form" and has resulted from a search for the simplest form of coordinate system possible in terms of which certain geometric properties of the transformation may be displayed.

References, Part 1:

1. Graustein, Introduction to Higher Geometry, Chap. XI.
2. Pincherle, Le Operatoni Distributive, p. 5.

3. Klein, Vorlesungen über Höhere Geometrie, p. 97
4. Cf. Kowalewski, Berichte der Königlich-Sächsischen Gesellschaft der Wissenschaften, vol. 69 (1917), p.325
5. Bôcher, Higher Algebra, p. 37
6. Dickson, Modern Algebraic Theories, p. 47
7. Bôcher, Higher Algebra, p. 269

Part 2.
Invariant Configurations and a Real Canonical Form.

1. Introduction. In a comparatively recent publication by Weyl[1] in which he set himself the same problem as that undertaken in Part 1 the same result was arrived at as there exhibited but by quite a different method of approach. Both of these methods, however, having arrived at the simplest form for exhibiting a linear transformation, namely, the classical canonical form, immediately become involved in the use of complex elements both in the algebra and the geometry of certain transformations. This is obvious from the fact that the solution of the equation formed by setting the characteristic determinant[2] $|A - \lambda I|$ equal to zero will in many cases contain complex roots. Given then any transformation $x_i' = \sum_{j=1}^{n} a_{ij} x_j$, $i = 1, 2, \ldots n$, which we write for brevity $x' = A(x)$, where the elements of A are in the field of real numbers, we seek in the second part of this paper the simplest form, $x' = A_r(x)$, $A_r = BAB^{-1}$, for its geometric characterization, such that the elements of A_r, B, and B^{-1} are each also in the field of real numbers.

We shall assume the following:

If the characteristic determinant $D(\lambda) = |A - \lambda I|$ set equal to zero has roots $\lambda_1, \lambda_2, \ldots \lambda_\omega$, of multiplicity

$n^{(1)}$, $n^{(2)}$, ... $n^{(\omega)}$, arranged for convenience so that $n^{(i)} \geq n^{(i+1)}$, and if G_j is the highest common factor of all minors of order n-j of this determinant, then

$$(1) \qquad D_j(\lambda) = \frac{G_{j-1}}{G_j} = (\lambda - \lambda_1)^{n_j^{(1)}} (\lambda - \lambda_2)^{n_j^{(2)}} \ldots (\lambda - \lambda_\omega)^{n_j^{(\omega)}}$$

$$j = 1,2, \ldots n.$$

$$= \lambda^{n_j} - \alpha_{1j} - \alpha_{2j}\lambda - \alpha_{3j}\lambda^2 \ldots\ldots \alpha_{nj}\lambda^{n_j - 1}$$

defines the j-th invariant factor of the matrix A.[3] The polynomial formed by replacing λ by A in (1) and equating to zero, if not a mere constant, is called the j-th characteristic polynomial of the transformation.

Let A be the classic canonical form of the matrix. Let $\lambda_2 = \lambda_1 + \lambda'_2$, $\lambda_3 = \lambda_1 + \lambda'_2 + \lambda'_3$, etc, and define the principal minor of $(A - \lambda_1 I)$ for which each $a_{ii} = 0$ as A_1, the principal minor of $(A - \lambda'_2 I)$ as A_2, etc. Then it can be readily shown[4] that the rank of the product $\prod_{i=1}^{\omega} (A - \lambda_i I)^{K_i}$ is equal to

$$(2) \qquad n - \sum_{i=1}^{\omega} \sum_{K=1}^{p} h_i^{(p)}$$

where $h_i^{(p)}$ is the number of invariant factors of A_i which are divisible by $(\lambda - \lambda_i)^p$.

Some of the above and other properties are easily followed from an example. For this purpose, let the matrix from a transformation in which

$$D_1(A) = (A - \lambda_1 I)^4 (A - \lambda_2 I)^2 (A - \lambda_3 I)$$

$$D_2(A) = (A - \lambda_1 I)^2 (A - \lambda_2 I)$$

$$D_3(A) = (A - \lambda_1 I)$$

$$D_4(A) = D_5(A) = \ldots\ldots \quad D_n(A) = 1$$

be written in the classic canonical form

$$(3) \qquad A = \left\| \begin{array}{ccccccc|ccc} \lambda_1 & & & & & & & & & \\ 1 & \lambda_1 & & & & & & & & \\ & 1 & \lambda_1 & & & & & & & \\ & & 1 & \lambda_1 & & & & & & \\ & & & 0 & \lambda_2 & & & & & \\ & & & & 1 & \lambda_2 & & & & \\ & & & & & 0 & \lambda_3 & & & \\ \hline & & & & & & & \lambda_1 & & \\ & & & & & & & 1 & \lambda_1 & \\ & & & & & & & & 0 & \lambda_2 \\ \hline & & & & & & & & & \lambda_1 \end{array} \right\|$$

A transformation $x' = A(x)$ is said to define a linear n-space and is designated a zero space or a value space according as x' is equal to or different from zero. In the notation above, A_1, A_2, ... A_ω, are designated as the successive zero spaces of A, of dimensions $n^{(1)}$, $n^{(2)}$, etc., and the corresponding complementary minors are designated the successive value spaces.

Any vector x can be very simply expressed in the vector coordinate system in terms of n independent vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$, and this will be the notation of this paper, rather than denoting x as a point as in the study of collineations.[5]

2. Some Theorems on Invariant Configurations. Given a transformation of matrix A in n-space, any vector of the n-space is transformed by this transformation into a vector of the n-space, that is, the n-space is invariant under the transformation A. The transformation $P(A)\mathbf{x}$, where P(A) is a polynomial in A, in general defines a new matrix A', say, which when applied to any vector x gives again a vector of the n-space. Moreover, if P(A) = D(A), then since every square matrix satisfies its characteristic equation, A' is identically zero and consequently the n-space of A' is a zero space. The

equation of lowest degree of the form $P(A) \equiv 0$ is $D_1(A)$.[6] Let m denote the degree of $D_1(A)$. If $D_1(A) \neq D(A)$, then the space defined by the corresponding m independent coordinate vectors is an invariant m-space of the n-space. In particular, if $D_1 = D_2 = \ldots = D_\omega$ then every i-space, $i = 1, 2, \ldots \omega$ of this n-space is invariant.

In order to discuss further subspaces other theorems are necessary.

Theorem 1. To each real invariant direction of a given transformation there corresponds a linear factor $(A - \lambda_1 I)$, λ_1 real, of $D(A)$.

Given a linear transformation $x' = A(x)$ which in general transforms a vector x into a vector x' which differs from x in magnitude and direction. If there are vectors whose direction is invariant, then the components of x' must bear a fixed ratio, say λ, to the corresponding components of x. In order to find such vectors it is necessary to solve the equations $(A - \lambda I)x = 0$. If the solution is to be non-trivial, the determinant $|A - \lambda I|$ must be equal to zero, that is, λ must be a root of the characteristic equation

$$(4) \qquad D(\lambda) = (-1)^n \lambda^n - a_1' \lambda^{n-1} + a_2' \lambda^{n-2} \ldots\ldots a_n = 0$$

The equation

$$(5) \qquad A(x) - \lambda_i(x) = 0$$

is the matrix equation defining an invariant direction corresponding to λ_i and any vector x of invariant direction must satisfy this relation. Obviously, any scalar multiple of x will also

lie along this invariant direction.

Theorem 2. Corresponding to each linear factor $(A - \lambda_i I)$ of D(A) there is an invariant direction.

Making use of the fact that the square matrix of a transformation satisfies its characteristic equation, it follows from (4) that

(6) $\quad (A^n - a_1 A^{n-1} + a_2 A^{n-2} - \cdots\cdots \quad a_n I) = 0$

If there is a real root of the characteristic equation, this equation can be written

(7) $\quad (A - \lambda_i I)\, F(A) = 0$

Assume, for the present, that each of these factors is different from zero. Applying F(A) to a vector x' and representing this vector F(A)x' by x, (5) becomes

(8) $\quad (A - \lambda_i I)x_i = 0 \quad$ or $\quad A(x_i) - \lambda_i(x_i) = 0$

from which one concludes that there is at least one real invariant direction corresponding to λ_i.

In equation (8) the rank of the matrix $(A - \lambda_i I)$ is, from (2), at most equal to n-1, being exactly n-1 in case the characteristic equation has distinct roots. When the rank of the matrix is n-r, then there are r independent equations in n unknowns defining an r-space. The space defined always contains an invariant direction since equation (8) is satisfied.

Since equation (8), which is true for some x, is also satisfied by A(x), the above argument covers the cases 0·r and 0·0, that is, those cases in which either one or both factors are zero.

In the case r=n, which could happen only in case $D_1 = D_2 =$

... D_ω, the characteristic equation has an n-tuple root, and hence every vector of the n-space satisfies equation (8).

Theorem 3. There is at least one invariant plane corresponding to each quadratic factor $(A^2 - pA + qI)$ of $D(A)$.

Let equation (6) be written as the product of two factors

$$(9) \qquad (A^2 - pA + qI)\cdot F(A) = 0$$

and assume, for the present, that each factor is different from zero. Let $F(A)$ be applied to any vector x_1' and represent the vector $F(A)x_1'$ by x_1. The resulting transformation

$$(10) \qquad (A^2 - pA + qI)\cdot x_1 = 0$$

gives n equations in n unknowns. The rank of the matrix formed by the quadratic factor is at most equal to n-2, being exactly n-2 in case the roots of the characteristic equation are real and distinct. When the rank is exactly n-2 the number of equations is n-2, and n-2 independent equations in n-space define an ordinary plane. The plane can also be shown to be invariant, for equation (10) is also satisfied by $A(x)$, the vector into which x is transformed by A.

When the rank of the matrix is less than n-2, the number of equations is less than n-2, and hence defines a hyperplane. This hyperplane is invariant by the same argument as above. Moreover, it contains an invariant plane, for take a vector x which does not belong to an invariable direction and the vector into which x is transformed by A, and form the sums $\mu x + \rho A(x)$, μ and ρ being variable scalars. These determine a plane. On applying A to $\mu x + \rho A(x)$, one obtains

$$(11) \qquad A[\mu x + \rho A(x)] = \mu A(x) + \rho A^2(x)$$

which, on substituting the value of $A^2(x)$ from (10), becomes

(12) $$(\nu + \rho p)\cdot A(x) - \rho q x = o$$

Since this expression is still a linear combination of x and A(x) the vector x remains in the same plane and the plane is therefore invariant.

More generally, when the rank of the matrix formed by the quadratic factor of (9) is r, where r is less than n, the matrix equation defines an invariant r-space. Beginning with an R-space for R = r-2, interpretation of the (r-1)-space can be given, from this the (r-2)-space can be characterized, and so on up to [(r+n) - (r+1)]-space.

Consider now the case in which $F(A) \equiv 0$ and the rank of the quadratic factor is r, $r \leq (n-2)$. Since $F(A) \equiv 0$, and consequently $F(A)x_1' = 0$, choose any vector $x \equiv F(A)x_1'$, and here again, on applying to x, the transformation defined by the quadratic factor of (9), there are r independent equations defining an n-r invariant space, where $n-r \geq 2$, thus reducing to the case already discussed.

Finally, since the equation of minimum degree satisfied by the matrix A is $D_1(A)$, any factor containing $D_1(A)$ will be of rank zero. Hence, either $D_1 = D_2 = \ldots D_\omega$ or else the quadratic factor must be identical with D_1 and in either case, as has already been shown, there is an invariant plane.

The converse of Theorem 3 is also true.

Theorem 4. Every vector which lies in an invariant plane must satisfy an equation

(13) $$(A^2 - pA + qI)x = 0.$$

Choose any vector x_1, not of invariant direction. Then x_1 and $A(x_1)$ determine a plane. By hypothesis, the plane is invariant. Hence, upon operating on x_1 a second time, one has $A[A(x_1)]$ which must again define a vector lying in the same plane, namely

$$A[A(x_1)] = pA(x_1) - q(x_1). \tag{14}$$

However, any vector x can be expressed in terms of these two vectors, and therefore one can write the three relations

$$\begin{aligned} x &= \delta(x_1) + \sigma A(x_1) \\ A(x) &= \delta A(x_1) + \sigma A^2(x_1) \\ A^2(x) &= \delta A^2(x_1) + \sigma A^3(x_1) \end{aligned} \tag{15}$$

from which one obtains

$$A^2(x) - pA(x) + q(x) = \delta[A^2(x_1) - pA(x_1) + q(x_1)] + \sigma A[A^2(x_1) - pA(x_1) + q(x_1)]. \tag{16}$$

The right member of this equation, by virtue of (14), is zero, from which it follows that

$$(A^2 - pA + qI)x = 0. \tag{17}$$

If every vector of the n-space belongs to an invariant direction, the theorem still holds since, in this case

$$A(x) - p(x) = 0 \tag{18}$$

and therefore

$$A^2(x) - pA(x) = 0. \tag{19}$$

3. Transformations in which the Roots of $(A - \lambda I) = 0$ are Complex. It is desirable to present first the simple case of 2-space.

If the transformation is written in the classic canonical form, λ, and $\bar{\lambda}$, denoting the conjugate complex roots of the characteristic equation, then it is immediately evident that cor-

responding to each root there would be an imaginary invariant direction. However, since it is possible to obtain a similar transformation in which all of the coefficients are real, it follows that any vector of the real plane is always transformed into a vector of the real plane. The geometric significance of such a transformation is seen by establishing

Theorem 5. A linear transformation in 2-space, the roots of whose characteristic equation are complex, can always be represented by a similar transformation which produces a stretching and a rotation.

If a vector whose components are (x,y) is transformed by a rotation

$$x_1 = x\cos\theta - y\sin\theta \tag{20}$$
$$y_1 = x\sin\theta + y\cos\theta$$

followed by a stretching (or vice versa) given by

$$x' = \rho x_1 \tag{21}$$
$$y' = \rho y_1$$

the combined transformation can be written

$$x' = \rho\cos\theta\cdot x - \rho\sin\theta\cdot y \tag{22}$$
$$y' = \rho\sin\theta\cdot x + \rho\cos\theta\cdot y$$

The characteristic equation of this transformation has the roots

$$\lambda_1 = \rho(\cos\theta + i\sin\theta); \qquad \bar{\lambda}_1 = \rho(\cos\theta - i\sin\theta) \tag{23}$$

which may also be written

$$\lambda_1 = \rho e^{i\theta}; \qquad \bar{\lambda}_1 = \rho e^{-i\theta} \tag{24}$$

Let then

$$x' = a_1 x + b_1 y \tag{25}$$
$$y' = a_2 x + b_2 y$$

be the equations of transformation whose characteristic equation has complex roots, say

(26) $\lambda_1' = \rho' e^{i\theta'}$ and $\bar{\lambda}_1' = \rho' e^{-i\theta'}$

Let the values of ρ' and θ' be identified with transformation (22). Denoting the matrices of (22) and (24) by R and M respectively, then, since the corresponding characteristic equations have the same distinct roots, their invariant factors will be equal, and consequently, the two matrices are similar[7]. But if two matrices R and M are similar, a transformation can be found such that $R = B^{-1}MB$, thus establishing the truth of the theorem.

It is helpful, both in determining results as well as in checking results found directly by the theorems presented, to observe that:

Two invariant lines determine an invariant plane.

A plane of invariant lines and in invariant line not of this plane, determine a pencil of invariant planes.

If $D_1(A) = (A - \lambda_1 I)^3$, and we write

$$(27) \quad \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 1 & \lambda_1 \end{Vmatrix} = \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{Vmatrix} + \begin{Vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{Vmatrix}$$

it follows that every vector not of the invariant plane is first multiplied by λ_1 and then its points sheared parallel to the x_2x_3 axis.

The complete results of 2-space and 3-space are most easily presented in the following tabular form.

7. Bocher. Higher Algebra 1907 p 283

2-Space

Transformation	Invariant Factors	Factor $A-\lambda_i I$	Rank	Equation of Invariant Space
a. $x_1' = \lambda_1 x_1$ $x_2' = \lambda_2 x_2$	$D_1 = (\lambda-\lambda_1)(\lambda-\lambda_2)$ $D_2 = 1$	$A-\lambda_1 I$ $A-\lambda_2 I$	1 1	$x_2 = 0$ $x_1 = 0$	x_1 axis x_2 "
b. $x_1' = \lambda_1 x_1$ $x_2' = x_1 + \lambda_1 x_2$	$D_1 = (\lambda-\lambda_1)^2$ $D_2 = 1$	$A-\lambda_1 I$	1	$x_1 = 0$	x_2 "
c. $x_1' = \lambda_1 x_1$ $x_2' = \lambda_1 x_2$	$D_1 = (\lambda-\lambda_1)$ $D_2 = (\lambda-\lambda_1)$	$A-\lambda_1 I$	0		Every direction invariant.
d. $x_1' = \rho\cos\theta - \rho\sin\theta$ $x_2' = \rho\sin\theta + \rho\cos\theta$	$D_1 = (\lambda-\lambda_1)(\lambda-\bar{\lambda}_1)$ $D_2 = 1$				A rotation and stretching of every vector of plane

3-Space

Transformation	Invariant Factors	Factors of $D(A)$	Rank	Equation of Invariant Space	
1. $X_1' = \lambda_1 X_1$ $X_2' = \lambda_2 X_2$ $X_3' = \lambda_3 X_3$	$D_1 = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$ $D_2 = 1$ $D_3 = 1$	$A-\lambda_1 I$ $A-\lambda_2 I$ $A-\lambda_3 I$ $(A-\lambda_1 I)(A-\lambda_2 I)$ $(A-\lambda_2 I)(A-\lambda_3 I)$ $(A-\lambda_2 I)(A-\lambda_3 I)$	2 2 2 1 1 1	$x_2 = x_3 = 0$ $x_1 = x_3 = 0$ $x_1 = x_2 = 0$ $x_3 = 0$ $x_2 = 0$ $x_1 = 0$	X_1 axis X_2 " X_3 " X_1X_2 Plane X_1X_3 " X_2X_3 "
2. $X_1' = \lambda_1 X_1$ $X_2' = X_1 + \lambda_1 X_2$ $X_3' = \lambda_2 X_3$	$D_1 = (\lambda-\lambda_1)^2(\lambda-\lambda_2)$ $D_2 = 1$ $D_3 = 1$	$A-\lambda_1 I$ $A-\lambda_2 I$ $(A-\lambda_1 I)^2$ $(A-\lambda_1 I)(A-\lambda_2 I)$	2 2 1 1	$x_1 = x_3 = 0$ $x_1 = x_2 = 0$ $x_3 = 0$ $x_1 = 0$	X_2 axis X_3 " X_1X_2 Plane X_2X_3 "
3. $X_1' = \lambda_1 X_1$ $X_2' = X_1 + \lambda_1 X_2$ $X_3' = X_2 + \lambda_1 X_3$	$D_1 = (\lambda-\lambda_1)^3$ $D_2 = 1$ $D_3 = 1$	$(A-\lambda_1 I)$ $(A-\lambda_1 I)^2$	2 1	$x_1 = x_2 = 0$ $x_1 = 0$	X_3 axis X_2X_3 Plane
4. $X_1' = \lambda_1 X_1$ $X_2' = \lambda_2 X_2$ $X_3' = \bar{\lambda}_2 X_3$	$D_1 = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\bar{\lambda}_2)$ $D_2 = 1$ $D_3 = 1$	$(A-\lambda_1 I)$ $(A-\lambda_2 I)(A-\bar{\lambda}_2 I)$	2 1	$x_2 = x_3 = 0$ $x_1 = 0$	X_1 axis X_2X_3 Plane
5. $X_1' = \lambda_1 X_1$ $X_2' = \lambda_2 X_2$ $X_3' = \lambda_1 X_3$	$D_1 = (\lambda-\lambda_1)(\lambda-\lambda_2)$ $D_2 = (\lambda-\lambda_1)$ $D_3 = 1$	$(A-\lambda_1 I)$ $(A-\lambda_2 I)$ $(A-\lambda_1 I)(A-\lambda_2 I)$ $(A-\lambda_1 I)^2$	1 2 0 1	$x_2 = 0$ $x_1 = x_3 = 0$ — $x_2 = 0$	X_1X_3 Plane [of invariant lines] X_2 axis Pencil of invariant planes
6. $X_1' = \lambda_1 X_1$ $X_2' = X_1 + \lambda_1 X_2$ $X_3' = \lambda_1 X_3$	$D_1 = (\lambda-\lambda_1)^2$ $D_2 = (\lambda-\lambda_1)$ $D_3 = 1$	$(A-\lambda_1 I)$ $(A-\lambda_1 I)^2$	1 0	$x_1 = 0$ —	X_2X_3 plane of invariant lines X_1X_2 plane
7. $X_1' = \lambda_1 X_1$ $X_2' = \lambda_1 X_2$ $X_3' = \lambda_1 X_3$	$D_1 = (\lambda-\lambda_1)$ $D_2 = (\lambda-\lambda_1)$ $D_3 = (\lambda-\lambda_1)$	$A-\lambda_1 I$	0	—	Every direction invariant

4. Two special problems in 4-space. Transformations involving multiple imaginary roots will be treated by presenting two special problems in 4-space.

Consider the transformation

$$(28)\qquad \begin{aligned} x_1' &= x_1 - x_2 \\ x_2' &= x_1 + x_2 \\ x_3' &= x_3 - x_4 \\ x_4' &= x_3 + x_4 \end{aligned}$$

in which

$$(29)\qquad \begin{aligned} D_1(A) &= [A - (1 + i)I][A - (1 - i)I] = (A^2 - 2A + 2I) \\ D_2(A) &= (A^2 - 2A + 2I) \\ D_3(A) &= D_4(A) = 1 \end{aligned}$$

It follows readily either from the form of the transformation itself or from Theorem 3. that there are two invariant planes, namely the x_1x_2 plane and the x_3x_4 plane. It is also obvious that these planes are mutually perpendicular. Vectors in each of these planes have the properties defined in 2-space. Moreover, every vector of the 4-space belongs to an invariant plane, since a plane defined by the vector x, x being determined by the points (0,0,0,0) and (x_1,x_2,x_3,x_4), and the vector into which x is transformed by A always contains the vector into which A(x) is transformed by A, namely $A^2(x)$. The truth of this statement is made evident by the fact that the two independent equations determined from the matrix[8]

$$(30)\qquad \begin{Vmatrix} x & y & z & k & 1 \\ 0 & 0 & 0 & 0 & 1 \\ x_1 & x_2 & x_3 & x_4 & 1 \\ x_1 - x_2 & x_1 + x_2 & x_3 - x_4 & x_3 + x_4 & 1 \end{Vmatrix}$$

are satisfied by the point $(-2x_2, 2x_1, -2x_4, 2x_3)$ which together with the origin determines the vector $A^2(x)$.

The angle between any vector (x_1, x_2, x_3, x_4) and the transformed vector is

$$(31) \qquad \theta = \cos^{-1} \frac{x_1(x_1-x_2)+x_2(x_1+x_2)+x_3(x_3-x_4)+x_4(x_3+x_4)}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}\ \sqrt{(x_1-x_2)^2+(x_1+x_2)^2+(x_3-x_4)^2+(x_3+x_4)^2}}$$

$$= \cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

and each vector is stretched in length by the factor $\sqrt{2}$.

The example given is of the form

$$(32) \qquad \begin{aligned} x_1' &= \alpha x_1 - \beta x_2 \\ x_2' &= \beta x_1 + \alpha x_2 \\ x_3' &= \alpha x_3 - \beta x_4 \\ x_4' &= \beta x_3 + \alpha x_4 \end{aligned}$$

where α and β are defined by $\theta = \tan^{-1}\frac{\beta}{\alpha}$, θ being the angle through which any vector is rotated when subjected to the transformation. Moreover, any transformation in 4-space whose characteristic equation has only two distinct roots, and whose invariant factors are

$$(33) \qquad \begin{aligned} D_1(A) &= [A - (\alpha + i\beta)I][A - (\alpha - i\beta)I] = [A^2 - 2\alpha A + (\alpha^2+\beta^2)I] \\ D_2(A) &= [A^2 - 2\alpha A + (\alpha^2+\beta^2)I] \\ D_3(A) &= D_4(A) = 1 \end{aligned}$$

can be written in form (32). Menge has given direct simple methods for the construction of transformations to both the rational and classic canonical forms.[9] The invariant factors (33) are those of each of these forms. The matrix of the corresponding rational canonical[10] form is

$$(34)\qquad \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -\alpha^2-\beta^2 & 2\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\alpha^2-\beta^2 & 2\alpha \end{Vmatrix}$$

There is a matrix B such that $BA_rB^{-1} = A_R$ where A_R and A_r are the matrices of (32) and (34) respectively. Such a matrix B of very simple form is readily found which will always perform this transformation whatever be the values of α and β, and is

$$(35)\qquad B = \begin{Vmatrix} \alpha & 1 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & \beta & 0 \end{Vmatrix} \qquad B^{-1} = \begin{Vmatrix} 0 & 1/\beta & 0 & 0 \\ 1 & \alpha/\beta & 0 & 0 \\ 0 & 0 & 0 & 1/\beta \\ 0 & 0 & 1 & \alpha/\beta \end{Vmatrix}$$

Any transformation then, whose invariant factors are those of (33) may be written in the form (32) and hence every vector in the 4-space belongs to an invariant plane and when subjected to this transformation is rotated through an angle $\Theta = \tan^{-1}\frac{\beta}{\alpha}$ and stretched by the factor $\sqrt{\alpha^2+\beta^2}$.

A second important problem arises in case the roots of the characteristic polynomial are as before, but with the difference that

$$(36)\qquad D_1(A) = [A^2 - 2\alpha A + (\alpha^2+\beta^2)I]^2$$

$$D_2(A) = D_3(A) = D_4(A) = 1$$

The rational canonical form of the transformation is, in this case

$$(37)\qquad \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha^2-\beta^2 & 4\alpha(\alpha^2+\beta^2) & -2(3\alpha^2+\beta^2) & 4\alpha \end{Vmatrix}$$

As before, a simple form is sought for the transformation in which all coefficients are real. One of the simplest is

$$(38)\qquad \begin{aligned} x_1' &= \alpha x_1 - \beta x_2 + x_4 \\ x_2' &= \beta x_1 + \alpha x_2 \\ x_3' &= \alpha x_3 - \beta x_4 \\ x_4' &= \beta x_3 + \alpha x_4 \end{aligned}$$

Denoting, as before, the matrix of (38) by A_R and the corresponding rati nal canonical form by A_r, a matrix B such that $BA_rB^{-1} = A_R$ is

$$(39)\qquad B = \begin{Vmatrix} \alpha^3+\alpha\beta^2+2\alpha^2 & -3\alpha^2-\beta^2-4\alpha & 3\alpha+2 & -1 \\ -\alpha^2\beta-\beta^3-2\alpha\beta & 2\alpha\beta+2\beta & -\beta & 0 \\ -2\alpha^2\beta & -2\beta^3 & 4\alpha\beta & -2\beta \\ -2\alpha^3 & -2\alpha\beta^2 & 6\alpha^2+2\beta^2 & -6\alpha \\ & & & 2 \end{Vmatrix}$$

$$B^{-1} = \begin{Vmatrix} \frac{-1}{2\beta^2} & 0 & \frac{1}{2\beta^3} & \frac{-1}{4\beta^2} \\ \frac{-\alpha}{2\beta^2} & \frac{1}{2\beta} & \frac{-(2\alpha+\beta^2)}{4\beta^3} & \frac{-\alpha}{4\beta^2} \\ \frac{-(\alpha^2-\beta^2)}{2\beta^2} & \frac{\alpha}{\beta} & \frac{-\alpha^2-\alpha\beta^2}{2\beta^3} & \frac{-(\alpha^2-\beta^2)}{4\beta^2} \\ \frac{-\alpha^3+3\alpha\beta^2}{2\beta^2} & \frac{3\alpha^2-\beta^2}{2\beta} & \frac{-(2\alpha^3+3\alpha^2\beta^2-\beta^4)}{4\beta^3} & \frac{2\beta^2-\alpha^3+3\alpha\beta^2}{4\beta^2} \end{Vmatrix}$$

The matrix defined by the quadratic factor

$$(40)\qquad [A^2 - 2\alpha A + (\alpha^2+\beta^2)I]$$

is of rank two, determining the x_1, x_2 plane as invariant. More-

over expressing the matrix of (39) in the form

$$(41)\qquad \begin{Vmatrix} \alpha & -\beta & 0 & 1 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{Vmatrix} = \begin{Vmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{Vmatrix} + \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}$$

a complete geometric characterization is readily made.

(a) Every vector in the x, x_2 plane is rotated through the angle $\theta = \tan^{-1}\frac{\beta}{\alpha}$ and has its points stretched by the factor $\sqrt{\alpha^2+\beta^2}$.

(b) Every vector not in the x, x_2 plane is not only rotated through the angle $\theta = \tan^{-1}\frac{\beta}{\alpha}$ and stretched by the factor $\sqrt{\alpha^2+\beta^2}$ but also all points of the vector are sheared parallel to the $x,$ axis by an amount proportional to x_4.

5. Real Canonical Form. Having obtained very simple forms whose matrices contain only real elements for the two problems in four-space we make the following definitions.

(a) The Real Canonical form of a transformation in which all roots of the characteristic equation are real will be defined as identical with the Classic Canonical Form.

(b) In 2-space the Real Canonical Form for the case in which $\alpha - i\beta$ and $\alpha + i\beta$ occur as the roots of the characteristic equation will be defined by the matrix

$$(42)\qquad \begin{Vmatrix} \alpha & -\beta \\ \beta & \alpha \end{Vmatrix}$$

(c) The matrix of the Real Canonical Form for any transformation in 3-space in which

$$(43)\qquad D(\lambda) = (\lambda - \lambda_1)[\lambda - (\alpha + i\beta)][\lambda - (\alpha - i\beta)]$$

will be defined as

$$(44) \qquad \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \alpha & -\beta \\ 0 & \beta & \alpha \end{Vmatrix}$$

(d) In 4-space the Real Canonical Form will be defined as that of the Classical Canonical Form for those elements corresponding to the real roots of the characteristic equation. Where complex elements occur in the classic canonical form they are replaced by second order non-overlapping minors, one such minor corresponding to each pair of complex roots. In case of repeated complex roots in the first invariant factor, the constant unity appears as the upper right element of the matrix.

(e) The extension of this definition to higher spaces follows directly from the definitions in (d).

By means of the theorems and principles given in this discussion it is possible to give in complete detail the invariant configurations of any n-space.

References, Part 2.

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