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MINIMUM FUEL TRAJECTORIES FOR THE SYNERGETIC PLANE CHANGE MANEUVER

by
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Preface

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Abstract

MINIMUM FUEL TRAJECTORIES FOR THE SYNERGETIC PLANE CHANGE MANEUVER

By M. Michael Bonner

The synergetic plane change maneuver is a method of changing the orbital plane of an earth satellite utilizing both aerodynamic forces and propulsive forces. A satellite vehicle is assumed to be designed so as to be able to generate aerodynamic forces with lift-to-drag ratios greater than one together with rocket propulsion forces in an arbitrary direction. This study discusses optimal (in the sense of minimum fuel or ideal ΔV) trajectories for synergetic plane changes in the presence of state and control variable constraints representing instantaneous stagnation point temperature and thrust acceleration.

The trajectories considered all begin and terminate in a circular orbit at an altitude of 700 nautical miles. The over-all trajectory consists of an arbitrary two-impulse co-planar maneuver resulting in arrival at 300,000 feet altitude with arbitrary velocity, followed by a synergetic turn giving the desired orbital inclination change terminating at 300,000 feet, followed by an optimal two-impulse co-planar transfer back to the initial orbital altitude. Trajectories are found which result in minimum fuel expenditure for the entire maneuver for plane changes of 60° and 90°. Quantities sought in the optimization are the initial vehicle velocity at 300,000 feet, defining the initial two-impulse maneuver and five functions of time, (angle of attack, bank angle, and three thrust components) which define the balance of the maneuver.

The optimization problem is solved by the gradient or steepest descent method utilizing a digital computer for the necessary calculations. The results are compared to the ideal ΔV required for single impulse plane change using rocket thrust alone as an upper bound, and an idealized synergetic maneuver for which an analytic solution can be found as a lower bound.

Qualitative results of the study include the following:

a. Operation at angles of attack higher than the angle for the maximum lift/drag ratio is required.

b. A side component of thrust is not required, hence a fixed rocket engine aligned at an appropriate angle with the vehicle axis will be sufficient.

c. At thrust acceleration constraint levels of 3-g's and greater the constraint does not affect the cost of the maneuver.

Quantitative results include the following:

a. Vehicle L/D Max = 4, 60° Plane Change

Single Impulse Rocket Maneuver,	ΔV = 23, 400 ft/sec
Unconstrained Synergetic Maneuver,	ΔV = 8, 500 ft/sec
Peak Temperature Constraint 4700°F,	ΔV = 11, 600 ft/sec

b. Vehicle L/D Max = 2, 60° Plane Change

Single Impulse Rocket Maneuver,	ΔV = 23, 400 ft/sec
Unconstrained Synergetic Maneuver,	ΔV = 13, 900 ft/sec
Peak Temperature Constraint 4700°F,	ΔV = 17, 800 ft/sec

In general, the optimum synergetic maneuver is better than the rocket-only maneuver for inclination changes greater than 20° for the range of vehicle parameters investigated, and for 60° plane changes the synergetic maneuver can be better than the rocket-only maneuver by a factor of 2.7 in ΔV. When one considers the effect of changes in vehicle wing-loading due to fuel burned, it appears that up to three 60° synergetic maneuvers can be made with the fuel required to make one rocket-only maneuver.

CHAPTER 1

THE DESCRIPTION OF THE PROBLEM

1.1

This study is concerned with finding an optimal method of making large changes in the orbital plane of a rocket-propelled earth satellite capable of generating aerodynamic lift when in the earth's atmosphere. The criterion of optimality is the minimization of the fuel required to accomplish the plane change.

1.2

In order to provide a clear understanding of the problem, this chapter will be devoted to a qualitative description of the problem in non-mathematical terms.

1.3

The problem begins with the satellite, equipped with a rocket engine and capable of generating aerodynamic lift, in circular orbit well outside the earth's atmosphere. By means of rocket thrust the satellite is placed in a coplanar elliptical orbit which has a perigee low enough to intersect the earth's atmosphere.

1.4

When the satellite enters the atmosphere a combination aerodynamic-propulsive (Synergetic*) maneuver is carried out which ends with the satellite in an elliptical orbit of a different inclination than the first. The satellite is then maneuvered in such a way as to place it in a new circular orbit at the same attitude as the original orbit but with a different inclination. Figure 1.1 illustrates this maneuver.

1.5

The problem consists of discovering, within the bounds of the physical and mathematical constraints imposed, just what sort of a maneuver in the atmosphere results in the minimum expenditure of

*The term Synergetic was apparently first used by Nyland (Ref. 2), who states that it means working together or cooperating.

fuel in transferring from orbit 1 to orbit 2.

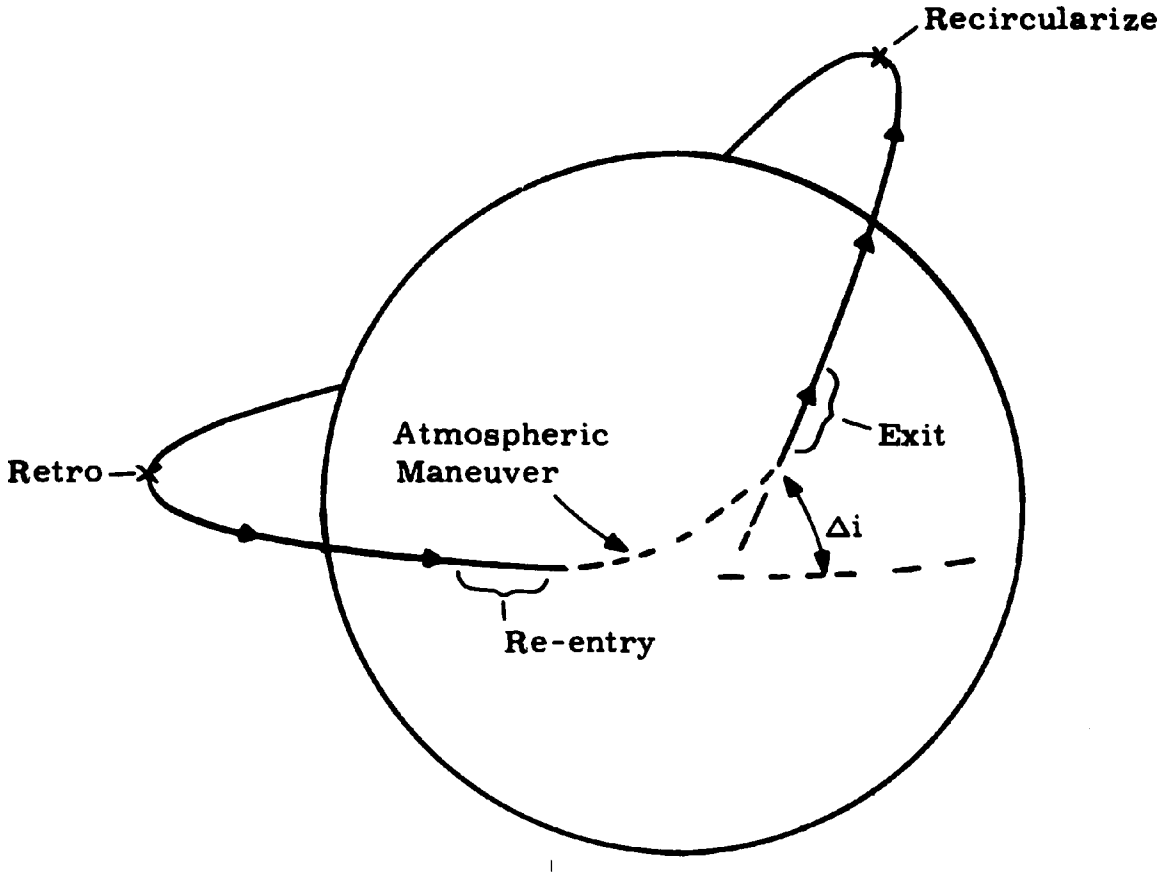


Illustration of Synergetic Maneuver

Figure 1.1

CHAPTER 2

BACKGROUND

2.1

The idea of being able to change the orbital plane of an earth satellite has received considerable attention in the literature because of its obvious usefulness for many civilian and military space missions. For orbits near the earth, (altitudes of less than 1 earth's radius) the fuel requirements for substantial orbital plane changes, using state of the art rocket propulsion systems only, are such that a prohibitively large proportion of the total weight of the satellite must be allocated to fuel, leaving little capability for payload if the plane change is made using rocket power alone.

2.2

As an example of what is meant by a prohibitive fuel requirement, consider a vehicle in a 700 N mile altitude circular orbit and let us make a rough calculation of the fuel required to make a 60° plane change using rocket thrust alone and assuming a conventional rocket engine with a vacuum specific impulse of 400 sec. It can be shown that the optimum single impulse maneuver requires that at least 83% of the total vehicle weight in orbit be expendable fuel.

2.3

Based on the results of the research described herein, it appears that it is possible to accomplish the same plane change with the Synergetic plane change maneuver using less than 60% of the total weight as expendable fuel. In fact, if physical restrictions such as "G" loads and aerodynamic heating are ignored, it can be shown that one can accomplish this plane change using less than 50% of the vehicle weight for fuel. The amount of fuel required depends on the vehicle's aerodynamic characteristics and the initial orbital altitude.

2.4

The Synergetic plane change maneuver was apparently first considered by H. S. London (Ref. 1) in 1962. F. S. Nyland (Ref. 2) published the paper that first brought the maneuver to the attention of the author. Both London and Nyland developed mathematical models simple enough to permit hand calculation of performance, but which were not very realistic. London assumes the vehicle is unpowered, and flies at the constant angle of attack and constant bank angle. Nyland assumes the vehicle follows a minor circle trajectory at constant angle of attack during the turning phase.

2.5

Bell and Hanke (Ref. 3) published a paper in 1963 in which they claim to "optimize" this maneuver. They assume (but do not prove) that thrust should be equal to drag, and that the turn should be flown at constant angle of attack giving the maximum ratio of lift to drag and constant bank angle. Further, Bell and Hanke consider the problem of changing the vehicle's heading which is not the same thing as changing its orbital inclination.

2.6

A serious attempt to optimize the Synergetic plane change maneuver was made by R. O. Shaver (Ref. 4). Shaver formulates an optimal control problem which is similar in some respects to the problem considered in this paper and obtains solutions by the Gradient Method. Shaver's work represents a large step forward in the analysis of the benefits to be obtained from this maneuver. However, in the opinion of the author, Shaver's problem is still not sufficiently general to give a complete picture of the benefits to be gained through use of this maneuver. Shaver uses as the cost function to be minimized, the total energy of the vehicle (Kinetic plus potential) at the end of the plane change. He allows bank angle and angle of attack to vary, but does not allow the use of thrust while in the atmosphere. Shaver's

problems were all started from fixed initial conditions. The solutions resulting from this type of problem typically terminate at relatively low altitudes and the drag losses associated with atmosphere exit are not considered.

2.7

In this study we will consider what the author belives to be a much more general problem. We will attempt to optimize the total fuel required to de-orbit, make a plane change and re-orbit. The use of continuously variable thrust in all directions during the atmospheric portion of the maneuver will be allowed, as well as changes in angle of attack and bank angle. Constraints on aerodynamic heating and allowable thrust accelerations will be included. It is the author's hope that the results of the problems considered and the mathematical model developed will provide a framework which will permit complete exploration of the benefits to be derived from the Synergetic Plane-change Maneuver.

CHAPTER 3

THE MATHEMATICAL MODEL

The development of a mathematical model for a real-world physical problem requires that many assumptions be made, and the validity of the model is strongly dependent on these assumptions. In the subsequent development an attempt to identify all assumptions will be made and in some cases their use justified. In other cases these assumptions are well known enough as to require no comment other than a statement that they were made.

3.1 The Co-ordinate System

The Co-ordinate system used to describe the motion of the space vehicle in this problem was proposed by Fogarty and Howe (Ref. 5) and is called the "H Frame" system. Before describing this system let us make the first assumption:

Assumption #1 The earth is non-rotating.

This assumption is made for the sake of simplicity in the mathematical model and its effect is to introduce errors of up to 1550 ft/sec. in the velocity of the vehicle relative to the earth's atmosphere. Corresponding errors are introduced in the computation of the aerodynamic forces involved, inasmuch as it will be assumed later that dynamic forces vary as the square of the vehicle's velocity relative to the air. It will turn out that the minimum inertial velocity is on the order of $\frac{1}{2}$ local circular and therefore the maximum error in aerodynamic forces introduced by this assumption could be as much as $\pm 12\%$ which is certainly not negligible. This assumption is justified on the basis that deletion of the assumption requires only addition of extra terms in some equations of motion and the methods of solution that are developed will still work. Also, it can be argued that the qualitative nature of the optimum trajectories found will be the same in either case.

3.1.1 Position Coordinates

The vehicle position is described in spherical coordinates using

conventional navigational latitude (L) and longitude (λ) as the two angles and the distance from the center of the earth (R) as the length of the radius vector. (See Figure 3.1) Because of the assumption that the earth is non-rotating, this system can be considered an inertial frame for this problem.

3.1.2 Velocity Co-ordinates

The unique feature of the "H Frame" Coordinate system is the manner in which the velocity vector and Force components are described. The direction of motion is described by specifying the navigational heading of the vehicle (ψ), the angle between North and the horizontal projection of the velocity vector (measured clockwise). The vertical and horizontal components of the velocity vector are measured in a vertical plane oriented in the instantaneous direction of motion. Stated in other words, a moving coordinate system is used to specify the velocity vector as follows: The X - Y plane is always perpendicular to the local vertical. The X axis is aligned in the direction of instantaneous motion and Z axis points vertically upward. (See Figure 3.2). Vehicle body axis orientation to this frame is described in section 3.4.2.

It is occasionally convenient to describe the vertical orientation of the velocity vector by a flight path angle γ, defined as the angle between the velocity vector and the local horizontal measured in the instantaneous plane of motion. (See Figure 3.3).

3.1.3 External Forces

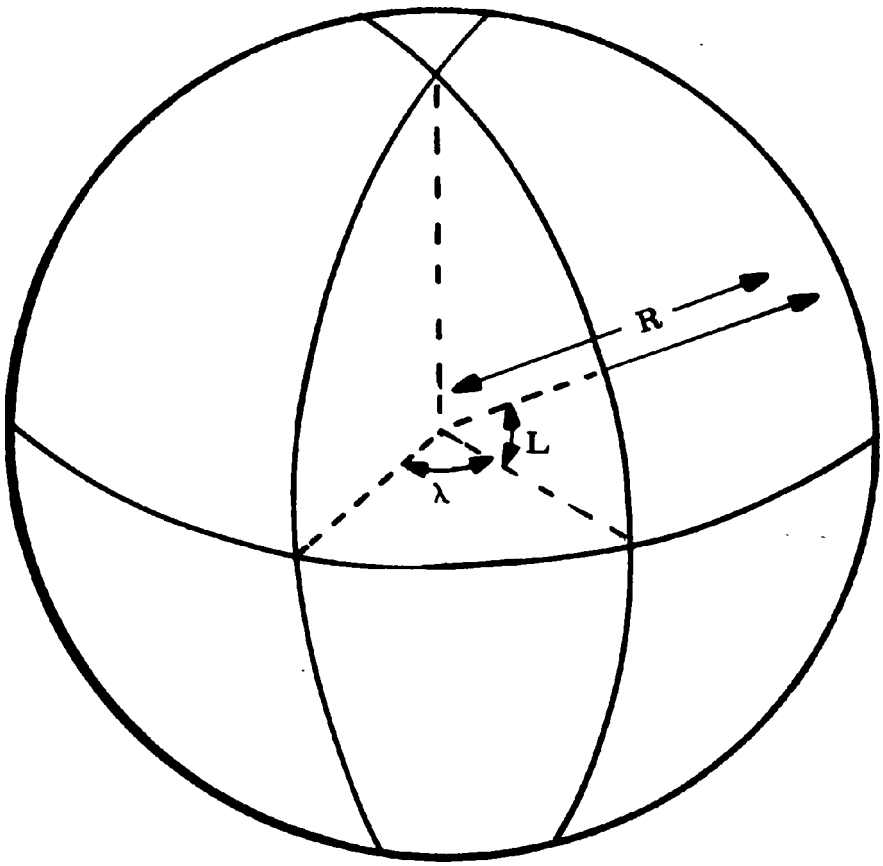
All external forces are referred to the moving X Y Z coordinate system defined in 3.1.2.

3.1.4 Summary

The instantaneous motion of the vehicle can be described by specifying:

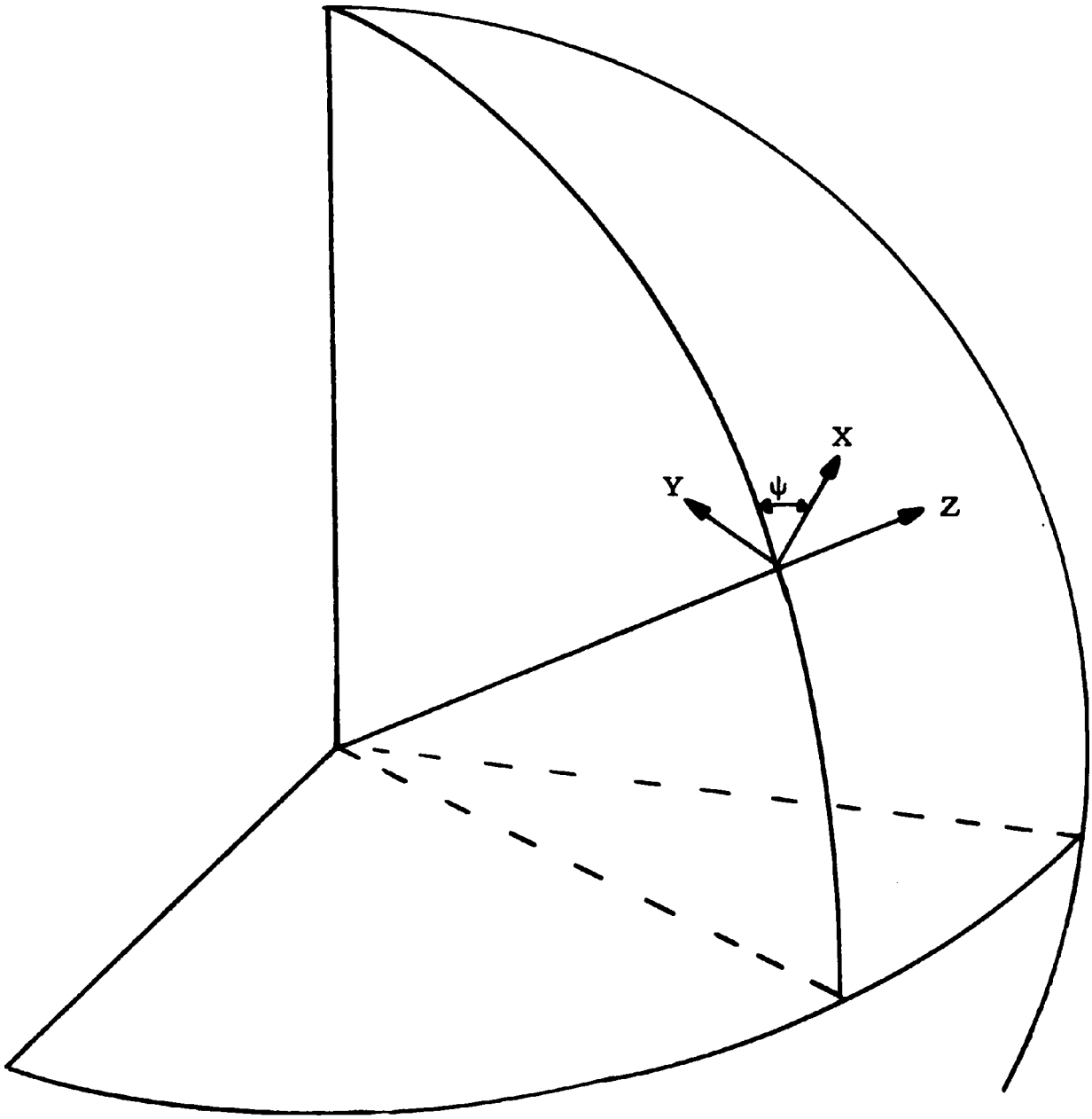
A. Latitude: L

B: Longitude: λ



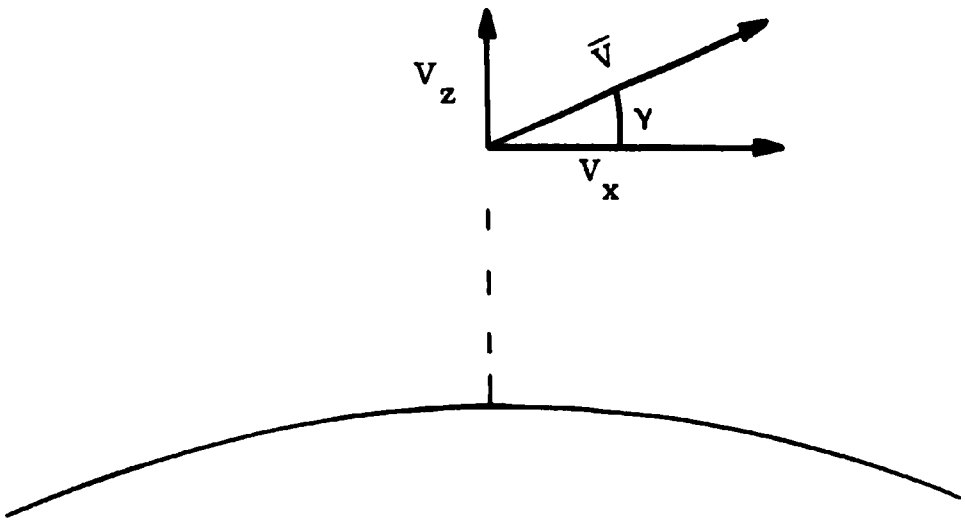
Position Coordinates

Figure 3.1



Velocity Coordinate Axis System

Figure 3.2



Velocity Coordinate Resolution

Figure 3.3

C. Distance from the center of the earth: R

D. Heading: ψ

E. The orientation of the velocity vector in a vertical plane in the direction of motion by specifying either

$$V_x \text{ and } V_z \quad \text{or} \quad |V| \text{ and } \gamma$$

3.2 The Equations of Motion

Before writing down the equations of motion, it is convenient to make assumption #2 at this point:

Assumption #2 The earth's graviational field can be described by the inverse square law and no other field forces are acting in the vehicle. Using assumptions 1 and 2, it can be shown, (see Ref. 5, Fogarty and Howe) that the motion of the vehicle can be represented by the solution of the following differential equations with the appropriate boundary values:

$$\frac{d}{dt}(V_x) = -\frac{V_x V_z}{R} + \frac{F_x}{M} \tag{3.2.1}$$

$$\frac{d}{dt}(R) = V_z \tag{3.2.2}$$

$$\frac{d}{dt}(V_z) = \frac{V_x^2}{R} - \frac{\mu}{R^2} + \frac{F_z}{M} \tag{3.2.3}$$

$$\frac{d}{dt}(\psi) = \frac{V_x}{R}\sin\psi \tan L + \frac{F_y}{V_x M} \tag{3.2.4}$$

$$\frac{d}{dt}(\lambda) = \frac{V_x \sin\psi}{R \cos L} \tag{3.2.5}$$

$$\frac{d}{dt}(L) = \frac{V_x}{R}\cos\psi \tag{3.2.6}$$

where: (1) The F_j are the resultant of the aerodynamic and propulsive forces in the jth direction.

(2) M is the instantaneous mass of the satellite.

(3) μ is the earth's gravitational constant.

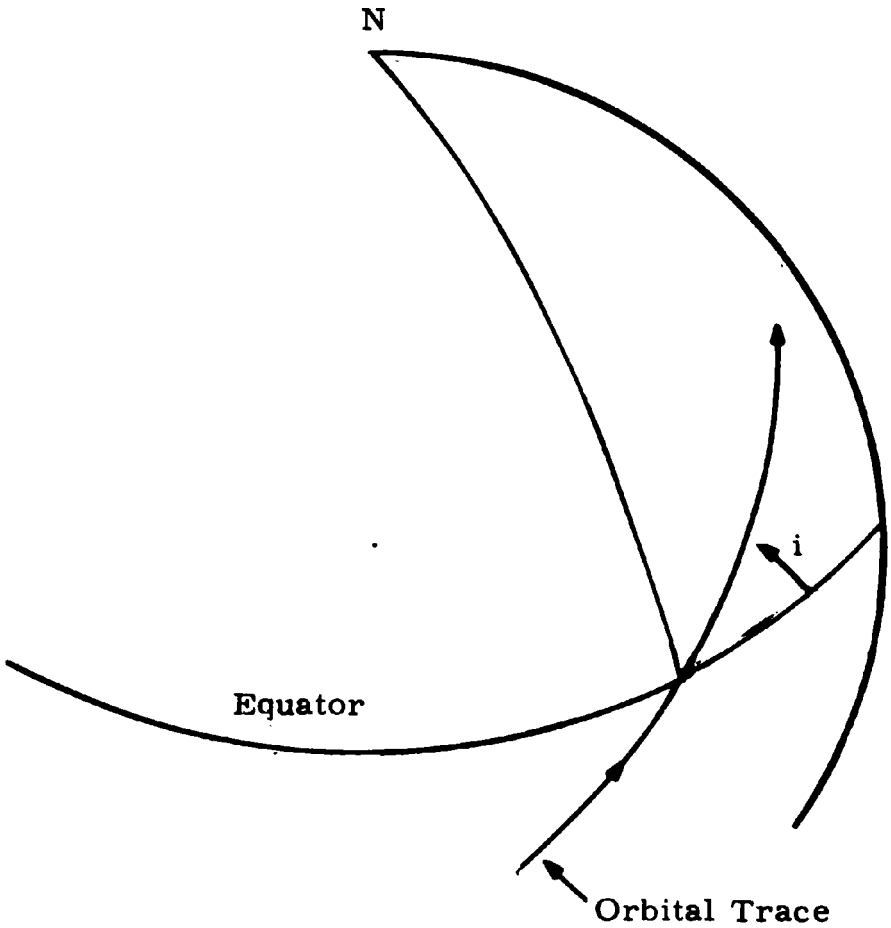
3.3 The Orbital Inclination

Inasmuch as we are interested in changing the orbital inclination of our space vehicle and this quantity does not appear explicitly in the equations of motion it is appropriate to define what is meant by orbital inclination at this point and derive an expression for it in terms of the state variables used in the equations of motion.

3.3.1 Definition of Orbital Inclination

This definition is the standard definition used in celestial mechanics.

Definition: The orbital inclination (i) is the angle between the earth's equatorial plane and the plane defined by the velocity vector and the earth's center measured counterclockwise at the ascending node. To be completely rigorous, one should now define the ascending node, but reference to Figure 3.4 will make clear which angle (i) represents.



Orbital Inclination Illustration

Figure 3.4

3.3.2 Derivation of the Relationship Between i and the State Variables

To derive the relationship between i and the state variables, consider the case where the satellite is moving in a plane, i.e., $F_y \equiv 0$.

In this case equations (3.2.4) and (3.2.6) are:

$$\frac{d}{dt}(\psi) = \frac{V_x}{R} \sin\psi \tan L \qquad (3.2.4a)$$

$$\frac{d}{dt}(L) = \frac{V_x}{R} \cos\psi \qquad (3.2.6a)$$

Dividing (3.2.4a) by (3.2.6a)

$$\frac{d/dt\,(\psi)}{d/dt\,(L)} = \frac{d\psi}{dL} = \frac{\sin\psi \tan L}{\cos\psi} \qquad (3.3.1)$$

(3.3.1) can be integrated by separation of variables giving:

$$\int_{\psi_0}^{\psi} \cot u \, du = \int_{L_0}^{L} \tan s \, ds \qquad (3.3.2)$$

which reduces to:

$$\frac{\sin\psi}{\sin\psi_0} = \frac{\cos L_0}{\cos L} \qquad (3.3.3)$$

Now as a consequence of the definition of i, we have the following relationship: When $L = 0$ at the ascending node, $\psi = 90° - i$. Applying this as a boundary value to (3.3.3), we get:

$$\frac{\sin\psi}{\sin(90° - i)} = \frac{1}{\cos L} \qquad (3.3.4)$$

and after a little trigonometry this gives:

$$\cos i = \sin\psi \cos L \qquad (3.3.5)$$

which defines i in terms of the state variables ψ and L. Note that this relationship is only valid for North latitudes and $270° \le \psi \le 360°$ or $0° \le \psi \le 90°$. However, this restriction will cause no problems because of the way the boundary values of the problem will be defined.

3.4 Derivation of the Force Terms in the Equations of Motion

So far, the existence of aerodynamic and propulsive forces have

been indicated only in a very general way by the inclusion of the quantities F_x, F_y, and F_z in the equations of motion. These quantities are in fact functions of the state variables and the control variables, and the determination of their time history is one of the goals of this problem. Let us now split these terms into two parts, aerodynamic and propulsive forces and adopt the following notation:

$$F_j = T_j + F_{aj} \qquad j = X, Y, Z \tag{3.4.1}$$

where the T_j are the propulsive force components and the F_{aj} are the aerodynamic force components.

3.4.1 Mathematical Model of the Propulsion System

It will be assumed that our vehicle is equipped with a rocket engine capable of exerting thrust in any direction and that this thrust can be continuously controlled from zero to any desired value. The three thrust components will be three of the five control variables in our optimal control problem. It will be further assumed that the rate of change of mass of our satellite is proportional to the Euclidean norm of the thrust vector and is governed by the following equations:

$$\frac{d}{dt}(M) = -\frac{\sqrt{T_x^2 + T_y^2 + T_z^2}}{IG_e} \tag{3.4.1.1}$$

where I is a parameter (assumed constant) called specific impulse and G_e is the acceleration of gravity at the earth's surface ($32.2 ft/sec^2$).

3.4.2 Mathematical Model for Aerodynamic Forces

It is assumed that the aerodynamics of the problem can be approximated by Newtonian flow. The relationships developed subsequently were derived from Chapman's work (Ref. 6). This model was used in the work in reference 3 (Bell and Hanke), and in reference 4 (Shaver).

3.4.2.1 The Vehicle as an Airplane

Assume that our vehicle bears some relationship to the conventional airplane and that it will always be flown with zero sideslip. Aerodynamic forces will be developed as a result of combinations of

pitching and banking maneuvers as appropriate. Let us first consider the aerodynamic force diagram in the plane of symmetry of the airplane. (See Figure 3.5.)

The angle of attack, (α), is defined as the angle between the zero lift line of the aircraft and the relative wind vector, positive as indicated in Figure 3.5. α will be one of the control variables in this problem and can have values in the range $-90° \leq \alpha \leq 90°$. Define lift (L) and drag (D) as the components of aerodynamic force perpendicular to and parallel to the relative wind as indicated.

Assume that the lift and drag forces can be represented by the following equations:

$$L = \tfrac{1}{2}\rho\, V_w^2\, S\, C_L(\alpha) \qquad (3.4.2.1)$$

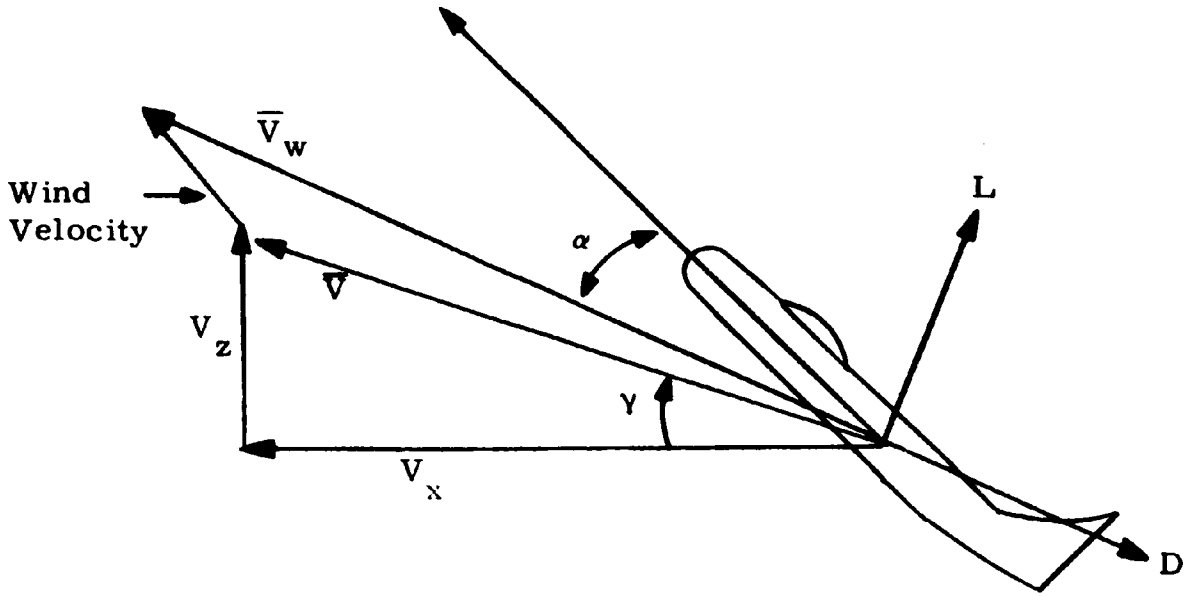
$$D = \tfrac{1}{2}\rho\, V_w^2\, S\, C_d(\alpha) \qquad (3.4.2.2)$$

where: ρ is the atmospheric density

S is the effective wing area of the vehicle

$C_L(\alpha)$ and $C_d(\alpha)$ are non-dimensional force coefficients depending only on α

V_w is the norm of the velocity of the vehicle with reference to the atmosphere.



Aerodynamic Force Diagram

Figure 3.5

In all that follows it is assumed that the wind velocity is 0 and that $\bar{V}_w = \bar{V}$. Assume, in accordance with reference 6 (Chapman), that C_L and C_d are related to α by the following functions

$$C_L(\alpha) = (C_{dm} - C_{d_0}) \sin\alpha|\sin\alpha| \cos\alpha \qquad (3.4.2.3)$$

$$C_d(\alpha) = (C_{dm} - C_{d_0})[b + |\sin^3\alpha|] \qquad (3.4.2.4)$$

where: C_{dm} is C_d at $\alpha = 90°$

C_{d_0} is C_d at $\alpha = 0°$

3.4.2.2 Variation of Atmospheric Density with Altitude

Let us further assume an exponential atmospheric model and thus arrive at Assumption #3.

Assumption #3 Atmospheric density (ρ) is governed by the following equations:

$$\rho(h) = \rho_e e^{-h/h_s} \qquad (3.4.2.5)$$

where: ρ_e is atmospheric density at sea-level

h is height above sea-level

h_s is the scale height

3.4.2.3 Resolution of Lift and Drag Forces into the XYZ Coordinate System

Let us first consider the drag force. Referring to Figure 3.5 we see that:

$$D_x = -D \cos\gamma \qquad (3.4.2.6)$$

$$D_z = -D \sin\gamma \qquad (3.4.2.7)$$

$$D_y = 0$$

Resolution of the lift force is not quite so simple. Before we are able to accomplish this it is necessary to define the bank angle, (θ), which will be our fifth control variable in this problem.

Definition: Bank angle θ is defined in a plane perpendicular to the velocity vector $\overline{V}$. It is the angle between the intersection of the XZ plane with the θ plane and the intersection of the vehicle plane of symmetry with the θ plane. $\theta > 0$ for aircraft banked to the left.

With the use of this definition it is evident that the L vector has the following components:

$$L_x = -L \cos\theta \sin\gamma \tag{3.4.2.8}$$

$$L_y = -L \sin\theta \tag{3.4.2.9}$$

$$L_z = L \cos\theta \cos\gamma \tag{3.4.2.10}$$

3.4.3 Summary of Section 3.4

Based on the results of section 3.4, we have the following relationships for the force terms:

$$F_x = T_x - D \cos\gamma - L \cos\theta \sin\gamma \tag{3.4.3.1}$$

$$F_y = T_y - L \sin\theta \tag{3.4.3.2}$$

$$F_z = T_z - D \sin\gamma + L \cos\theta \cos\gamma \tag{3.4.3.3}$$

$$L = \tfrac{1}{2}\rho(R) V_w^2 S C_L(\alpha) \tag{3.4.3.4}$$

$$D = \tfrac{1}{2}\rho(R) V_w^2 S C_d(\alpha) \tag{3.4.3.5}$$

$$C_L = (C_{dm} - C_{d_0}) \sin\alpha \,|\sin\alpha|\, \cos\alpha \tag{3.4.3.6}$$

$$C_d = (C_{dm} - C_{d_0})[b + |\sin^3\alpha|] \tag{3.4.3.7}$$

$$\rho(R) = \rho_e \, e^{-(R-R_e)/h_s} \tag{3.4.3.8}$$

By virtue of Assumption #1 we also have:

$$V^2 = (V_x^2 + V_z^2) \tag{3.4.3.9}$$

$$\sin\gamma = \frac{V_z}{\sqrt{V^2}} \tag{3.4.3.10}$$

$$\cos\gamma = \frac{V_x}{\sqrt{V^2}} \tag{3.4.3.11}$$

and we also have another differential equation to add to those in section 3.2.

$$\dot{M} = -\frac{\sqrt{T_x^2 + T_y^2 + T_z^2}}{IG_e} \qquad (3.4.3.12)$$

3.5 Non-Dimensionalization of the Equations of Motion

Following the paper by Fogarty and Howe, (Ref. 5), the equations of motion will be put into non-dimensional form by defining new variables in terms of values pertaining to a reference circular orbit at distance R_0, velocity $V_0 = \sqrt{\frac{\mu}{R_0}}$ and orbital angular frequency $\omega_0 = \frac{G_0}{V_0}$ $= \sqrt{\frac{\mu}{R_0^3}} = \frac{V_0}{R_0} = \frac{\mu}{R_0^2 V_0}$.

3.5.1 Let us introduce new dimensionless variables and convert to subscript notation by making the following definitions:

a. A Non-Dimensional time unit τ specified by:

$$\tau = \omega_0 t \qquad (3.5.1.1)$$

b. A Non-Dimensional radius measurement x_3 defined by:

$$R_0(1 + x_3) = R \qquad (3.5.1.2)$$

c. A Non-Dimensional measure of horizontal velocity, x_1 defined by:

$$V_0(1 + x_1) = V_x \qquad (3.5.1.3)$$

d. A Non-Dimensional vertical velocity x_2 defined by:

$$V_0 x_2 = V_z \qquad (3.5.1.4)$$

e. A Non-Dimensional measure of fuel used, x_7 defined by:

$$M_0(1 - x_7) = M \qquad (3.5.1.5)$$

f. A conversion of the position angles defined into subscript notation as follows:

$$\psi = x_4 \qquad (3.5.1.6)$$

$$\lambda = x_5 \qquad (3.5.1.7)$$

$$L = x_6 \qquad (3.5.1.8)$$

3.5.3

With the above definitions our seven differential equations become:

$$[\dot{x}_j \triangleq \frac{d}{d\tau}(x_j)]$$

$$\dot{x}_1 = -\frac{(1 + x_1)x_2}{(1 + x_3)} + \frac{F_x}{M_0G_0(1 - x_7)} \qquad (3.5.2.1)$$

$$\dot{x}_2 = \frac{2x_1 + x_1^2}{(1 + x_3)} + \frac{x_3}{(1 + x_3)^2} + \frac{F_z}{M_0G_0(1 - x_7)} \qquad (3.5.2.2)$$

$$\dot{x}_3 = x_2 \qquad (3.5.2.3)$$

$$\dot{x}_4 = \frac{(1 + x_1)}{(1 + x_3)} \sin x_4 \tan x_6 + \frac{F_y}{(1+x_1)M_0G_0(1 - x_7)} \qquad (3.5.2.4)$$

$$\dot{x}_5 = \frac{(1 + x_1)}{(1 + x_3)} \frac{\sin x_4}{\cos x_6} \qquad (3.5.2.5)$$

$$\dot{x}_6 = \frac{(1 + x_1)}{(1 + x_3)} \cos x_4 \qquad (3.5.2.6)$$

$$\dot{x}_7 = \frac{V_0\sqrt{T_x^2 + T_y^2 + T_z^2}}{IG\, M_0G_0} \qquad (3.5.2.7)$$

3.5.3

Now let us examine the members of equations (3.5.2.1), (3.5.2.2), and (3.5.2.4) containing the force terms in detail, with the purpose of defining non-dimensional forces and parameters. We have from these equations and sections 3.4.2,

$$\frac{F_x}{M_0G_0} = \frac{T_x}{M_0G_0} - \frac{\frac{1}{2}\rho V^2S}{M_0G_0}[C_d(\alpha)\cos\gamma + C_L(\alpha)\cos\theta\sin\gamma] \qquad (3.5.3.1)$$

$$\frac{F_y}{M_0G_0} = \frac{T_y}{M_0G_0} - \frac{\frac{1}{2}\rho V^2S}{M_0G_0} C_L(\alpha)\sin\theta \qquad (3.5.3.2)$$

$$\frac{F_z}{M_0G_0} = \frac{T_z}{M_0G_0} + \frac{\frac{1}{2}\rho V^2S}{M_0G_0}[C_L(\alpha)\cos\theta\cos\gamma - C_d(\alpha)\sin\gamma] \qquad (3.5.3.3)$$

At this point it is convenient to introduce a result derived from the analog investigation of the problem (which will be discussed later) in order to simplify the three expressions above. It turns out that for

all trajectories of interest, γ is small, and the following approximations can be made without significant loss of accuracy:

$$\cos\gamma \approx 1 \tag{3.5.3.4}$$

$$C_L(\alpha)\cos\theta\sin\gamma \ll C_d\cos\gamma \tag{3.5.3.5}$$

$$C_d(\alpha)\sin\gamma \ll C_L\cos\theta\cos\gamma \tag{3.5.3.6}$$

$$V^2 \approx V_x^2 = V_0^2(1+x_1)^2 \tag{3.5.3.7}$$

Using (3.4.3.7) and (3.4.2.5) defining ρ as a function of altitude, one can write the dynamic pressure term (q) as:

$$q = \tfrac{1}{2}\rho_e V_0^2(1+x_1)^2 \exp\left\{-\frac{h_0}{h_s}\right\}\exp\left\{-\frac{R_0x_3}{h_s}\right\} \tag{3.5.3.8}$$

Let us now define a dimensionless dynamic pressure, (Q) as:

$$Q = Q_0(1+x_1)^2\exp\left\{-\frac{R_0x_3}{h_s}\right\} \tag{3.5.3.9}$$

where

$$Q_0 = \frac{\tfrac{1}{2}\rho_e V_0^2\exp\left\{-\frac{h_0}{h_s}\right\}}{W_L} \tag{3.5.3.10}$$

and W_L is a parameter called wing loading defined by:

$$W_L = \frac{M_0G_0}{S} \tag{3.5.3.11}$$

Let us define the dimensionless control variables in subscript notation as U_j, $j = 1 -- 5$ as follows:

$$U_1 = \alpha \tag{3.5.3.12}$$

$$U_2 = \theta$$

$$U_3 = \frac{T_x}{M_0G_0}$$

$$U_4 = \frac{T_z}{M_0G_0}$$

$$U_5 = \frac{T_y}{M_0G_0}$$

3.5.4 Summary of Section 3.5

Using the work of section 3.5.3, the basic differential equations of motion can now be written as:

$$\dot{x}_1 = -\frac{x_2(1+x_1)}{(1+x_3)} - \frac{QC_d(U_1)}{(1-x_7)} + \frac{U_3}{(1-x_7)} \qquad (3.5.4.1)$$

$$\dot{x}_2 = \frac{2x_1 + x_1^2}{(1+x_3)} + \frac{x_3}{(1+x_3)^2} + \frac{QC_L(U_1)\cos U_2}{(1-x_7)} + \frac{U_4}{(1-x_7)} \qquad (3.5.4.2)$$

$$\dot{x}_3 = x_2 \qquad (3.5.4.3)$$

$$\dot{x}_4 = \frac{(1+x_1)}{(1+x_3)} \sin x_4 \tan x_6 - \frac{QC_L(U_1)\sin U_2}{(1+x_1)(1-x_7)} + \frac{U_5}{(1-x_7)(1+x_1)} \qquad (3.5.4.4)$$

$$\dot{x}_5 = \frac{(1+x_1)}{(1+x_3)} \frac{\sin x_4}{\cos x_6} \qquad (3.5.4.5)$$

$$\dot{x}_6 = \frac{(1+x_1)}{(1+x_3)} \cos x_4 \qquad (3.5.4.6)$$

$$\dot{x}_7 = \frac{V_0}{IG_e} \sqrt{U_3^2 + U_4^2 + U_5^2} \qquad (3.5.4.7)$$

These then are the seven differential equations which, together with the appropriate boundary values, will be used to describe the motion of the space vehicle. In order to keep the physical meaning of the seven state variables and five control variables in mind, they are listed below with a brief description of what they correspond to:

x_1 ⟶ Horizontal velocity

x_2 ⟶ Vertical velocity

x_3 ⟶ Altitude

x_4 ⟶ Heading

x_5 ⟶ Longitude

x_6 ⟶ Latitude

x_7 ⟶ Fuel used

U_1 ⟶ Angle of attack

U_2 ⟶ Bank angle

U_3 ⟶ Horizontal thrust component in direction of motion

U_4 ⟶ Vertical thrust component

U_5 ⟶ Horizontal thrust component at right angles to direction of motion

This completes the derivation of the equations of motion to be used.

CHAPTER 4

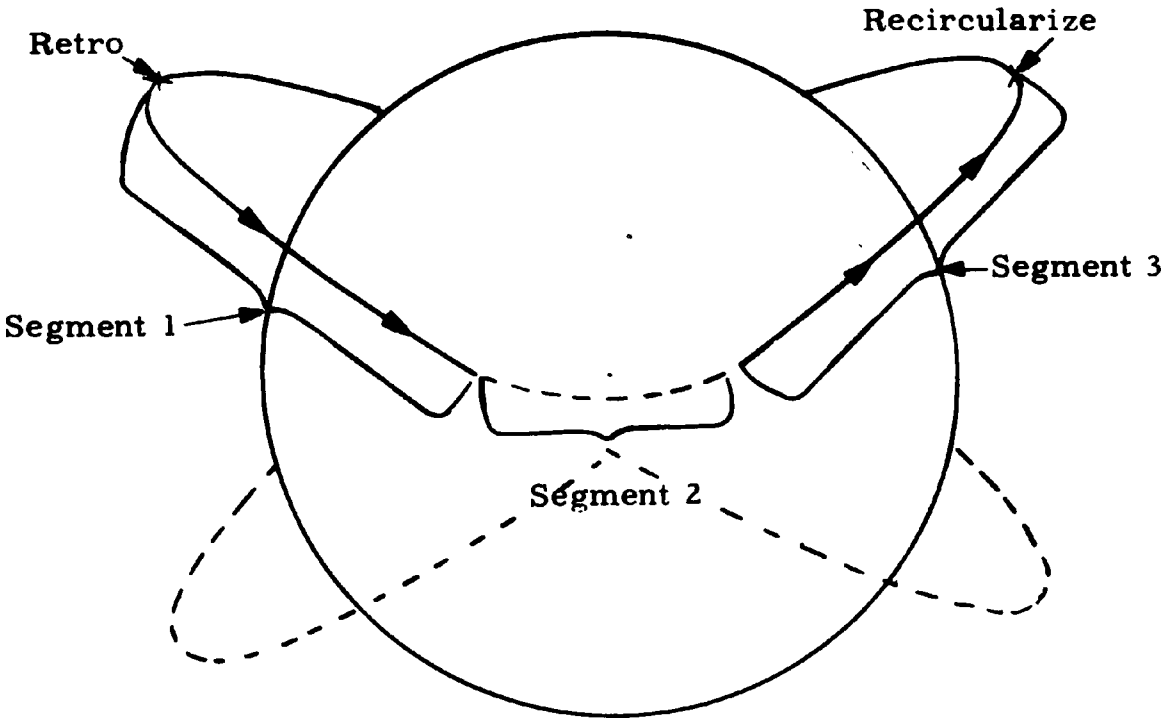
THE DERIVATION OF THE COST FUNCTIONAL

4.1 Introduction

As indicated in Chapter 1, we are looking for an optimal (in the sense of minimum fuel) trajectory for our space vehicle within the class of trajectories which are solutions to the Synergetic Plane Change Problem.

Actually there are two equivalent questions that could be asked: One could ask: A. "For a fixed amount of fuel, what is the maximum plane change one can achieve?" or B. "For a fixed plane change, what trajectory uses the minimum amount of fuel?" The trajectories resulting as answers to the two questions must have a one to one correspondence for corresponding amounts of plane change so the selection between which of the two questions to ask is dictated largely by the choice of the optimizer. In this study we have chosen to look for the answer to question B.

Having derived a mathematical model of the problem, it is now necessary to transfer the statement, "I want a minimum fuel trajectory," into a cost functional to be minimized. As the reader has no doubt observed, there is no known general analytic solution to equations (3.5.4.1) through (3.5.4.7) for arbitrary U_j (j = 1 -- 5) and thus this problem must of necessity be solved with the aid of a computer. The method of computer solution chosen has an influence on the selection of the cost functional, so let us digress a moment and discuss this subject briefly. For purposes of discussion the Synergetic plane change maneuver can be broken into three segments (See Fig. 4.1). Let us call Segment 1 the descent from the original orbit to the region where atmospheric effects become appreciable. Segment 2 is the part of the trajectory where atmospheric effects are appreciable, and where the plane change is done, Segment 3 is the part of the trajectory transversed after the vehicle leaves the effective



Synergetic Plane Change Trajectory Sketch

Figure 4.1

atmosphere and is returning to its new orbit. We are primarily interested in Segment 2 of this trajectory, and therefore, based on an analysis of the problem parameters, some distance R_0 will be selected at which the atmospheric effects begin to become noticeable, and the computer solutions will be started and stopped at the point where the vehicle first crosses this altitude on Segment 1 and leaves it after the end of Segment 2. The cost functional should, therefore, contain an expression that is a measure of the fuel used on Segment 1 and Segment 3 as well as on Segment 2, in order that a minimum fuel trajectory can be found for the whole maneuver.

It has become common in the astronautics field to speak about fuel expended in maneuvering in terms of "the ΔV equivalent." The ΔV equivalent is defined as the increase in magnitude of total velocity of a vehicle if we imagine that thrust is applied in the form of a Dirac delta function of strength F. Under these conditions, the equations of motion of the vehicle can be written as:

$$\dot{\overline{V}} = \frac{\overline{F}\delta(t)}{M} \qquad (4.1.1)$$

$$\dot{M} = \frac{|F|\delta(t)}{IG_e} \qquad (4.1.2)$$

These equations can be solved to yield:

$$|\Delta\overline{V}| = IG_e \ln\left(\frac{M_1}{(M_1 - M_f)}\right) \qquad (4.1.3)$$

where M_1 is the initial Mass, and M_f is the mass of fuel burned. Thus it can be seen that ΔV is a measure of the fuel burned.

4.2 Definition of the Cost Functional

In light of the discussion in section 4.1 above, let us define the "cost" function to be minimized, (J) as follows:

$$J = \Delta V_1 + \Delta V_2 + \Delta V_3 \qquad (4.2.1)$$

where ΔV_1 is the "ΔV equivalent" of the fuel used on Segment 1 of the trajectory, ΔV_2 is the ΔV equivalent for Segment 2, etc. Now the problem is to express J in terms of the state variables of the problem.

We can immediately write down

$$\Delta V_2 = \frac{IG_e}{V_0} \ln\left(\frac{1}{1 - x_{7f}}\right) \qquad (4.2.2)$$

where x_{7f} is the value of x_7 at the end of Segment 2.

4.2.1 Derivation of ΔV_1 and ΔV_3

For purposes of this problem, ΔV_1 will be defined as the ΔV required to make a minimum fuel two impulse transfer from circular orbit at radius R_a to radius R_0 with velocity components V_{x0} and V_{z0}. Similarly, ΔV_3 will be defined as the ΔV required to make a two impulse transfer from radius R_0 with velocity components V_{xf} and V_{zf} to a circular orbit at radius R_a. Since we are seeking an overall minimum fuel trajectory, it is appropriate to define ΔV_1 and ΔV_3 in relation to the optimum impulsive maneuver consistent with the constraints of the problem. The constraint in the exit maneuver is that no trajectory that results from this exit maneuver can come inside radius R_0 again. Because of symmetry of the vacuum trajectories, it is evident that the optimum 2 impulse exit maneuver, carried out in reverse, will also be the optimum entry maneuver. In view of this, let us consider the exit maneuver in some detail. The exit phase starts, by definition, when the trajectory crosses $R = R_0$ after completion of the required plane change. At this time we have vertical velocity $V_{zf} > 0$, and some horizontal velocity V_{xf}. We are looking for the optimum two-impulse maneuver that results in the vehicle being placed in a circular orbit at radius R_a, subject to the constraint that $R \geq R_0$ at all times. It can be shown (see Ref. 8) that the first impulse should be applied immediately in such a way as to produce a new elliptical orbit with apogee at R_a, followed by a second impulse to circularize the orbit at radius R_a. Let us call the first impulse $\Delta V_3'$. Let us designate the x and z components of $\Delta V_3'$ as I_x and I_z respectively. R. H. Munick et al (Ref. 8) have shown that the impulses that result in minimization of:

$$\Delta V_3 = \sqrt{I_x^2 + I_z^2} + \left[\frac{1}{\sqrt{R_a}} - \frac{V_x + I_x}{R_a} \right] \qquad (4.2.1.2)$$

under the conditions of this problem must satisfy the following relationships:

$$\frac{I_z}{I_x} = \frac{V_{zf}}{V_{xf} + 1/R_a \sqrt{\frac{2R_a}{R_a + 1}}} \qquad (4.2.1.2)$$

$$(V_{xf} + I_x)^2 \frac{(R_a^2 - 1)}{R_a^2} + (V_{zf} + I_z)^2 = \frac{2(R_a - 1)}{R_a} \qquad (4.2.1.3)$$

Unfortunately, use of (4.2.1.2) and (4.2.1.3) to write ΔV_3 as a function of V_{xf} V_{zf} and R_a results in a very unwieldy expression. For this reason one is induced to introduce an approximation at this point. Referring to (4.2.1.2) one observes that in the type of solutions which typically result from the atmospheric maneuver, $V_{zf} \ll V_{xf}$. Thus the optimum cost can be closely approximated by one in which $I_z = 0$. Increases in ΔV_3 using the approximation are less than 0.5% for typical trajectories encountered in this problem.

In view of the foregoing, we will define ΔV_3 by the following expressions: (R_0 is taken to be unity)

$$\Delta V_3 = |I_3| + \left[\frac{1}{\sqrt{R_a}} - \frac{V_{xf} + I_3}{R_a} \right] \qquad (4.2.1.4)$$

$$\frac{(R_a^2 - 1)}{R_a^2} (V_{xf} + I_3)^2 + V_{zf}^2 = \frac{2(R_a - 1)}{R_a} \qquad (4.2.1.5)$$

In a similar manner ΔV_1 will be defined by:

$$\Delta V_1 = |I_1| - \left[\frac{1}{\sqrt{R_a}} - \frac{V_{x0} + I_1}{R_a} \right] \qquad (4.2.1.6)$$

$$\frac{(R_a^2 - 1)}{R_a^2} (V_{x0}^2 + I_1)^2 + V_{z0}^2 = \frac{2(R_a - 1)}{R_a} \qquad (4.2.1.7)$$

At this juncture the need for a second approximation becomes evident. We propose to attack this problem by means of the gradient method. Use of the gradient method requires that the cost functional posses continuous first partial derivations with respect to its arguments, and thus use of the absolute value function is barred. It is therefore necessary to replace $|I_1|$ and $|I_3|$ with an approximate representation. The approximation we chose is:

$$|I_j| \approx \sqrt{I_j^2 + \epsilon} \qquad (4.2.1.8)$$

where ϵ is a small positive number.

Finally, after making the appropriate subsitution for I_1' and I_3 we have

$$J = \left[\frac{1}{\sqrt{R_a}} - \sqrt{A_1 - A_2 V_{z0}^2} \right] + \sqrt{I_1^2 + \epsilon} + A_4 \ln \frac{1}{1 - x_{7f}}$$

$$+ \left(\frac{1}{\sqrt{R_a}} - \sqrt{A_1 - A_2 V_{zf}^2} \right) + \sqrt{I_3^2 + \epsilon} \qquad (4.2.1.9)$$

$$I_1 = R_a \sqrt{A_1 - A_2 V_{z0}^2} - V_{x0} \qquad (4.2.1.10)$$

$$I_3 = R_a \sqrt{A_1 - A_2 V_{zf}^2} - V_{xf} \qquad (4.2.1.11)$$

$$A_1 = \frac{2}{R_a(1 + R_a)} \qquad (4.2.1.12)$$

$$A_2 = \frac{1}{R_a^2 - 1} \qquad (4.2.1.13)$$

(4.2.1.9) now represents the basic cost function to be minimized in this problem.

Let us now redefine (4.2.1.9) as:

$$J = \sum_{i=1}^{5} \Delta V_i \qquad (4.2.1.14)$$

where the ΔV's correspond to the 5 terms in (4.2.1.9). ΔV_1 is the first impulse required to de-orbit. ΔV_2 is a second impulse just before the start of the Synergetic maneuver. ΔV_3 is the fuel used in

the Synergetic maneuver. ΔV_4 is the impulse added at the end of the Synergetic maneuver to get back to distance R_a, and ΔV_5 the impulse to circularize at R_a.

CHAPTER 5

CONSTRAINTS AND BOUNDARY VALUES

In Chapter 3 we derived the equations of motion for the problem, and in Chapter 4 the cost functional. Two more basic elements are required before the complete optimal control problem can be formulated: the constraints and the boundary values. These elements are the subject of this chapter.

5.1 Constraints

As mentioned briefly in Chapter 1, if this maneuver is to be of any practical value in applied astronautics, the optimum found must be one which recognizes that in our current state of engineering knowledge, machinery and people are subject to limitations as to how hot they can get and how much acceleration they can withstand. Further, rocket engines cannot be designed to produce unlimited amounts of thrust, and in particular delta functions. It turns out that mathematical expressions for temperature and acceleration constraints are functions of the instantaneous values of the control and state variables and as such introduce serious difficulties into the optimal control problem. It is the presence of these constraints which makes this problem particularly interesting from a control engineering viewpoint.

5.1.1 Heating Constraint

There are three classes of constraints in this category that are of interest to the vehicle designer, total heat input, instantaneous heating rate, and surface temperature. The first of these constraints reduces to what is known as a terminal constraint in optimal control jargon and the theory for handling this type of constraint is well developed. Therefore the total heat input constraint will not be treated explicitly in the problem, although it is evident that if the instantaneous heating rate is kept below some given value at all times, the total heat input is automatically limited also. To get a mathematical expression

for surface temperature or heating rate let us assume that heating rate is a function of the 4th power of the stagnation temperature. An approximate expression for the stagnation temperature is given by Loh (See Ref. 9) as:

$$T^8 = C_1 \rho V^6 \qquad (5.1.1.1)$$

where C_1 is a constant of proportionality depending on the configuration of the body and the units of the variables. Thus, if one requires in this problem that:

$$\rho(t) V^6 \leq \beta_1, \quad 0 \leq t \leq t_f \qquad (5.1.1.2)$$

where β_1 is a constant, one will have limited both the stagnation temperature and the heating rate to be less than some value. Rewriting (5.1.1.2) in terms of the non-dimensional variables of Chapter 3, and using Assumption 1 and the result that $\gamma << 1$ or $V \approx V_x$, we have for our heating constraint equation:

$$Q_0(1 + x_1)^6 \exp\{-\frac{R_0 x_3}{h_s}\} \leq \beta \qquad (5.1.1.3)$$

5.1.2 Acceleration Constraints

While acceleration constraints are usually related to the vehicle axis system and may have different values in different directions, for purposes of this problem a constraint on the Euclidean norm of the resultant acceleration vector will be imposed. This constraint can be written in terms of the notation of section 3.2 as:

$$\sqrt{\left(\frac{dV_x}{dt}\right)^2 + \left(\frac{dV_z}{dt}\right)^2 + \left(\frac{F_y}{M}\right)^2} \leq A'_{max} \qquad (5.1.2.1)$$

Unfortunately, when this expression is expanded it becomes very complex and increases the volume of the computations considerably. As will become evident later, this entire problem is non-trivial from a computational viewpoint and severely strains the capacity of an IBM 7094 class computer. Let us, therefore, choose the simpler thrust acceleration constraint for the actual computations, to wit:

$$\sqrt{\left(\frac{T_x}{M}\right)^2 + \left(\frac{T_y}{M}\right)^2 + \left(\frac{T_z}{M}\right)^2} \leq A_{max} \qquad (5.1.2.2)$$

This constraint limits the acceleration due to the thrust of the rocket engine only. It will turn out in most cases that if the heating constraint is satisfied and the thrust acceleration constraint is satisfied, (5.1.2.1) is also satisfied, so one achieves the desired result with a substantial saving in the number of computations required. Note that (5.1.2.2) is a control variable inequality constraint rather than a state variable constraint since the control variables appear explicitly in the constraint equation. Rewriting (5.1.2.2) in terms of the non-dimensional variables of Chapter 3 we have for our thrust acceleration constraint:

$$\frac{\sqrt{U_3^2 + U_4^2 + U_5^2}}{(1 - x_7)} \leq A_{max} \qquad (5.1.2.3)$$

5.2 Boundary Values

Most of the boundary values for the problem have been implicitly defined by the discussion in section 4.1, the manner in which the non-dimensional state variables were defined and the derivation of the cost functional.

5.2.1 Initial Values

The initial boundary values not implicitly defined already are those for the navigation angles, Heading, (ψ or x_4), Latitude, (L or x_6) and Longitude, (λ or x_5). Without any loss in generality and at a considerable advantage in simplicity, one can take $\psi = \frac{\pi}{2}$, $L = 0$, $\lambda = 0$. This corresponds to starting in an equatorial orbit at the Greenwich Meridian, headed East. For an arbitrary starting condition one can consider a rotation of the inertial reference frame described in section 3.1.1 to align the equatorial plane with the initial orbital plane and take $\lambda = 0$ to occur at the time $R = R_0$. Thus the position coordinates presented in the solutions to be developed can be thought of as referred to any initial orbital plane whatsoever. Hence, we have for

initial values: $x_1(0)$ and $x_2(0)$ (not specified since they are variables in the cost functional to be found as part of the solution) and

$$x_3(0) = x_5(0) = x_6(0) = x_7(0) = 0 \qquad (5.2.1.1)$$

$$x_4(0) = \frac{\pi}{2} \qquad (5.2.1.2)$$

5.2.2 Final Values

$x_1(t_f)$, $x_2(t_f)$ and $x_7(t_f)$ are to be found as part of the optimization procedure. $x_3(t_f)$ is specified to be zero as one of the conditions defining the end of Segment 2 of the trajectory (see section 4.1). $x_5(t_f)$ (Longitude) is free since we do not care what its final value is. x_4 (Heading) and x_6 (Latitude) must be specified as an implicit relationship that provides the desired amount of plane change (Δi). Since we have chosen our initial values so as to be in an equatorial orbit, ($i = 0$), we can use equation (3.3.5) to give:

$$\cos \Delta i = \sin x_4(t_f) \cos x_6(t_f) \ . \qquad (5.2.2.1)$$

This together with

$$x_3(t_f) = 0 \qquad (5.2.2.2)$$

gives our final boundary value specifications.

CHAPTER 6

FORMULATION OF THE OPTIMAL CONTROL PROBLEM

In Chapters 3, 4, and 5 the necessary relationships have been developed to enable a formal formulation of the optimal control problem to be made. In the interests of compact notation let us use vector notation wherein $\bar{S}$ refers to a vector or a vector function and S_j refers to the jth component of this vector or vector function.

6.1 The Optimal Control Problem

Given the Cost Functional:

$$J = J(\bar{x}(t_f),\ \bar{x}(0)) \tag{6.1.1}$$

where $\bar{x}(t)$ is a solution of the differential equations:

$$\dot{\bar{x}}(t) = \bar{f}[\bar{x}(t),\ \bar{U}(t)] \quad 0 \le t \le t_f \tag{6.1.2}$$

and $\bar{U}(t)$ is a control vector. Find $\bar{U}(t)$ $(0 \le t \le t_f)$, $x_1(0)$, $x_2(0)$ and t_f, such that J is a minimum and:

A. The following boundary values are satisfied:

$$x_3(0) = x_5(0) = x_6(0) = x_7(0) = 0 \tag{6.1.3}$$

$$x_4(0) = \frac{\pi}{2} \tag{6.1.4}$$

$$x_3(t_f) = 0 \tag{6.1.5}$$

$$\sin x_6(t_f)\cos x_6(t_f) = \cos \Delta i \tag{6.1.6}$$

B. The following equations of constraint are satisfied:

$$Q_0(1 + x_1)^6 \exp\left\{-\frac{R_0 x_3}{h_s}\right\} \le \beta \quad 0 \le t \le t_f \tag{6.1.8}$$

$$\frac{\sqrt{U_3^2 + U_4^2 + U_5^2}}{(1 - x_7)} \le A_{max} \quad 0 \le t \le t_f \tag{6.1.9}$$

CHAPTER 7

ANALYTIC INVESTIGATION OF THE SYNERGETIC OPTIMAL CONTROL PROBLEM

If one ignores the heating constraint, the optimal control problem as formulated falls within the class of problems to which the Pontryagin Maximum Principle applies. (See Ref. 7.) By use of the maximum principle one can transform the problem from one of finding an unknown vector function, $(\overline{U}(t))$, to one of finding a set of boundary values to the differential equations of constraint and a set of auxiliary differential equations derived from them. Unfortunately in this problem we have no effective way to find the unknown boundary values , so use of the Maximum Principle will not be of any direct help. However, use of this technique will help us gain an insight into the nature of the optimal solution. In addition, through use of this principle, a solution to a different problem will be developed that can be considered a limiting case of the actual problem and thus provide a measure of the "goodness" of the approximate solutions to be found. Further, by investigation of the dynamic behavior of the inclination angle we will gain still further insight into the optimal control strategy.

7.1 Application of the Maximum Principle to the Problem of Chapter 6 Minus the Heating Constraints

We assume the reader is familiar with the Maximum Principle and will proceed to apply it directly with a minimum of explanation.

7.1.1 The Auxiliary Differential Equations

Direct use of solutions to these equations is not required in this chapter. However, their solution will be required to apply the gradient method in the subsequent work, so let us take this opportunity to write down their definitions.

The auxiliary equations are defined as follows:

Given the differential equations:

$$\dot{\overline{x}}(t) = \overline{f}(\overline{x}, \overline{u}) \qquad 0 \le t \le t_f \qquad (7.1.1.1)$$

the auxiliary differential equations are defined by

$$\dot{\bar{y}}(t) = -\left[\frac{\partial \bar{f}}{\partial \bar{x}}\right]^T \bar{y}(t) \qquad 0 \le t \le t_f \tag{7.1.1.2}$$

where $\left[\frac{\partial \bar{f}}{\partial \bar{x}}\right]$ is a time-varying 7 × 7 matrix where elements are made up of the indicated terms evaluated along a solution of (7.1.1.1).

For convenience let us re-name $\left[\frac{\partial \bar{f}}{\partial \bar{x}}\right]$ as A(t).

These equations are variously referred to as adjoint equations, costate equations, Lagrange multipliers, auxiliary equations, etc., depending on the use to which they are put and the boundary values used. In the work on gradient solutions these equations will be referred to as adjoint equations. In the remainder of this chapter we will use the same terminology, but the reader should bear in mind that the evaluation of the matrix A(t) is defined along an optimal solution, and the boundary values are different than those specified in subsequent chapters.

7.1.2 Transversality Conditions

Let us now examine the transversality conditions which serve to partially determine the boundary values on the adjoint equations. If we write the boundary conditions as a vector function: $(\overline{M})$

$$\overline{M} = \overline{M}[\bar{x}(0),\ \bar{x}(t_f)] = 0 \tag{7.1.2.1}$$

the transversality conditions can be written as

$$\bar{y}^T(0) = -\left[\frac{\partial J}{\partial \bar{x}(0)} + \bar{\gamma}^T \frac{\partial \overline{M}}{\partial \bar{x}(0)}\right] \tag{7.1.2.2}$$

and

$$\bar{y}^T(t_f) = \left[\frac{\partial J}{\partial \bar{x}(t_f)} + \bar{\gamma}^T \frac{\partial \overline{M}}{\partial \bar{x}(t_f)}\right] \tag{7.1.2.3}$$

where $\bar{\gamma}$ is a vector of undetermined constants of the same dimension as $\overline{M}$.

In addition, it can be shown that the Hamiltonian function for

this problem, defined as:

$$H(t) = \sum_{i=1}^{7} y_i(t)\,\dot{x}_i(t) \qquad (7.1.2.4)$$

must satisfy the following requirements along an optimal trajectory:

$$H(t) = H(0) = H(t_f) = 0 \qquad (7.1.2.5)$$

Applying (7.1.2.2) and (7.1.2.3) to the appropriate expressions, one obtains:

$$y_1(0) = \frac{I_1}{\sqrt{I_1^2 + \epsilon}} \qquad (7.1.2.6)$$

$$y_2(0) = -\frac{A_2 x_2(0)}{\sqrt{A_1 - A_2 x_2^2(0)}}\left[1 - \frac{R_a I_1}{\sqrt{I_1^2 + \epsilon}}\right] \qquad (7.1.2.7)$$

where

$$I_1 = R_a\sqrt{A_1 - A_2 x_2^2(0)} - (1 + x_1(0)) \qquad (7.1.2.8)$$

$$y_3(0) = -\gamma_1 \qquad (7.1.2.9)$$

$$y_4(0) = -\gamma_2 \qquad (7.1.2.10)$$

$$y_5(0) = -\gamma_3 \qquad (7.1.2.11)$$

$$y_6(0) = -\gamma_4 \qquad (7.1.2.12)$$

$$y_7(0) = -\gamma_5 \qquad (7.1.2.13)$$

$$y_1(t_f) = -\frac{I_3}{\sqrt{I^2 + \epsilon}} \qquad (7.1.2.14)$$

$$y_2(t_f) = \frac{A_2 x_2(t_f)}{\sqrt{A_1 - A_2 x_2^2(t_f)}}\left[1 - \frac{R_a I_3}{\sqrt{I_3^2 + \epsilon}}\right] \qquad (7.1.2.15)$$

where

$$I_3 = R_a\sqrt{A_1 - A_2 x_2^2(t_f)} - (1 + x_1(t_f)) \qquad (7.1.2.16)$$

$$y_3(t_f) = \gamma_6 \qquad (7.1.2.17)$$

$$y_4(t_f) = \gamma_7 \cos x_4(t_f) \cos x_6(t_f) \qquad (7.1.2.18)$$

$$y_5(t_f) = 0 \qquad (7.1.2.19)$$

$$y_6(t_f) = -\gamma_7 \sin x_4(t_f) \sin x_6(t_f) \qquad (7.1.2.20)$$

$$y_7(t_f) = \frac{A_4}{[1 - x_7(t_f)]} \qquad (7.1.2.21)$$

A comparison of these 14 equations, considering that a detailed examination of the adjoint equations will show that $y_5(t) = 0$, $0 \le t \le t_f$, yields the following unknown quantities to be found:

$x_1(0)$	γ_4
$x_2(0)$	γ_5
γ_1	γ_7
γ_2	t_f

We will show shortly that the use of the maximum principle will permit the control vector $\bar{U}(t)$ to be expressed as a function of the adjoint and state variables, so we have succeeded in transforming the problem of finding 5 unknown functions of time, $x_1(0)$, $x_2(0)$, and t_f into the problem of finding 8 unknown constants:

$$\gamma_1,\ x_1(0),\ x_2(0),\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_7 \text{ and } t_f \text{ itself.}$$

From a theoretical point of view we have accomplished a great deal. As a practical matter, the computational problem of finding the right set of values to satisfy the boundary values of the problem is a non-trivial task.

Now let us derive some relationships between $\bar{U}(t)$, $\bar{y}(t)$ and $\bar{x}(t)$ along an optimal trajectory.

7.1.3 The Variational Hamiltonian

The Maximum Principle gives the following relationship applicable to this problem:

Given the variational Hamiltonian function H(t) defined by (7.1.2.4) and the sign convention chosen for (7.1.2.2) and (7.1.2.3), a necessary condition for a trajectory to be optimal is that $\bar{U}(t)$ be chosen so that H(t) is a minimum, $(0 \le t \le t_f)$ and the transversality conditions are satisfied.

To find the $\overline{U}(t)$ that satisfies the statement above, we can apply the necessary conditions of the theory of ordinary minima to (7.1.2.4), and we find that for $\overline{U}(t)$ to be optimal,

$$\overline{y}^T \left[\frac{\partial \overline{f}}{\partial \overline{U}} \right] = 0, \quad \text{subject to the constraints on } \overline{U} \qquad (7.1.3.1)$$

An expansion of (7.1.2.4) will show that the Hamiltonian can be separated into three parts relative to the control vector $\overline{U}$, a part which does not depend on $\overline{U}$, a part which depends on U_1 and U_2 (angle of attack and bank) and a part which depends on U_3, U_4 and U_5 (the three thrust components). Let us designate these three parts of the Hamiltonian H_0, H_1 and H_2 respectively and examine H_1 and H_2 individually.

7.1.3.1 Minimization of H_1

For convenience in notation in this chapter, we will define the following:

$$(1 + x_1) = v > 0 \qquad 0 \le t \le t_f$$

$$(1 - x_7) = m > 0 \qquad 0 \le t \le t_f$$

The reader can verify from Chapter 3 that v is the non-dimensional horizontal velocity component and m is the non-dimensional mass. With these definitions H_1 is:

$$H_1 = -\frac{y_1 Q C_d(U_1)}{m} + \frac{y_2 Q C_L(U_1) \cos U_2}{m} - \frac{y_4 Q C_L(U_1) \sin U_2}{mv} \qquad (7.1.3.1.1)$$

For greater insight let us assume that U_1 can be eliminated from the expressions given in Chapter 3 for $C_L(U_1)$ and $C_d(U_1)$ and one can write $C_d = C_d(C_L)$. The work that follows is not mathematically rigorous for a number of reasons, including the fact that C_d is a two-valued function of C_L and does not have continuous derivatives with respect to C_L at two points. Since there is no closed form solution for this problem and the purpose of this chapter is to gain an insight into some of the behavior to be expected from the gradient solutions described

subsequently, we will proceed without attempting to introduce the mathematical rigor (and its attendant complications) which would be expected for the development of an analytic solution.

Because of the definition of the aerodynamic model and the angles U_1 and U_2 there are two possible values of U_1 and U_2 for each orientation of the lift vector. To limit the consideration of possible combinations, let us introduce a constraint on C_L (or U_1) as follows:

$$C_{L(Max)} > C_L > 0 \qquad (7.1.3.1.2)$$

Further, in view of the initial boundary values specified in Chapter 5 one can argue from physical reasons that if the plane change is limited to going from $i = 0°$ to $i \leq 90°$ one can expect U_2 (bank angle) to be limited to:

$$0 \leq U_2 \leq 180° \qquad (7.1.3.1.3)$$

Keeping these restrictions in mind, let us formally take the partial derivatives of H_1 with respect to C_L and U_2:

$$\frac{\partial H_1}{\partial U_2} = -\frac{y_2 Q C_L \sin U_2}{m} - \frac{y_4 Q C_L \cos U_2}{mv} \qquad (7.1.3.1.4)$$

$$\frac{\partial H_1}{\partial C_L} = -\frac{y_1 Q \frac{dC_d}{dC_L}}{m} + \frac{y_2 Q \cos U_2}{m} - \frac{y_4 Q \sin U_2}{mv} \qquad (7.1.3.1.5)$$

Subject to several restrictions on variables not being zero, etc., one may conclude that to minimize H_1, the following relationships must hold:

$$\sin U_2 = -\frac{y_4 \cos U_2}{y_2 v} \qquad (7.1.3.1.6)$$

$$\frac{dC_d}{dC_L} = \frac{y_2 \cos U_2 - \frac{y_4}{v} \sin U_2}{y_1} \qquad (7.1.3.1.7)$$

(7.1.3.1.6) can be manipulated in several ways. In particular, for (7.1.3.1.3) to be true:

$$\sin U_2 = \frac{\left|\frac{y_4}{v}\right|}{\sqrt{\frac{y_4^2}{v^2} + y_2^2}} \qquad (7.1.3.1.8)$$

$$\cos U_2 = \pm \frac{|y_2|}{\sqrt{\frac{y_4^2}{v^2} + y_2^2}} \qquad (7.1.3.1.9)$$

Using (7.1.3.1.6) directly in (7.1.3.1.7) one gets:

$$\frac{dC_d}{dC_L} = \frac{\cos U_2 [y_2^2 + \frac{y_4^2}{v^2}]}{y_1 y_2} \qquad (7.1.3.1.10)$$

Using (7.1.3.1.8) and (7.1.3.1.9) in (7.1.3.1.7) one gets:

$$\frac{dC_d}{dC_L} = \pm \frac{y_2 |y_2| - \frac{y_4}{v}\left|\frac{y_4}{v}\right|}{y_1 \sqrt{\frac{y_4^2}{v^2} + y_2^2}} \qquad (7.1.3.1.11)$$

Now with the transversality conditions on y_1 and y_2 given in paragraph 7.1.2, one can reach certain conclusions about C_L and U_2 at the initial and final times. Let us consider these transversality conditions. One notes that $y_1(0)$, $y_2(0)$, $y_1(t_f)$ and $y_2(t_f)$ are functions of the initial and final values of the state variables x_1 and x_2 which are included in the cost functional. Hence one cannot make absolute judgments on their values without having solved the problem. We note, however, that $y_1(0)$ and $y_1(t_f)$ are derivatives of approximations of $|I_1|$ and $|I_3|$ respectively. Hence if one assumes that I_1 and I_3 are not 0, and since the number ϵ is arbitrary, one may say:

$$y_1(0) \approx \text{sgn}(I_1) \qquad (7.1.3.1.12)$$

$$y_1(t_f) \approx -\text{sgn}(I_3) \qquad (7.1.3.1.13)$$

Further, the definition of the boundary values of the problem are such that:

$$x_2(0) < 0$$

$$x_2(t_f) > 0$$

$$R_a > 1$$

Thus one may conclude, subject to $I_1 \neq 0$ and $I_3 \neq 0$, that:

$$y_1(0)\, y_2(0) < 0 \qquad (7.1.3.1.14)$$

$$y_1(t_f)\, y_2(t_f) > 0 \qquad (7.1.3.1.15)$$

Applying these conclusions to (7.1.3.1.10) and recalling that (7.1.3.1.2) implies that $\frac{dC_L}{dC_d} > 0$, one concludes that $\cos U_2(0) < 0$ or that $U_2 > 90°$ at $t = 0$. By similar reasoning one can conclude that $U_2 < 90°$ at $t = t_f$.

Using the above conclusions, one can show additional conclusions about (7.1.3.1.11) at the initial and final times as well as conclusions about the sign of the initial and final boundary value of y_4 in relation to y_2.

At $t = 0$, the requirement for $U_2 > 90°$ taken with (7.1.3.1.6) implies that $y_1(0)$ and $y_4(0)$ must be of the same sign. Further, the sign ambiguity of (7.1.3.9) is now resolved. Thus at $t = 0$ and $t = t_f$ we have:

$$\frac{dC_d}{dC_L} = \frac{\sqrt{\frac{y_4^2}{v^2} + y_2^2}}{|y_1|} \qquad (7.1.3.1.16)$$

In fact (7.1.3.1.16) must be true for all time, barring the singularities implicit in the derivation. Further, one concludes that:

$$y_2(0)\, y_4(0) > 0 \qquad (7.1.3.1.17)$$

$$y_2(t_f)\, y_4(t_f) < 0 \qquad (7.1.3.1.18)$$

Next, let us examine the part of H depending on U_3, U_4 and U_5.

7.1.3.2 Minimization of H_2

H_2 is in detail:

$$H_2 = \frac{y_1 U_3}{m} + \frac{y_2 U_4}{m} + \frac{y_4 U_5}{m} + \frac{y_7 V_0 \sqrt{U_3^2 + U_4^2 + U_5^2}}{IG_e} \qquad (7.1.3.2.1)$$

We have a constraint on U_3, U_4 and U_5, to wit:

$$\frac{\sqrt{U_3^2 + U_4^2 + U_5^2}}{m} \leq A_{max} \qquad (7.1.3.2.2)$$

For convenience in notation, let us lump the constant $\frac{V_0}{IG_e}$ into a single number, P. One can proceed formally using the techniques of constrained minimization to solve for the thrust components. However, the author feels that the following heuristic derivation will give more insight into the behavior of these controls on an optimal trajectory. If one proceeds to take the partial derivatives of (7.1.3.2.1) with respect to the thrust controls when the constraint is not active and set them equal to zero one obtains:

$$U_3 = - \frac{y_1 P \sqrt{U_3^2 + U_4^2 + U_5^2}}{y_7 m} \qquad (7.1.3.2.3)$$

$$U_4 = - \frac{y_2 P \sqrt{U_3^2 + U_4^2 + U_5^2}}{y_7 m} \qquad (7.1.3.2.4)$$

$$U_5 = - \frac{y_4 P \sqrt{U_3^2 + U_4^2 + U_5^2}}{y_7 mv} \qquad (7.1.3.2.5)$$

Substituting these into the equation for H_2 the result is:

$$H_2 = - \frac{P \sqrt{U_3^2 + U_4^2 + U_5^2}}{m} \left[\frac{(y_1^2 + y_2^2 + \frac{y_4^2}{v^2}) - \frac{y_7^2}{m}}{y_7 m} \right] \qquad (7.1.3.2.6)$$

Let us define a discriminant function from (7.1.3.2.6) above:

$$K(\bar{x}, \bar{y}) = \frac{(y_1^2 + y_2^2 + \frac{y_4^2}{v^2}) - \frac{y_7^2}{m}}{y_7 m} \qquad (7.1.3.2.7)$$

Now (7.1.3.2.6) can be written:

$$H_2 = - \frac{P \sqrt{U_3^2 + U_4^2 + U_5^2}}{m} K(\bar{x}, \bar{y}) \qquad (7.1.3.2.8)$$

Now, in view of the thrust acceleration constraint, it is evident that to minimize H_2, the following rules apply:

A. For $K(x, y) > 0$

$$\frac{\sqrt{U_3^2 + U_4^2 + U_5^2}}{m} = A_{max} \qquad (7.1.3.2.9)$$

B. For $K(x, y) < 0$

$$\frac{\sqrt{U_3^2 + U_4^2 + U_5^2}}{m} = 0 \qquad (7.1.3.2.10)$$

and finally we come to an interesting third circumstance.

If $K = 0$ for a finite time interval, there exists a possiblity for thrust at some intermediate level. Miele, (Ref. 11) shows that in some cases of rocket powered flight in the presence of drag forces it is possible to have $K = 0$, $\dot{K} = 0$ simultaneously and for an optimal solution to exist with thrust at some intermediate level. To see why intermediate thrust is possible in this case and not in the case of trajectories in a vacuum, let us consider a generalized flight mechanics problem for motion of a point mass. We go to a generalized problem at this point because the particular coordinate system selected for the state equations in this problem do not lend themselves to easy insight into the mechanisms involved.

By proper choice of a coordinate system, any flight mechanics problem utilizing rocket propulsion can be written in the form:

$$\dot{V}_i = I_i(\bar{S}, \bar{V}) + \frac{F_i}{M}(\bar{S}, \bar{V}) + \frac{T_i}{M}, \quad i = 1...3 \qquad (7.1.3.2.11)$$

$$\dot{S}_i = V_i, \quad i = 1...3 \qquad (7.1.3.2.12)$$

$$\dot{M} = -C\sqrt{\Sigma T_i^2} \qquad (7.1.3.2.13)$$

where $\bar{V}$ is an acceleration vector, $\bar{S}$ a displacement vector, $\bar{F}$ an aerodynamic force vector such that $|\bar{F}| \neq 0$ and $\bar{I}$ is a vector function of field and inertial forces.

Now let us consider an optimal control problem constrained by this system of 7 equations where the cost is any function whatsoever of initial and final values of the state variable. The variational Hamiltonian in such cases will be of the form:

$$H = \sum_{i=1}^{3} \lambda s_i V_i + \sum_{i=1}^{3} \lambda a_i I_i + \sum_{i=1}^{3} \lambda v_i \frac{F_i}{M} + \sum_{i=1}^{3} \lambda v_i \frac{T_i}{M} - C\lambda m \sqrt{\Sigma T_i^2} \qquad (7.1.3.2.14)$$

where $\bar{\lambda}s$, $\bar{\lambda}v$ and λm are the Lagrange multipliers adjoint to the displacement, velocity and mass equations, respectively. For systems of this form, one always arrives at the necessary condition on the thrust components as follows:

$$T_i = \frac{\lambda v_i \sqrt{\Sigma T_j^2}}{C \lambda m M} \quad i = 1 \ldots 3 \qquad (7.1.3.2.15)$$

when there is a constraint on thrust, or thrust acceleration of the form:

$$\frac{\sqrt{\Sigma T_j^2}}{M} \leq A_{max} \quad \text{or} \quad \sqrt{\Sigma T_j^2} \leq T_{max}$$

one ends up with a generalized equation of the form (7.1.3.2.14)

$$H_2 = -C \sqrt{\Sigma T_j^2}\ K_T(\lambda v, \lambda m, M) \qquad (7.1.3.2.16)$$

where

$$K_T = \frac{\Sigma \lambda v^2 - C^2 M^2 \lambda m^2}{M^2 \lambda m} \qquad (7.1.3.2.17)$$

and we conclude that full thrust is required when $K_T > 0$, no thrust when $K_T < 0$. Thus we are invariably faced with the interesting question as to what goes on when $K_T = 0$. If $K_T = 0$ only at isolated instants of time, the question becomes moot. However, if it is possible for $K_T = 0$, and $\dot{K}_T = 0$ simultaneously, then the possibility for intermediate thrust levels exists. To look into this possibility let us consider the differential equations the Lagrange multipliers must satisfy. These are of the form:

$$\dot{\lambda} v_i = -\sum_{j=1}^{3} \frac{\partial [I_i + F_i/m]}{\partial V_j} \lambda v_j - \lambda s_i \qquad (7.1.3.2.18)$$

$$\dot{\lambda} s_i = -\sum_{j=1}^{3} \frac{\partial [I_i + F_i]}{\partial S_j} \lambda v_j \qquad (7.1.3.2.19)$$

$$\dot{\lambda} m = \sum_{j=1}^{3} \frac{(F_j + T_j)}{M^2} \lambda v_j \qquad (7.1.3.2.20)$$

The important point to notice here is that the equations for λv and λs do not depend on λm, or explicitly on $\bar{T}$, and that the equation for λm

does not depend on λ_m or on $\bar{\lambda}_s$.

Now let us substitute in equation (7.1.3.2.20) from the necessary conditions on the T_j, equation (7.1.3.2.15). Then on an optimal trajectory λ must be:

$$\dot{\lambda}_m = \sum_{j=1}^{3} \frac{F_j \lambda_{v_j}}{M^2} + \frac{\sqrt{\Sigma T_j^2}}{C \lambda_m M^3} \sum_{j}^{3} \lambda_{v_j}^2 \qquad (7.1.3.2.21)$$

Now let us consider the case where no aerodynamic forces are acting: If $F \equiv 0$, then we have from (7.1.3.2.21) a necessary condition for an optimal trajectory:

$$\lambda_m \dot{\lambda}_m \geq 0 \qquad (7.1.3.2.22)$$

If, in addition, we require that $K_T = 0$ from (7.1.3.2.16) in order to allow intermediate thrust, and apply the consequences of this relationship to the equation for $\dot{\lambda}_m$ with $\bar{F} \equiv 0$ we get:

$$\dot{\lambda}_m = - \frac{C \sqrt{\Sigma T_j^2}}{M} \lambda_m \qquad (7.1.3.2.23)$$

which is clearly inconsistent with equation (7.1.3.2.21) above. Hence, we conclude: the possibility of intermediate thrust cannot occur in the absence of aerodynamic forces.

Next let us consider the case where aerodynamic forces are present. Application of the criteria for optimality and intermediate thrust leads to:

$$\dot{\lambda}_m = \sum_{j=1}^{3} \frac{F_j \lambda_{v_j}}{M^2} - \frac{C \sqrt{\Sigma T_j^2}}{M} \lambda_m \qquad (7.1.2.3.24)$$

and this equation would lead one to suspect that there are some cases wherein it would be possible to have λ_m vary in such a way as to satisfy equation (7.1.3.2.22) for a finite period of time. Let us consider the complete range of information available to us now. Let us designate the part of the Hamiltonian not depending on $\bar{T}$ as H_1 and the balance by H_2. For the condition of intermediate thrust to occur we have shown

that:

$$H_2 = 0 \qquad (7.1.2.3.25)$$

It follows that on an optimal trajectory with intermediate thrust the following relationships must apply:

$$H_1(\bar{\lambda}v, \bar{\lambda}s, \bar{V}, \bar{S}, M) = 0 \qquad (7.1.3.2.26)$$

$$\dot{H}_1(\bar{\lambda}v, \bar{\lambda}s, \bar{V}, \bar{S}, \bar{T}, M) = 0 \qquad (7.1.3.2.27)$$

$$\frac{\partial H_2}{\partial T_1}(\bar{\lambda}v, \bar{\lambda}m, M, \bar{T}) = 0 \qquad (7.1.3.2.28)$$

$$\frac{\partial H_2}{\partial T_2}(\bar{\lambda}v, \lambda m, M, \bar{T}) = 0 \qquad (7.1.3.2.29)$$

$$\frac{\partial H_3}{\partial T_3}(\bar{\lambda}v, \lambda m, M, \bar{T}) = 0 \qquad (7.1.3.2.30)$$

$$K_T(\bar{\lambda}v, \lambda m, M) = 0 \qquad (7.1.3.2.31)$$

$$\dot{K}_T(\bar{\lambda}v, \bar{\lambda}s, \bar{v}, \bar{s}, \bar{T}, M, \lambda m) = 0 \qquad (7.1.3.2.32)$$

Thus we have a system of 7 equations. If one of the components of $\bar{\lambda}s$ is a constant, these seven equations may, under some conditions, be combined to yield a single scalar function, $G(\bar{V}, \bar{S}, M, \bar{T}) = 0$, which will provide a necessary condition for an optimal trajectory which will allow intermediate thrust. The Synergetic problem does in fact have a component of $\bar{\lambda}s$ constant, and thus is a candidate for an intermediate thrust solution under some conditions.

Now let us again particularize to our problem. It is interesting to see if the y_7 adjoint differential equation can have the required behavior to allow for intermediate thrust. The equation is:

$$\dot{y}_7 = -A_{17}y_1 - A_{27}y_2 - A_{47}y_4 \qquad (7.1.3.2.33)$$

Substituting for the elements of the A(t) matrix one obtains:

$$\dot{y}_7 = \frac{QC_d y_1}{m^2} - \frac{QC_L \cos U_2 y_2}{m^2} + \frac{QC_L \sin U_2 y_4}{m^2} - \frac{U_3 y_1}{m^2} - \frac{U_4 y_2}{m^2} - \frac{U_5 y_4}{m^2 v} \qquad (7.1.3.2.34)$$

Substituting from the previous work for the optimal values of U_2, U_3, U_4 and U_5, one obtains:

$$\dot{y}_7 = \frac{QC_d y_1}{m^2} + \frac{QC_L\left[\frac{y_4|y_4|}{v^2} \pm y_2|y_2|\right]}{m^2\sqrt{y_2^2 + \frac{y_4^2}{v^2}}}$$

$$+ \frac{\sqrt{\sum_{i=3}^{5} U_i^2}}{y_7 m^3}\left[y_1^2 + y_2^2 + \frac{y_4^2}{v^2}\right] \qquad (7.1.3.2.35)$$

where the sign in the second term is minus for $0 \leq U_2 \leq 90°$ and plus for $90° \leq U_2 \leq 180°$.

Taken with the restriction $C_L > 0$, it can be seen that it is possible for $\dot{y}_7$ to be of either sign and thus possible for y_7 to vary in an arbitrary manner so as to make the discriminant function (7.1.3.2.7) zero for a finite length of time.

7.2 A Limiting Case Unconstrained Solution

One result of the analog investigation of this problem (of which more will be said later) was that, in the absence of heating and acceleration constraints, trajectories flown at progressively higher levels of dynamic pressure showed progressively lower and lower costs. This leads one to speculate that if one could obtain very high levels of dynamic pressure one could get very low cost. This in turn leads to the formulation of a new optimal control problem which can be viewed as a limiting case of the actual problem: the equations of motion of this problem can be written in the following form:

$$\dot{x}_1 = f_1(\bar{x}) - g_1(\bar{x})C_d(U_1) + h_1(\bar{x})U_3 \qquad (7.2.1)$$

$$\dot{x}_2 = f_2(\bar{x}) - g_1(\bar{x})C_L(U_1)\cos U_2 + h_1(\bar{x})U_4 \qquad (7.2.2)$$

$$\dot{x}_3 = f_3(\bar{x}) \qquad (7.2.3)$$

$$\dot{x}_4 = f_4(\bar{x}) - \frac{g_1(\bar{x})}{(1+x_1)}C_L(U_1)\sin U_2 + \frac{h_1(\bar{x})}{(1+x_1)}U_5 \qquad (7.2.4)$$

$$\dot{x}_5 = f_5(\bar{x}) \tag{7.2.5}$$

$$\dot{x}_6 = f_6(\bar{x}) \tag{7.2.6}$$

$$\dot{x}_7 = C\sqrt{U_3^2 + U_4^2 + U_5^2} \tag{7.2.7}$$

where the $f_i(\bar{x})$ ($i = 1 \ldots 6$), $h_1(\bar{x})$ and $g_1(\bar{x})$ are functions of the state variables whose exact definition may be found in Chapter 3.

Now, if there is any pay-off in this problem over using propulsion alone, it lies in the proposition that one can get aerodynamic side force to drag ratios greater than 1. For this limiting solution let us ignore the propulsion terms, and set up the following set of differential equations:

$$\dot{x}_1 = f_1(\bar{x}) - A_1 \tag{7.2.8}$$

$$\dot{x}_2 = f_2(\bar{x}) \tag{7.2.9}$$

$$\dot{x}_4 = f_4(\bar{x}) - \frac{A_2}{(1 + x_1)} \tag{7.2.10}$$

Equations for $\dot{x}_3$, $\dot{x}_5$, and $\dot{x}_6$ are as before. Assume the following constraints on A_1 and A_2, which are our aerodynamic force terms:

$$A_1 \geq 0$$

$$A_2 \geq 0$$

$$E_2 A_1 \geq A_2 \quad E_2 \text{ a constant, } 0 \leq E_2 \leq 1 \tag{7.2.11}$$

This set of relationships describes a problem similar to the problem described in Chapter 3 with the following changes:

A. The rocket thrust terms have been deleted from the state equations.

B. The aerodynamic force terms have been replaced with a two dimensional control vector $\bar{A}$ lying in a horizontal plane with one component in the x (drag) direction and the other component in the y (side force) direction.

C. The constraints on the aerodynamic force vector $\bar{A}$ are an idealization of the real world situation in that it is possible to have zero drag when there is zero side force acting.

D. The drag force (A_1) can be of any value as long as it is greater than a fixed fraction (E_2) of the side force (A_2). This is in contrast to the aerodynamic model used in Chapter 3 where the

drag force was directly related to the side or lift force. Note that $1/E_2$ corresponds to L/D max in the model used in Chapter 3.

Let us specify boundary values: $M[x_1(0), x_2(0), x_3(0)] = 0$ where M is the energy integral with a value consistent with an elliptical orbit. Also let $x_4(0) = \pi/2$, $x_6(0) = 0$. (Equatorial orbit).

Now let us specify a cost functional to be minimized:

$$J = \int_0^{t_f} \dot{i}\, dt + \int_0^{t_f} A_1\, dt \qquad (7.2.12)$$

We note that with the restrictions we have placed on A_2 and the boundary values selected, $\dot{i}$ will always be negative and minimization of J will correspond to the largest possible plane change with the smallest possible drag loss.

Now at this moment our problem is meaningless inasmuch as A_1 and A_2 are not bounded. The control variables must belong to a bounded set of some kind, so let us specify that:

$$\int_0^{t_f} \sqrt{A_2^2}\ dt = E_1 \qquad (7.2.13)$$

i.e., we have a fixed total impulse of side force available.

7.2.2 Now let us put all these elements together in an optimal control problem. Noting that $\dot{i}$ can be written in the form:

$$\dot{i} = -f_0(\bar{x})A_2 \qquad (7.2.1.1)$$

where $f_0(\bar{x})$ is a function of the state variables alone, we have the following optimal control problem.

Given the cost functional, J,

$$J = -\int_0^{t_f} f_0(x)\ A_2 + \int_0^{t_f} A_1\, dt \qquad (7.2.1.2)$$

where $\bar{x}(t)$ is a solution to the following differential equations for $0 \le t \le t_f$,

$$\dot{x}_1 = f_1(\bar{x}) - A_1 \qquad (7.2.1.3)$$

$$\dot{x}_2 = f_2(\bar{x}) \qquad (7.2.1.4)$$

$$\dot{x}_3 = f_3(\bar{x}) \qquad (7.2.1.5)$$

$$\dot{x}_4 = f_4(\bar{x}) - \frac{A_2}{(1 + x_1)} \tag{7.2.1.6}$$

$$\dot{x}_5 = f_5(\bar{x}) \tag{7.2.1.7}$$

$$\dot{x}_6 = f_6(\bar{x}) \tag{7.2.1.8}$$

where the $f_i(\bar{x})$, $i = 1 \ldots 6$, can be deduced from chapter 3 with the following boundary values:

$$M[x_1(0), x_2(0), x_3(0)] = 0 \tag{7.2.1.9}$$

$$x_4(0) = \frac{\pi}{2} \tag{7.2.1.10}$$

$$x_5(0) = x_6(0) = 0 \tag{7.2.1.11}$$

where A_1 and A_2 are control variables subject to the following algebraic and integral constraints:

$$A_1 \geq 0 \tag{7.2.1.12}$$

$$A_2 \geq 0 \tag{7.2.1.13}$$

$$E_2 A_1 \geq A_2, \; E_2 \text{ a constant}, \quad 0 \leq E_2 \leq 1 \tag{7.2.1.14}$$

$$\int_0^{t_f} \sqrt{A_2^2}\, dt = E_1 \tag{7.2.1.15}$$

Find $A_1(t)$ and $A_2(t)$ $(0 \leq t \leq t_f)$ such that J is a minimum.

7.2.2 Solution of Problem 7.2.1

To adapt the problem as formulated to the maximum principle, let us adjoin to the equations of motion the following additional equation and associated boundary value:

$$\dot{x}_7 = \sqrt{A_2^2}$$
$$x_7(0) = 0, \; x_7(t_f) = E_1 \tag{7.2.2.1}$$

(Note that x_7 does not have the same meaning as used in chapter 3 and elsewhere in this study.)

Now let us form the variational Hamiltonian. It can be written as:

$$H(t) = \sum_{i=1}^{6} y_i(t)\, f_i(\bar{x}) - y_1 A_1 - \frac{y_4 A_2}{(1+x_1)} + y_7\sqrt{A_2^2} + y_0[A_1 - f_0(\bar{x})A_2] \qquad (7.2.2.2)$$

where the y_j ($j = 0 \ldots 7$) are the solutions to the adjoint equations along an optimal trajectory.

We are looking for A_1 and A_2 which make H a maximum subject to the algebraic and integral constraints on A_1 and A_2.

Following the usual procedure for constrained minimization problems we will adjoin to H the following relationships:

$$\eta_1(A_1 - R_1^2) = 0 \qquad (7.2.2.3)$$

$$\eta_2(A_2 - R_2^2) = 0 \qquad (7.2.2.4)$$

$$\eta_3(E_2A_1 - A_2 - R_3^2) = 0 \qquad (7.2.2.5)$$

where the η_j ($j = 1 \ldots 3$) are constant Lagrange multipliers, and the R_j ($j = 1 \ldots 3$) are slack variables, and proceed to take the appropriate partial derivatives. We get as necessary conditions:

$$\frac{\partial H}{\partial A_1} = -y_1 + y_0 + \eta_1 + \eta_3 E_2 = 0 \qquad (7.2.2.6)$$

$$\frac{\partial H}{\partial A_2} = -\frac{y_4}{(1+x_1)} + y_7\frac{A_2}{\sqrt{A_2^2}} - y_0 f_0(\bar{x}) + \eta_2 - \eta_3 = 0 \qquad (7.2.2.7)$$

$$\frac{\partial H}{\partial R_1} = 2\eta_1 R_1 = 0 \qquad (7.2.2.8)$$

$$\frac{\partial H}{\partial R_2} = 2\eta_2 R_2 = 0 \qquad (7.2.2.9)$$

$$\frac{\partial H}{\partial R_3} = -2\eta_3 R_3 = 0 \qquad (7.2.2.10)$$

Observe that $A_1 = A_2 = 0$ is a solution. However, since this results in no plane change, one deduces that there must be other admissible values for A_1 and A_2. If A_1 and $A_2 \neq 0$, then:

$$A_1 = R_1^2 > 0 \qquad (7.2.2.11)$$

$$A_2 = R_2^2 > 0 \qquad (7.2.2.12)$$

$$\eta_1 = \eta_2 = 0 \qquad (7.2.2.13)$$

Then equations (7.2.2.6) through (7.2.2.10) and the constraint equations reduce to:

$$- y_1 + y_0 = \eta_3 E_2 = 0 \qquad (7.2.2.14)$$

$$- \frac{y_4}{(1 + x_1)} + y_7 - y_0 f_0(\bar{x}) - \eta_3 = 0 \qquad (7.2.2.15)$$

$$\eta_3 R_3 = 0 \qquad (7.2.2.16)$$

$$E_2 R_1^2 - R_2^2 - R_3^2 = 0 \qquad (7.2.2.17)$$

Now either η_3 or $R_3 = 0$. If $\eta_3 = 0$, there is no unique solution possible for R_1^2, R_2^2 and R_3^2. If $R_3 = 0$, the following equations are obtained:

$$E_2 A_1 = A_2 \qquad (7.2.2.18)$$

$$- \frac{y_4}{(1 + x_1)} + y_7 - y_0 f_0(\bar{x}) + \frac{y_0 - y_1}{E_2} = 0 \qquad (7.2.2.19)$$

The meaning of (7.2.2.18) is obvious; the ratio of side force to lift force must be L/D_{max}. The meaning of (7.2.2.19) is not clear at the moment. To see what this equation tells us, let us rewrite the Hamiltonian using the information we have up to and including (7.2.2.18):

$$H = \sum_{i=1}^{6} y_i(t) f_i(\bar{x}) + \left[- \frac{y_1}{E_2} - \frac{y_4}{(1+x_1)} + y_7 + \frac{y_0}{E_2} - y_0 f_0(\bar{x}) \right] A_2 \qquad (7.2.2.20)$$

We note that the term in brackets multiplied by A_2 is precisely the left hand side of equation (7.2.2.19) which is alleged to have to be 0 for the Hamiltonian to be maximized when $A_2 \neq 0$.

Now we have as a necessary condition that if $A_2 \neq 0$, its coefficient = 0 or vice-versa, thus:

$$H = \sum_{i=1}^{6} y_i(t) f_i(x) \qquad (7.2.2.21)$$

Let us consider this a minute. In the absence of control forces, our system of differential equations represents a conservative autonomous system. Ergo, its motion is an extremal by definition and $H \equiv 0$ which is its maximum value. In the absence of control forces, and with the

boundary values specified, x_1, x_2, and x_3 will exhibit periodic motion; furthermore, it can be shown that y_1 will do so also. Now a little work with the transversality conditions applied to this problem will show that equation (7.2.2.19) can be satisfied at most at isolated instants of time. Hence, we are looking for a function $A_2(t)$ which is zero except at isolated instants of time, but which has a bounded integral. [See equation (7.2.1.15).] Thus one can argue that $A_1(t)$ and hence $A_2(t)$ should be positive Dirac delta functions at the time or times (7.2.2.19) is satisfied. If the state equations were linear, this argument could be easily proved by a limiting process. Later in this chapter (section 7.4) we will develop other relationships which lend more weight to this argument. Thus one can conclude, that the limiting case solution for this problem will be of the form

$$A_1(t) = \frac{B}{L/D_{max}} \delta(t - \tau) \qquad (7.2.2.22)$$

$$A_2(t) = B\,\delta(t - \tau) \qquad (7.2.2.23)$$

where B is a constant whose value depends on the boundary values of the problem.

7.2.3 Application of the Limiting Case Solution to the Problem Formulated in Chapter 6

To apply the results of (7.2.2) to our actual problem, let us imagine that we can enter the earth's atmosphere without drag as long as we are using no lift. We set up an elliptical transfer orbit such that we just graze the earth's surface at perigee. At this altitude we assume the dynamic pressure is high enough that one can effectively get unbounded aerodynamic forces. At perigee, we apply our δ function of side force (together with the corresponding δ function of drag) of such magnitude that we get our desired amount of plane change and then compute our cost J (which we will call J_{Lim}) at this point. It is claimed here that the resulting cost, (J_{Lim}) will be less than the minimum cost that can be obtained in the absence of heating and acceleration constraints for the actual problem, and will represent a

sort of goal that one may strive for but will never quite reach. Thus one can use J_{Lim} as a yardstick to measure the relative pay-off possible by relaxing the constraints in the constrained problem. Let us therefore compute J_{Lim} for later use.

The equations to be solved are:

$$\dot{x}_1 = \frac{B\,\delta(t - t_f)}{L/D_{max}} \tag{7.2.3.1}$$

$$\dot{x}_4 = -\frac{B\,\delta(t - t_f)}{(1 + x_1)} \tag{7.2.3.2}$$

where B is a constant to be determined.

Equation (7.2.3.2) can be rearranged to give:

$$-B\,\delta(t - t_f) = \dot{x}_4 (1 + x_1) \tag{7.2.3.3}$$

Substituting this in (7.2.3.1), one gets:

$$\dot{x}_1 = \frac{\dot{x}_4 (1 + x_1)}{L/D_{max}} \tag{7.2.3.4}$$

This can be integrated by separation of variables to give:

$$x_4(t_f) - x_4(0) = L/D_{max} \ln\left(\frac{1 + x_1(t_f)}{1 + x_1(0)}\right) \tag{7.2.3.5}$$

If we start on an equitorial orbit, $x_4(t_f) - x_4(0) = -\Delta i$ and we have:

$$x_1(t_f) = [1 + x_1(0)]\, e^{-\frac{\Delta i}{L/D_m}} - 1 \tag{7.2.3.6}$$

$$J_{Lim} = R'_a \sqrt{A'_1}\,[1 - e^{-\frac{\Delta i}{L/D_m}}] + 2\left(\frac{1}{\sqrt{R'_a}} - \sqrt{A'_1}\right) \tag{7.2.3.7}$$

where the primes on the constants indicate that they are evaluated with relation to a reference orbit at the earth's surface rather than at the edge of the atmosphere as is the case for the corresponding constants in the regular cost function.

7.3 Impulsive Plane Change Using Rocket Thrust

Equation (7.2.3.7) above opens the door to many interesting

parametric studies. One of the questions one can ask that will be of interest in the subsequent work is: "What is the highest initial altitude ($R_{a_{max}}$) at which the limiting synergetic maneuver ceases to become competitive with the usual impulsive maneuver using rocket thrust?"

To discuss this question we need to know what the ΔV requirements are, using rocket thrust alone. There are two cases to consider for the problem in question. For plane changes of less than 39° it can be shown that a single impulse is best with the ΔV required, given by:

$$\frac{\Delta V}{V_c} = 2 \sin \frac{i}{2} \qquad (7.3.1)$$

For $i > 39°$ it can be shown that a 3-impulse maneuver is better, where one transfers into an elliptical orbit with an apogee higher than the original altitude, makes the plane change at apogee, and then transfers back to a circular orbit at perigee again. For the three impulse plane-change the pertinent formulae are:

$$\frac{\Delta V}{V_c} = 2\Delta V_1 + \left[-\frac{2}{(1 + \Delta V_1)} - (1 + \Delta V_1) \right] 2 \sin \frac{\Delta i}{2} \qquad (7.3.2)$$

$$\Delta V_1 = \sqrt{\frac{2 \sin \frac{\Delta i}{2}}{1 - \sin \frac{\Delta i}{2}}} - 1 \qquad (7.3.3)$$

It should be noted that for $\Delta i \geq 60°$ the optimum three impulse transfer takes an infinite length of time and is not a very practical maneuver in the real world. Hence, if one really wanted to use a 3-impulse maneuver for a Δi of 60° or more, one would settle for something less than optimum.

The computer work described in the subsequent chapters uses a Δi of 60° so let us look at this case. Our two impulsive ΔV's then become, (in terms of the non-dimensional units derived in chapter 3)

$$\Delta V \text{ (1 impulse)} = \frac{1}{\sqrt{R_a}} \qquad (7.3.4)$$

$$\Delta V \text{ (3 impulse)} = \frac{.828}{\sqrt{R_a}} \qquad (7.3.5)$$

Now by setting (7.3.4) or (7.3.5) equal to (7.2.3.6) and solving for R_a, one can answer the question posed above. A little computation produces the following numbers (to slide-rule accuracy) for a Δi of 60°:

L/D_{max}	4	2	1
$R_{a_{max}}$ (1 impulse)	2.80	1.93	1.40
$R_{a_{max}}$ (3 impulse)	2.71	1.565	1.201

The figures for a L/D_{max} of 1 are a little surprising at first glance, since one's intuition might lead one to believe that at a L/D of 1 there is no advantage to the synergetic maneuver at any altitude, whereas these computations show that the synergetic maneuver is better for altitudes of less than about 700 N.M. even for L/D = 1.

7.4 A Note on Plane Change Efficiency

Let us now direct our attention to the equation which relates our change in inclination Δi to the state variables. Writing this equation in terms of instantaneous values, we have

$$\cos \Delta i(t) = \sin x_4(t) \cos x_6(t) \qquad (7.4.1)$$

(Note: Recall that this equation is only valid for an orbit which is initially equatorial. However, the definition of the angles x_4 and x_6 can be made with respect to an arbitrary plane and the comments which follow referring to the equator can be taken to mean the original orbital plane.)

Let us differentiate this equation with respect to time, and solve for $\dot{\Delta i}$.

$$\dot{\Delta i} \sin \Delta i = -\dot{x}_4 \cos x_4 \cos x_6 + \dot{x}_6 \sin x_4 \sin x_4 \qquad (7.4.2)$$

Substituting for $\dot{x}_4$ and $\dot{x}_6$ from the equations of motion developed in chapter 3,

$$\dot{\Delta i} = -\frac{\cos x_6 \cos x_4}{\sin \Delta i} \left[\frac{F_y}{(1 + x_1)(1 - x_7)} \right] \qquad (7.4.3)$$

By use of a little trigonometry we can write sin Δi as

$$\sin \Delta i = \sqrt{1 - \sin^2 x_4 \cos^2 x_6} \qquad (7.4.4)$$

which leads to the equation

$$\dot{\Delta i} = \frac{\cos x_4}{\sqrt{\frac{1}{\cos^2 x_6} - \sin^2 x_4}} \left[- \frac{F_y}{(1 + x_1)(1 - x_7)} \right] \qquad (7.4.5)$$

Now, realizing that any use of side force, either lift or thrust, results in an increase of cost, we can immediately conclude that, other things being equal, if we are interested in maximum plane change for minimum cost, we will want to keep $\dot{\Delta i}$ as large as possible for a given side force. Thus we can look at the terms involving trigonometric functions in equation (7.4.5) as a sort of gain factor which limits the $\dot{i}$ available for a fixed amount of side force. Let us define a quantity, to be called "plane change efficiency", E from (7.4.5) by:

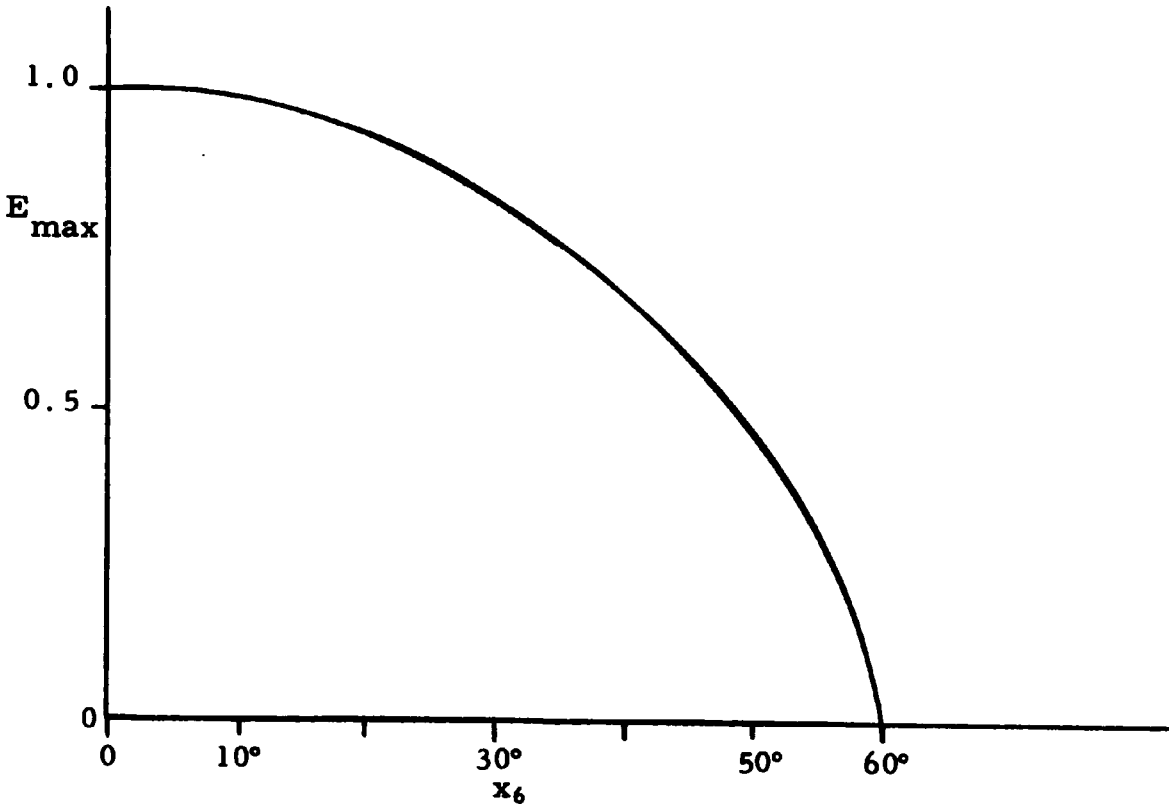
$$E = \frac{\cos x_4}{\sqrt{\frac{1}{\cos^2 x_6} - \sin^2 x_4}} \qquad (7.4.6)$$

Let us recall that x_6 corresponds to longitude and x_4 to heading. We note that $E = 1$ for $x_6 = 0$ (on the equator) for all values of x_4. We also note that $E < 1$ for $x_6 \neq 0$ regardless of x_4. Let us assume we are interested in a 60° plane change and compute E_{max} for various values of x_6. Figure 7-1 shows the results. E_{max} occurs when x_4 just matches the value required for $i = 60$ at the appropriate value of x_6. Note that $E = 0$ for $x_4 = \frac{\pi}{2}$ <u>unless</u> $x_6 = 0$.

Thus we conclude that it is worthwhile to do our turning as close to the initial orbital plane as possible, which is precisely the conclusion of section 7.2 above where we concluded that as a limiting case we wanted to do all our turning on the equator using a delta function control. In fact, as the subsequent work will show, with more realistic conditions it pays to use C_L values larger than those giving L/D_{max} and take some extra drag penalties in order to do the turning

close to the initial orbital plane.

This result is one of the reasons the author questions the validity of the results of Bell and Hanke (Ref. 3). One of the steps in Ref. 3 involves maximizing $\dot{x}_4$ (Rate of change of heading) and this is clearly not the same thing as maximizing $\dot{i}$ which is the object of the problem.



Variation of Maximum Plane Change Efficiency with Latitude

Figure 7.1

CHAPTER 8

THE ANALOG COMPUTER INVESTIGATION

As has been implied earlier, to finally solve the problem defined in Chapter 6, one must resort to approximate methods. In this problem, a variety of the gradient or steepest descent method has been chosen. To use this method, it is necessary to start with some sort of nominal, (non-optimal) trajectory. The closer the nominal is to the optimal, the fewer computations are required to get the optimal. For this reason it was felt that it would be useful to simulate the equations of motion on an analog computer in order to get the nominal solutions required. In addition, the analog work provides a physical "feel" for the dynamics of the problem that one cannot get easily using a digital computer.

8.1 The Simulation

Because of the non-linear nature of the equations of motion their implementation on an analog computer is a considerable task in itself. For a simulation assuming that the flight path angle γ and the altitude perturbation x_3 are not small, over 50 multiplication/division operations on two variables are required. In addition, between 85 to 120 operational amplifiers are required depending on the particular implementation chosen. We do not propose to tax the reader with analog circuit diagrams; however, it is felt there are a few features of the simulation that are worth mentioning.

8.1.1 Time Scaling

In developing the non-dimensional equations of motion in Chapter 3, we have introduced a time scale parameter as part of the non-dimensionalization; to wit: $\tau = \frac{V_0}{R_0} t$. Now it turns out that solution to this problem in the real world typically takes 20-30 minutes which results in a range of τ_f from about 1.5 to 2.2 units. If one implemented the equations on the analog computer without further time scaling, this would result in analog solution time of 1.5 to 2.2 seconds. This

order of solution time is perfectly satisfactory provided one uses solid-state elements for the non-linear operations such as multiplication and division. However, it is much too short to introduce the computer operator in the control loop. At the other end of the scale, to revert to real-time solutions results in solution times much too long for computer accuracy, and longer than necessary for operator intervention. Since the analog equipment available for this problem entailed the use of servo multipliers for computation and servo-driven plotters for data read-out, a multiple time scale simulation was implemented where the following time scales were readily available:

A. $\bar{\tau}$ (Computer time) = 200τ (Non-dimensional Problem time)

B. $\bar{\tau} = 100\tau$

C. $\bar{\tau} = 20\tau$

D. $\bar{\tau} = 10\tau$

This set of scales provided a very satisfactory range of solution times for all purposes and in addition provided favorable amplitude scaling within the computer.

8.1.2 Controls

As mentioned above, one of the purposes of the analog simulation was to provide a physical "feel" for the dynamics of the problem. For this reason the analog computer was effectively converted into a flight simulator with an instrument panel providing indications of the instantaneous values of various variables, an aircraft control stick which provided rate changes in U_1 and U_2 (angle of attack and bank) and suitable controls for the rocket engine. Thus, it was possible to "fly" the vehicle by hand through the plane change maneuver while observing the effects of various control inputs. Later an autopilot was designed and installed to control U_2 (bank) according to two different control laws.

8.1.3 Computer Requirements

The final simulation configuration used for most of the work required the following numbers of components:

A. Operational Amplifiers : 76 (Including 15 integrators)

B. Servo Multipliers : 9 (4 multiplying potentiometers per servo)

C. Diode Multipliers : 4

D. Diode Function Generators : 5

8.1.4 Type of Equipment Used

The basic computer used for this work was a 90 amplifier Hybrid computer built by the University of Michigan Department of Instrumentation Engineering and generally similar in design to equipment manufactured commercially by Applied Dynamics, Inc. For the reader not familiar with the term "Hybrid Computer", this machine has a total of 36 amplifiers available for use as integrators. Each of these integrating amplifiers can be individually controlled in the Hold-Operate-Reset modes through use of integral or external digital logic elements, providing a great deal of flexibility in the operations possible.

8.2 The Equations Solved

The equations 3.5.4.1 through 3.5.4.7 were used for the analog solution with one additional assumption, $x_3 << 1$, which in turn implies:

$$(1 + x_3) \approx 1$$

The validity of this assumption can be seen by noting that if one starts the problem at about 300,000 ft for $x_1 = 0$, and descends all the way to the surface of the earth, the maximum of $|x_3|$ will be about 0.014.

Additionally, since the primary interest was in the response to aerodynamic maneuvering, it was assumed that:

$$U_4 = U_5 \equiv 0$$

In addition to the state equations, 5 auxiliary differential equations were solved. Four of these arise from an implicit-high accuracy computational method for generating $\sin(x)$ and $\cos(x)$ given the angular rate $\dot{x}$. (See Ref. 12 for details.) The four equations are:

$$\frac{d}{d\tau}(\sin x_4) = \dot{x}_4 \cos x_4 + \mu\epsilon \sin x_4 \qquad (8.2.1)$$

$$\frac{d}{d\tau}(\cos x_4) = -\dot{x}_4 \sin x_4 + \mu\epsilon \cos x_4 \qquad (8.2.2)$$

where $\epsilon = 1 - \cos^2 x_4 - \sin^2 x_4$ and μ is an arbitrary gain constant. There are two corresponding differential equations for $\sin x_6$ and $\cos x_6$.

The 5th equation arises from an implicit computation for the non-dimensional dynamic pressure Q. It can be readily shown that

$$\dot{Q} = Q\left[\frac{2\dot{x}_1}{1 + x_1} - \frac{R_0}{h_s} x_2\right] \qquad (8.2.3)$$

The three extra integrators indicated as requirements in Par. 8.1.3 are for the purpose of generating U_1 and U_2 from control stick or autopilot rate inputs and for generating a voltage proportional to real time for reference purposes.

8.3 Parameter Selection

There are a number of parameters involved in the problem. For convenience they are summarized here and their selection discussed. The major parameters are:

R_a	the size of the initial circular orbit assumed
h_s	the scale factor in the experimental atmospheric density approximation
W_L	the vehicle wing loading
L/D_{max}	the maximum lift/drag ratio
Δi	the orbital inclination change desired
A_{max}	the value of the acceleration constraint
B_{max}	the value of the temperature constraint
I_{sp}	the specific impulse of the rocket engine

Let us start out with the easy parameters first. $\Delta i = 60°$ was selected

for the reason that this is a substantial plane change where the ΔV requirement for a single impulse maneuver at that altitude is easy to compute, to wit: exactly local circular. This value turns out to be $\frac{1}{\sqrt{R_a}}$.

A_{max} was selected as 10 g's because this is near the maximum short-period acceleration a human could be expected to tolerate.

W_L was selected as $\frac{40 \text{ lbs}}{\text{ft}^2}$.

h_s is taken to be 24,000 feet.

R_a was selected as $1.2 R_0$ since this corresponds to orbital altitudes near 700 miles and represents a sort of average altitude for military space operations as currently envisioned.

The selection of R_0 is a slightly more complicated problem. Having fixed W_L, a selection of R_0 essentially defines $Q(0)$ (the initial condition for the implicit computation of Q, the dimensionless dynamic pressure). Ideally one would like to allow Q a range of at least two orders of magnitude in order to start the problem where drag forces are negligible and to be able to fly into regions where substantial aerodynamic accelerations are possible. Because of problems in setting initial conditions accurately, a Q range of about 1 order of magnitude was used. Since peak lift forces on the order of 10 g's were desired a Q_{max} of about 20 was selected, and then R_0 was specified to be an altitude that gives Q_{min} of about 1. This gives a minimum drag acceleration at this altitude of from 8×10^{-3} g's to 20×10^{-3} g's depending on L/D_{max} selected. Having decided the approximate value of $Q(0)$, R_0 was selected as a convenient number so as to simplify computation of altitude in feet. Thus after a little trial and error an R_0 corresponding to an altitude of 225,000 feet was used. This resulted in $Q(0) = 1.672(1 + x_1(0))^2$.

Specific impulse was taken to be 400 sec which results in a value of $\frac{V_0}{IGe}$ of 2.0.

L/D_{max} must be investigated for a range of values. Most of the analog work was done at a L/D_{max} of 4, since this apparently represents a sort of upper limit on reasonable re-entry vehicles and went down to a L/D_{max} of 2.

8.4 Initial Condition Selection

As mentioned previously, there are only two initial conditions to select, i.e., $x_1(0)$, and $x_2(0)$. There is a limit on how large $x_2(0)$ can be in absolute value as a result of the assumption that $\gamma << 1$. As it turned out, severe control difficulties developed if re-entry was attempted with $\gamma < -3°$ or $\gamma > -1°$. Within this range, variation in γ did not have any appreciable effect on the cost, so a fixed γ of $-1.5°$ was selected for most of the analog work. This corresponds to an $x_2(0) \approx -2.5 \times 10^{-2}$ and an $x_1(0) = 4.4 \times 10^{-2}$. Further, it developed that γ never got appreciably larger than $\gamma(0)$ on any trajectories flown, thus the assumption that $\gamma << 1$ turns out to be justified.

8.5 Results of the Pre-Autopilot Phase

Initial attempts to "fly" the re-entry part of the maneuver, i.e., from $x_3 = 0$ till x_2 was stabilized near 0, indicated that the control problems associated with this maneuver were quite critical. Two undesirable trajectories were the general results of these attempts.

A. An immediate skip back out of the atmosphere

B. An uncontrollable dive to beyond the allowable range of Q, - equivalent to burning up.

These maneuvers were more frequent when the conventional technique of trying to re-enter with wings level and controlling vertical lift by changes in angle of attack was used. It was soon learned that a more feasible method of control was to re-enter at constant U_1 (angle of attack) and provide variations of vertical force by changes in U_2 (bank angle). It is interesting to note that the vehicle is generally still at super-circular velocity after pullout and it is necessary to fly more or less upside-down for a few minutes until x_1 becomes negative to

keep from skipping out of the atmosphere. After pullout and the decay of x_1 to negative values, it proved reasonably easy to fly the vehicle around a curved path by attempting to maintain constant dynamic pressure with changes in bank angle while keeping a constant angle of attack.

In spite of learning how to control the vehicle by hand, it soon became apparent that hand flying was not the answer. Because of the extreme sensitivity of the early portions of the trajectory it was impossible to get the control history of any two runs to agree. Thus the need for some sort of automatic control became evident.

8.6 Development of an Autopilot

Based on the knowledge of the control techniques gained in the pre-autopilot phase, it was decided to design an autopilot which would attempt to maintain a command dynamic pressure (Q_{comd}) by changes in bank angle. The first attempt consisted of trying to generate U_2 (bank) by a solution to the following equations:

$$\dot{U}_2 = K[Q_{comd} - Q] \quad 0 \leq U_2 \leq U_{2\,max} \tag{8.6.1}$$

$$\dot{U}_2 = 0, \text{ for } U_2 = U_{2\,max} \text{ and } (Q_{comd} - Q) > 0 \tag{8.6.2}$$

$$\dot{U}_2 = 0, \text{ for } U_2 = 0 \text{ and } (Q_{comd} - Q) \leq 0 \tag{8.6.3}$$

(Equations 8.6.2 and 8.6.3 are a way of expressing the fact that the integrator generating U_2 was limited to a dynamic range of 0 - $U_{2\,max}$ and this will be indicated henceforth by the statement "U_2 Limited ($0 \leq U_2 \leq U_{2\,max}$)").

It can be seen that if $Q < Q_{comd}$, $\dot{U}_2 > 0$ and U_2 increases pulling the vehicle into a region of denser air and vice-versa. Unfortunately, this equation did not provide satisfactory control. For $K < 50$ the response was too sluggish and the vehicle either skipped or "burned up". For larger values of K, Q oscillated in a slightly divergent manner about Q_{comd}. Next, to provide some damping, this equation was used to generate U_2:

$$\dot{U}_2 = K[Q_{comd} - Q + N_1 x_2], \quad (U_2 \text{ limited } 0 < U_2 < U_{2max}) \tag{8.6.4}$$

This equation provided satisfactory control of Q, but U_2 and x_2 exhibited a limit cycle-type oscillation which was considered undesirable, so an additional term was added to the equation for $\dot{U}_2$ giving:

$$\dot{U}_2 = K[Q_{comd} - Q + N_1 x_2 + N_2 \dot{x}_2], \quad (U_2 \text{ limited } 0 \leq U_2 \leq U_{2max}) \tag{8.6.5}$$

This equation, after a little trial and error experimenting on the three gain constants, gave excellent control. A typical time history of U_2 and Q is given in Fig. 8.1.

Not only was the response excellent during the re-entry maneuver, but this autopilot compensated for changes in U_1 during the course of a maneuver, essentially without error, and showed very quick recovery from artificially induced disturbances in Q. A value of K = 350 was selected. N_1 and N_2 had to be adjusted depending on Q_{comd} programs and re-entry angles. The range used was:

$$200 \leq N_1 \leq 400$$

$$1 \leq N_2 \leq 5$$

8.7 Results of the Analog Work Subsequent to the Development of an Autopilot

With the autopilot installed one has, in effect, traded one control variable for another. The control variables now are U_1 (angle of attack), Q_{comd} and U_3 (Horizontal Thrust). Using these variables several classes of trajectories were investigated.

Class A. Constant Q_{comd} Constant U_1, $U_3 = 0$ until turn is complete

Class B. Constant Q_{comd} Variable U_1, $U_3 = 0$ until turn is complete

Class C. Constant Q_{comd} Constant U_1, Variable U_3

Class D. $Q_{comd} = \frac{\beta}{(1 + x_1)^4}$ Constant U_1, $U_3 = 0$ until turn is complete

Class E. $Q_{comd} = \frac{\beta}{(1 + x_1)^4}$, Variable U[, $U_3 = 0$ until turn is complete

Class F. $Q_{comd} = \frac{\beta}{(1 + x_1)^4}$ Constant U_1, Variable U_3

All these classes had a common feature: the autopilot remained in control until i = 60°. This means that in each case, at some time $t_1 < t_f$ where t_f is defined as the time the vehicle returns to its initial altitude, it was necessary to turn off the autopilot, roll the wings level, and execute some kind of combined aerodynamic-propulsive maneuver to get back to the proper altitude to terminate the problem. It was found that if one exercised a little discretion in the type of maneuver employed to get back to $x_3 = 0$, the value of the cost functional changed very little from its value at $t = t_1$. Since a fixed $x_2(0)$ was used for all trajectories, this means that the relative "goodness" of the various trajectories can be measured quite closely by the value of $x_1(t_1)$. For this reason this number will be used to compare the results of the various classes. The smaller $|x_1(t_1)|$ is, the better the trajectory is, since this means a smaller velocity loss during the turning maneuver.

Let us now proceed to discuss the results obtained from the various trajectory classes individually.

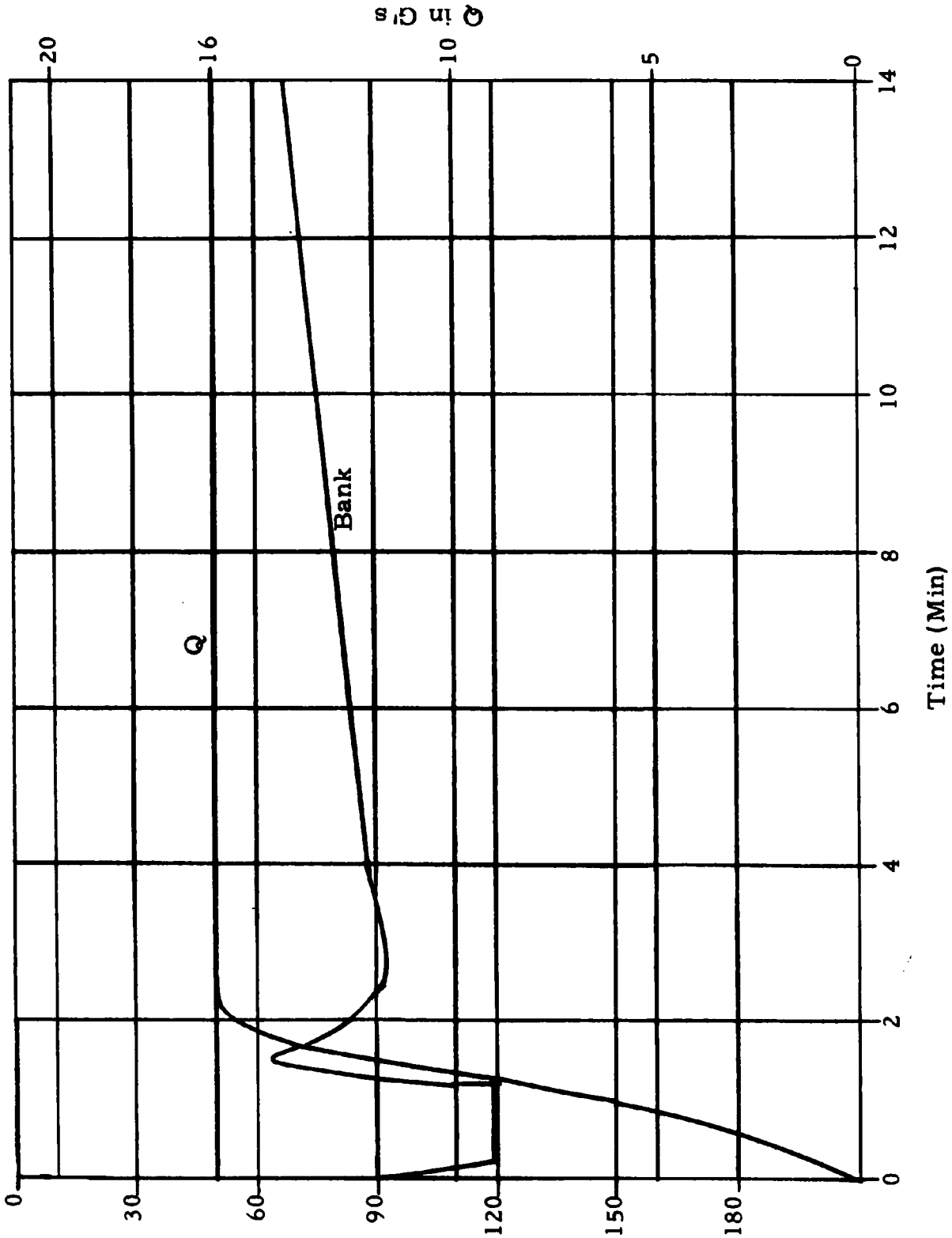
8.7.1 Class A, Constant Q_{comd}, Constant U_1, $U_3 = 0$

Typical bank angle history for this type of trajectory consisted of an immediate roll to $U_2 = 120°$, which was held until Q approached Q_{comd}, a roll back to $U_2 \approx 30°$ for a very short time, then a quick roll back to $U_2 \approx 95°$ and then a gradual decrease of U_2 to the end of the turn with $U_2(t_1) \approx 50°$ giving a time history similar to Fig. 8.1.

General Results for Class A Trajectories:

A. For each Q_{comd} there exists a unique constant U_1 which produces minimum cost.

B. The optimum U_1 for each Q_{comd} is not the U_1 corresponding to L/D_{max} but is a larger value. The optimum U_1 is largest



Typical History of Bank Angle and Dynamic Pressure
For Synergetic Maneuver

Figure 8.1

for small Q_{comd} and approaches U_1 for L/D_{max} as Q_{comd} becomes large.

C. For trajectories flown at U_1 opt., cost decreases as Q_{comd} becomes larger.

D. Lower values of L/D_{max} produce higher costs for the same values of Q_{comd}, when flown at U_1 opt., for the value of Q_{comd} concerned.

Figure 8.2 illustrates results A - C above.

Figure 8.3 illustrates result D.

8.7.2 Class B, Constant Q_{comd}, Variable U_1, $U_3 = 0$

This class of trajectories was flown while trying simple programs for varying U_1 around the optimum value found in the work on Class A. Programs tried included trying $U_1 < U_1$ opt., during the early part of the trajectory, then $U_1 > U_1$ opt., during later parts of trajectory, and the converse case.

Result of Class B Trajectories:

No program of variable U_1 used resulted in any significant decrease in cost.

8.7.3 Class C, Constant Q_{comd}, Constant U_1, Variable U_3

This class of trajectories was flown with $U_1 = U_1$ opt found for Class A trajectories. Various U_3 programs were tried including one that has been used in several earlier studies of this problem. (Ref. 2 - Nyland, and Ref. 3 - Bell and Hanke) i.e., Thrust = drag. Other programs included thrust = 0 until x_1 reached some arbitrary value below local circular, and then thrust = drag, and pulses of thrust at various intervals during the trajectory.

Result for Class C Trajectories:

All programs of U_3 tried, resulted in higher costs than those obtained in Class A Trajectories with the thrust = drag program being the worst.

Synergetic Cost for $L/D_{max} = 4$

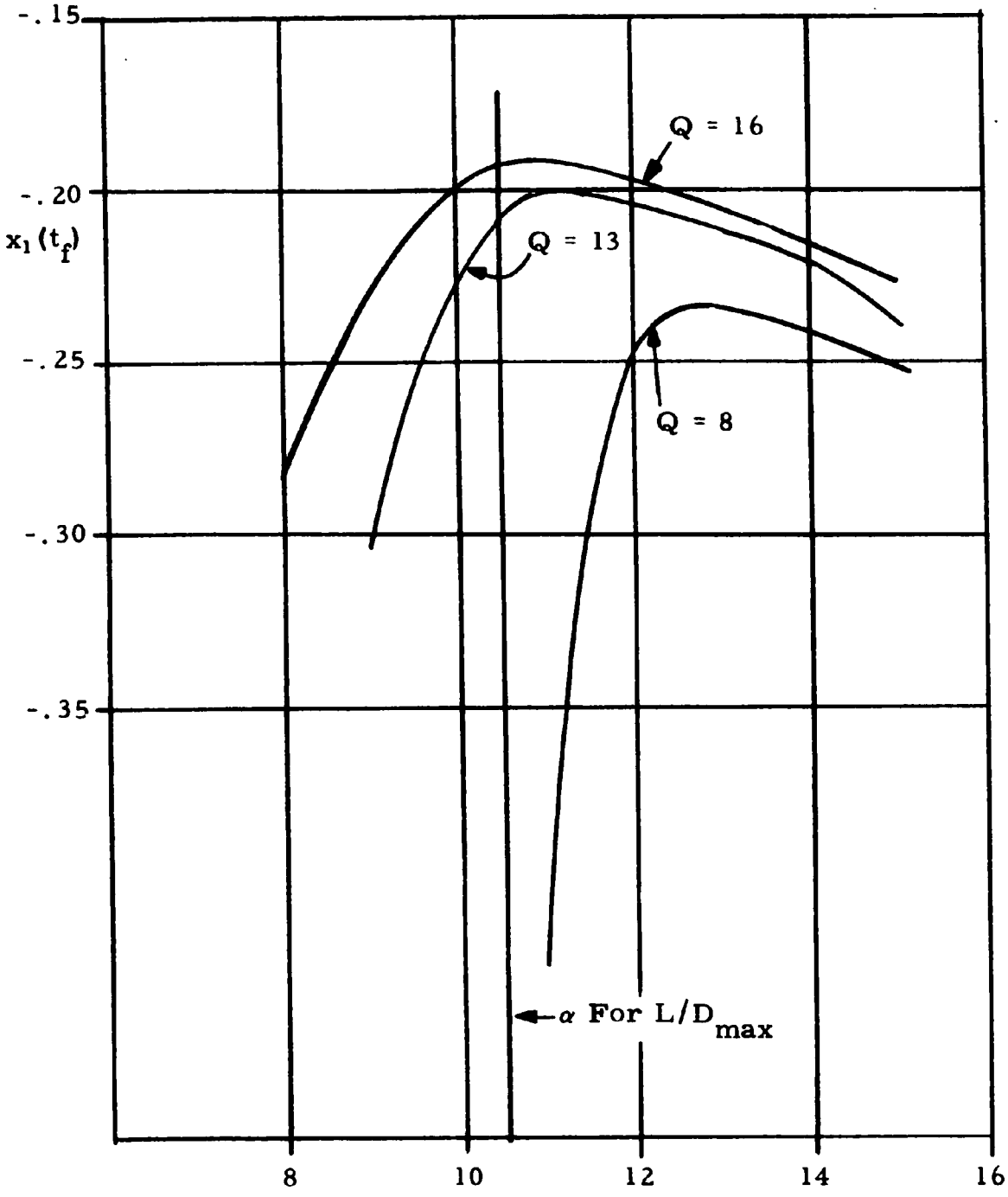


Figure 8.2

Synergetic Cost for $L/D_{max} = 3$

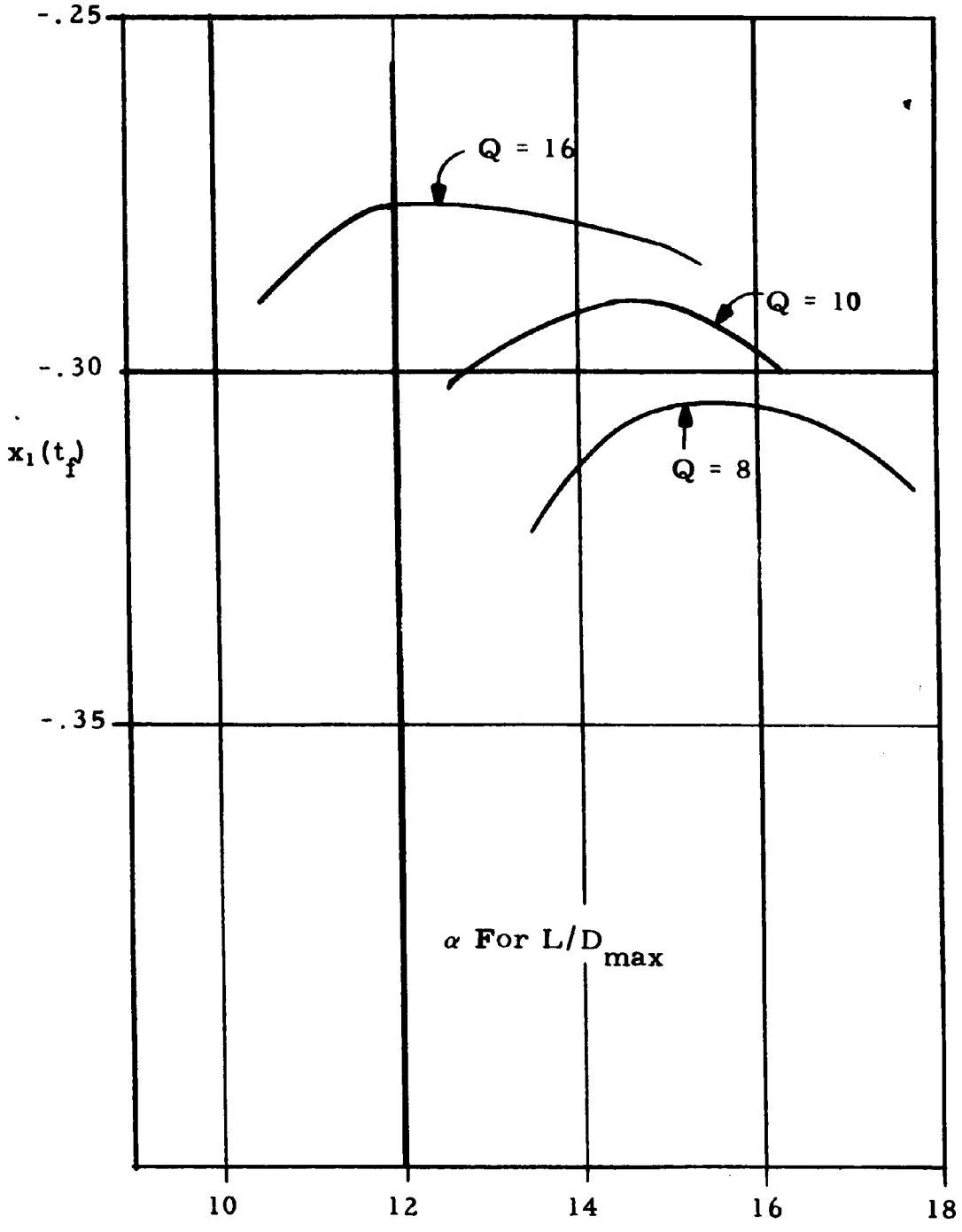


Figure 8.3

8.7.4 Summary of Results of Class A and B Trajectories

In general the results of the work done on these sections indicates that for minimum cost one wants to:

A. Fly at as high a dynamic pressure as possible.

B. Make the necessary turning as close to the equator as possible even at the expense of flying at L/D ratios lower than the configuration permits.

C. Not use any appreciable thrust while aerodynamic forces are available.

These results with the exception of C correspond in a qualitative way to those predicted by the theoretical work of sections 7.2 and 7.4. It is of interest to note that the total ΔV for the best Class A trajectory flown was 0.373. The solution of J_{Lim} for the same altitude gives a $J_{Lim} = 0.323$ indicating that we have approached 86% of the limiting solution already.

One last general result worth mentioning is that at no time did the peak aerodynamic acceleration exceed 1 G in these classes of trajectories.

8.7.5 Class C, D, and E Trajectories

Because of scaling problems in the analog computer it was not possible to investigate a large range of values of β. For reasons of scaling, a value of $\beta = 3.34$ was chosen which corresponds to a stagnation point temperature of 4690°F or a heating rate input of 250 BTU/ft^{2}sec. The results of Classes D and E generally parallel the applicable results of Classes A, B, and C. They are:

A. There exists a unique constant $U_1 > U_1(L/D_{max})$ which results in a minimum cost.

B. Lower values of L/D_{max} result in higher costs.

C. No program of varying U_1 that was tried resulted in significantly lower costs.

D. Any thrust program tried resulted in higher costs.

8.7.6 General Remarks

Trajectories flown at $U_1 = U_1(L/D_{max})$ at low Q's ($Q_{comd} < 10$) and Class C trajectories resulted in a phenomenon which we refer to as "going over the top." This can best be understood by referring to Fig. 8.4 which shows on a Mercator projection a sketch of such a trajectory.

At point 1 the heading becomes 90° and it is necessary to reverse the direction of bank in order to continue to change plane in the same direction. Needless to say this type of behavior results in spending a long time in the region of low plane-change efficiency and consequently results in very high cost trajectories.

Now, armed with an understanding of the problem and a set of nominal trajectories, let us proceed to the solution of this problem using a digital computer and the gradient method.

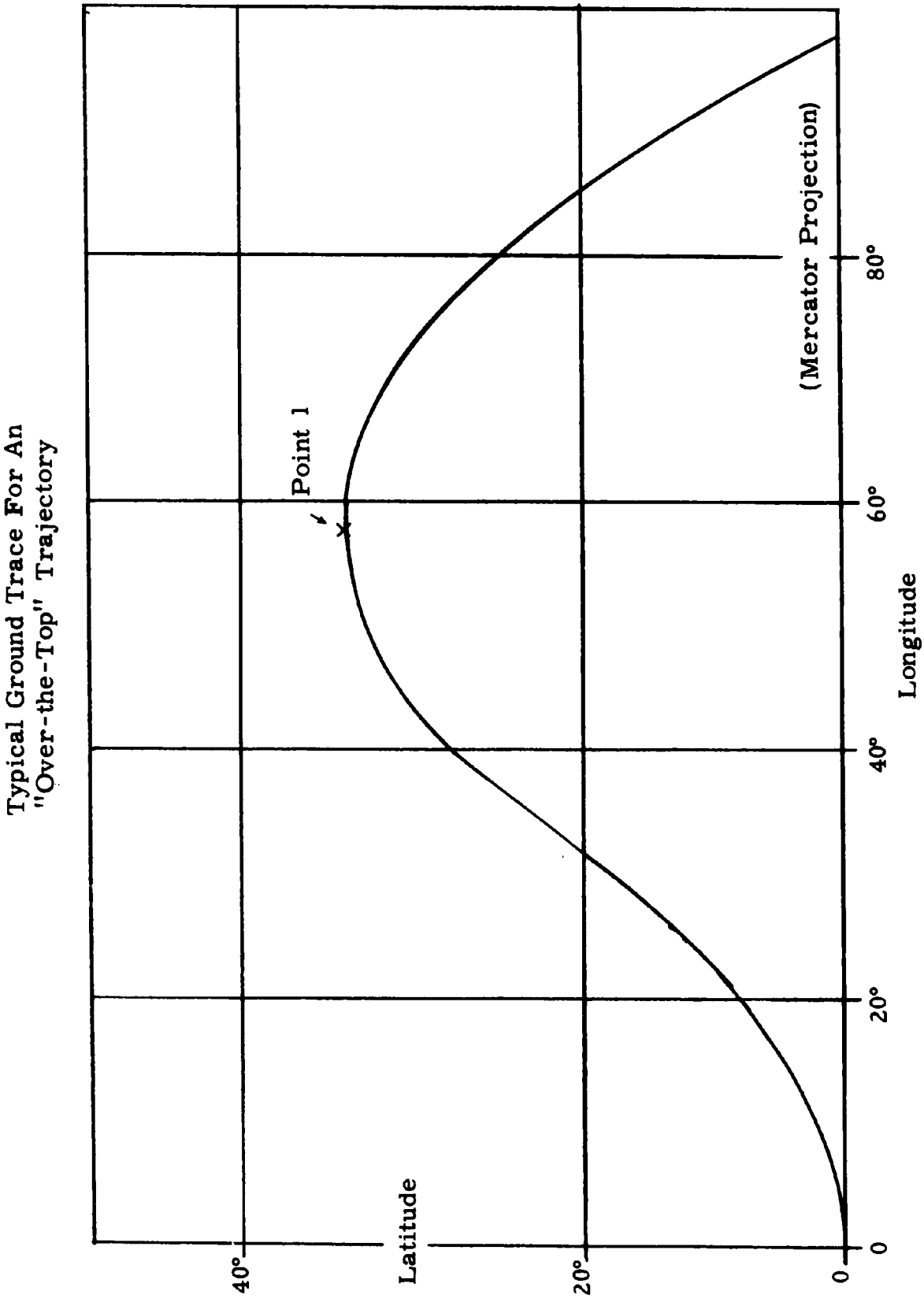

Typical Ground Trace For An
"Over-the-Top" Trajectory
Point 1
(Mercator Projection)
Latitude
Longitude
0
20°
40°
0
20°
40°
60°
80°

Figure 8.4

CHAPTER 9

THEORETICAL DEVELOPMENT OF THE GRADIENT METHOD FOR SOLUTION OF THE PROBLEM OF CHAPTER 6

We saw in Chapter 7 how the use of indirect methods of optimal control theory led us to a two-point boundary value problem which we were unable to solve directly because of the nonlinear nature of the equations of motion. Thus we are led to look for another method of finding our optimal controls and trajectories. A method which has been successful in simpler problems is the "gradient method" or "Method of Steepest Ascent." One of the clearest expositions of the technique can be found in Ref. 10 - Kelly. Also useful is Ref. 12-14 - Bryson and Denham. While in the previous chapters we have tried to avoid reproducing the underlying theory behind the applications made when this theory is presented elsewhere, it is felt a thorough understanding of the gradient method is essential to an appreciation of the results obtained from it, and for this reason a fairly detailed description of the method is given here.

9.1 Qualitative Description of the Gradient Method

Use of the gradient method has often been explained by analogy to a problem in mountain-climbing and it is felt that this is indeed a close analogy. Let us imagine ourselves on the side of a strange mountain in a thick fog so that we can only see for a very few feet in any direction. We are equipped with a plumb-bob, a protractor, and a compass. Our problem is to climb to the top. One way to solve this problem with our equipment is to determine the direction of the steepest upward slope at the point we happen to be (the maximum gradient) and, using our compass, walk a certain number of steps in this direction. Next, we stop, look for the direction of the steepest slope a second time, and again walk in this direction. Hopefully, by repeating this process over and over, we will eventually arrive at a point where the ground is level or slopes downward in all directions and we would be tempted to

conclude that we have in fact reached the top of the mountain. There are a number of hazards associated with this method of solving the problem. We may run into the base of a vertical cliff or fall off a cliff. (Discontinuous Functions). We may find the surface so rough that we have to make so many measurements and take so many small steps that we run out of patience. (Slow Convergence). And last but not least, we may finally attain our summit and sit there boasting loudly about our prowess when the fog lifts and we find ourselves sitting on a small knob on the side of the mountain gazing up at some other fellow sitting on the real peak looking down on us.

Nevertheless, if we have access to a little extra information, such as: knowing in advance that we're close to the peak, being reasonably sure there are no cliffs around and knowing that the surface is fairly smooth, we have a good chance of reaching the top. Here, however, is where the analogy breaks down. In the world of optimal control problems, the fog never lifts, and we can never say for sure that we have done more than reach the top of some knob. (Local Maximum). In fact, if we pose a slightly different problem, like "Find the highest point in a radius of 10 miles" it may develop that we should have first gone downhill into a valley and then up a neighboring mountain. In fact, if we used as a nominal trajectory for this problem a circular orbit well outside the atmosphere, the gradient method would never find the particular mountain we are starting on as a result of the work in Chapter 8.

While we are still discussing our analogy, there is one more point worth mentioning. Earlier in this section it was mentioned that after one found the direction of the steepest gradient one walked for an indefinite distance and then looked for a new gradient direction. The question arises as to just how far one should walk before changing direction. There are two schools of thought on this subject, which we will refer to as the "Kelly School" and the "Bryson-Denham School."

Kelly (Ref. 10) advocates walking in a single direction until we no longer have any upward component of motion in that direction. Bryson and Denham (Ref. 13) advocate moving only a short way in any one direction before changing course. If we call each trip in a fixed direction a "step" the Kelly method in general requires fewer "steps", but the Bryson-Denham method generally follows a more direct route. In this problem we shall use the Bryson-Denham method since there is reason to believe that we are already close to the top and not many steps will be required in any case.

9.2 Gradient Method Development for a Simple Case

Because the gradient method as applied to this problem becomes rather involved, let us start with a simple case and then proceed to add the complications a step at a time. Let us first consider this optimal control problem:

$$J = J[\bar{x}(t_f)] \text{ to be minimized} \tag{9.2.1}$$

$$\dot{\bar{x}}(t) = \bar{f}[\bar{x}(t)\, \bar{U}(t)] \qquad 0 \le t \le t_f \tag{9.2.2}$$

$$\bar{x}(0) \text{ - all components specified} \tag{9.2.3}$$

$$\bar{x}(t_f) \text{ - all components free} \tag{9.2.4}$$

$$t_f \text{ fixed} \tag{9.2.5}$$

We assume we have a solution to the differential equations for some $\bar{U}(t)$, $0 \le t \le t_f$ which is not optimal, and we are looking for some changes in $\bar{U}(t)$ (which we will call $\delta\bar{U}(t)$) which will lower the cost J. Now J is a function only of the final values of the solution $\bar{x}(t)$, so we need to derive a relationship between $\delta\bar{U}(t)$ and $J[\bar{x}(t_f)]$. To do this, we linearize the problem. For "small" changes in J, (we will studiously avoid now and hereafter specifying just what a "small" change is) we can write, for fixed t_f,

$$dJ = \frac{\partial J}{\partial \bar{x}(t_f)}\, \delta\bar{x}(t_f) \tag{9.2.6}$$

where $\delta\bar{x}(t_f)$ is a small change in $\bar{x}(t_f)$.

The linearized equations of motion are:

$$\delta\dot{\bar{x}} = \frac{\partial \bar{f}}{\partial \bar{x}}\,\delta\bar{x} + \frac{\partial \bar{f}}{\partial \bar{U}}\,\delta\bar{U} \qquad (9.2.7)$$

where $\frac{\partial \bar{f}}{\partial \bar{x}}$ and $\frac{\partial \bar{f}}{\partial \bar{U}}$ are evaluated along the nominal trajectory.

Next, one defines another set of differential equations variously called Adjoint Equations, Costate equations or Influence functions. These are:

$$\dot{\bar{y}} = -\left[\frac{\partial \bar{f}}{\partial \bar{x}}\right]^T \bar{y}, \quad \frac{\partial \bar{f}}{\partial \bar{x}} \text{ evaluated along the nominal.} \qquad (9.2.8)$$

By virtue of the properties of linear differential equations, the following relationship between the adjoint solution $\bar{y}(t)$ and the solution of the linearized equations is true:

$$\bar{y}^T(t_f)\,\delta\bar{x}(t_f) = \bar{y}^T(0)\,\delta\bar{x}(0) + \int_0^{t_f} \bar{y}^t(s)\,\frac{\partial \bar{f}(s)}{\partial \bar{U}(s)}\,\delta\bar{U}(s)\,ds \qquad (9.2.9)$$

Now, looking at (9.2.9) we see that for $\delta\bar{x}(0) = 0$ we have a relationship between $\delta\bar{x}(t_f)$, and $\delta\bar{U}$ through the solution of the adjoint equations. In particular, if we require that:

$$\bar{y}_j^T(t_f) = \frac{\partial J}{\partial \bar{x}(t_f)} \quad \text{and } \delta\bar{x}(0) = 0 \qquad (9.2.10)$$

we can write:

$$dJ = \int_0^{t_f} \bar{y}_j^T(s)\,\frac{\partial \bar{f}(s)}{\partial \bar{U}(s)}\,\delta\bar{U}(s)\,ds \qquad (9.2.11)$$

and we now have an equation relating small changes in $J[\bar{x}(t_f)]$ to small changes in $\bar{U}$, $\delta\bar{U}(t) \quad 0 \le t \le t_f$.

Next we pose the question, for $\delta\bar{U}(t)$ belonging to some bounded space, what is the $\delta\bar{U}(t)$ which gives the largest negative change in dJ? A convenient definition of bounded space for $\delta\bar{U}(t)$ is the following:

$$\int_0^{t_f} \delta\bar{U}^T(s)\,\delta\bar{U}(s)\,ds = \epsilon^2 \qquad (9.2.12)$$

If we use (9.2.12) as an equality constraint, we can now define an

isoparametric problem in the calculus of variations to find $\delta\bar{U}$. One adjoins (9.2.12) to (9.2.11) with a constant Lagrange multiplier η.

$$dJ = \int_0^t \bar{y}_j^T \frac{\partial \bar{f}}{\partial \bar{U}} \delta\bar{U}\, ds + \eta \left[\epsilon^2 - \int_0^{t_f} \delta\bar{U}^T \delta\bar{U}\, ds\right] \tag{9.2.13}$$

For convenience in notation let us define:

$$\bar{y}_j^T(t) \frac{\partial \bar{f}(t)}{\partial \bar{U}(t)} = \bar{\Lambda}_j^T(t) \tag{9.2.14}$$

A necessary condition for dJ to be a minimum (or maximum) is:

$$\frac{\partial[dJ]}{\partial(\delta\bar{U})} = 0 \tag{9.2.15}$$

Applying the condition to (9.2.13) with the substitution of (9.2.14) one obtains:

$$\int_0^{t_f} [\bar{\Lambda}_j^T - 2\eta\, \delta\bar{U}^T]\, ds = 0 \tag{9.2.16}$$

For this to be true for arbitrary $\bar{\Lambda}_j$ we require

$$\delta\bar{U} = \frac{1}{2\eta} \bar{\Lambda}_j \tag{9.2.17}$$

Using (9.2.17) in the equation of constraint (9.2.12)

$$\frac{1}{4\eta^2} \int_0^{t_f} \bar{\Lambda}_j^T \bar{\Lambda}_j\, ds = \epsilon^2 \tag{9.2.18}$$

Again for convenience of notation let us define:

$$\int_0^{t_f} \bar{\Lambda}_j^T \bar{\Lambda}_j\, ds = I_{jj} \tag{9.2.19}$$

(Note that I_{jj} is a fixed number, ≥ 0. Thus we have:

$$\eta = \frac{\pm \sqrt{I_{jj}}}{2\sqrt{\epsilon^2}} \tag{9.2.20}$$

and our desired $\delta\bar{U}$ for maximum or minimum dJ is:

$$\delta\bar{U}(t) = \frac{\pm\sqrt{\epsilon^2}}{\sqrt{I_{jj}}}\,\bar{\Lambda}_j(t) \tag{9.2.21}$$

It is of interest to verify that this $\delta\bar{U}$ does satisfy our constraint equation, (9.2.12). Substituting (9.2.21) into (9.2.12) we get:

$$\frac{\epsilon^2}{I_{jj}}\int_0^{t_f}\bar{\Lambda}_j^T\,\Lambda_j\,ds = \epsilon^2 \tag{9.2.22}$$

and by the definition of I_{jj} (9.2.19) we have an identity.

Now let us substitute our optimum $\delta\bar{U}$ in (9.2.13) and see if we can resolve the sign ambiguity.

$$dJ = \frac{\pm\sqrt{\epsilon^2}}{\sqrt{I_{jj}}}\int_0^{t_f}\bar{\Lambda}_j^T\,\bar{\Lambda}_j\,ds \tag{9.2.23}$$

$$= \pm\sqrt{\epsilon^2}\;\sqrt{I_{jj}}$$

Now, if we are interested in making dJ negative to minimize our cost, it is evident that the negative sign must be chosen. Thus our desired δU is:

$$\delta\bar{U}(t) = \frac{-\sqrt{\epsilon^2}}{\sqrt{I_{jj}}}\,\bar{\Lambda}_j(t) \tag{9.2.24}$$

It is of interest to look at the application of the Maximum Principle to this problem. The Maximum Principle requires that along an optimal trajectory:

$$\frac{\partial H}{\partial\bar{U}} = \frac{\partial}{\partial\bar{U}}\left[\bar{y}^T\bar{f}(\bar{x},\bar{U})\right] = \bar{y}^T\frac{\partial\bar{f}}{\partial\bar{U}} = \bar{\Lambda}_j^T = 0 \tag{9.2.25}$$

Looking at (9.2.23) we see that at the optimum, dJ = 0 which says that the steepest gradient is 0 which is just the condition that we mentioned in Section 9.1 as the criteria for having arrived at the top of our mountain. Further the application of the transversality conditions to the problem yield:

$$\bar{y}^T(t_f) = \frac{\partial J}{\partial\bar{x}(t_f)} \tag{9.2.26}$$

which is precisely the set of boundary values we used to define our adjoint solutions $\bar{y}_j(t)$.

As a practical matter we can never actually reach a minimum or maximum using the gradient method since for fixed ϵ^2, our "steps" in dJ get smaller and smaller as we get closer and closer to the optimum solution. However, one can in theory get as close as one likes to this optimum (within the constraints of the computer time available).

9.3 The Gradient For a More Complicated Case

Let us assume the same problem as in section 9.2 with one change. Let our cost J now be a function of the initial state as well as the final state.

$$J = J[\bar{x}(0),\ \bar{x}(t_f)] \tag{9.3.1}$$

Now our gradient dJ becomes:

$$dJ = \frac{\partial J}{\partial \bar{x}(0)}\,\delta\bar{x}(0) + \frac{\partial J}{\partial \bar{x}(t_f)}\,\delta\bar{x}(t_f) \tag{9.3.2}$$

By again requiring that the boundary values of the adjoint equations be:

$$\bar{y}_j^T(t_f) = \frac{\partial J}{\partial \bar{x}(t_f)} \tag{9.3.3}$$

we get

$$dJ = \left[\frac{\partial J}{\partial \bar{x}(0)} + \bar{y}^T(0)\right]\delta\bar{x}(0) + \int_0^{t_f} \bar{\Lambda}_j^T(s)\,\delta\bar{U}(s)\,ds \tag{9.3.4}$$

and we now have two vectors to find in order to define our minimum or maximum dJ; $\delta\bar{x}(0)$ and $\delta\bar{U}(t)$. It is apparent that we will have to define a new constraint equation to include $\delta\bar{x}(0)$ and while we are doing so, let us include some "weighting functions." These functions will enable us to shift emphasis from one variable to another and from one time period to another during our ascent if we desire to do so. Let us use as a constraint:

$$\delta\bar{x}^T(0)\,W_x\,\delta\bar{x}(0) + \int_0^{t_f} \delta\bar{U}^T(s)\,W_u(s)\,\delta\bar{U}(s)\,ds = \epsilon^2 \tag{9.3.5}$$

where W_x is an arbitrary positive constant matrix of weighting functions and $W_u(t)$ is an arbitrary matrix of positive time-varying weighting functions. For convenience in notation let us make the following definition:

$$\bar{L}_j^T = \left[\frac{\partial J}{\partial \bar{x}(0)} + \bar{y}^T(0) \right] \tag{9.3.6}$$

Further, let us change the definition of section 9.2 for I_{jj} to:

$$I_{jj} = \bar{L}_j^T W_x^{-1} \bar{L}_j + \int_0^{t_f} \bar{\Lambda}_j^T(s) W_u^{-1}(s) \bar{\Lambda}_j(s)\, ds \tag{9.3.7}$$

Now we are in a position to proceed in a manner entirely analogous to section 9.2 to find $\delta\bar{U}(t)$ and $\delta\bar{x}(0)$ which will minimize dJ. Our necessary conditions will be:

$$\delta\bar{x}(0) = \frac{W_x^{-1}}{2\eta} L_j \tag{9.3.8}$$

$$\delta\bar{U}(t) = \frac{W_u^{-1}(t)}{2\eta} \bar{\Lambda}_j(t) \tag{9.3.9}$$

After solving for η we will get:

$$\delta\bar{x}(0) = \frac{-\sqrt{\epsilon^2}}{\sqrt{I_{jj}}} W_x^{-1} \bar{L}_j \tag{9.3.10}$$

$$\delta\bar{U}(t) = \frac{-\sqrt{\epsilon^2}}{\sqrt{I_{jj}}} W_u^{-1}(t) \bar{\Lambda}_j(t) \tag{9.3.11}$$

and our optimum dJ will turn out to be:

$$dJ = -\sqrt{\epsilon^2}\ \sqrt{I_{jj}} = -\sqrt{\epsilon^2} \left[\bar{L}_j^T W_x^{-1} \bar{L}_j + \int_0^{t_f} \Lambda_j^T W_u^{-1} \bar{\Lambda}_j \right]^{\frac{1}{2}} \tag{9.3.12}$$

9.4 Development of the Gradient, Step 3

In this step let us introduce the following additional complications:

A. Let the final time t_f not be fixed, but be defined implicitly by some other event such as:

$$\Omega[\bar{x}(t)] = 0 \tag{9.4.1}$$

As an example, in our synergetic plane-change problem we define the final time as the time when $x_3(t) = 0$, $t > 0$.

B. Let there be some other terminal boundary values to be satisfied:

$$\bar{M}[\bar{x}(t_f)] = 0 \tag{9.4.2}$$

An example would be the requirement for the orbital plane change to be a certain value, expressed as:

$$\sin x_4(t_f) \cos x_6(t_f) - \cos \Delta i = 0 \tag{9.4.3}$$

Let us first consider complication A, above. Since the terminal time is no longer fixed, we must now write our expression for DJ as:

$$dJ = \frac{\partial J}{\partial \bar{x}(0)} \delta\bar{x}(0) + \frac{\partial J}{\partial \bar{x}(t_f)} \delta\bar{x}(t_f) + \left.\frac{dJ}{dt}\right|_{t=t_f} \delta t_f \tag{9.4.4}$$

There may be some question as to what is meant by the last term in (9.4.4) since, strictly speaking, J as it has been defined is not a function of t. $\frac{dJ}{dt}$ is defined to be:

$$\left.\frac{dJ}{dt}\right|_{t=t_f} = \dot{J}(t_f) = \frac{\partial J}{\partial \bar{x}(t_f)} \dot{\bar{x}}(t_f) \tag{9.4.5}$$

Now we define δt_f by the following relationship derived from (9.4.1):

$$d\Omega(t_f) = \left.\frac{\partial \Omega}{\partial \bar{x}}\right|_{t=t_f} \delta\bar{x}(t_f) + \dot{\Omega}(t_f)\, \delta t_f = 0 \tag{9.4.6}$$

Solving for δt_f, one gets:

$$\delta t_f = -\left[\frac{1}{\dot{\Omega}} \frac{\partial \Omega}{\partial \bar{x}}\right]_{t=t_f} \delta\bar{x}(t_f) \tag{9.4.7}$$

Substituting (9.4.7) into (9.4.4) we have:

$$dJ = \frac{\partial J}{\partial \bar{x}(0)} \delta\bar{x}(0) + \left[\frac{\partial J}{\partial \bar{x}(t_f)} - \left(\frac{\dot{J}}{\dot{\Omega}} \frac{\partial \Omega}{\partial \bar{x}}\right)_{t=t_f}\right] \delta\bar{x}(t_f) \tag{9.4.8}$$

Now based on the work in sections 9.2 and 9.3 it is easy to see

that if we require:

$$\bar{y}_j^T(t_f) = \left[\frac{\partial J}{\partial \bar{x}(t_f)} - \left(\frac{\dot{J}}{\dot{\Omega}} \frac{\partial \Omega}{\partial \bar{x}}\right)_{t=t_f}\right] \tag{9.4.9}$$

We can write dJ again as:

$$dJ = \bar{L}_j^T \delta\bar{x}(0) + \int_0^{t_f} \bar{\Lambda}_j^T(s)\, \delta\bar{U}(s)\, ds \tag{9.4.10}$$

Now let us turn our attention to complication B above. In keeping with our reasoning on changes in dJ we can linearize equation (9.4.2) to get

$$d\bar{M}(t_f) = \left\{\frac{\partial \bar{M}}{\partial \bar{x}(t_f)} - \left[\frac{1}{\dot{\Omega}} \frac{\partial \Omega}{\partial \bar{x}} \dot{\bar{M}}\right]_{t=t_f}\right\} \delta\bar{x}(t_f) \tag{9.4.11}$$

Note that in (9.4.10) $d\bar{M}$ is a vector, $\frac{\partial \bar{M}}{\partial \bar{x}}$ is a matrix and $\dot{\bar{M}}$ is a vector for the general case. In the particular case for this problem, $\bar{M}$ turns out to be a scalar function and $d\bar{M}$ degenerates to a scalar.

If one starts out with a nominal trajectory that exactly satisfies the terminal boundary values, in theory one could simply require that dM = 0 at each step. In practice one does not start out with the terminal boundary values satisfied, and if one did, since we are applying linear theory to predict changes in a set of nonlinear equations, there is little chance that one could keep $\bar{M} = 0$ throughout the course of the solution. Consequently let us require that

$$d\bar{M} = \delta\bar{M} \tag{9.4.12}$$

$\delta\bar{M}$ represents any small changes we may wish to make in our terminal constraints.

If we now choose a matrix solution to our adjoint equations $Y_m(t)$ with boundary values

$$[Y_m]^T(t_f) = \left\{\frac{\partial \bar{M}}{\partial \bar{x}(t_f)} - \left[\frac{1}{\dot{\Omega}} \frac{\partial \Omega}{\partial \bar{x}} \dot{\bar{M}}\right]_{t=t_f}\right\} \tag{9.4.13}$$

we can then relate $d\bar{M}(t_f)$ to $\delta\bar{U}(t)$ and $\delta\bar{x}(0)$ by:

$$d\bar{M}(t_f) = [Y_m(0)]^T \delta\bar{x}(0) + \int_0^{t_f} [Y_m(s)]^T \frac{\partial \bar{f}(s)}{\partial \bar{U}(s)} \delta\bar{U}(s)\, ds \tag{9.4.14}$$

One can begin to appreciate that if the dimension of $\bar{M}$ is very large, the computational task of finding the minimum dJ is going to get quite complicated. In view of this, let us forego generality at this point and recognize that in the problem we are working on, M is a scalar function and designate $\bar{y}_x(t)$ as the solution of the adjoint equations such that:

$$\bar{y}_x^T(t_f) = \left\{ \frac{\partial M}{\partial \bar{x}(t_f)} - \left[\frac{\dot{M}}{\dot{\Omega}} \frac{\partial \Omega}{\partial \bar{x}} \right]_{t=t_f} \right\} \tag{9.4.15}$$

By analogy to the previous work, let us make some definitions to simplify notation:

$$\bar{\Lambda}_x(t) = \bar{y}_x^T(t) \frac{\partial \bar{f}(t)}{\partial \bar{U}(t)} \tag{9.4.16}$$

$$\bar{L}_x(t) = \bar{y}_x(0) \tag{9.4.17}$$

$$I_{pq} = \bar{L}_p^T W_x^{-1} \bar{L}_q + \int_0^{t_f} \bar{\Lambda}_p^T(s) W_u^{-1}(s) \bar{\Lambda}_q(s)\, ds \tag{9.4.18}$$

We are now in a position to add a second equation of constraint to our problem of finding dJ min:

$$\eta_2 [\bar{L}_x^T \delta\bar{x}(0) + \int_0^{t_f} \bar{\Lambda}_x^T(s) \delta\bar{U}(s)\, ds - \delta M] = 0 \tag{9.4.19}$$

and proceed to solve the problem.

By the same process we used before in (9.2) and (9.3) we will obtain:

$$\delta\bar{x}(0) = \frac{-\sqrt{\epsilon^2 - I_{xx}^{-1}(\delta M)^2}}{\sqrt{I_{jj} - I_{xj}^2 I_{xx}^{-1}}} W_x^{-1} (\bar{L}_j - I_{xj} I_{xx}^{-1} \bar{L}_x)$$

$$+ I_{xx}^{-1} W_x^{-1} \bar{L}_x \delta M \tag{9.4.20}$$

$$\delta\bar{U}(t) = \frac{-\sqrt{\epsilon^2 - I_{xx}^{-1}(\delta M)^2}}{\sqrt{I_{jj} - I_{xj}^2 I_{xx}^{-1}}} W_u^{-1}(t)[\bar{\Lambda}_j(t) - I_{xj} I_{xx}^{-1} \bar{\Lambda}_x(t)]$$

$$+ I_{xx}^{-1} W_u^{-1}(t)\bar{\Lambda}_x(t)(\delta M) \qquad (9.4.21)$$

$$dJ = -\sqrt{\epsilon^2 - I_{xx}^{-1}(\delta M)^2}\ \sqrt{I_{jj} - I_{xj}^2 I_{xx}^{-1}} + I_{jx} I_{xx}^{-1}(\delta M)^2 \qquad (9.4.22)$$

9.5 Development of the Gradient Method, Step 4

There is one final complication which must be considered. The development of section 9.4 assumes that the perturbation in the final time resulting from changes in the controls and initial conditions will be small. This assumption may not turn out to be true. There is a method for handling this problem also. Returning to Equation (9.4.7) we recall that we defined changes in the final time t_f by the following relationship:

$$\delta t_f = -\left(\frac{1}{\dot{\Omega}}\right) \frac{\partial \Omega}{\partial \bar{x}}_{t=t_f} \delta \bar{x}_f \qquad (9.5.1)$$

If one defines a third set of solutions to the adjoint equations with boundary values:

$$\bar{y}_t^T(t_f) = -\left(\frac{1}{\dot{\Omega}}\right) \frac{\partial \Omega}{\partial \bar{x}}_{t=t_f} \qquad (9.5.2)$$

one can write:

$$\delta t_f = \bar{y}_t^T(0)\, \delta\bar{x}(0) + \int_0^{t_f} \bar{\Lambda}_t^T(s)\, \delta\bar{U}(s)\, ds \qquad (9.5.3)$$

In theory one could use (9.5.3) as an additional constraint to the problem discussed in section 9.4 by requiring, for example that: $(\delta t_f)^2 \leq K$ where K is some small number. In practice, it turns out that the addition of a third constraint to the problem of section 9.4 results in a set of equations from which one cannot derive neat closed

form solutions for $\delta\bar{U}(t)$ and $\delta\bar{x}(0)$.

There still exists a method for handling this situation indirectly. If one substitutes the expressions for $\delta\bar{U}(t)$ and $\delta\bar{x}(0)$ obtained in section 9.4 in equation (9.5.3) one obtains after simplification:

$$\delta t_f = \frac{-\sqrt{\epsilon^2 - I_{xx}^{-1}(\delta M)^2}}{\sqrt{I_{jj} - I_{xj}^2 I_{xx}^{-1}}} [I_{tj} - I_{xj} I_{xx}^{-1} I_{tx}] + I_{tx} I_{xx}^{-1}(\delta M) \qquad (9.5.4)$$

This expression may now be used to assist in the choice of an appropriate value of the constant ϵ^2 which will keep δt_f of reasonable size. However, use of this method creates some practical problems associated with the computational aspects of the problem. These problems will be discussed in a subsequent chapter.

9.6 Inclusion of the Control and State Variable Constraints

So far, our development of the gradient method as assumed that there are no constraints on the control variables or the state variables. We have to introduce the constraints of the problem into the gradient method. Denham (Ref. 13) proposes a method which sounds very attractive in many respects and which he claims is superior to the penalty function method proposed by Kelley (Ref. 10). While there is no doubt that Denham's claims may be correct for problems with single control variables, we believe it would be very dangerous to apply Denham's methods to problems with multiple control variables. To see where the danger lies, let us first present a brief summary of Denham's Method.

9.6.1 The Denham Method of Handling State Variable Constraints

In an optimization problem involving state variable constraints, in general, one expects that the optimal solution will be broken up into segments, in which some segments are in regions where the constraints are not acting and other segments involve trajectories which must follow the boundaries of the constraints. Let us consider the case

where the constraint function is a single scalar function of the state variables which we may write as:

$$C[\bar{x}(t)] \leq 0 \qquad 0 \leq t \leq t_f \tag{9.6.1.1}$$

Now for the solution to follow the constraint for a finite interval, $(t_1 - t_2)$ where $0 \leq t_1, t_2 \leq t_f$,

$$C[\bar{x}(t)] = 0 \qquad t_1 \leq t \leq t_2 \tag{9.6.1.2}$$

For (9.6.1.2) to hold, the following equation must also hold:

$$\frac{dC}{dt} = \dot{C}(t) = \frac{\partial C}{\partial \bar{x}} \dot{\bar{x}}(t) = 0 \quad t_1 \leq t \leq t_2 \tag{9.6.1.3}$$

Further for (9.6.1.3) to hold

$$\ddot{C}(t) = 0 \qquad t_1 \leq t \leq t_2 \tag{9.6.1.4}$$

and in general

$$C^{(n)} = 0 \quad t_1 \leq t \leq t_2$$

where (n) denotes the nth derivative with respect to time.

To apply Denham's method one proceeds with the indicated differentiation process until one reaches a term where the control variable (or variables) appear explicitly. Let us assume this happens on the nth differentiation, then we have the following equations:

$$C^{(j)}[\bar{x}(t)] = 0 \qquad j = \ldots(n-1) \tag{9.6.1.6}$$

$$C^{(n)}[\bar{x}(t)\, \bar{U}(t)] = 0 \tag{9.6.1.7}$$

Equations (9.6.1.6) give a set of terminal boundary values on the unconstrained segment which must be satisfied at the time t_1 in order for the solution to follow the constraint boundary subsequently to time t_1.

Equation (9.6.1.7) is supposed to provide information to modify the equations of motion while in the boundary. To see how this works let us suppose that our control $\bar{U}(t)$ is a scalar, $U(t)$. Then from (9.6.1.7), one can say that for small perturbations

$$\frac{\partial C^{(n)}}{\partial \bar{x}} \delta\bar{x} + \frac{\partial C^{(n)}}{\partial U} \delta U = 0 \tag{9.6.1.8}$$

From this one deduces that on the constraint boundary our perturbation equations of motion and their adjoints become:

$$\dot{\delta\bar{x}} = \left[\frac{\partial \bar{f}}{\partial \bar{x}} - \frac{\partial \bar{f}}{\partial U}\left(\frac{\partial C^{(n)}}{\partial U}\right)^{-1} \frac{\partial C^{(n)}}{\partial \bar{x}}\right] \delta\bar{x} \qquad (9.6.1.9)$$

$$\dot{\bar{y}} = -\left[\frac{\partial \bar{f}}{\partial \bar{x}} - \frac{\partial \bar{f}}{\partial U}\left(\frac{\partial C^{(n)}}{\partial U}\right)^{-1} \frac{\partial C^{(n)}}{\partial \bar{x}}\right]^{T} \bar{y} \qquad (9.6.1.10)$$

Thus we see that for the scalar control case while on the boundary, we have no freedom of choice for our control. Our only choice is in varying our times t_1 and t_2 to enter and leave the boundary.

Now for a vector control it may happen that one of the control components appears explicitly before the others, in which case (9.6.1.7) might look like:

$$\frac{\partial C^{(n)}}{\partial \bar{x}} \delta\bar{x} + \frac{\partial C^{(n)}}{\partial U_k} \delta U_k = 0 \qquad (9.6.1.11)$$

Denham advocates that we now consider δU_k to be determined by (9.6.1.11) and that we consider ourselves free to select the other components of $\delta\bar{U}$ arbitrarily. We consider this proposal to be unwise. In the synergetic plane change problem for example, application of this proposal to the heating constraint would imply that we stay on the constraint boundary by changing U_1 (angle of attack) and one is free to vary U_2 (bank angle) arbitrarily. There is nothing in the problem that tells one that this is the optimum way to fly along the boundary. In fact, one suspects that just the opposite is true and it may be better to use bank control to stay on the boundary and leave the angle of attack free to be changed as desired.

Let us now consider the case where p of our r control variables appear explicitly in (9.6.1.7). Then for small perturbations we have:

$$\sum_{i=1}^{k} \frac{\partial C^{(n)}}{\partial x_i} \delta x_i + \sum_{j=1}^{p} \frac{\partial C^{(n)}}{\partial U_j} \delta U_j = 0 \qquad (9.6.1.12)$$

giving a single equation in p unknowns so that the solution is indeterminate.

9.6.2 Other Methods of Handling State Variable Constraints

There is another method of handling state variable constraints, the Penalty function method proposed by Kelly (Ref. 10).

9.6.2.1 The Penalty Function Method

To apply the penalty function method to state variable (or control variable) constraints, one does the following:

$$C[\bar{x}(t), \bar{U}(t)] \leq 0 \qquad 0 \leq t \leq t_f \tag{9.6.2.2}$$

One defines as an additional differential equation to adjoin to the set of n differential equations of motion as follows:

$$\dot{x}_{n+1} = \tfrac{1}{2}\{C[\bar{x}(t)\,\bar{U}(t)]\}^2 \quad h\{C[\bar{x}(t)\,\bar{U}(t)]\}$$

$$x_{n+1}(0) = 0 \tag{9.6.2.3}$$

where $h(s)$ is the unit step function, $h(s) = 0$ $s \leq 0$, $h(s) = 1$ $s > 0$. Then one adds an additional term to the cost functional:

$$J_1 = J[\bar{x}(0)\,\bar{x}(t_f)] + K x_{n+1}(t_f) \tag{9.6.2.4}$$

where K is an arbitrary positive constant.

Now it can be seen that (9.6.2.3) has the necessary continuity properties for the gradient method to work, i.e.:

$$\frac{\partial f_{n+1}}{\partial \bar{x}} \text{ and } \frac{\partial f_{n+1}}{\partial \bar{\bar{U}}} \text{ exist and are continuous.}$$

$K x_{n+1}(t_f)$ is a measure of the seriousness of our violations of the constraint boundaries. By making K sufficiently large we can insure that on an optimal trajectory we violate the constraint boundaries by as little as we wish. Note that the penalty function method does not "favor" one control variable over another but treats them all on the same basis. This method is the one we will use to solve our problem.

One can use as many additional penalty functions as there are constraints.

9.7

Once one has taken the initial step in the use of the penalty functions for state variable constraints, one can consider using penalty

functions also for constraints on the final boundary values of the state equations. Kelley (Ref. 10) shows that one can also handle final boundary value constraints by an analogous method. Recalling that the final boundary value constraints can be written as

$$\overline{M}[x(t_f)] = 0 \tag{9.7.1}$$

If one writes an augmented cost functional J_2 as:

$$J_2 = J[(\bar{x}(0), \bar{x}(t_f)] + \sum_{i=1}^{p} K_i x_{n+i}(t_f) + \sum_{i=1}^{n} \tfrac{1}{2} P_i [M_i(x)]^2 \tag{9.7.2}$$

where the P_i are arbitrary positive constants analogous to the K_i used for state variable constraints, one can develop a gradient algorithm to minimize J_2 which, with proper selection of the constants P_i, will minimize the components of $\overline{M}$ as well as the other factors involved. This method offers certain advantages in that only one set of adjoint equations need be solved. In contrast, the Bryson-Denham method requires one set of adjoint equations for the cost functional and an additional set for each component of the vector $\overline{M}$.

9.7.1

The equations for the Kelley algorithm are derived in a manner entirely enalogous to section 9.4 with the exception that there now is no set of adjoint equations $\bar{y}_x(t)$, since there is no explicit terminal boundary value constraint. The corresponding expressions for $\delta\bar{x}(0)$, $\delta\overline{U}(t)$ and dJ are:

$$\delta\bar{x}(0) = -\frac{\sqrt{\epsilon^2}}{I_{jj}} W_x^{-1} \bar{L}_j \tag{9.7.1.1}$$

$$\delta\overline{U}(t) = -\frac{\sqrt{\epsilon^2}}{I_{jj}} W_u^{-1}(t) \bar{\Lambda}_j(t) \tag{9.7.1.2}$$

$$dJ = -\sqrt{\epsilon^2}\sqrt{I_{jj}} \tag{9.7.1.3}$$

Note that equations (9.7.1.1), (9.7.1.2), and (9.7.1.3) are identical to the corresponding equations developed in section 9.3. The only difference is in the boundary values used for the adjoint equations.

9.8 Summary of the Gradient Development

In this chapter we have seen how, by use of solutions of the adjoint equations with special sets of terminal boundary values, we can computer small perturbations in control functions and initial boundary values in such a way as to give the maximum change to the cost functional J while still satisfying the terminal boundary value constraints on the differential equations of motion. We have also seen how state and control variable constraints can be handled by the use of penalty functions.

In the next chapter we will apply this development to our problem.

CHAPTER 10

APPLICATION OF THE GRADIENT METHOD TO THE SYNERGETIC PLANE CHANGE PROBLEM

Let us now consider in some detail the application of the theory outlined in chapter 9 to the problem to be solved.

10.1 Outline of the Solution Method

In outline form the steps necessary to generate a solution to the problem are as follows:

a. Produce an initial, non-optimal set of initial boundary values and control functions. In this case, these boundary values and control functions were derived from the work done on the analog computer described in chapter 8.

b. Generate solutions to the system state equations using the appropriate control functions and boundary values. At the same time, generate and store the information necessary to form the two time varying matrices, $\frac{\partial \bar{f}}{\partial \bar{x}}$ and $\frac{\partial \bar{f}}{\partial \bar{U}}$ which we will henceforth denote as A(t) and B(t). At the end of the solution, save the final values of the state equations necessary to compute the value of the cost function and the boundary values for the adjoint equations.

c. Compute the boundary values for the adjoint equations required, and generate solutions to them, backwards in time. At the same time compute and store the functions $\bar{\Lambda}(t)$ and the integrals necessary to compute the various constants I_{pq} defined in chapter 9.

d. Using the results of c above, generate the desired changes to the boundary values for the state equations $\delta\bar{x}(0)$ and the changes to the control functions, $\delta\bar{U}(t)$. Add these changes to their previous values and return to Step b.

This conceptually simple process outlined above is repeated until convergence is attained.

10.2 General Remarks about Application of the Gradient Method

If the reader is familiar with the literature on this subject, he will have noted that most of the literature ends discussion of method at about the point we have now reached and begins discussion of answers obtained. The reader will also note that most of the solutions displayed in the literature are for problems depending only on a single control function. Solution of the Synergetic Plane Change problem requires one to find five control functions and two initial boundary values. It turns out that the practical difficulties in obtaining a solution increase in proportion to the dimensionality of the state equations and also in proportion to the dimensionality of the optimal quantities being solved for.* Insofar as the author has been able to determine, no solution to a problem of equal or greater difficulty than the subject has ever been generated. For these reasons considerable attention will be devoted to the practical aspects of implementing the steps outlined in section 10.1 above.

10.3 Specific Details on the Magnitude of the Computational Problem

The complete details on the specific equations to be solved are developed in Appendix A. In this section a summary of information from this Appendix will be presented to afford an appreciation of the computational task associated with generation of a solution to this problem.

10.3.1 The State Equations

With the addition of two state equations used for generation of the penalty functions for the heating and thrust acceleration constraints, the state equations form a ninth order system of differential equations.

10.3.2 The Adjoint Equations

Two sets of ninth order adjoint equations are required for this problem. In addition, three integrals of the form:

*The exact nature of this proportion is not known, but the author suspects a relationship of the form: Difficulty = e^{N-1}.

$$I'_{pq} = \int_0^{t_f} \bar{\Lambda}_p^T W_u^{-1} \bar{\Lambda}_q \, dt \qquad (10.3.2.1)$$

must be generated. This results in the equivalent of a twenty-first order system to be solved. Fortunately, because of the coupling between the equations and the boundary values used, the solutions of eight of the components of the adjoint equations required are trivial, (e.g., a constant) leaving a total of 13 differential equations to be integrated for solution of the adjoint system. In section 9.5 it was pointed out that generation of a predicted change in the end-time, δt_f, for this problem created some practical problems. These problems involve the addition of another six equations to be integrated, since the computation time required for numerical integration of differential equations depends, to first order accuracy, on the order of the system being integrated. Addition of the extra six equations would extend the computer time required to generate the adjoint solutions by at least 25 percent.

10.3.3 The Storage of Functions of Time

Generation of time-varying functions for computation of problems of this type requires storage of tables of values over the time span involved with some sort of interpolation to find values between the tabular points. The size of this task is another measure of the magnitude of the computational problem. For this problem, we have first of all the five components of the vector $\bar{U}(t)$. Next, we have the time-varying 5×5 matrix $W_u^{-1}(t)$. For this problem, only the diagonal terms were used, giving five more functions. Thirdly, we have the problem of the two matrices of partial derivatives, A(t), (9×9) and B(t) (9×5). There are 32 non-zero elements of these matrices used. The storage requirements for these functions can be reduced by recognizing that they can be generated by algebraic combinations of the nine state variables and five control variables. Thus storage of the nine state variable solutions results in a substantial saving.

Fourth, we have the two five dimensional vectors $\bar{\Lambda}_j(t)$ and $\bar{\Lambda}_x(t)$. In summary, one finds a requirement for storage of tables for at least 29 time-varying functions in order to solve this problem. If one uses 150-200 points for each function (a practical minimum for acceptable accuracy), this implies a minimum of 4500-6000 numbers to be stored for generation of these functions.

10.4 The Computer Program

The solutions to this problem were generated using a generalized computer program devised by the author (Ref. 16). This program, called MOUNTAINEER, is capable of solving a wide class of optimal control problems by the gradient method. MOUNTAINEER was originally written in the MAD (Michigan Algorithm Decoder) programming language for the IBM 7090 Computer. It has subsequently been rewritten in FORTRAN IV for the CDC 3600 Computer and also in CHIPPEWA FORTRAN for the CDC 6600 Machine.

10.4.1 Program Organization

MOUNTAINEER is organized on a building block philosophy for convenience in maintenance and to facilitate changes in the problem being solved. It consists of the following major parts:

a. The Main Program: This program is a problem-independent executive routine that controls the solution of the problem. It directs continuous progress through the general steps outlined in section 10.1 for any desired number of iterations.

b. The Data Input Sub-Routine. This subroutine reads in the numerical data necessary to start a series of iterations. This data includes the tabular data for the controls, $\bar{U}(t)$, the weighting functions, W(t), the boundary values for the state equations and other numerical data necessary for the solution. This subroutine is essentially independent of the problem being solved.

c. A Numerical Integration Sub-Routine. This subroutine performs all computations necessary for numerical integration of an

arbitrary set of simultaneous first order differential equations of arbitrary order and is essentially independent of the problem to be solved. There are actually two interchangeable integration subroutines available for MOUNTAINEER. Both subroutines feature continuous automatically variable step size, variable accuracy criteria, and double precision accumulation. One subroutine uses a Milne-Reynolds fifth order Predictor-Corrector Algorithm. The other uses a fourth order Runge-Kutta Algorithm.

d. Derivative Evaluation Sub-Routine. There are two of these subroutines, one for the state equations and one for the adjoint equations. These subroutines, while having a similar logical organization, are peculiar to the equations being solved. They accomplish the derivative evaluation function required for numerical integration, control the print-out of the solutions of the differential equations at specified intervals, store tabular values of various variables required to generate functions of time and do the necessary interpolation to derive functions of time from previously stored tabular values.

e. The Gradient Sub-Routine. This subroutine performs the necessary operations to connect the solutions of the state and adjoint equations in the desired manner so as to implement the desired gradient algorithm. It consists of two sections. These sections are dependent on the problem being solved and the particular gradient algorithm (Bryson or Kelley Method), but have a logical organization which is not strongly problem dependent.

Section one is entered after solution of the state equations. It computes the current value of the cost functional and the boundary values for the adjoint equations.

Section two is entered after solution of the adjoint equations. It computes the various parameters required to generate the new control functions and generates the new control functions and state equation boundary values.

f. A Data Output Sub-Routine. This subroutine is entered at the

conclusion of a cycle of solution iterations. It computes the value of the cost functional and produces a punched card deck with the current control law table and boundary value set for use in re-starting the problem at a later time.

10.4.2 Execution Time

Inasmuch as computer time is expensive, this subject is always of interest and must be considered when working on a problem of this type. Because of the many variables affecting this item one cannot give definite rules for time requirements; however, some general information may be of value.

First of all, since most of the time involved is used for performing numerical integration, execution time is highly dependent on the integration accuracy specified and the maximum and minimum integration step size selected.

Secondly, execution time is dependent on the method employed for solution. Both the Bryson and Kelley algorithms have been employed on this problem. The reader will note from the work in Appendix A that the Bryson method requires integration of a thirteenth order system of adjoint equations and the Kelley method a seventh order system. It turns out that this makes at least a 25 percent difference in the time required for integration of the adjoint equations.

Thirdly, computation time depends strongly on the equipment used. Most of the work on this problem was done on a CDC 3600 computer with on-line input and output. The average time per iteration (State Equation - Adjoint Equation) was five minutes. The same program on a CDC 6600 with off-line input-output reduced this time by a factor of ten.

Lastly, the factors affecting program efficiency become quite important for a program and problem of the size involved here. Such items as the amount of printed output required can make substantial differences in execution time.

Some of these factors will be discussed again in the context of accuracy of the solutions obtained.

10.5 Accuracy and Convergence

In any problem being solved by an iterative technique a fundamental question to be answered is: "When does one stop iterating?" This question is closely related to another problem. Since the cost of obtaining a solution is directly related to the number of iterations required, one must also consider techniques to minimize the number of iterations required. Discussion of these questions is related to the equations derived for generating the new control functions and initial conditions at each step, so let us review these equations in some detail for both the Bryson and Kelley Algorithms.

10.5.1

For the Bryson algorithm one has the following relationships:
The change to the control functions,

$$\delta\bar{U}(t) = -\frac{\sqrt{\epsilon^2 - I_{xx}^{-1}(\delta M)^2}}{\sqrt{I_{jj} - I_{xj}^2 I_{xx}^{-1}}}\,[W_u^{-1}(t)][\bar{\Lambda}_j(t) - I_{xx}^{-1} I_{xj}\bar{\Lambda}_x(t)]$$

$$+ I_{xx}^{-1}[W_u^{-1}(t)]\,\bar{\Lambda}_x(t)(\delta M) \qquad (10.5.1.1)$$

The change to the initial boundary values,

$$\delta\bar{x}(0) = -\frac{\sqrt{\epsilon^2 - I_{xx}^{-1}(\delta M)^2}}{\sqrt{I_{jj} - I_{xj}^2 I_{xx}^{-1}}}\,[W_x^{-1}][\bar{l}_j - I_{xx}^{-1} I_{xj}\bar{l}_x]$$

$$+ I_{xx}^{-1}[W_x^{-1}]\,\bar{l}_x(\delta M) \qquad (10.5.1.2)$$

The predicted change in the cost functional at each step,

$$dJ = \sqrt{\epsilon^2 - I_{xx}^{-1}(\delta M)^2}\,\sqrt{I_{jj} - I_{xx}^{-1} I_{xj}} + I_{xj} I_{xx}^{-1}(\delta M) \qquad (10.5.1.3)$$

The definition of the constants I_{jj}, I_{xx} and I_{xj},

$$I_{pq} = \bar{\ell}_p^T [W_x^{-1}] \bar{\ell}_q + \int_0^{t_f} \bar{\Lambda}_p^T(t)[W_u^{-1}(t)] \bar{\Lambda}_q(t)\, dt \qquad (10.5.1.4)$$

10.5.1.1

Let us next review the source of the various terms in these relationships. The terms arise from two sources, those controlled by the programmer and those which are generated as a consequence of the equations involved in the problem.

10.5.1.2

The terms over which the programmer has direct control are:

(a) ϵ^2, an arbitrary constant which partially controls the so-called "step size".

(b) W_x, an arbitrary constant matrix (2×2 in this problem) which controls to some degree the relationship between the components of $\delta\bar{x}(0)$ and also the relative effect of the changes in boundary values versus changes in control functions on the change in cost at each step, dJ.

(c) $W_u(t)$, an arbitrary time-varying matrix (5×5 in this problem) which controls the relationship between the components of $\delta\bar{U}(t)$ and also influences the relative effect of changes in boundary values versus changes in control functions on dJ.

(d) δM, a constant which represents the desired change in orbital inclination at each step in the problem.

Note that ϵ^2, W_x and $W_u(t)$ are related by an equality that must be satisfied at each step:

$$\epsilon^2 = \delta\bar{x}(0)^T W_x\, \delta\bar{x}(0) + \int_0^{t_f} \delta\bar{U}^T(t)[W_u(t)]\delta\bar{U}(t)\, dt \qquad (10.5.1.5)$$

We will discuss the choice of these terms in detail later in this chapter.

10.5.1.3

The terms which are generated as a consequence of the equations involved in the problem are:

(a) The time-varying vectors $\bar{\Lambda}_j(t)$ and $\bar{\Lambda}_x(t)$. These vectors are defined as:

$$\bar{\Lambda}_j(t) = B^T(t)\,\bar{y}_j(t)$$

$$\bar{\Lambda}_x(t) = B^T(t)\,\bar{y}_x(t)$$

where B(t) is the matrix of partial derivatives of the state equations with respect to the control variables and $\bar{y}_j(t)$ and $\bar{y}_x(t)$ are the two sets of solutions of the adjoint equations generated with boundary values defined by equations (9.4.9) and (9.4.15).

(b) The constant vectors $\bar{\ell}_j$ and $\bar{\ell}_x$. These vectors are defined as:

$$\bar{\ell}_j = \left[\frac{\partial J}{\partial \bar{x}(0)}\right]^T + \bar{y}_j(0)$$

$$\bar{\ell}_x = \bar{y}_x(0)$$

where $\bar{y}_j(0)$ and $\bar{y}_x(0)$ are the solution of the j and x adjoint equations at t = 0. These really represent the final values of these equations since their boundary values are defined at $t = t_f$ and they are integrated backward in time.

10.5.2

Now let us discuss what is meant by convergence for this problem. To do this let us look at equation (10.5.1.3). Assume that the boundary values are satisfied exactly, ($\delta M = 0$). We now recall that this relationship was derived from the following type of proposition: "Find the $\delta\bar{U}(t)$ and $\delta\bar{x}(0)$ that gives the largest possible change in dJ subject to the constraint that M not change." It can be shown that a necessary condition for a constrained local minimum (or maximum) is that there is no combination of $\delta\bar{U}(t)$ and $\delta\bar{x}(0)$ satisfying the constraint which results in a change of cost (dJ) in the desired direction. For this problem the condition for a local minimum is that

$$\sqrt{I_{jj} - I_{xx}^{-1} I_{xj}^2} = 0 \qquad (10.5.2.1)$$

Thus, one measures the degree of convergence by the size of the term $\sqrt{I_{jj} - I_{xx}^{-1} I_{xj}^2}$, which we will henceforth refer to as "GRAD", recognizing that it is analogous to the norm of a gradient vector in simpler problems. For this problem one must consider the size of GRAD in a relative manner since the numbers I_{jj}, I_{xx} and I_{xj} (see equation (10.5.1.4) depend on the matrices W_x and $W_u(t)$ which can be varied arbitrarily. The theory of the gradient method indicates that by performing enough iterations one can make GRAD as small as one wishes, thereby getting an answer as close to the actual solution as one desires. The results of the work on this problem show that as a practical matter, for fixed W_x, $W_u(t)$ and ϵ^2, GRAD approaches a non-zero limit dictated by the errors inherent in the numerical computation process.

10.5.3

The accuracy limitations for the application of the gradient method are primarily a consequence of the dimensionality of the solution. The fundamental problem encountered is a direct consequence of the fact that we are looking for a vector control law and a vector initial condition. The discussion that follows will be limited to the five dimensional vector function $\delta\bar{U}(t)$ but the same problems also apply to generation of the two dimensional constant $\delta\bar{x}(0)$.

10.5.3.1

Let us consider equation (10.5.1.1) with the assumption that $\delta M = 0$ and $W_u(t) = [1]$ (the identity matrix). This equation then becomes:

$$\delta\bar{U}(t) = -\frac{\sqrt{\epsilon^2}}{GRAD}\left[\bar{\Lambda}_j(t) - I_{xj} I_{xx}^{-1} \bar{\Lambda}_x(t)\right] \qquad (10.5.3.1)$$

Now it can be seen that $\delta\bar{U}$ is the sum of two vectors multiplied by a constant. In this problem, it turns out that these two vectors, $\bar{\Lambda}_j$ and $I_{xj} I_{xx}^{-1} \bar{\Lambda}_x$ are almost 180 degrees apart in a five dimensional sense.

Furthermore, as one gets closer and closer to convergence, the two vectors get closer and closer to being exactly 180 degrees out of phase. As a consequence, the resultant becomes very sensitive, in both magnitude and direction, to errors in the components.

Let us consider a specific example taken from one of the problems worked. The norm of a k dimensional vector $\bar{V}$ is defined as:

$$|\bar{V}| = \sqrt{\sum_{i=1}^{i=k} V_i^2}$$

We have the following norms:

$$|\bar{\Lambda}_j| = 127.06$$

$$|I_{xx}^{-1} I_{xj} \bar{\Lambda}_x| = 127.85$$

$$|\bar{\Lambda}_j - I_{xx}^{-1} I_{xj} \bar{\Lambda}_x| = 1.085$$

Physical reasoning supplies the cause of this problem. One can think of $\bar{\Lambda}_j$ as the component of $\delta\bar{U}$ that is trying to decrease the cost. $I_{xx}^{-1} I_{xj} \bar{\Lambda}_x$ is the component that is trying to keep the orbital inclination constant. One recognizes that the minimum fuel plane change maneuver is one which accomplishes 0° plane change. Thus the fact that the two components are in effect working against each other is not surprising. Furthermore, it becomes obvious why the effect becomes more pronounced as the optimum control law is approached, since there is progressively less and less fredom of choice of a $\delta\bar{U}(t)$ which will decrease the cost and at the same time not decrease the inclination change.

10.5.3.2

As a result of the phenomenon described in 10.5.3.1 above, as one proceeds through the solution of the problem the predicted change in cost, dJ, at each iteration departs further and further from the actual change. At the same time the number GRAD stops decreasing monotonically and begins to exhibit an oscillating behavior. Finally,

one reaches a point where the errors in $\delta\bar{U}$ completely overshadow the desired effect and convergence stops for all practical purposes.

We would like to point out that this phenomenon is independent of the choice of the "step size" parameter, ϵ^2, but arises largely as a result of inaccuracies in the generation of the functions $\bar{\Lambda}_j(t)$, $\bar{\Lambda}_x(t)$, and their integrals. The author feels that the type of results obtained under these circumstances are not particularly satisfying from an intellectual point of view considering the behavior one expects based on the mathematical theory used. One feels a bit presumptuous to claim a "Near-Optimal" trajectory has been generated without a good appreciation of the quantitative nature of the term "Near". The severity of this behavior is not usually present in problems with scalar control functions. Although the fundamental error is present, it is manifested in errors in magnitude of a single function rather than uncertainty about the magnitude and direction of a vector function. Having discussed the symptom, let us next consider the disease.

10.5.3.3

Inasmuch as the errors limiting convergence arise from inaccuracies in functions generated as a result of numerical integration, one is naturally tempted to suspect the integration subroutine. Numerical integration subroutines have notoriously bad reputations. The subroutines used for this problem however are high order and by virtue of the variable step size feature can be made highly accurate. Considerable effort (and computer time) was spent in verifying that the integration subroutines were not at fault in this case. In particular, the following tests were performed with no significant change in the results:

a. Variation of the accuracy criteria up and down by a factor of 10.

b. Use of double precision accumulation in the integration subroutine.

c. Change from a 36 bit word length (CDC 3600) to a 55 bit word length (CDC 6600).

Having found that integration accuracy was not causing the observed inaccuracies, an attempt was made to circumvent the problem by use of the Kelley Gradient Algorithm. Recall that the corresponding formulae for $\delta\bar{U}(t)$ and $\delta\bar{x}(0)$ using the Kelley Algorithm are:

$$\delta\bar{U}(t) = -\frac{\sqrt{\epsilon^2}}{\sqrt{I_{jj}}}\left[W_u^{-1}(t)\right]\bar{A}_j(t) \qquad (10.5.3.3.1)$$

$$\delta\bar{x}(0) = -\frac{\sqrt{\epsilon^2}}{\sqrt{I_{jj}}}\left[W_x^{-1}\right]\bar{\ell}_j \qquad (10.5.3.3.2)$$

These indicate that one is no longer faced with the problem of forming the sum of two vectors to generate the control changes. It turns out, however, that one gets the same symptoms. The reason for this is that although one is now solving only one set of adjoint equations, the vector summing is taking place implicitly by virtue of the boundary values placed on the single set of equations. Thus it must be concluded that the errors arise from some other source.

At this time it appears that the errors arise from the most inconvenient possible source, the requirements for use of functions derived from tabular data stored in the computer. As we mentioned in section 10.3.3, this problem requires storage of a minimum of 29 functions of time in tabular form. (The actual program uses 37 for reasons we will discuss later.)

Generation of intermediate values must be done by some form of interpolation. In the state equations, the functions $\bar{U}(t)$ are generated in this manner. In the adjoint equations, the matrices A(t) and B(t) are generated by interpolation. There are a number of possible solutions for this problem area. Unfortunately, they all involve the requirement for more computer capacity and/or more computing time. Both of these requirements are hard to satisfy in the real world. Consequently one is forced to live with this problem at some

level, and accept its influence on the end product.

As an example of the sensitivity of this problem to the use of interpolated data consider the following: for one of the problems solved an initial trajectory was generated on the digital computer using the "Autopilot" described in chapter 8. This trajectory resulted in a 60° plane change. The control $\overline{\overline{U}}(t)$ for this trajectory was stored as a 150 point per component table and trajectory re-run using interpolated values. The resulting plane change was 23 degrees!

This completes the discussion on accuracy and convergence. Next, let us consider use of the weighting matrices W_x and $W_u(t)$.

10.6 Use of the Weighting Matrices W_x and $W_u(t)$

Denham (Ref. 16) points out that the arbitrary matrices can be used by the programmer to improve convergence. However, he is rather vague as to just how this is done. In single control variable problems where W_x is not employed the use is fairly obvious. $W_u(t)$ as a scalar can control the relative amount of "action" at various points in the trajectory.

In a problem where $\delta\overline{U}$ is a vector, the effect of changes in W_u is not so clear-cut. The author has succeeded in developing a concept for the use of this programming aid which he feels is more definite than Denham's and may assist others working on problems involving vector controls.

10.6.1

Before proceeding to discuss the subject of weighting matrices further, it is necessary to define more precisely some qualitative statements about the relative goodness of the convergence characteristics of this problem. The author judges this quality with the aid of equation (10.5.1.3) which gives the predicted change in cost for each iteration. Once the problem has reached the point where the final boundary conditions are very nearly satisfied ($\delta M \approx 0$), one of the goals a programmer strives for is to find a combination of W matrices

and ϵ^2 such that the predicted change in cost, dJ, is "close" to the actual change. This provides some assurance that one is actually following the desired path of steepest descent. Based on a comparison of predicted and actual changes in cost, the author makes the following evaluation of the problem. If one denotes the actual change in cost by ΔJ, the evaluation is as follows:

$dJ/\Delta J < 1.1$	Excellent
$1.1 < dJ/\Delta J < 2$	Good
$2 < dJ/\Delta J < 10$	Fair
$dJ/\Delta J > 10$	Poor

Having pinned down convergence quality, we now proceed to confuse the issue by pointing out that this definition has some weak points. It is sometimes possible to find combinations of the various numbers under programmer control that give larger values of dJ per iteration at "fair" quality than one can get at "good" or "excellent" quality. Since computer time is expensive it behooves one, other things being equal, to try for the largest practicable value of dJ per iteration. Thus, pursuit of quality must be tempered by cost-consciousness.

10.6.2

Let us now consider the use of $W_u(t)$. In the absence of prior information as to what to do with this matrix, the natural course of action is to select the identity matrix and observe the result. In this problem the result is fair to poor convergence. Further, the author was unable to find any simple logic for changing $W_u(t)$ which would substantially improve convergence. A more careful study of the problem yielded some interesting insight and a logically justifiable answer as to how to choose $W_u(t)$.

Two of the five elements of $\delta\overline{U}$ are angles and the remaining three are forces. Consequently, it is very difficult to visualize the relative effect of differentially weighting these two classes of controls. Further, the two angles are quite different in their influence on the

trajectory. Thus one is led to seek a method of weighting the influence of the control vector changes in a manner which provides the programmer with more insight. Since this problem is a flight mechanics problem dealing with forces and accelerations it is natural to seek to tie the control changes to these quantities.

If one examines the state equations for this problem one finds that the control variables affect the trajectory in a first order fashion in three of the seven state equations, corresponding to the first, second and fourth equations as defined in chapter 3. Thus a possible approach is to replace the integral term of the control constraint (10.5.1.5) with a term of the form:

$$\int_0^t \delta\bar{A}_u^T(t)\,\delta\bar{A}_u(t)\,dt = \text{Const.} \qquad (10.6.2.1)$$

where $\delta\bar{A}_u$ now represents a small change in the accelerations in the three state equations identified above.

Next, we identify the three component vector $\delta\bar{A}_u$ as follows:

$$\delta\bar{A}_u = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial \bar{U}} \\ \frac{\partial \dot{x}_2}{\partial \bar{U}} \\ \frac{\partial \dot{x}}{\partial \bar{U}} \end{bmatrix} \delta\bar{U} \triangleq A_c(t)\,\delta\bar{U}(t) \qquad (10.6.2.2)$$

where the matrix $A_c(t)$ is 3×5.

Substituting (10.6.2.2) into (10.6.2.1) one obtains:

$$\int_0^{t_f} \delta\bar{U}^T(t)[W_c(t)]\,\delta\bar{U}(t)\,dt = \text{Const.} \qquad (10.6.2.3)$$

where

$$W_c(t) = A_c^T(t)\,A_c(t) \qquad (10.6.2.4)$$

Thus, one returns to an expression identical in form to that in equation (10.5.1.5). Now, however, we have a new weighting matrix W_c which has converted the control variable constraint into an acceleration constraint. If one proceeds to use this development in the derivation of the gradient algorithm the same results as presented in chapter 9 emerge except that the matrix W_u is now replaced with W_c.

At this point a very disconcerting difficulty arises. The formulae for the various quantities to be computed to generate the changes in $\bar{U}$ involve the inverse of W_c and it turns out that W_c is singular!! A little thought will convince one that this is only natural since in effect we are looking for a unique solution to three equations in five unknowns.

10.6.2.1

Several schemes were considered to get around the difficulty described above. The most appealing from a programmer viewpoint is to separate the gradient constraints into several terms. An example is the following set:

$$\epsilon_b^2 = \delta\bar{x}^T(0)\, W_x\, \delta\bar{x}(0) \qquad (10.6.2.1.1)$$

$$\epsilon_\ell^2 = \int_0^{t_f} \delta\bar{L}^T\, \delta\bar{L}\, dt \qquad (10.6.2.1.2)$$

$$\epsilon_t^2 = \int_0^{t_f} \delta\bar{T}^T \delta\bar{T}\, dt \qquad (10.6.2.1.3)$$

where $\delta\bar{L}$ is the change in the two components of the lift acceleration and $\delta\bar{T}$ is the change in the three components of the thrust acceleration.

Unfortunately, it turns out that one cannot write down an explicit solution to a gradient problem having more than two equality constraints. Since the Bryson method requires one equality constraint to handle the terminal boundary values, direct application of the scheme is impractical. One can, however, add the three equations (10.6.2.1.1), (10.6.2.1.2), and (10.6.2.1.3) together to get a combined

equation:

$$\epsilon^2 = \delta\bar{x}^T(0)W_x\delta\bar{x}(0) + \int_0^{t_f}\delta\bar{L}^T\delta\bar{L}\,dt + \int_0^{t_f}\delta\bar{T}^T\delta T\,dt \qquad (10.6.2.1.4)$$

This approach does provide an explicit solution and its use does improve the convergence qualities of the solution.

The details of the implementation of equation (10.6.2.1.4) will be found in Appendix A.

10.6.3

Now we will discuss the use of the matrix W_x and a problem we will call the "Balance" problem. In chapter 9 an expression for the predicted change in cost was developed (9.4.10). For convenience, it is repeated here:

$$dJ = \bar{L}_j^T\delta\bar{x}(0) + \int_0^{t_f}\bar{\lambda}_j^T(t)\,\delta\bar{U}(t)\,dt \qquad (10.6.3.1)$$

If one substitutes the expressions for $\delta\bar{x}(0)$ and $\delta\bar{U}(t)$ from equations (10.5.1.2) and (10.5.1.1) the contributions to dJ from the change in boundary values and from the change in control variables can be computed separately. It is these contributions which, when combined, produce equation (10.5.1.3). By computing and looking at these contributions separately, one frequently notes some behavior of the problem that is not intuitively obvious from the theory. For example, it can turn out that the change in cost due to $\delta\bar{x}(0)$ and the change due to $\delta\bar{U}(t)$ are opposite in sign. It may also turn out that one is significantly larger than the other in absolute value. Furthermore, the relationships can change considerably in a few iterations. A specific example follows.

10.6.3.1

The numbers to be presented were taken from one of the problems solved. Instead of the matrix W_x, we will discuss W_x^{-1} since this is the quantity actually used in the computations. W_x^{-1} in all cases

is taken to be the identity matrix multiplied by a scalar constant. We designate as dJ_u the predicted change in cost due to $\delta\bar{U}$ and dJ_x as the change due to $\delta x(0)$. The behavior of the critical numbers has been worked out for a range of values of the matrix W_x^{-1} and a fixed value of the step size ϵ^2 and matrix $W_u(t)$. Before writing down the numbers, it will be beneficial to consider the formulae for them.

$$dJ_b = -\frac{\sqrt{\epsilon^2}}{GRAD}\,\bar{\ell}_j^T\,W_x^{-1}\,[\bar{\ell}_j - I_{xx}^{-1}I_{xj}\bar{\ell}_x] \qquad (10.6.3.1.1)$$

$$dJ_u = -\frac{\sqrt{\epsilon^2}}{GRAD}\int_0^{t_f}\bar{\Lambda}_j^T\,W_u^{-1}[\bar{\Lambda}_j - I_{xx}^{-1}I_{xj}\bar{\Lambda}_x]dt \qquad (10.6.3.1.2)$$

The results of the computation are:

W_x^{-1}	dJ_b	dJ_u	dJ	GRAD
0.	-0-	-5.43×10^{-3}	-5.43×10^{-3}	2.38
0.01	1.02×10^{-3}	-7.23×10^{-3}	-6.21×10^{-3}	2.48
0.1	6.10×10^{-3}	-1.38×10^{-2}	-7.74×10^{-3}	3.09
1.0	9.72×10^{-3}	-2.12×10^{-2}	-1.15×10^{-2}	4.61
10.0	-8.40×10^{-3}	-1.35×10^{-2}	-2.19×10^{-2}	8.76
25.0	-2.30×10^{-2}	-9.33×10^{-3}	-3.24×10^{-2}	12.86
100.0	-5.67×10^{-2}	-4.90×10^{-3}	-6.16×10^{-2}	24.67

Note that while casual inspection of equation (10.6.3.1.1) seems to show that for the type of matrix used for W_x^{-1}, dJ_b should vary directly with W_x^{-1}. However, this matrix is also used in computation of the numbers I_{xx}, etc., and as a result the complex relationship shown in the table above results. To show how tricky this selection can be, let us look at some of these quantities taken from the same problem four iterations later.

W^{-1}	dJ_b	dJ_u	dJ	GRAD
0	-0-	-2.64×10^{-3}	-2.64×10^{-3}	1.056
0.01	-1.59×10^{-4}	-2.11×10^{-3}	-2.27×10^{-3}	1.045
1.0	-4.31×10^{-3}	1.60×10^{-3}	-2.71×10^{-3}	1.084
10.0	-5.56×10^{-3}	2.43×10^{-3}	-3.13×10^{-3}	1.259
25	-5.84×10^{-3}	2.15×10^{-3}	-3.69×10^{-3}	1.469

Note that in this case the sign change occurs in dJ_u rather than dJ_b.
10.6.3.2

In summary, the selection of W_x for proper "balance" is a tricky proposition. For this problem it turns out that better convergence is obtained if $|dJ_b|$ is smaller than $|dJ_u|$. For this reason a W_x is selected to produce this effect. In addition, it seems pointless to have the $\delta\bar{U}$ and $\delta\bar{x}(0)$ providing opposite effects to the cost, so the computer program was written so as to make $W_x^{-1} = 0$ whenever $dJ_b > 0$ occurred.

One could carry the process of computing separate effects farther and compute the effect of each component of $\delta\bar{x}(0)$ and $\delta\bar{U}(t)$ on dJ, and come up with still more sophisticated methods of correcting both W_x and W_u at each step. This has not been done for this problem at present.

10.7 Use of the Penalty Function Gains

The state variable constraints are handled by the use of penalty functions added to the basic cost functional. These take the form of $K_i x_i(t_f)$ where K is an arbitrary gain factor. If the constraint is violated use of this method always results in some violation throughout the rest of the problem The K's are used to get the violation to an acceptably "small" value, with the size of "Small" being at the discretion of the programmer.

Similarly to the situation in regard to the weighting matrices, the use of these factors in a multi-dimensional problem has many

complex interactions. For example, an increase in one of the K's changes the J-adjoint solution, which changes the numbers I_{jj}, etc., which in turn may lead to a requirement to change the matrices W_x and W_u, which may require a change in step size, etc.

As Kelley points out in Ref. 10, excessively large values of the K's can lead to unstable behavior of the convergence process. In view of this, conservatism is essential in their choice. From the experience on this problem it appears that the following general rules are appropriate:

a. For initial corrections to large constraint violations, use a K such that the cost contribution due to the constraint is 10 to 20 times larger than the basic cost.

b. After the violations have been reduced to a reasonable level, reduce K so that the constraint contribution is about the same size as the basic cost.

c. When the constraint violations have been reduced to an acceptable degree of "smallness", adjust K in such a manner as to keep the violation approximately constant.

The procedure outlined above effectively distributes the control effort in different ways, ranging from almost complete concentration on the constraint to nearly complete concentration on the basic cost.

10.8 Use of the Boundary Value Correction δM

The number δM is the easiest and most straightforward to use of all of the programmer controlled parameters in this problem. In this problem δM is defined as:

$$\delta M = \cos \Delta i_{desired} - \cos \Delta i \qquad (10.8.1)$$

where Δi is the change in orbital inclination.

For the cases considered, δM varies between 0 and 1 in absolute value. Once the desired Δi trajectory has been achieved, (10.8.1) is used directly to generate δM. At each iteration, the Δi changes slightly from the desired and δM introduces a correction which attempts to restore it to its proper value.

For generating initial trajectories, where δM could be too large to satisfy the linearity assumption in the gradient method, a limiting algorithm is used which limits $|\delta M|$ to 0.05 maximum. This amount of change can be easily achieved in one iteration. Thus a trajectory can be "bent" from 10° to 60°, for example, in 10 iterations.

Typical performance after achieving the desired nominal plane change can result in $|\delta M|$ being held to less than 5×10^{-3}, corresponding to $\pm 1/3$ of a degree at 60°. Actually the variation of δM turns out to depend on the step size ϵ, and thus can be controlled to any desired value by proper choice of ϵ. We will discuss this further in the next paragraph.

10.9 Use of the Step Size Parameter ϵ

The last of the major programmer controlled parameters in this problem is the step size, ϵ. Its effect can most easily be seen from the equation for the predicted change in cost, dJ:

$$dJ = -\sqrt{\epsilon^2 - I_{xx}^{-1}(\delta M)^2}\,[GRAD] + I_{xx}^{-1} I_{xj}\,\delta M \qquad (10.9.1)$$

If one assumes $\delta M = 0$, we see that dJ varies directly with ϵ. None of the other factors in this equation depend on ϵ. In actuality, δM is never zero and ϵ^2 can be thought of as dividing the control effort between satisfying the boundary values and reducing the cost. For example, if one makes

$$\epsilon^2 = I_{xx}^{-1}(\delta M)^2 \qquad (10.9.2)$$

all the control effort will go toward making $\delta M = 0$ and the change in cost will be that associated with this activity, to wit: $I_{xx}^{-1} I_{xj}\,\delta M$.

10.9.1

The choice of the proper value for ϵ is relatively straightforward. First of all, there is a minimum value which is specified by (10.9.2) above.

In this problem the maximum size of ϵ is dictated by the fact that ΔM, the actual change in orbital inclination, never matches δM. In

fact, it may turn out to be in the opposite direction. In general, the larger one makes ϵ, the worse these excursions get, finally reaching a point where the major observable phenomenon as the iterations progress in an oscillation of ΔM.

It must be pointed out, however, that this behavior can also occur for other reasons than a bad choice of ϵ. Specifically, it can be a symptom of a poor choice of any one of the other programmer controlled parameters discussed previously. This oscillating behavior can also signal that the problem has converged as much as it is going to.

The basic procedure the author uses to choose ϵ is as follows: (assuming K, W_x and W_u are held fixed)

a. Determine approximately how much variation in ΔM is "acceptable." (e.g., $\pm 0.5°$)

b. Gradually increase ϵ until the variation is exceeded, then cut by 1/2 to be safe.

There are other tools available for guesses on ϵ which are quite valuable. These are based on equation (10.9.1). For example let us suppose the following:

$$J = 7.6, \quad GRAD = 100.0$$

We know that $J_{min} < 1.0$. Thus it is evident that unless $\epsilon > 10^{-3}$ one can expect to need over 100 iterations (at 5 minutes computer time each) to get convergence.

As a second example, consider:

$$J = 0.7, \quad GRAD = 1.0$$

Assume that we have previously found that the maximum ϵ we can use and maintain stability is .005. Further assume that ΔJ, the actual change in J, has been averaging 50% of dJ. Then it is apparent that we are getting about 0.35% improvement per iteration and it will require over 15 iterations to get at most another 5% cost decrease. This may be enough to convince one that the point of diminishing

returns has been reached and it is time to terminate the problem.

As a third example, assume we desire to get a 2% improvement each iteration. Then one immediately arrives at an algorithm for automatic adjustment of ϵ at each iteration:

$$\epsilon = \frac{0.02J}{GRAD}$$

There are several other ways that (10.9.1) can serve as a useful tool in the selection of ϵ. We leave this subject with a final warning: If the problem has been ill-conditioned by poor choices of other arbitrary prameters, there may be no possible selection of ϵ that will give satisfactory behavior.

10.10 Summary

In sections 10.5-10.9 we have attempted to give an appreciation for the effects and interactions of the various arbitrary parameters available for programmer control of this type of problem. In working a problem of this type the engineer cannot hope to turn over a set of equations to a programming specialist and expect to see results in due course without further effort. Active participation by an individual with a thorough appreciation of the characteristics of the system represented by the state equations is essential. It has been the author's experience that 5 - 6 iterations are the most that should be run at a time before re-examination of the problem is made to verify its progress.

Next, we will begin examination of the actual solutions obtained.

CHAPTER 11

SPECIFIC RESULTS

In this chapter we discuss the specific cases of the Synergetic Plane Change Problem which have been worked out to date.

11.1 General

When this project began, the author hoped to generate enough solutions over a range of interesting cases to form a rather complete picture about the optimal trajectories for this problem. It has turned out to be very expensive in terms of computer time to get a solution for a single case. Time required per case varies with the number of iterations, but has varied 7 to 15 hours. As a result we have been forced for the moment to stop at $5\frac{1}{2}$ cases.

11.2

The cases solved for are outlined in the following subparagraphs. Most of the constants in the State equations were the same for all cases. These will be discussed separately.

11.2.1 Case 1 and Case 2

These are unconstrained problems, i.e., the heating and thrust acceleration constraints are not active. A plane change of 60° was required and a max L/D of 4 was used. The difference between Case 1 and Case 2 is only in the initial nominal solution used. These cases started from widely separated (in cost) solutions. Case 1, the high cost solution, was carried only for enough to assure that it would converge (in cost) to the Case 2. Case 2 was carried to completion. This provided a partial check that the optimal trajectory for this problem was unique.

11.2.2 Case 3

In this case the thrust acceleration constraint was introduced. The problem was started with the solution for Case 2 and carried to completion.

11.2.3 Case 4

Case 4 introduced the heating constraint. This case was started from a new nominal which almost satisfied the constraint and was carried to completion.

11.2.4 Case 5

Case 5 is a solution for a 90° plane change with all constraints effective. This solution was started from the results of Case 4.

11.2.5 Case 6

Case 6 is a solution for a 60° plane change with all constraints active. The aerodynamic model was changed to give a L/D_{max} of 2.

11.3 Parameters Common to all Cases

The numerical value of parameters appearing in the state equations which were not varied are discussed below.

11.3.1

R_0 is the reference radius. It is at this distance, corresponding to an altitude of about 300,000 ft, at which the solution of the state equation is started and stopped.

$$R_0 = 2.122 \times 10^7 \text{ ft}$$

11.3.2

R_a is the radius of the initial and final circular orbits.

$$R_a = 1.2 R_0$$

This corresponds to an orbital altitude of about 750 n.mi.

11.3.3

h_s is the scale height used in the exponential atmospheric density model.

$$h_s = 24,000 \text{ ft}$$

11.3.4

B_{max} is the number defining the maximum heating rate in stagnation point temperature. The non-dimensional value used was:

$$B_{max} = 3.34$$

This corresponds to a stagnation point temperature of 4690°F or a heating rate of 250 BTU/ft^{2}sec.

11.3.5

A_{max} is the number defining the maximum thrust acceleration.

$$A_{max} = 3.0 \text{ g's}$$

11.3.6

Q_0 is a number combining several parameters. In equation (3.5.3.11) Q_0 was defined as:

$$Q_0 = \frac{\frac{1}{2} \rho_e V_0^2 e^{-h_0/h_s}}{W_L}$$

The symbols not mentioned previously in this chapter are:

(a) ρ_e, the atmospheric density at the earth's surface

$$\rho_e = 2.378 \times 10^{-3} \text{ slugs/ft}^3$$

(b) V_0, the circular velocity at radius R_0

$$V_0 = 25,750 \text{ ft/sec}$$

(c) M_0G_e, the initial vehicle weight at the start of the state equation solution

(d) S, the effective surface area of the vehicle.

The wing loading, W_L, was taken to be:

$$W_L = \frac{M_0G_e}{S} = 40 \frac{\text{lbs}}{\text{ft}^2}$$

resulting in a value of Q_0 of:

$$Q_0 = 0.0737$$

11.3.7

A final parameter is the drag coefficient at zero lift, C_{d_0}. This number is changed to define L/D_{max}. The two values used were:

$$\text{For } L/D_{max} = 4, \; C_{d_0} = 3.79 \times 10^{-3}$$

$$\text{For } L/D_{max} = 2, \; C_{d_0} = 2.708 \times 10^{-2}$$

For readers accustomed to thinking in terms of ballistic coefficient, $B = W/C_dA$, the vehicle defined has a minimum B of about 10,000 for

L/D_{max} = 4 and about 1500 for L/D_{max} = 2.

11.4 Cases 1 and 2.

These are unconstrained trajectories for a 60° Plane Change with L/D_{max} = 4. The pertinent numerical data for the initial solutions is:

	J	i	GRAD
Case 1	0.373078	60°01'	423.2
Case 2	0.354377	59°46'	1.02

Case 1 was run for 60 iterations with the results at that time being:

	J	i	GRAD
Case 1	0.3582074	60°57'	1.06

At this time, being assured that Case 1 was converging toward Case 2, it was abandoned and Case 2 was initiated. Case 2 was run for 50 iterations with the following values at termination:

	J	i	GRAD
Case 2	0.347440	59°57'	0.455

This value of cost is within 4% of the limiting case cost of 0.3347. (Ref. chapter 7 for development of limiting case.) The Case 2 cost, when compared to the ΔV required for a single impulse rocket plane change at radius R_a, represents a reduction by a factor of 2.63. When one considers the reductions in wing loading, it is evident that three 60° synergetic plane changes can be accomplished for the cost of a single-impulse rocket maneuver.

In terms of fuel required expressed as a percentage of the original weight for a I_{sp} of 400, we have the following situation:

Single-Impulse Rocket	83.8%
Case 2 Result	50.1%
Limiting Case Synergetic Maneuver	47.8%

Thus it can be seen that an optimal unconstrained synergetic maneuver is certainly much more attractive than the Rocket-only maneuver.

11.4.1

Some of the features of the final Case 2 trajectory which may be of interest are:

a. Time for the synergetic maneuver 17.4 min.

b. Minimum Altitude 129,000 feet

c. Maximum Non-Dimensional Dynamic Pressure $Q = 69.5$

d. Coordinates of the Final Point, Latitude = 34.8° Longitude = 56.6°

e. Maximum Lift Force, $QC_L/(1 - x_7) = 2.68$ g's

f. Minimum Velocity - 80% of Orbital

g. Maximum Stagnation Point Temperature - 9,366°F

11.5 Case 3

Case 3 introduced the 3 g thrust acceleration constraint which was violated by the Case 2 trajectory. This case was stated with a Case 2 solution which was near the final one. In discussing costs for this case, as well as for all subsequent ones, we will present two cost numbers, J and J1. J is the actual cost of the trajectory in terms of ΔV and J1 is the cost including the value of the penalty functions added.

11.5.1

This case is of interest because problems of this type may tend toward the impulsive thrust case in the absence of constraints. The reader may recall that in chapter 7 we showed that this problem could be an exception to the general trend. In Case 2 the nominal used a near-impulse of about 10 g's in the U_3 (Horizontal Thrust) control at the termination of the atmospheric turn in order to gain enough velocity to climb back to 300,000 feet. This did not show any tendency to grow during the course of the problem. However, this behavior is not conclusive since this near-impulse was of 0.2 min. duration and the nature of the gradient solution on a computer is such that short duration events are more difficult to change than longer ones.

11.5.2

The nominal for this case had the following characteristics:

J	J1	i	GRAD
0.347532	0.876282	60°	4.8

Twenty iterations were made at which time we had:

J	J1	i	GRAD
0.34861	0.865248	60°05'	4.7

This was enough to show that the problem was converging. However, based on the rate of convergence, it appeared that at least 1,000 iterations would be required to satisfy the constraint!! In view of this, a different approach was taken. The control U_3 was modified to a value which did not violate the constraint (3 g's) and the problem was restarted. If the constraint has an influence on the problem, subsequent iterations should alter the controls in such a manner that eventually the constraint will be violated. After the modification the situation was:

J	J1	i	GRAD
0.349536	0.349536	60°01'	1.518

Next, 70 more iterations were performed. There was no tendency for the thrust acceleration to increase markedly, and the constraint was never violated. In addition, during the progress of this problem, some of the techniques described in chapter 10 for improving convergence were implemented. As a result, we were able to produce a better trajectory than the final result of Case 2!! The final result was:

J	J1	i	GRAD
0.343756	0.343756	60°	0.356

This represents a 1% improvement over the Case 2 results, and is within 2% of the limiting case solution. Thus, we conclude that a thrust acceleration constraint of 3.0 g's or more has no effect on this problem.

11.5.3

The features of the final trajectory include:

a.	Time for the Synergetic Maneuver	16.6 min.
b.	Minimum Altitude	217,700 feet
c.	Maximum Dynamic Pressure	73.9 g's
d.	Coordinates of the Final Point, Latitude 31.62°, Longitude 53.96°	
e.	Maximum Lift Force	2.8 g's
f.	Maximum Thrust Acceleration	2.65 g's
g.	Minimum Velocity	80% of Orbital
h.	Maximum Stagnation Point Temperature 9,371°F	

11.5.4

Before presenting any graphic portrayals of the trajectories, let ust devote a little time at this point to discussion of features common to all of the solutions which will be prominent in the figures.

11.5.4.1

The reader will notice a considerable amount of "wildness" in the control functions, particularly in the thrust controls. This occurs because gradient problems typically converge much faster in cost than in the controls. This situation gets worse as one approaches the optimum cost since, by definition, the optimum cost is insensitive to variations in the controls. In simple problems with a scalar control the cost function can frequently be smoothed by use of the weighting function. In a multi-dimensional problem, as we have pointed out in chapter 10, use of the weighting function is a tricky proposition. For this reason no attempt has been made to smooth out the control functions in these problems.

11.5.4.2

A second feature common to all the problems is the qualitative behavior of two of the three thrust controls, U_4 and U_5. While these

controls vary by as much as three orders of magnitude over a trajectory, they are generally small when compared to the aerodynamic accelerations in the same directions. Consequently, one concludes that these controls could well be omitted from the problem. While mentioning the wide variations in thrust, we apologize for presenting the thrust controls in tabular form. Unfortunately, a convenient method of graphing a function which can be of either sign and varies by as much as three orders of magnitude is not readily apparent at this time.

11.5.5

Figure 11.1 shows a time history of altitude, horizontal velocity, stagnation temperature and inclination change for Case 3. Note that the trajectory is a rapid descent to a rather low altitude, where most of the inclination change takes place in a relatively short time (7 minutes). This behavior approaches that predicted for the limiting case solution, as one would expect. Figure 11.2 shows a ground trace of the trajectory. Note here that most of the turning takes place close to the initial orbital plane, again predicted from theory developed in chapter 7. Figure 11.3 shows a time history of angle of attack (U_1) and bank angle (U_2). Note that the angle of attack averages about 9.5° giving a L/D of about 3.9. Figure 11.4 shows, in tabular form, the thrust history. One must conclude that U_4 is mostly "noise" generated due to errors in computation. Note that U_3 helps counteract drag slightly, but not significantly since drag averages 0.5 g's on this trajectory. U_5 is in general small, but in a direction to assist the plane change. Figure 11.4 also gives the initial conditions $x_1(0)$ and $x_2(0)$ solved for. There was no significant change (less than 4%) in the part of the cost depending on the initial boundary values and the cost contribution due to the initial boundary values is about 13% of the total.

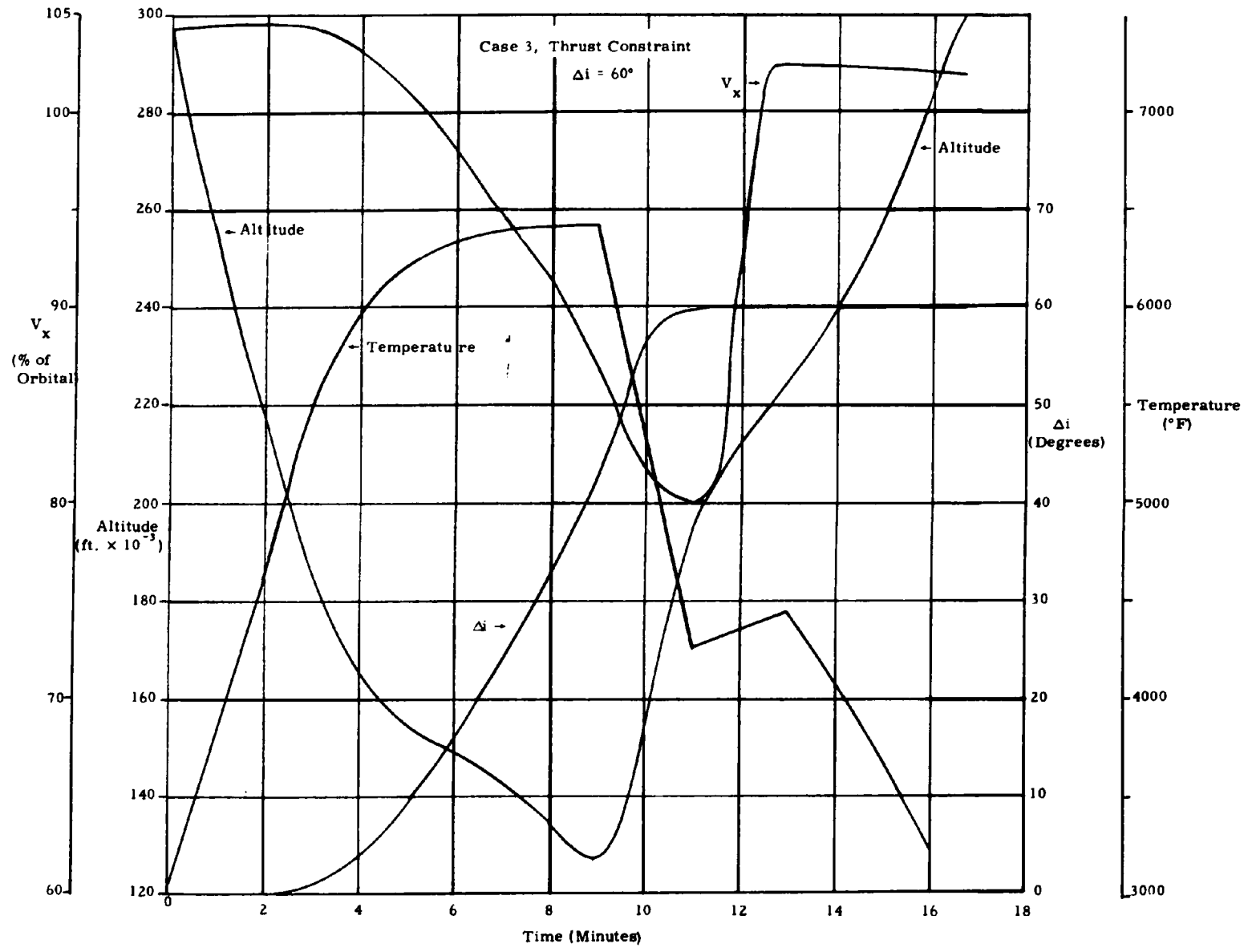

Case 3, Thrust Constraint
Δi = 60°
V_x →
← Altitude
← Altitude
← Temperature
Δi →
V_x
(% of Orbital)
Altitude
(ft. × 10^{-3})
Δi
(Degrees)
Temperature
(°F)
Time (Minutes)
105
100
90
80
70
60
300
280
260
240
220
200
180
160
140
120
70
60
50
40
30
20
10
0
7000
6000
5000
4000
3000
0
2
4
6
8
10
12
14
16
18

Figure 11.1

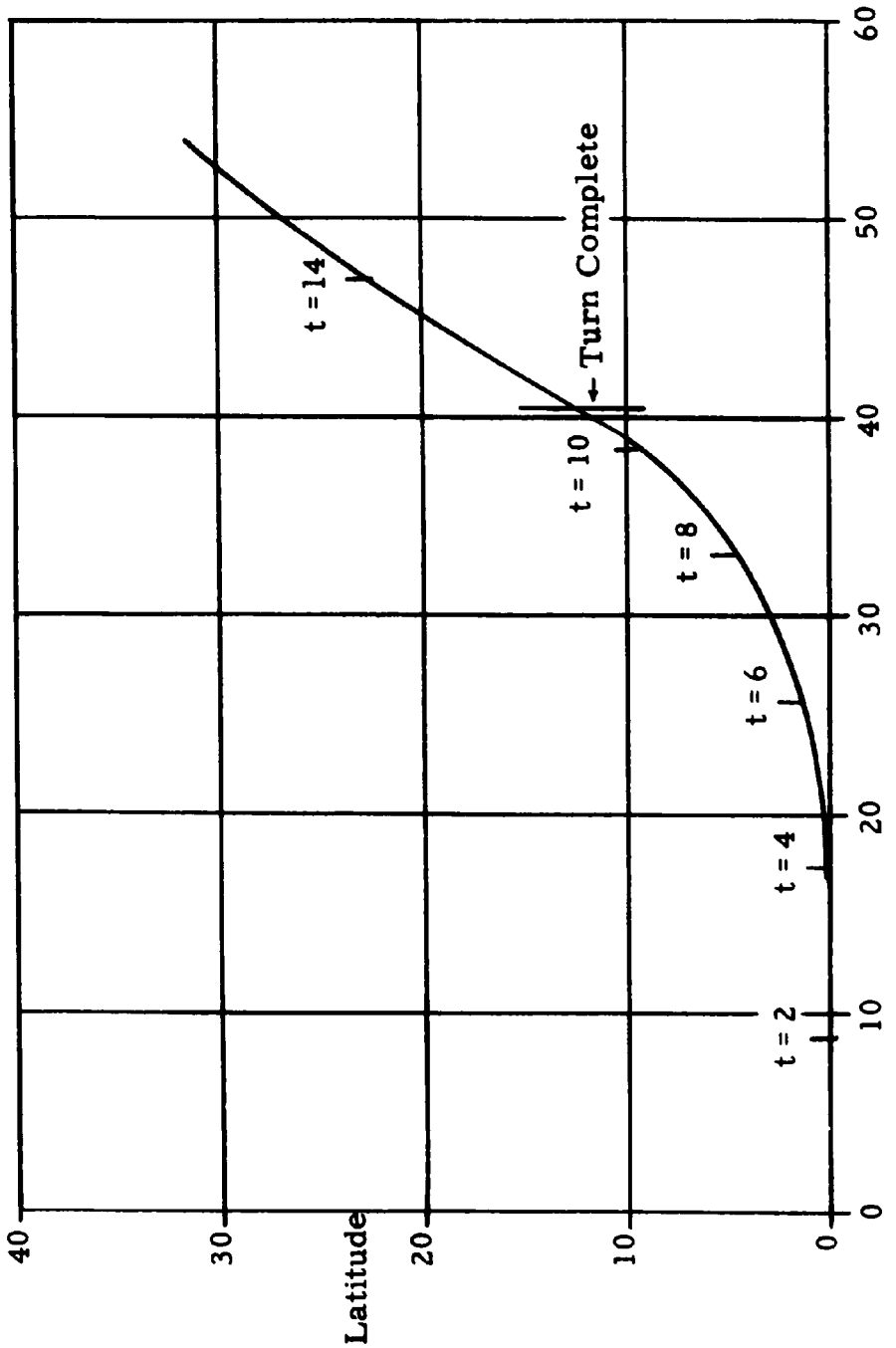

Case 3, Thrust Acceleration
Constraint, Δi = 60°
Latitude
Longitude
0
10
20
30
40
50
60
t = 2
t = 4
t = 6
t = 8
t = 10
t = 14
Turn Complete

Figure 11.2

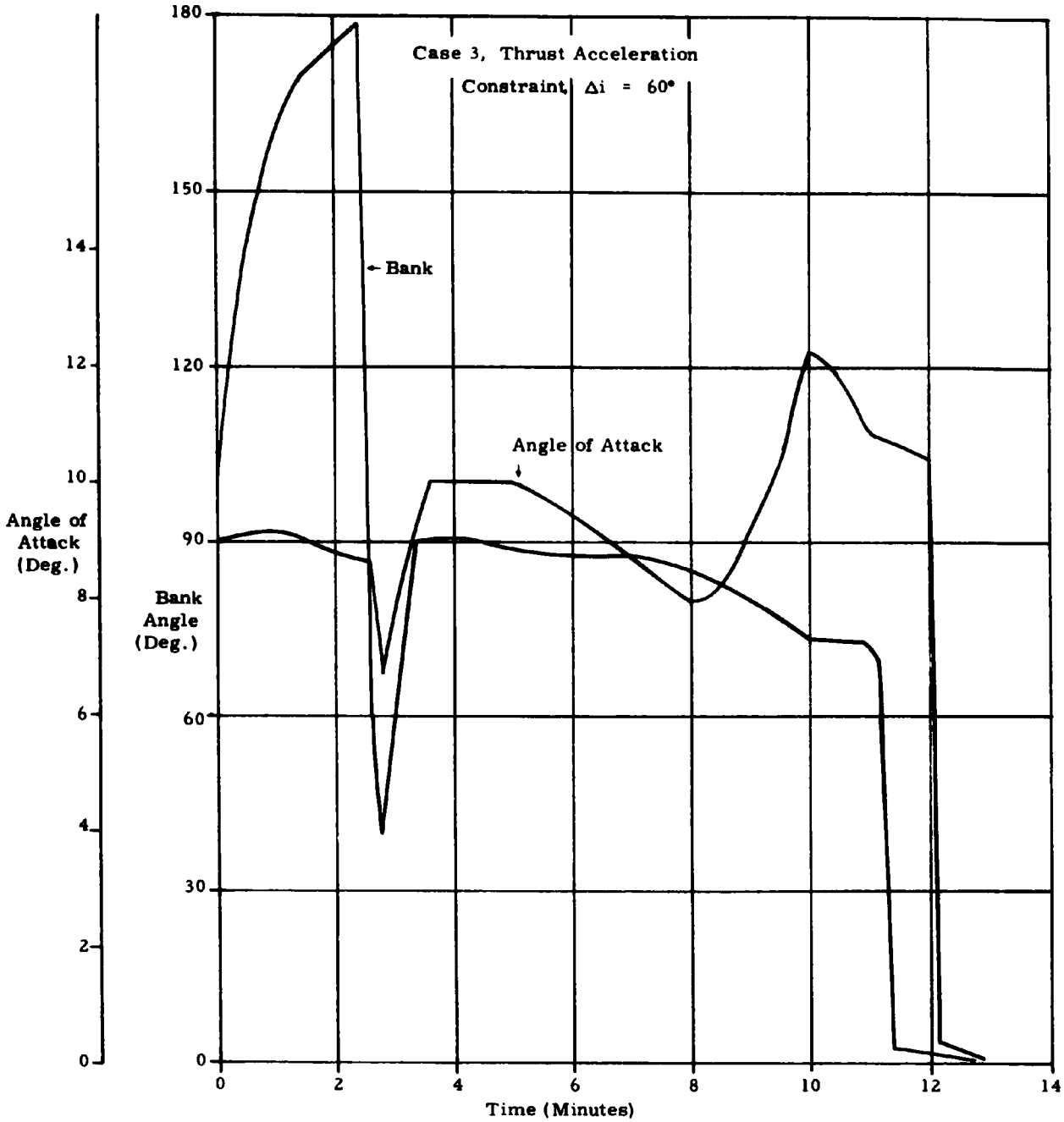

Case 3, Thrust Acceleration
Constraint, Δi = 60°
← Bank
Angle of Attack
Angle of Attack (Deg.)
Bank Angle (Deg.)
180
150
120
90
60
30
0
14
12
10
8
6
4
2
0
0
2
4
6
8
10
12
14
Time (Minutes)

Figure 11.3

Case 3, Thrust Control

Time	U_3	U_4	U_5
0.0	7.43×10^{-5}	8.04×10^{-6}	-3.93×10^{-5}
1.0	1.22×10^{-4}	-1.13×10^{-5}	-3.89×10^{-5}
2.0	1.46×10^{-4}	-3.02×10^{-5}	4.47×10^{-5}
3.0	6.39×10^{-5}	-7.83×10^{-6}	-2.47×10^{-5}
4.0	4.09×10^{-5}	-5.26×10^{-5}	-6.83×10^{-5}
5.0	1.14×10^{-4}	-1.14×10^{-5}	-2.99×10^{-5}
6.0	7.05×10^{-5}	8.82×10^{-6}	-3.39×10^{-5}
7.0	5.48×10^{-4}	-1.33×10^{-4}	-1.40×10^{-4}
8.0	2.32×10^{-3}	-5.91×10^{-4}	-5.69×10^{-4}
9.0	4.49×10^{-3}	3.33×10^{-5}	-1.06×10^{-3}
10.0	7.04×10^{-3}	2.43×10^{-3}	-1.72×10^{-3}
11.0	6.22×10^{-3}	1.58×10^{-3}	-1.24×10^{-3}
12.0	2.00×10^{-0}	4.57×10^{-3}	-9.77×10^{-3}
13.0	1.35×10^{-3}	1.13×10^{-4}	-3.00×10^{-4}
14.0	1.00×10^{-3}	3.43×10^{-5}	-1.94×10^{-4}
15.0	9.61×10^{-4}	1.52×10^{-5}	-1.78×10^{-4}
16.0	9.58×10^{-4}	1.02×10^{-5}	-1.70×10^{-4}

Initial Conditions

$x_1(0) = 0.0445387$ $\quad$ $x_2(0) = -0.0321334$

Figure 11.4

11.6 Case 4

In Case 4 we introduce a heating constraint that will limit the maximum stagnation point temperature to about 4690°F, about 50% of that encountered in the unconstrained problem. This problem was started with a nominal trajectory that had the following characteristics:

J	J1	i	GRAD
0.646833	0.64964	22°	10.85

After twelve iterations, including considerable manual editing of the U_3 control component we had:

J	J1	i	GRAD
0.500837	0.500841	57°	5.28

This illustrates the changes possible in boundary values.

At the end of 68 iterations the problem had converged to the following final result:

J	J1	i	GRAD
0.450674	0.450710	60°03'	0.678

Note that as explained earlier there is some violation of the heating constraint, which can never be reduced to zero. For this trajectory the maximum stagnation point temperature was 4699°F, a 9° violation.

This trajectory is within 35% of the limiting case solution and within 31% of the best unconstrained solution. One concludes that at this heating level one could get two 60° synergetic plane changes for the cost of one rocket maneuver.

In terms of fuel used, the cost is 59% of the original mass.

11.6.1

The principal features of this trajectory include:

a.	Time for the maneuver	34 minutes
b.	Minimum altitude	161,000 feet
c.	Maximum Dynamic Pressure	12.2 g's
d.	Maximum Lift Force	1.4 g's
e.	Coordinates of the Final Point Latitude 48.5°, Longitude 123.0°	
f.	Minimum Velocity	70% of Orbital

11.6.2

Figure 11.5 shows a time history of altitude, velocity and inclination change for Case 4. As one would expect with the constraint active, the inclination change is spread over a longer time (24 minutes), and the trajectory does not dip as deeply into the atmosphere. In fact, the maximum density reached is less by a factor of 4 than the Case 3 trajectory. Figure 11.6 shows the ground trace of the maneuver. Again, most of the turn is made relatively close to the equator. However, this trajectory goes 1/3 of the way around the earth, as contrasted to Case 3 which only covered 55° of longitude. Figure 11.7 gives the time history of U_1 (Angle of Attack) and U_2 (Bank). Note that this trajectory is flown at an average angle of attack of about 15°, corresponding to an L/D of 3.2. This again shows agreement with our qualitative conclusions in chapter 8 that maneuvering at L/D_{max} is not optimal in the presence of constraints. Note also the high angle of attack in the first minute of the trajectory. This phenomenon will be much more pronounced in Case 6 ($L/D_{max} = 2$).

The thrust control history is tabulated in Figure 11.8 along with the initial conditions. There are no new features observed in these controls that have not been discussed under Case 3.

11.7 Case 5

This case requires a 90° plane change while satisfying all constraints. This problem was started with a trajectory near the final Case 4 trajectory:

J	J1	i	GRAD
0.445595	0.445676	59°	1.22×10^4

The Kelley algorithm was used for this case for the first 19 iterations. At this time the costs were:

J	J1	i	GRAD
0.559621	0.704906	86°	84.0

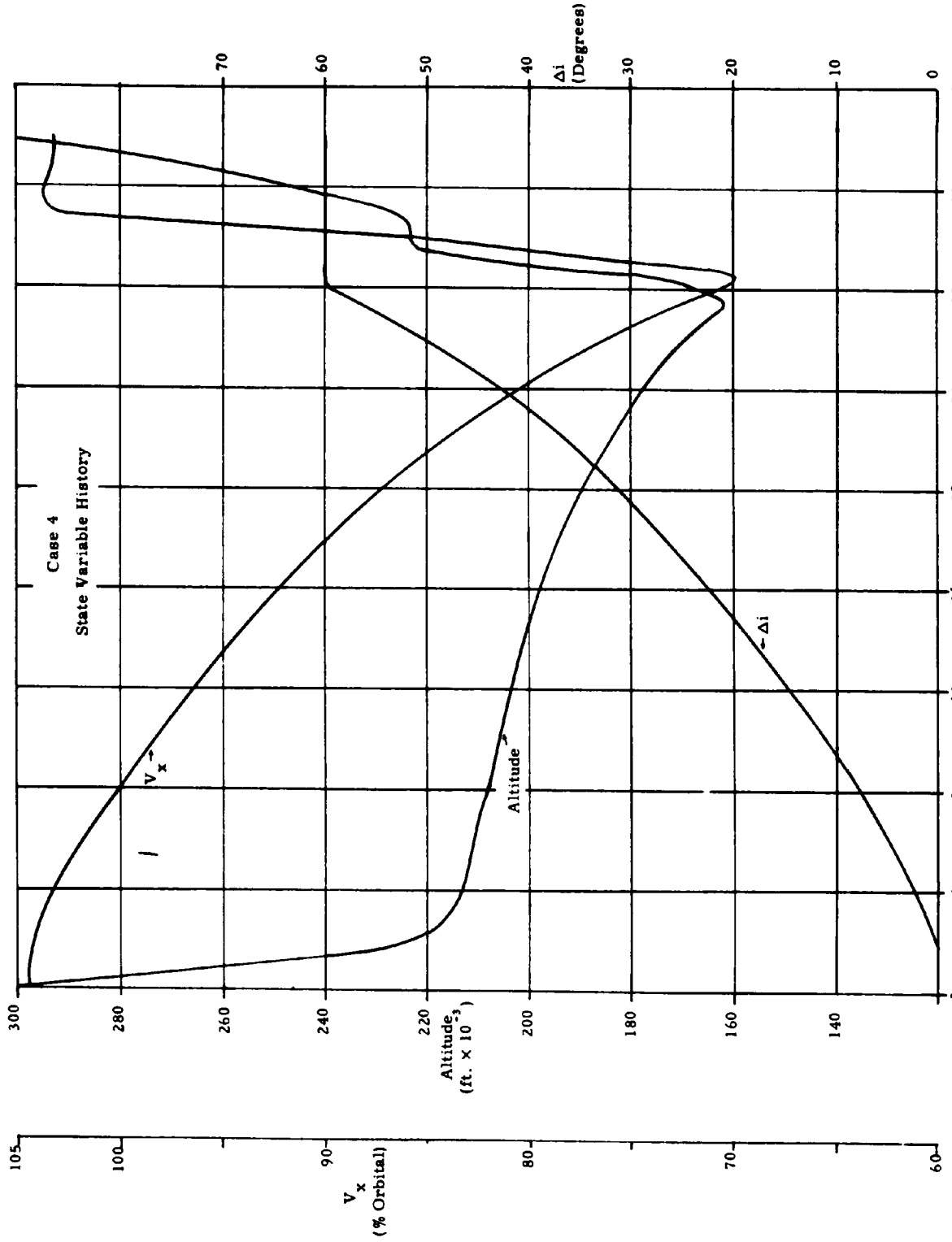
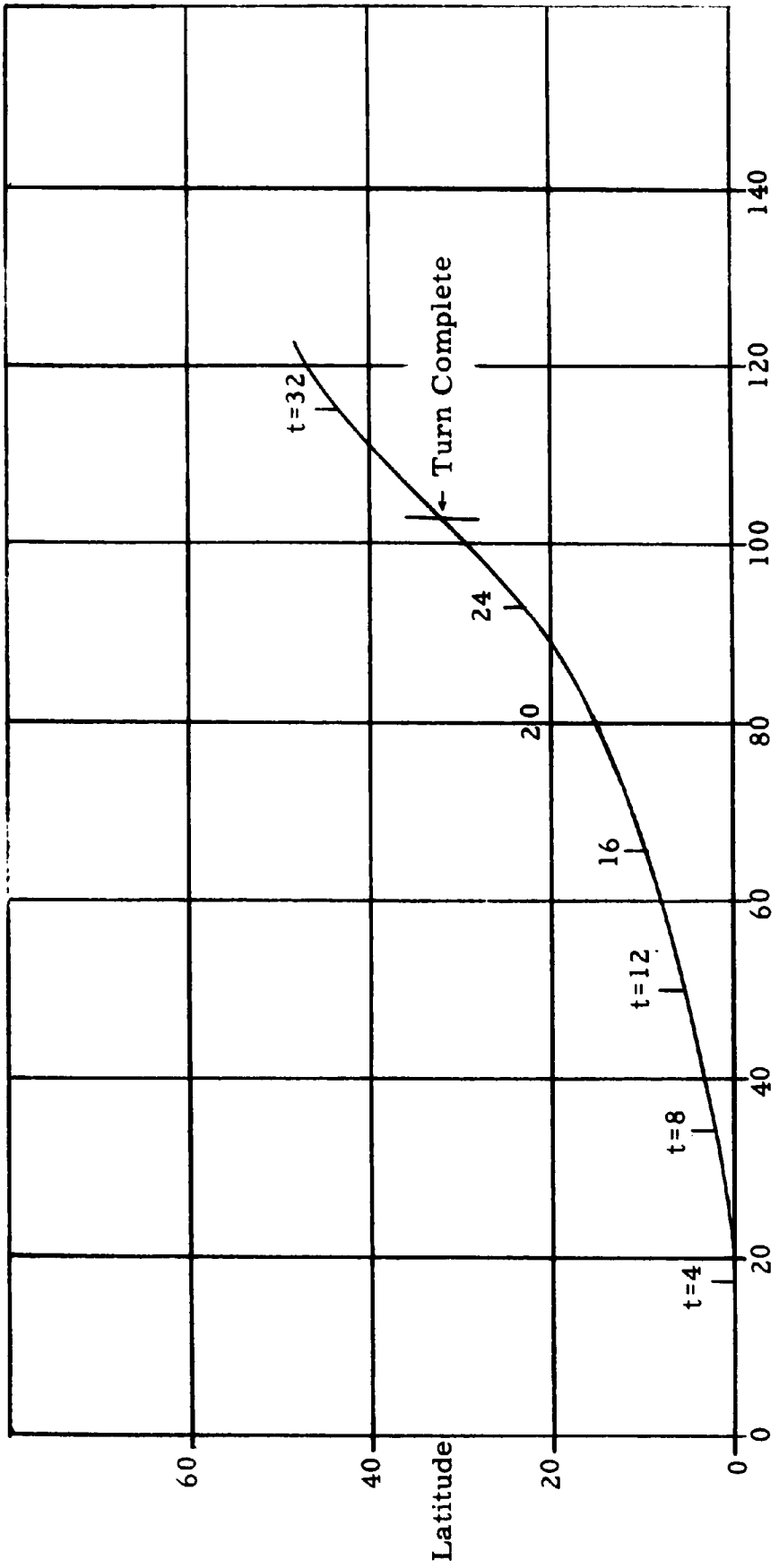

Case 4
State Variable History
V_x
Altitude
Δi
V_x (% Orbital)
Altitude (ft. x 10^{-3})
Δi (Degrees)
Time (Minutes)

Figure 11.5



Ground Trace For Case 4 Trajectory

Figure 11.6

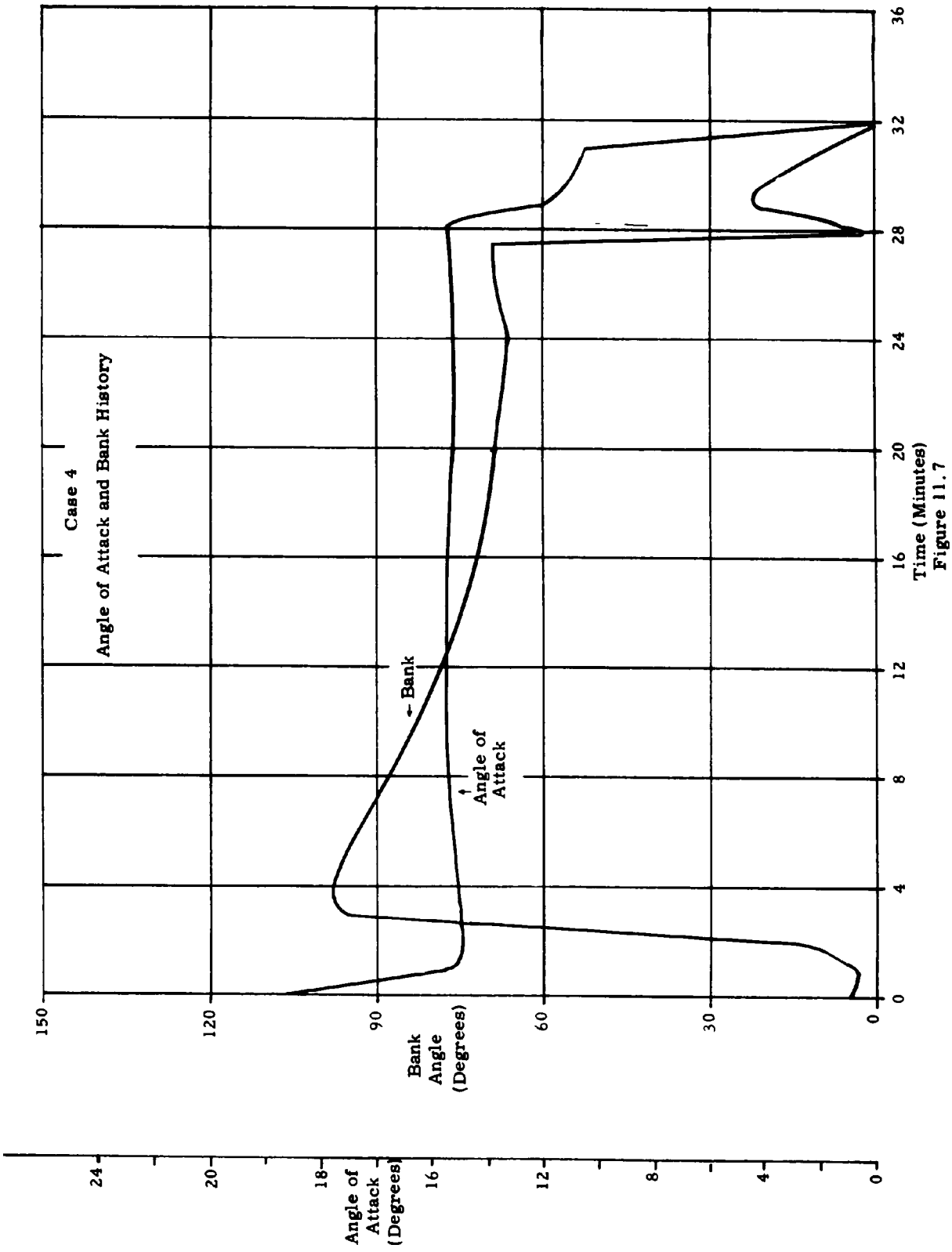

Case 4
Angle of Attack and Bank History
Bank
Angle of Attack
Bank Angle (Degrees)
Angle of Attack (Degrees)
Time (Minutes)
150
120
90
60
30
0
24
20
18
16
12
8
4
0
0
4
8
12
16
20
24
28
32
36

Figure 11.7

Case 4, Thrust Control

Time	U_3	U_4	U_5
0.0	3.33×10^{-5}	4.09×10^{-5}	-5.98×10^{-5}
2.0	2.58×10^{-4}	1.29×10^{-4}	-5.79×10^{-5}
4.0	4.64×10^{-4}	-7.88×10^{-6}	-7.86×10^{-5}
6.0	1.59×10^{-4}	-8.82×10^{-6}	-5.54×10^{-6}
8.0	-2.88×10^{-5}	4.17×10^{-6}	5.63×10^{-6}
10.0	1.37×10^{-4}	-4.26×10^{-5}	-2.75×10^{-5}
12.0	3.34×10^{-4}	-8.49×10^{-5}	-1.22×10^{-4}
14.0	4.52×10^{-4}	6.54×10^{-6}	-2.21×10^{-4}
16.0	4.22×10^{-4}	-1.38×10^{-5}	-1.67×10^{-4}
18.0	2.41×10^{-3}	-3.83×10^{-4}	-9.89×10^{-4}
20.0	1.43×10^{-3}	-2.31×10^{-4}	-5.48×10^{-4}
22.0	2.81×10^{-3}	-9.37×10^{-5}	-1.01×10^{-3}
24.0	4.80×10^{-3}	1.44×10^{-4}	-1.64×10^{-3}
26.0	7.09×10^{-3}	-1.93×10^{-4}	-2.23×10^{-3}
28.0	1.22×10^{-2}	3.78×10^{-3}	-3.10×10^{-3}
30.0	1.49×10^{-0}	8.17×10^{-3}	-7.85×10^{-3}
32.0	8.04×10^{-4}	2.16×10^{-5}	-1.38×10^{-4}
34.0	8.38×10^{-4}	1.06×10^{-5}	-1.20×10^{-4}

Initial Conditions

$x_1(0) = 0.0427679$ $\quad$ $x_2(0) = -0.0350433$

Figure 11.8

The iterations were continued using the Bryson algorithm and a total of 122 iterations made.

The final values of cost were:

J	J1	i	GRAD
0.606186	0.612597	90°04'	1.68

The maximum stagnation temperature was 4721°F, a violation of 31°F. This trajectory is within 40% of the limiting case unconstrained cost of 0.4343, and is better than the one-impulse rocket cost by a factor of 2.1. These numbers correspond closely to the corresponding numbers for the 60° plane change.

11.7.1

The principal features of th 90° constrained trajectory are:

a.	Time for the Maneuver	34.5 minutes
b.	Minimum Altitude	145, 400 feet
c.	Maximum Dynamic Pressure	16.94 g's
d.	Maximum Lift Force	1.9 g's
e.	Coordinates of the Final Position Latitude 60.49° Longitude 91.53°	
f.	Minimum Velocity	58% of Orbital

11.7.2

Figures 11.9, 11.10, 11.11, and 11.12 provide graphical information on the Case 5 trajectory. It is qualitatively similar to the comparable information for Case 4 and no further discussion is required.

11.8 Case 6

The final Case solved was for a constrained 60° trajectory with a $L/D_{max} = 2$. For this case, a nominal was chosen which violated the heating constraint severely. The nominal was:

J	J1	i	GRAD
0.522685	6.96196	52°	78.6

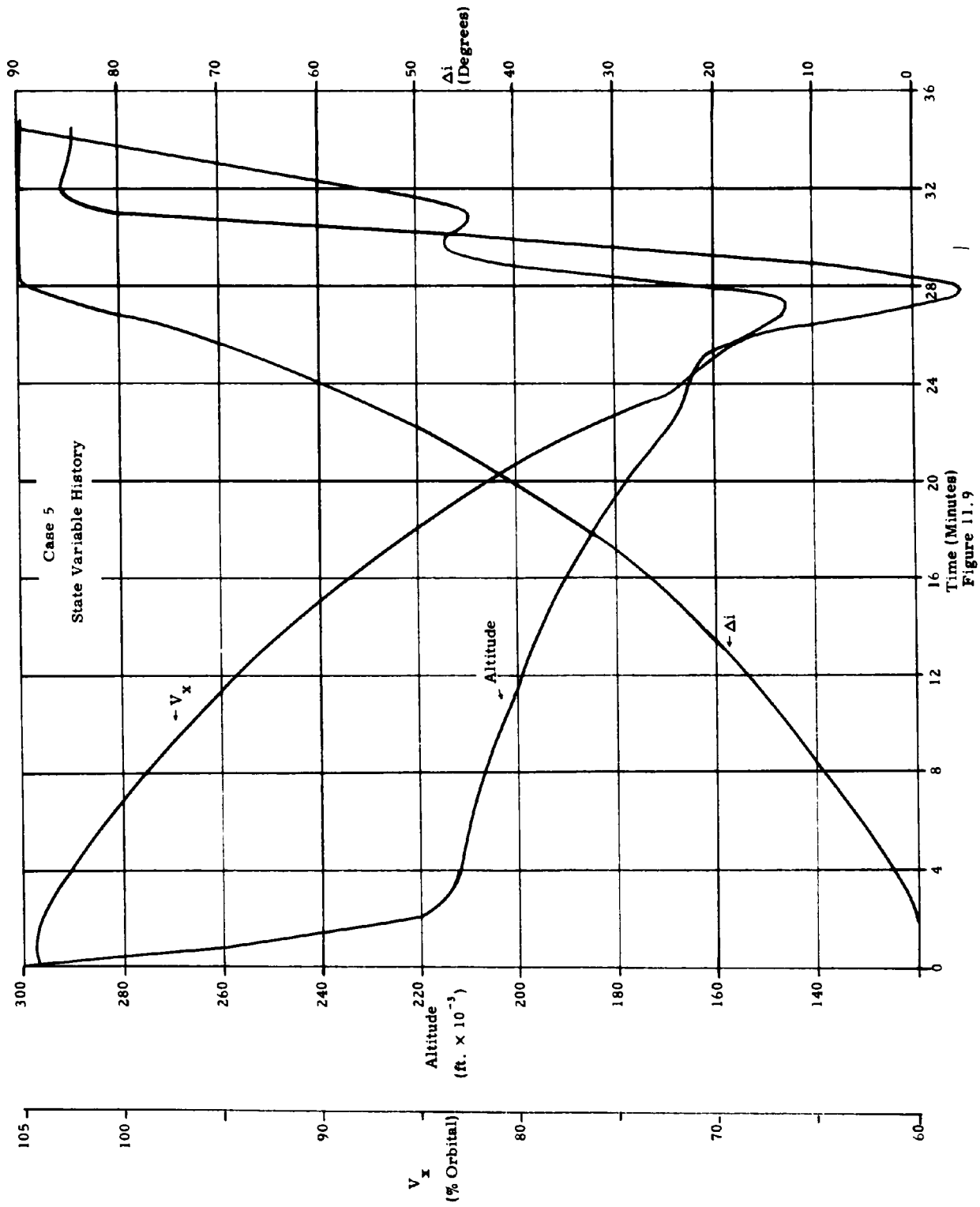
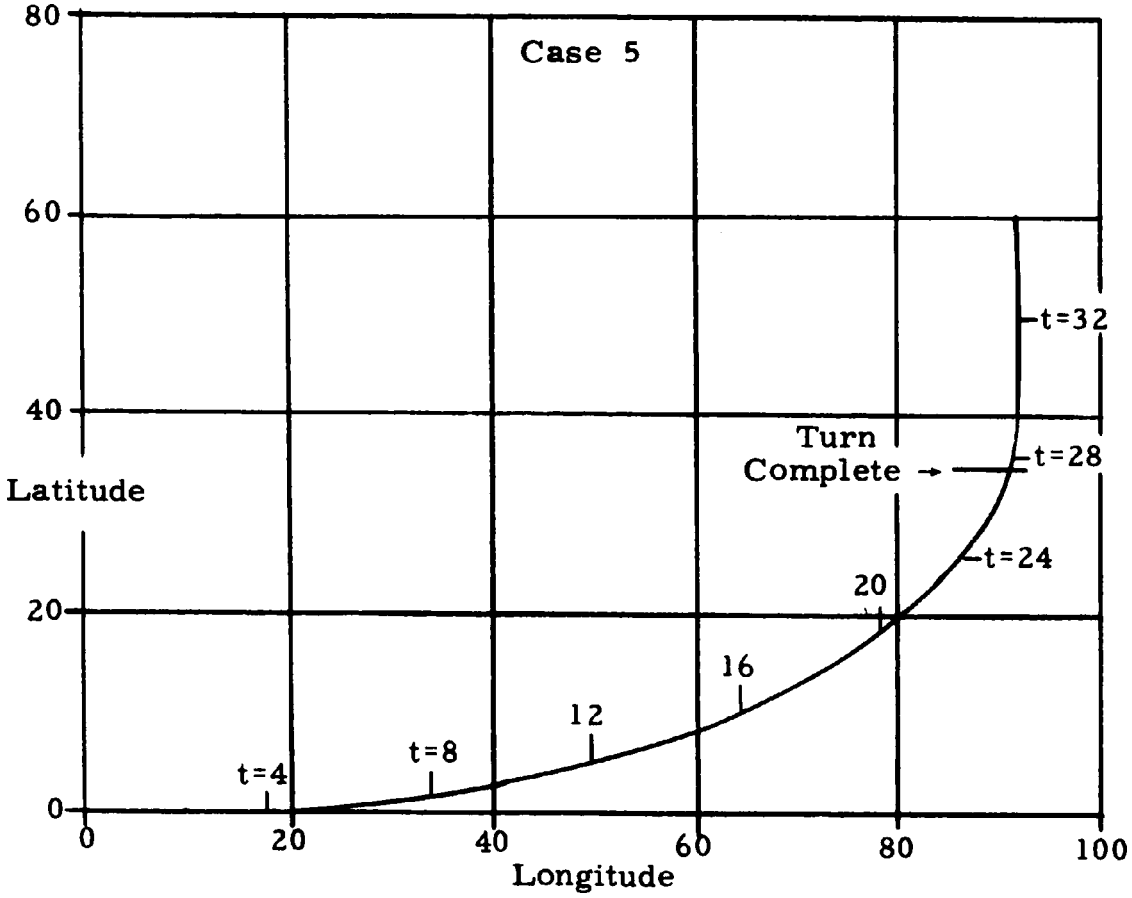

Case 5
State Variable History
$\leftarrow V_x$
Altitude
$\leftarrow \Delta i$
V_x (% Orbital)
Altitude (ft. × 10^{-3})
Δi (Degrees)
Time (Minutes)

Figure 11.9



Ground Trace For Case 5 Trajectory

Figure 11.10

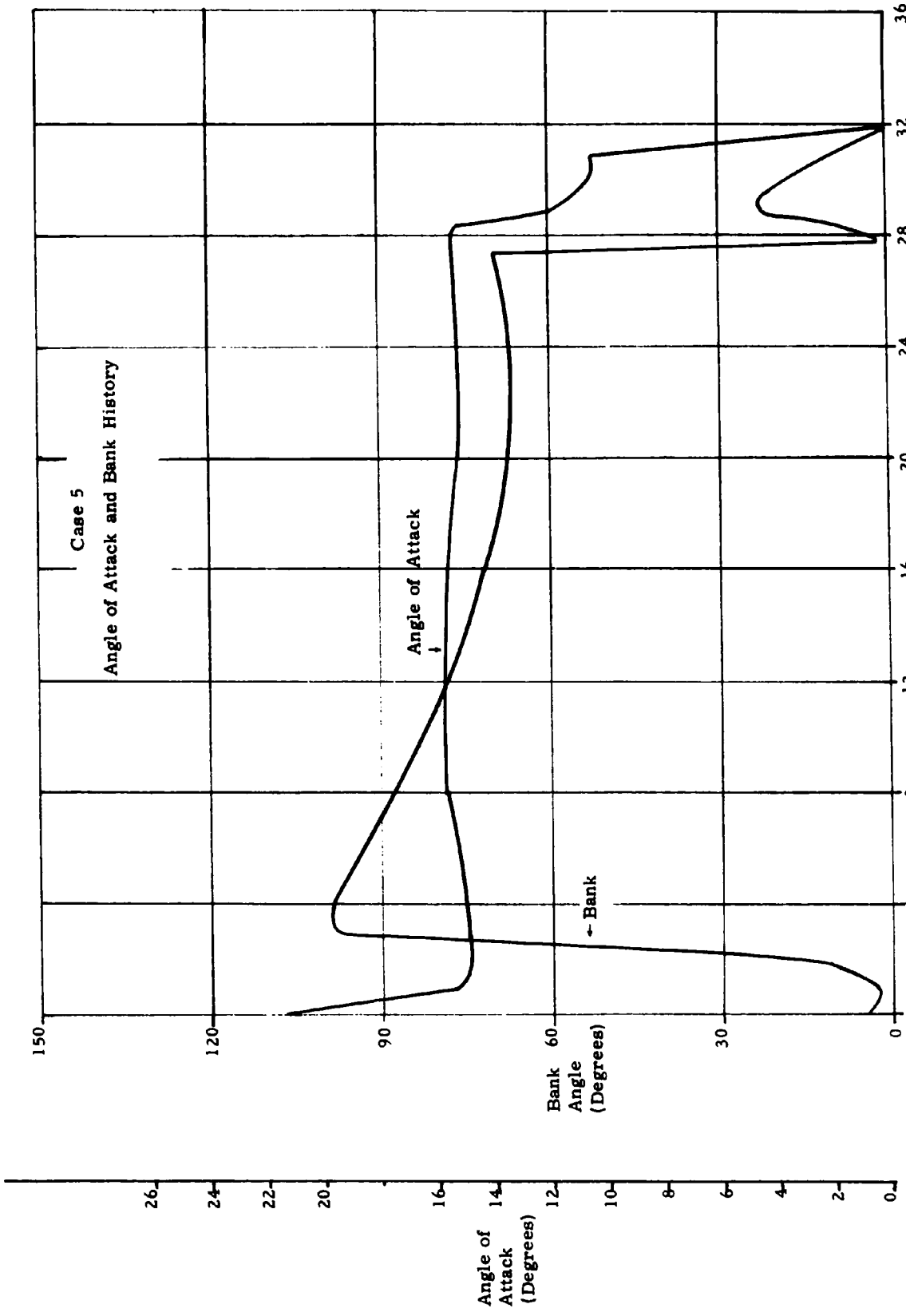

Case 5
Angle of Attack and Bank History
Angle of Attack
← Bank
Bank Angle (Degrees)
Angle of Attack (Degrees)
Time (Minutes)
0
30
60
90
120
150
0
2
4
6
8
10
12
14
16
20
22
24
26
0
4
8
12
16
20
24
28
32
36

Figure 11.11

Case 5, Thrust Control

Time	U_3	U_4	U_5
0.0	-1.69×10^{-2}	-1.75×10^{-2}	3.94×10^{-5}
2.0	-5.47×10^{-3}	5.16×10^{-3}	-2.06×10^{-3}
4.0	-1.24×10^{-2}	8.83×10^{-3}	-1.48×10^{-3}
6.0	-1.47×10^{-2}	1.01×10^{-2}	-2.93×10^{-3}
8.0	-1.56×10^{-2}	1.14×10^{-2}	-3.75×10^{-3}
10.0	-1.34×10^{-2}	1.18×10^{-2}	-4.29×10^{-3}
12.0	-9.30×10^{-3}	1.07×10^{-2}	-4.62×10^{-3}
14.0	-4.36×10^{-3}	8.45×10^{-3}	-4.42×10^{-3}
16.0	-8.12×10^{-4}	4.26×10^{-3}	-2.72×10^{-3}
18.0	-9.11×10^{-4}	1.60×10^{-4}	-3.46×10^{-4}
20.0	-1.01×10^{-4}	1.61×10^{-4}	-2.96×10^{-4}
22.0	3.99×10^{-3}	-4.51×10^{-3}	-2.35×10^{-3}
24.0	4.89×10^{-4}	2.09×10^{-6}	-1.63×10^{-4}
26.0	1.31×10^{-2}	-8.68×10^{-3}	-4.89×10^{-3}
28.0	1.97×10^{-2}	4.48×10^{-3}	-6.42×10^{-3}
30.0	1.49×10^{-0}	9.84×10^{-3}	-1.06×10^{-2}
32.0	1.23×10^{-3}	5.88×10^{-5}	-2.21×10^{-4}
34.0	1.09×10^{-3}	1.32×10^{-5}	-1.67×10^{-4}

Initial Conditions

$x_1(0) = 0.0427507$ $\quad$ $x_2(0) = -0.0350623$

Figure 11.12

Because of the severity of the constraint violation, an unusually large number of iterations was required for this case. After a total of 195 iterations our final result was

J	J1	i	GRAD
0.690540	0.697640	59°58'	1.04

This cost is within 33% of the limiting case unconstrained solution for L/D = 2, (0.5219). This corresponds closely with the results for Case 4. The maximum temperature encountered was 4751°, a violation of 61° which is more serious than the previous constrained case, but not a source for concern since it could be worked down with a few more iterations. Two interesting features of this case are the short time required for the maneuver and the relatively high aerodynamic forces generated. These two features are not surprising when one considers that to operate at L/D_{max} for this case requires values of C_L about 4.5 times as large as those required for the $L/D_{max} = 4$ cases. Thus, other things being equal, forces are larger and times for changes to occur become shorter.

11.8.1

Principal features for this case include

a.	Time Required for the Maneuver	8.5 minutes
b.	Minimum Altitude	135,000 feet
c.	Maximum Dynamic Pressure	Q = 24.6 g's
d.	Maximum Lift Force	4.64 g's
e.	Coordinates of the Final Position Latitude 10.9°, Longitude 23.8°	
f.	Minimum Velocity	58% of Orbital

Another interesting feature of the final trajectory for this case is that the final velocity at 300,000 feet is about 80% of orbital. All other cases arrived at 300,000 feet with slightly more than orbital velocity.

11.8.2

Figure 11.13 shows the time history of altitude, velocity and inclination change for Case 6. This trajectory is qualitatively similar to Case 4, with the major difference being a time compression of the entire history by a factor of 4. Figure 11.14 gives the ground trace, which is also subject to a space compression by a factor of about 5. Figure 11.15 shows the time history of bank and angle of attack. Note the very high angle of attack attained for the first minute of the trajectory. At its peak (71°) this angle corresponds to a L/D of 0.23!!! During the stabilized portion of the flight, the vehicle is operating at a L/D of 1.98. A phenomenon which may be related to the initial high angle of attack was the behavior of the U_3 thrust component. In section 11.7 we mentioned that the original trajectory for this case violated the heating constraint severely. As a reaction to this, a considerable history of negative (in the same direction as the drag) U_3 thrust was built up. This reached magnitudes of 0.3 g's which was nearly 1/3 of the drag force at times. This behavior is evidently the gradient method's way of trying to slow the vehicle down and thus lessen the temperature. Figure 11.16 shows that this effect has been largely cancelled out by the final series of iterations. Further iterations would probably cancel it out completely. Figure 11.16 also shows rather substantial amounts of side thrust (U_5) (in a direction to assist in turning) when compared to the other cases. However, this case also exhibited substantial aerodynamic force levels, and the ratio of side thrust to the corresponding lift component is still small.

11.9 Analysis and Summary

Let us review the data we have obtained:

For (L/D) max = 4, we have

A.	60°, No Constraint	J = 0.343756
B.	60°, Constrained	J = .450674
C.	90°, Constrained	J = 0.606186

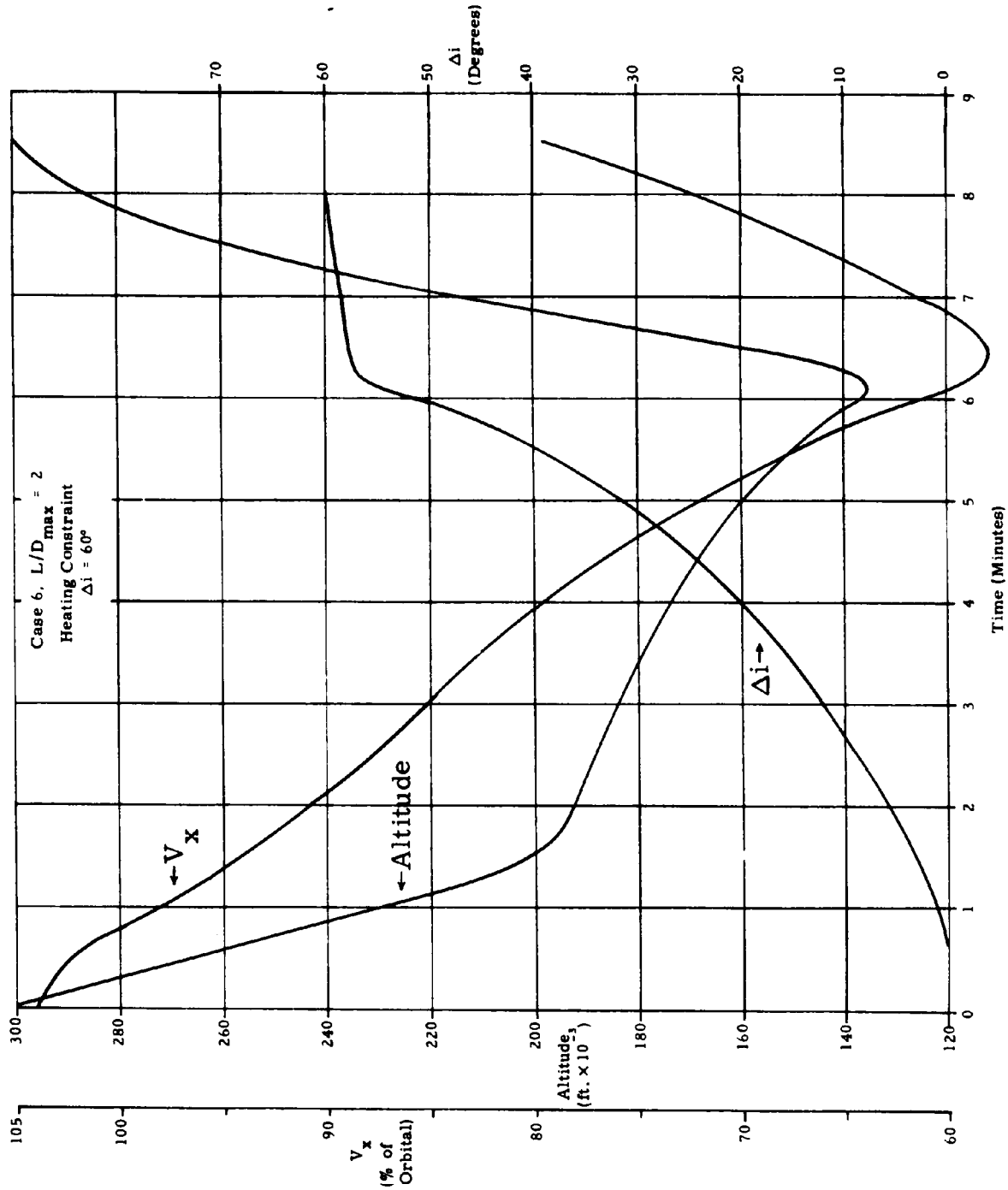
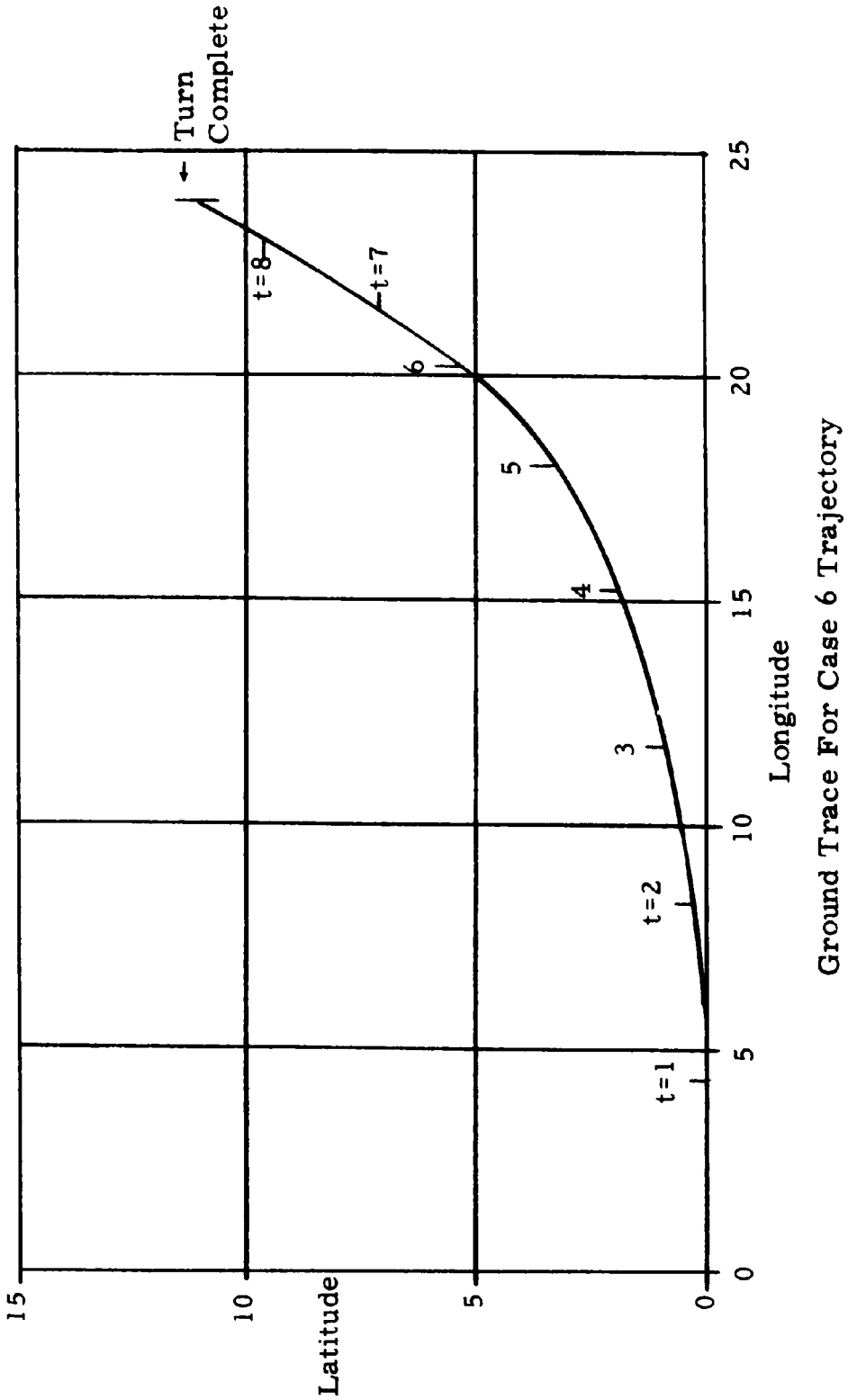

Case 6, L/D max = 2
Heating Constraint
Δi = 60°
←Vx
←Altitude
Δi→
Vx (% of Orbital)
Altitude (ft. × 10^3)
Δi (Degrees)
Time (Minutes)

Figure 11.13



Ground Trace For Case 6 Trajectory

Figure 11.14

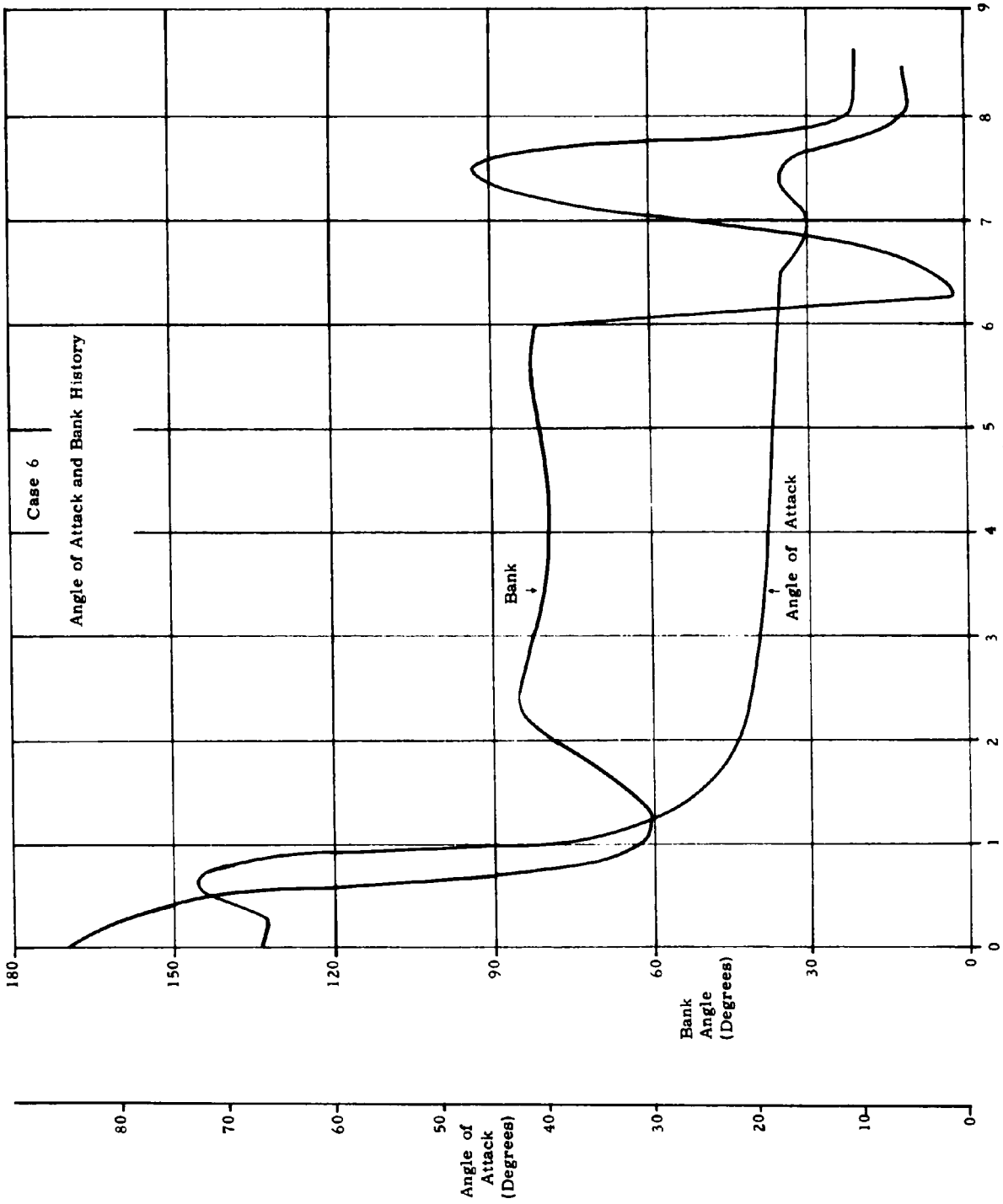

Case 6
Angle of Attack and Bank History
Bank
Angle of Attack
Bank Angle (Degrees)
Angle of Attack (Degrees)
Time (Minutes)
0
30
60
90
120
150
180
0
10
20
30
40
50
60
70
80
0
1
2
3
4
5
6
7
8
9

Figure 11.15

Case 6, Thrust Controls

Time	U_3	U_4	U_5
0.0	-3.95×10^{-1}	-5.47×10^{-1}	-7.97×10^{-2}
0.5	-2.77×10^{-1}	-1.46×10^{-1}	-7.50×10^{-2}
1.0	-2.75×10^{-1}	1.26×10^{-1}	-7.62×10^{-2}
1.5	-3.08×10^{-1}	2.37×10^{-1}	-7.69×10^{-2}
2.0	-2.02×10^{-1}	1.27×10^{-1}	-7.25×10^{-2}
2.5	-4.78×10^{-2}	5.59×10^{-2}	-6.13×10^{-2}
3.0	-1.52×10^{-2}	6.07×10^{-2}	-5.42×10^{-2}
3.5	3.65×10^{-4}	5.71×10^{-2}	-4.87×10^{-2}
4.0	1.71×10^{-2}	3.59×10^{-2}	-3.74×10^{-2}
4.5	2.21×10^{-2}	2.78×10^{-2}	-2.97×10^{-2}
5.0	1.35×10^{-2}	1.34×10^{-2}	-1.24×10^{-2}
5.5	1.54×10^{-2}	1.16×10^{-2}	-9.87×10^{-3}
6.0	5.05×10^{-2}	2.66×10^{-2}	-2.68×10^{-2}
6.5	9.63×10^{-1}	8.04×10^{-3}	-1.21×10^{-1}
7.0	9.95×10^{-1}	-6.21×10^{-3}	-1.04×10^{-1}
7.5	9.99×10^{-1}	-3.70×10^{-3}	-8.72×10^{-2}
8.0	9.99×10^{-1}	-1.69×10^{-3}	-4.55×10^{-2}
8.5	9.99×10^{-1}	-9.36×10^{-4}	4.37×10^{-2}

Initial Conditions

$x_1(0) = 0.042245$ $\quad$ $x_2(0) = -0.035037$

Figure 11.16

For (L/D) max = 2, we have

D. 60°, Constrained $J = 0.690540$

This appears to be a small amount of data from which to draw any conclusions. Nevertheless, with the aid of the limiting case solution developed in chapter 7, it turns out one can get a lot of information from this data without excessive loss of creditability.

11.9.1 Extension of Computer Solutions

In chapter 7 the following formula for the limiting case cost was developed:

$$J_{Lim} = 2I_1 + V_0\left(1 - e^{-\frac{\Delta i}{L/D}}\right) \qquad (11.9.1.1)$$

where

$$I_1 = \frac{1}{\sqrt{R_a}} - \sqrt{\frac{2}{R_a(R_a - 1)}} \qquad (11.9.1.2)$$

$$V_0 = \sqrt{\frac{2R_a}{(R_a + 1)}} \qquad (11.9.1.3)$$

Using (11.9.1.1) as a basis, let us hypothesize an approximation to the actual constrained solution of the form:

$$J_c \approx 2I_1 + K_s V_0\left(1 - e^{-\frac{\Delta i}{L/D}}\right) \qquad (11.9.1.4)$$

where we pick the number K_s to fit the data. Applying this hypothesis to the two data points for L/D = 4, one gets the following results:

$K_s = 1.47757$ for the 60° solution

$K_s = 1.501781$ for the 90° solution.

These numbers are within 1.6%, and give costs that agree with 1.1% when applied in either direction. When plotted graphically, as we will do subsequently, these numbers applied to (11.9.1.4) give results

which fall within the line width. Clearly one can use (11.9.1.1) as a basis for more sophisticated approximations. However, adhering to the principles of Occam's razor, we will use (11.9.1.4) as the simplest hypothesis which will fit the data.

11.9.2

Applying (11.9.1.4) to the case for $(L/D)_{max} = 4$, Figure 11.17 was produced. This figure shows the cost for various amounts of plane change for rocket propulsion alone, the optimal constrained Synergetic maneuver, and the limiting case Synergetic maneuver. This figure indicates that the Synergetic maneuver becomes competitive with the rocket maneuver for $\Delta i > 10°$ and becomes better by a factor of 2 from $\Delta i > 60°$ for $(L/D)_{max} = 4$.

11.9.3

Situations of more immediate practical interest are the various maneuver costs for $(L/D)_{max} = 2$. Applying (11.9.1.4) to our single data point for this case, one gets Figure 11.18, showing the same relationships outlined in section 11.9.2 above. In this case, we see that the constrained Synergetic maneuver becomes competitive for $\Delta i > 20°$ and is better by a factor of 1.4 for $\Delta i \geq 90°$.

11.9.4 Comparison with Other Work

It is of interest to compare the results obtained from this problem with some of the results obtained by others who have worked on the Synergetic maneuver. Direct comparisons are difficult because of differences in parameters. However, with the use of (11.9.1.4) a few comparisons can be made.

11.9.4.1 Comparison with Nyland's Solution

First consider the results of Nyland in Ref. 2. Nyland's solution for $(L/D)_{max} = 4$, $\Delta i = 60°$ and no constraints was:

$$J = 0.2965$$

and for $(L/D)_{max} = 2$, $\Delta i = 60°$

$$J = 0.6651.$$

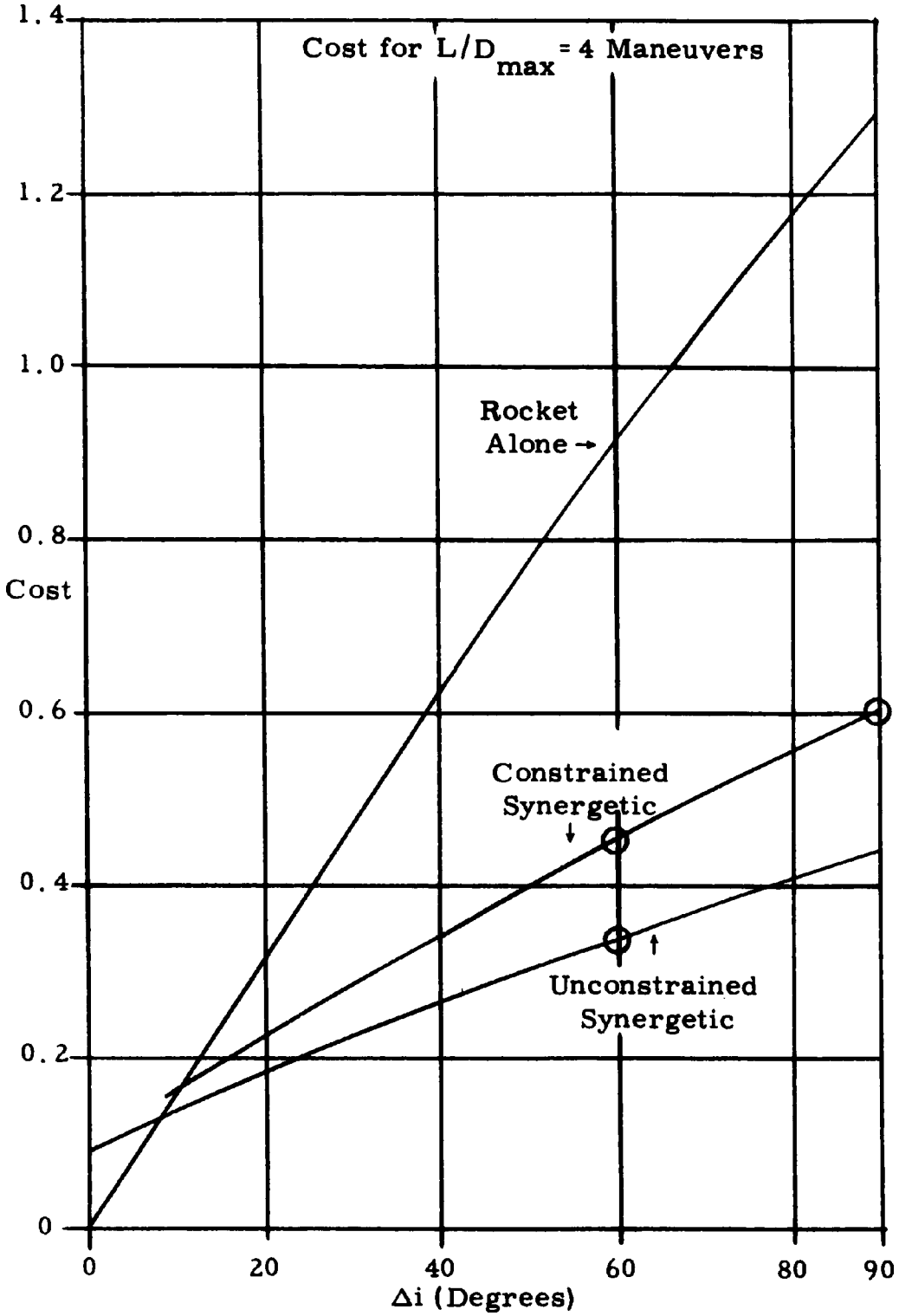

Cost for L/D_{max} = 4 Maneuvers
Rocket Alone
Constrained Synergetic
Unconstrained Synergetic
Cost
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0
20
40
60
80
90
Δi (Degrees)

Figure 11.17

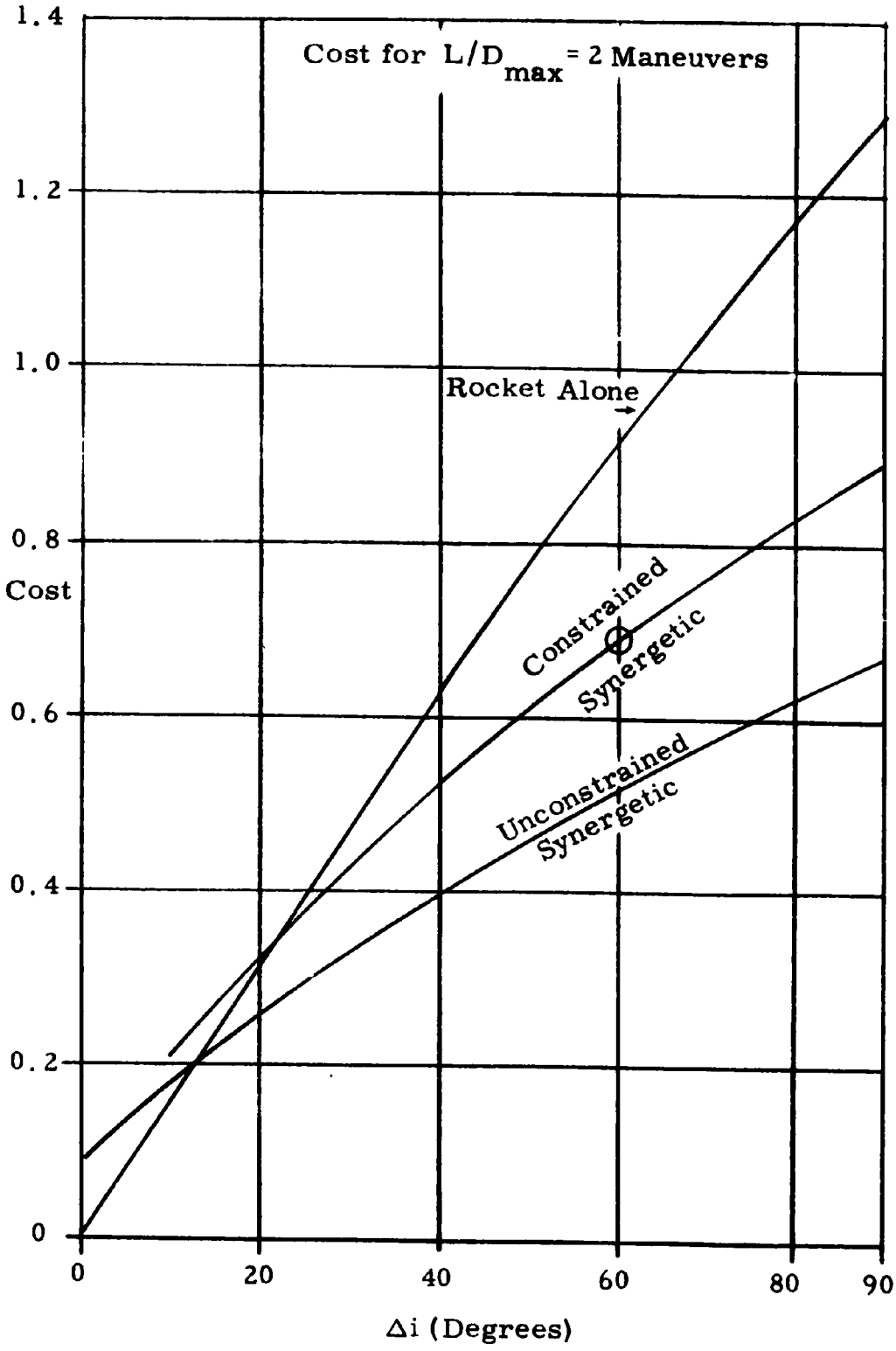

Cost for L/D_{max} = 2 Maneuvers
Rocket Alone
Constrained Synergetic
Unconstrained Synergetic
Cost
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0
20
40
60
80
90
Δi (Degrees)

Figure 11.18

These solutions were for an initial orbit of 300 miles, corresponding to $R_a = 1.0716$. Applying the appropriate corrections for altitude to the three 60° solutions of this paper one obtains:

$(L/D)_{max} = 4$,

Limiting Case	J = 0.2773
Unconstrained	J = 0.2862
Constrained	J = 0.3903

$(L/D)_{max} = 2$

Limiting Case	J = 0.4597
Constrained	J = 0.6240

Thus it appears that Nyland's approximate solution is overly conservative in all cases.

11.9.4.2 Comparison with Shaver's Solution

Next let us look at the results of Shaver in Ref. 4. Shaver's work was application of the gradient method to a slightly different Synergetic problem (See chapter 2 for more details). Shaver obtained an unconstrained cost of 0.5138 for a $(L/D)_{max} = 2$, 60° maneuver from 300 miles altitude. The solutions obtained here corrected to the same conditions are:

Limiting Case	J = 0.4597
Constrained	J = 0.6240

If the data for the L/D = 4 unconstrained trajectory can be taken as indicating that at a L/D = 2 the unconstrained optimum will be within 2% of the limiting case, it would appear that Shaver's restrictions on the problem he worked resulted in conservative answers, also. This cannot be concluded with certainty since Shaver used a wing loading of 30 lbs/ft^2 in contrast to the 40 lbs/ft used here. We are unable to predict the effect of a change in wing loading on our problem.

11.9.4.3 Comparison with the Solution of Bell and Hanke

Comparison with the work of Bell and Hanke (Ref. 3) is even more difficult than with others, primarily because Bell and Hanke restricted their work to small heading changes. Recall that heading changes are not the same as inclination changes. Nonetheless, if one assumes that for small changes $\Delta\psi \approx \Delta i$, one extracts from Ref. 3 for $L/D = 4$, $\Delta i = 15°$ an unconstrained cost of about 0.0856. The corresponding solutions for this problem are:

Limiting Case	J = .0785
Unconstrained	J = .0810
Constrained	J = 0.1093

Thus it would also appear that the results obtained in Ref. 3 are also unduly conservative for optimal maneuvers.

11.10

In summary, it appears that the results obtained from the problem formulated here certainly indicate the attractiveness of the optimized Synergetic maneuver for large plane changes. Further, a comparison with other work done on related problems shows that the results obtained are significantly better. It is the author's hope that this work will stimulate others to continue analysis of this maneuver in order to gain a more complete understanding of its potentialities.

Appendix A

In this appendix the detailed derivation of the adjoint equations and related quantities for this problem are presented.

A.1

All of the work in this appendix is based on quantities derived from the state equations developed in chapter 3 and the cost function developed in chapter 4. For convenience, these equations are repeated here. The state equations are:

$$\dot{x}_1 = -\frac{(1 + x_1)}{(1 + x_3)} - \frac{(QC_d - U_3)}{(1 - x_7)} \tag{A.1.1}$$

$$\dot{x}_2 = \frac{2x_1 + x_1^2}{(1 + x_3)} + \frac{x_3}{(1 + x_3)^2} + \frac{(QC_L \cos U_2 + U_4)}{(1 - x_7)} \tag{A.1.2}$$

$$\dot{x}_3 = x_2 \tag{A.1.3}$$

$$\dot{x}_4 = \frac{(1 + x_1)}{(1 + x_3)} \sin x_4 \tan x_6 - \frac{(QC_L \sin U_2 - U_5)}{(1 - x_7)(1 + x_1)} \tag{A.1.4}$$

$$\dot{x}_5 = \frac{(1 + x_1) \sin x_4}{(1 + x_3) \cos x_6} \tag{A.1.5}$$

$$\dot{x}_6 = \frac{(1 + x_1)}{(1 + x_3)} \cos x_4 \tag{A.1.6}$$

$$\dot{x}_7 = \frac{V_0}{IG_e} \sqrt{U_3^2 + U_4^2 + U_5^2} \tag{A.1.7}$$

$$\dot{x}_8 = \tfrac{1}{2} C_1^2 \, h(C_1) \tag{A.1.8}$$

$$\dot{x}_9 = \tfrac{1}{2} C_2^2 h(C_2) \tag{A.1.9}$$

where

$$Q = Q_0 \, e^{-\frac{R_0}{h_s} x_3} (1 + x_1)^2 \tag{A.1.10}$$

$$C_d = C_{d_0} + 1.7 \left|\sin^3 U_1\right| \tag{A.1.11}$$

$$C_L = 1.7 \cos U_1 \sin U_1 \left|\sin U_1\right| \tag{A.1.12}$$

$$C_1 = Q(1 + x_1) - B_m \tag{A.1.13}$$

$$C_2 = \frac{\sqrt{U_3^2 + U_4^2 + U_5^2}}{(1 - x_7)} - A_{max} \tag{A.1.14}$$

h(x) is the unit step function with argument x.

The cost function for the Bryson method solution can be broken into a sum of terms of the form:

$$J = \sum_{i=1}^{5} \Delta V_i + K_1 x_8(t_f) + K_2 x_9(t_f) \qquad (A.1.15)$$

For convenience we will write out each of the ΔVi:

$$\Delta V_1 = \frac{1}{\sqrt{R_a}} - \sqrt{\frac{2}{R_a(1+R_a)} - \frac{x_2(0)^2}{(R_a^2-1)}} \qquad (A.1.16)$$

$$\Delta V_2 = \left\{ \left[R_a \sqrt{\frac{2}{R_a(1+R_a)} - \frac{x_2(0)^2}{(R_a^2-1)}} - (1+x_1(0)) \right]^2 + P \right\}^{\frac{1}{2}} \qquad (A.1.17)$$

$$\Delta V_3 = \frac{IG_e}{V_0} \; \mathrm{Ln}\left(\frac{1}{1 - x_7(t_f)} \right) \qquad (A.1.18)$$

$$\Delta V_4 = \left\{ \left[R_a \sqrt{\frac{2}{R_a(1+R_a)} - \frac{x_2(t_f)^2}{(R_a^2-1)}} - (1+x_1(t_f)) \right]^2 + P \right\}^{\frac{1}{2}} \qquad (A.1.19)$$

$$\Delta V_5 = \frac{1}{\sqrt{R_a}} - \sqrt{\frac{2}{R_a(1+R_a)} - \frac{x_2(t_f)^2}{(R_a^2-1)}} \qquad (A.1.20)$$

where P is a small positive constant.

These relationships plus the appropriate boundary values for the state equations form the basis for all subsequent work.

A.2 Boundary Values for the Adjoint Equations

A.2.1 Boundary values for the J-adjoint equations.

From paragraph 9.4 we have:

$$\bar{y}_j^T(t_f) = \left[\frac{\partial J}{\partial \bar{x}(t_f)} - \left(\frac{\dot{J}}{\dot{\Omega}} \frac{\partial \Omega}{\partial \bar{x}} \right)_{t=t_f} \right] \qquad (A.2.1.1)$$

where:

$$\Omega = x_3(t_f) = 0 \qquad (A.2.1.2)$$

Performing the indicated operations one obtains:

$$y_{j_1}(t_f) = \frac{-1}{\Delta V_4}\left[R_a\sqrt{\frac{2}{R_a(1+R_a)} - \frac{x_2(t_f)^2}{(R_a^2-1)}} - (1+x_1(t_f))\right] \quad (A.2.1.3)$$

$$y_{j_2}(t_f) = \frac{x_2(t_f)(1 + R_a y_{j_1}(t_f)}{(R_a^2-1)\sqrt{\frac{2}{R_a(1+R_a)} - \frac{x_2(t_f)^2}{(R_a^2-1)}}} \quad (A.2.1.4)$$

$$y_{j_7}(t_f) = \frac{IG_e}{V_0(1 - x_7(t_f)} \quad (A.2.1.5)$$

$$y_{j_3}(t_f) = -\frac{1}{x_2(t_f)}\left[y_{j_1}(t_f)\dot{x}_1(t_f) + y_{j_2}(t_f)\dot{x}_2(t_f)\right. \quad (A.2.1.6)$$

$$\left. + y_{j_7}(t_f)\dot{x}_7(t_f) + K_1\dot{x}_8(t_f) + K_2\dot{x}_9(t_f)\right]$$

$$y_{j_8}(t_f) = K_1 \quad (A.2.1.7)$$

$$y_{j_9}(t_f) = K_2 \quad (A.2.1.8)$$

$$y_{j_4}(t_f) = y_{j_5}(t_f) = y_{j_6}(t_f) = 0 \quad (A.2.1.9)$$

A.2.2 Boundary values for the x-adjoint equations

Again from paragraph 9.4 we have:

$$\bar{y}_x^T(t_f) = \frac{\partial M}{\partial \bar{x}(t_f)} - \left[\frac{\dot{M}}{\dot{\Omega}} \quad \frac{\partial \Omega}{\partial \bar{x}}\right]_{t=t_f} \quad (A.2.2.1)$$

where:

$$M = \sin x_4(t_f)\cos x_6(t_f) - \cos \Delta i = 0 \quad (A.2.2.2)$$

Performing the indicated operations one obtains

$$y_{x_4}(t_f) = \cos x_4(t_f)\ \cos x_6(t_f) \quad (A.2.2.3)$$

$$y_{x_6}(t_f) = -\sin x_4(t_f)\ \sin x_6(t_f) \quad (A.2.2.4)$$

$$y_{x_3}(t_f) = -\frac{1}{x_2(t_f)}\left[y_{x_4}(t_f)\dot{x}_4(t_f) + y_{x_6}(t_f)\dot{x}_6(t_f)\right] \quad (A.2.2.5)$$

$$y_{xi}(t_f) = 0, \quad i = 1, 2, 5, 7, 8, 9. \quad (A.2.2.6)$$

A.3 The A(t) and B(t) matrices

From Chapter 9 we recall that to solve the adjoint equations and generate the influence functions $\bar{\Lambda}(t)$ one requires two time varying matrices defined as:

$$A(t) = \frac{\partial \dot{\bar{x}}(t)}{\partial \bar{x}(t)} \tag{A.3.1}$$

$$B(t) = \frac{\partial \dot{\bar{x}}(t)}{\partial \bar{U}(t)} \tag{A.3.2}$$

where A(t) is 9×9 and B(t) is 9×5.

A.3.1 The A(t) Matrix

Performing the operation indicated in (A.3.1) on the state equations, one obtains for the elements of A(t) the following non-zero terms:

$$A_{11} = -\frac{x_2}{(1+x_3)} - \frac{2QC_d}{(1-x_7)(1+x_1)} \tag{A.3.1.1}$$

$$A_{12} = -\frac{(1+x_1)}{(1+x_3)} \tag{A.3.1.2}$$

$$A_{13} = \frac{(1+x_1)x_2}{(1+x_3)^2} + \frac{R_0 QC_d}{h_s(1-x_7)} \tag{A.3.1.3}$$

$$A_{17} = -\frac{QC_d}{(1-x_7)^2} + \frac{U_3}{(1-x_7)^2} \tag{A.3.1.4}$$

$$A_{21} = \frac{2(1+x_1)}{(1+x_3)} + \frac{2QC_L \cos U_2}{(1-x_7)(1+x_1)} \tag{A.3.1.5}$$

$$A_{23} = -\frac{(2x_1 + x_1^2)}{(1+x_3)^2} + \frac{(1-x_3)}{(1+x_3)^3} - \frac{R_0 QC_L \cos U_2}{h_s(1-x_7)} \tag{A.3.1.6}$$

$$A_{27} = \frac{QC_L \cos U_2 + U_4}{(1-x_7)^2} \tag{A.3.1.7}$$

$$A_{31} = 1 \tag{A.3.1.8}$$

$$A_{41} = \frac{\sin x_4 \tan x_6}{(1+x_3)} - \frac{(QC_L \sin U_2 + U_5)}{(1+x_1)^2(1-x_7)} \tag{A.3.1.9}$$

$$A_{43} = -\frac{(1 + x_1)\sin x_4 \tan x_6}{(1 + x_3)^2} + \frac{R_0 Q C_L \sin U_2}{h_s(1 + x_1)(1 - x_7)} \qquad (A.3.1.10)$$

$$A_{44} = \frac{(1 + x_1)\cos x_4 \tan x_6}{(1 + x_3)} \qquad (A.3.1.11)$$

$$A_{46} = \frac{(1 + x_1)\sin x_4}{(1 + x_3)\cos^2 x_6} \qquad (A.3.1.11a)$$

$$A_{47} = -\frac{(QC_L \sin U_2 - U_5)}{(1 + x_1)(1 - x_7)^2} \qquad (A.3.1.12)$$

$$A_{51} = \frac{\sin x_4}{(1 + x_3)\cos x_6} \qquad (A.3.1.13)$$

$$A_{53} = -\frac{(1 + x_1)\sin x_4}{(1 + x_3)^2 \cos x_6} \qquad (A.3.1.14)$$

$$A_{54} = \frac{(1 + x_1)\cos x_4}{(1 + x_3)\cos x_6} \qquad (A.3.1.15)$$

$$A_{56} = \frac{(1 + x_1)\sin x_4 \sin x_6}{(1 + x_3)\cos^2 x_6} \qquad (A.3.1.16)$$

$$A_{61} = \frac{\cos x_4}{(1 + x_3)} \qquad (A.3.1.17)$$

$$A_{63} = -\frac{(1 + x_1)\cos x_4}{(1 + x_3)^2} \qquad (A.3.1.18)$$

$$A_{64} = -\frac{(1 + x_1)\sin x_4}{(1 + x_3)} \qquad (A.3.1.19)$$

$$A_{81} = 6C_1 h(C_1) Q(1 + x_1)^3 \qquad (A.3.1.20)$$

$$A_{83} = -\frac{R_0 C_1 h(C_1) Q(1 + x_1)^4}{h_s} \qquad (A.3.1.21)$$

$$A_{97} = \frac{C_2 h(C_2) \sqrt{U_3^2 + U_4^2 + U_5^2}}{(1 - x_7)^2} \qquad (A.3.1.22)$$

Thus we have 23 non-zero elements out of a possible 81 in the A(t) matrix. In particular, one finds that columns 5, 8 and 9 and row 7 are identically 0.

A.3.2 The B(t) Matrix

Performing the operations indicated in (A.3.2) on the state equations, one obtains for the non-zero elements of B(t) the following:

$$B_{11} = -\frac{Q}{(1 - x_7)} \frac{\partial C_d}{\partial U_1} \qquad (A.3.2.1)$$

$$B_{13} = \frac{1}{(1 - x_7)} \qquad (A.3.2.2)$$

$$B_{21} = \frac{Q \cos U_2}{(1 - x_7)} \frac{\partial C_L}{\partial U_1} \qquad (A.3.2.3)$$

$$B_{22} = -\frac{Q C_L \sin U_2}{(1 - x_7)} \qquad (A.3.2.4)$$

$$B_{24} = \frac{1}{(1 - x_7)} \qquad (A.3.2.5)$$

$$B_{41} = -\frac{Q \sin U_2}{(1 + x_1)(1 - x_7)} \frac{\partial C_L}{\partial U_1} \qquad (A.3.2.6)$$

$$B_{45} = \frac{1}{(1 - x_7)(1 + x_1)} \qquad (A.3.2.7)$$

$$B_{73} = \frac{V_0 U_3}{IG_e \sqrt{U_3^2 + U_4^2 + U_5^2}} \qquad (A.3.2.8)$$

$$B_{74} = \frac{V_0 U_4}{IG_e \sqrt{U_3^2 + U_4^2 + U_5^2}} \qquad (A.3.2.9)$$

$$B_{75} = \frac{V_0 U_5}{IG_e \sqrt{U_3^2 + U_4^2 + U_5^2}} \qquad (A.3.2.10)$$

$$B_{93} = \frac{C_2 h(C_2) U_3}{(1 - x_7) \sqrt{U_3^2 + U_4^2 + U_5^2}} \qquad (A.3.2.11)$$

$$B_{94} = \frac{C_2 h(C_2) U_4}{(1 - x_7) \sqrt{U_3^2 + U_4^2 + U_5^2}} \qquad (A.3.2.12)$$

$$B_{95} = \frac{C_2 h(C_2) U_5}{(1 - x_7) \sqrt{U_3^2 + U_4^2 + U_5^2}} \qquad (A.3.2.13)$$

where:

$$\frac{\partial C_L}{\partial U_1} = 3.4|\sin U_1|\cos U_1 - 1.7|\sin^3 U_1| \qquad (A.3.2.14)$$

$$\frac{\partial C_d}{\partial U_1} = 3C_L \qquad (A.3.2.15)$$

Thus we see that B(t) has 13 non-zero elements out of a possible 45, and that rows 5, 6 and 8 are identically zero.

A.4 The Adjoint Equations

The adjoint equations are defined by:

$$\dot{\bar{y}}(t) = -A^T(t)\,\bar{y}(t), \quad 0 \leqq t \leqq t_f \qquad (A.4.1)$$

$$\bar{y}(t_f) = \text{As Specified} \qquad (A.4.2)$$

Since this equation must be solved backwards in time and numerical integration subroutines are normally written for positive incrementing of the independent variable, it is convenient to define a new independent variable:

$$\tau = t_f - t \qquad (A.4.3)$$

It can be shown that equations (A.4.1) and (A.4.2) can then be written as:

$$\dot{\bar{y}}(\tau) = A^T(t_f - \tau)\,\bar{y}(\tau) \qquad 0 \leqq \tau \leqq t_f \qquad (A.4.4)$$

$$\bar{y}(0) = \text{As Specified} \qquad (A.4.5)$$

Using the results of paragraph A.3.1 we have for the elements of $\bar{y}(t)$ the following:

$$\dot{y}_1 = A_{11}y_1 + A_{21}y_2 + A_{41}y_4 + A_{51}y_5 + A_{61}y_6 + A_{81}y_8 \qquad (A.4.6)$$

$$\dot{y}_2 = A_{12}y_1 + A_{13}y_3 \qquad (A.4.7)$$

$$\dot{y}_3 = A_{13}y_1 + A_{23}y_2 + A_{43}y_4 + A_{53}y_5 + A_{63}y_6 + A_{83}y_8 \qquad (A.4.8)$$

$$\dot{y}_4 = A_{44}y_4 + A_{54}y_5 + A_{64}y_6 \qquad (A.4.9)$$

$$\dot{y}_5 = 0 \qquad \text{(A.4.10)}$$

$$\dot{y}_6 = A_{46} y_4 + A_{56} y_5 \qquad \text{(A.4.11)}$$

$$\dot{y}_7 = A_{17} y_1 + A_{27} y_2 + A_{42} y_4 + A_{97} y_9 \qquad \text{(A.4.12)}$$

$$\dot{y}_8 = 0 \qquad \text{(A.4.13)}$$

$$\dot{y}_9 = 0 \qquad \text{(A.4.14)}$$

A.4.1 The J-Adjoint Equations

Applying the boundary values of paragraph A.2.1, the J-adjoint equations simplify to the following:

$$\dot{y}_{j_1} = A_{11} y_{j_1} + A_{21} y_{j_2} + A_{81} K_1 \qquad \text{(A.4.1.1)}$$

$$\dot{y}_{j_2} = A_{12} y_{j_1} + A_{13} y_{j_3} \qquad \text{(A.4.1.2)}$$

$$\dot{y}_{j_3} = A_{13} y_{j_1} + A_{23} y_{j_2} + A_{83} K_1 \qquad \text{(A.4.1.3)}$$

$$\dot{y}_{j_7} = A_{17} y_{j_1} + A_{27} y_{j_2} + A_{97} K_2 \qquad \text{(A.4.1.4)}$$

A.4.2 The X-Adjoint Equations

The boundary values of paragraph A.2.2 lead to the following set of X-adjoint equations to be integrated:

$$\dot{y}_{x_1} = A_{11} y_{x_1} + A_{21} y_{x_2} + A_{41} y_{x_4} + A_{61} y_{x_6} \qquad \text{(A.4.2.1)}$$

$$\dot{y}_{x_2} = A_{12} y_{x_1} + A_{13} y_{x_3} \qquad \text{(A.4.2.2)}$$

$$\dot{y}_{x_3} = A_{13} y_{x_1} + A_{23} y_{x_2} + A_{43} y_{x_4} + A_{63} y_{x_6} \qquad \text{(A.4.2.3)}$$

$$\dot{y}_{x_4} = A_{44} y_{x_4} + A_{64} y_{x_6} \qquad \text{(A.4.2.4)}$$

$$\dot{y}_{x_6} = A_{46} y_{x_4} \qquad \text{(A.4.2.5)}$$

$$\dot{y}_{x_7} = A_{17} y_{x_1} + A_{27} y_{x_2} + A_{47} y_{x_4} \qquad \text{(A.4.2.6)}$$

A.5 The Influence Functions

The influence functions $\bar{\Lambda}(\tau)$ are defined by:

$$\bar{\Lambda}(\tau) = B(t_f - \tau)^T y(\tau) \qquad \text{(A.5.1)}$$

Applying the definition to the results obtained in paragraphs A.4 and A.3.2, one gets:

$$\Lambda_{j_1} = B_{11} y_{j_1} + B_{21} y_{j_2} \qquad (A.5.2)$$

$$\Lambda_{j_2} = B_{22} y_{j_2} \qquad (A.5.3)$$

$$\Lambda_{j_3} = B_{13} y_{j_1} + B_{73} y_{j_7} + B_{93} K_2 \qquad (A.5.4)$$

$$\Lambda_{j_4} = B_{24} y_{j_2} + B_{74} y_{j_7} + B_{94} K_2 \qquad (A.5.5)$$

$$\Lambda_{j_5} = B_{75} y_{j_7} + B_{95} K_2 \qquad (A.5.6)$$

$$\Lambda_{x_1} = B_{11} y_{x_1} + B_{21} y_{x_2} + B_{41} y_{x_4} \qquad (A.5.7)$$

$$\Lambda_{x_2} = B_{22} y_{x_2} + B_{42} y_{x_4} \qquad (A.5.8)$$

$$\Lambda_{x_3} = B_{13} y_{x_1} + B_{73} y_{x_7} \qquad (A.5.9)$$

$$\Lambda_{x_4} = B_{24} y_{x_4} + B_{74} y_{x_7} \qquad (A.5.10)$$

$$\Lambda_{x_5} = B_{45} y_{x_4} + B_{75} y_{x_7} \qquad (A.5.11)$$

A.6 The Influence Numbers

In Chapter 9, three influence numbers were defined which must be computed. These are:

$$I_{jj} = \bar{L}_j^T W_x^{-1} \bar{L}_j + \int_0^{t_f} \bar{\Lambda}_j^T W_u^{-1} \bar{\Lambda}_j \, dt \qquad (A.6.1)$$

$$I_{xx} = \bar{L}^T W_x^{-1} \bar{L}_x + \int_0^{t_f} \bar{\Lambda}_j^T W_u^{-1} \bar{\Lambda}_x \, dt \qquad (A.6.2)$$

$$I_{xj} = \bar{L}_x^T W_x^{-1} \bar{L}_j + \int_0^{t_f} \bar{\Lambda}_x^T W_u^{-1} \bar{\Lambda}_j \, dt \qquad (A.6.3)$$

A.6.1 We designate the integral terms on the right hand sides of (A.6.1), (A.6.2) and (A.6.3) as I'_{jj}, I'_{xx} and I'_{xj} respectively. Clearly I'_{jj}, etc., can be generated backward in time while the adjoint equations are being integrated and the influence functions computed by solution of differential equations of the form

$$\dot{I}'_{pq}(s) = \bar{\Lambda}_p^T(s) W_u^{-1}(t_f - s) \bar{\Lambda}_q(s) \qquad (A.6.1.1)$$

$$I'_{pq}(0) = 0 \qquad \text{(A.6.1.2)}$$

If the matrix W_u^{-1} is specified to be diagonal, we can then write in detail:

$$\dot{I}_{jj} = \sum_{i=1}^{5} W_{u_{ii}}^{-1} (\Lambda_{ji})^2 \qquad \text{(A.6.1.3)}$$

$$\dot{I}'_{xx} = \sum_{i=1}^{5} W_{u_{ii}}^{-1} (\Lambda_{xi})^2 \qquad \text{(A.6.1.4)}$$

$$\dot{I}'_{xj} = \sum_{i=1}^{5} W_{u_{ii}}^{-1} \Lambda_{xi} \Lambda_{ji} \qquad \text{(A.6.1.5)}$$

A.6.2 The vectors $\bar{L}_j$ and $\bar{L}_x$ were defined as:

$$\bar{L}_j = \left(\frac{\partial J}{\partial \bar{x}} + \bar{y}_j \right)_{t=0 \text{ or } \tau=t_f} \qquad \text{(A.6.2.1)}$$

$$\bar{L}_x = (\bar{y}_x)_{t=0 \text{ or } \tau = t_f} \qquad \text{(A.6.2.2)}$$

Note that the matrix W_x is defined in the context of equation (9.3.5). Since in this problem we are only free to vary the first two components of $\bar{x}(0)$, $\delta\bar{x}(0)$ is a 2-vector, and W_x a 2×2 matrix. Consequently, only the first two components of y_j and y_x need be used.

Referring to paragraph A.1 it can be seen that:

$$\frac{\partial J}{\partial x_1(0)} = \frac{1}{\Delta V_2} \left[-R_a \sqrt{\frac{2}{R_a(1+R_a)} - \frac{x_2(0)^2}{(R_a^2 - 1)}} + [1 + x_1(0)] \right] \qquad \text{(A.6.2.3)}$$

$$\frac{\partial J}{\partial x_2(0)} = \frac{x_2(0)\ [1 + R_a(\frac{\partial J}{\partial x_1(0)})]}{(R_a^2 - 1)\sqrt{\frac{2}{R_a(1+R_a)} - \frac{x_2(0)^2}{R_a^2 - 1}}} \qquad \text{(A.6.2.4)}$$

A.7 The Detailed Equations for the Kelley Algorithm

Use of the Kelley Algorithm involves addition of another term to the cost function given as equation (A.1.15). If one defines the new cost as J_1, one has:

$$J_1 = J + \tfrac{1}{2} K_3 M^2[x(t_f)] \qquad (A.7.1)$$

where M is defined by equation (A.2.2.2).

A.7.1 Use of J_1 in place of J results in two new boundary values for the J-Adjoint equations. These are:

$$y_{j_4}(t_f) = K_3 M(\bar{x}) \cos x_4 \; \text{xos} \; x_6 \qquad (A.7.1.1)$$

$$y_{j_6}(t_f) = K_3 M(\bar{x}) \sin x_4 \; \sin x_6 \qquad (A.7.1.2)$$

A.7.2 Because of the new boundary values above, the J-Adjoint equations are different than those given in paragraph A.4.1. The equations for the Kelley Algorithm are:

$$\dot{y}_{j_1} = A_{11} y_{j_2} + A_{21} y_{j_2} + A_{41} y_{j_4} + A_{61} y_{j_6} + A_{81} K_1 \qquad (A.7.2.1)$$

$$\dot{y}_{j_2} = A_{12} y_{j_1} + A_{13} y_{j_3} \qquad (A.7.2.2)$$

$$\dot{y}_{j_3} = A_{13} y_{j_1} + A_{23} y_{j_2} + A_{43} y_{j_4} + A_{63} y_{j_6} + A_{83} y_{j_8} \qquad (A.7.2.3)$$

$$\dot{y}_{j_4} = A_{44} y_{j_4} + A_{64} y_{j_6} \qquad (A.7.2.4)$$

$$\dot{y}_{j_6} = A_{46} y_{j_4} \qquad (A.7.2.5)$$

$$\dot{y}_{j_7} = A_{17} y_{j_1} + A_{27} y_{j_2} + A_{47} y_{j_4} + A_{97} K_2 \qquad (A.7.2.6)$$

A.7.3 The functions $\bar{\Lambda}_j$ are also different, becoming:

$$\Lambda_{j_1} = B_{11} y_{j_1} + B_{21} y_{j_2} + B_{41} y_{j_4} \qquad (A.7.3.1)$$

$$\Lambda_{j_2} = B_{22} y_{j_2} + B_{42} y_{j_4} \qquad (A.7.3.2)$$

$$\Lambda_{j_3} = B_{13} y_{j_1} + B_{73} y_{j_7} + B_{93} K_2 \qquad (A.7.3.3)$$

$$\Lambda_{j_4} = B_{24} y_{j_2} + B_{74} y_{j_7} + B_{94} K_2 \qquad (A.7.3.4)$$

$$\Lambda_{j_5} = B_{45} y_{j_4} + B_{75} y_{j_7} + B_{95} K_2 \qquad (A.7.3.5)$$

A.7.4 Using the Kelley Algorithm, there are no X-Adjoint equations, no $\bar{\Lambda}_x$ functions, and no numbers I_{xj} and I_{xx}.

A.8 The Time-Varying Weighting Matrix $W_u(t)$

In paragraph 10.6.2.1 a concept was outlined for more efficient use of the arbitrary weighting matrix $W_u(t)$ in this problem. In this section the concept will be developed in detail and the terms for the weighting matrix derived.

In paragraph 10.6.2.1 it was pointed out that a control variable constraint of the form:

$$\int_0^{t_f} \delta\bar{L}^T(t)\, \delta\bar{L}(t)dt + \int_0^{t_f} \delta\bar{T}^T(t)\, \delta\bar{T}(t)dt \qquad (A.8.1)$$

was desired where:

$\delta\bar{L}$ is the incremental change in the lift acceleration vector due to changes in bank and angle of attack;

$\delta\bar{T}$ is the incremental change in the thrust acceleration vector due to changes in the thrust vector.

A.8.1 First consider the integrand of the form $\delta\bar{L}^T\delta\bar{L}$. $\delta\bar{L}$ will be defined as:

$$\delta\bar{L} = \begin{bmatrix} \frac{\partial \dot{x}_2}{\partial U_1} & \frac{\partial \dot{x}_2}{\partial U_2} \\ \frac{\partial \dot{x}_4 (1+x_1)}{\partial U_1} & \frac{\partial \dot{x}_4 (1+x_1)}{\partial U_2} \end{bmatrix} \begin{bmatrix} \delta U_1 \\ \delta U_2 \end{bmatrix} \qquad (A.8.1.1)$$

The factor $(1 + x_1)$ in the second row arises because the $\dot{x}_4$ equation (A.1.4) is an angular rate equation and the factor is required to convert the appropriate moment terms to force.

For compactness, let us denote the matrix of equation (A.7.1.1) as **F**, and the vector as $\delta\bar{\theta}$. The elements of **F** are:

$$\mathbf{F} = \begin{bmatrix} -\frac{Q}{M}\sin U_2 \frac{dC_L}{dU_1} & -\frac{Q}{M}C_L\cos U_2 \\ \frac{Q}{M}\cos U_2 \frac{dC_L}{dU_1} & -\frac{Q}{M}C_L\sin U_2 \end{bmatrix} \qquad (A.8.1.2)$$

where $M = (1 - x_7)$ and the product $\mathbf{F}^T\mathbf{F}$ is:

$$\mathbf{F}^T\mathbf{F} = \begin{bmatrix} \frac{Q^2}{M^2}\left(\frac{dC_L}{dU_1}\right)^2 & 0 \\ 0 & \frac{Q^2}{M^2}C_L^2 \end{bmatrix} \qquad (A.8.1.3)$$

This is a particularly pleasing result because eventually matrix inversion will be required and diagonal matrices are particularly easy to invert.

A.8.2 Now let us consider the second integrand, $\delta\bar{T}^T\delta\bar{T}$. $\delta\bar{T}$ is defined as:

$$\delta\bar{T} = \begin{bmatrix} \frac{\partial\dot{x}_1}{\partial U_3} & 0 & 0 \\ 0 & \frac{\partial\dot{x}_2}{\partial U_4} & 0 \\ 0 & 0 & \frac{\partial\dot{x}_4}{\partial U_5}(1 + x_1) \end{bmatrix} \begin{bmatrix} \delta U_3 \\ \delta U_4 \\ \delta U_5 \end{bmatrix} \qquad (A.8.2.1)$$

Again for compactness let us denote the matrix as **R** and the vector as $\delta\bar{R}$. The elements of **R** are:

$$\mathbf{R} = \begin{bmatrix} \frac{1}{M} & 0 & 0 \\ 0 & \frac{1}{M} & 0 \\ 0 & 0 & \frac{1}{M} \end{bmatrix} \qquad (A.8.2.2)$$

and the integrand $\delta\bar{T}^T\delta\bar{T}$ becomes: $\delta\bar{R}^T R^T R\delta\bar{R}$. The product $R^T R$ is:

$$R^T R = \begin{bmatrix} \frac{1}{M^2} & 0 & 0 \\ 0 & \frac{1}{M^2} & 0 \\ 0 & 0 & \frac{1}{M^2} \end{bmatrix} \qquad (A.8.2.3)$$

Again a pleasing result in that one has a diagonal matrix to invert.

A.8.3 Inspection of the integrals given in (A.7.1) together with the results of paragraphs A.8.1 and A.8.2 indicates that if one defines a new 5 × 5 matrix G as:

$$G = \begin{bmatrix} \frac{Q^2}{M^2}\left(\frac{\partial C_L}{\partial U_1}\right) & 0 & 0 & 0 & 0 \\ 0 & \frac{Q^2}{M^2}C_L^2 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{M^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{M^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{M^2} \end{bmatrix} \qquad (A.8.3.1)$$

The two integrals are equivalent to a single integral:

$$\int_0^{t_f} \delta\bar{U}^T G\,\delta\bar{U} \qquad (A.8.3.2)$$

(A.8.3.2) is identical in form to the constraint we started out with, except that we have acceleration-weighted our control variables. Now let us proceed one step further and multiply (A.8.3.1) by a second diagonal matrix W(t), and let W(t) be programmer controlled. Thus we have

$$W_u(t) = G(t)\,W(t) \qquad (A.8.3.3)$$

and the matrix $W_u(t)$ now combines the best features of Denham's proposal and the author's.

There is one note of caution appropriate here. G(t) is singular when $U_1 = 0$, which can occur. In addition, for combinations of small Q and small U_1, the inverse of G_{11} and G_{22} can become very large. For this reason the author built limiting features into the generation of these terms which for purposes of generation of G alone:

a. Keeps $|U_1| \geq 1^\circ$

b. Keeps G_{11} and $G_{22} \geq 0.001$.

A.8.4 The inverse of the new weighting matrix (A.8.3.3) is generated for computation by computing and storing $1/G_{11}$ and $1/G_{22}$ each time the state equations are solved.

The remaining elements of G^{-1} are available simply since $x_7(t)$, from which M is formed, is already stored in tabular form.

A.9 Interpolation

In paragraph 10.5.3.3 it was pointed out that inaccuracies introduced by the interpolation process are one of the major error sources in the computer solution of this problem. There are two classes of functions for which interpolation introduces errors: the control variables U(t) and the state variables used to generate the matrices A(t) and B(t).

A.9.1 Insofar as the control variables are concerned, straight linear interpolation was used. Second order schemes were tried, but gave worse results than the linear algorithm.

A.9.2 In the case of the state variables, their derivatives are also available explicitly and a third degree interpolation scheme using derivatives was used on some of the later problem cases worked. The algorithm is as follows:

Given: x(t) and $\dot{x}(t)$ stored at equal intervals of t, Δt, together with the appropriate values of t. Let x_n, $\dot{x}_n$, t_n be the values at the beginning of an interval inside of which an estimate of x at time t_1 is required. Let x_{n+1}, $\dot{x}_{n+1}$, be the set of values at the end of the interval. Then:

$$x(t_1) = q^2[(1 + 2P)x_n + \Delta t P\dot{x}_n] + P^2[(1 + 2q)x_{n+1} - \Delta t q\dot{x}_{n+1}] \quad (A.9.2.1)$$

where:

$$p = (t - t_n)/\Delta t \quad (A.9.2.2)$$

$$q = 1 - P \quad (A.9.2.3)$$

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