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QUASI-SIMILARITY AND SUBNORMAL OPERATORS

by
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INTRODUCTION

In trying to understand the structure of (bounded linear) operators on a Hilbert space one is sometimes led to consider various equivalence relations among operators. For example, the familiar theorem on reducing a Hermitian matrix to diagonal form asserts that every matrix (finite-dimensional operator) which is Hermitian is unitarily equivalent to a diagonal matrix. This generalizes to normal operators, and it also generalizes nicely to infinite-dimensional spaces (the spectral theorem). In the absence of normality it is still possible to get a sharp theorem in the finite-dimensional case, provided one replaces unitary equivalence with the weaker equivalence relation of similarity. This is the theorem on the Jordan canonical form: Every matrix is similar to a matrix in Jordan form. In contrast to the spectral theorem, very little of this theorem seems to survive the passage to infinite-dimensional spaces, and as a result the structure of the general non-normal operator on an infinite-dimensional Hilbert space is very poorly understood, although for certain classes of operators some results have been obtained. (See, for example, [11] , [24] , and[25] .) In particular, Sz.-Nagy and Foiaş have introduced an equivalence relation called quasi-similarity, which is weaker than similarity, and have used it to obtain, among other things, a generalization of the Jordan form for a certain restricted class of infinite-dimensional operators.

Considerations such as those above indicate that quasi- similarity may be of some importance in the study of non-normal operators. This paper investigates a few questions concerning quasi-similarity within the class of subnormal operators. The reasons for selecting subnormal

operators may be summarized in the following four points.

(1) Although quasi-similar normal operators are always unitarily equivalent, the relation of quasi-similarity for subnormal operators is genuinely weaker than unitary equivalence or similarity.

(2) Subnormal operators are not totally intractable. In particular, the existence of a normal extension makes measure-theoretic techniques available, as in [2].

(3) Despite point (2), the structure of subnormal operators is not very well understood. For example, it is apparently not yet known whether subnormal operators must have non-trivial invariant subspaces. It is perhaps conceivable that the study of quasi-similar subnormal operators might contribute something to the solution of such (presumably) difficult problems. The present paper, however, contains little or nothing in this direction. (For some results on the existence of invariant subspaces for certain subnormal operators see the work of Brennan, for instance [3].)

(4) Some positive results have been obtained. The most important of these are perhaps the theorem on equality of spectra (Section 3), the characterization of all subnormal operators quasi-similar or similar to the unilateral shift (Sections 4 and 6, respectively), and the determination of the invariant subspaces for such operators (Section 7).

The following is a brief outline of the results in this paper.

Chapter I deals with quasi-similarity in general, with the emphasis on relationships between spectra of quasi-similar operators. Quasi-similarity is defined in Section 1, and a simple example of Hoover [16] showing that quasi-similar operators may have unequal spectra is presented.

Section 2 begins with a slight generalization of the Putnam-Fuglede

commutativity theorem, which is used to generalize the theorem that quasi-similar normal operators are unitarily equivalent. Then an example is given to show that nothing like this happens for subnormal operators: Quasi-similar subnormal operators need not even be similar.

On the other hand, there is at least one thing which can be said about quasi-similar subnormal operators which does not hold in general: They have equal spectra. This is proven in Section 3, not only for subnormal operators, but for the larger class of hyponormal operators.

Chapter II is devoted to the study of one equivalence class of quasi-similar subnormal operators, namely the equivalence class containing the unilateral shift. (Some of these results have been announced in [6].) In Section 4 it is shown that a subnormal operator is quasi-similar to the shift if and only if it is unitarily equivalent to an operator U_μ: $H^2(\mu) \to H^2(\mu)$ where μ is a measure in the closed unit disc in the complex plane satisfying certain specified conditions. Here $H^2(\mu)$ is the closure of the polynomials in $L^2(\mu)$, and U_μ denotes "multiplication by z." The proof that U_μ is quasi-similar to the shift provided μ satisfies these conditions is based on "sweeping out" to the boundary that part of μ which is in the interior of the disc.

In preparation for Sections 6 and 7, in Section 5 a functional Hilbert space $\mathfrak{H}^2(u)$ of analytic functions of bounded characteristic is defined for each measure u for which U_μ is quasi-similar to the shift, and it is shown that U_μ is unitarily equivalent to "multiplication by z" on $\mathfrak{H}^2(u)$.

If U_μ and U_ν are quasi-similar to the shift they are, of course, quasi-similar to each other. Section 6 gives necessary and sufficient

conditions that they be similar or unitarily equivalent to each other. By means of a theorem of Carleson [5] a more easily verified condition is obtained, which is necessary and sufficient for U_μ to be similar to the shift. It is also shown that the Fredholm essential spectrum of any subnormal operator quasi-similar to the shift is the unit circle, and then a theorem in [4] is used to infer that every such operator is unitarily equivalent to a compact perturbation of the shift.

Section 7 deals with the commutant and the invariant subspaces of any subnormal operator quasi-similar to the shift. It is shown that the commutant is always isometrically isomorphic to H^∞ and that every invariant subspace is hyperinvariant. Beurling's theorem is used to set up a correspondence between inner functions and invariant subspaces, from which it follows that the lattice of invariant subspaces is isomorphic to the invariant subspace lattice of the shift.

Finally, Section 8 points out a few open questions related to the results of this paper.

CHAPTER I

QUASI-SIMILARITY AND SPECTRA

The purpose of this chapter is to establish terminology and to illustrate the properties of the relation of quasi-similarity through a discussion of the spectra of quasi-similar operators. The main new result is the proof, given in Section 3, that quasi-similar hyponormal operators have equal spectra.

SECTION 1 General operators

In this section we define quasi-similarity and present Hoover's simple example of two quasi-similar operators with unequal spectra.

Throughout this paper the term operator will be used to mean a bounded linear transformation mapping a complex Hilbert space into itself (or into another complex Hilbert space), and the term subspace will always mean a closed linear manifold. Unless the contrary is explicitly stated, the phrase $A: H \to K$ will imply that A is an operator.

If $A: H \to H$ and $B: K \to K$ are operators we say that A is unitarily equivalent to B, and write $A \cong B$, iff there exists a unitary operator $V: H \to K$ such that $VA = BV$. When we want to indicate the so-called intertwining map V implementing the unitary equivalence we will write $A \cong B$ (via V) or, equivalently, $B \cong A$ (via V^*).

We say that A is similar to B, and write $A \sim B$, iff there exists an invertible operator $S: H \to K$ such that $SA = BS$. To indicate the role of S we will write $A \sim B$ (via S), or $B \sim A$ (via S^{-1}).

Various weakenings of the notion of similarity have been studied. In this paper we will be concerned with one called quasi-similarity, which was introduced by Sz.-Nagy and Foiaş (see, for example, [24]). An operator $X: H \to K$ is called quasi-invertible (or a quasi-affinity) iff X is one-to-one and has dense range. We say that A is a quasi-affine transform of B iff there exists a quasi-invertible operator $X: H \to K$ such that $XA = BX$, and that A is quasi-similar to B (denoted $A \sim\sim B$) iff each is a quasi-affine transform of the other. Thus $A \sim\sim B$ iff there exist quasi-invertible operators $X: H \to K$ and $Y: K \to H$ such that $XA = BX$ and $YB = AY$. To indicate the intertwining maps X and Y we will write $A \sim\sim B$ (via X,Y), or $B \sim\sim A$ (via Y,X).

Certain things about the relation of quasi-similarity are clear. It is an equivalence relation in the class of all operators, as are similarity and unitary equivalence. Also, $A \cong B \Rightarrow A \sim B \Rightarrow A \sim\sim B$; we shall see that neither implication can be reversed in general. If $A \sim\sim B$ (via X,Y), then $A^* \sim\sim B^*$ (via Y^*,X^*), $\alpha A - \lambda \sim\sim \alpha B - \lambda$ (via X,Y), and $A^n \sim\sim B^n$ (via X,Y) for $n \geq 0$ (and for $n < 0$, too, if both A and B are invertible). If $A: H \to H$ is a quasi-affine transform of $B: K \to K$, then $\dim H = \dim K$ (see [14 , Problem 42]), where "dim" means "orthogonal dimension".

We next define certain Hilbert spaces and operators which we will use throughout this paper, especially in Chapter II. If μ is a finite positive (countably additive) Borel measure in the complex plane $\mathbf{C}$ we will say that a Borel set S carries μ iff $\mu(\mathbf{C} \setminus S) = 0$. There exists a smallest closed set which carries μ; this set is called the support of μ and is denoted $\operatorname{supp} \mu$. With the exception of Lemma 4.5α and some possibly infinite measures $\check{\mu}$ in Section 6, we will henceforth use the

term "measure" to mean "finite positive Borel measure in C having compact support."

Let P denote the set of polynomials in one variable z, with complex coefficients. For any measure μ, let $P(\mu)$ be P regarded as a subspace of $L^2(\mu)$, where $L^2(\mu)$ is the Hilbert space of (equivalence classes of) complex-valued functions f such that $\int |f|^2 d\mu < \infty$, with the inner product $(f,g) = \int f\bar{g}d\mu$, and let $H^2(\mu)$ be the norm-closure of $P(\mu)$ in $L^2(\mu)$. (If supp μ is a finite set many distinct members of P will, of course, yield the same member of $P(\mu)$.) Then $H^2(\mu)$ is a Hilbert space and $z \cdot H^2(\mu) \subseteq H^2(\mu)$, where z denotes, as usual, the polynomial p defined by $p(z) = z$. Define $W_\mu : L^2(\mu) \to L^2(\mu)$ to be multiplication by z, that is, $(W_\mu f)(z) = zf(z)$, and let $U_\mu : H^2(\mu) \to H^2(\mu)$ be the restriction of W_μ to $H^2(\mu)$. Clearly W_μ and U_μ are operators.

Let T be the unit circle $\{ z \in C: |z| = 1 \}$. We will always use m to denote normalized Lebesque measure "$d\theta/2\pi$" on T (considered as a Borel measure, of course). The operator U_m is called the <u>unilateral shift</u> (or simply "the shift"), and W_m is called the <u>bilateral shift</u>. (See [14], Problems 67 and 68, and Chapter 14.)

One operator can be a quasi-affine transform of another without being quasi-similar to it, as illustrated by the unilateral and bilateral shifts in our first proposition.

<u>PROPOSITION 1.1</u> (1) There exists a quasi-invertible operator $X: H^2(m) \to L^2(m)$ such that $XU_m = W_m X$.

(2) If $V: H \to H$ is any unitary operator and $Y: H \to H^2(m)$ satisfies $YV = U_m Y$, then $Y = 0$.

(3) The unilateral shift U_m is a quasi-affine transform of the bilateral shift W_m, but U_m and W_m are not quasi-similar to each other.

PROOF (1) This follows from Szegö's theorem. Let $\varphi \in L^\infty(m)$ be a function which does not vanish but $\int \log \varphi \cdot dm = -\infty$, and define $X: H^2(m) \to L^2(m)$ to be multiplication by φ. Then X has dense range [15, page 59, problem 6], and clearly X is one-to-one and satisfies $XU_m = W_m X$.

(2) If $YV = U_m Y$ with V unitary, then $Y^* = V^n Y^* U_m^{*n}$ for $n \geq 0$, and therefore $||Y^* f|| = ||V^n Y^* U_m^{*n} f|| \leq ||Y^*|| \cdot ||U_m^{*n} f||$ for every $f \in H^2(m)$. But U_m^{*n} tends strongly to 0 as $n \to \infty$, so $Y^* = 0$, and thus $Y = 0$.

(3) Immediate from (1) and (2). ∎

The next result, due to Hoover [16, Theorem 1.7], is useful for constructing examples of "bad" properties of quasi-similarity.

PROPOSITION 1.2 (Hoover) For each j in some index set J let $A_j: H_j \to H_j$ and $B_j: K_j \to K_j$ be quasi-similar. Assume $\sup \{||A_j|| : j \in J\} < \infty$ and $\sup \{||B_j|| : j \in J\} < \infty$. Then the direct sums $A = \oplus_j A_j$ and $B = \oplus_j B_j$ exist, and A and B are quasi-similar.

PROOF SKETCH Assume with no loss of generality that $\dim H_j = \dim K_j > 0$ for every $j \in J$. If $A_j \sim\sim B_j$ (via X_j, Y_j), then it is easily verified that $A \sim\sim B$ (via X,Y) where

$$X = \oplus_j \frac{1}{||X_j||} X_j \quad \text{and} \quad Y = \oplus_j \frac{1}{||Y_j||} Y_j . \; ∎$$

Note that Proposition 1.2 remains true (trivially) if "quasi-similar"

is replaced at each occurrence by "unitarily equivalent" but not if "quasi-similar" is replaced by "similar" (see Example 1.3 below). Thus quasi-similarity is better behaved with respect to direct sums than is similarity.

Some of the results in this paper (such as those in Section 3) assert the equality of the spectra (or parts of the spectra) of quasi-similar operators, assuming the operators satisfy some other conditions. In order to put such results in perspective we now present an example of two quasi-similar operators with unequal spectra. (For an earlier example constructed in a different way, see [24, page 262].)

EXAMPLE 1.3 (Hoover [16]) For $n = 1,2,3,\ldots$ let A_n be the operator on n-dimensional Hilbert space defined by the matrix all of whose entries are 0 except on the subdiagonal where all the entries are 1 (thus A_n is an n-dimensional shift operator), and let $B_n = \frac{1}{n} A_n$. Let

$A = \bigoplus_{n=1}^{\infty} A_n$ and $B = \bigoplus_{n=1}^{\infty} B_n$. Then $A \sim\sim B$ but

$\sigma(A) = \overline{D}$, $\sigma(B) = \{0\}$, where σ denotes spectrum and $\overline{D} = \{z \in C : |z| \leq 1\}$

PROOF SKETCH For each n, $A_n \sim B_n$ (A_n being the Jordan form for B_n), so $A \sim\sim B$ by Proposition 1.2. Since A and B are weighted shifts their spectra can be calculated by techniques in [17]. ∎

As Hoover remarks, the spectra of two quasi-similar operators must have at least one point in common (unless the underlying spaces are 0-dimensional), and so Example 1.3 is an extreme case. The proof is implicit in [22].

Note that when quasi-similar operators have unequal spectra this is necessarily because they have unequal approximate point spectra, for it is easy to see that quasi-similarity preserves both point spectrum

(eigenvalues) and compression spectrum. (For terminology, see [14 , Problem 58]). Actually, we can say somewhat more.

PROPOSITION 1.4 Let $A: H \to H$ and $B: K \to K$.

(1) If $X: H \to K$ is one-to-one and satisfies $XA = BX$, then dim ker $A \leq$ dim ker B, where "ker" denotes kernel (null space).

(2) If $A \sim\sim B$, then dim ker A = dim ker B.

PROOF: (1) Given such an X, $f \in \ker A \Rightarrow Af = 0$

$\Rightarrow XAf = 0$

$\Rightarrow BXf = 0$

$\Rightarrow Xf \in \ker B$,

so $X(\ker A) \subseteq \ker B$. Since X is one-to-one it follows that dim ker $A \leq$ dim ker B. (See [14, Problem 42].)

(2) Apply (1) twice. ∎

SECTION 2 Normal operators

Quasi-similarity of normal operators turns out not to be very interesting, for it is known that if two normal operators are quasi-similar, they must be unitarily equivalent. In this section we will see that this theorem can be generalized slightly, but that it cannot be generalized to pairs of subnormal operators. First, the relevant definitions.

Recall that an operator $A: H \to H$ is normal iff $AA^* = A^*A$ (or equivalently, iff $||A^*f|| = ||Af||$ for every $f \in H$). An operator $A: H \to H$ is said to be subnormal iff there exists a Hilbert space $K \supseteq H$ and a normal operator $B: K \to K$ such that $BH \subseteq H$ and $Af = Bf$ for every $f \in H$ (that is, iff A has a normal extension). If $B: K \to K$ is a normal extension of the subnormal operator $A: H \to H$ such that the only subspace of K which contains H and reduces B is K itself, then B is called the minimal normal extension of A; the use of the definite article is justified since any two minimal normal extensions of a subnormal operator are unitarily equivalent [14 , Problem 155] . Clearly every subnormal operator has a minimal normal extension.

An operator $A: H \to H$ is called hyponormal iff $AA^* \leq A^*A$ (or equivalently, iff $||A^*f|| \leq ||Af||$ for every $f \in H$). Every subnormal operator is hyponormal [14 , Problem 160] , but not conversely [14, Solutions 160 and 164].

Finally, an operator A is co-subnormal iff A^* is subnormal, A is co-hyponormal iff A^* is hyponormal, and A is semi-normal iff A is either hyponormal or co-hyponormal.

The concepts "subnormal" and "hyponormal" were both introduced by

Halmos in[12], although his terminology was different from what is now used.

We now present a generalization of the Putnam-Fuglede theorem, the proof of which is a simple adaptation of Rosenblum's elegant proof in [23].

PROPOSITION 2.1 Assume A: $H \to H$ is co-subnormal, B: $K \to K$ is subnormal, and X: $H \to K$ satisfies $XA = BX$. Then $XA^* = B^*X$.

PROOF: For $n = 0,1,2, \ldots$ we have $XA^n = B^nX$, so $Xe^{i\bar{z}A} = e^{i\bar{z}B}X$ for every complex number z. Thus $X = e^{i\bar{z}B} Xe^{-i\bar{z}A}$, and so

$$e^{izB^*} Xe^{-izA^*} = e^{izB^*} e^{i\bar{z}B} Xe^{-i\bar{z}A} e^{-izA^*}.$$

Now let S denote either $-izA^*$ or $i\bar{z}B$. In either case S is subnormal and so has some normal extension N. Since e^{-N} is a normal extension of e^{-S}, e^{-S} is subnormal and so, in particular, e^{-S} is hyponormal. Thus $||e^{-S^*}e^{S}f|| \leq ||e^{-S}e^{S}f|| = ||f||$, that is, $e^{-S^*}e^{S}$ is a contraction. Thus $e^{izB^*} e^{i\bar{z}B}$ and $e^{-i\bar{z}A} e^{-izA^*}$ are both contractions.

Hence the operator-valued entire function $F(z) = e^{izB^*}Xe^{-izA^*}$ satisfies $||F(z)|| \leq ||X||$ for all complex z. Forming the scalar-valued entire functions $(F(z)f, g)$ for all choices of $f \in H$, $g \in K$, and applying Liouville's theorem on bounded entire functions, we conclude that F(z) is constant, that is, $e^{izB^*}Xe^{-izA^*} = X$ for all z. Differentiating with respect to z and then setting $z = 0$, we see that $XA^* = B^*X$. ∎

I do not know whether the conclusion of Proposition 2.1 remains true if the hypotheses are weakened so that A^* and B are assumed to be merely hyponormal instead of subnormal. The above proof would generalize, of course, if it could be shown that e^B is hyponormal whenever B is hyponormal. That would be surprising enough to be interesting in its own right.

PROPOSITION 2.2 If the co-subnormal operator A: $H \to H$ is a quasi-affine transform of the subnormal operator B: $K \to K$, then A and B are normal and are unitarily equivalent to each other.

PROOF: Assume X: $H \to K$ is quasi-invertible and satisfies $XA = BX$. By Proposition 2.1 we also know $XA^* = B^*X$. Let $X = PV$ be the polar decomposition, so $P \geq 0$ and V: $H \to K$ is unitary. Substituting $X = PV$ in $XA = BX$ and multiplying both sides of the resulting equation by V^*P yields $PVAV^*P = BPVV^*P = BP^2$. But $AV^*P = V^*PB$ by taking adjoints in $XA^* = B^*X$, and so $BP^2 = PVAV^*P = PVV^*PB = P^2B$. Thus B commutes with P^2, and so B commutes with P. Therefore $PVA = BPV = PBV$. Since P is one-to-one we have $VA = BV$ with V unitary. ∎

Of course "co-subnormal" and "subnormal" cannot be interchanged in Proposition 2.2, because of Proposition 1.1.

COROLLARY 2.2A (Putnam [19], Douglas [8]). Quasi-similar normal operators are unitarily equivalent. ∎

Proposition 2.2 speaks of a co-subnormal operator and a subnormal operator. There is no analogous result for two subnormal operators, for there is a simple example due to D.E. Sarason [14 , Solution 156] of two subnormal operators which are similar but not unitarily equivalent. We now present Sarason's example and then use it to construct a pair of subnormal operators which are quasi-similar but not similar.

EXAMPLE 2.3 (Sarason) Let δ be the unit point mass at zero and for any $c \geq 0$ let $A_c = U_{m+c\delta}$. Then A_c is subnormal and similar to U_m, but A_c is not unitarily equivalent to U_m if $c > 0$.

PROOF: Clearly A_c is subnormal since $W_{m+c\delta}$ is normal. The fact that A_c is similar to U_m is a special case of the results in Section 6, but

can be proven more simply by means of the theory of weighted shifts. Using the orthonormal basis $\frac{1}{\sqrt{1+c}}$, z, z^2, z^3, ... in $H^2(m+c\,\delta)$, we see that A_c is unitarily equivalent to the unilateral weighted shift with weights $\frac{1}{\sqrt{1+c}}$, 1, 1, 1, ... By [14, Problem 76], A_c is similar to U_m. But if $c > 0$, then A_c is not an isometry, so it is not unitarily equivalent to U_m. ∎

EXAMPLE 2.4 Let A_c be as in Example 2.3, let $A = A_1 \oplus A_2 \oplus A_3 \oplus \ldots$, and let $U_\infty = U_m \oplus U_m \oplus U_m \oplus \ldots$. Then A and U_∞ are subnormal and $A \sim\sim U_\infty$. However, U_∞ is an isometry while A is not bounded below, so A and U_∞ have unequal approximate point spectra and thus are not similar to each other.

PROOF: Any direct sum of subnormal operators is itself subnormal, and $A \sim\sim U_\infty$ by Example 2.3 and Proposition 1.2. It is obvious that U_∞ is an isometry, and A cannot be bounded below since the "greatest lower bound" of A_n is $\frac{1}{\sqrt{1+n}}$, which tends to 0 as $n \to \infty$. (More detailed calculations show that the approximate point spectrum of A is exactly the set $T \cup \{0\}$.) ∎

We close this section by mentioning two results which, while not dealing with normal operators, give further instances where one or both of the implications $A \cong B \Rightarrow A \sim B \Rightarrow A \sim\sim B$ may be reversed. Hoover [16 , Theorem 1.9] has shown that quasi-similar isometries are unitarily equivalent, and Lambert [18 , Theorem I.1] has shown that if two unilateral weighted shifts with non-zero weights are quasi-similar then

they must be similar. Note that Proposition 1.1 shows that Hoover's theorem (unlike Corollary 2.2A) becomes false if we assume merely that one operator is a quasi-affine transform of the other instead of being quasi-similar to it. Note also that the operators A and B in Example 1.3 are unilateral weighted shifts, but Lambert's theorem does not apply to them since they have many weights equal to zero.

SECTION 3 Hyponormal operators

We have already seen that quasi-similar operators need not have equal spectra (Example 1.3). Quasi-similar normal operators do, of course, have equal spectra, since they have to be unitarily equivalent (Corollary 2.2A). Since quasi-similar subnormal operators need not be unitarily equivalent, or even similar, and since they may have unequal approximate point spectra (Example 2.4), it is remarkable that they must, in fact, have equal spectra. Equally remarkable is the extreme simplicity of the proof of this fact. The proof works just as well for hyponormal operators, and so we present it in that generality.

LEMMA 3.1α Let $B: K \to K$ be hyponormal and let $\{g_n\}_{n=0}^{\infty}$ be a sequence in K such that $Bg_{n+1} = g_n$ for all $n \geq 0$. Then either $||g_0|| \geq ||g_1|| \geq ||g_2|| \geq \ldots$ or else $||g_n|| \to \infty$ as $n \to \infty$.

PROOF: For any $g \in K$, $||Bg|| = \sqrt{(Bg, Bg)} = \sqrt{(B^*Bg, g)} \leq \sqrt{||B^*Bg|| \cdot ||g||}$ $\leq \sqrt{||B^2g|| \cdot ||g||} \leq \frac{1}{2}(||B^2g|| + ||g||)$. Letting $g = g_{n+2}$ we see that $||g_{n+1}|| \leq \frac{1}{2}(||g_n|| + ||g_{n+2}||)$, so the sequence $\{||g_n||\}_{n=0}^{\infty}$ is convex and the conclusion follows at once. ∎

LEMMA 3.1β Let $A: H \to H$ be invertible, $B: K \to K$ be hyponormal, and $X: H \to K$ satisfy $XA = BX$. Then $||XA^{-1}f|| \leq ||A^{-1}|| \cdot ||Xf||$ for all $f \in H$.

PROOF: Assume with no loss of generality that $\dim H \geq 1$, and let $c = ||A^{-1}|| > 0$. Fix $f \in H$ and define $g_n = c^{-n} XA^{-n}f$ for $n \geq 0$. Then

$cBg_{n+1} = g_n$ and $||g_n|| \leq ||X|| \cdot ||f||$ for all $n \geq 0$. Since cB is hyponormal, $||g_1|| \leq ||g_0||$ by Lemma 3.1α. That is,

$$||c^{-1}XA^{-1}f|| \leq ||Xf|| . \blacksquare$$

THEOREM 3.1 If A: H → H is a quasi-affine transform of B: K → K and B is hyponormal, then $\sigma(B) \subseteq \sigma(A)$.

PROOF: Again assume dim H ≥ 1. Let XA = BX, where X is quasi-invertible, and suppose $\lambda \notin \sigma(A)$. Then $A-\lambda$ is invertible, $B-\lambda$ is hyponormal, and $X(A-\lambda) = (B-\lambda)X$, so by Lemma 3.1 β, $||(B-\lambda)X(A-\lambda)^{-1}f|| = ||Xf||$

$\geq ||X(A-\lambda)^{-1}f|| / ||(A-\lambda)^{-1}||$ for all f ∈ H. Thus $B-\lambda$ is bounded below on the range of X, and hence on all of K. And $B-\lambda$ clearly has dense range. Hence $B-\lambda$ is invertible and $\lambda \notin \sigma(B)$. ∎

Various corollaries follow easily from Theorem 3.1.

COROLLARY 3.1A Quasi-similar hyponormal operators have equal spectra. ∎

COROLLARY 3.1B Quasi-similar semi-normal operators have equal spectra.

PROOF: In case both operators are hyponormal or both are co-hyponormal this follows from Corollary 3.1A. In case one operator, say A, is co-hyponormal and the other, say B, is hyponormal, then Theorem 3.1 implies $\sigma(A) = \sigma(B)$ if we merely assume A is a quasi-affine transform of B. ∎

In light of Proposition 2.2 there seems some hope that if A is co-hyponormal, B is hyponormal, and A is a quasi-affine transform of B, then A and B must actually be unitarily equivalent. While I cannot prove this (or present a counter-example), some special cases are easy. One of the easiest is:

COROLLARY 3.1C If the unitary operator A is a quasi-affine transform of the hyponormal operator B, then B is unitary and $A \cong B$.

PROOF: By Theorem 3.1, $\sigma(B) \subseteq T$. Thus B is invertible and $||B|| \leq 1$, since the norm of a hyponormal operator is equal to its spectral radius [14 , Problem 162] . By the spectral mapping theorem $\sigma(B^{-1}) \subseteq T$, so $||B^{-1}|| \leq 1$ too, since B^{-1} is hyponormal [20 , Theorem 3.10.1] . The two inequalities $||B|| \leq 1$ and $||B^{-1}|| \leq 1$ imply that B is unitary, so $A \cong B$ by Proposition 2.2. ∎

In Example 1.3 the operator B is compact but A is not. Thus quasi-similarity need not, in general, preserve compactness. But for semi-normal operators it does.

COROLLARY 3.1D If A and B are quasi-similar semi-normal operators, then A is compact if and only if B is compact.

PROOF: Suppose one of the operators, say B, is compact. Then $\sigma(B)$ has planar Lebesque measure 0, and so does $\sigma(A)$ since $\sigma(A) = \sigma(B)$ by Corollary 3.1B. By a theorem of Putnam [21] this implies that A and B are normal, so $A \cong B$ by Corollary 2.2A. Thus A is compact. ∎

It seems only fair to point out that Corollary 3.1D has a more elementary proof, based on the fact that if a hyponormal operator A is quasi-similar to a diagonal operator B, then A must be unitarily equivalent to B.

CHAPTER II

SUBNORMAL OPERATORS QUASI-SIMILAR TO THE SHIFT

In this chapter our perspective changes from the general to the special, and the techniques required begin to draw on measure theory and the theory of analytic functions. The subject matter is the class of subnormal operators quasi-similar to the unilateral shift, and the main results are these: This class of operators is completely characterized in measure theoretic terms (Section 4), the similarity subclasses are distinguished (Section 6), and the invariant subspaces of each such operator are determined (Section 7).

SECTION 4 Operators of type S

In this section we will give a necessary and sufficient condition for a subnormal operator to be quasi-similar to the unilateral shift. (Note that the corresponding problem for the bilateral shift is solved by Proposition 2.2 or, better, by Corollary 3.1C: If B is hyponormal, $B \sim\sim W_m$ if and only if $B \cong W_m$.) We first show that it suffices to consider only operators of the form U_μ, and then we express the condition for quasi-similarity to U_m in terms of μ.

If $A: H \to H$ and $f \in H$, then f is called a cyclic vector for A iff the smallest closed subspace of H which contains f and is invariant under A is H itself. An operator is cyclic iff it has a cyclic vector.

PROPOSITION 4.1 For any measure μ, U_μ is a cyclic subnormal operator and W_μ is its minimal normal extension.

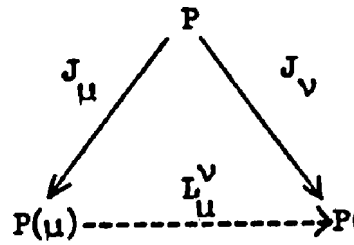
PROOF: It is clear that W_μ is normal and thus U_μ is subnormal, and $1 \in H^2(\mu)$ is a cyclic vector for U_μ. To see that W_μ is the minimal normal extension of U_μ, assume that M is a reducing subspace of W_μ such that $H^2(\mu) \subseteq M \subseteq L^2(\mu)$. Since $1 \in M$ and $z \cdot M \subseteq M$ and $\bar{z} \cdot M \subseteq M$, it follows that $p(z,\bar{z}) \in M$ for any polynomial p of two variables. By the Stone-Weierstrass theorem, M contains all functions continuous on supp μ, and thus all of $L^2(\mu)$, so $M = L^2(\mu)$. ∎

It is easy to see that if two operators are quasi-similar and one is cyclic, then so is the other. Thus in the problem of characterizing all subnormal operators quasi-similar to the shift it will suffice to consider only cyclic subnormal operators. We next show that it suffices to consider only operators of the form U_μ.

PROPOSITION 4.2 (Bram [2]) If A is a cyclic subnormal operator, then there exists a measure μ such that $A \cong U_\mu$.

PROOF: Assume A: $H \to H$ is cyclic, $f_0 \in H$ is one of its cyclic vectors, and B: $K \to K$ is its minimal normal extension. By minimality, f_0 is *-cyclic for B (that is, the smallest closed subspace of K which contains f_0 and reduces B is K), so the spectral theorem [13] implies the existence of a measure μ and a unitary map V: $K \to L^2(\mu)$ such that $Vf_0 = 1$ and $VB = W_\mu V$. Now consider $VH \subseteq L^2(\mu)$. Clearly $1 \in VH$ and $W_\mu(VH) \subseteq VH$ (since $BH \subseteq H$), so $H^2(\mu) \subseteq VH$. From the fact that f_0 is cyclic for A it follows easily that $H^2(\mu) = VH$, and so $A \cong U$ (via V_1) where V_1: $H \to H^2(\mu)$ is the restriction of V. ∎

For measures μ and ν let L_μ^ν : $P(\mu) \to P(\nu)$ be the (not necessarily bounded) linear transformation, if such exists, which makes the diagram



commute, where J_μ and J_ν are the maps taking each polynomial in P to the equivalence class it determines. Thus if p is a polynomial, L_μ^ν takes "p as a member of $P(\mu)$" to "p as a member of $P(\nu)$." Clearly ker J_μ [ker J_ν] is the set of polynomials vanishing on supp μ [supp ν], so L_μ^ν exists if and only if supp μ is an infinite set or supp $\mu \supseteq$ supp ν, and if it exists it is uniquely determined. If L_μ^ν exists let $||L_\mu^\nu||$ be its norm (possibly $= \infty$) computed using the inner-product-derived norms on $P(\mu)$ and $P(\nu)$, so (if $\mu \neq 0$)

$$||L_\mu^\nu||^2 = \sup \frac{\int |p|^2 d\nu}{\int |p|^2 d\mu}$$, where the supremum is taken over all polynomials p such that $\int |p|^2 d\mu \neq 0$. Define $\Lambda_\mu^\nu : H^2(\mu) \to H^2(\nu)$ to be the (unique) continuous extension of L_μ^ν, if such exists. Thus Λ_μ^ν exists if and only if $||L_\mu^\nu|| < \infty$. (The statement "$||L_\mu^\nu|| < \infty$" is always to be construed as an abbreviation for "L_μ^ν exists and $||L_\mu^\nu|| < \infty$.")

PROPOSITION 4.3 For any measures μ and ν :

(1) If $\nu \leq \mu$, then $\| L_\mu^\nu \| \leq 1$.

(2) If Λ_μ^ν exists, then $\Lambda_\mu^\nu U_\mu = U_\nu \Lambda_\mu^\nu$.

(3) If it exists, Λ_μ^ν has dense range.

(4) If Λ_μ^ν exists and $\nu << \mu$, then $\Lambda_\mu^\nu f = f$ a.e. $d\nu$ for every $f \in H^2(\mu)$.

PROOF: Items (1), (2), and (3) follow easily from the definitions. To prove (4), let $f \in H^2(\mu)$ and let $\{p_n\}$ be a sequence of polynomials such that $p_n \to f$ in $H^2(\mu)$-norm. By the continuity of Λ_μ^ν and the fact that $\Lambda_\mu^\nu p_n = p_n$, $p_n \to \Lambda_\mu^\nu f$ in $H^2(\nu)$-norm. There is a subsequence $\{p_{n_k}\}$ such that $p_{n_k} \to f$ a.e. $d\mu$ and $p_{n_k} \to \Lambda_\mu^\nu f$ a.e. $d\nu$, and since $\nu << \mu$ this implies that $\Lambda_\mu^\nu f = f$ a.e. $d\nu$. ∎

Let D be the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and let $\bar{D}$ be the closed unit disc, $\bar{D} = D \cup T = \{ z \in \mathbb{C} : |z| \leq 1 \}$. For $0 < p \leq \infty$ let H^p denote the Hardy space of certain analytic functions on D; in particular, H^∞ is the Banach space of all bounded analytic functions on D, with the supremum norm, and H^2 is the Hilbert space of

analytic functions f such that $\lim_{r \to 1-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty$,

with the inner product $(f,g) = \lim_{r \to 1-} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} d\theta$.

References [9] and [15] may be consulted for the theory of these spaces, which is assumed known for much of what follows. Here are some of the results we will use: all functions in any H^p space may be considered

defined, not only on D, but also a.e. dm on T, via radial limits; each function is uniquely determined by its radial limits, thus making it possible to "identify" H^p as a subspace of $L^p(m)$; in the case p =2, H^2 corresponds to the space $H^2(m)$, and so U_m is (unitarily equivalent to) multiplication by z on H^2. We will also use the theory of the so-called "inner" and "outer" functions.

For any measure τ carried by D, define a function R_τ on T by

$$R_\tau(e^{i\theta}) = \int_D \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) d\tau(z)$$

$$= \int_D P_{|z|}(\theta - \arg z)\, d\tau(z),$$

where "P" denotes the Poisson kernel, $P_r(\theta) \quad \dfrac{1-r^2}{1-2r\cos\theta + r^2}$

for $0 \leq r < 1$.

The next result shows how R_τ may be used to replace a measure τ carried by D with a measure on T. This is the idea of balayage familiar in the theory of harmonic functions. Here we have inequality instead of equality since $|f|^2$ need not be harmonic, but only subharmonic.

LEMMA 4.4α If τ is carried by D, then $0 \leq R_\tau < \infty$ a.e. dm on T and $R_\tau \in L^1(m)$, and

$$\int_D |f|^2 d\tau \leq \int_T |f|^2 R_\tau\, dm < \infty \quad \text{for every } f \in H^\infty .$$

PROOF: Using the positivity of the Poisson kernel and a form of Fubini's theorem, we have

$$\int_T R_\tau \, dm = \int_T \int_D \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right) d\tau(z) \, dm(\zeta)$$

$$= \int_D \int_T \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right) dm(\zeta) \, d\tau(z)$$

$$= \int_D 1 \, d\tau(z) = \tau(D) < \infty,$$

so $R_\tau \in L^1(m)$, and thus $0 \leq R_\tau < \infty$ a.e. dm.

Now for any $f \in H^\infty$,

$$\int_D |f|^2 d\tau = \int_D \left| \int_T f(\zeta)^2 \, \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right) dm(\zeta) \right| d\tau(z)$$

$$\leq \int_D \int_T |f(\zeta)|^2 \, \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right) dm(\zeta) \, d\tau(z)$$

$$= \int_T |f(\zeta)|^2 \int_D \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right) d\tau(z) \, dm(\zeta)$$

$$= \int_T |f|^2 R_\tau dm,$$

and this last integral is finite since $|f|^2 \in L^\infty(m)$ and $R_\tau \in L^1(m)$. ∎

If μ is any measure and E is a Borel set, let $\mu|E$ denote the restriction of μ to E, that is, the measure μ' is defined by $\mu'(F) = \mu(F \cap E)$ for every Borel set $F \subseteq \mathbb{C}$.

We shall say that a measure μ is <u>of type S</u> (for "shift") iff

(i) supp $\mu \subseteq \overline{D}$;

(ii) $\mu_0 << m$, where $\mu_0 = \mu|T$;

and (iii) $\log \frac{d\mu_0}{dm} \in L^1(m)$.

We shall say that an operator A is <u>of type S</u> iff there exists a measure μ of type S such that $A \cong U_\mu$. In particular, every operator of type S is a cyclic subnormal operator. Furthermore, every operator of type S is quasi-similar to the shift. Before proving this it is convenient to state two small lemmas.

<u>LEMMA 4.4β</u> If μ is of type S, then $H^\infty \subseteq H^2(\mu)$.

<u>PROOF:</u> Assume μ is of type S and let $\varphi \in H^\infty$. Define $\varphi_n(z) = \varphi((1-\frac{1}{n})z)$ for n = 1,2,3, ... and then choose polynomials p_n such that $|\varphi_n(z)-p_n(z)| \leq \frac{1}{n}$ for all $z \in \bar{D}$. (This is possible since the power series expansion of φ_n about 0 has radius of convergence > 1.) By Fatou's theorem [15 , page 34] and the bounded convergence theorem, $p_n \to \varphi$ in $L^2(\mu)$-norm, and so $\varphi \in H^2(\mu)$. ∎

<u>LEMMA 4.4γ</u> If μ is of type S, $g \in H^2(\mu)$, and $g = 0$ a.e. dm, then $g = 0$ a.e. $d\mu$.

<u>PROOF:</u> Write $\mu = \mu_0 + \tau$ where $\mu_0 = \mu|T$ and $\tau = \mu|D$, and let $\{p_n\}$ be a sequence of polynomials such that $p_n \to g$ in $L^2(\mu)$-norm and also a.e. $d\mu$. Since $\mu_0 \leq \mu$, $p_n \to g$ in $L^2(\mu_0)$-norm, which means that $p_n \to 0$ in $L^2(\mu_0)$-norm (since $g = 0$ a.e. dm and $\mu_0 << m$). Hence $p_n(\zeta) \to 0$ for every $\zeta \in D$ by [15 , pages 58-59, problem 4] . That implies that $g = 0$ a.e. $d\tau$, and we already knew $g = 0$ a.e. $d\mu_0$, so $g = 0$ a.e. $d\mu$. ∎

THEOREM 4.4 If u is of type S, then U_μ is quasi-similar to U_m.

PROOF: Assume μ is of type S and write $d\mu = h\,dm + d\tau$ where τ is carried by D, $h \in L^1(m)$, and $\log h \in L^1(m)$.

We have $\frac{1}{2}\log h = \log\sqrt{h} \leq \log\sqrt{1+h+R_\tau} \leq \frac{1}{2}(h+R_\tau)$,

and $\frac{1}{2}\log h$ and $\frac{1}{2}(h+R_\tau)$ are both in $L^1(m)$, so $\log\sqrt{h}$ and $\log\sqrt{1+h+R_\tau}$ are in $L^1(m)$. Therefore we can choose outer functions ψ and φ whose boundary values satisfy $|\psi| = \sqrt{h}$ and $|\varphi| = \frac{1}{\sqrt{1+h+R_\tau}}$ a.e. dm on T.

Define $\Psi : L^2(\mu) \to L^2(m)$ to be multiplication by ψ. For any $g \in L^2(\mu)$, $\|\Psi g\|^2_{L^2(m)} = \int_T |\psi \cdot g|^2\,dm = \int_T |g|^2 h\,dm$

$\leq \int_{\overline{D}} |g|^2\,d\mu = \|g\|^2_{L^2(\mu)}$, so Ψ is well-defined and $\|\Psi\| \leq 1$.

Now ψ is an outer function and $\int_T |\psi|^2\,dm = \int_T h\,dm < \infty$, so

$\psi = \Psi 1 \in H^2(m)$, so $\Psi p \in H^2(m)$ for all $p \in P(\mu)$. Since $P(\mu)$ is dense in $H^2(\mu)$ it follows that $\Psi H^2(\mu) \subseteq H^2(m)$. Let $X: H^2(\mu) \to H^2(m)$ be the restriction of Ψ.

Define $\Phi : H^2 \to L^2(\mu)$ to be multiplication by φ. For any $f \in H^2$, $\|\Phi f\|^2_{L^2(\mu)} = \int_T |\varphi \cdot f|^2 h\,dm + \int_D |\varphi \cdot f|^2\,d\tau$

$\leq \int_T |\varphi|^2 \cdot |f|^2 \cdot (h+R_\tau)\,dm$ (by Lemma 4.4 a)

$= \int_T |f|^2 \frac{h+R_\tau}{1+h+R_\tau}\,dm \leq \int_T |f|^2\,dm = \|f\|^2_{H^2}$,

so Φ is well-defined and $||\Phi|| \leq 1$. Now $\varphi \in H^\infty$, so by Lemma 4.4 β, $\varphi = \Phi 1 \in H^2(\mu)$, so $\Phi p \in H^2(\mu)$ for every $p \in P$. Since P is dense in H^2, $\Phi H^2 \subseteq H^2(\mu)$. Let Y: $H^2 \to H^2(\mu)$ be the restriction of Φ.

It is clear that $XU_\mu = U_m X$ and $YU_m = U_\mu Y$, where in the latter equation we are identifying U_m as multiplication by z on H^2. To finish it will suffice to show that X and Y are quasi-invertible.

If $g \in H^2(\mu)$ and $Xg = 0$, then $\psi \cdot g = 0$ a.e. dm, so $g = 0$ a.e. dm, and thus $g = 0$ a.e. $d\mu$ by Lemma 4.4 γ. Therefore X is one-to-one. That the range of X is dense in $H^2(m)$ follows from the fact that ψ is an outer function in H^2.

If $f \in H^2$ and $Yf = 0$, then $\varphi \cdot f = 0$ a.e. $d\mu$, so $f = 0$ a.e. dm, whence $f \equiv 0$ on D. This shows that Y is one-to-one. Since φ is outer and $\int_T |\frac{1}{\varphi}|^2 dm = \int_T (1 + h + R_\tau) dm < \infty$, it follows from [9 , Theorem 2.11] that $1/\varphi \in H^2$. Thus $p/\varphi \in H^2$ for every $p \in P$ and so the range of Y is dense in $H^2(\mu)$, since $Y(p/\varphi) = p$. ∎

I would like to quote the next lemma as "a standard result in topological measure theory", but being unable to find it in the standard references, I present it with proof. The proof below is essentially the proof of Lemma II. 7.4 in [10].

LEMMA 4.5α Let μ and ν be positive Baire measures (or regular Borel measures) on a compact Hausdorff space S, and assume $\mu \perp \nu$. Then there exists an F_σ Baire set E such that $\mu(S \setminus E) = \nu(E) = 0$.

PROOF: Since $\mu \perp \nu$ there exists a Baire set E_0 such that $\mu(S \setminus E_0) = \nu(E_0) = 0$. By Lusin's theorem C(S) is dense in $L^1(\mu + \nu)$, and in fact we can get functions $f_n \in C(S)$, n = 1,2,3, ..., such that

$0 \le f_n \le 1$ on S and $\| f_n - g \|_{L^1(\mu+\nu)} \le \frac{1}{n^2}$, where g is the characteristic function of E_0. From the calculation

$$\int \sum_{n=1}^{\infty} (1-f_n)d\mu + \int \sum_{n=1}^{\infty} f_n d\nu = \sum_{n=1}^{\infty} \left(\int (1-f_n)d\mu + \int f_n d\nu\right)$$

$$= \sum_{n=1}^{\infty} \| f_n - g \|_{L^1(\mu+\nu)} \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

we see that $f_n \to 1$ a.e. $d\mu$ and $f_n \to 0$ a.e. $d\nu$.

Now let $E = \{ x \in S: \liminf_{n \to \infty} f_n(x) > 0 \}$

$$= \bigcup_{j \ge 1} \bigcup_{k \ge 1} \bigcap_{n \ge k} \{x \in S: f_n(x) \ge \frac{1}{j}\},$$

so E is an F_σ Baire set, and $\mu(S \setminus E) = \nu(E) = 0$ since $f_n \to 1$ a.e. $d\mu$ and $f_n \to 0$ a.e. $d\nu$. ∎

LEMMA 4.5 β Let supp $\mu \subseteq \overline{D}$ and let μ_s be the singular part of $\mu | T$ with respect to m. Then there exists an F_σ-set $E \subseteq T$ such that $m(E) = 0$ and μ_s is carried by E. If g is the characteristic function of E, then $\bar{z}^N g \in H^2(\mu)$ for each $N \ge 0$.

PROOF: The existence of the F_σ-set E follows from Lemma 4.5α ; it does not matter that μ_s and m are literally measures in the non-compact space $\mathbb{C}$, since they have compact supports contained in T. Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ be closed subsets of T with $E = \bigcup_{n=1}^{\infty} E_n$, and let g_n denote the characteristic function of E_n. Since g_n is upper semicontinuous on T there is a real-valued continuous function h_n on T such that $g_n \le h_n \le 1$ on T and $\int (h_n - g_n)\, dm \le \frac{1}{n^3}$, that is, $\int h_n\, dm \le \frac{1}{n^3}$. Since Re P is uniformly

dense in the space of real continuous functions on T, there exist $p_n \in P$ such that $\sup_T |\operatorname{Re} p_n - h_n| \leq \frac{1}{n^3}$, so $g_n - \frac{1}{n^3} \leq \operatorname{Re} p_n \leq h_n + \frac{1}{n^3}$ on T and $-\frac{1}{n^3} \leq \int (\operatorname{Re} p_n)\, dm \leq \frac{2}{n^3}$, that is, $-\frac{1}{n^3} \leq \operatorname{Re} p_n(0) \leq \frac{2}{n^3}$. By adding a pure imaginary constant we shall assume with no loss of generality that $p_n(0)$ is real, so $-\frac{1}{n^3} \leq p_n(0) \leq \frac{2}{n^3}$.

Let $f_n = e^{-np_n}$. We need four facts about the f_n's:

(1) $f_n \in H^2(\mu)$; (2) $|f_n| \leq e^{1/n^2} \leq e$; (3) $f_n \to 0$ on E;

(4) $f_n \to 1$ a.e. dm on T. Fact (1) is clear since f_n is an entire function; (2) follows from the maximum modulus principle and the fact that $|f_n| = e^{-n \operatorname{Re} p_n} \leq e^{1/n^2}$ on T; (3) is clear since $|f_n(z)| = e^{-n \cdot \operatorname{Re} p_n(z)} \leq e^{-n(1-\frac{1}{n^3})}$ for all $z \in E_k$ for $k \leq n$; and (4) may be proven as follows. Since $f_n(0) = e^{-np_n(0)} \geq e^{-2/n^2} \geq 1 - \frac{2}{n^2}$, we have

$$\begin{aligned}\int |1-f_n|^2\, dm &= \int (1 - f_n - \bar{f}_n + |f_n|^2)\, dm \\ &\leq 1 - 2\operatorname{Re} \int f_n\, dm + e^{2/n^2} \\ &= 2 - 2f_n(0) + (e^{2/n^2} - 1) \\ &\leq 2 - 2(1 - \tfrac{2}{n^2}) + \tfrac{7}{n^2} = \tfrac{11}{n^2}.\end{aligned}$$

Thus $\int \sum_{n=1}^{\infty} |1-f_n|^2\, dm = \sum_{n=1}^{\infty} \int |1-f_n|^2\, dm \leq \sum_{n=1}^{\infty} \frac{11}{n^2} < \infty$, so $\sum_{n=1}^{\infty} |1-f_n|^2 < \infty$ a.e. dm and therefore $f_n \to 1$ a.e. dm.

Using (2), (4), and the dominated convergence theorem we see that

$f_n \to 1$ in $L^2(m)$-norm, that is, in H^2-norm, and so $f_n \to 1$ on D as well as a.e. dm. Combining this with (3) and using the dominated convergence theorem again, we see that $f_n \to 1 - g$ in $H^2(\mu)$-norm, so $1 - g \in H^2(\mu)$ by (1). Thus $g \in H^2(\mu)$.

Now fix $N \geq 0$ and use Kolmogorov and Kreĭn's generalization of Szegö's theorem to get polynomials g_n such that $g_n \to \bar{z}^N$ in $L^2(\mu_s)$-norm. Since $g_n g \in H^2(\mu)$ and $\|g_n g - \bar{z}^N g\|^2_{L^2(\mu)} = \int |g_n - \bar{z}^N|^2 |g|^2 d\mu = \int |g_n - \bar{z}^N|^2 d\mu_s$ $= \|g_n - \bar{z}^N\|^2_{L^2(\mu_s)} \to 0$ as $n \to \infty$, we have $\bar{z}^N g \in H^2(\mu)$. ∎

Parts of the proof of Lemma 4.5β, and also of Theorem 5.3 and Lemma 7.3α below, are based on [10 , pages 42-43].

Note that Lemma 4.5β essentially shows that the space $H^2(\mu)$ can be regarded as the direct sum $H^2(\mu-\mu_s) \oplus L^2(\mu_s)$, and that the operator U_μ has a corresponding direct sum decomposition.

LEMMA 4.5γ Let supp $\mu \subseteq \bar{D}$ and let $\mu_0 = \mu | T$, and assume $X: H^2(\mu) \to H^2(m)$ satisfies $XU_\mu = U_m X$. Then there exists a unique $X_0: H^2(\mu_0) \to H^2(m)$ such $X = X_0 \Lambda_\mu^{\mu_0}$. Also, $X_0 U_{\mu_0} = U_m X_0$ and $\|X_0\| = \|X\|$.

PROOF: Letting $\tau = \mu | D$ we have $\|Xp\|^2_{H^2(m)} = \|U_m^n Xp\|^2_{H^2(m)}$ $= \|XU_\mu^n p\|^2_{H^2(m)} \leq \|X\|^2 \|U_\mu^n p\|^2_{H^2(\mu)} = \|X\|^2 \int |z^n p|^2 d\mu$ $= \|X\|^2 (\int |p|^2 d\mu_0 + \int |z|^{2n} |p|^2 d\tau)$ for all $p \in P$. Letting $n \to \infty$ and applying the dominated convergence theorem we see that $\|Xp\|_{H^2(m)} \leq \|X\| \cdot \|p\|_{H^2(\mu_0)}$. Thus the map $X_1: P(\mu_0) \to H^2(m)$ defined by $X_1 p = Xp$ for every $p \in P$ is well-defined and bounded, and so has a

continuous extension X_0: $H^2(\mu_0) \to H^2(m)$, and $||X_0|| \leq ||X||$. It is clear that $X = X_0 \Lambda_\mu^{\mu_0}$, and so $||X|| = ||X_0 \Lambda_\mu^{\mu_0}|| \leq ||X_0|| \cdot ||\Lambda_\mu^{\mu_0}|| \leq ||X_0||$ by Proposition 4.3(1). Thus $||X_0|| = ||X||$. That X_0 is the only bounded operator satisfying $X = X_0 \Lambda_\mu^{\mu_0}$ follows from Proposition 4.3(3). Finally for every $p \in P$ we have $X_0 U_{\mu_0} p = X_0(zp) = X(zp) = XU_\mu p = U_m Xp = U_m X_0 p$, so $X_0 U_{\mu_0} = U_m X_0$. ∎

THEOREM 4.5 The operator U_μ is quasi-similar to U_m if and only if μ is of type S.

PROOF: Theorem 4.4 shows that if μ is of type S, then $U_\mu \sim\sim U_m$. Conversely, assume that $U_\mu \sim\sim U_m$. By Corollary 3.1A and the equality of norm and spectral radius, $||U_\mu|| \leq 1$. In particular, $||U_\mu^n 1||$ remains bounded as $n \to \infty$. Thus supp $\mu \subseteq \overline{D}$, since otherwise Fatou's lemma would imply that $||U_\mu^n 1|| \to \infty$ as $n \to \infty$.

From here on we need only assume that U_μ is a quasi-affine transform of U_m and that supp $\mu \subseteq \overline{D}$.

Write $\mu = \mu_0 + \tau$ where $\mu_0 = \mu | T$ and $\tau = \mu | D$, and let $\mu_0 = \mu_a + \mu_s$ where $\mu_a \ll m$ and $\mu_s \perp m$.

We must show that $\mu_s = 0$ and $\log \frac{d\mu_a}{dm} \in L^1(m)$.

Let X: $H^2(\mu) \to H^2(m)$ be quasi-invertible and satisfy $XU_\mu = U_m X$. Let $E \subseteq T$ be an F_σ-set carrying μ_s and such that $m(E) = 0$, as in Lemma 4.5β, and let $g \in H^2(\mu)$ be the characteristic function of E. Using Lemma 4.5β we compute $||Xg|| = \lim_{n\to\infty} ||Xg||$

$$= \lim_{n\to\infty} ||XU_\mu^n \bar{z}^n g|| = \lim_{n\to\infty} ||U_m^n X \bar{z}^n g||$$

$$= \lim_{n\to\infty} ||X \bar{z}^n g|| = \lim_{n\to\infty} ||U_m^{*n} U_m^n X \bar{z}^n g||$$

$$= \lim_{n\to\infty} ||U_m^{*n} X U_\mu^n \bar{z}^n g|| = \lim_{n\to\infty} ||U_m^{*n} Xg|| = 0,$$

so $Xg = 0$. Since X is one-to-one, $g = 0$, and therefore $\mu_s = 0$ and $\mu_a = \mu_0$.

Now pretend that $\int \log \frac{d\mu_0}{dm} \cdot dm = -\infty$. Then by Szegö's theorem $H^2(\mu_0) = L^2(\mu_0)$, so $U_{\mu_0} = W_{\mu_0}$, a unitary operator. By Lemma 4.5γ there exists X_0: $L^2(\mu_0) \to H^2(m)$ such that $X_0 W_{\mu_0} = U_m X_0$ and $||X_0|| = ||X|| \neq 0$, but by Proposition 1.1(2), $X_0 = 0$, a contradiction. It follows that $\log \frac{d\mu_0}{dm} \in L^1(m)$. ∎

COROLLARY 4.5A A subnormal operator A is quasi-similar to the unilateral shift if and only if A is of type S.

PROOF: Immediate from Proposition 4.2 and Theorem 4.5. ∎

SECTION 5 Some functional Hilbert spaces

In our investigation of similarity and invariant subspaces in Sections 6 and 7 it will be useful to work with operators of type S which act on spaces of analytic functions. Therefore in this section we define, for each measure μ of type S, a functional Hilbert space $\tilde{H}^2(\mu)$ of functions analytic on D, and we prove that U_μ is unitarily equivalent to "multiplication by z" on $\tilde{H}^2(\mu)$.

A functional Hilbert space [14 , Problem 29] on a set Δ is a Hilbert space H whose elements are functions $f: \Delta \to \mathbb{C}$ such that, for every $\zeta \in \Delta$, the evaluation map $f \to f(\zeta)$ is a bounded linear functional. This implies that the linear structure of H is the natural one for a set of functions.

For any measure μ of type S let $\tilde{H}^2(\mu)$ be the set of all functions $f: D \to \mathbb{C}$ in the class N^+ such that $\int_{\overline{D}} |f|^2 d\mu < \infty$. Here N^+ is the set of all analytic functions of the form φ/ψ where φ and ψ are in H^∞ and ψ is outer; see [9]. The above integral makes sense since f necessarily has boundary values a.e. dm on T. For $f, g \in \tilde{H}^2(\mu)$ define the inner product $(f,g) = (f,g)_{\tilde{H}^2(\mu)} = \int_{\overline{D}} f \bar{g}\, d\mu$.

Note that $H^2 = \tilde{H}^2(m)$. In the proof of Theorem 4.4 we have already used the fact that U_m is unitarily equivalent to multiplication by z on H^2.

Clearly $\tilde{H}^2(\mu)$ is an inner product space. We next show that it is actually a functional Hilbert space on D.

PROPOSITION 5.1 If μ is of type S, then $\tilde{H}^2(\mu)$ is a functional Hilbert space on D.

PROOF: Write $\mu = \mu_0 + \tau$ where $\mu_0 = \mu \mid T$ and $\tau = \mu \mid D$, and note that μ_0

is also of type S. We will first show that the space $\tilde{H}^2(\mu_0)$ is complete. Let ψ be an outer function such that $|\psi|^2 = d\mu_0/dm$ a.e. dm on T. If $f \in \tilde{H}^2(\mu_0)$, then $\psi f \in N^+$ and $||\psi f||^2_{L^2(m)} = \int |\psi f|^2 dm = \int |f|^2 d\mu_0$ $= ||f||^2_{\tilde{H}^2(\mu_0)}$, so $\psi f \in H^2$ (see [9, Theorem 2.11]) and the map $X: \tilde{H}^2(\mu_0) \to H^2$ defined as multiplication by ψ is an isometry. For any $g \in H^2$, $g/\psi \in \tilde{H}^2(\mu_0)$ and $X(g/\psi) = g$, so X is onto. Since H^2 is complete, that implies that $\tilde{H}^2(\mu_0)$ is complete.

Let $f \in \tilde{H}^2(\mu_0)$ and write $(\psi f)(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in D$, so $||\psi f||^2_{H^2} = \sum_{n=0}^{\infty} |a_n|^2$. By the Cauchy-Schwarz inequality

$$|\psi(z)||f(z)| = |(\psi f)(z)| = \left|\sum_{n=0}^{\infty} a_n z^n\right| \leq \sqrt{\sum_{n=0}^{\infty} |a_n|^2}\ \sqrt{\sum_{n=0}^{\infty} |z|^{2n}} =$$

$$= \frac{||\psi f||_{H^2}}{\sqrt{1-|z|^2}} = \frac{||f||_{\tilde{H}^2(\mu_0)}}{\sqrt{1-|z|^2}}, \text{ so for any } z \in D \text{ the map } f \to f(z)$$

is a bounded linear functional on $\tilde{H}^2(\mu_0)$ whose norm is $\leq \dfrac{1}{|\psi(z)|\sqrt{1-|z|^2}}$.

It is now easy to see that $\tilde{H}^2(\mu)$ itself is complete, for if $\{f_n\}$ is a Cauchy sequence in $\tilde{H}^2(\mu)$ the equation

$$||f_j - f_k||^2_{\tilde{H}^2(\mu)} = ||f_j - f_k||^2_{\tilde{H}^2(\mu_0)} + ||f_j - f_k||^2_{L^2(\tau)} \text{ shows that}$$

it is a Cauchy sequence in both $\tilde{H}^2(\mu_0)$ and $L^2(\tau)$, which are complete. Say $f_n \to g$ in $\tilde{H}^2(\mu_0)$-norm and $f_n \to h$ in $L^2(\tau)$-norm. By the boundedness of the evaluation functionals in $\tilde{H}^2(\mu_0)$-norm, $f_n \to g$ pointwise on D. But

some subsequence tends to h a.e. $d\tau$, so $h = g$ a.e. $d\tau$. Thus $f_n \to g$ in $\tilde{H}^2(\mu_0)$-norm and in $L^2(\tau)$-norm, and therefore in $\tilde{H}^2(\mu)$-norm. It follows that $\tilde{H}^2(\mu)$ is complete.

Finally, the evaluation functionals at points of D are bounded functionals on $\tilde{H}^2(\mu)$ since they are bounded on $\tilde{H}^2(\mu_0)$ and

$$\|f\|_{\tilde{H}^2(\mu_0)} \leq \|f\|_{\tilde{H}^2(\mu)} . \quad \blacksquare$$

For any μ of type S define $\tilde{U}_\mu : \tilde{H}^2(\mu) \to \tilde{H}^2(\mu)$ to be multiplication by z. It is clear that $\tilde{U}_\mu$ is an operator. We will show that $\tilde{U}_\mu$ is unitarily equivalent to U_μ by proving that P is dense in $\tilde{H}^2(\mu)$.

PROPOSITION 5.2 If μ is of type S and M is an invariant subspace of $\tilde{U}_\mu$, then $H^\infty \cdot M = M$.

PROOF: Since $1 \in H^\infty$ it is clear that $H^\infty \cdot M \supseteq M$.

To prove the reverse inclusion take any $\varphi \in H^\infty$, let $\varphi_n(z) = \varphi((1 - \frac{1}{n})z)$ for $n = 1,2,3, \ldots$, and choose polynomials p_n such that $|p_n(z) - \varphi_n(z)| \leq \frac{1}{n}$ for all $z \in \bar{D}$. Clearly $|p_n(z)| \leq \|\varphi\|_{H^\infty} + 1$ and $p_n(z) \to \varphi(z)$ as $n \to \infty$ for all $z \in D$. By Fatou's theorem, $p_n \to \varphi$ a.e. dm on T also. It follows from the dominated convergence theorem that $(p_n f, g)_{\tilde{H}^2(\mu)} \to (\varphi f, g)_{\tilde{H}^2(\mu)}$ for every $f, g \in \tilde{H}^2(\mu)$.

Now take any $f \in M$. Then $p_n f \in M$, so $(p_n f, g) = 0$ for every $g \in M^\perp$. Letting $n \to \infty$ we see that $(\varphi f, g) = 0$ for every $g \in M^\perp$, so $\varphi f \in M$. $\blacksquare$

THEOREM 5.3 If μ is of type S and M is an invariant subspace of $\tilde{U}_\mu$, then $[M \cap H^\infty]_{\tilde{H}^2(\mu)} = M$, where the brackets denote closure in $\tilde{H}^2(\mu)$-norm.

PROOF: The inclusion "$\subseteq$" is clear since M is closed. To prove the reverse inclusion we take any $f \in M$, assuming with no loss of generality that $f \not\equiv 0$, and then construct a sequence $\{f_n\}_{n=1}^{\infty}$ of bounded functions in M such that $f_n \to f$ in $\tilde{H}^2(\mu)$-norm.

The functions f_n may be defined as follows. Since $f \in N^+$, factor $f = \varphi \cdot g$ where φ is inner and g is the outer function $g = e^h$, with $h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log |f(e^{i\theta})| \cdot d\theta$. Write $H(e^{i\theta}) = \log |f(e^{i\theta})| = \log |g(e^{i\theta})|$. Since $H \in L^1(m)$ we can define $H_n(e^{i\theta}) = \min (c_n, H(e^{i\theta}))$ for $n = 1,2,3, \ldots$, where the constants c_n are chosen so that $||H-H_n||_{L^1(m)} \leq 2^{-n}$. We then define

$h_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} H_n(e^{i\theta})d\theta$, $g_n = e^{h_n}$, and finally $f_n = \varphi \cdot g_n$.

It remains only to show that $f_n \in M \cap H^\infty$ and that $f_n \to f$ in $\tilde{H}^2(\mu)$-norm.

Since $H_n \leq H$ on T, Re $h_n(z) \leq$ Re $h(z)$ for all $z \in D$, by the positivity of the Poisson kernel. This implies that $g_n/g = \exp(h_n-h)$ is an H^∞-function bounded by 1, whence $|g_n| \leq |g|$ on D and a.e. dm on T, and so $|g_n| \leq |g|$ a.e. $d\mu$. Multiplying by $|\varphi|$ we see that $|f_n| \leq |f|$ a.e. $d\mu$.

Also $H_n \leq c_n$ on T, so Re $h_n(z) \leq c_n$ on D and a.e. dm on T, so $|f_n| \leq |g_n| \leq e^{c_n}$ a.e. $d\mu$ and f_n, $g_n \in H^\infty$. Since $f_n = f \cdot (g_n/g)$ with $f \in M$ and $g_n/g \in H^\infty$, Proposition 5.2 implies that $f_n \in M \cap H^\infty$.

If we knew that $g_n \to g$ a.e. dm on T it would follow that $f_n \to f$ in $\tilde{H}^2(\mu)$-norm. For we know that $g \in L^2(\mu_0)$, where $\mu_0 = \mu|T$, and that $|g_n| \leq |g|$ a.e. $d\mu_0$. Thus if $g_n \to g$ a.e. dm, then $g_n \to g$ in $L^2(\mu_0)$-norm

by the dominated convergence theorem and the fact that $\mu_0 << m$, and so $g_n \to g$ in $\tilde{H}^2(\mu_0)$-norm. By Proposition 5.1 that implies that $g_n \to g$ pointwise on D and therefore $g_n \to g$ a.e. $d\mu$. Multiplying by φ we see that $f_n \to f$ a.e. $d\mu$. But $f \in L^2(\mu)$ and $|f_n| \le |f|$ a.e. $d\mu$, so $f_n \to f$ in $L^2(\mu)$-norm (by the dominated convergence theorem again), which means that $f_n \to f$ in $\tilde{H}^2(\mu)$-norm.

Thus we need only show that $g_n \to g$ a.e. dm on T. Since

$$0 \le h(0) - h_n(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (H(e^{i\theta}) - H_n(e^{i\theta}))\, d\theta = \| H - H_n \|_{L^1(m)} \le 2^{-n},$$

we have $g_n(0)/g(0) = \exp(h_n(0)-h(0)) \ge \exp(-2^{-n}) \ge 1 - 2^{-n}$. Now g_n/g is an H^∞-function bounded by 1, so $\| 1 - \frac{g_n}{g} \|^2_{L^2(m)}$

$$= \int (1 - 2\,\mathrm{Re}\,\frac{g_n}{g} + \left|\frac{g_n}{g}\right|^2)\, dm$$

$$\le 1 - 2\cdot\frac{g_n(0)}{g(0)} + 1 \le 2^{-n+1},$$

and therefore $\int \sum_{n=1}^{\infty} \left|1 - \frac{g_n}{g}\right|^2 dm = \sum_{n=1}^{\infty} \int \left|1 - \frac{g_n}{g}\right|^2 dm$

$$= \sum_{n=1}^{\infty} \| 1 - \frac{g_n}{g} \|^2_{L^2(m)}$$

$$\le \sum_{n=1}^{\infty} 2^{-n+1} = 2 < \infty.$$

Thus $\sum_{n=1}^{\infty} \left|1 - \frac{g_n}{g}\right|^2 < \infty$ a.e. dm on T, and that implies that $g_n \to g$ a.e. dm on T. ∎

Recall that P denotes the set of polynomials in one variable.

COROLLARY 5.3A If μ is of type S, then P is dense in $\tilde{H}^2(\mu)$.

PROOF: For any $f \in H^\infty$ we can construct polynomials p_n which tend to f boundedly a.e. $d\mu$, as in the proof of Proposition 5.2, and then the bounded convergence theorem implies that $p_n \to f$ in $\tilde{H}^2(\mu)$-norm. Thus $H^\infty \subseteq [P]_{\tilde{H}^2(\mu)}$, so it will suffice to show that H^∞ is dense in $\tilde{H}^2(\mu)$. This follows immediately upon taking $M = \tilde{H}^2(\mu)$ in Theorem 5.3. ∎

We remark in passing that other types of "H^2-spaces" are studied, and in some of them the polynomials are not dense.

COROLLARY 5.3B If μ is of type S, then U_μ and $\tilde{U}_\mu$ are unitarily equivalent.

PROOF: Let V: $H^2(\mu) \to \tilde{H}^2(\mu)$ be defined by letting $Vp = p$ for polynomials p and extending by continuity. Clearly V is an isometry and $VU_\mu = \tilde{U}_\mu V$. By Corollary 5.3A, V is unitary, so $U_\mu \cong \tilde{U}_\mu$ (via V). ∎

SECTION 6 Similarity and unitary equivalence

This section contains four theorems, each giving a necessary and sufficient condition for certain operators to be similar or unitarily equivalent. There is one theme common to the four theorems, namely, that there is one "natural" candidate to try to use as the intertwining map, and if it doesn't work (that is, if it is unbounded, or not invertible, or, in the case of unitary equivalence, not isometric), then nothing can work.

Note especially the crucial use of Lebesgue's dominated convergence theorem in Theorems 6.1,6.2, and 6.3, and compare with Lemma 4.5γ.

Throughout this section we will let $\mu_0 = \mu | T$, for any measure μ.

THEOREM 6.1 The operator U_μ is similar to an isometry if and only if $||L^\mu_{\mu_0}|| < \infty$, in which case U_μ is similar to the isometry U_{μ_0}.

PROOF: By Proposition 4.3, $\Lambda^{\mu_0}_\mu$ exists, has dense range, and satisfies $\Lambda^{\mu_0}_\mu U_\mu = U_{\mu_0} \Lambda^{\mu_0}_\mu$. If $||L^\mu_{\mu_0}|| < \infty$ then $\Lambda^{\mu_0}_\mu$ is bounded below and therefore U_μ is similar to the isometry U_{μ_0}.

Conversely, assume U_μ is similar to some isometry $V: H \to H$, say $U_\mu \sim V$(via S). Let $a = ||S^{-1}||$, $b = ||S||$. Then $||U^n_\mu f|| = ||S^{-1}V^nSf|| \leq ab||f||$ for all $n \geq 0$ and all $f \in H^2(\mu)$. Setting $f = 1$, this means that $\int |z|^{2n} d\mu(z) \leq a^2b^2 \int d\mu(z)$. Letting $n \to \infty$ and applying Fatou's lemma we see that supp $\mu \subseteq \overline{D}$. Now the

calculation $\|f\|_{H^2(\mu)} = \|S^{-1}Sf\|_{H^2(\mu)} \leq a\|Sf\|_H = a\|V^nSf\|_H$

$= a\|SU_\mu^n f\|_H \leq ab\|U_\mu^n f\|_{H^2(\mu)}$ is valid for all $n \geq 0$ and all

$f \in H^2(\mu)$, and $\|f\| \geq \|U_\mu f\| \geq \|U_\mu^2 f\| \geq \dots$, so

$\|f\|_{H^2(\mu)} \leq ab \lim_{n\to\infty} \|U_\mu^n f\|_{H^2(\mu)}$ for every $f \in H^2(\mu)$, and in particular for every polynomial. Therefore for every $p \in P$ we have

$$\|p\|^2_{H^2(\mu)} \leq a^2b^2 \lim_{n\to\infty} \|U_\mu^n p\|^2_{H^2(\mu)} = a^2b^2 \lim_{n\to\infty} \int_{\bar{D}} |z^n p|^2 d\mu$$

$$= a^2b^2 \lim_{n\to\infty} \left(\int_T |p|^2 d\mu_0 + \int_D |z|^{2n} |p|^2 d\mu\right)$$

$$= a^2b^2 \int_T |p|^2 d\mu_0 = a^2b^2 \|p\|^2_{H^2(\mu_0)}$$

by the dominated convergence theorem, and thus $\|L^\mu_{\mu_0}\| \leq ab < \infty$. ∎

<u>COROLLARY 6.1A</u> The operator U_μ is similar to U_m if and only if μ is of type S and $\|L^\mu_{\mu_0}\| < \infty$.

<u>PROOF</u>: If $U_\mu \sim U_m$ then μ is of type S by Theorem 4.5 and $\|L^\mu_{\mu_0}\| < \infty$ by Theorem 6.1.

Conversely, if $\|L^\mu_{\mu_0}\| < \infty$ then $U_\mu \sim U_{\mu_0}$ by Theorem 6.1, and if μ is of type S then $U_{\mu_0} \cong U_m$ by [15 , page 59, problem 7]. Thus $U_\mu \sim U_m$. ∎

It was shown in Example 2.4 that quasi-similar subnormal operators need not be similar, but the operators used there are not cyclic. We are now in a position to construct examples of cyclic subnormal operators which are quasi-similar, but not similar.

COROLLARY 6.1B Let $F: [0,1) \to [0,\infty)$ be an increasing function (that is, $x_1 \le x_2 \Rightarrow F(x_1) \le F(x_2)$) such that $\int_0^1 F(x)dx < \infty$, define $d\tau = F(x)dx$ on $0 \le x < 1$, and let $\mu = m + \tau$. Then $U_\mu \sim\sim U_m$, but $U_\mu \sim U_m$ if and only if $\lim_{x\to 1-} F(x) < \infty$.

PROOF: Since μ is of type S, Theorem 4.5 implies that $U_\mu \sim\sim U_m$. Now suppose $\lim_{x\to 1-} F(x) = c < \infty$, which implies that $F(x) \le c$ on $0 \le x < 1$. Then for any $p \in P$,

$$\begin{aligned} \|p\|^2_{H^2(\mu)} &= \int |p|^2 d\mu \\ &= \int |p|^2 dm + \int_0^1 |p|^2 F dx \\ &\le \int |p|^2 dm + c\int_{-1}^1 |p|^2 dx \\ &\le (1 + \tfrac{c}{2}) \int |p|^2 dm \\ &= (1 + \tfrac{c}{2}) \|p\|^2_{H^2(m)} , \end{aligned}$$

the second inequality being a consequence of the Fejer-Riesz inequality [9 , Theorem 3.13]. Thus $\|L^\mu_m\| \le \sqrt{1 + \frac{c}{2}} < \infty$, so $U_\mu \sim U_m$ by Corollary 6.1A.

Next, suppose $\lim_{x\to 1-} F(x) = \infty$. For $n \ge 1$ let

$p_n(z) = 1 + z + z^2 + \ldots + z^{n-1}$, so $p_n(z) = \frac{1 - z^n}{1-z}$ for $z \neq 1$,

$p_n(1) = n$, and $\| p_n \|^2_{H^2(m)} = n$ by the orthonormality of $\{z^k\}_{k=0}^{\infty}$.

Note that p_n is increasing on $[0,1]$ and $p_n(1 - \frac{1}{n}) = n[1-(1-\frac{1}{n})^n] \geq n(1-e^{-1})$.

Thus $\| p_n \|^2_{H^2(\mu)} = \| p_n \|^2_{H^2(m)} + \int_0^1 p_n(x)^2 F(x)dx$

$$\geq \int_{1-\frac{1}{n}}^{1} p_n(x)^2 F(x)dx$$

$$\geq \frac{1}{n} p_n(1-\tfrac{1}{n})^2 F(1-\tfrac{1}{n})$$

$$\geq n(1-e^{-1})^2 F(1-\tfrac{1}{n}),$$

so $\| L^{\mu}_{m} \| \geq \frac{\| p_n \|_{H^2(\mu)}}{\| p_n \|_{H^2(m)}} \geq (1-e^{-1}) \sqrt{F(1-\frac{1}{n})} \to \infty$ as $n \to \infty$.

Therefore $\| L^{\mu}_{m} \| = \infty$, so U_{μ} is not similar to U_m, by Corollary 6.1A. ∎

The next theorem gives a complete set of similarity invariants for operators U_{μ} where μ is of type S. For any measure μ of type S we will let ψ_{μ} denote the (unique) outer function (necessarily in H^2) such that $\psi_{\mu}(0) > 0$ and $d\mu_0/dm = |\psi_{\mu}|^2$ a.e. dm on T.

THEOREM 6.2 If μ and ν are of type S, then $U_{\mu} \sim U_{\nu}$ if and only if

$$\psi_{\mu} \cdot \widetilde{H^2}(\mu) = \psi_{\nu} \cdot \widetilde{H^2}(\nu).$$

PROOF: Assume that $\psi_\mu \cdot \tilde{H}^2(\mu) = \psi_\nu \cdot \tilde{H}^2(\nu)$ and define $S: \tilde{H}^2(\mu) \to \tilde{H}^2(\nu)$ by $Sf = \frac{\psi_\mu}{\psi_\nu} f$. Since $\tilde{H}^2(\nu) = \frac{\psi_\mu}{\psi_\nu} \cdot \tilde{H}^2(\mu)$, S is well-defined and one-to-one and onto. The boundedness of S (and of S^{-1}) follows from the closed graph theorem. For if $f_n \to f$ in $\tilde{H}^2(\mu)$-norm and $Sf_n \to g$ in $\tilde{H}^2(\nu)$-norm, then $f_n \to f$ and $\frac{\psi_\mu}{\psi_\nu} f_n \to g$ pointwise on D by Proposition 5.1 so $g = \frac{\psi_\mu}{\psi_\nu} f = Sf$. Clearly $S\tilde{U}_\mu = \tilde{U}_\nu S$, so $U_\mu \sim U_\nu$ by Corollary 5.3B.

Conversely, assume that $\tilde{U}_\mu \sim \tilde{U}_\nu$ (via S), and let $\varphi = S1 \in \tilde{H}^2(\nu) \subseteq N^+$. The calculation $Sp = Sp(\tilde{U}_\mu)1 = p(\tilde{U}_\nu)S1 = p(\tilde{U}_\nu)\varphi = \varphi \cdot p$, valid for every $p \in P$, actually proves that $Sf = \varphi \cdot f$ for every $f \in \tilde{H}^2(\mu)$. For if $f \in \tilde{H}^2(\mu)$, then by Corollary 5.3A there exist polynomials p_n such that $p_n \to f$ in $\tilde{H}^2(\mu)$-norm, and by the boundedness of S we also have $\varphi \cdot p_n = Sp_n \to Sf$ in $\tilde{H}^2(\nu)$-norm. Then Proposition 5.1 implies that $p_n \to f$ and $\varphi \cdot p_n \to Sf$ pointwise on D, whence $Sf = \varphi \cdot f$.

Note that φ is an outer function, since $\varphi \in N^+$ and $\frac{1}{\varphi} = S^{-1}1 \in \tilde{H}^2(\mu) \subseteq N^+$. Thus $h = \frac{\varphi\psi_\nu}{\psi_\mu}$ is outer. Since S is onto, $\varphi \cdot \tilde{H}^2(\mu) = \tilde{H}^2(\nu)$, that is, $\psi_\mu h \cdot \tilde{H}^2(\mu) = \psi_\nu \cdot \tilde{H}^2(\nu)$. To finish it will suffice to show that $h \cdot \tilde{H}^2(\mu) = \tilde{H}^2(\mu)$.

For any $f \in \tilde{H}^2(\mu)$, $a \|f\|_{\tilde{H}^2(\mu)} \leq \|Sf\|_{\tilde{H}^2(\nu)} \leq b \|f\|_{\tilde{H}^2(\mu)}$ where $a = \frac{1}{\|S^{-1}\|}$ and $b = \|S\|$, $0 < a \leq b < \infty$. That is,

$a^2\int |f|^2 d\mu \le \int |\varphi f|^2 d\nu \le b^2 \int |f|^2 d\mu$. Replacing f with $z^n f$ and letting $n \to \infty$, the dominated convergence theorem implies that

$$a^2 \int_T |f|^2 d\mu_0 \le \int_T |\varphi|^2 |f|^2 d\nu_0 \le b^2 \int_T |f|^2 d\mu_0,$$

that is,

$$a^2 \int_T |\psi_\mu|^2 |f|^2 dm \le \int_T |\varphi\psi_\nu|^2 |f|^2 dm \le b^2 \int_T |\psi_\mu|^2 |f|^2 dm$$

for all $f \in \tilde{H}^2(\mu)$, and in particular for all $f \in H^\infty$. Since functions on T which are boundary values of functions of the form $|f|^2$ with $f \in H^\infty$ are dense in the positive functions in $L^\infty(m)$, it follows that $a|\psi_\mu| \le |\varphi\psi_\nu| \le b|\psi_\mu|$ a.e. dm on T, that is, $a \le |h| \le b$ a.e. dm on T. Since h is outer, $a \le |h| \le b$ on D, and thus from the definition of $\tilde{H}^2(\mu)$ it is clear that $h \cdot \tilde{H}^2(\mu) = \tilde{H}^2(\mu)$. ∎

COROLLARY 6.2A If μ and ν are of type S and $\mu_0 = \nu_0$, then the following are equivalent:

(i) $\tilde{H}^2(\mu) = \tilde{H}^2(\nu)$;

(ii) $||L_\mu^\nu|| < \infty$ and $||L_\nu^\mu|| < \infty$;

(iii) $U_\mu \sim U_\nu$.

PROOF: (i) ⇒ (ii) Assume $\tilde{H}^2(\mu) = \tilde{H}^2(\nu)$ and let J: $\tilde{H}^2(\mu) \to \tilde{H}^2(\nu)$ be the "identity" map. If $f_n \to f$ in $\tilde{H}^2(\mu)$-norm and $Jf_n = f_n \to g$ in $\tilde{H}^2(\nu)$-norm, then for each $z \in D$ we have both $f_n(z) \to f(z)$ and $f_n(z) \to g(z)$ by Proposition 5.1, and so $f = g$, that is, $Jf = g$. Thus J has a closed graph, so by the closed graph theorem $||J|| < \infty$ and $||J^{-1}|| < \infty$. Considering

only the polynomials, that says $\|L_\mu^\nu\| < \infty$ and $\|L_\nu^\mu\| < \infty$.

(ii) $\Rightarrow$ (iii) If $\|L_\mu^\nu\| < \infty$ and $\|L_\nu^\mu\| < \infty$, then Λ_μ^ν exists and is invertible, so $U_\mu \sim U_\nu$ (via Λ_μ^ν) by Proposition 4.3(2).

(iii) $\Rightarrow$ (i) If $U_\mu \sim U_\nu$, then $\phi_\mu \cdot \tilde{H}^2(\mu) = \phi_\nu \cdot \tilde{H}^2(\nu)$ by Theorem 6.2. But $\phi_\mu = \phi_\nu \neq 0$, so $\tilde{H}^2(\mu) = \tilde{H}^2(\nu)$. ∎

As an application of Corollary 6.2A, let $0 < \beta < 1$, define $d\tau = (1-x)^{-\beta}\,dx$ on $0 \leq x < 1$, and let $\mu = m + \tau$. An easy calculation shows that the outer function $(1-z)^{-\alpha}$ (with α real) is in $\tilde{H}^2(\mu)$ if and only if $\alpha < \frac{1-\beta}{2}$. Thus distinct values of β $(0 < \beta < 1)$ yield unequal sets $\tilde{H}^2(\mu)$ and therefore non-similar operators U_μ.

Contrast the next corollary with Example 2.4.

COROLLARY 6.2B If A is a subnormal operator quasi-similar to U_m then the approximate point spectrum of A is T. In particular, A is bounded below.

PROOF: Write $\pi(B)$ for the approximate point spectrum of an operator B. By Corollary 4.5A there exists a measure μ of type S such that $A \cong U_\mu$. Thus it will suffice to prove that $\pi(U_\mu) = T$.

For each ε satisfying $0 < \varepsilon < 1$ let R_ε denote the closed annulus $\{ z \in C: 1-\varepsilon \leq |z| \leq 1\}$, and let $\mu_\varepsilon = \mu | R_\varepsilon$. It is clear that $\tilde{H}^2(\mu_\varepsilon) = \tilde{H}^2(\mu)$ since μ_ε and μ agree except on $|z| < 1-\varepsilon$, and every

function in N^+ is bounded there. Thus $U_{\mu_\varepsilon} \sim U_\mu$ by Corollary 6.2A, and so $\pi(U_{\mu_\varepsilon}) = \pi(U_\mu)$. But it is clear that $\pi(U_{\mu_\varepsilon}) \subseteq \text{supp } \mu_\varepsilon \subseteq R_\varepsilon$. Letting $\varepsilon \to 0+$, we see that $\pi(U_\mu) \subseteq T$.

The reverse inclusion follows from the fact that $\sigma(U_\mu) = \overline{D}$, since the boundary of $\sigma(B)$ is a subset of $\pi(B)$ for any operator B. (See [14 , Problem 63]) ∎

An operator $B: H \to H$ is a Fredholm operator iff B has closed range and ker B and ker B* are both finite-dimensional. The index of a Fredholm operator B is the integer ind B = dim ker B - dim ker B*.

For any operator $B: H \to H$ the essential spectrum of B is $\sigma_e(B) = \{\lambda \in \mathbb{C} : B - \lambda \text{ is not Fredholm}\}$.

COROLLARY 6.2C If A is subnormal and quasi-similar to U_m, then $\sigma_e(A) = T$ and

$$\text{ind } (A - \lambda) = \begin{cases} -1 & \text{if } |\lambda| < 1, \\ 0 & \text{if } |\lambda| > 1. \end{cases}$$

PROOF: If $|\lambda| \neq 1$ then $A - \lambda$ is bounded below by Corollary 6.2B and therefore $A - \lambda$ has closed range. Since $A - \lambda \sim\sim U_m - \lambda$ and $(A - \lambda)^* \sim\sim (U_m - \lambda)^*$ it follows from Proposition 1.4 that dim ker $(A - \lambda)$ = dim ker $(U_m - \lambda) < \infty$ and dim ker $(A - \lambda)^*$ = dim ker $(U_m - \lambda)^* < \infty$. Thus $A - \lambda$ is Fredholm, ind $(A - \lambda)$ = ind $(U_m - \lambda)$ (which = -1 if $|\lambda| < 1$ and = 0 if $|\lambda| > 1$), and $\sigma_e(A) \subseteq T$. If $\sigma_e(A)$ were a proper subset of T, then $\mathbb{C} \setminus \sigma_e(A)$ would be connected and so ind $(A - \lambda)$ would have to be a constant function of λ (since ind is continuous on the set of Fredholm operators; see [7, Theorem 5.36]), which it is not. ∎

COROLLARY 6.2D If A is subnormal and quasi-similar to U_m, then there exists a compact operator K such that $A \cong U_m + K$.

PROOF: It is a special case of a theorem in [4] that if A is a hyponormal operator such that $\sigma_e(A) = T$ and ind $A = -1$, then there exists an isometry V and a compact operator K such that $V \cong U_m$ and $A = V + K$. Thus Corollary 6.2D follows from Corollary 6.2C. ∎

Could it be that the operator K in the conclusion of Corollary 6.2D can always be chosen to be of trace class?

Note that there are subnormal operators unitarily equivalent to $U_m + K$, K compact, which are not quasi-similar to U_m. An example is U_μ, where μ is two-dimensional Lebesque measure on D, $d\mu = dxdy|D$.

We next give a complete set of unitary invariants for operators U_μ where μ is of type S. Recall the outer functions Ψ_μ defined before Theorem 6.2, and, for any measure μ of type S, let $\check{\mu}$ be the (possibly infinite) measure defined by $d\check{\mu} = \dfrac{d\mu}{|\Psi_\mu|^2}$, supp $\check{\mu} \subseteq \bar{D}$. Note that $\check{\mu} | T = m$, $\check{\mu} | D$ is a σ-finite Borel measure in the locally compact space D, and $\check{\mu}$ is of type S if it is finite.

THEOREM 6.3 If μ and ν are of type S, then $U_\mu \cong U_\nu$ if and only if $\check{\mu} = \check{\nu}$.

PROOF: Assume $\check{\mu} = \check{\nu}$ and define V: $\tilde{H}^2(\mu) \to \tilde{H}^2(\nu)$ by $Vf = \dfrac{\Psi_\mu}{\Psi_\nu} f$. The calculation

$$\|Vf\|^2_{\tilde{H}^2(\nu)} = \int_{\bar{D}} \left|\frac{\Psi_\mu}{\Psi_\nu}\right|^2 |f|^2 d\nu = \int_{\bar{D}} |f|^2 d\mu = \|f\|^2_{\tilde{H}^2(\mu)}$$

shows that V is well-defined is an isometry. Repeating the calculation with the roles of μ and ν reversed shows that the map $g \to \dfrac{\Psi_\nu}{\Psi_\mu} g$ from

$\tilde{H}^2(\nu)$ to $\tilde{H}^2(\mu)$ is also an isometry; it is clearly V^{-1}, and so V is unitary. It is clear that $V\tilde{U}_\mu = \tilde{U}_\nu V$, so $\tilde{U}_\mu \cong \tilde{U}_\nu$. By Corollary 5.3B, $U_\mu \cong U_\nu$.

Conversely, assume that $\tilde{U}_\mu \cong \tilde{U}_\nu$ (via V), and let $\varphi = V1 \in \tilde{H}^2(\nu) \subseteq N^+$. As in the proof of Theorem 6.2 we see that φ is outer and $Vf = \varphi \cdot f$ for every $f \in \tilde{H}^2(\mu)$. Since V is unitary, $\int_{\bar{D}} |\varphi f|^2 d\nu = \int_{\bar{D}} |f|^2 d\mu$ for all $f \in \tilde{H}^2(\mu)$. If we replace f with $z^n f$ and let $n \to \infty$, the dominated convergence theorem implies that $\int_T |\varphi|^2 |f|^2 |\psi_\nu|^2 \, dm = \int_T |f|^2 |\psi_\mu|^2 dm$ for all $f \in \tilde{H}^2(\mu)$, and in particular for all $f \in H^\infty$. By the density of $|H^\infty|^2$ in the positive functions in $L^\infty(m)$ it follows easily that $|\varphi| = |\psi_\mu/\psi_\nu|$ a.e. dm on T. Since φ and ψ_μ/ψ_ν are both outer, there is a constant c such that $|c| = 1$ and $\varphi = c\,\frac{\psi_\mu}{\psi_\nu}$. Thus $Vf = c\,\frac{\psi_\mu}{\psi_\nu} f$ for all $f \in \tilde{H}^2(\mu)$.

Now $(Vf, Vg)_{\tilde{H}^2(\nu)} = (f,g)_{\tilde{H}^2(\mu)}$ for all $f,g \in \tilde{H}^2(\mu)$ since V is unitary, so we have $\int \left|\frac{\psi_\mu}{\psi_\nu}\right|^2 f\bar{g}\, d\nu = \int f\bar{g}\, d\mu$ for all $f,g \in \tilde{H}^2(\mu)$. If we let $\mathcal{A}$ denote the set of all finite linear combinations of functions of the form $f\bar{g}$ where f and g are in P, then clearly $\int F \cdot |\psi_\mu/\psi_\nu|^2 \, d\nu = \int F d\mu$ for all $F \in \mathcal{A}$. By the Stone-Weierstrass theorem $\mathcal{A}$ is dense in $C(\bar{D})$, and so $\int F \cdot |\psi_\mu/\psi_\nu|^2 d\nu = \int F \, d\mu$ for all $F \in C(\bar{D})$. Therefore $|\psi_\mu/\psi_\nu|^2 d\nu = d\mu$, that is, $\breve{\nu} = \breve{\mu}$. ∎

COROLLARY 6.3A If μ and ν are of type S and $\mu_0 = \nu_0$, then $U_\mu \cong U_\nu$ if and only if $\mu = \nu$. ∎

Following [9] we define a finite measure τ to be a

Carleson measure iff τ is carried by D and there is a constant c such that $\tau(E) \le ct$ for every set E of the form

$E = \{z = re^{i\theta}: 1-t \le r < 1,\ \theta_0 \le \theta \le \theta_0 + t\}$, $0 < t \le 1$, $-\pi \le \theta_0 \le \pi$.

L. Carleson has shown that a measure τ carried by D is a Carleson measure if and only if $||L_m^{m+\tau}|| < \infty$. See [5] or [9, Theorem 9.3].

THEOREM 6.4 The operator U_μ is similar to U_m if and only if μ is of type S and $\check{\mu}|D$ is a Carleson measure.

PROOF: After Theorem 4.5 we may assume that μ is of type S, and so $\check{\mu}$ is defined. If $\check{\mu}$ is a finite measure, we have

$\check{\mu}|D$ is a Carleson measure iff $||L_m^{\check{\mu}}|| < \infty$ (by Carleson's theorem)

iff $U_{\check{\mu}} \sim U_m$ (by Corollary 6.1 A)

iff $U_\mu \sim U_m$ (by Theorem 6.3).

To finish we need only show that if $U_\mu \sim U_m$, then $\check{\mu}$ is finite. This follows from Theorem 6.2, for if $U_\mu \sim U_m$, then $\psi_\mu \cdot \tilde{H}^2(\mu) = H^2$, so $\frac{1}{\psi_\mu} \in \tilde{H}^2(\mu)$. That is, $\int d\check{\mu} = \int \frac{d\mu}{|\psi_\mu|^2} < \infty$. ∎

Note that Theorem 6.4 does for similarity what Theorem 4.5 does for quasi-similarity. The following alternate form is probably worth stating separately.

COROLLARY 6.4 A A subnormal operator A is similar to the unilateral shift if and only if there exists a Carleson measure τ such that $A \cong U_{m+\tau}$.

PROOF: Combine Proposition 4.2 and the proof of Theorem 6.4. ∎

SECTION 7 Invariant subspaces

In this section we determine the commutant and the invariant subspace lattice for each subnormal operator quasi-similar to the unilateral shift. The results are always the same, that is, isomorphic to the commutant and invariant subspace lattice of the shift.

Yoshino [26] has shown that, for any measure μ, the commutant of U_μ: $H^2(\mu) \to H^2(\mu)$ is the set of all multiplications, restricted to $H^2(\mu)$, by functions in $H^2(\mu) \cap L^\infty(\mu)$. Theorem 7.1 says practically the same thing for a restricted class of measures, but says it in terms of the spaces $\tilde{H}^2(\mu)$ and H^∞ of analytic functions. (Theorem 7.1 can be derived easily from Yoshino's theorem, and Yoshino's theorem can be proved by a simple variation on the proof below.)

THEOREM 7.1 If μ is of type S, then the commutant of $\tilde{U}_\mu$ is the set of all multiplications, restricted to $\tilde{H}^2(\mu)$, by functions in H^∞. The norm of an operator in the commutant of $\tilde{U}_\mu$ is equal to the H^∞-norm of the corresponding function, and so the map which takes each H^∞-function to the corresponding operator on $\tilde{H}^2(\mu)$ is an isometric isomorphism of the Banach algebra H^∞ onto the commutant of $\tilde{H}^2(\mu)$.

PROOF: Clearly $\tilde{H}^2(\mu)$ is invariant under multiplication by H^∞-functions and every such multiplication commutes with $\tilde{U}_\mu$.

Conversely, let B be an operator which commutes with $\tilde{U}_\mu$ and let $\varphi = B1 \in \tilde{H}^2(\mu)$. As in the second paragraph of the proof of Theorem 6.2 we see that $Bf = \varphi \cdot f$ for all $f \in \tilde{H}^2(\mu)$. To finish we need only show that $\varphi \in H^\infty$ and $\|B\| = \|\varphi\|_{H^\infty}$. For $n \geq 0$ we have

$||\varphi^n||_{\tilde{H}^2(\mu)} = ||B^n 1||_{\tilde{H}^2(\mu)} \leq ||B||^n ||1||_{\tilde{H}^2(\mu)}$, so

$\int_{\overline{D}} \left(\frac{|\varphi|}{||B||} \right)^{2n} d\mu \leq \mu(\overline{D})$. Letting $n \to \infty$, Fatou's lemma implies that

$|\varphi| \leq ||B||$ a.e. $d\mu$, so $|\varphi| \leq ||B||$ a.e. dm on T. Since $\varphi \in N^+$, that means $\varphi \in H^\infty$ and $||\varphi||_{H^\infty} \leq ||B||$. The reverse inequality is obvious, so $||B|| = ||\varphi||_{H^\infty}$. ∎

If A: $H \to H$ and M is a subspace of H, then M is called a <u>hyperinvariant</u> subspace of A iff M is invariant under every operator which commutes with A. In particular, every hyperinvariant subspace of an operator is invariant.

<u>COROLLARY 7.1A</u> If A is subnormal and quasi-similar to U_m, then every invariant subspace of A is hyperinvariant.

<u>PROOF</u>: Immediate from Corollary 4.5A, Theorem 7.1, and Proposition 5.2. ∎

Note that other cyclic subnormal operators can have invariant subspaces which are not hyperinvariant. A simple example is the bilateral shift $W_m : L^2(m) \to L^2(m)$. Since W_m is unitary and *-cyclic, it is cyclic [2, Theorem 6]. (For an explicit cyclic vector, see the function φ used in the proof of Proposition 1.1(1).) However, $H^2(m)$ is an invariant subspace of W_m which is not hyperinvariant, since it is not invariant under W_m^*.

We next determine the invariant subspace lattices for operators of type S. These lattices all turn out to be isomorphic to the invariant subspace lattice of the unilateral shift, which was determined by

Beurling [1], and the isomorphism is produced by showing that every invariant subspace of $\tilde{U}_\mu$ is of the form $[\varphi \cdot \tilde{H}^2(\mu)]_{\tilde{H}^2(\mu)}$ for some inner function φ. The following example shows that multiplication by an inner function on an $\tilde{H}^2(\mu)$-space need not have closed range, so it really is necessary to take the closure of $\varphi \cdot \tilde{H}^2(\mu)$.

EXAMPLE 7.2 Let $\mu = m+\tau$ where $d\tau = \frac{dx}{\sqrt{1-x}}$ on $0 \le x < 1$, and let s be the singular inner function $s(z) = \exp\left(-\frac{1+z}{1-z}\right)$. Then $s \cdot \tilde{H}^2(\mu)$ is not closed in $\tilde{H}^2(\mu)$.

PROOF: An easy calculation shows that $s \cdot H^2 \subseteq \tilde{H}^2(\mu)$, so we may define an operator $S: H^2 \to \tilde{H}^2(\mu)$ by $Sf = s \cdot f$. If $g(z) = (1-z)^{-1/3}$, then $g \in H^2$ but $g \notin \tilde{H}^2(\mu)$. Choose polynomials p_n such that $p_n \to g$ in H^2-norm. Then $Sp_n \to Sg$ in $\tilde{H}^2(\mu)$-norm. Each $Sp_n \in s \cdot H^2(\mu)$ but $Sg \notin s \cdot \tilde{H}^2(\mu)$, and therefore $s \cdot \tilde{H}^2(\mu)$ is not closed. ∎

LEMMA 7.3a Let $\sum_{n=1}^{\infty} \beta_n < \infty$ where each $\beta_n \ge 0$, and assume $g \in H^\infty$, $g_n \in H^2$, and $\|g-g_n\|_{H^2} \le \beta_n$ for each n. If $g \not\equiv 0$, then there exist outer functions $\psi_n \in H^\infty$ such that $\|\psi_n\|_{H^\infty} \le 1$, $\|\psi_n g_n\|_{H^\infty} \le \|g\|_{H^\infty}$, and $\psi_n g_n \to g$ both in H^2-norm and a.e. dm on T.

PROOF: Assume with no loss of generality that $\|g\|_{H^\infty} = 1$. Since $\log^+ |g_n| \in L^1(m)$, let $h_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log^+ |g_n(e^{i\theta})|\, d\theta$, and let $\psi_n = \exp(-h_n)$, so $\psi_n \in H^\infty$ and ψ_n is outer. Since $|\psi_n| = \min\left(1, \frac{1}{|g_n|}\right)$ a.e. dm on T, $\|\psi_n\|_{H^\infty} \le 1$ and $\|\psi_n g_n\|_{H^\infty} \le 1$,

as desired.

If u and v are complex numbers such that $|v| \leq 1$, then

$$\log^+|u| = \max(0, \log|u|)$$

$$\leq \max(0, |u| - 1)$$

$$\leq \max(0, |u| - |v|)$$

$$\leq \max(0, |u-v|)$$

$$= |u-v|,$$

so we have $h_n(0) = \int \log^+|g_n| \cdot dm$

$$\leq \int |g_n - g| \cdot dm$$

$$\leq ||g_n - g||_{H^2}$$

$$\leq \beta_n,$$

and therefore $\psi_n(0) = e^{-h_n(0)} \geq e^{-\beta_n} \geq 1 - \beta_n$.

It follows that $\int |1-\psi_n|^2 dm = \int (1 - \psi_n - \overline{\psi}_n + |\psi_n|^2) dm$

$$= 1-2 \operatorname{Re} \int \psi_n dm + \int |\psi_n|^2 dm$$

$$\leq 1-2\psi_n(0) + 1$$

$$\leq 2-2(1-\beta_n) = 2\beta_n,$$

so $\int \sum_{n=1}^{\infty} |1-\psi_n|^2 dm = \sum_{n=1}^{\infty} \int |1-\psi_n|^2 dm \leq 2 \sum_{n=1}^{\infty} \beta_n < \infty$. Thus

$\sum_{n=1}^{\infty} |1-\psi_n|^2 < \infty$ a.e. dm, so $\psi_n \to 1$ a.e. dm on T.

Similarly, $\int \sum_{n=1}^{\infty} |g-g_n|^2 dm = \sum_{n=1}^{\infty} \|g-g_n\|_{H^2}^2 \leq \sum_{n=1}^{\infty} \beta_n^2 < \infty$, so $g_n \to g$ a.e. dm on T. Thus $\psi_n g_n \to g$ a.e. dm on T. Since $|\psi_n g_n| \leq 1$, the bounded convergence theorem implies that $\psi_n g_n \to g$ also in $L^2(m)$-norm, that is, in H^2-norm. ∎

LEMMA 7.3β If μ is of type S and M is a non-zero invariant subspace of $\tilde{U}_\mu$ then there exists an inner function φ such that $M \cap H^\infty = \varphi \cdot H^\infty$.

PROOF: Let $M_1 = [M \cap H^\infty]_{H^2}$. Clearly $\tilde{U}_m M_1 \subseteq M_1$ and, by Theorem 5.3, $M_1 \neq (0)$. By Beurling's theorem on the invariant subspaces of the shift [1], there is an inner function φ such that $M_1 = \varphi \cdot H^2$.

Clearly $M \cap H^\infty \subseteq M_1 \cap H^\infty = \{\varphi \cdot H^2\} \cap H^\infty = \varphi \cdot H^\infty$, so we need only show that $\varphi \cdot H^\infty \subseteq M \cap H^\infty$, for which purpose it will suffice to show that $\varphi \cdot H^\infty \subseteq M$. So let $g \in \varphi \cdot H^\infty$, and assume with no loss of generality that $g \neq 0$. Then $g \in M_1$, so there exist $g_n \in M \cap H^\infty$ such that $g_n \to g$ in H^2-norm, and we can even require that $\|g-g_n\| \leq 2^{-n}$. By Lemma 7.3α there exist (outer) functions $\psi_n \in H^\infty$ such that $\|\psi_n\|_{H^\infty} \leq 1$, $\|\psi_n g_n\|_{H^\infty} \leq \|g\|_{H^\infty}$, and $\psi_n g_n \to g$ both in H^2-norm and a.e. dm on T. Thus $\psi_n g_n \to g$ pointwise on D and a.e. dm on T, so $\psi_n g_n \to g$ boundedly a.e. $d\mu$. By the bounded convergence theorem $\psi_n g_n \to g$ in $H^2(\mu)$-norm. Each $g_n \in M$, so $\psi_n g_n \in M$ by Proposition 5.2, and so $g \in M$ since M is

closed in $\tilde{H}^2(\mu)$. Thus $\varphi \cdot H^\infty \subseteq M$. ∎

THEOREM 7.3 Let μ be of type S and for each inner function φ define $M(\varphi) = [\varphi \cdot \tilde{H}^2(\mu)]_{\tilde{H}^2(\mu)}$. Then each $M(\varphi)$ is a non-zero invariant subspace of $\tilde{U}_\mu$. If M is a non-zero invariant subspace of $\tilde{U}_\mu$, then $M = M(\omega)$ for some inner function ω, and φ is uniquely determined by M except for a scalar factor of modulus 1. Furthermore, $M(\varphi_1) \subseteq M(\varphi_2)$ if and only if $\varphi_2 \mid \varphi_1$, that is, if and only if φ_1/φ_2 is an inner function.

PROOF: It is clear that each $M(\omega)$ is an invariant subspace of $\tilde{U}_\mu$. Now suppose that $M(\varphi_1) \subseteq M(\varphi_2)$. Then $\varphi_1 \in M(\varphi_2)$, so there exist functions $f_n \in \tilde{H}^2(\mu)$ such that $\varphi_2 f_n \to \varphi_1$ in $\tilde{H}^2(\mu)$-norm, hence also in $\tilde{H}^2(\mu_0)$-norm, where $\mu_0 = \mu \mid T$. But $\varphi_2 \cdot \tilde{H}^2(\mu_0)$ is closed in $\tilde{H}^2(\mu_0)$ since multiplication by φ_2 is an isometry on $\tilde{H}^2(\mu_0)$, so $\varphi_1 \in \varphi_2 \cdot \tilde{H}^2(\mu_0)$, that is, $\varphi_1/\varphi_2 \in \tilde{H}^2(\mu_0) \subseteq N^+$. Since $|\varphi_1/\varphi_2| = 1$ a.e. dm on T, that implies φ_1/φ_2 is an inner function.

Let M be a non-zero invariant subspace of $\tilde{U}_\mu$. The uniqueness (up to scalar factors) of the inner function ω such that $M = M(\varphi)$ follows from the previous paragraph, and the existence may be proven as follows. By Lemma 7.3β there exists an inner function φ such that $M \cap H^\infty = \varphi \cdot H^\infty$. Thus $M = [M \cap H^\infty]_{\tilde{H}^2(\mu)} = [\varphi \cdot H^\infty]_{\tilde{H}^2(\mu)} = M(\varphi)$, where the first and last equalities follow from Theorem 5.3 and Corollary 5.3A respectively. ∎

COROLLARY 7.3A If A is subnormal and quasi-similar to U_m, then the in-

variant subspace lattices of A and U_m are isomorphic. In particular, A is irreducible.

PROOF: The fact that the lattices are isomorphic follows at once from Corollary 4.5A and Theorem 7.3. If A were reducible it would then have an invariant subspace M such that $M^{\perp}$ would also be invariant, with both M and $M^{\perp}$ non-zero. Using Theorem 7.3, say $M = M(\varphi_1)$ and $M^{\perp} = M(\varphi_2)$. Then $\varphi_1\varphi_2 \in M \cap M^{\perp}$, which is a contradiction since $\varphi_1\varphi_2 \neq 0$. ∎

SECTION 8 Open questions

We close by listing a few of the questions which were raised but not answered during the course of the present investigation.

As shown in Section 3, quasi-similar hyponormal operators must have equal spectra. Their approximate point spectra do not have to be equal, as shown by Example 2.4, but even in Example 2.4 the two operators have equal essential spectra. Is this the case in general?

Question 8.1 Must quasi-similar hyponormal operators have equal essential spectra?

An idea for a proof of a "yes" answer for Question 8.1 is to note that Corollary 3.1A and the theory of (abstract) C*-algebras imply that in any C*-algebra quasi-similar hyponormal elements have equal spectra, where "quasi-similar" and "hyponormal" have natural algebraic definitions (for example, "quasi-invertible" means "left and right cancellable"). Since the Calkin algebra (the algebra of all operators on a Hilbert space modulo the ideal of compact operators) is a C*-algebra and the essential spectrum of an operator is just the spectrum of the corresponding element of the Calkin algebra, it would suffice to show that quasi-similar hyponormal operators are quasi-similar when regarded as members of the Calkin algebra. However it is not clear how one might prove this; the most obvious approach would be to show that every quasi-invertible operator is quasi-invertible in the Calkin algebra, but the existence of compact quasi-invertible operators shows that this is false.

There are at least two directions in which one might try to generalize the results of Chapter II. One of these would be to try to develop a theory of subnormal operators similar or quasi-similar to a given cyclic subnormal operator other than the shift, say to the so-called Bergman shift U_μ, where μ is planar Lebesque measure restricted to D. This is likely to be much more difficult than the corresponding theory for the shift. Consider the following simple test question:

Question 8.2 Let μ be a planar Lebesque measure restricted to D, let τ be linear Lebesque measure restricted to [0,1], and let $\nu = \mu + \tau$. Are U_μ and U_ν similar?

Although U_μ and U_ν are quasi-similar, I would guess that the answer to Question 8.2 is "no", since $||L_\mu^\nu|| = \infty$, but an actual proof using the methods of Section 6 is probably not possible because here all the measure is in the interior of the disc. Perhaps a theorem of Phragmen-Lindelöf type might be applicable.

The other natural generalization of the results of Chapter II would be to try to determine the subnormal operators quasi-similar (or similar) to unilateral shifts of multiplicity higher than 1, that is, to the operators $U_m \oplus U_m \oplus \cdots \oplus U_m$, or even to shifts of infinite multiplicity.

Question 8.3 Can the problem of characterizing the subnormal operators quasi-similar to $U_m \oplus U_m \oplus \cdots \oplus U_m$ be reduced to the corresponding problem for U_m ?

Adapting a proof of Sz.-Nagy and Foiaş [24], Hoover [16] has

shown that for arbitrary quasi-similar operators A and B, if A has a non-trivial hyperinvariant subspace, then so does B. However, it does not seem to be known whether A and B must have isomorphic lattices of hyperinvariant subspaces. After the results of Section 7 the following is a natural question to ask.

Question 8.4 Must quasi-similar subnormal operators have isomorphic lattices of hyperinvariant subspaces?

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