DOCTORAL DISSERTATION SERIES
Publication No.: 11,263

AUTHOR: Earl Hicks Crisler, Ph. D., 1955
University of Michigan

TITLE: ON PARTITION SYSTEMS, CHORDAL SYSTEMS AND THEIR APPLICATION TO REGULAR CURVE-FAMILIES FILLING THE PLANE

University Microfilms, Ann Arbor, Michigan

ON PARTITION SYSTEMS, CHORDAL SYSTEMS AND THEIR APPLICATION TO REGULAR CURVE-FAMILIES FILLING THE PLANE

by
E. H. Crisler

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1954

Doctoral Committee:

Associate Professor Wilfred Kaplan, Chairman
Associate Professor E. E. Moise
Associate Professor George Piranian
Professor L. L. Rauch
Doctor C. J. Titus

ON PARTITION SYSTEMS, CHORDAL SYSTEMS AND THEIR APPLICATION TO REGULAR CURVE-FAMILIES FILLING THE PLANE

By E. Earl H. Michael Crisler

The purpose of this paper is to investigate chordal systems and a weaker system called a partition system. The paper consists of four chapters. The first serves as an introduction. It discusses the relationship of chordal systems to regular curve-families filling the oriented plane and gives some geometric background for the work to follow. In addition, some results of W. Kaplan's investigations on regular curve-families are presented without proof.

The second chapter deals with partition systems. Two sets of axioms for partition systems are given and their equivalence demonstrated.

In the third chapter, an intrinsic topology is introduced for partition systems. In terms of the intrinsic topology, a necessary and sufficient condition is determined, so that, if a partition system satisfies this condition, then it is normal, that is, it has a particularly simple structure consisting of a denumerable number of half-open intervals.

In the last chapter, the notion of discontinuity element is introduced. The various types of discontinuity elements that a partition system and chordal system may have are classified. The main theorem of the chapter shows that a necessary and sufficient condition for a normal chordal system to be isomorphic to the chordal system defined by the level curves of the real part of a schlicht function in $|z| < 1$ is that it contains no Y-discontinuities.

The writer wishes to acknowledge the great amount of assistance and encouragement given to him by Professor Wilfred Kaplan, faculty advisor in the Department of Mathematics at the University of Michigan. This dissertation pursues some of the avenues of investigation that Professor Kaplan's researches have opened and is dependent upon them, not only for basis, but also for method.

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CHAPTER I

INTRODUCTION

1.01 The Nature of the Investigation. W. Kaplan in [1]* , [2] , [3] , [4] and other papers has investigated the topological properties of regular curve-families filling a simply connected set of the plane. A regular curve-family is a family of curves which is locally homeomorphic to a family of parallel lines. As a tool for studying the structure of these families, he has associated with each regular curve-family an algebraic structure called a chordal system. Employing the notion of a chordal system, it is possible to decompose a regular curve-family into a denumerable number of simple curve-families each of which is topologically equivalent to the vertical lines filling the nonnegative half-plane. The result of such a process is called a normal subdivision of the chordal system. A chordal system for which such a subdivision exists is said to be normal.

In this paper, we investigate chordal systems and a weaker algebraic system, called a partition system, from an abstract viewpoint, that is, without associating these systems with a particular regular curve-family. Chapter II deals with partition systems and demonstrates that they have most of the essential properties of chordal systems. They are the counterpart of chordal systems in the study of regular families of surfaces filling a simply connected set of Euclidean three space. We introduce in Chapter III a topology for the systems discussed in

*Numbers in brackets refer to the bibliography at the end of the paper.

Chapter II and in terms of this topology determine a necessary and sufficient condition for such a system to be normal. Chapter IV deals with chordal systems. In it we determine a necessary and sufficient condition for a given normal chordal system to be isomorphic to the chordal system of the level curve-family generated by the real part of a one-to-one conformal mapping defined on the interior of the unit circle. This in conjunction with a fundamental result of [2] allows us to characterize those regular curve-families which are equivalent to the level curve-family of the real part of a schlicht function defined on the interior of the unit circle. The chapter concludes by showing that a restricted class of partition systems can be made into chordal systems.

The remaining paragraphs of this chapter are devoted to a discussion of some of the results of the previously mentioned papers. These results will be of use later on and in addition give some geometric background for the work to follow.

1.02 Definition[∂]. A set of points of the plane is a curve if and only if it is the homeomorphic image of the open unit interval.

1.03 Definition. A family F of curves { C } is said to fill a region R if and only if each curve of F is contained in R and each point of R lies on one and only one C.

1.04 Definition. Let the family F of curves { C } fill the region R and let P be a point of R. Then F will be termed regular at P if and only if there exists a neighborhood N of P and a homeomorphism which maps N onto the plane and carries the curves of F filling N onto the vertical lines of the plane.

[∂] The term "curve" is used for convenience. A more appropriate term would be "open arc" (c.f. [5] p. 27).

1.05 Definition. Two curve-families F and F' filling the plane are O-equivalent if and only if there exists an orientation preserving homeomorphism of the plane onto itself which carries the curves of F into the curves of F'. Throughout, the plane is assumed oriented.

1.06 Theorem. If F is a regular curve-family filling the plane and C is a curve of F, then C extends to infinity in both directions, i.e., it separates the plane into exactly two components.

1.07 Remark. It is this property of curves of regular curve-families that suggests the notion of chordal system. It is clear that if given any triple of curves C_1, C_2, C_3 of a regular curve-family, then either of the curves separates the other two or that none of the three separates the other two. In the latter case, it is possible to assign an orientation to the triple (see 1.09). The five possible cases that can occur for a triple of curves of F are illustrated in Figure 1-1.

1.08 Definition. Let C_2 belong to the regular curve-family F filling the plane. Let $\delta(C_2)$ denote one of the two components into which C separates the plane and $\delta^*(C)$ the other. Then the distinct triple of curves C_1, C_2, C_3 of F satisfy the parallel chordal relation $C_1|C_2|C_3$ if and only if it is possible to select $\delta(C_2)$ and $\delta^*(C_2)$ so that $C_1 \subset \delta(C_2)$ and $C_3 \subset \delta^*(C_2)$.

1.09 Definition. A triple C_1, C_2, C_3 of distinct curves of the regular curve-family F filling the plane, satisfy the circular chordal relation $|C_1, C_2, C_3|^+$ if and only if there exists a closed curve $P_1P_2P_3P_1$ meeting the curves C_1, C_2, C_3 only at the points P_1 on C_1, P_2 on C_2, and

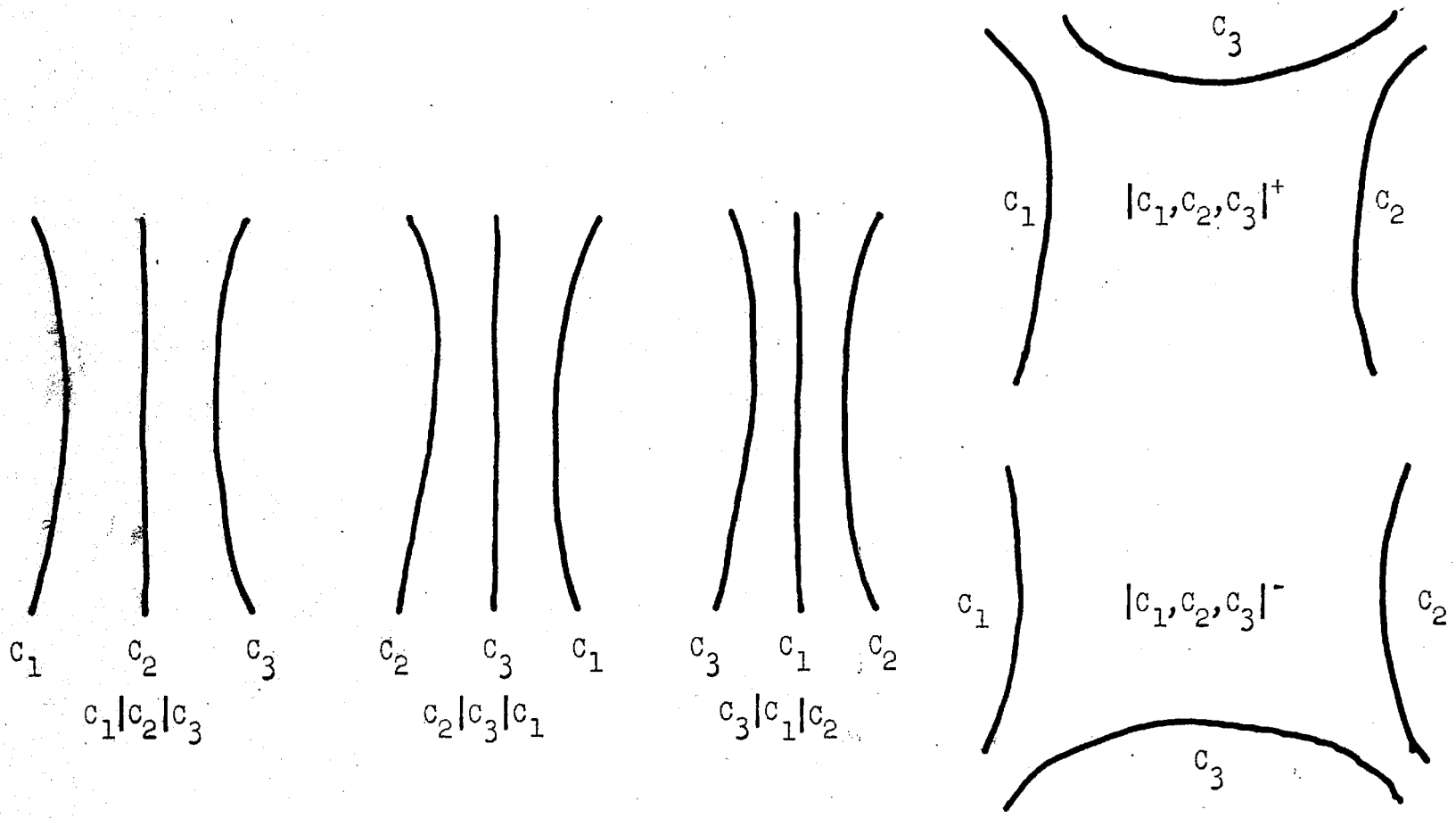

c1
c2
c3
c1|c2|c3
c2
c3
c1
c2|c3|c1
c3
c1
c2
c3|c1|c2
c3
c1
|c1,c2,c3|+
c2
c1
|c1,c2,c3|-
c2
c3

Figure 1-1

P_3 on C_3 and such that the orientation of $P_1P_2P_3P_1$ is positive.

1.10 Remarks. It is shown in [1] that a regular curve-family filling the plane with chordal relations defined by 1.08 and 1.09 satisfy the axioms of a chordal system and that such a chordal system is normal. The axioms for a chordal system and a normal chordal system are given in paragraph 2.27 and 3.18 respectively. In the future when we speak of the chordal system of a regular curve-family, it will be understood that the relations of the system are defined by 1.08 and 1.09. The theorem stated in the next paragraph from [3] reduces the study of regular curve-families filling the plane to the study of the level curve-families of harmonic functions.

1.11 Theorem. Let F be a regular curve-family filling the plane. Then there exists a homeomorphism of the plane onto a domain D of the (x,y)-plane such that F is transformed onto the family of level curves of a function $u(x,y)$ which is harmonic in D.

CHAPTER II

PARTITION SYSTEMS AND NONORIENTED CHORDAL SYSTEMS

2.01 **Definition.** If E is a nonempty set and $\underline{a}$ is an element of E, then a partition of E relative to $\underline{a}$ will be a subdivision of E into three mutually disjoint sets, one of which consists solely of the element $\underline{a}$. The sets of a partition of E relative to $\underline{a}$ shall be denoted by $\Delta(\underline{a})$, $\underline{a}$, and $\Delta^*(\underline{a})$. The symbol $\underline{a}$ will be used to denote both the element $\underline{a}$ and the set consisting solely of the element $\underline{a}$. The context in which it is used will make clear which meaning is intended.

2.02 **Definition.** A set E will be called a partition system or a PS if and only if it satisfies the following two axioms.

(PS1) Corresponding to each element $\underline{a}$ in E there is given a unique partition of E relative to $\underline{a}$.

Notation. If b is an element of E distinct from $\underline{a}$, then $\Delta_b(\underline{a})$ will denote the set of the partition of E relative to $\underline{a}$ that contains b and $\Delta^*_b(\underline{a})$ will denote the partition set relative to $\underline{a}$ not containing b.

(PS2) If $\underline{a}$ and b are distinct elements of E, then $\Delta^*_{\underline{a}}(b) \subset \Delta_b(\underline{a})$.

2.03 **Remark.** In this section the symbol E will stand for a partition system unless specified otherwise. Throughout the symbol "iff" will stand for the phrase "if and only if" and will be used as the logical symbol of equivalence.

2.04 **Lemma.** If $\underline{a}$,b,c are distinct elements of E, then:

(1) $c \in \Delta_{\underline{a}}(b)$ iff $\Delta_{\underline{a}}(b) = \Delta_c(b)$ iff $\underline{a} \in \Delta_c(b)$;

(2) $c \in \Delta^*_{\underline{a}}(b)$ iff $\Delta^*_a(b) = \Delta_c(b)$ iff $\Delta_{\underline{a}}(b) = \Delta^*_c(b)$ iff $a \in \Delta_c(b)$;

(3) $c \in \Delta^*_{\underline{a}}(b)$ implies $\underline{a} \in \Delta_b(c)$ and $c \in \Delta_b(\underline{a})$.

Proof. (1) By (PS1) either $\Delta_c(b) = \Delta_{\underline{a}}(b)$ or $\Delta_c(b) \cap \Delta_{\underline{a}}(b) = 0$.

(2) As in (1).

(3) $c \in \Delta^*_{\underline{a}}(b)$ implies $c \in \Delta_b(\underline{a})$ by (PS2). $c \in \Delta^*_{\underline{a}}(b)$ implies $\underline{a} \in \Delta^*_c(b)$ by (2). $\underline{a} \in \Delta^*_c(b)$ implies $\underline{a} \in \Delta_b(c)$ by (PS2).

2.05 Definition. The elements $\underline{a}, b, c$ of E satisfy the relation $\underline{a}|b|c$ iff $c \in \Delta^*_{\underline{a}}(b)$.

2.06 Theorem. $\underline{a}|b|c$ iff $c|b|\underline{a}$.

Proof. This is an immediate consequence of 2.05 and (2) of 2.04.

2.07 Theorem. $\underline{a}|b|c$ implies not $\underline{a}|c|b$ and not $b|\underline{a}|c$.

Proof. $\underline{a}|b|c$ implies $c \in \Delta^*_{\underline{a}}(b)$. By 2.04 this implies $\underline{a} \in \Delta_b(c)$ and $c \in \Delta_b(\underline{a})$. Hence, $\underline{a} \notin \Delta^*_b(c)$ and $c \notin \Delta^*_b(\underline{a})$. Thus, not $\underline{a}|c|b$ and not $b|\underline{a}|c$.

2.08 Theorem. $\underline{a}|b|c$ and $b|c|d$ imply $\underline{a}|b|d$ and $\underline{a}|c|d$.

Proof. $\underline{a}|b|c$ implies $c|b|\underline{a}$ which implies $\underline{a} \in \Delta^*_c(b)$. $b|c|d$ implies $d \in \Delta^*_b(c)$. $\Delta^*_b(c) \subset \Delta_c(b)$ by (PS2). Thus, $\Delta_d(b) = \Delta_c(b)$ and, consequently, $\Delta^*_d(b) = \Delta^*_c(b)$. Hence, $\underline{a} \in \Delta^*_d(b)$ and $\underline{a}|b|d$.

$b|c|d$ implies $d|c|b$ and $a|b|c$ implies $c|b|a$. $d|c|b$ and $c|b|\underline{a}$ imply $d|c|\underline{a}$ by the above result. Hence, $\underline{a}|c|d$ and the theorem is proved.

2.09 Theorem. Of the three relations $b|\underline{a}|c$, $b|\underline{a}|d$, and $c|\underline{a}|d$ at most two can hold.

Proof. If all three held, then since $b|\underline{a}|c$ iff $c|\underline{a}|b$, we would obtain: $b \in \Delta^*_c(\underline{a})$, $d \in \Delta^*_b(\underline{a})$, and $d \in \Delta^*_c(\underline{a})$. Thus by the uniqueness

feature of (PS1), we would have $b \in \Delta_c^*(\underline{a}) = \Delta_b^*(\underline{a})$. This result, however, contradicts the definition of $\Delta_b^*(\underline{a})$.

2.10 Theorem. If $b \neq d$ and $\underline{a}|b|c$, then either $\underline{a}|b|d$ or $d|b|c$ but not both.

Proof. $\underline{a}|b|c$ iff $c \in \Delta_{\underline{a}}^*(b)$ iff $\Delta_{\underline{a}}(b) = \Delta_c^*(b)$. If $b \neq d$, then either $d \in \Delta_{\underline{a}}^*(b)$ or $d \in \Delta_{\underline{a}}(b) = \Delta_c^*(b)$. Hence, either $d|b|\underline{a}$ or $d|b|c$ but not both.

2.11 Theorem. $\underline{a}|b|c$ and $\underline{a}|c|d$ imply $\underline{a}|b|d$ and $b|c|d$.

Proof. If $d = b$, then $\underline{a}|b|c$ and $\underline{a}|c|b$ which contradicts 2.07. Hence, $d \neq b$. $d \neq b$ implies by 2.10 either $\underline{a}|b|d$ or $d|b|c$, if $\underline{a}|b|c$ and either $\underline{a}|c|d$ or $b|c|d$, if $\underline{a}|c|d$. But $\underline{a}|b|c$ implies not $\underline{a}|c|b$. Hence, $b|c|d$. $b|c|d$ implies not $d|b|c$ by 2.07. Therefore, $\underline{a}|b|d$.

2.12 Definition. The elements $\underline{a},b,c$ of a partition system E satisfy the relation $|\underline{a},b,c|^{\pm}$ iff $\Delta_b(\underline{a}) = \Delta_c(\underline{a})$, $\Delta_{\underline{a}}(b) = \Delta_c(b)$, and $\Delta_{\underline{a}}(c) = \Delta_b(c)$.

2.13 Theorem. $|\underline{a},b,c|^{\pm}$ iff $|e,f,g|^{\pm}$ where e,f,g is any permutation of the symbols $\underline{a},b,c$.

Proof. This is an immediate consequence of 2.12.

2.14 Theorem. For each triple $\underline{a},b,c$ of distinct elements of E one and only one of the following hold: $\underline{a}|b|c$, $b|c|\underline{a}$, $c|\underline{a}|b$, $|\underline{a},b,c|^{\pm}$.

Proof. (at least one) By (PS1) we obtain: either $c \in \Delta_{\underline{a}}^*(b)$ or $c \in \Delta_{\underline{a}}(b)$, either $\underline{a} \in \Delta_b^*(c)$ or $\underline{a} \in \Delta_b(c)$, and either $b \in \Delta_c^*(\underline{a})$ or $b \in \Delta_c(\underline{a})$. Thus, either one of the first three relations holds, or we obtain:

$\Delta_{\underline{a}}(b) = \Delta_c(b)$, $\Delta_{\underline{a}}(c) = \Delta_b(c)$, and $\Delta_b(\underline{a}) = \Delta_c(\underline{a})$ and, consequently, $|\underline{a},b,c|^{\pm}$.

(at most one) If $|\underline{a},b,c|^{\pm}$, then from the above none of the first three can hold. For the same reasons any one of the first three denies the possibility of the fourth while by 2.07 any one of the first three denies the possibility of the remaining two.

2.15 Theorem. $|\underline{a},b,c|^{\pm}$ and $\underline{a}|b|d$ imply $c|b|d$ and $|\underline{a},d,c|^{\pm}$.

Proof. $|\underline{a},b,c|^{\pm}$ implies $\Delta_{\underline{a}}(b) = \Delta_c(b)$. $\underline{a}|b|d$ implies $d \in \Delta^*_{\underline{a}}(b)$. $\Delta_{\underline{a}}(b) = \Delta_c(b)$ iff $\Delta^*_{\underline{a}}(b) = \Delta^*_c(b)$ by (PS1). Therefore, $d \in \Delta^*_c(b)$ and $c|b|d$.

$c|b|d$ implies $d \in \Delta_b(c)$ and $c \in \Delta_b(d)$. $\underline{a}|b|d$ implies $d \in \Delta_b(\underline{a})$ and $\underline{a} \in \Delta_b(d)$. $|\underline{a},b,c|^{\pm}$ implies $\Delta_b(c) = \Delta_{\underline{a}}(c)$ and $\Delta_b(\underline{a}) = \Delta_c(\underline{a})$. Hence, $\Delta_d(c) = \Delta_{\underline{a}}(c)$, $\Delta_d(\underline{a}) = \Delta_c(\underline{a})$, and $\Delta_c(d) = \Delta_{\underline{a}}(d)$. Thus, $|\underline{a},d,c|^{\pm}$.

2.16 Definition. If E is a nonnull class of elements $\underline{a},b,c,\cdots$, then E is called a nonoriented chordal system or briefly a NCS iff there exists two trinary relations $\underline{a}|b|c$ and $|\underline{a},b,c|^{\pm}$ defined for the distinct triples of E such that the following axioms hold:

(NCS1) For each triple of distinct elements $\underline{a},b,c$, one and only one of the following relations hold: $\underline{a}|b|c$, $b|c|\underline{a}$, $c|\underline{a}|b$, $|\underline{a},b,c|^{\pm}$.

(NCS2) $\underline{a}|b|c$ iff $c|b|\underline{a}$.

(NCS3) If $|\underline{a},b,c|^{\pm}$, then $|e,f,g|^{\pm}$ provided e,f,g is a permutation of $\underline{a},b,c$.

(NCS4) $|\underline{a},b,c|^{\pm}$ and $\underline{a}|b|d$ imply $c|b|d$ and $|\underline{a},d,c|^{\pm}$.

(NCS5) $\underline{a}|b|c$ and $b|c|d$ imply $\underline{a}|b|d$ and $\underline{a}|c|d$.

(NCS6) Of the three relations $b|\underline{a}|c$, $b|\underline{a}|d$, and $c|\underline{a}|d$ at most two can hold.

2.17 **Theorem.** Every partition system satisfies the axioms of a nonoriented chordal system under the definitions 2.05 and 2.12.

Proof. The theorem is an immediate consequence of 2.06, 2.08, 2.09, 2.13, 2.14, and 2.15.

2.18 **Theorem.** If F is a NCS, then $\underline{a}|b|c$ and $\underline{a}|c|d$ imply $\underline{a}|b|d$ and $b|c|d$.

Proof. $d|\underline{a}|b$ and $\underline{a}|b|c$ imply by (NCS5) $d|\underline{a}|c$. But, $d|\underline{a}|c$ and $\underline{a}|c|d$ is impossible by (NCS1). Hence, (1) not $d|\underline{a}|b$. $|\underline{a},b,d|^{\pm}$ and $\underline{a}|b|c$ imply by (NCS4) $|\underline{a},c,d|^{\pm}$. Hence, (2) not $|\underline{a},b,d|^{\pm}$. $\underline{a}|c|d$ and $c|d|b$ imply by (NCS5) $\underline{a}|c|b$. Hence, by (NCS1) and (NCS2) (3) not $c|d|b$. $|b,c,d|^{\pm}$ and $\underline{a}|b|c$ imply by (NCS4) $|\underline{a},c,d|^{\pm}$. Hence by (NCS1), (4) not $|b,c,d|^{\pm}$. $c|b|d$ and $\underline{a}|b|d$ together contradict $\underline{a}|b|c$ by (NCS6). Hence, (5) not both $c|b|d$ and $\underline{a}|b|d$. $c|b|d$ and $b|d|\underline{a}$ imply by (NCS5) $c|d|\underline{a}$. Hence by (NCS1), (6) not both $c|b|d$ and $b|d|\underline{a}$. $b|c|d$ and $\underline{a}|d|b$ imply by (NCS5) $\underline{a}|b|d$. Hence by (NCS1) and (NCS2), not both $b|c|d$ and $\underline{a}|d|b$.

In summary we have:

(1)	not $d|\underline{a}|b$	and	(3)	not $c|d|b$;
(2)	not $|\underline{a},b,d|^{\pm}$	and	(4)	not $|b,c,d|^{\pm}$;
(5)	not both $\underline{a}|b|d$	and		$c|b|d$;
(6)	not both $b|d|\underline{a}$	and		$c|b|d$;
(7)	not both $\underline{a}|d|b$	and		$b|c|d$.

By (NCS1) and the above, we obtain either $\underline{a}|b|d$ or $b|d|\underline{a}$ and either $b|c|d$ or $d|b|c$. If $b|d|\underline{a}$, then not $b|c|d$ and not $c|b|d$ which contradicts (NCS1). If $d|b|d$, then not $\underline{a}|b|d$ and not $b|d|\underline{a}$ which contradicts (NCS1). Hence, $\underline{a}|b|d$ and $b|c|d$ for they are the only remaining possibilities.

2.19 Theorem. If F is a NCS, then $d \neq b$ and $\underline{a}|b|c$ imply either $d|b|c$ or $\underline{a}|b|d$ but not both.

Proof. By (NCS6), at most two of the above three hold. Suppose not $d|b|c$. $\underline{a}|b|c$ and $b|c|d$ imply by (NCS5) $\underline{a}|b|d$. $|d,b,c|^{\pm}$ and $\underline{a}|b|c$ imply $\underline{a}|b|d$ by (NCS3), (NCS4), and (NCS2). $c|d|b$ and $\underline{a}|b|c$ imply by (NCS2) and (NCS5) $\underline{a}|b|d$. Therefore, $\underline{a}|b|c$ and not $d|b|c$ imply $\underline{a}|b|d$. In the same way, $\underline{a}|b|c$ and not $\underline{a}|b|d$ imply $d|b|c$.

2.20 Theorem. If $\underline{a}$ and b are distinct elements of a NCS F, then the sets $\Delta_b(\underline{a})$, $\underline{a}$, $\Delta_b(\underline{a})$ form a partition of F relative to $\underline{a}$ where $\Delta_b^*(\underline{a}) = \{c: b|\underline{a}|c\}$ and $\Delta_b(\underline{a}) = F - [\underline{a} \cup \Delta_b^*(\underline{a})]$.

Proof. Since the chordal relations hold only for distinct triples of elements, it follows immediately from (NCS1) that the sets are mutually disjoint and thus partition F relative to $\underline{a}$.

2.21 Theorem. If $\underline{a}$ and b are distinct elements of a NCS, then $c \in \Delta_b(\underline{a})$ iff $\Delta_c(\underline{a}) = \Delta_b(\underline{a})$.

Proof. Suppose $c \neq b$ for otherwise the theorem is immediate. By 2.20, it will suffice to show $\Delta_c^*(\underline{a}) = \Delta_b^*(\underline{a})$ iff $c \in \Delta_b(\underline{a})$. $c \in \Delta_b(\underline{a})$ implies not $b|\underline{a}|c$ and $x \in \Delta_b^*(\underline{a})$ iff $b|\underline{a}|x$. But, $b|\underline{a}|x$ and not $b|\underline{a}|c$ imply $c|\underline{a}|x$ by 2.19. Therefore, $\Delta_b^*(\underline{a}) \subset \Delta_c^*(\underline{a})$ and similarly, $\Delta_c^*(\underline{a}) \subset \Delta_b^*(\underline{a})$.

If $\Delta_c(\underline{a}) = \Delta_b(\underline{a})$, then since not $c|a|c$ by (NCS1), we have $c \in \Delta_c(\underline{a}) = \Delta_b(\underline{a})$.

2.22 Theorem. If $\underline{a}$ and b are distinct elements of a NCS F, then $\Delta_{\underline{a}}^*(b) \subset \Delta_b(\underline{a})$.

Proof. $c \in \Delta^{**}_{\underline{a}}(b)$ iff $\underline{a}|b|c$. $\underline{a}|b|c$ implies not $b|\underline{a}|c$ by (NCS1) and, consequently, $c \in \Delta_b(\underline{a})$. Therefore, $\Delta^{**}_{\underline{a}}(b) \subset \Delta_b(\underline{a})$.

2.23 Theorem. If F is a nonoriented chordal system, then F satisfies the axioms of a partition system under the definition of 2.20.

Proof. 2.20 shows that there is a partition of F relative to each element of F. 2.21 shows that this partition is unique and hence that (PS1) is satisfied. 2.22 states that (PS2) is fulfilled.

2.24 Notation. If E is a partition system, then $R^{\pm}$ will denote the subset of E^3 such that $(\underline{a},b,c) \in R^{\pm}$ iff $|\underline{a},b,c|^{\pm}$.

2.25 Definition. If E is a partition system, then E is said to be orientable iff $R^{\pm}$ can be decomposed into two disjoint sets R^+ and R^- such that:

(O1) $(\underline{a},b,c) \in R^+$ iff $(\underline{a},c,b) \in R^-$;

(O2) $(b,c,\underline{a}) \in R^+$ if $(\underline{a},b,c) \in R^+$;

(O3) $(\underline{a},b,d) \in R^+$ if $(\underline{a},b,c)$ and $(\underline{a},c,d)$ in R^+.

2.26 Notation. For any orientation of E $(\underline{a},b,c) \in R^+(R^-)$ shall be denoted by $|\underline{a},b,c|^+(|\underline{a},b,c|^-)$.

2.27 Definition. Let E be a nonnull class of elements $\underline{a},b,c,\cdots$. E is called a chordal system or a CS iff there exists two trinary relations $\underline{a}|b|c$ and $|\underline{a},b,c|^+$ defined for the triples of distinct elements of E such that the following axioms hold.

(CS1) For each triple $\underline{a},b,c$ of distinct elements one and only one of the following five relations holds: $\underline{a}|b|c$, $b|c|\underline{a}$, $c|\underline{a}|b$, $|\underline{a},b,c|^+$,

$|\underline{a},c,b|^+$.

Notation. $|\underline{a},b,c|^-$ shall mean $|\underline{a},c,b|^+$. $|\underline{a},b,c|^\pm$ shall mean $|\underline{a},b,c|^+$ or $|\underline{a},b,c|^-$. $[\underline{a},b,c] \sim [e,f,g]$ shall mean that the relation holding for $\underline{a},b,c$ holds for e,f,g when $\underline{a}$ is replaced by e, b by f, and c by g.

(CS2) $|\underline{a},b,c|^+$ iff $|b,c,\underline{a}|^+$.

(CS3) $|\underline{a},b,c|^+$ and $|\underline{a},c,d|^+$ imply $|\underline{a},b,d|^+$ and $|b,c,d|^+$.

(CS4) $\underline{a}|b|c$ iff $c|b|\underline{a}$.

(CS5) $|\underline{a},b,c|^\pm$ and $\underline{a}|b|d$ imply $c|b|d$ and $|\underline{a},d,c|^\pm$ where $[\underline{a},b,c] \sim [a,d,c]$.

(CS6) $\underline{a}|b|c$ and $b|c|d$ imply $\underline{a}|b|d$ and $\underline{a}|c|d$.

(CS7) Of the three relations $b|\underline{a}|c$, $b|\underline{a}|d$, and $c|\underline{a}|d$ at most two can hold.

2.28 Theorem. If E is an oriented partition system, then E satisfies the axioms of a chordal system.

Proof. By 2.23, we have that E satisfies the axioms of a NCS. Therefore, every triple of distinct elements $\underline{a},b,c$ of E must satisfy one and only one of the following four relations: $\underline{a}|b|c$, $b|c|\underline{a}$, $c|\underline{a}|b$, $|\underline{a},b,c|^\pm$. Suppose $|\underline{a},b,c|^\pm$, then since R^+ and R^- are disjoint, we have either $|\underline{a},b,c|^+$ or $|\underline{a},b,c|^-$. If $|\underline{a},b,c|^-$, then by (O1) $|\underline{a},c,b|^+$. Thus, (CS1) is satisfied.

(CS2) is equivalent to (O2).

By (O3), $|\underline{a},b,c|^+$ and $|\underline{a},c,d|^+$ imply $|\underline{a},b,d|^+$. By (O2), $|\underline{a},b,c|^+$ implies $|c,\underline{a},b|^+$ and $|\underline{a},c,d|^+$ implies $|c,d,\underline{a}|^+$. $|c,d,\underline{a}|^+$ and $|c,\underline{a},b|^+$ imply $|c,d,b|^+$ by (O3). $|c,d,b|^+$ implies $|b,c,d|^+$ by (O2). Therefore,

$|\underline{a},b,c|^+$ and $|\underline{a},c,d|^+$ imply $|\underline{a},b,d|^+$ and $|b,c,d|^+$ and (CS3) is satisfied.

(CS4), (CS6), and (CS7) are among the NCS axioms.

It remains only to show that (CS5) is satisfied. Suppose $|\underline{a},b,c|^+$ and $\underline{a}|b|d$. $|\underline{a},b,c|^+$ implies $|\underline{a},b,c|^{\pm}$, hence by (NCS4), we obtain $c|b|d$ and $|\underline{a},d,c|^{\pm}$. $|\underline{a},d,c|^{\pm}$ implies by (O1) either $|\underline{a},d,c|^+$ or $|\underline{a},c,d|^+$. If $|\underline{a},c,d|^+$, then $|\underline{a},b,c|^+$ and $|\underline{a},c,d|^+$ imply by (O3) $|b,c,d|^+$ which is impossible. Similarly, $|\underline{a},b,c|^-$ and $\underline{a}|b|d$ imply $c|b|d$ and $|\underline{a},d,c|^-$.

2.29 Theorem. The notions of partition system and nonoriented chordal system are equivalent with respect to the definitions of paragraphs 2.05, 2.12, and 2.20.

Proof. Let E be a partition system. By 2.17, definitions 2.05 and 2.12 make E into a NCS E' and by the definition of 2.20, E' can be made into a partition system E". We must show that E and E" have the same structure, i.e., that if $\Delta_b(\underline{a})$ denotes a partition set in E and $\Delta_b(\underline{a})"$ denotes the corresponding partition set in E", then $\Delta_b(\underline{a}) = \Delta_b(\underline{a})"$ or equivalently $\Delta_b^*(\underline{a}) = \Delta_b^*(\underline{a})"$. Let $x \in \Delta_b(\underline{a})"$, then $b|\underline{a}|x$ by 2.05. $b|\underline{a}|x$ implies $x \in \Delta_b^*(\underline{a})"$ by 2.20. Hence, $\Delta_b^*(\underline{a}) \subset \Delta_b^*(\underline{a})"$. Since the above argument is reversible, we have also $\Delta_b^*(\underline{a})" \subset \Delta_b^*(\underline{a})$ and therefore, E and E" are equivalent.

Let F be a NCS. By 2.23, the definition of 2.20 makes F into a partition system F' and by 2.17 the definitions 2.05 and 2.17 make F' into a NCS F". Hence, we must show that a chordal relation holds for a triple $\underline{a},b,c$ in F iff the same chordal relation holds in F". In light of (NCS1), it will suffice to show $\underline{a}|b|c$ iff $\underline{a}/b/c$ where $\underline{a}|b|c$ is the parallel relation in F and $\underline{a}/b/c$ is the parallel relation in F".

Suppose $\underline{a}|b|c$, then $c \in \Delta^{*}_{\underline{a}}(b)$ by 2.20. $c \in \Delta^{*}_{\underline{a}}(b)$ implies $\underline{a}/b/c$ by 2.05. Since this argument is reversible, we have $\underline{a}|b|c$ iff $\underline{a}/b/c$ and the equivalence is established.

2.30 Remarks. As a consequence of 2.29, we can employ the results on nonoriented chordal systems and partition systems interchangably. This will be done in the succeeding paragraphs, the particular choice of notation being made for the sake of simplicity.

From the preceding work, it is clear that it can be demonstrated that the concepts of oriented partition system and chordal system are equivalent. The proof of this fact is omitted because of its similarity to the proof of 2.29.

In view of 2.28, the question arises as to when it is possible to assign an orientation to a nonoriented chordal system. Intuitively, it seems reasonable to expect that this can be accomplished without additional restrictions, or under very mild ones, at most. However, my best result to date is for the case where the NCS is normal. Normality is defined and discussed in Chapter III and the orientability theorem given at the close of Chapter IV.

In the next three paragraphs, we give a method of viewing nonoriented chordal systems in terms of partially ordered sets. This viewpoint is not exploited but is presented to illustrate the general nature of nonoriented chordal systems.

2.31 Definition. A collection of elements P form a partially ordered set with respect to the relation $\succcurlyeq$ iff:

1) $a \succcurlyeq a$ for no a in P;

2) if a,b,c are in P such that $a > b$ and $b > c$, then $a > c$.

2.32 Theorem. If P is a partially ordered set such that:

(P1) there exists $o \in P$ such that $a \neq o$ implies $o > \underline{a}$ or $\underline{a} > o$,

(P2) $\underline{a} > b > d$ and $\underline{a} > c > d$ implies either $b = c$, $b > c$, or $c > b$,

then P can be made into a partition system by letting $\Delta(\underline{a}) = \{x : x > \underline{a}\}$ and $\Delta^*(\underline{a}) = P - [\underline{a} \cup \Delta(\underline{a})]$.

Proof. The sets $\Delta(\underline{a})$, $\underline{a}$, and $\Delta^*(\underline{a})$ give a unique partition of P relative to $\underline{a}$. Hence (PS1).

The proof that (PS2) holds will be broken into two cases.

Case 1. ($\underline{a} > b$ or $b > \underline{a}$.) If $b > \underline{a}$, then $b \in \Delta(\underline{a})$ or $\Delta_b(\underline{a}) = \Delta(\underline{a})$. $x \in \Delta(b)$ implies $x > b$. $x > b$ and $b > \underline{a}$ implies $x > \underline{a}$. Therefore, $\Delta(\underline{a}) \supset \Delta(b)$ and consequently, $\Delta_b(\underline{a}) \supset \Delta^*_{\underline{a}}(b)$. If $\underline{a} > b$, then from the above we have $\Delta_{\underline{a}}(b) \supset \Delta^*_b(\underline{a})$. From the definition of $\Delta^*(\underline{a})$ this implies, $\Delta_b(\underline{a}) \supset \Delta^*_{\underline{a}}(b)$.

Case 2. ($\underline{a}$ and b are not comparable.) $\underline{a}$ and b not comparable implies by transitivity and (P1) that either $\underline{a} > o$ and $b > o$ or $o > \underline{a}$ and $o > b$. Consider the former situation, the latter will follow analogously. $\underline{a}$ and b not comparable implies $\underline{a} \in \Delta^*(b)$ and $b \in \Delta^*(\underline{a})$ or that $\Delta_b(\underline{a}) = \Delta^*(\underline{a})$ and $\Delta^*_{\underline{a}}(b) = \Delta(b)$. If $x \in \Delta(b)$, then $x > b$. If also $x > \underline{a}$, then $x > b > o$ and $x > \underline{a} > o$ implies by (P2) that $\underline{a}$ and b are comparable. Therefore, $\Delta(\underline{a}) \supset \Delta(b)$ or $\Delta_b(\underline{a}) \supset \Delta^*_{\underline{a}}(b)$.

2.33 Theorem. If E is a partition system, then E can be made into a partially ordered set satisfying conditions (P1) and (P2).

Proof. Let o be any element of E. Fix $\Delta(o)$. If $\underline{a} \in \Delta(o)$, let $\Delta(\underline{a}) = \Delta_o(\underline{a})$ and if $\underline{a} \in \Delta^*(o)$, let $\Delta(\underline{a}) = \Delta_o(\underline{a})$. By (PS2) we have either

$\Delta(o) \supset \underline{a} \cup \Delta(\underline{a})$ or $\Delta(\underline{a}) \supset o \cup \Delta(o)$.

Let $\underline{a} > b$ iff $\Delta(b) \supset \underline{a} \cup \Delta(\underline{a})$. $\underline{a} > \underline{a}$ for no $\underline{a}$ in E by (PS1). If $\underline{a} > b$ and $b > c$, then $\Delta(b) \supset \underline{a} \cup \Delta(\underline{a})$ and $\Delta(c) \supset b \cup \Delta(b)$. Thus, $\Delta(c) \supset \underline{a} \cup \Delta(\underline{a})$ and $\underline{a} > c$. If $\underline{a} \neq o$, then from above we have either $\underline{a} > o$ or $o > \underline{a}$. Consequently, it remains only to show that (P2) holds.

Suppose $\underline{a} > b > d$ and $\underline{a} > c > d$. If $b \neq c$, then $\Delta_c(b) \supset \Delta^*_b(c)$. Hence one of the following four statements is true: $\Delta^*(b) \supset c \cup \Delta^*(c)$, $\Delta(b) \supset c \cup \Delta(c)$, $\Delta^*(b) \supset c \cup \Delta(c)$, or $\Delta(b) \supset c \cup \Delta^*(c)$. If $\Delta^*(b) \supset c \cup \Delta(c)$, then since $\underline{a} > c$ implies $\underline{a} \in \Delta(c)$, we obtain $\underline{a} \in \Delta^*(b)$ and $\underline{a} \not> b$. If $\Delta(b) \supset c \cup \Delta^*(c)$, then since $c > d$ implies $\Delta(d) \supset c \cup \Delta(c)$ or $\Delta^*(c) \supset d \cup \Delta(d)$ and $b > d$ implies $\Delta(d) \supset b \cup \Delta(b)$ or $\Delta^*(b) \supset d \cup \Delta^*(d)$, we obtain a contradiction of (PS1). Hence, either $\Delta^*(b) \supset c \cup \Delta^*(c)$ or $\Delta(b) \supset c \cup \Delta(c)$ and thus either $b > c$ or $c > b$.

CHAPTER III

A TOPOLOGY FOR NONORIENTED CHORDAL SYSTEMS

3.01 Remark. In this chapter we will be concerned with nonoriented chordal systems or equivalently partition systems. The first goal will be to define a "natural" topology for a NCS. To reduce the number of trivial remarks that are required when the NCS can contain only one element, we assign the discrete topology to such a NCS and in the future restrict our attention to those systems which contain two or more elements. ff

After defining an intrinsic topology for each NCS, we discuss the notion of normal subdivision. It is shown that necessary and sufficient conditions can be had in terms of the intrinsic topology of E for a NCS to be capable of being normally subdivided.

A number of the basic notions and theorems of elementary point set topology are used in the succeeding paragraphs without specific reference. For full details on these, the reader is referred to [5].

Throughout this chapter we assume a NCS E given.

3.02 Definition. A subset V of E is cyclic iff V contains at least one element and $|\underline{a},b,c|^{+}_{-}$ for every distinct triple $\underline{a},b,c$ in V.

3.03 Definition. Let V be a cyclic set. If V contains only one element x, then $\theta(V) = \Delta(x)$ for some choice of $\Delta(x)$. If V contains more than one element, then

$$\theta(V) = \Delta_x(y) \cap \{\bigcap_{z \in V-y} \Delta_y(z)\}$$

where x and y are distinct elements of V.

3.04 Remark. If V consists of a single element, then $\theta(V)$ is double valued. If V consists of more than one element, then $\theta(V)$ is well defined. This follows immediately from the definition of cyclic set and the properties of $|x,y,z|^{\pm}$. In the important case where $\theta(V)$ is known to contain an element $\underline{a}$, then

$$\theta(V) = \bigcap_{x \in V} \Delta_{\underline{a}}(x).$$

V can be intuitively thought of as the "boundary" of $\theta(V)$.

3.05 Theorem. If V is a cyclic set which contains at least two elements, then an element x of E is a member of $\theta(V)$ iff either $\underline{a}|x|b$ or $|\underline{a},x,b|^{\pm}$ for every distinct pair $\underline{a},b$ of V.

Proof. Let $x \in \theta(V)$, then $x \notin \Delta_b(\underline{a})$ and $x \notin \Delta_{\underline{a}}(b)$ by 2.03. Hence, $x \notin \Delta^*_b(\underline{a})$ and $x \notin \Delta^*_{\underline{a}}(b)$ or by 2.05 not $b|\underline{a}|x$ and not $\underline{a}|b|x$. Thus by 2.16, we have either $\underline{a}|x|b$ or $|\underline{a},x,b|^{\pm}$. Conversely, if either $\underline{a}|x|b$ or $|\underline{a},x,b|^{\pm}$, then $x \notin \Delta^*_b(\underline{a})$ and $x \notin \Delta^*_{\underline{a}}(b)$. Hence, $x \in \Delta_b(\underline{a}) \cap \Delta_{\underline{a}}(b)$. Since this is true for all $\underline{a},b$ of V, we have by 3.03 $x \in \theta(V)$.

3.06 Definition. If V is a cyclic subset of E and $\theta(V)$ is well defined, then $\lambda(V) = V \cup \theta(V)$.

3.07 Definition. Two sets $\lambda(V_1)$ and $\lambda(V_2)$ are said to be adjacent iff V_1 and V_2 have exactly one element x in common and $\Delta(x)$ can be chosen so that $\Delta(x) \supset \theta(V_1)$ and $\Delta^*(x) \supset \theta(V_2)$.

3.08 Theorem If U and V are cyclic sets such that $\theta(U) \cap \theta(V) \neq 0$, then there exists a cyclic set $W = \{\lambda(U) \cap V\} \cup \{\lambda(V) \cap U\}$ such that $\theta(W) = \theta(U) \cap \theta(V)$.

Proof. $W = \{\theta(U) \cap V\} \cup \{\theta(V) \cap U\} \cup \{U \cap V\}$. Let x,y,z be any three elements of W. If all three belong either to U or to V, then $|x,y,z|^{+}_{-}$. If two belong to each set, say x and y in U and x and z in V, then since z is in $\theta(U)$, we have either $x|z|y$ or $|x,y,z|^{+}_{-}$ and since y is in $\theta(V)$, we have either $x|y|z$ or $|x,y,z|^{+}_{-}$. But $x|z|y$ and $x|y|z$ are contradictory and thus $|x,y,z|^{+}_{-}$. The remaining case is where only one cyclic set contains two elements, say x and y are in U and z is in V. Then since z is in $\theta(U)$, we have either $x|z|y$ or $|x,y,z|^{+}_{-}$. But x and y are in $\theta(V)$ and hence the former is impossible. Thus, $|x,y,z|^{+}_{-}$ in all cases and W is cyclic.

If $\underline{a}$ is in $\theta(U) \cap \theta(V)$, then $\theta(U) = \bigcap_{x \in U} \Delta_{\underline{a}}(x)$ and $\theta(V) = \bigcap_{x \in V} \Delta_{\underline{a}}(x)$. Consequently, $\theta(U) \cap \theta(V) = \bigcap_{x \in U \cup V} \Delta_{\underline{a}}(x)$. But $W \subset U \cup V$ and therefore $\theta(U) \cap \theta(V) \subset \theta(W)$.

x not in $\theta(U)$ implies the existence of z in U such that x is not in $\Delta_{\underline{a}}(z)$. If z is in W, then x is not in $\theta(W)$. If z is not in W, then z is not in $\lambda(V)$ by the definition of W and, consequently, there exists a Z in V such that $z \in \Delta^{*}_{\underline{a}}(Z)$ or $z|Z|\underline{a}$. If $Z \notin W$, then $Z \notin \lambda(U)$ and, consequently, there exists $z' \in U$ such that $Z \in \Delta^{*}_{\underline{a}}(z')$ or $Z|z'|\underline{a}$. However, $z|Z|\underline{a}$ and $Z|z'|\underline{a}$ imply $z|z'|\underline{a}$ and this implies $\underline{a} \notin \theta(U)$. Therefore, $Z \in W$. $x \notin \Delta_{\underline{a}}(z)$ implies either $x = z$ or $x \in \Delta^{*}_{\underline{a}}(z)$. If $x = z$, then $z|Z|\underline{a}$ implies $x \notin \theta(W)$. If $x \in \Delta^{*}_{\underline{a}}(z)$, then $x|z|\underline{a}$ and $z|Z|\underline{a}$ imply $x|Z|\underline{a}$ which in turn implies $x \notin \theta(W)$. Hence, $x \notin \theta(U)$ implies $x \notin \theta(W)$ and $\theta(W) \subset \theta(U)$. Similiarly, $\theta(W) \subset \theta(V)$. Hence, $\theta(W) \subset \theta(U) \cap \theta(V)$ and the theorem is established.

3.09 Definition. Let x be an element of E. The "intrinsic"

neighborhoods of x will consist of all the sets $\theta(V)$ where V is cyclic and $x \in \theta(V)$.

3.10 Remark. The topology that is imposed on E by intrinsic neighborhoods will be called the intrinsic topology of E. It is clear that each element of E has a neighborhood since E contains at least two elements and a one-element set is trivially cyclic.

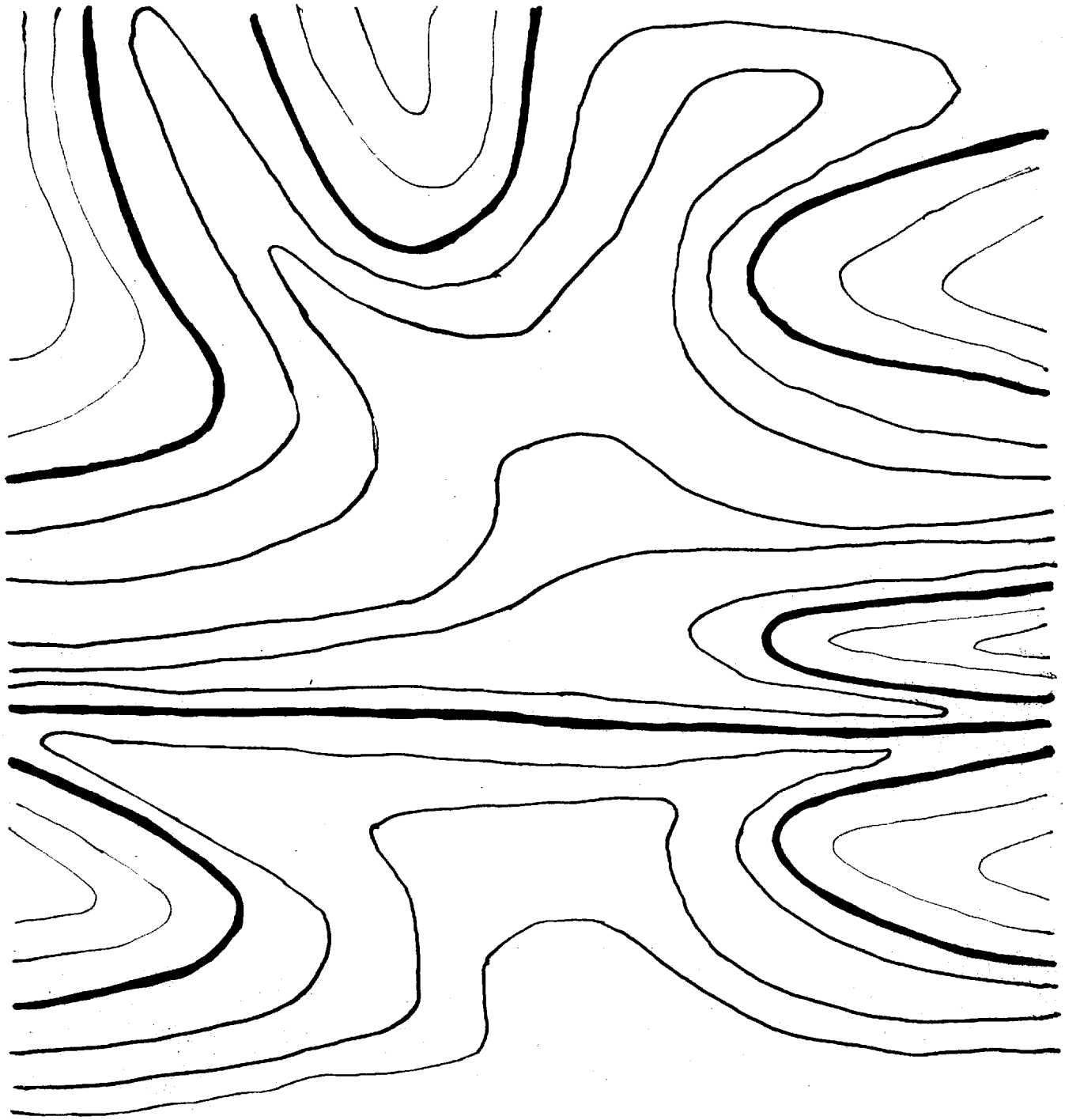
3.11 Theorem. E is a T_1 space with respect to its intrinsic topology.

Proof. By definition, every point is contained in each of its neighborhoods and every neighborhood is a neighborhood of each point that it contains. If $N_1(\underline{a})$ and $N_2(\underline{a})$ are two neighborhoods of $\underline{a}$, then by 3.08, their intersection is again a neighborhood of $\underline{a}$.

If $\underline{a}$ and b are distinct elements of E, then $\Delta_b(\underline{a})$ and $\Delta_{\underline{a}}(b)$ are, respectively, neighborhoods of $\underline{a}$ and b which do not contain, respectively, b and $\underline{a}$.

3.12 Remark. In general a nonoriented chordal system need not be a T_2 topological space. This is exemplified by the interpretation of a regular curve-family as a chordal system as described in the introduction. Figure 3-1 illustrates such a family and points out the elements where the chordal system fails to be T_2 with respect to its intrinsic topology.

3.13 Definition. A subset of E will be said to be half-parallel iff it is homeomorphic to the nonnegative half of the real axis. A subset of E will be said to be parallel iff it is homeomorphic to the



Heavy Lines Represent

Discontinuity Elements

Figure 3-1

real line. From the cut point property of real numbers, it is seen that only the parallel chordal relation can hold for triples of these sets.

3.14 Remark. The terms parallel and half-parallel are carried over from the application of the notion of a chordal system to regular curve families in [1] and succeeding papers. For regular curve families, a set S of curves is half-parallel iff there exists a one-to-one correspondence between the elements of S and the lines y = constant in $0 \leq y < \infty$ which preserves chordal relations. It is readily seen that the above definition and 3.13 coincide. The symbol R_1 will be used to denote the real numbers in the following paragraphs. A R_1 neighborhood of a NCS will be a neighborhood that is homeomorphic to an open interval of R_1.

3.15 Definition. Let α be a sequence of integers p_i $(i = 1,2,\cdots,n)$ with $p_i > 0$. Then for any integer $k > 0$, α,k will denote the sequence $p_1,p_2,\cdots,p_n,k$.

3.16 Definition. Let A be a class of finite sequences of integers greater than zero. A will be termed admissible iff the following conditions hold:

(1) $\underline{a}$ contains the sequence 1 and no other one element sequence;

(2) if α,k is in A and $k > 1$, then $\alpha,k-1$ is in A;

(3) if $\alpha,1$ is in A, then α is in A.

3.17 Definition. A subset E_0 of E is seminormal iff there exists an admissible class A of finite sequences and a corresponding set of distinct elements a_α of E_0 such that:

(1) $E_0 = a_1 \vee \Delta(a_1)$ for appropriate choice of $\Delta(a_1)$.

(2) The sets $V_\alpha = a_\alpha \cup \{ \cup a_{\alpha,k} \}$ are cyclic.

(3) The $\theta(V_\alpha)$ can be uniquely determined so that if α and α,k are in A then $\lambda(V_\alpha)$ and $\lambda(V_{\alpha,k})$ are adjacent.

(4) $a_\alpha \cup \theta(V_\alpha)$ is half-parallel.

(5) $E_0 = \bigcup_\alpha \lambda(V_\alpha)$.

(6) $x \in V_\alpha - a_\alpha$ implies the existence of b in $\theta(V_\alpha)$ such that $|a_\alpha, b, x|^+_-$.

The V_α are said to determine the seminormal subdivision of E_0.

3.18 Definition. A partition system E is normal iff there exists an element $\underline{a}$ of E such that $\underline{a} \cup \Delta(\underline{a})$ and $\underline{a} \cup \Delta^*(\underline{a})$ are seminormal.

3.19 Remark. In the following paragraphs we shall investigate normal partition systems with respect to their intrinsic topology. The symbol E_N shall be used to denote a normal partition system.

3.20 Theorem. Every E_N is separable.

Proof. By hypothesis, there exists an element a_1 of E_N such that $a_1 \cup \Delta(a_1)$ and $a_1 \cup \Delta^*(a_1)$ are seminormal. Consider $a_1 \cup \Delta(a_1)$. Since $a_1 \cup \Delta(a_1) = \bigcup_\alpha [a_\alpha \cup \theta(V_\alpha)]$, E_N is the union of a denumerable number of half-parallel sets. Each half-parallel set is homeomorphic to the nonnegative real axis and, consequently, separable. Thus E_N is separable.

3.21 Theorem. If $\theta(V_\alpha)$ is any parallel set obtained by normal subdivision, then every element $\underline{a}$ of V_α is a limit point of $\theta(V_\alpha)$.

Proof. Suppose the contrary, then there exists V_α, $\underline{a} \in V_\alpha$, and $\theta(V)$ ($\theta(V)$ a neighborhood of $\underline{a}$) such that $\theta(V_\alpha) \cap \theta(V) = 0$. Since

$\theta(V) \cap \theta(V_\alpha) = 0$, we have for each x in $\theta(V_\alpha)$ a z in V such that $x \in \Delta^*_{\underline{a}}(z)$ or $\underline{a}|z|x$. $z \notin V_\alpha$ for then $x \notin \theta(V_\alpha)$ by 3.05. Hence, either $z \in \theta(V_\alpha)$ or there exists y in V_α such that $z \in \Delta^*_x(y)$ or $x|y|z$. But $\underline{a}|z|x$ and $z|y|x$ imply by (NCS5) $\underline{a}|y|x$ which in turn implies $x \notin \theta(V_\alpha)$ by 3.05. Hence $z \in \theta(V_\alpha)$.

Let V' denote the set of z's described above. V' is not empty since $\theta(V_\alpha)$ is not empty. V' can contain at most two elements for it is a subset of V and therefore cyclic and is also contained in $\theta(V_\alpha)$ which is parallel. If V' contains two elements z_1 and z_2, then since $\theta(V_\alpha)$ is parallel, there exists $x \in \theta(V_\alpha)$ such that $z_1|x|z_2$. From the above, we must have either $\underline{a}|z_1|x$ or $\underline{a}|z_2|x$. But $\underline{a}|z_1|x$ and $z_1|x|z_2$ implies $\underline{a}|z_1|z_2$ and $\underline{a}|z_2|x$ and $z_2|x|z_1$ implies $\underline{a}|z_2|z_1$ by (NCS5) while either $\underline{a}|z_1|z_2$ or $\underline{a}|z_2|z_1$ implies $a \notin \theta(V)$ by 3.05. Thus, V must consist of a single element z which separates $\theta(V_\alpha)$ from $\underline{a}$. This is impossible since z is an interior point of $\theta(V_\alpha)$. Hence, a contradiction and the theorem is proved.

3.22 Lemma. If $\theta(U)$ and $\theta(V)$ are neighborhoods of $\underline{a}$ and $\theta(U)$ is R_1, then $\theta(U) \cap \theta(V)$ is a R_1 neighborhood of $\underline{a}$.

Proof. $\theta(W) = \theta(U) \cap \theta(V)$ is a neighborhood of $\underline{a}$ by 3.08, where

$$W = \{\theta(U) \cap V\} \cup \{\theta(V) \cap U\} \cup \{U \cap V\}$$
$$= \{\lambda(U) \cap V\} \cup \{\lambda(V) \cap U\}$$

Hence,

$$\theta(W) = \left\{\bigcap_{x \in \overline{\lambda(V)}} \Delta_{\underline{a}}(x)\right\} \cap \left\{\bigcap_{x \in \overline{\lambda(U)}} \Delta_{\underline{a}}(x)\right\} = \theta(U) \cap \left\{\bigcap_{x \in \overline{\theta(U)}} \Delta_{\underline{a}}(x)\right\}$$

$\theta(U)$ a R_1 neighborhood implies that each of its distinct triples satisfies a parallel relation. Consequently, $V \cap \theta(U)$ can contain at most

two elements and $\theta(W)$ is an R_1 neighborhood of $\underline{a}$.

3.23 Notation. If the set $a_\alpha \cup \theta(V_\alpha)$ is half-parallel, its elements can be represented as d_t for $0 \leq t < \infty$, where $d_r|d_s|d_t$ is equivalent to the condition that s lies between r and t. If $\theta(V_\alpha)$ is a half-parallel set obtained by normal subdivision and $a_{\alpha,k} \in V_\alpha - a_\alpha$, then $B(a_{\alpha,k})$ shall denote the set of all d_t such that $|a_\alpha, a_{\alpha,k}, d_t|^+_-$. By condition (6) of normality, $B(a_{\alpha,k}) \neq 0$.

3.24 Lemma. Let $a_\alpha \cup \theta(V_\alpha)$ be a half-parallel set obtained by normal subdivision whose elements are represented as described in 3.23. Then $a_{\alpha,k} \in V_\alpha - a_{\alpha,k}$ implies the existence of a unique number $t_{\alpha,k} \geq 0$ such that

$$\text{either (1) } B(a_{\alpha,k}) = \bigcup_{t \leq t_{\alpha,k}} d_t$$

$$\text{or (2) } B(a_{\alpha,k}) = \bigcup_{t \geq t_{\alpha,k}} d_t$$

Proof. If $d_t \in B(a_{\alpha,k})$ and $s > t$, then $a_\alpha|d_t|d_s$. Hence, by (NCS4) $|a_\alpha, a_{\alpha,k}, d_s|^+_-$ and $d_s \in B(a_{\alpha,k})$. Therefore, $B(a_{\alpha,k})$ must have one of the above forms.

3.25 Notation. $\theta(V^k_\alpha)$ shall denote the set $\Delta_{a_{\alpha,\kappa}}(d_{t_{\alpha,\kappa}}) \cap \theta(V_\alpha)$ where $a_\alpha \cup \theta(V_\alpha)$ is a set obtained by normal subdivision and $t_{\alpha,k}$ is the number defined in the preceding lemma. V^k_α is well defined by 3.08.

3.26 Lemma. $d_t \in \theta(V^k_\alpha)$ iff $|a_{\alpha,k}, d_{t_{\alpha,\kappa}}, d_t|^+_-$.

Proof. $|a_{\alpha,k}, d_{t_{\alpha,\kappa}}, d_t|^+_-$ implies $d_t \in \Delta_{a_{\alpha,\kappa}}(d_{t_{\alpha,\kappa}})$ which in turn implies $d_t \in \theta(V^k_\alpha)$.

If $d_t = a_{\alpha,k}$, then $\theta(V^k_\alpha) = \theta(V_\alpha)$ and the lemma holds as a consequence

of 3.23. Hence suppose $a_\alpha \neq d_t$.

If $\theta(V_\alpha^k) = \{ d_t : t > t_{\alpha,k} \}$, then $a_\alpha | d_{t_{\alpha,k}} | d_t$. $a_\alpha | d_{t_{\alpha,k}} | d_t$ and $d_{t_{\alpha,k}} | d_t | a_{\alpha,k}$ implies $a_\alpha | d_t | a_{\alpha,k}$ by (NCS5). $a_\alpha | d_t | a_{\alpha,k}$ and $a_\alpha | d_{t_{\alpha,k}} | d_t$ contradict the conclusion of 3.24. Therefore, $|a_{\alpha,k}, d_{t_{\alpha,k}}, d_t|^{\pm}$ by 3.05.

If $\theta(V_\alpha^k) = \{ d_t : t \geq t_{\alpha,k} \}$, then $a_\alpha | d_t | d_{t_{\alpha,k}}$. $a_\alpha | d_t | d_{t_{\alpha,k}}$ and $a_{\alpha,k} | d_t | d_{t_{\alpha,k}}$ imply $\Delta_{a_\alpha}(d_t) = \Delta_{a_{\alpha,k}}(d_t)$. $d_t \in \theta(V_\alpha^k)$ and $\Delta_{a_\alpha}(d_t) = \Delta_{a_{\alpha,k}}(d_t)$ imply $|a_{\alpha,k}, a_\alpha, d_t|^{\pm}$ by 3.05. $|a_{\alpha,k}, a_\alpha, d_t|^{\pm}$ and $a_\alpha | d_t | d_{t_{\alpha,k}}$ imply a contradiction of 3.24. Therefore, $|a_{\alpha,k}, d_{t_{\alpha,k}}, d_t|^{\pm}$ by 3.05.

3.27 Remark. As a consequence of the last lemma, we have that $a_{\alpha,k}$ and $d_{t_{\alpha,k}}$ are not separated by any of the elements of $\theta(V_\alpha^k)$. By 3.21, $a_{\alpha,k}$ and $d_{t_{\alpha,k}}$ are limit points of $\theta(V_\alpha^k)$. Because of the parallel nature of $\theta(V_\alpha^k)$ and the lack of separation just mentioned, we must have that every sequence of $\theta(V_\alpha^k)$ that approaches $a_{\alpha,k}$ must also approach $d_{t_{\alpha,k}}$ and vice versa.

3.28 Lemma. $a_{\alpha,k} \cup \theta(V_\alpha^k)$ is half-parallel.

Proof. Since $a_\alpha \cup \theta(V_\alpha)$ is half-parallel, we have immediately that $d_{t_{\alpha,k}} \cup \theta(V_\alpha^k)$ is half-parallel. By the preceding remark, we have that a sequence of elements of $\theta(V_\alpha^k)$ converges to $d_{t_{\alpha,k}}$ iff it converges to $a_{\alpha,k}$. Thus, $a_{\alpha,k} \cup \theta(V_\alpha^k)$ is half-parallel. For if H is a homeomorphism which carries $d_{t_{\alpha,k}} \cup \theta(V_\alpha^k)$ onto $0 \leq t < \infty$ where $H(d_{t_{\alpha,k}}) = 0$, then H^* where $H^*(b) = H(b)$ for $b \in \theta(V_\alpha^k)$ and $H^*(a_{\alpha,k}) = 0$, is also a homeomorphism with the desired properties.

3.29 Theorem. Let $E_\alpha = a_\alpha \cup \Delta(V_\alpha)$, then $\alpha \neq \beta$ implies $E_\alpha \cap E_\beta = 0$.

Proof. $\theta(V_1) = \Delta(a_1) \cap \{ \bigcap_k \Delta_a (a_{1,k}) \}$ by the definition of normality and 3.03 and since $\lambda(V_\alpha)$ and $(V_{\alpha,k})$ are adjacent,

$$\theta(V_{\alpha,k}) = \Delta^*_{a_\alpha}(a_{\alpha,k}) \cap \{ \bigcap_n \Delta_a (a_{\alpha,k,n}) \} .$$

If $j \neq k$, then

$$E_{\alpha,j} \cap E_{\alpha,k} \subset \{ a_{\alpha,j} \cup \Delta^*_{a_\alpha}(a_{\alpha,j}) \} \cap \{ a_{\alpha,k} \cup \Delta^*_{a_\alpha}(a_{\alpha,k}) \} .$$

Since a_α, $a_{\alpha,j}$, $a_{\alpha,k} \in V_\alpha$, we have by 2.12 and (PS2)

$$\Delta_{a_\alpha}(a_{\alpha,j}) = \Delta_{a_{\alpha,k}}(a_{\alpha,j}) \supset \Delta^*_{a_{\alpha,j}}(a_{\alpha,k}) = \Delta^*_{a_\alpha}(a_{\alpha,k}).$$

Consequently, $\Delta^*_{a_\alpha}(a_{\alpha,j}) \cap \Delta^*_{a_\alpha}(a_{\alpha,k}) = 0$ and $E_{\alpha,j} \cap E_{\alpha,k} = 0$.

Consider E_α and E_β with $\alpha \neq \beta$, let α_0 be the largest sequence such that $\alpha = \alpha_0, p_1, \cdots, p_m$ and $\beta = \alpha_0, p_1', \cdots, p_n'$ with $p_1 \neq p_1'$. We have just treated the case where $\alpha = \alpha_0, p_1$ and $\beta = \alpha_0, p_1'$. Suppose $m > 1$ and $n \geq 1$, then $E_\alpha \subset a_\alpha \cup \Delta^*_{a_{\alpha_0,\cdots,p_{m-1}}}(a_\alpha) \subset \cdots \subset \Delta^*_{a_{\alpha_0}}(a_{\alpha_0,p_1})$ while $E_\beta \subset a_{\alpha_0,p_1'} \cup \Delta^*_{a_{\alpha_0}}(a_{\alpha_0,p_1'})$, hence $E_\alpha \cap E_\beta = 0$. If $m \geq 1$ and $\beta = \alpha_0$, then $E_\alpha \subset a_{\alpha_0,p_1} \cup \Delta^*_{a_{\alpha_0}}(a_{\alpha_0,p_1})$ while $E_\beta \subset \Delta_{a_{\alpha_0}}(a_{\alpha_0,p_1})$ and again $E_\alpha \cap E_\beta = 0$. Hence the theorem follows.

3.30 Theorem. Every E_N is locally R_1.

Proof. Let $c \in \Delta(a_1)$ and $\theta(V)$ be any neighborhood of c. By 3.22, it will suffice to show that there exists a parallel neighborhood of c. By 3.29, c a member of $\Delta(a_1)$ implies the unique existence of $\alpha, k \in A$ such that c belongs to $a_{\alpha,k} \cup \theta(V_{\alpha,k})$. If $c \in \theta(V_{\alpha,k})$, we are through for $\theta(V_{\alpha,k})$ is parallel.

Let $c = a_{\alpha,k}$. $\theta(V^k_\alpha)$ and $\theta(V_{\alpha,k})$ are adjacent by (6) of normality and $a_{\alpha,k}$ is a limit point of each by 3.27. $a_{\alpha,k} \cup \theta(V^k_\alpha)$ and $a_{\alpha,k} \cup \theta(V_{\alpha,k})$ are half-parallel by 3.28 and normality.

Let H and H' be homeomorphisms which map $a_{\alpha,k} \cup \theta(V^k_\alpha)$ and $a_{\alpha,k} \cup \theta(V_{\alpha,k})$

onto $-\infty < t \leq 0$ and $0 < t \leq \infty$, respectively. Let H^* equal H on $\theta(V_\alpha^k)$ and H' on $a_{\alpha,k} \cup \theta(V_{\alpha,k})$. Clearly, H^* is one-to-one and continuous.

$\theta(V_\alpha^k) \cup a_{\alpha,k} \cup \theta(V_{\alpha,k}) = \theta(V')$ where V' is the cyclic set $V_\alpha^k \cup V_{\alpha,k} - a_{\alpha,k}$. Hence the image of $-\infty < t < \infty$ under H^{-1} is a neighborhood of $a_{\alpha,k}$. If (s,t) is an open interval containing the origin, then its image under H^{*-1} is $\theta(V') \cap \Delta_{a_{\alpha,k}}(H^{*-1}(s)) \cap \Delta_{a_{\alpha,k}}(H^{*-1}(t))$. By 3.08, this set is a neighborhood of $a_{\alpha,k}$. Therefore, H^* is a homeomorphism and $\theta(V')$ is an R_1 neighborhood of $a_{\alpha,k}$.

The cases where $c = a_1$ and $c \in \Delta^*(a_1)$ are similar to the ones treated above. It follows that every E_N is locally R_1.

3.31 Theorem. Every E_N is connected.

Proof. Since $a_\alpha \cup \theta(V_\alpha)$ is half-parallel, it is connected. By 3.21, $a_{\alpha,k}$ is in the closure of $\theta(V_\alpha)$. Hence, $\lambda(V_\alpha)$ is connected. By condition (3) of normality, $\lambda(V_\alpha)$ and $\lambda(V_{\alpha,k})$ have $a_{\alpha,k}$ in common. Hence, their union is connected. By (5) of normality, $a_1 \cup \Delta(a_1) = \bigcup_\alpha \lambda(V_\alpha)$. Thus, from the definition of the admissible class A it is seen that $a_1 \cup \Delta(a_1)$ is connected. Similarly, $a_1 \cup \Delta^*(a_1)$ is connected. It follows that E_N is connected.

3.32 Remark. Summarizing paragraphs 3.19 - 3.31, we have that every partition system that can be normally subdivided is connected, separable, and locally R_1 with respect to its intrinsic topology. In the remaining paragraphs of the chapter, it is assumed that our partition system E enjoys these three properties. The main goal then will be to show that these conditions are sufficient to permit E to be normally subdivided.

Since the partition system E is connected, and $\Delta(\underline{a})$ and $\Delta(\underline{a})$ are disjoint open sets, we have that every element $\underline{a}$ of E is a cut point.

3.33 Theorem. If S is a parallel set and $V \neq 0$ is the boundary of S, then V is a denumberable cyclic set.

Proof. Let $\underline{a}$,b,c be any three elements of V. $\underline{a}|b|c$ implies $\Delta_{\underline{a}}(b) \cap \Delta_c(b) = 0$ and consequently, that S is disconnected. This is impossible. Hence, not $a|b|c$. Similarly, not $b|c|\underline{a}$ and not $c|\underline{a}|b$. Thus by elimination, $|\underline{a},b,c|^{+}_{-}$ and V is cyclic.

Let $x \in S$. Then from the definition of cyclic set, it is seen that $\Delta^{*}_{x}(\underline{a}) \cap \Delta^{*}_{x}(b) = 0$ for every pair $\underline{a}$,b in V. Since E is locally R_1, every $\Delta_x(c)$ contains inner points. Hence by the separability of E, V is at most countable.

3.34 Theorem. If S is a parallel set and $V \neq 0$ is the boundary of S, then $S = \theta(V)$.

Proof. If V = a, then since S is connected, $\Delta(\underline{a})$ can be selected to contain S. Thus, $S \subset \theta(V)$. Let c be in S and $\underline{a}$,b be in V. Since $\underline{a}$ and b are boundary points of S either $c|\underline{a}|b$ or $\underline{a}|b|c$ would deny the connectedness of S. Hence, either $\underline{a}|c|b$ or $|\underline{a},c,b|^{+}_{-}$ and by 3.05, c is in $\theta(V)$. Thus, $S \subset \theta(V)$.

Suppose there exists $c \in \theta(V)$ not in S. Let $\underline{a} \in V$. E is connected and locally R_1. Hence, there exists a simple chain of R_1 neighborhoods $\{N_i\}_{i=1}^{n}$ from c to $\underline{a}$ (c.f. [5] p. 34). Since the chain is simple and $\underline{a}$ is a cut point of E, the only set of the chain that can meet $\Delta^{*}_{c}(\underline{a})$ is N_n. Since $\underline{a} \in V$, $N_n \cap \Delta_c(\underline{a})$ must contain points of S. Consequently, there must be another boundary point of S somewhere in the chain.

Each of the N_i can contain two points of V, for if any were to contain three or more, then S could not be connected. Thus, the chain contains only a finite number of the points of V. Let b denote the first point of V occuring in the chain as one proceeds from c to $\underline{a}$. Let N_k be the neighborhood of the chain with smallest index containing b. Then $\{N_i\}_{i=1}^{k}$ is a simple chain from c to b containing no other points of V in between. But by a repetition of the argument used above, there must be a point of V in this chain between b and c. Therefore, a contradiction. Hence, $c \in S$ and $\theta(V) \subset S$.

3.35 Theorem. If $\underline{a}$ is any point of E, $\Delta(\underline{a})$ chosen arbitrarily, and K any compact set of E containing $\underline{a}$, then there exists a half-parallel set $S(\underline{a})$ which is contained in $\underline{a} \cup \Delta(\underline{a})$ and which has points that do not lie in K.

Proof. Since E is locally R_1, we can assume the existence of a homeomorphism h which maps a neighborhood N_0 of $\underline{a}$ onto the interval $(-1,1)$ such that $h(\underline{a}) = 0$ and the points of $N_0 \cap \Delta(\underline{a})$ are carried onto $(0,1)$. Let each point of K be covered by one of its R_1 neighborhoods. Since K is compact, we can find a finite number of these neighborhoods $N_1,\cdots,N_r$ such that they cover K and each is homeomorphic to $(0,1)$.

Let S_0 be the half-parallel set obtained by cutting down h to h_0 so that its image set is $(0,1)$. Suppose $S_0 \subset K$. Let P_n be a sequence of points of S_0 such that $\lim h_0(P_n) = 1$. From the compactness of K, there exists a point a_1 of K such that a subsequence of $\{P_n\}$ approaches a_1. Let N_k be a neighborhood of the finite covering which contains a_1. Let h_1 denote the homeomorphism which carries N_k onto

$(0,1)$. From the parallel nature of N_0 and N_k, we must have that either $h_1(P_n) > h_1(a_1)$ or $h_1(P_n) < h_1(a_1)$ for P_n in $N_0 \cap N_k$. In case the former condition holds, we can employ the mapping $x' = 1 - x$ to $(0,1)$ after h_1 so that the latter can be assumed to always hold. We now map the half open interval $[h_1(a_1),1)$ homeomorphically onto $[1,2)$ so that $h_1(a_1)$ is carried into 1. Let h_1' denote the composite homeomorphism which maps $N_k \cap \{a_1 \cup \Delta_{\underline{a}}(a_1)\}$ onto $[1,2)$ and h_1 the mapping consisting of h_0 and h_1'. As in the proof of 3.28, it can be shown that h_1 is a homeomorphism which maps

$$S_1 = [\{\underline{a} \cup \Delta(\underline{a})\} \cap N_0] \cup [\{a_1 \cup \Delta_{\underline{a}}(a_1)\} \cap N_k]$$

onto $[0,2)$.

Thus S_1 is half-parallel and either S_1 has points which are not contained in K and we are through or the process can be repeated. Since K is covered by r neighborhoods, the process can be repeated at most r times. If K should contain any one of the covering neighborhoods, then it must contain a boundary point. But each of K's boundary points which is contained in K is covered by some neighborhood. Hence, if the process could be carried on for r times, then S_r could not be contained in K.

3.36 Lemma. For each $\underline{a}$ of E there exists a sequence of closed, compact, connected sets K_n such that: (1) $\underline{a}$ is in K_1, (2) $K_n \subset K_{n+1}$, and (3) $\lim K_n = E$.

Proof. Since E is locally R_1 for each point b of E, there exists a compact connected set which contains b in its interior. Assume that such a set has been determined for each point of E. The collection of

these sets form a covering for E and since E is separable there exists a denumberable subcollection which covers E. Let $\{ N_i \}_{i=1}^{\infty}$ denote such a subcollection. We can assume $\underline{a} \in N_1$. Let K_1 be the closure of N_1. K_1 is a closed, connected, compact set. Assume that K has been defined. $K_n \neq E$ as a consequence of 3.35. Therefore, due to the connectedness of E there exists a N_k not previously used that meets K_n. Let K_{n+1} be K_n plus the closure of N_k. K_{n+1} is a closed connected compact set. The sequence $\{ K_n \}$ is now defined for all n and has E as a limit.

3.37 Definition. If $\underline{a}$ is an element of E and $\{ K_n \}$ is a sequence satisfying the conclusion of 3.37, then a half-parallel set S is said to extend to infinity from $\underline{a}$ in $\Delta(\underline{a})$ iff (1) $\underline{a}$ is in S and (2) $S \cap \Delta(\underline{a})$ is not contained in any K_n.

3.38 Lemma. If K is a compact set of E and $\{ K_n \}$ a sequence of sets satisfying the conclusion of 3.36, then there exists a n such that K_n contains K.

Proof. Suppose not, then there exists P_n of K such that $P_n \in K_{n+1} - K_n (n = 1,2,3,\cdots)$. K compact implies there exists a subsequence $\{ P_{n_i} \}$ of $\{ P_n \}$ such that $\{ P_{n_i} \}$ converges to a point P of K. Since $P_{n_i} \notin K_n$ for $n_i > n$, it follows that $P \notin K_n$ for all n. Since $\lim K_n = E$, $P \notin E$. This is impossible since $K \subset E$.

3.39 Theorem. The concept of a half-parallel set extending to infinity in $\Delta(\underline{a})$ is independent of the sequence of sets $\{ K_n \}$ satisfying 3.37.

Proof. Suppose S extends to infinity in $\Delta(\underline{a})$ with respect to $\{ K_n \}$ and $\{ K_n' \}$ is another sequence of sets satisfying the conclusion of 3.37. Suppose further, $S \cap \Delta(\underline{a})$ is contained in K_n' for some n. It then follows from 3.38 that $S \cap \Delta(\underline{a})$ is contained in some K_n. This is impossible. Hence, S also extends to infinity in $\Delta(\underline{a})$ with respect to $\{ K_n' \}$.

3.40 Theorem. If $\underline{a}$ is an arbitrary point of E and $\Delta(\underline{a})$ is arbitrary, then there exists a half-parallel set $S(\underline{a})$ which extends from $\underline{a}$ to infinity in $\Delta(\underline{a})$.

Proof. Since E is locally R_1, there exists a neighborhood N of $\underline{a}$ such that $\underline{a} \cup \{ N \cap \Delta(\underline{a}) \}$ is half-parallel. By 3.35 and its proof, $\underline{a} \cup \{ N \cap \Delta(\underline{a}) \}$ can be extended to $S_1(\underline{a})$ so that $S_1(\underline{a}) \cap \Delta(\underline{a})$ is not contained in K_1. Suppose $S_n(\underline{a})$ has been defined so that $S_n(\underline{a}) \cap \Delta(\underline{a})$ is not contained in K_n. By application of 3.35, we can obtain a half-parallel set $S_{n+1}(\underline{a})$ such that $S_{n+1}(\underline{a}) \cap \Delta(\underline{a})$ is not contained in K_n. Thus we obtain a sequence of half-parallel sets $\{ S_n(\underline{a}) \}$ which approaches a limiting half-parallel set $S(\underline{a})$ that extends from $\underline{a}$ to infinity.

3.41 Theorem. If E is a separable, connected, locally R_1 topological space with respect to its intrinsic topology, then E is normal.

Proof. Take any point a_1 of E and choose $\Delta(a_1)$ fixed. We shall then carry out a semi-normal subdivision of $a_1 \cup \Delta(a_1)$ corresponding to an admissible class A of finite sequences of positive integers. A_n shall denote the subset of A whose sequences have n terms ($n = 1,2,3,\cdots$).

For every $\underline{a}$ of $\Delta(a_1)$, we can find by 3.40 a half-parallel set $S^*(\underline{a})$

extending to infinity in $\Delta^*_{a_1}(\underline{a})$ and another $S(\underline{a})$ extending to infinity in $\Delta_{a_1}(\underline{a})$. $S(\underline{a}) \cup S^*(\underline{a})$ is open since it is parallel.

Assume now a fixed choice of some $N(c)$ where $N(c)$ is an R_1 neighborhood of c for each c of $a_1 \cup \Delta(a_1)$. Let $\{K_n\}$ be a sequence of closed, compact, connected sets such that $K_n \subset K_{n+1}$ and $\lim K_n = E$. By the Borel covering theorem, a finite number of the $N(c)$'s cover the set $K_n \cap \{a_1 \cup \Delta(a_1)\}$. We can thus select a sequence $\{N(c_i)\}$ such that for sufficiently large r the set $K_r \cap \{a_1 \cup \Delta(a_1)\}$ is included in $\bigcup_{i=1}^{r} N(c_i)$. We can assume c_1 is a_1. If c is any point of $a_1 \cup \Delta(a_1)$, $N(c_i,c)$ shall denote the $N(c_i)$ of smallest index which contains c. $S(c_i,c)$ shall then denote the half-parallel set on $S(c) \cup S^*(c)$ consisting of $\{S(c_i) \cup S^*(c_i)\} \cap \{c \cup \Delta^*_{a_1}(c)\}$. $S(c_i,c)$ consists of the part of $S(c_i) \cup S^*(c_i)$ joining c to infinity in $\Delta^*_{a_1}(c)$.

The class A_1 will now consist of the one element sequence 1. a_1 will be the corresponding element of E. Suppose now the classes $A_1,\cdots,A_r$ have been defined and also the points a_α for α in $A_1,\cdots,A_r$. Then for α in A_r consider a_α. The boundary of the set $S(c_i,a_\alpha)$ is by 3.33 a cyclic set which is at most countable infinite. Its points can be given as $a_{\alpha,1}, a_{\alpha,2},\cdots$. The indices $\alpha,1;\alpha,2;\cdots$; with α varying over A_r then form the class A_{r+1}. The class A of all A_n is defined and admissible and the elements a_α are defined for all α in A.

Condition (2) is now satisfied. If we determine $\theta(V_\alpha)$ in the case of a one element set V_α as $\Delta^*_{a_1}(a_\alpha)$, then, we shall have $\theta(V_\alpha) = \Delta^*_{a_1}(a_\alpha) \cap \{\bigcap \Delta_{a_1}(a_{\alpha,k})\}$ for $a_\alpha \neq a_1$ and $\theta(V_\alpha) = \Delta(a_1) \cap \{\bigcap \Delta_{a_1}(a_{1,k})\}$ for $a_\alpha = a_1$. By 3.41, $S(c_i,a_\alpha) = a_\alpha \cup \theta(V_\alpha)$

and hence, $a_\alpha \cup \theta(V_\alpha)$ is half-parallel. Hence, condition (4) holds.

Next we verify condition (5). Suppose $\underline{a}$ is in $\Delta(a_1)$ but not in $\bigcup_\alpha \lambda(V_\alpha)$. Then $\underline{a}$ cannot lie in $\lambda(V_1)$ and hence lies in some $\Delta^*_{a_1}(a_{1,k})$. $\underline{a}$ cannot lie in $\lambda(V_{1,k})$; hence it lies in some $\Delta^*_{a_1}(a_{1,k,j})$. Repeating this argument, we find there is an expanding sequence of finite sequences α^n, with α^n in A_n and $\alpha^{n+1} = \alpha^n, k$ for some k and such that $\underline{a}$ lies in each $\Delta^*_{a_1}(a_{\alpha^n})$. Let K_n be the compact, connected set of least index which contains $\underline{a}$. $\Delta^*_{a_1}(a_{\alpha^n})$ must meet K_n for $\underline{a}$ is in $\Delta^*_{a_1}(a_{\alpha^n})$. Since $\Delta^*_{a_1}(a_{\alpha^n})$ does not contain a_1, a_{α^n} is in K_n for otherwise a_{α^n} would disconnect K_n. Now $K_n \cap \{a_1 \cap \Delta(a_1)\}$ is covered by a finite number of the neighborhoods $N(c_1), N(c_2), \cdots, N(c_s)$ of the sequence $\{N(c_i)\}$. $N(c_i, a_{\alpha^n})$ was chosen with minimum index to contain $\underline{a}$. Hence $1 \leq i \leq s$. Since there are infinitely many a_{α^n}'s, we must have for some $n < p$ $N(c_i, a_{\alpha^n}) = N(c_i, a_{\alpha^p})$ with the same i. But $N(c_i, a_{\alpha^n})$ lies in the open set $S(c_i) \cup S^*(c_i)$ while $N(c_i, a_{\alpha^p})$ contains boundary points or other points not in $S(c_i) \cup S^*(c_i)$. For if $p = n + 1$, then a_{α^p} is in the boundary of $S^*(c_i)$ and otherwise $a_{\alpha^p} \subset \Delta^*_{a_1}(a_{\alpha^{p-1}}) \subset \cdots \subset \Delta^*_{a_1}(a_{\alpha^{n+1}}) \subset \Delta^*_{a_1}(a_{\alpha^n})$ and $S^*(c_i)$ is contained in $\Delta_{a_1}(a_{\alpha^n})$. Hence, a contradiction. Thus, $a_1 \cup \Delta(a_1) = \bigcup_\alpha \lambda(V_\alpha)$ and (5) is established.

Condition (1) is immediate. It remains to establish condition (6). Let $a_{\alpha,k}$ be in V_α and c in $\theta(V_\alpha)$. Then not $c|a_\alpha|a_{\alpha,k}$ and not $c|a_{\alpha,k}|a_\alpha$ by 3.05. If for every c in $\theta(V_\alpha)$, $a_\alpha|c|a_{\alpha,k}$, then the set $a_\alpha \cup a_{\alpha,k} \cup \theta(V_\alpha)$ is homeomorphic to $[0,1]$ and hence compact. By 3.40, this is impossible. Hence, there exists an element $\underline{a}$ in $\theta(V_\alpha)$ such that $|a_\alpha, \underline{a}, a_{\alpha,k}|^{\pm}$.

Thus, $a_1 \cup \Delta(a_1)$ is seminormal. Similarly, $a_1 \cup \Delta^*(a_1)$ is seminormal.

Hence, E is normal.

3.42 Main Theorem. If E is a nonoriented chordal system, then E is capable of normal subdivision if and only if E is separable, connected, and locally R_1 with respect to its intrinsic topology.

Proof. 3.41 states that E is capable of normal subdivision if separable, connected, and locally R_1 while 3.20, 3.30, and 3.31 state that the converse holds.

CHAPTER IV

DISCONTINUITY ELEMENTS OF CHORDAL SYSTEMS

4.01 Remarks. In this chapter, we shall determine a necessary and sufficient condition that a normal chordal system be isomorphic to the chordal system generated by the level curves of the real part of a complex valued function that is schlicht on the interior of the unit circle. From an intuitive point of view, this condition is the requirement that the situation illustrated in Figure 4-1 does not occur in the chordal system when a curve-family is used as a model.

The proof of the sufficiency of the condition will be primarily of a constructional nature but will utilize the Riemann mapping theorem. In brief outline, we will construct a Jordan domain J in the w-plane such that the lines u = constant filling J will be isomorphic to the given normal chordal system. By the Riemann mapping theorem, there then exists a one-to-one conformal map of J onto the interior of the unit circle. The real part of the inverse of this function will be the desired harmonic function.

Our first step towards attaining this goal will be to define a discontinuity element of a chordal system. The reason for calling a particular element of a chordal system a discontinuity element will become evident later in the chapter. Perhaps a more appropriate term could have been selected, for it is at its discontinuity elements that a chordal system fails to be a T_2 topological space.

4.02 Definition. Let c be any element of a NCS F, then d in F - c

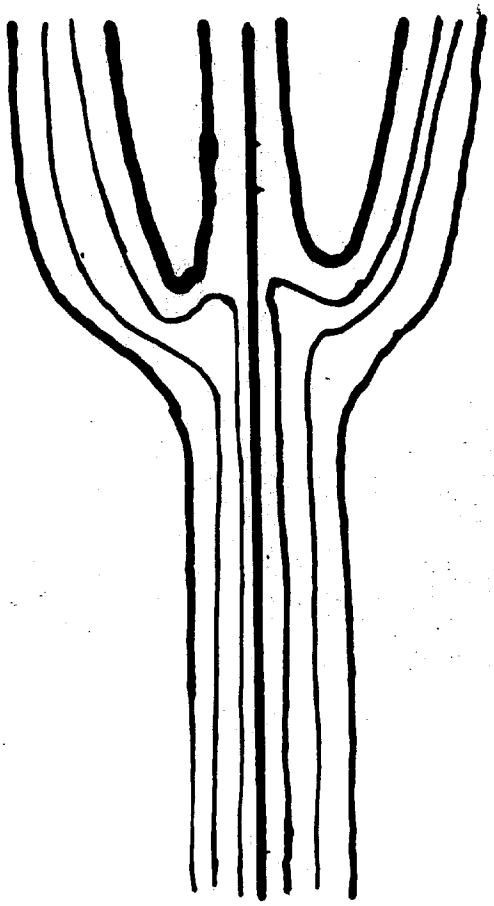


Figure 4-1

belongs to the discontinuity set V(c) of E iff there do not exist intrinsic neighborhoods of c and d which are disjoint.

4.03 Definition. Let c be any element of a NCS F, then c is a discontinuity element of E iff $V(c) \neq 0$.

4.04 Theorem. If c is a discontinuity element of a NCS F and d and e belong to V(c), then $d|x|e$ implies $x = c$, i.e., c is the only element of F which can separate elements of its discontinuity set.

Proof. Suppose $d|x|e$ and $x \neq c$. Then by 2.10, we have either $c|x|e$ or $d|x|c$. Hence, either $\Delta_e(x)$ and $\Delta_c(x)$ are disjoint neighborhoods of e and c respectively or $\Delta_d(x)$ and $\Delta_c(x)$ are disjoint neighborhoods of d and c respectively. Thus, either $d \notin V(c)$ or $e \notin V(c)$ and we have a contradiction. Therefore, a necessary condition for $d|x|e$ to hold is that $x = c$.

4.05 Theorem. If c is a discontinuity element of a NCS F, then V(c) is a cyclic set.

Proof. If V(c) consists of one or two elements, it is trivially cyclic. If d,e,f are in V(c), then not $d|e|f$, not $f|d|e$, and not $e|f|d$ by 4.04. Hence, $|d,e,f|^{+}$ by (NCS1).

4.06 Theorem. If c is a discontinuity element of a NCS F and if there exists $\underline{a} \neq c$ such that $\underline{a} \in \theta(V(c))$ where $\theta(V(c)) \supset c$, then $\underline{a} \cup V(c)$ is a cyclic set.

Proof. If V(c) consists of a single element, the theorem is immediate. Suppose d,e in V(c), then by 3.05 either $d|\underline{a}|e$ or $|d,\underline{a},e|^{+}$. Since $\underline{a} \neq c$, $d|\underline{a}|e$ is impossible by 4.03. Hence, $|d,\underline{a},e|^{+}$ and $\underline{a} \cup V(c)$

is cyclic.

4.07 Theorem. If c is a discontinuity element of a NCS F and if $a_1|c|a_2$ holds where $a_1, a_2 \in \theta(V(c))$ and $\theta(V(c)) \supset c$, then $a_1 \cup a_2 \cup V(c)$ is a cyclic set.

Proof. By 4.06, $a_1 \cup \theta(V(c))$ and $a_2 \cup \theta(V(c))$ are cyclic sets. It remains only to show that $|d,a_1,e|^{\pm}$ and $|d,a_2,e|^{\pm}$ for d,e in V(c). However, these conclusions follow immediately from (NCS1) and 4.03.

4.08 Definition. Let $\underline{a} \neq c$, let $V(c;\underline{a}) = \Delta_{\underline{a}}(c) \cap V(c)$, and let $V^*(c;a) = \Delta_{\underline{a}}(c) \cap V(c)$, then c is said to be a left (right) discontinuity with respect to $\underline{a}$ iff $V(c;\underline{a}) \neq 0 (V^*(c;\underline{a}) \neq 0)$.

4.09 Remark. It is immediate from 4.08 that if b is in $\Delta_{\underline{a}}(c)$, then $V(c;b) = V(c;a)$ and $V^*(c;b) = V^*(c;\underline{a})$ while if b is in $\Delta^*_{\underline{a}}(c)$, then $V(c;b) = V^*(c;\underline{a})$ and $V^*(c;b) = V(c;\underline{a})$.

4.10 Definition. Let c be a discontinuity element of a chordal system E. Let a_1 and a_2 be any two elements of E satisfying the hypothesis of 4.07, then the set $V^+(c;a_1,a_2)$ ($V^-(c;a_1,a_2)$) will consist of the elements d of V(c) such that $|a_2,d,a_1|^+$ ($|a_2,d,a_1|^-$).

4.11 Remark. Since $|a_2,d,a_1|^+$ iff $|a_1,d,a_2|^-$, $V^+(c;a_1,a_2) = V^-(c;a_2,a_1)$ and $V^-(c;a_1,a_2) = V^+(c;a_2,a_1)$.

4.12 Theorem. If c is a discontinuity element of the chordal system E, $V^+(c;a_1,a_2)$ and $V^-(c;a_1,a_2)$ are defined, and there exists an element b of E such that either $c|a_2|b$ or $c|b|a_2$ holds, then

$V^+(c;a_1,a_2) = V^+(c;a_1,b)$ and $V^-(c;a_1,a_2) = V^-(c;a_1,b)$.

Proof. Suppose $c|a_2|b$. $c|a_2|b$ and $a_1|c|a_2$ imply by (CS3) $a_1|c|b$ and $a_1|a_2|b$. Hence, $V^+(c;a_1,b)$ is defined by 4.10 and 4.07. Suppose $|a_2,d,a_1|^+$ where $d \in V(c)$. $|a_2,d,a_1|^+$ and $a_1|a_2|b$ imply by (CS5) $|b,d,a_1|^+$. Hence, $V^+(c;a_1,a_2) \subset V^+(c;a_1,b)$.

Suppose $|b,d,a_1|^+$ where $d \in V(c)$. Since $d \neq a_2$, $a_1|a_2|b$ implies either $d|a_2|b$ or $a_1|a_2|d$ but not both by 2.10. However, $V^+(c;a_1,a_2)$ being defined implies $|a_1,a_2,d|^+_-$. Hence, $d|a_2|b$. $|a_1,d,a_2|^+$ and $d|a_2|b$ imply by (CS5) $|a_1,d,b|^+$, a contradiction. Therefore, $|a_2,d,a_1|^+$ and $V^+(c;a_1,a_2) = V^+(c;a_1,b)$. $V^-(c;a_1,a_2) = V^-(c;a_1,b)$ since $V^-(c;a_1,a_2) = V(c) - V^+(c;a_1,a_2)$.

The case where $c|b|a_2$ holds is demonstrated in a similar manner.

4.13 Remark. If $V^+(c;a_1,a_2)$ is defined and there exists b_1,b_2 in E such that either $b_1|a_1|c$ or $a_1|b_1|c$ and either $c|a_2|b_2$ or $c|b_2|a_2$ holds, then $V^+(c;a_1,a_2) = V^+(c;a_1,b_2) = V^+(c;b_1,b_2)$ by 4.12. Hence $V^+(c;a_1,a_2)$ depends on a_1 alone for given a_1',a_2'', there exists $b \in V(c)$ so that $c|b|a_2'$, $c|b|a_2''$.

4.14 Definition. If c is a discontinuity element of the chordal system E and sets $V^+(c;a_1,a_2)$ and $V^-(c;a_1,a_2)$ are defined, then let

$$V^{++}(c;a_1) = V^+(c;a_1,a_2) \cap V^*(c;a_1);$$
$$V^{-+}(c;a_1) = V^+(c;a_1,a_2) \cap V(c;a_1);$$
$$V^{+-}(c;a_1) = V^-(c;a_1,a_2) \cap V^*(c;a_1);$$
$$V^{--}(c;a_1) = V^-(c;a_1,a_2) \cap V(c;a_1).$$

If $V^{++}(c;a_1) \neq 0$ and $V^{-+}(c;a_1) = 0$ ($V^{+-}(c;a_1) \neq 0$ and $V^{-+}(c;a_1) = 0$), then c is said to be a right discontinuity with respect to $V^+(c;a_1)$ ($V^-(c;a_1)$). If $V^{-+}(c;a_1) \neq 0$ and $V^{++}(c;a_1) = 0$ ($V^{--}(c;a_1) \neq 0$ and $V^{+-}(c;a_1) = 0$), then c is said to be a left discontinuity with respect

to $V^+(c;a_1)$ ($V^-(c;a_1)$). If c is simultaneously a right (left) discontinuity with respect to $V^+(c;a_1)$ and a left (right) discontinuity with respect to $V^-(c;a_1)$, then c is said to be a Z-discontinuity. If either $V^{++}(c;a_1) \neq 0$ and $V^{-+}(c;a_1) \neq 0$ or $V^{--}(c;a_1) \neq 0$ and $V^{+-}(c;a_1) \neq 0$, then c is said to be a Y-discontinuity.

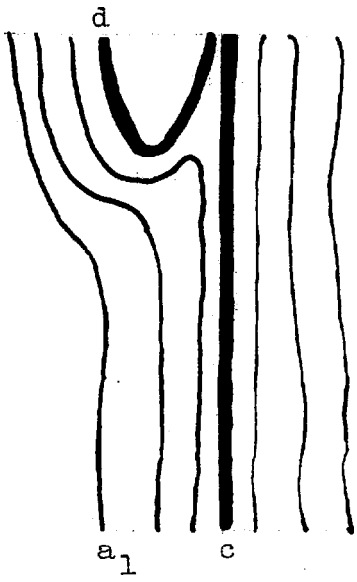
4.15 Remarks. Figure 4-2 illustrates the various types of discontinuities defined in 4.13 where a regular curve family is thought of as the model for the chordal system.

Throughout the remainder of the chapter it will be assumed that E denotes an abstract normal chordal system with semi-normal subdivision of the sets $E_0 = c_1 \cup \Delta(c_1)$ and $E_0 = c_1 \cup \Delta^*(c_1)$ by the sets V_α of E_0, $\alpha \in A$ and sets V^*_α of E^*_0, $\alpha \in A^*$ where A and A^* are admissible classes of sequences.

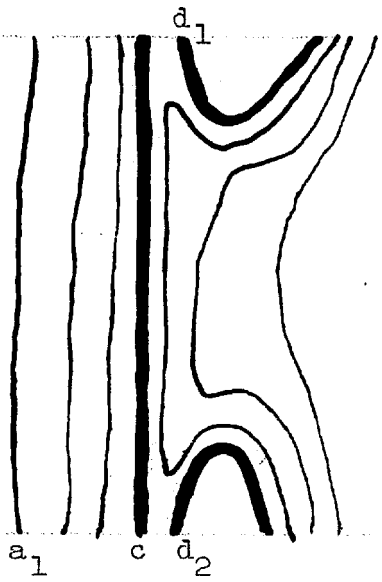
Since a normal chordal system is locally R_1, it is clear that the sets $V^+(c;a_1,a_2)$ and $V^-(c;a_1,a_2)$ are defined for each element c of E. In the future, if $c \in \theta(V_\alpha)$, then c_α will be used as a_1 to give the orientation of $V(c)$ while if $c = c_\alpha$ where $E_\alpha = c_\alpha \cup \theta(V_\alpha)$, then some point a_1 of an R_1 neighborhood of c_α, $a_1 \notin E_\alpha$ will be used. It is clear from 4.12 and 4.13 that the discontinuity set of each element of E now has a well-defined orientation.

4.16 Definition. Let α denote a finite sequence of positive integers $p_1, p_2, \cdots p_k$, then the norm of α (denoted: $||\alpha||$) shall be the natural number $\sum_{i=1}^{k} p_i$.

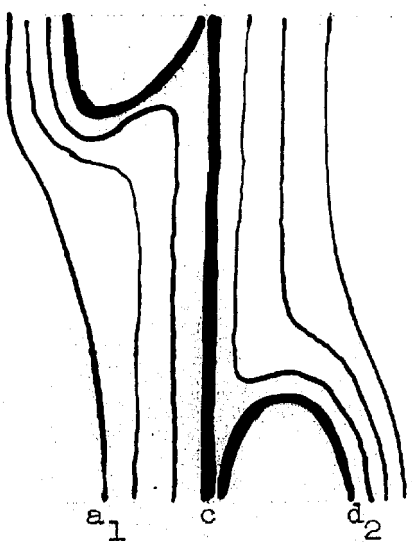
4.17 Definition. Let A be any admissible class of sequences.



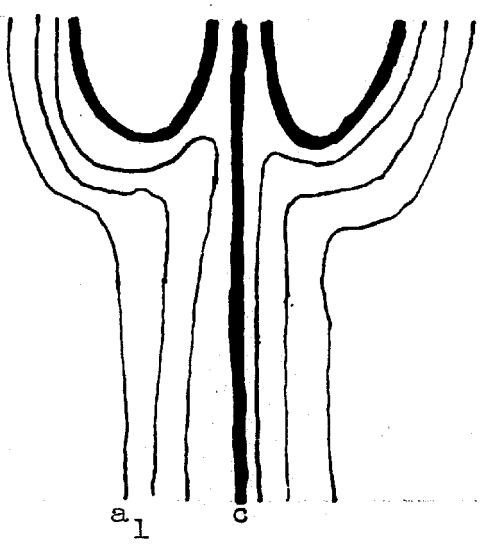
Left Discontinuity



Right Discontinuity



Z-discontinuity



Y-discontinuity

Figure 4-2

We define the following order relation for the elements of A: $\alpha < \beta$ iff either 1) $||\alpha|| < ||\beta||$ or 2) the lexicographic order of α is less than the lexicographic order of β and $||\alpha|| = ||\beta||$.

4.18 Remarks. Since A is countable and there are only a finite number of α such that $||\alpha||$ equals some fixed constant, it is clear that the admissible class A is well-ordered with respect to the symbol $<$ defined above and thus is in one-to-one order preserving correspondence with the natural numbers. Hence, the rth element of A under this ordering corresponds to r and conversely. Consequently, the elements of E_0 that have been tagged with α can be denoted by a natural number r. In general, we shall denote such an element by any one of the following: c_α, c^r, or c^r_α. Similarly, the sequences of A^* can be ordered and the same notational conventions made.

4.19 Remark. In the next four paragraphs, we restate from [1] and [2] a number of definitions and theorems of W. Kaplan which are fundamental in the work that follows.

In the preceding chapter, we defined a set $c \cup \theta(V)$ to be half-parallel iff it was homeomorphic to the nonnegative real axis with c being mapped into zero. In this chapter, we replace $[0,\infty)$ by the interval $[0,2)$. Clearly, there is no loss.

4.20 Theorem. [c.f. 3.23, 3.24]. If $c_{\alpha,k}$ is in V_α, then there exist a unique number $u_{\alpha,k} \geq 0$ such that either $B(c_{\alpha,k}) = \bigcup_u d^u$ for $u \geq u_{\alpha,k}$ or $B(c_{\alpha,k}) = \bigcup_u d^u$ for $u > u_{\alpha,k}$ where

$$B(c_{\alpha,k}) = \{ d^u : |c_\alpha, c_{\alpha,k}, d^u|^+_- \} .$$

4.21 Notation. V_α^+ (V_α^-) will denote the set of elements $c_{\alpha,k}$ of V_α such that $|d_\alpha^u, c_{\alpha,k}, c_\alpha|^+$ ($|d_\alpha^u, c_{\alpha,k}, c_\alpha|^-$) for some d_α^u of E_α.

4.22 Theorem. $V_\alpha^+ \cap V_\alpha^- = 0$ and $V_\alpha^+ \cup V_\alpha^- = V_\alpha - c$.

4.23 Theorem. The definition $c_{\alpha,i} < c_{\alpha,j}$ iff $|c_{\alpha,i}, c_\alpha, c_{\alpha,j}|^+$ ($|c_{\alpha,i}, c_\alpha, c_{\alpha,j}|^-$) induces a linear ordering in $V_\alpha^+(V_\alpha^-)$. Further, $c_{\alpha,i} < c_{\alpha,j} < c_{\alpha,k}$ implies $|c_{\alpha,k}, c_{\alpha,j}, c_{\alpha,i}|^+$ ($|c_{\alpha,k}, c_{\alpha,j}, c_{\alpha,i}|^-$) for $c_{\alpha,i}, c_{\alpha,j}, c_{\alpha,k}$ in $V_\alpha^+(V_\alpha^-)$.

4.24 Notation. $V_\alpha^{++}(s)$ ($V_\alpha^{+-}(s)$) will denote the set of all $c_{\alpha,k}$ in $V_\alpha^+(V_\alpha^-)$ such that $u_{\alpha,k} = s$ and $d_\alpha^s \notin B(c_{\alpha,k})$. $V_\alpha^{-+}(s)$ ($V_\alpha^{--}(s)$) will denote the set of $c_{\alpha,k}$ in $V_\alpha^+(V_\alpha^-)$ for which $u_{\alpha,k} = s$ and $d_\alpha^s \in B(c_{\alpha,k})$.

4.25 Theorem. $V_\alpha^{ij}(s) = V^{ij}(d_\alpha^s; c_\alpha)$ where $(i, j = +, -)$ and $d_\alpha^s \neq c_\alpha$.

Proof. We show that $V_\alpha^{++}(s) = V_\alpha^{++}(d^s; c_\alpha)$. The other cases follow analogously. Let $c_{\alpha,k} \in V_\alpha^{++}(s)$, then by 3.27, $c_{\alpha,k} \in V(d_\alpha^s)$. Since $d_\alpha^s \in B(c_{\alpha,k})$, we have $c_\alpha | d_\alpha^s | c_{\alpha,k}$ by 3.05. Therefore, $c_{\alpha,k} \in V^*(d_\alpha^s; c_\alpha)$. Since $|d_\alpha^u, c_{\alpha,k}, c_\alpha|^+$ for $u > u_{\alpha,k} = s$, we have $c_{\alpha,k} \in V^+(d_\alpha^s; c_\alpha)$ by 4.10. Thus, $V_\alpha^{++}(s) \subset V^{++}(d_\alpha^s; c_\alpha)$.

If $c \in V^{++}(d_\alpha^s; c_\alpha)$, then $c \in V^+(d_\alpha^s; c_\alpha)$ and $c \in V^*(d_\alpha^s; c_\alpha)$. By 4.02, $c \in \bar{E}_\alpha = \lambda(V_\alpha)$. Since E_α is half-parallel, $c \in V_\alpha$. By 4.20, $B(c) = \bigcup_u d_\alpha^u$ for $u > t$ or $B(c) = \bigcup_u d_\alpha^u$ for $u \geq t$. $c \in V^+(d_\alpha^s; c_\alpha)$ implies $|d_\alpha^u, c, c_\alpha|^+$ for $u > s$. Hence, $t \not\leq s$. $c \in V^*(d_\alpha^s; c_\alpha)$ implies $c \in \Delta_c^*(d_\alpha^s)$ or $c_\alpha | d_\alpha^s | c$. Therefore, $t = s$ and $d_\alpha^s \notin B(c)$. Thus, $V^{++}(d_\alpha^s; c_\alpha) \subset V_\alpha^{++}(s)$.

4.26 Remark. For $d_\alpha^s = c_\alpha$, we can demonstrate as in 4.25 that $V_\alpha^{++}(0) = V^{+-}(c_\alpha; a_1)$ and $V_\alpha^{+-}(0) = V^{+-}(c_\alpha; a_1)$ where a_1 is chosen according

to 4.15. The sets $V_\alpha^{-+}(0)$ and $V_\alpha^{--}(0)$ are empty.

4.27 Definition. Two chordal system F and F' are isomorphic iff there exists a one-to-one correspondence between the elements of F and F' that preserves chordal relations.

4.28 Remark. The definition and two theorems stated below from [2] will greatly simplify the problem of showing the curve-family E' which we will construct later is isomorphic to a given normal chordal system E without Y-discontinuities. The definition of a triple of type (β) given below differs from the one stated in [2] due to the fact that the proof of Theorem 4 in [2] is inaccurate and the corrected version (see [4]) requires the definition of a triple of type (β) to be given as stated below.

4.29 Theorem. Let E be a normal chordal system subdivided normally by the sets V_α for α in A and V_α^* for α in A^*; let E' be a normal chordal system subdivided normally by sets V_α' for α in A and $V_\alpha^{*\prime}$ for α in A^* with

$$\bigcup \lambda(V_\alpha) = a_1 \cup \Delta(a_1) = E_0, \qquad \bigcup \lambda(V_\alpha^*) = a_1 \cup \Delta^*(a_1) = E_0^* \text{ and}$$

$$\bigcup \lambda(V_\alpha) = a_1' \cup \Delta(a_1') = E_0', \quad \bigcup \lambda(V_\alpha^{*\prime}) = a_1' \cup \Delta^*(a_1') = E_0^{*\prime}.$$

Let $f(a) = a'$ be a one-to-one transformation of E onto E' such that f is an isomorphism of each $\lambda(V_\alpha)$ onto $\lambda(V_\alpha')$ of each $\lambda(V_\alpha^*)$ onto each $\lambda(V_\alpha^{*\prime})$. Then E is isomorphic to E' under f.

4.30 Definition. A triple of type (α) of $\lambda(V_\alpha)$ consists of three elements $c_\alpha, c_{\alpha,k}, d_\alpha^u$ where $c_\alpha, c_{\alpha,k} \in V_\alpha$ and $d_\alpha^u \in \theta(V_\alpha)$. A triple of type ($\beta$) consists of three elements $c_\alpha, c_{\alpha,i}, c_{\alpha,j}$ where $B(c_{\alpha,i}) = B(c_{\alpha,j})$ and either $c_{\alpha,i}, c_{\alpha,j} \in V_\alpha^+$ or $c_{\alpha,i}, c_{\alpha,j} \in V_\alpha^-$.

4.31 Theorem. Let $\lambda(V_\alpha) = c_\alpha \cup \{\bigcup c_{\alpha,k}\}$ and $\lambda(V_{\alpha'})$ be subsets in the normal subdivision of E and E', respectively. Let $f(a) = a'$ be a one-to-one transformation of $\lambda(V_\alpha)$ onto $\lambda(V_{\alpha'})$ with $f(c_\alpha) = c_{\alpha'}$, $f(c_{\alpha,k}) = c'_{\alpha,k}$, and $f(d^u_\alpha) = d^{u'}_\alpha$. If further, $[a_1, a_2, a_3,] \sim [a_1', a_2', a_3']$ for each triple a_1, a_2, a_3 of types (α) and (β), then f maps $\lambda(V_\alpha)$ isomorphically onto $\lambda(V_{\alpha'})$.

4.32 Construction of F^+_α and F^-_α. Let E be a normal chordal system without Y-discontinuities. Let $E_\alpha = \bigcup_u d^u_\alpha$ $(0 \leqq u < 2)$ be any half-parallel set of E obtained by normal subdivision. We construct real valued functions F^+_α and F^-_α each of total variation less than or equal to 2^{-3}, defined on $[0,2)$ whose discontinuities are precisely at those values of u where d^u_α is a discontinuity of E_α.

There are only a denumerable number of elements d^u_α such that the set $V^{++}_\alpha(u) \cup V^{-+}_\alpha(u)$ is not null. Number the corresponding set of values of u for these elements $u^r (r = 1, 2, 3, \cdots)$ so that if $c_\alpha = d^0_\alpha$ is one of them, 0 will bear the superscript 1. Let $g^+_\alpha(u^r) = -2^{-(r+3)}$ or $2^{-(r+3)}$ according to whether $V^{-+}_\alpha(u^r) \neq 0$ or $V^{++}_\alpha(u^r) \neq 0$. $g^+_\alpha(u^r)$ is single valued since we have assumed that E contains no Y-discontinuities. Let $F^+_\alpha(u) = 2^{-1} - \sum_u g^+_\alpha(u^r)$ for $u^r \leqq u$. It is readily seen that F^+_α has the properties mentioned above and that $2^{-2} \leqq F^+_\alpha(u) \leqq 3/2^2$. To obtain F^-_α, we replace $V^{++}_\alpha(u) \cup V^{-+}_\alpha(u)$ by $V^{-+}_\alpha(u) \cup V^{--}_\alpha(u)$ and proceed analogously. The bounds for F^-_α are $-3/2^2$ and -2^{-2}.

4.33 The Construction of $\bar{E}_{\alpha'}$. We again impose the restriction on E that it contain no Y-discontinuities and proceed to construct for each $\bar{E}_\alpha$ of E a corresponding set of curves $\bar{E}_{\alpha'}$. These $\bar{E}_{\alpha'}$ will then

be fitted together to form the regular curve-family E' which will consist solely of vertical line segments filling a Jordan domain D.

Let $E_\alpha = c_\alpha \cup \theta(V_\alpha) = \bigcup_u d_\alpha^u$ for $0 \leqq u < 2$ be any half-parallel set of E obtained by normal subdivision. We shall construct in the w-plane a Jordan domain J_α such that: a) the set of open vertical intervals $\theta(V_\alpha)'$ filling J_α corresponds in a prescribed manner to the elements of $\theta(V_\alpha)$; b) K_α, the boundary of J_α, contains a set of vertical line segments V_α' which correspond in a prescribed manner to the elements of V_α; c) K_α is of length less than 5 while $K_\alpha \cup J_\alpha$ is contained in a square whose vertices are (0,1), (0,-1), (2,-1) and (2,1).

Consider F_α^+ determined in 4.32 for E_α. Adjoin segments of the lines $u = u^r$ to the graph of F_α^+ in the (u,v)-plane so that the modified graph of F_α^+ (denoted: mgF_α^+) becomes a simple path. This is always possible due to the manner in which F_α^+ was defined. Carry out the same procedure on the graph of F_α^-. Construct next the simple path K_α consisting of mgF_α^+, mgF_α^-, and the segments of the lines $u = 0$ and $u = 2$ joining $F_\alpha^-(0+)$ to $F_\alpha^+(0+)$ and $F_\alpha^-(2-)$ to $F_\alpha^+(2-)$.

Referring to the construction of F_α^+ and F_α^- again, it is seen that the length of mgF_α^+ and mgF_α^- are each less than $2 + 2^{-3}$ and consequently, that the length of K_α is less than $9/4 + 5/2 < 5$.

Let J_α denote the interior of the path K_α. $E_\alpha = \bigcup_u d_\alpha^u$ for $0 \leqq u < 2$, therefore to each d_α^t for $0 < t < 2$, we correspond the open interval $d_\alpha^{t'}$ consisting of the intersection of J_α with the line $u = t$.

We next designate open intervals in the line segments adjoined to the graphs of F_α^+ and F_α^- which will correspond to the elements of V_α.

The open segment whose end points are (0,1/4) and (0,-1/4) will

correspond to the element $c_\alpha = d^0_\alpha$. Assume for the sake of definiteness that $V^{++}_\alpha(u^r) \neq 0$, then since E has no Y-discontinuities, $V^{-+}_\alpha(u^r) = 0$ and by the construction of F^+_α, it is seen that $F^+_\alpha(u^r+) - F^-_\alpha(u^r-) = 2^{-r-3}$. Since the left hand limit of F^+_α exists at u^r, there exists $T > 0$ such that $|F^+_\alpha(u) - F^+_\alpha(u^r-)| < 2^{-r-5}$ for $T < u < u^r$. Set $L(r,+)$ $= \min \{ u^r - T, 2^{-r-5} \}$ and let $I(r,+)$ denote the line segment whose end points are $P = (u^r, F^+_\alpha(u^r+))$ and $P' = (u^r, F^+_\alpha(u^r+) - L(r,+))$ (See Figure 4-3).

By 4.23, the elements of $V^{++}_\alpha(u^r)$ form a linearly ordered subset of the elements of V^+_α. If $V^{++}_\alpha(u^r)$ has a last element with respect to this order, the last element is made to correspond to an open interval such that one-fourth of the interval lies above P. Every other element $c_{\alpha,i}$ of $V^{++}_\alpha(u^r)$ is made to correspond in a one-to-one manner with an open interval $C_{\alpha,i}$ of $I(r,+)$ so that if these intervals are ordered by their midpoints, then $c_{\alpha,i} < c_{\alpha,j}$ for $c_{\alpha,i}, c_{\alpha,j}$ in $V^{++}_\alpha(u^r)$ iff for the corresponding intervals $C_{\alpha,i}, C_{\alpha,j}$ in $I(r,+)$, $C_{\alpha,i}$ lies below $C_{\alpha,j}$. Further, these designated intervals are so restricted in length that the distance between any two exceeds twice the maximum of the lengths of the two intervals. Finally, let $c_{\alpha,k}'$ denote the middle half of each $C_{\alpha,k}$ of $I(r,+)$. If $c_{\alpha,k}$ is in $V^{++}_\alpha(u^r)$, then $c_{\alpha,k}'$ will be the corresponding vertical interval in K_α.

If a $V^{+-}_\alpha(u^r)$ had been used above, the procedure would have been the same except that if $V^{+-}_\alpha(u^r)$ had a first element, it would have been treated as a last element of $V^{++}_\alpha(u^r)$ would be. If a $V^{--}_\alpha(u^r)$ had been used, the order of the $C_{\alpha,i}$'s would have been reversed, i.e., $c_{\alpha,i} < c_{\alpha,j}$ iff the midpoint of $C_{\alpha,j}$ lies below the midpoint of $C_{\alpha,i}$. Finally, if

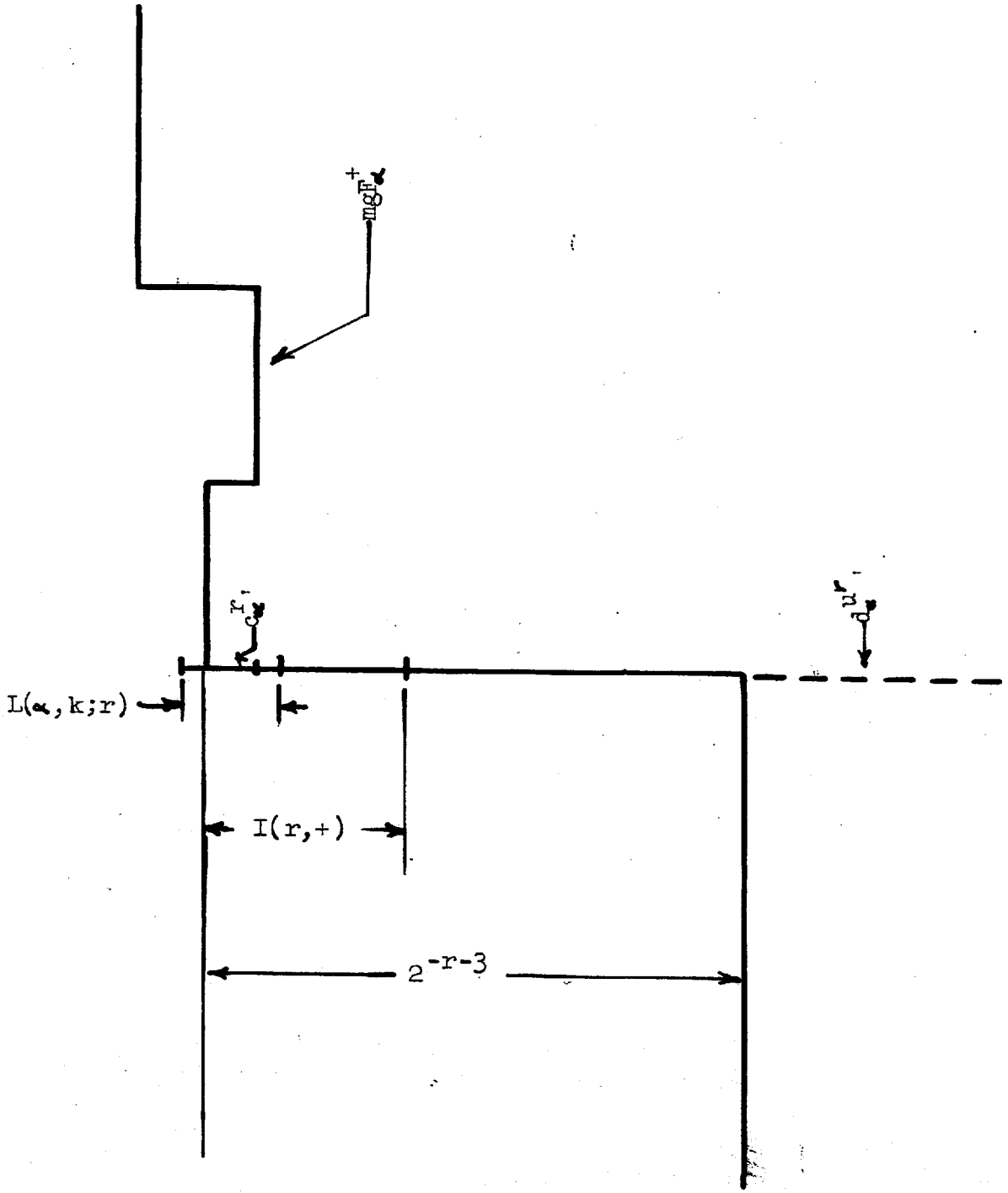


Figure 4-3

a $V_\alpha^{-+}(u^r)$ had been used, the order of $C_{\alpha,i}$'s would have been reversed and if V_α^{-+} had a first element, it would have been treated in the manner of a last element of a $V_\alpha^{++}(u^r)$. Thus, we can assume that the construction has been carried out for the sets $V_\alpha^{++}(u^r)$, $V_\alpha^{-+}(u^r)$, $V_\alpha^{--}(u^r)$, and $V_\alpha^{+-}(u^r)$ and all discontinuities u^r so that we have a collection of vertical intervals V_α' corresponding to the elements of V_α.

4.34 Remarks. Corresponding to E, we now construct a curve family E' consisting of vertical line segments filling a simply connected region D of the w-plane. This will be accomplished in two steps. We first define inductively an expanding sequence of Jordan domains D_n whose limit D_0 will be a Jordan domain with the property that the lines u = constant filling D_0 will correspond to the elements of $E_0 - c_1$ in a prescribed manner.

To simplify the second stage of the construction, we also assume that the element c_1 of E was so chosen that it has an empty discontinuity set. There is no loss in generality in this assumption since all but a denumerable number of the elements of E have vacuous discontinuity sets and any element can be made the initial element of a normal subdivision.

4.35 The Construction of E_0'. By 4.33, we have corresponding to each $\lambda(V_\alpha)$ of E_0, a set $V_\alpha' \cup E_\alpha'$ of vertical line segments of the w-plane. Using the notation established in 4.18, let $D_1 = J^1$ and $B_1 = K^1$.

B_1 is a simple closed path of length less than 5 containing a set of vertical intervals $\{C_{1,k}^r\}$ in one-to-one correspondence with the

elements of V_1^1. If necessary, we reduce the lengths $L(1,k;r)$ of these $C_{1,k}^r$ while holding their midpoints fixed (except if $C_{1,k}^r$ contains one of the points $(u^r, F_1^+(u^{r+}))$, $(u^r, F_1^-(u^{r+}))$ in which case this point is held fixed) so that: 1) $L(1,k;r) < 2^{-r}$ and 2) $L(1,j;r) \leq L(1,k;s)$ for $r < s$ and $C_{1,j}^r, C_{1,k}^s$ in B_1. In the construction of V_α', the intervals $\{c_1, c_{1,k}\}$ were so located that if one side of a square of side length $L(1,k;r)$ was made to coincide with $C_{1,k}^r$, then the other three sides could be located so that they did not meet $D_1 = J^1$. Indeed, the construction was such that if a square of this kind was attached to each of the $C_{1,k}^r$ and V_1^1, deleted from B_1, the resulting set made up of $B_1 - V_1^1$, and the three intact sides of each square would still separate the plane into two simply connected regions. Let $Sq(1,k;r)$ $(Sq(1;1))$ denote a square of the type described above attached to $C_{1,k}^r$ (C_1^1) and further restrict the $C_{1,k}^r$ so that: 3) $\rho(C_{1,k}^r, Sq(1,j;s)) \geq 10L(1,k;r)$ for all $s < r$ where $\rho(A,B)$ denotes the distance between the sets A and B.

Finally, let $h_1(t)$ denote a homeomorphism of the unit circle onto B_1 so that the arc length of $h^{-1}(c_{1,k}^r{}')$ is less than $L(1,k;r)$.

Suppose that D_n has been defined so that it is a Jordan domain with boundary B_n whose length is less than $5 + (5/2)\sum_{k=2}^{n} 2^{-k}$; that the lines $u =$ constant filling D_n correspond in a prescribed manner to the elements of $\bigcup_{k=1}^{n} E^k - c_1'$; that B_n contains vertical line segments $\{C_\alpha^r\}$ that correspond in a prescribed manner to the elements of $\bigcup_{k=1}^{n}(V^k - c^k)$ so that if squares $Sq(\alpha;r)$ are attached to the C_α^r as previously described, then the set $B_n - \bigcup_{k=1}^{n} V_\alpha^k{}'$ plus the three sides of the $Sq(\alpha;r)$ separate the w-plane into two simply connected regions; and that the $L(\alpha;r)$ of C_α^r in B_n has been restricted so that: 1) $L(\alpha;r) < 2^{-r}$;

2) $L(\rho;s) \leq L(\alpha;r)$ for $r < s$; and 3) $\rho(C^r, Sq(\rho;s)) \geq 10L(\alpha;r)$ for $s < r$.

Suppose further that h_n has been defined so that it is a homeomorphism of the unit circle onto B_n with the property that $h_n^{-1}(c_\alpha^{r'})$ has an arc length less than $L(\alpha;r)$ for $c_\alpha^{r'}$ in B_n.

Consider the interval C_ρ^{n+1} in B_n. From 4.33 we have a set $E_\rho^{n+1'} \cup V_\rho^{n+1'}$ such that $J^{n+1} \cup K^{n+1}$ is contained in a square of side length 2 with vertices (0,1), (0,-1), (2,-1), and (2,1). We shrink the square described above uniformly in both the u and v dimensions holding the origin fixed so that the square described above becomes a square of side length $L(\rho;n+1)$ and then rotate and translate this set so that the composite image of the side that contained d_ρ^0 coincides with C_ρ^{n+1} and the composite image of J^{n+1} does not meet $D_n \cup B_n$. This operation is always possible by the induction hypothesis.

We assume hereafter that this mapping has been performed but use the same notation for the image sets as was used in 4.33. It follows from the definitions of F_α^+ and F_α^- and the definition of $c_\rho^{n+1'}$ in terms of C_ρ^{n+1} that $d_\rho^{0'}$ coincides with $c_\rho^{n+1'}$. Further, we have that $B_{n+1} = (B_n - C_\rho^{n+1'}) \cup K_\rho^{n+1}$ is a simple closed curve of length less than $5 + (5/2)\sum_{k=2}^{n+1} 2^{-k}$; and that $D_{n+1} = D_n \cup C_\rho^{n+1'} \cup J_\rho^{n+1}$ is a Jordan domain. Figure 4-4 illustrates how $K^{n+1} \cup J^{n+1}$ is adjoined to $D_n \cup B_n$.

B_{n+1} contains vertical line segments C_α^r corresponding to the elements of $\bigcup_{k=1}^{n+1}(V_\alpha^k - c_\alpha^k)$. We restrict the length $L(\alpha;r)$ of C_α^r, holding the same type of points fixed as in the case of $V_1^{1'}$ so that: 1) $\rho(C_\alpha^r;B_n) \geq 10L(\alpha;r)$ for each C_α^r in $K_\rho^{n+1} - C_\rho^{n+1}$; 2) $L(\alpha;r) \leq 2^{-r}$; 3) $L(\rho;s) \leq L(\alpha;r)$ for $r < s$; and 4) $\rho(C_\alpha^r;Sq(\rho;s)) \geq L(\alpha;r)$ for all $s < r$.

Due to condition 1) above, it is clear that if squares $Sq(\alpha;r)$

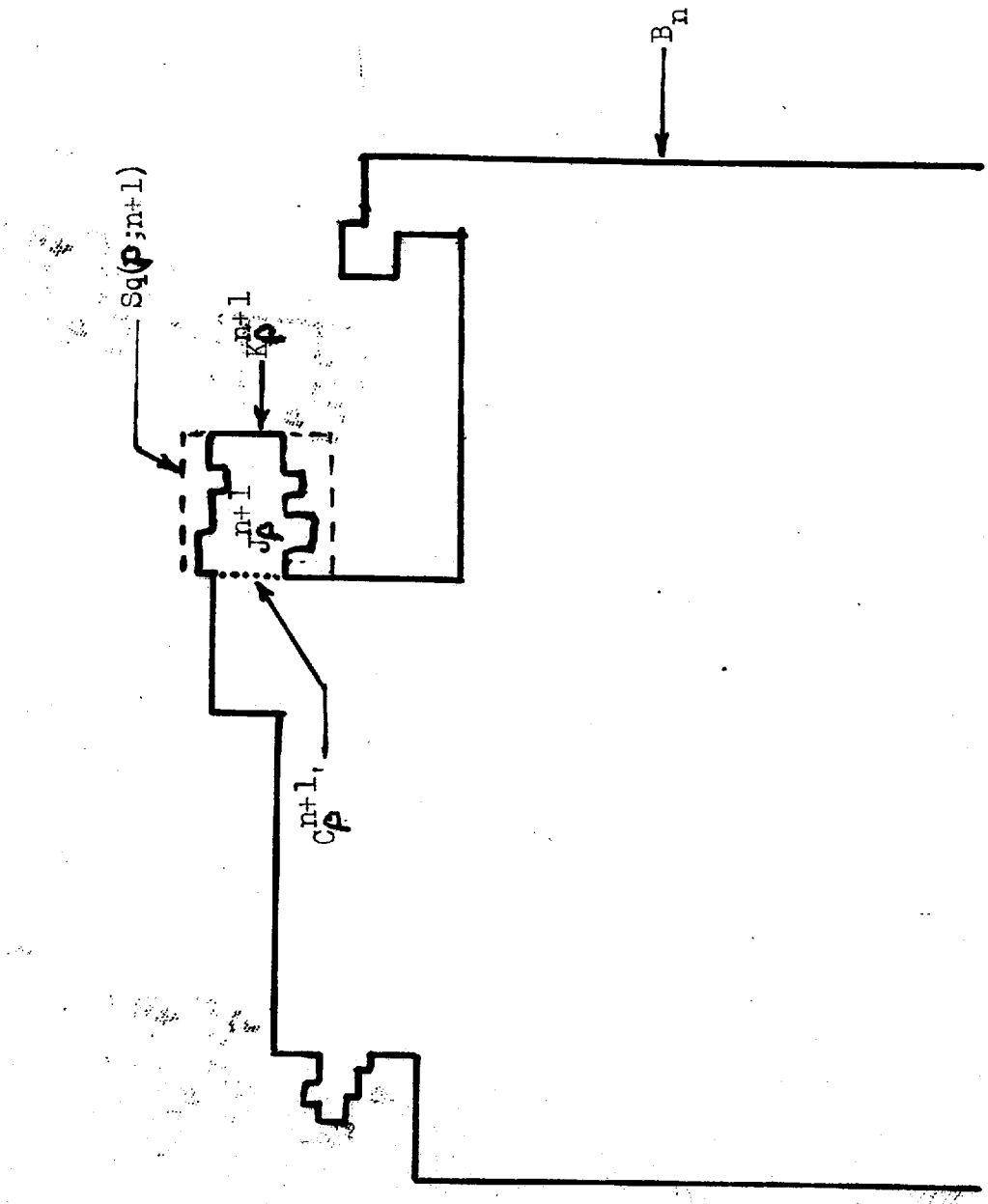


Figure 4-4

are affixed to the C_{α}^{r} in B_{n+1} as previously described, and elements of $\bigcup_{k=1}^{n+1}(V_{\alpha}^{k,} - c_{\alpha}^{k,})$ deleted, then the resulting set separates the w-plane into simply connected regions.

Let $h_{n+1}(t) = h_n(t)$ on $h_n^{-1}(B_n - c_{\beta}^{n+1,})$. On $h_n^{-1}(c_{\beta}^{n+1,})$, let $h_{n+1}(t)$ be a homeomorphism onto $K_{\beta}^{n+1} - c_{\beta}^{n+1,}$ so that the arc length of $h_{n+1}^{-1}(c_{\alpha}^{r})$ is less than $L(\alpha;r)$ for the $c_{\alpha}^{r,}$ in $K_{\beta}^{n+1} - c_{\beta}^{n+1,}$ and h_{n+1} maps the unit circle homeomorphically onto B_{n+1}.

Hence D_n, B_n, and h_n are defined for all n. Let $D_0 = \bigcup_{n=1}^{\infty} D_n$, $h_0(t) = \lim h_n(t)$, and B_0 denote the image of the unit circle under h_0. The $\{h_n(t)\}$ converges uniformly for $|h_n(t) - h_{n+1}(t)| \leqq \sqrt{2}/2^{n+1}$ by the definition of $h_{n+1}(t)$ in terms of $h_n(t)$ and condition 2) above. B_0 is the boundary of D_0. For if P is in B_0, there exists a T such that $h_0(T) = P$. Since $\lim h_n(T) = h_0(T)$, we have that every every neighborhood of P contains points of B_n for n sufficiently large and consequently, points of D_n. Hence, P is in the boundary of D_0. Conversely, since D_n is an expanding sequence of domains, P in the boundary of D_0 implies that every neighborhood of P meets all but a finite number of D_n and consequently, all but a finite number of B_n. Hence, there exists $\{h_n(t_n)\}$ such that $\lim h_n(t_n) = P$. But then the uniform convergence of the h_n implies $\lim h_0(t_n) = P$ and therefore, P is in B_0. E_0' is now defined and consists of the vertical line segments filling D_0 plus the open line segment whose end points are $(0,1/8)$ and $(0,-1/8)$.

4.36 Theorem. B_0 is a rectifiable Jordan curve.

Proof. Since the h_n converge uniformly, we have that h_0 maps the unit circle continuously onto B_0. Since each B_n was of length less

than $5 + (5/2)\sum_{k=2}^{n} 2^{-k}$, we have that B_0 is rectifiable and of length less than 8. Thus it remains to show that h_0 is a homeomorphism.

Let $t_1 \neq t_2$ be any two points of $0 \leqq t < 2\pi$, and $V_0' = \bigcup V_\alpha^r{}'$. Then from the construction of B_n we see that one of the following must hold.

Case 1. There exists a N such that neither $h_N(t_1)$ nor $h_N(t_2)$ belongs to any interval $c_\alpha^r{}'$ of V_0'.

Case 2. There exists a N such that for $n \geqq N$ one, say $h_n(t_1)$, belongs to some interval $c_\alpha^r{}'$ of V_0' while the other remains fixed.

Case 3. There exists a N such that for $n \geqq N$, $h_n(t_1)$ and $h_n(t_2)$ belong to distinct elements of V_0'.

Case 4. For all n, $h_n(t_1)$ and $h_n(t_2)$ belong to the same element of V_0'.

Suppose Case 1 holds. Since $h_{n+1}(t) = h_n(t)$, if $h_n(t)$ is not a member of some element of V_0', we have: $h_0(t_1) = h_N(t_1) \neq h_N(t_2) = h_0(t_2)$.

If Case 2 holds, then $h_0(t_2) = h_N(t_2)$ and $h_N(t_1)$ belongs to some $c_\alpha^r{}'$ of B_N. By the construction at the r + 1 stage, $h_{r+1}(t_2)$ belongs to some $c_{\alpha,k}^R{}'$ where $\rho(c_{\alpha,k}^R{}', B_r) \geqq 10L(\alpha, k; R)$. But also by the construction and the above inequality, we have that $\rho(h_0(t_2), h_{r+1}(t_2)) < 2\sqrt{2}L(\alpha, k; R)$ since $Sq(\alpha, k; R)$ is of diameter $\sqrt{2}L(\alpha, k; R)$. Hence, by successive applications of the triangle inequality, we obtain $h(t_1) \neq h(t_2)$.

Suppose Case 3 holds. In this case $h_N(t_1)$ is in a $c_\alpha^r{}'$ and $h_N(t_2)$ is in a $c_\beta^s{}'$ where $r \neq s$. Suppose $r < s$. At the r+1 stage of the construction, we have that $h_{r+1}(t) \in c_{\alpha,k}^R{}'$ while $h_{r+1}(t_2) = h_N(t_2) \in c_\beta^s{}'$. If $R < s$, then $L(\alpha, k; R) \geqq L(\beta, s)$. Thus $\rho(h_{r+1}(t_2), h_0(t_2)) \leqq 2\sqrt{2}L(\beta, s)$ $\leqq 2\sqrt{2}L(\alpha, k; R)$ and $\rho(h_{r+1}(t_1), h_0(t_1)) \leqq 2\sqrt{2}L(\alpha, k; R)$ while $\rho(c_\beta^s{}', c_{\alpha,k}^R{}')$ $\geqq \rho(c_{\alpha,k}^R{}', B_r) \geqq 10L(\alpha, k; R)$ and consequently, $h_0(t_1) \neq h_0(t_2)$. If

$R > s$, then since $\rho(C^s_\beta, Sq(\alpha,r)) \geqq 10L(\beta;s)$, we have $\rho(C^s_\beta, C^R_{\alpha,k}) \geqq 10L(\beta;s)$ while $\rho(h_0(t_1), h_{r+1}(t_1)) \leqq 2\sqrt{2}L(\alpha,k) \leqq 2\sqrt{2}L(\beta;s)$ and $\rho(h_0(t_2), h_{r+1}(t_2)) \leqq 2\sqrt{2}L(\beta,s)$. Hence, $h_0(t_1) \neq h_0(t_2)$.

If Case 4 holds, then $h_n(t_1)$ and $h_n(t_2)$ converge to the same point since $L(\alpha;r) < 2^{-r}$. But by the construction of h_n, this can only occur if $t_1 = t_2$. Hence, $t_1 \neq t_2$ implies the impossibility of Case 4.

Thus $t_1 \neq t_2$ implies $h_0(t_1) \neq h_0(t_2)$. Hence, h_0 is a homeomorphism, B_0 a Jordan curve, and D_0 a Jordan domain.

4.37 The Construction of E'. At each stage of the construction of $B_0 \cup D_0$ a square $Sq(1;1)$ was adjoined to the vertical interval $C^1_1, -(1/2) \leqq v \leqq (1/2)$ so that $Sq(1;1) \cap D_n = 0$. It follows from this and the restrictions on the C^r_α in B_n that B_0 and $Sq(1;1)$ have at most boundary points in common. Since c^1_1 has a vacuous discontinuity set, we can adjoin after shrinking and rotating the set $K^1_1 \cup J^1_1$, $1 \varepsilon A^*$. As in the inductive step of the construction of D_0, we obtain a Jordan domain D^1_0 with rectifiable boundary B^1_0. Proceeding as before, only now adjoining $J^n_\alpha \cup K^n_\alpha$ for α in A^*, we obtain an expanding sequence of Jordan domains D^n_0. We define $D = \bigcup_{n=1}^{\infty} D^n_0$. Continuing as in the case of D_0, we then can show that D is a Jordan domain with rectifiable boundary B. The open vertical line segments filling D constitute the set E' in correspondence with E. It follows immediately from the definition of regular curve family that E' is a regular curve family.

4.38 Chordal Relations in E'. The family E' has been constructed as a set of intervals filling a Jordan domain D where the end points of each of the intervals lie on the boundary B of D. Consequently,

each c' of E' partitions D into two components $\delta(c')$ and $\delta^*(c')$. Thus, if we define chordal relations for E' as in 1.10, we have that E' is a normal chordal system.

4.39 Theorem. If c_i' is a interval of E' with end points P_i, Q_i on B($i = 1,2,3$), then $c_1'|c_2'|c_3'$ is equivalent to the condition that the end points of c_i', if properly designated, lie in the order $P_3P_1P_2Q_2Q_1Q_3$ on B and $|c_1', c_2', c_3'|^+$ is equivalent to the condition that the end points can be named so that they lie in the order $P_1Q_1P_2Q_2P_3Q_3$ and this order determines the positive orientation of B.

Proof. If $c_2'|c_1'|c_3'$, then $c_2' \subset \delta(c_1')$ and $c_3' \subset \delta^*(c_1')$ for proper choices of $\delta(c_1')$ and $\delta^*(c_1')$. If we fix P_1, Q_1, then the end points of c_2' must lie on one part of B - $\{P_1, Q_1\}$ and c_3' on the other. Hence, the other points can be designated so that the final order is $P_1P_2P_3Q_3Q_2Q_1$ as desired. Conversely, if such a designation is possible, then $c_2' \subset \delta(c_1')$ and $c_3' \subset \delta^*(c_3')$ or $c_2'|c_1'|c_3'$.

Suppose $|c_1', c_2', c_3'|^+$. There exists then a positively oriented closed curve $M_1M_2M_3$ passing through the points M_1 on c_1', M_2 on c_2', and M_3 on c_3' and only through these points of c_1', c_2', c_3'. Such a curve determines arcs of B P_1Q_1, P_2Q_2, and P_3Q_3 which do not overlap and which in the order given follow the positive orientation of B. Hence, the end points can be named in the order $P_1Q_1P_2Q_2P_3Q_3$. Conversely, if they can be so named, then $c_1'|c_2'|c_3'$, $c_3'|c_1'|c_2'$, and $c_2'|c_3'|c_1'$ are all impossible by the first part of the theorem. Hence, $|c_1', c_2', c_3'|^+$. $|c_1', c_2', c_3'|^-$ implies by previous reasoning that c_1', c_2', c_3' determines arcs P_1Q_1, P_2Q_2, and P_3Q_3 following the negative orientation of B. This

contradicts the given naming of the points. Therefore, $|c_1', c_2', c_3'|^+$ and the theorem is established.

4.40 Theorem. The sets $V_\alpha' = c_\alpha' \cup \{ \bigcup c_{\alpha,k}' \}$ and $V_\alpha^{*\prime} = c_\alpha^{*\prime} \cup \{ \bigcup c_{\alpha,k}^{*\prime} \}$ determine a normal subdivision of E'.

Proof. Consider the vertical line segments filling B_0. These form a set $E_0' - c_1' = \delta(c_1')$. The curves c_α' and $c_{\alpha,k}'$ are vertical line segments with end points on B and are homeomorphic images of line segments contained in the Jordan curve K_α. It follows from the way that E was defined that no one of these intervals separates any two others. Thus by (CS1), $|c_1', c_2', c_3'|^\pm$ for any three elements of V_α' and each V_α' is cyclic.

For each $c_{\alpha,k}'$, $\delta(c_{\alpha,k}')$ denotes the vertical line segments filling one of the two components into which $c_{\alpha,k}'$ separates D. We fix the choice by requiring that $\delta(c_{\alpha,k}')$ does not include c_α'. We then have $\theta(V_\alpha') = \delta(c_\alpha') \cap \{ \bigcap_k \delta^*(c_{\alpha,k}') \}$ where this equation serves as definition of $\theta(V_\alpha')$ in the event that V_α' contains only c_α'. Clearly, $\lambda(V_\alpha')$ and $\lambda(V_{\alpha,k}')$ are adjacent.

Each set E_α' is half-parallel by the definition of the chordal relations and the construction. The original $E_\alpha' - c_\alpha' = \theta(V_\alpha')$ fill J_α. Hence by the construction, $E_0' = \bigcup \lambda(V_\alpha')$.

Conditions (1) - (5) of seminormality are satisfied. To verify condition (6), we take any interval $c_{\alpha,k}' \; V_\alpha' - c_\alpha'$, then by the construction of the first $\lambda(V_\alpha') = E_\alpha' \cup V_\alpha'$, c_α' and $c_{\alpha,k}'$ are intervals in the boundary K_α of J_α where c_α' lies on $u = 0$ and $c_{\alpha,k}'$ lies on some line $u = u' < 2$. If $d_\alpha^{u'}$ is any element of E_α' with $u' < u < 2$, then $d_\alpha^{u'}$ cannot separate c_α' and $c_{\alpha,k}'$ and consequently, $|c_\alpha', c_{\alpha,k}', d_\alpha^{u}|^\pm$.

Similarly, $c_1' \cup \delta(c_1')$ is seminormal. It follows that E' is normal.

4.41 Theorem. E is isomorphic to E'.

Proof. The elements of E' have been constructed to correspond in a one-to-one manner with the elements of E'. It remains to show that this correspondence preserves chordal relations. By 4.29 we need only show that chordal relations are preserved in each $\lambda(V_\alpha)$, and by 4.31 it is seen that we need only show that chordal relations of types (α) and (β) are preserved in $\lambda(V_\alpha)$. Since E' was defined by fitting together the $\lambda(V_\alpha')$ by means of shrinkings, translations, and rotations all of which preserve chordal relations as defined for E', we can confine our attention to the $\lambda(V_\alpha')$ in the form that they were originally constructed.

Consider a triple of type (α). Suppose for example that $c_{\alpha,k}$ is in V_α^+. The interval $c_{\alpha,k}'$ then lies on $mgF_\alpha^+ \subset K_\alpha$. Further, an end point of $d_\alpha^u{}'$ lies between the upper end points of c_α' and $c_{\alpha,k}'$ depending upon whether or not d_α^u does or does not belong to $B(c_{\alpha,k})$. Hence, by the definition of $B(c_{\alpha,k})$ and 4.33 we have that either $c_\alpha | c_{\alpha,k} | d_\alpha^u$ and $c_\alpha' | c_{\alpha,k}' | d_\alpha^u{}'$ or $|d_\alpha^u, c_{\alpha,k}, c_\alpha|^+$ and $|d_\alpha^u{}', c_{\alpha,k}', c_\alpha'|^+$. Thus chordal relations are preserved for triples of type (α).

Consider a triple of type (β): $c_\alpha, c_{\alpha,i}, c_{\alpha,j}$. Assume for example that $c_{\alpha,i}$ and $c_{\alpha,j}$ belong to some $V_\alpha^{++}(u^r)$. By 4.33 $c_{\alpha,i}, c_{\alpha,j}$ are represented by intervals $c_{\alpha,i}'$ and $c_{\alpha,j}'$ in mgF_α^+ in the same order, that is, $c_{\alpha,i}'$ lies between c_α' and $c_{\alpha,j}'$ on mgF_α^+ iff $c_{\alpha,i} < c_{\alpha,j}$. Thus, by 4.23 and 4.39 $|c_{\alpha,i}, c_\alpha, c_{\alpha,j}|^+$ iff $|c_{\alpha,i}', c_\alpha', c_{\alpha,j}'|^+$. Hence, chordal relations for triples of type (β) are preserved in this case. Similar reasoning holds for the other cases.

Thus, each $\lambda(V_\alpha)$ is isomorphic to the corresponding $\lambda(V_\alpha')$ and similarly, each $\lambda(V_\alpha^*)$ is isomorphic to the corresponding $\lambda(V_\alpha^{*'})$. Hence, E is isomorphic to E'.

4.42 Theorem. Let E be a normal chordal system without Y-discontinuities. Then there exists a function f(z) defined in $|z| \leq 1$ such that:

1) f(z) is schlicht on $|z| < 1$;

2) f(z) is a homeomorphism on $|z| \leq 1$;

3) Ref(z) and Imf(z) are of bounded variation on $|z| = 1$;

4) the regular curve-family consisting of the level curves of Ref(z) generate a normal chordal system E" which is isomorphic to E.

Proof. By the Riemann mapping theorem, there exists a one-to-one conformal function f^{-1} which maps the simply connected region D that we have constructed onto the interior of the unit circle of the z-plane. By the Osgood-Caratheodory* extension of this theorem, we have that f^{-1} can be extended so that it maps $B \cup D$ homeomorphically onto $|z| \leq 1$.

Since f^{-1} is an orientation preserving homeomorphism of D onto $|z| < 1$, we have that the regular curve-family consisting of the level curves of Ref(z) when considered as a chordal system E" is isomorphic to E' and consequently to E by 4.41. Since B is rectifiable, it follows that both the real and imaginary parts of f(z) on $|z| = 1$ are of bounded variation.

4.43 Theorem. Let f(z) be schlicht on $|z| < 1$. Let F denote the

* See Caratheodory, C, Mathematische Annalen, vol. 73 (1913) for this theorem and other extension theorems when the boundary is not a Jordan

regular curve family consisting of the level curves of $u(z) = \mathrm{Re}f(z)$ filling the interior of $|z| < 1$. Then F when considered as a chordal system is without Y-discontinuities.

Proof. Let E be the regular curve-family that is the image of F under the mapping f. By remarks of 4.42, it will suffice to show that E has no Y-discontinuities. From the nature of F and f, it is clear that the elements of E are open intervals of the lines u = constant filling a simply connected region of the w-plane.

E being the chordal system of a regular curve-family implies the existence of a R_1 neighborhood of each element of E. Let N be a R_1 neighborhood of c where c is a Y-discontinuity of E. By 4.11 and 4.14, we can then find a a_1 in N, $a_1 \neq c$ such that $V^{++}(c;a_1)$ and $V^{-+}(c;a_1)$ are not null. Let $c_1 \in V^{++}(c;a_1)$ and $c_2 \in V^{-+}(c;a_1)$. From 4.14, 4.10 and 4.08 we obtain: $|a_2,c_1,a_1|^+$, $|a_2,c_2,a_1|^+$, and $c_1|c|c_2$ where a_2 is some element of N such that $a_1|c|a_2$. It follows from the nature of E and the definition of discontinuity set in 4.02 that c and its discontinuity elements must all lie on the same vertical line of the w-plane, say $u = u_0$. $|a_2,c,a_1|^+$ and $|a_2,c_2,a_1|^+$ imply by the definition of $|\ ,\ ,\ |^+$ for regular curve-families that either c_1 and c_2 lie above c on $u = u_0$ or c_1 and c_2 lie below c on $u = u_0$. Suppose c_1 lies between c_2 and c on $u = u_0$. By the definition of $V^{-+}(c;a_1)$, c_2 lies in $\delta_{a_1}(c)$ and every sequence of intervals of E that approach c in $\delta_{a_1}(c)$ must also approach c_2. Since c_1 lies between c_2 and c on $u = u_0$, it is clear that such a sequence of intervals must also approach c_2. This is impossible for $c_1|c|c_2$ and $c_2 \subset \delta_{a_1}(c)$ imply $\delta_{a_1}(c) = \delta_{c_1}(c) \cap \delta_{c_2}(c) = 0$, i.e. there exists a neighborhood of c_2 which does not meet $\delta_{a_1}(c)$. The

case where c_2 lies between c_1 and c is handled in a similar manner by interchanging the roles of a_1 and a_2. It follows that F can contain no Y-discontinuities.

4.44 Remark. Collecting together the last two results, we can now state the principal theorem of the chapter.

4.45 Main Theorem. A necessary and sufficient condition that an abstract normal chordal system E be isomorphic to the chordal system defined by the level curves of the real part of a function f(z) schlicht in $|z| < 1$ is that E contain no Y-discontinuities. If E does contain no Y-discontinuities, the corresponding function f can be chosen to map $|z| = 1$ onto a domain bounded by a rectifiable Jordan curve.

4.46 Theorem. A necessary and sufficient condition that a regular curve-family F filling a simply connected region be O-equivalent to the level curves of the real part of a function f(z) schlicht in $|z| < 1$ is that the chordal system of F contain no Y-discontinuities. If the chordal system of F contains no Y-discontinuities, then the corresponding function f can be chosen to map $|z| = 1$ onto a domain bounded by a rectifiable Jordan curve.

Proof. This follows immediately from the preceding result and the fundamental theorem of Kaplan establishing the relationship between O-equivalent curve-families filling a simply connected region of the plane and their chordal systems. This theorem is established in [2] and states that a necessary and sufficient condition for two regular curve-families filling a simply connected region of the plane to be O-equivalent is that their chordal systems be isomorphic.

4.47 Remark. In the remaining paragraphs of the chapter, we demonstrate that every normal nonoriented chordal system (or normal partition system) is orientable.

4.48 Theorem. If V is a cyclic set of a NCS, then V is orientable.

Proof. Let $<$ be any order relation such that V is linearly ordered with respect to $<$. The well ordering principle assures us that such a relation exists.

If $\underline{a},b,c$ is any triple of V^3, then $(\underline{a},b,c)$ is defined to belong to R^+ iff $a < b < c$ or $c < a < b$ or $b < c < a$ and to R^- iff either $c < b < a$ or $a < c < b$ or $b < a < c$.

Clearly $R^+ \cap R^- = 0$ and $R^+ \cup R^- = R^{\pm}$. It is also readily seen that (01) and (02) are satisfied. Suppose (a,b,c) and (a,c,d) are in R^+. If $a < c$ then $a < b < c$ and either $a < c < d$ or $d < a < c$. Thus, either $a < b < c < d$ or $d < a < b < c$ and $(a,b,d) \in R^+$. If $c < a$ then either $b < c < a$ or $c < a < b$ and $c < d < a$. Thus, either $b < c < d < a$ or $c < d < a < b$ and again $(a,b,d) \in R^+$. Hence, (03) is satisfied and V is orientable.

4.49 Theorem. If F is a normal nonoriented chordal system, then F is orientable.

Proof. Kaplan has shown in [2] that corresponding to every normal chordal system E there exists a regular curve-family G filling the plane whose chordal system is isomorphic to E. In constructing G, the only properties of chordal systems that were used that are not common to both nonoriented chordal systems and chordal systems are that the sets V_α and V_α^* normally subdividing E are oriented and that triples of types (α) and (β) are defined for each $\lambda(V_\alpha)$ and $\lambda(V_\alpha^*)$. Thus, if

we take a normal subdivision of F by the sets V_α, $\alpha \in A$ and V_α^*, $\alpha \in A^*$, then according to 4.47, we can assign orientations to the sets V_α and V_α^*. Assuming this accomplished, we see from their definitions in 3 that triples of types (α) and (β) are now defined in each $\lambda(V_\alpha)$ and $\lambda(V_\alpha^*)$. Hence by following the procedure in [2], we can construct a regular curve family G filling the plane whose chordal system when considered as a nonoriented chordal system is isomorphic to F. Let f denote the isomorphism between F and the chordal system of G. If for each triple $\underline{a},b,c$ in F such that $|\underline{a},b,c|^{\pm}$ holds, we define $|\underline{a},b,c|^{+}$ iff $|f(a),f(b),f(c)|^{+}$ for $f(a),f(b),f(c)$ in G, it is immediate that F is oriented since it is then isomorphic to the chordal system of G.

4.50 Remark. It is fairly evident that 4.48 can be demonstrated without appealing to curve families. However, such a proof would probably repeat much of the work of [2] in only a slightly modified form. It is expected that eventually orientability can be shown for partition systems under much weaker restrictions than normality.

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