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A MIXED PROBLEM FOR HYPERBOLIC EQUATIONS WITH TIME-DEPENDENT DOMAIN.

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A MIXED PROBLEM FOR HYPERBOLIC EQUATIONS WITH TIME-DEPENDENT DOMAIN

by

King Mei Lee

**A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
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CHAPTER I

INTRODUCTION

This thesis deals with the mixed initial-boundary value problem for linear second order hyperbolic equations in a time-dependent domain, or more briefly, the mixed problem. This problem may be described as follows:

Let Q be an unbounded region in R^{n+1} whose boundary B is homeomorphic to a cylinder. We shall denote the intersection of Q (of B) with $[t_1, t_2] \times R^n$ by $Q(t_1, t_2)$ (by $B(t_1, t_2)$) and the intersection of the hyperplane $t = t_0$ with Q by $A(t_0)$. A point in R^n with coordinates $(x_1, \ldots, x_n)$ will be denoted by x, and a point in R^{n+1} with coordinates $(t, x_1, \ldots, x_n)$ will be denoted by (t, x). The indices i, j will always have range $1, \ldots, n$ and the indices r, s will always have range $0, 1, \ldots, n$ (the index 0 refers to the t coordinate).

We seek a weak solution (Definition 2.2.8 below) in Q to the following problem:

$$(1.1.1) \quad u_{tt} - (a_{ij}(t, x) u_{x_i})_{x_j} + a_i(t, x) u_{x_i} + a(t, x) u = p(t, x)$$

$$u(0, x) = f(x) \qquad u_t(0, x) = g(x)$$

$$u(t, x) = 0, \quad (t, x) \in B.$$

We shall call this problem the mixed problem. It is one of the more intractable problems of mathematical physics. The solution describes a diffracted wave in the region Q of space-time. If the diffracting body is in motion, then the spatial domain of the wave, A(t), varies with time. The cases that have been treated for the most part have been those in which the diffracting body is stationary, and even in this case, beyond the fact that a unique solution exists, little is known except for so called separable bodies.

In two dimensions, existence, uniqueness, and regularity theorems have been obtained for the mixed problem with time-dependent space domain [2, pp. 471-475]. N. Balazs [1] has obtained an explicit solution for the two dimensional wave equation with certain restrictions on the shape of the boundary. For three or four dimensions, many special cases of (1.1.1) have been investigated (for a few of these, see [8, Chapter 11]). It is only in the last decade that results have been obtained for the more general versions of the mixed problem. Among the recent contributers is O. Ladyzhenskaya [5]. She obtained existence and uniqueness of a weak solution for a class of operator equations that includes (1.1.1) if A(t) does not depend on t. E. Rojak [9] obtained existence, uniqueness, and regularity theorems for the wave equation in the case where the diffracting body is in rigid motion. For an extensive bibliography of the literature, see [6] and [9]. In this thesis, existence and uniqueness of a weak solution are proved under rather weak restrictions upon A(t).

All constants and functions appearing in this thesis are real-valued. and the coefficients in equation (1.1.1) are bounded together with their first and second derivatives. We assume that for any n-tuple $\xi_1, \ldots, \xi_n$ of real numbers the following inequality holds:

$$(1.1.2) \qquad c_m^2 \xi_i \xi_i \leq a_{ij} \xi_i \xi_j \leq c_M^2 \xi_i \xi_i, \quad 0 < c_m \leq c_M .$$

Without loss of generality, we may take $c_m = 1$. It will be convenient to define the operators Λ_0, Λ_1, and Λ_2 by

$$(1.1.3) \qquad \begin{aligned} \Lambda_0 u &= (a_{ij} u_{x_i})_{x_j} - u , \\ \Lambda_1 u &= a_i u_{x_i} + (a-1)u \\ \Lambda_2 u &= u_{tt} - \Lambda_0 u + \Lambda_1 u, \end{aligned}$$

and the time derivatives of Λ_0 and Λ_1 by

$$\begin{aligned} (\Lambda_0)_t u &= ((a_{ij})_t u_{x_i})_{x_j} \\ (\Lambda_1)_t u &= (a_i)_t u_{x_i} + a_t u \end{aligned}$$

The inequalities (1.1.2) express the condition that Λ_0 be strictly elliptic. Note that equation (1.1.1) may be written as $\Lambda_2 u = p$.

We assume that, for some α $(0<\alpha<1)$, the components, n_r, of the unit normal, $\underset{\sim}{n}$, to B satisfy the inequalities

$$(1.1.4) \qquad n_0^2 \leq (1-\alpha)^2 n_i n_i \leq (1-\alpha)^2 a_{ij} n_i n_j .$$

This requirement implies that each point on the diffracting body moves with velocity less than unity.

The conditions on p, f, and g are that $p \in L_2(Q(0,T))$, $f \in H^1(A(0))$ (see Definition 2.2.7), and $f \in L_2(A(0))$.

In Chapter II, we present preliminary lemmas needed in the sequel. Various function spaces are introduced, their properties are stated, and, for these functions, variants of the divergence theorem are proved. A vector $\underset{\sim}{b}$ is introduced and some of its properties proved. Results on weak regularity of strictly elliptic equations are also proved. Chapter III contains our main results. Using the machinery developed in Chapter II, we obtain an energy inequality, which shows that a solution, in some sense, depends continuously on the functions p, f, and g in (1.1.1). To prove this inequality we have developed a method that reminds one of the (a, b, c)-method of Friedrichs [3, pp. 425]. Then by extending methods developed by O. Ladyzhenskaya [5] we obtain existence and uniqueness of a weak solution.

CHAPTER II

FUNCTIONS SPACES AND VARIANTS OF THE DIVERGENCE THEOREM

This chapter deals with theorems needed in the sequel. Section 2.1 contains basic definitions and smoothness conditions on B. In Section 2.2 we state some theorems first proved by Sobolev which are used in Section 2.3 to prove variants of the divergence theorem. Section 2.4 contains some results concerning strictly elliptic equations. Theorem 2.4.1 extends well-known results of solutions in bounded regions to unbounded regions.

2.1 Definitions. Let X be a region in R^n or R^{n+1} with boundary ∂X. The space $C^m(X)(C_0^m(X))$ is the set of functions on X (with compact support in X) having continuous partial derivatives up to order m inclusive. If $u, v \in L_p(X)$, then $u_{x_{i_1}, \ldots, x_{i_m}}$, the generalized derivative with respect to $x_{i_1}, \ldots, x_{i_m}$, exists and is equal to v if

$$\int_X u \phi_{x_{i_1}, \ldots, x_{i_m}} dX = (-1)^m \int_X u_{x_{i_1}, \ldots, x_{i_m}} \phi \, dX$$

for every $\phi \in C_0^m(X)$. The Sobolev space $W_p^m(X)$ is the set of functions possessing all generalized derivatives of order m or less in $L_p(X)$. In Chapter III we deal only with the case $p = 2$. However, we introduce the spaces $W_p^m(X)$ because we believe that the Sobolev theorems in Section 2.3 and better understood in this more general context.

We assume that B satisfies the following smoothness conditions:

(i) there exists a 1-1 transformation ϕ of R^{n+1} onto R^{n+1} mapping B onto the surface of the right circular cylinder $R^1 \times S^{n-1}$, where S^{n-1} is the boundary of the n-ball;

(ii) ϕ and its inverse are given by

$$(2.1.1) \qquad y_r = y_r(t, x_1, \ldots, x_n), \quad x_r = x_r(y_0, y_1, \ldots, y_n),$$

where $y_r(t, x_1, \ldots, x_n)$ and $x_r(y_0, y_1, \ldots, y_n)$ are functions in $C^2(R^{n+1})$ with bounded first derivatives

Let $\underset{\sim}{b}$ be a vector defined on B with components

$$(2.1.2) \qquad b_0 = n_j n_j$$
$$b_i = -n_0 n_i$$

It is easily verified that, for some $\beta > 0$,

$$(2.1.3) \qquad \text{(i)} \quad 0 < \beta < b_0 < 1$$
$$\text{(ii)} \quad (b_j b_j)^{1/2} b_0^{-1} = |n_0| (n_j n_j)^{1/2} < 1 - \alpha$$
$$\text{(iii)} \quad b_r n_r = 0$$

The inequality (i) follows from (2.12) and the fact that $\underset{\sim}{n}$ is a unit vector; condition (ii) follows from (1.1.4).

The vector $\underset{\sim}{b}$ will play an important role in proving the energy inequality. For the case of cylindrical boundary this theorem is proved, for sufficiently smooth u vanishing on the boundary, by integrating the expression $u_t \wedge_2 u$ over a "light cone", applying the

divergence theorem, and using the fact that u_t and n_0 vanish on B. This method fails if the boundary is time-dependent for then u_t will not vanish on B. In this case, we note that since u is constant (and equal to zero) on B, gradu = ∇u is either the null vector or is normal to B. In either case $\underset{\sim}{b} \cdot \nabla u = 0$ on B, since $\underset{\sim}{b}$ is "tangential" (2.1.3, iii). This observation suggests that the energy inequality may be obtained by extending $\underset{\sim}{b}$ into Q and applying the divergence theorem to $(\nabla \underset{\sim}{u} \cdot \underset{\sim}{b}) \wedge_2 u$. We note that if B is cylindrical, then $n_0 = 1$, and $\nabla \underset{\sim}{u} \cdot \underset{\sim}{b}$ reduces to u_t. We now prove that it is possible to extend $\underset{\sim}{b}$ into Q having certain properties.

Lemma 2.1.1. Given a compact set K containing B(0,T) in its interior, $\underset{\sim}{b}$ may be extended to a vector in Q, again denoted by $\underset{\sim}{b} = (b_r)$, whose components have the following properties in Q:

(2.1.5)
(i) The support of b_i in Q is contained in $K \cap Q$.
(ii) b_r and $(b_r)_{x_s}$ are bounded in Q,
(iii) $0 < \beta < b_0 < 1$,
(iv) $(b_j b_j)^{1/2} b_0^{-1} < 1-\alpha$.

Proof. Let Q', $B' = R^1 \times S^{n-1}$ (S^{n-1} is the boundary of the n-ball), and K' denote the image of Q, B, and K, respectively, under ϕ (2.1.1). A point in Q' with coordinates $y_0, y_1, \ldots, y_n$ will be denoted by (y_0, y). If $u(t,x) \in C^2(Q)$, then $u'(y_0, y) = u(t, x_1(y_0, y), \ldots, x_n(y_0, y))$ will be in $C^2(Q')$. Let $h(y_0, y)$ be a function in $C^2(R^{n+1})$ that satisfies the conditions

(i) $h(y_0, y) = 1$ $\qquad (y_0, y) \in B'(0, T)$,

(ii) $h(y_0, y) = 0$ $\qquad (y_0, y) \notin K'$,

(iii) $0 \leq h(y_0, y) \leq 1$.

Since $\underset{\sim}{b} = (b_r(t, x))$ is defined on B, the vector $\underset{\sim}{b'} = (b'(y_0, y))$ is defined on B'. We extend $b'_r(y_0, y)$ to Q' as follows: on the portion of any hyperplane $y_0 =$ constant exterior to B', $b'_0(y_0, y)$ is to be constant along each ray through the origin of that hyperplane. The values of $b'_0(y_0, y)$ on B' thus determine the values of $b_0(y_0, y_s)$ on Q'. We similarily define $b'_i(y_0, y)$ along each ray from the origin of the hyperplane $y_0 =$ constant to be the value of $b'_i(y_0, y)$ at B' times $h(y_0, y)$.

If we set $|y| = (y_j y_j)^{1/2}$, then $b'_r(y_0, y_s)$ can be expressed analytically as:

$$
\begin{aligned}
b'_0(y_0, y) &= b'_0(y_0, y_1 |y|^{-1}, \ldots, y_n |y|^{-1}), \\
b'_i(y_0, y) &= h(y_0, y)\, b'_i(y_0, y_1 |y|^{-1}, \ldots, y_n |y|^{-1}).
\end{aligned}
\tag{2.1.6}
$$

We define $b_r(t, x)$ on Q by

$$b_r(t, x) = b'_r(y_0(t, x), y_1(t, x), \ldots, y_n(t, x)).$$

It is evident that $b_r(t, x)$ is bounded and satisfies conditions (i), (iii) and (iv) of (2.1.5). It remains to show all partial derivatives of $b_r(t, x)$ are bounded in Q. We calculate $(b'_0(y_0, y))_{y_r}$ at the point $(\bar{y}_0, \bar{y})$ in Q'. From the definition of derivatives and from (2.1.6) we see that

$$(b_0(\bar{y}_0, \bar{y}))_{y_r} = \lim_{h\to 0} \frac{1}{h} (b_0'(\bar{y}_0, \ldots, \bar{y}_r + h, \ldots, \bar{y}_n) - b_0'(\bar{y}_0, \bar{y}))$$

$$= \lim_{h\to 0} \frac{1}{h} \{b_0'(\bar{y}_0, \frac{y_1}{|\bar{y}_h|}, \ldots, \frac{y_r + h}{|\bar{y}_h|}, \ldots, \frac{y_n}{|\bar{y}_h|}) \tag{2.1.7}$$

$$- b_0'(y_0, \frac{y_1}{|\bar{y}|}, \ldots, \frac{y_n}{|\bar{y}|})\},$$

where

$$|\bar{y}_h|^{-1} = (\bar{y}_1^2 + , \ldots, (\bar{y}_r + h)^2 + , \ldots, \bar{y}_n^2)^{1/2},$$

$$|\bar{y}| = (\bar{y}_1^2 + , \ldots, + \bar{y}_n^2)^{1/2}.$$

From (2.1.2) and the condition that the $y_r(t, x)$ have bounded derivatives, it follows that $b_0'(y_0, y)|_{B'} \in C^1(B')$ and that $b_0'(y_0, y)$, considered as a function on B', has bounded derivatives. Since the limit in the last line of (2.1.7) is evaluated on B', the limit exists (as a "tangential derivative") and is bounded. Hence $(b_0')_{y_r}$ exists at $(\bar{y}_0, \bar{y})$ and is bounded. It follows that

$$(b_0(t, x))_{x_s} = (b_0'(y_0, y))_{y_r} \frac{\partial y_r}{\partial x_s}$$

is bounded.

Similarily, we can show that the derivatives of other components of $\underset{\sim}{b}$ are bounded.

2.2 Theorems from Functional Analysis. In this section, theorems giving properties of the Sobolev spaces $W_p^m(X)$ are stated. The proofs

of these theorems may be found in [10, pp. 338-375] and [11, pp. 55-85].

Theorem 2.2.1. $W_p^m(X)$ is a Banach space with norm

$$\|u\|_{W_p^m(X)} = [\int_X \Sigma_{k=0}^m (\Sigma_{i_1,\ldots,i_k=0}^n u_{x_{i_1}\ldots x_{i_k}}^2)^{p/2} dX]^{1/p}.$$

In the case $p = 2$, $W_2^m(X)$ is a Hilbert space with inner product

$$(2.2.1) \qquad (u, v) = \int_X \Sigma_{k=0}^m \Sigma_{i_1,\ldots,i_k=0}^n u_{x_{i_1}\ldots x_{i_k}} v_{x_{i_1}\ldots x_{i_k}} dX$$

Theorem 2.2.2. If u and v are in $W_2^m(X)$, then

$$(uv)_{x_r} = u_{x_r} v + v u_{x_r}.$$

A set D is a C^k-submanifold of dimension h in R^{n+1} if there exists an open set U containing D and a transformation of class C^k, with non-vanishing Jacobian, mapping U onto the cube $|y_r| \leq 1$ in such a way that D is mapped into the surface

$$y_{h+1} = y_{h+2} = \ldots = y_{n+1} = 0.$$

A C^k-cell is a compact set such that there exists a map of class C^k with non-vanishing Jacobian transforming K onto the cube $y_r \leq 1$.

Theorem 2.2.3. If K is a C^m-cell and $u \in W_p^m(K)$, then there exists a sequence of function $\{u^k\} \in C^\infty(K)$ converging to u in $W_p^m(K)$.

The following two "embedding theorems" are due to Sobolev.

Theorem 2.2.4. Let K be a C^m-cell in R^{n+1}, and let $u \in W_p^m(K)$. If $n + 1 < pm$, then u is equivalent to a continuous function

and

$$(2.2.2) \qquad \sup_{(t,x)\in K} |u(t,x)| \le c_1 \|u\|_{W_p^m(K)} .$$

Theorem 2.2.5. If $pm \le n+1$, if K is a C^m-cell in R^{n+1}, and if $u \in W_p^m(K)$, then for each C^m-submanifold D of dimension $s > n+1-mp$ and for any $q < ps(n+1-pm)^{-1}$ the restriction of u to D, denoted by $u|_D$, is in $L_q(D)$, and

$$(2.2.3) \qquad \|u|_D\|_{L_q(D)} \le c_2 \|u\|_{W_p^m(K)} .$$

Corollary. There exists a sequence $\{u^k\}$ of functions in $C^m(K)$ such that:

(i) $u^k \to u$ in $W_p^m(K)$

(ii) $u^k|_D \to u|_D$ in $L_q(D)$.

Proof of Corollary. Theorem 2.2.5 assures the existence of the sequence $\{u^k\}$ satisfying (i). Since $u^k \in C^m(K)$, $u^k|_D$ may be taken to be the usual restriction of u to D. By (2.2.3),

$$\|u^k|_D - u|_D\|_{L_q(D)} \le c_2 \|u^k - u\|_{W_p^m(K)} .$$

Therefore, as $u^k \to u$, in $W_p^m(K)$,

$$u^k|_D \to u|_D .$$

If u is continuous on K, then u takes on well-defined values on every submanifold in K. If $u \in W_p^m(K)$, then $u \in L_p(K)$ and,

since submanifolds have measure zero, we may ask: in what sense (if any) may u take on values on these lower dimensional surfaces? The Sobolev theorems supply the answer.

If u has sufficiently many generalized derivatives $(m > (n+1)p^{-1})$ then u takes on values on each submanifold since u is equivalent to a continuous function. If u does not possess $(n+1)p^{-1}$ generalized derivatives, then we whould not expect u to be defined on submanifolds of arbitrary dimension. Theorem 2.2.5 states that for a submanifold D of sufficiently high dimension s, or of sufficiently large p $(p > (n+1-s)m^{-1})$ u is defined on D as an element of $L_q(D)$ $(q < ps(n+1-pm)^{-1})$. Further, the higher the dimension of D (the larger s is), the better behaved u is (the larger q may be taken) on D.

Theorem 2.2 6. Let $u, D, s, q,$ and K be as in Theorem 2.2.5 Assume that the translation of D by the vector $x = (\Delta x_r)$ lies in K. Then

$$\int_D |u(t+\Delta x_0, \ldots, x_i + \Delta x_i, \ldots) - u(t, \ldots, x_i, \ldots)|^q dD \to 0$$

as $|\Delta x_r| \to 0$.

The proof of Theorem 2.2.6 is found within the proof of the Kondrasev Theorem [11, p. 85].

If D is a bounded submanifold, we shall always understand by $u|_D$ the map defined in Theorem 2.2.5. Suppose D and K are both unbounded sets. Let $\{K_h\}$ $(h = 1, 2, \ldots)$ be a family of C^m - cells with pairwise nonintersecting interiors, and such that

$$D \subset \bigcup_{h}^{\infty} K_h \subset K.$$

By $u|_D$ we shall always mean the measurable function on D equal to $u|_{D \cap K_h}$ on $D \cap K_h$.

Definition 2.2.7

$$H^m(A(t)) = \{u \mid u \in W_2^m(A(t)),\ u|_{\partial A(t)} = 0\}$$

$$H^m(Q(0,T)) = \{u \mid u \in W_2^m(Q(0,T)),\ u|_{B(0,T)} = 0\}$$

Definition 2.2.8. A function u is a weak solution to the mixed problem (see (1.1.1)) if

(i) for all $T > 0$, $u \in H^1(Q(0,T))$

(ii) $$\int_{Q(0,T)} (u_t \phi_t - a_{ij} u_{x_i} \phi_{x_j} - (a_i u_{x_i} + a u - p)\phi)\, dQ - \int_{A(0)} g \phi_t\, dA = 0$$

for all $\phi \in C^1(Q(0,T))$ that satisfy $\phi_t|_{t=T} = \phi|_{t=T} = \phi|_B = 0$ and that have compact support.

(iii) $u|_{A(0)} = f$.

It is easily seen that a classical solution (a solution in $C^2(Q(0,T))$) to the mixed problem is also a weak solution. The integral equation (ii) can be obtained by multiplying equation (1.1.1) by ϕ and applying the divergence theorem.

O. Ladyzhenskaya proved existence and uniqueness of a weak solution to a version of the mixed problem using several spaces of vector-valued functions. Below we point out some connections between these spaces and the Sobolev spaces. A more detailed exposition of such vector-valued functions may be found in [4, Ch. III].

Let H be a Hilbert space with norm $\| \ \|_H$. A map of $[t_1, t_2]$ into H is a vector-valued function of $[t_1, t_2]$ into H. We denote by $L_2(t_1, t_2; H)$ the set of maps $w(t)$ on $[t_1, t_2]$ and for which

$$\| w(t) \|^2_{L_2(t_1, t_2; H)} \equiv \int_{t_1}^{t_2} \| w(t) \|^2_H \, dt$$

exists; $L_2(t_1, t_2; H)$ is a Hilbert space with the above norm. We say that $w \in L_2(t_1, t_2; H)$ is continuous (has derivative $\frac{dw}{dt} \in L_2(t_1, t_2; H)$) if, for almost all $t \in [t_1, t_2]$,

$$\lim_{\Delta t \to \infty} \| w(t + \Delta t) - w(t) \|_H = 0$$

$$(\lim_{\Delta t \to 0} \| (w(t + \Delta t) - w(t))(\Delta t)^{-1} - \frac{dw}{dt}(t) \|_H = 0).$$

Let A be a region in R^n with boundary ∂A, and let $Q^* = [0, T] \times A$. Suppose $w \in L_2(Q^*)$; then by Fubini's theorem, we can consider $w \in L_2(0, T; L_2(A))$. Conversely, elements in $L_2(0, T; L_2(A))$ can be considered elements in $L_2(Q^*)$.

A function $w \in L_2(0, T; W_2^1(A))$ is said to vanish on $[0, T] \times \partial A$ if $w(t) \big|_{\partial A} = 0$ for almost all t. Further, if w has a derivative w', then w can be considered as an element of $W_2^1(Q^*)$. For if $\phi \in C_0^1(Q^*)$, then

$$\int_0^T \int_A w \phi_t \, dA \, dt$$

$$= \int_0^T \lim_{t \to 0} \left(\int_A w(t) (\phi(t + \Delta t, x) - \phi(t, x))(\Delta t)^{-1} \, dA \right) dt$$

$$= \lim_{t \to 0} \int_0^T \int_A (w(t - \Delta t) - w(t)(\Delta t)^{-1} \phi(t) \, dA \, dt$$

$$= -\int_0^T \int_A w'(t) \phi(t) \, dA \, dt.$$

Suppose $w \in W_2^1(Q^*)$. If A is bounded, then w, considered as an element of $L_2(0, T; L_2(A))$, is continuous by Theorem 2.2.6. If $w \in W_2^2(Q^*)$, then, by Theorem 2.2.5, w defines an element in $L_2(0, T; W_2^1(A))$.

In this thesis $A(t)$ depends on time. The maps of $[0, T]$ into the spaces $W_p^m(A(t))$ are maps of the interval, not into a single Hilbert space, but into the infinite family of spaces $W_p^m(A(t))$, one of each t. It turns out that the variants of the divergence theorem (see the next section) can be extended more easily relative to the Sobolev spaces, such as $W_p^m(Q(0, T))$, than in the setting of functions of $[0, T]$ into $W_p^m(A(t))$.

2.3. The Divergence Theorem. A crucial step in the proof of the energy inequality is the application of the divergence theorem. In the present section we extend this theorem to a theorem about elements of the space $W_2^m(X)$ and prove special formulas needed in the sequel.

Lemma 2.3.1 (Divergence Theorem). Let $\underset{\sim}{v}(\underset{\sim}{w})$ be a vector whose components $v_s(w_s)$ are in $W_2^1(K)$, where K is a finite union of C^1-cells with pairwise nonintersecting interiors, and let $\underset{\sim}{u}$ be the vector with components $u_s = v_s w_s$ (no sum on s). Then

$$(2.3.1) \qquad \int_K (u_s)_{x_s} \, dK = \int_{\partial K} u_s n_s \, d\partial K,$$

where $\underset{\sim}{n} = (n_s)$ is the unit normal to ∂K, the boundary of K.

Proof. First consider the case where K is a single cell. By the Corollary of Theorem 2.2.3, there exist sequences $\{v_s^k\}$ ($\{w_s^k\}$) with elements in $C^1(K)$ that converge to $v_s(w_s)$ in $W_2^1(K)$. For each k,

the divergence theorem holds, and

$$\int_K (u_s^k)_{x_s}\, dK = \int_{\partial K} u_s^k n_s\, d\partial K .$$

If we take the limit as $k \to \infty$, (2.3.1) follows since if $\{v_s^k\} \to v_s$ ($\{w_s^k\} \to w_s$) in $W_2^1(K)$, $\{(v_s^k)_{x_s}\} \to (v_s)_{x_s}$ ($\{(w_s^k)_{x_s}\} \to (w_s)_{x_s}$) in $L_2(K)$ and $\{v_s^k|_{\partial K}\} \to v_s|_{\partial K}$ ($\{w_s^k|_{\partial K}\} \to w_s|_{\partial K}$) in $L_2(\partial K)$.

If $K = \bigcup_h K_h$, where each K_h is a C^1-cell, we may apply the divergence theorem to each cell, and sum the results. We obtain the formula

$$\int_K (u_s)_{x_s}\, dK = \int_{\partial K} u_s n_s\, d\partial K + \Sigma_h \int_{\partial K_h'} u_s n_s\, d\partial K_h' ,$$

where $\partial K_h'$ represents the part of the boundary of K_h not appearing in ∂K. The sum on the right side of the equation is zero because of the opposite orientation of normal vectors of adjacent cells and because of Theorem 2.2.6.

In the next two lemmas, we assume the vector $\underset{\sim}{b}$ (2.1.2) has been extended into $Q(0, T)$ by Lemma 2.1.1.

Lemma 2.3.2. Let K be as in Lemma 2.3.1, and in addition, let K lie in $Q(0, T)$. If $u, w \in W_2^1(Q(0, T))$, then

$$\int_K (vw b_r)_{x_r}\, dK = \int_{\partial K'} vw b_r n_r\, d\partial K' ,$$

where $\partial K'$ is the part of ∂K that is not contained in $B(0, T)$.

Proof. Since each b_r has bounded first derivatives, $vb_r \in W_2^1(Q(0, T))$ for each r. We may apply Lemma 2.3.1 to the vectors $v = (vb_0, vb_1, \ldots, vb_n)$, $w = (w, \ldots, w)$. On $B(0, T)$, $n_r b_r = 0$ by (2.1.3, iii); hence, the surface integral reduces to an integral over ∂K.

Lemma 2.3.3. Let K and $\partial K'$ be as in Lemma 2.3.2. If a vector $\underset{\sim}{v}$ has components $v_s \in W_2^1(Q(0,T))$, and if $u \in H^2(Q(0,T))$, then

$$\int_K (b_r u_{x_r} v_s)_{x_s} \, dK = \int_{\partial K'} b_r u_{x_r} v_s n_s \, d\partial K' .$$

Proof. If we show that $\underset{\sim}{b} \cdot \underset{\sim}{\nabla} u \,|_{B(0,T) \cap K} = 0$ the lemma will follow from Lemma 2.3.1. We first assume that K is a single C^1 - cell. By definition of a C^1 - cell, there is a C^1 - map of K onto the cube $|y_r| \leq 1$. Suppose the map and its inverse are given by

$$y_r = y_r(t,x) \quad , \quad x_r = x_r(y_0, y) .$$

We may assume $B(0,T) \cap K$ is mapped onto

$$y_0 = y_0(t,x) = 1 .$$

Since the equation of $B(0,T) \cap K$ is $y_0(t,x) = 1$, we may assume that

$$\underset{\sim}{\nabla} y_0(t,x) = \underset{\sim}{n} \tag{2.3.1}$$

where $\underset{\sim}{n}$ is the unit normal to $B(0,T) \cap K$.

Set

$$u'(y_0, \ldots, y_n) = u(t(y_0, y), \ldots, x_n(y_0, y)) .$$

Since $u|_{B(0,T)} = 0$,

$$u'(1, y_1, \ldots, y_n) = 0$$

and therefore

$$(u')_{y_i} \,|_{y_0 = 1} = 0 . \tag{2.3.2}$$

Now

$$u_{x_r} = (u'(y_0(t,x), \ldots, y_n(t,x)))_{x_r} \tag{2.3.3}$$

$$= (u'(y_0(t,x),\ y(t,x))_{y_s} \frac{\partial y_s}{\partial x_r}$$

Since $y_0 = 1$ is the image of $B(0,T) \cap K$ we see from (2.3.2) that

$$u'(y_0(t,x),\ y(t,x))_{y_i} \Big|_{B(0,T)\cap K} = 0 .$$

Evaluating (2.3.3) on $B(0,T) \cap K$ we obtain the formula

$$u_{x_r} \Big|_{B(0,T)\cap K} = u'(y_0(t,x),\ y(t,x))_{y_0} \frac{\partial y_0}{\partial x_r} \Big|_{B(0,T)\cap K}$$

and since the u_{x_r} are just the components of ∇u,

$$\nabla u = u'_{y_0} \nabla y_0 .$$

By (2.3.1), ∇u is in the normal direction to $B(0,T) \cap K$, and by (2.1.3, iii), $b \cdot \nabla u = 0$.

The extension to the case where K is not a single cell is obvious.

2.4. Strictly Elliptic Equations. In this section we prove several properties of solutions of elliptic equations. Theorem 2.4.1 is a modification of an idea used by Lions and Nirenberg [7, pp. 96-98]. Lemma 2.4.5 is due to O. Ladyzhenskaya [5].

Let A denote the exterior of the unit sphere in R^n with center at the origin, and define

$$H^m(A) = \{u \mid u \in W_2^m(A),\ u \big|_{\partial A} = 0\}.$$

In this section we consider Λ_0 to be that operator on $H^2(A)$, depending on the parameter t, defined by

$$\Lambda_0(t): \; u \to (a_{ij}(t,x)\, u_{x_i}(x))_{x_j} \in L_2(A).$$

Theorem 2.4.1. If $p(x) \in L_2(A)$, the problem

$$\Lambda_0(t)\, u = p$$

has a unique solution in $H^2(A)$, and

$$\|u\|_{H^2(A)} \leq \gamma \|p\|_{L_2(A)}.$$

Corollary 1. For each $u \in H^2(A)$,

$$\|u\|_{H^2(A)} \leq \gamma \|\Lambda_0 u\|_{L_2(A)}. \tag{2.4.1}$$

Corollary 2. The conclusions of the theorem are valid if A is a region for which there exists a map, with bounded second derivatives, transforming A either into the exterior of the unit sphere or into an annulus.

Proof. For the sake of clarity several long calculations involved in this proof have been removed from it and are carried out in Lemmas 2.4.1 - 2.4.4. These are to be found following the main body of the proof, which is also long and involves many estimates.

By the strict ellipticity of Λ_0, the inner product in $H^1(A)$ given by (2.2.1) is equivalent to the inner product.

$$\{u, v\} = \int_A (a_{ij}\, u_{x_i}\, v_{x_j} + uv)\, dA. \tag{2.4.2}$$

The map $v \to \int_A p v \, dA$ defines a linear continuous functional on $H^1(A)$ equipped with the inner product (2.4.2) since

$$\int_A p v \, dA \leq \| p \|_{L_2(A)} \| v \|_{L_2(A)}$$

$$\leq \| p \|_{L_2(A)} \{v, v\}^{1/2} .$$

By the Riesz representation theorem, there exists a $u \in H^1(A)$ such that

$$(2.4.3) \qquad \int_A p v \, dA = \{u, v\}$$

$$= \int_A (a_{ij} u_{x_i} v_{x_j} + uv) \, dA$$

for all $v \in H^1(A)$ and further

$$(2.4.4) \qquad \| u \|_{H^1(A)} \leq \{u, u\}^{1/2}$$

$$\leq \| p \|_{L_2(A)} .$$

We next show that $u \in H^2(A)$. It is convenient to introduce the notation

$$\theta_k = \text{Arctan} \, (x_{k+1} \, x_k^{-1}) \qquad k < n$$

$$\theta_n = (\Sigma_i^n \, x_i^2)^{1/2}$$

For any number θ, $| \theta | < 1$, we define the map $\tau_\theta^{\#}$ of A onto A to be the rotation of each point of A by $\theta \, \theta_n^{-1}$ radians in the x_1, x_2 plane. Analytically, $\tau_\theta^{\#}$ is given by

$$z_1(\theta, x) = x_1 \cos \frac{\theta}{\theta_n} + x_2 \sin \frac{\theta}{\theta_n}$$

$$z_2(\theta, x) = -x_1 \sin \frac{\theta}{\theta_n} + x_2 \cos \frac{\theta}{\theta_n} \tag{2.4.5}$$

$$z_k(\theta, x) = x_k$$

We define a map τ_θ of $H^1(A)$ onto $H^1(A)$ by

$$\tau_\theta v(x) = v(\tau_\theta^{\#} x) = v(z) \quad (v \in H^1(A)).$$

We shall prove that the difference quotient

$$\theta^{-1} \| \tau_\theta u(x) - u(x) \|_{H^1(A)} < c_1$$

where c_1 is a constant independent of θ.

From (2.4.5), it can be shown (see Lemma 2.4.1) that

$$(\tau_\theta u)_{x_i} = \tau_\theta u_{x_i} + h_{ij}(x, \theta)\, \tau_\theta u_{x_j}, \tag{2.4.6}$$

where h_{ij} is a bounded continuous function in A and

$$\lim_{\theta \to 0} h_{ij}\, \theta^{-1} < c_2 .$$

The constant c_2 depends only on the dimension n. It is easily seen that, for v and w belonging to $H^1(A)$,

$$\tau_\theta(vw) = (\tau_\theta v)(\tau_\theta w)$$

and

$$\int_A v(\tau_\theta w)\, dA = \int_A (\tau_{-\theta} v)\, w\, dA .$$

Replacing v in (2.4.3) by $\tau_{-\theta} v$, we obtain

$$(2.4.7) \qquad \int_A a_{ij} u_{x_i} (\tau_{-\theta} v)_{x_j} dA + \int_A u \tau_{-\theta} v \, dA = \int_A p \tau_{-\theta} v \, dA .$$

If (2.4.6) is used twice, the first integral becomes

$$\begin{aligned} \int_A a_{ij} u_{x_i} (\tau_{-\theta} v)_{x_j} dA &= \int_A \{a_{ij} u_{x_i} \tau_{-\theta} v_{x_j} + a_{ij} u_{x_i} h_{jk} \tau_{-\theta} v_{x_k}\} dA \\ &= \int_A \{(\tau_\theta a_{ij})(\tau_\theta u_{x_i}) v_{x_j} + a_{ij} u_{x_i} h_{jk} \tau_{-\theta} v_{x_k}\} dA \qquad (2.4.8) \\ &= \int_A \{(\tau_\theta a_{ij})(\tau_\theta u)_{x_i} v_{x_j} + F(\theta, u, v)\} dA , \end{aligned}$$

where

$$F(\theta, u, v) = a_{ij} u_{x_i} h_{jk} \tau_{-\theta} v_{x_k} - \tau_\theta a_{ij} v_{x_j} h_{ik} \tau_\theta u_{x_k} , \quad (k = 1, 2, \ldots, n) .$$

Since each term in the expression for $F(\theta, u, v)$ contains an h_{ik}, $\lim_{\theta \to 0} \theta^{-1} F(\theta, u, v)$ exists in the sense of $L_1(A)$. Substituting (2.4.8) into (2.4.7), subtracting (2.4.3), and rearranging, we obtain the relation

$$\begin{aligned} &\int_A \{(\tau_\theta a_{ij})(\tau_\theta u)_{x_i} v_{x_j} - a_{ij} u_{x_i} v_{x_j} + (\tau_\theta u - u) v\} dA \\ &= \int_A p \{(\tau_\theta v - v) - F(\theta, u, v)\} dA . \end{aligned}$$

We add and subtract $a_{ij} (\tau_\theta u_{x_i}) v_{x_j}$ to the left integrand, divide by θ, introduce the notation $\Delta_\theta w = \theta^{-1} (\tau_\theta w - w)$, and rearrange the equation. The result is

$$(2.4.9) \qquad \int_A (a_{ij} (\Delta_\theta u)_{x_i} v_{x_j} + v \Delta_\theta u) dA = \int_A \{p \Delta_\theta v - \Delta_\theta a_{ij} (\tau_\theta u)_{x_i} v_{x_j} - \frac{1}{\theta} F\} dA .$$

The right side of (2.4.9) is bounded by $c_3 \|p\|_{L_2(A)} \|u\|_{H^1(A)}$, where c_3 is a constant independent of θ by Lemma 2.4.2. Now (2.4.9) can be rewritten as

$$\int_A (a_{ij} (\Delta_\theta u)_{x_i} v_{x_j} + v \Delta_\theta u)\, dA \le c_3 \|p\|_{L_2(A)} \|v\|_{H^1(A)}.$$

So far we have not made a specific choice of v in $H^1(A)$. We now choose $v = \Delta_\theta u$. Thus,

$$\|\Delta_\theta u\|^2_{H^1(A)} \le \int_A \{a_{ij} (\Delta_\theta u)_{x_i} (\Delta_\theta u)_{x_j} + (\Delta_\theta u)^2\}\, dA$$

$$\le c_3 \|p\|_{L_2(A)} \|\Delta_\theta u\|_{H^1(A)},$$

and therefore

$$\|\Delta_\theta u\|_{H^1(A)} \le c_3 \|p\|_{L_2(A)}. \tag{2.4.10}$$

Since $\Delta_\theta u$ is bounded uniformly for $|\theta| \le 1$, it follows from weak compactness of bounded sets in Banach spaces that for some sequence $\theta^m \to 0$, $\Delta_{\theta^m} u$ converges weakly to an element in $H^1(A)$. For $\phi \in C_0^1(A)$,

$$\lim_{m \to \infty} \int_A (\Delta_{\theta^m} u)\, \phi\, dA = \lim_{m \to \infty} \int_A u(-\Delta_{\theta^m} \phi)\, dA$$

$$= -\int_A u\, \theta_n^{-1} \phi_{\theta_1}\, dA.$$

Consequently, $\Delta_{\theta^m} u$ converges weakly to $\theta_n^{-1} u_{\theta_1}$ in $H^1(A)$; and, by (2.4.10),

$$\|\theta_n^{-1} u_{\theta_1}\|_{H^1(A)} \le c_3 \|p\|_{L_2(A)}.$$

Further, again for $\phi \in C_0^1(A)$,

$$\int_A (u_{x_i})_{\theta_1} \phi \, dA = -\int_A u_{x_i} \phi_{\theta_1} \, dA$$

$$= \int_A u \phi_{\theta_1 x_i} \, dA$$

$$= \int_A u \phi_{x_i \theta_1} \, dA$$

$$= \int_A u_{\theta_1 x_i} \phi \, dA ,$$

i.e., $u_{x_i \theta_1} = u_{\theta_1 x_i}$.

Similarily, $\theta_n^{-1} u_{\theta_k} \in H^1(A) \, (k < n)$, and

$$\| \theta_n^{-1} u_{\theta_k} \|_{H^1(A)} \leq c_3 \| p \|_{L_2(A)} \, (k < n) . \tag{2.4.11}$$

Let A' denote the set of points in A contained in the n-1 parameter family of rays:

$$x_i = y_i \xi \qquad (\xi \geq 0)$$

$$| y_i - 1 | < \frac{1}{2}$$

A' lies in the interior of the first quadrant and is the intersection of A with a cone having vertex at the origin. In A' the Jacobian associated with the coordinate transformation

$$(x_1, \ldots, x_n) \rightarrow (\theta_1, \ldots, \theta_n)$$

in nonvanishing By straight-forward calculations (Lemma 2.4.3), we can show that

$$\text{(i)} \qquad \theta_n^{-1} \frac{\partial x_i}{\partial \theta_n} < c_4 \qquad (k < n),$$

$$\text{(2.4.12)} \quad \text{(ii)} \qquad \frac{\partial x_i}{\partial \theta_n} < c_4$$

$$\text{(iii)} \qquad \theta_n^{-1} \frac{\partial \theta_k}{\partial x_i} < c_5 \qquad (k < n).$$

We know that $\theta_n^{-1} u_{\theta_k}$ $(k < n)$ and u_{θ_n} $(= u_{x_i} \frac{\partial \theta_n}{\partial x_i})$ are in $L_2(A')$. We shall show that each second partial derivative (with respect to the θ_i) of u, mulitplied by an appropriate factor of θ_n, is in $L_2(A')$.

Firstly,

$$\| \theta_n^{-2} u_{\theta_k \theta_m} \|_{L_2(A')} = \| \theta_n^{-1} u_{\theta_k x_i} (\theta_n^{-1} \frac{\partial x_i}{\partial \theta_m} \|_{L_2(A')} \qquad (k, m < n)$$

$$\leq c_6 \| p \|_{L_2(A')}$$

and

$$\| \theta_n^{-1} u_{\theta_k \theta_n} \|_{L_2(A')} = \| \theta_n^{-1} u_{\theta_k x_i} \frac{\partial x_i}{\partial \theta_n} \|_{L_2(A')} \qquad (k < n)$$

$$\leq c_6 \| p \|_{L_2(A')}$$

by (2.4.11) and (2.4.12, (i), (ii)).

It remains to show that $u_{\theta_n \theta_n} \in L_2(A')$. By the chain rule,

$$a_{ij} u_{x_i} v_{x_j} = a_{ij} \frac{\partial \theta_k}{\partial x_i} \frac{\partial \theta_m}{\partial x_j} u_{\theta_k} v_{\theta_m}$$

$$= a'_{km} u_{\theta_k} v_{\theta_m} \qquad (k, m = 1, \ldots, n).$$

By (2.4.12 (iii)), a'_{km} (k, m not both n) is dominated by θ_n^{-1} or θ_n^{-2}, depending on whether one or neither of the indices k, m is equal to n.

In either case, $(a'_{km}\, u_{\theta_k})_{\theta_m} \in L_2(A')$ (no sum on k, m; k, m not both equal to n) by the preceding remarks.

We may write (2.4.3), with $\phi \in C_0^1(A')$ in place of v, as follows:

$$\int_A a'_{nn}\, u_{\theta_n}\, \phi_{\theta_n}\, dA = \int_A \{(p-u)\,\phi - a'_{km}\, u_{\theta_k}\, \phi_{\theta_m}\}\, dA$$

$$= \int_A \{(p-u) + (a'_{km}\, u_{\theta_k})_{\theta_m}\}\, \phi\, dA$$

(n is fixed and we sum on k, m with k, m not both equal to n).

It follows that $(a'_{nn}\, u_{\theta_n})_{\theta_n}$ exists and

$$\|(a'_{nn}\, u_{\theta_n})_{\theta_n}\|_{L_2(A')} = \|p - u + (a'_{km}\, u_{\theta_k})_{\theta_m}\|_{L_2(A')}$$

$$\leq c_7 \|p\|_{L_2(A')}$$

(k, m not both equal to n).

An easy calculation (Lemma 2.4.4) shows that $u_{\theta_n \theta_n} \in L_2(A')$ and

$$\|u_{\theta_n \theta_n}\|_{L_2(A')} = \|\{(a'_{nn}\, u_{\theta_n})_{\theta_n} - (a'_{nn})_{\theta_n}\, u_{\theta_n}\}(a'_{nn})^{-1}\|_{L_2(A')}$$

$$\leq c_8 \|p\|_{L_2(A')}.$$

Finally, $u_{x_i x_j} \in L_2(A')$ and

$$(2.4.13) \quad \|u_{x_i x_j}\|_{L_2(A')} = \left\|u_{\theta_k \theta_m} \frac{\partial \theta_k}{\partial x_i} \frac{\partial \theta_m}{\partial x_j} + u_{\theta_k} \frac{\partial^2 \theta_k}{\partial x_i \partial x_j}\right\|_{L_2(A')}$$

$$\leq c_9 \|p\|_{L_2(A')} \qquad (k, m = 1, 2, \ldots, n).$$

We next show that $u \in H^2(A)$. Assume the boundary of ∂A is divided into subregions ∂A_k, and let A_k denote the intersection of A and the cone generated by rays from the origin through each point of ∂A_k. We assume that after a suitable rotation of coordinate axis, each A_k can be made to lie in the interior of the first quadrant. If we apply the above argument to A_k, we find that all second partials of u with respect to the x_j's are in $L_2(A_k)$. Since $A = \bigcup_k A_k$, we see that all second partials of u are in $L_2(A)$, and the second partials satisfy an inequality of the form (2.4.13). Thus

$$\|u\|_{H^2(A)} < \gamma \|p\|_{L_2(A)}, \qquad \text{qed}$$

Lemma 2.4.1.

$$(\tau_\theta u)_{x_i} = \tau_\theta u_{x_i} + h_{ij}(\theta, x)\, \tau_\theta u_{x_i},$$

and

$$\lim_{\theta \to 0} h_{ij}\, \theta^{-1} < c_2.$$

Proof. Using (2.4.5), we find that

$$\begin{aligned}
(\tau_\theta^{\#} u(x))_{x_1} &= (u(z))_{x_1} \\
&= u_{z_1}(z) \frac{\partial z_1}{\partial x_1} + u_{z_2} \frac{\partial z_2}{\partial x_1} \\
&= (\tau_\theta u_{x_1})(\cos \frac{\theta}{\theta_n} + x_1^2 \theta\, \theta_n^{-3} \sin \frac{\theta}{\theta_n} - x_2 x_1\, \theta\, \theta_n^{-3} \cos \frac{\theta}{\theta_n}) \\
&\quad + (\tau_\theta u_{x_2})(-\sin \frac{\theta}{\theta_n} + x_1^2\, \theta\, \theta_n^{-3} \cos \frac{\theta}{\theta_n} + x_2 x_1 \theta \theta_n^{-3} \sin \frac{\theta}{\theta_n});
\end{aligned}$$

the last equation follows because u_{x_1} and u_{z_1} both denote differentiation

with respect to the first argument. Similarily, $u_{x_2} = u_{z_2}$. Let

$$h_{11} = (\cos \frac{\theta}{\theta_n} - 1) + x_1^2 \theta \theta_n^{-3} \sin \frac{\theta}{\theta_n} - x_2 x_1 \theta \theta_n^{-3} \cos \frac{\theta}{\theta_n} ,$$

$$h_{12} = -\sin \frac{\theta}{\theta_n} + x_1^2 \theta \theta_n^{-3} \cos \frac{\theta}{\theta_n} + x_2 x_1 \theta \theta_n^{-3} \sin \frac{\theta}{\theta_n} .$$

Using similar calculations, we define

$$h_{21} = x_1 x_2 \theta \theta_n^{-3} \sin \frac{\theta}{\theta_n} + \sin \frac{\theta}{\theta_n} - x_2^2 \theta_n^{-3} \theta \cos \frac{\theta}{\theta_n} ,$$

$$h_{22} = (\cos \frac{\theta}{\theta_n} - 1) + x_2^2 \theta \theta_n^{-3} \sin \frac{\theta}{\theta_n} + x_1 x_2 \theta \theta_n^{-3} \cos \frac{\theta}{\theta_n} ,$$

$$h_{i1} = x_1 x_i \theta \theta_n^{-3} \sin \frac{\theta}{\theta_n} - x_2 x_i \theta \theta_n^{-3} \cos \frac{\theta}{\theta_n} ,$$

$$h_{i2} = x_1 x_i \theta \theta_n^{-3} \cos \frac{\theta}{\theta_n} + x_2 x_i \theta \theta_n^{-3} \sin \frac{\theta}{\theta_n} ,$$

and we define all other h_{ij}'s to be zero. The lemma follows by inspection.

Lemma 2.4.2

$$\int_A \{p \Delta_\theta v - (\Delta_\theta a_{ij})(\tau_\theta u)_{x_i} v_{x_j} - \frac{1}{\theta} F(\theta, u, v)\} dA \leq c_3 \|p\|_{L_2(A)} \|v\|_{H^1(A)}.$$

Proof. Recall $z(\theta, x) = \tau_\theta^{\#} x$. Then

$$\{\Sigma_{i=1}^n (z_i(\theta, x) - x_i)^2\}^{1/2} \leq \theta_n \frac{\theta}{\theta_n}$$

$$= \theta .$$

Since the coefficients of Λ_o have bounded first derivatives, they lie in $\text{Lip}^1(A)$, and

$$\Delta_\theta a_{ij} = \theta^{-1} (\tau_\theta a_{ij} - a_{ij})$$

is uniformly bounded in A. It easily follows that

$$(2.4.14) \qquad \int_A \Delta_\theta a_{ij}(\tau_\theta u_{x_i}) v_{x_j}\, dA \le c_{10} \|v\|_{H^1(A)} \|u\|_{H^1(A)}$$

$$\le c_{10} \|v\|_{H^1(A)} \|p\|_{L_2(A)} ;$$

(2.4.4) was used to obtain the last inequality.

Let ν be a parameter, $0 \le \nu \le 1$, and define the curve $z(\nu)$ with components $(\tau^{\#}_{\nu\theta} x_i)$, i.e., whose components are

$$z_1(\nu) = x_1 \cos \frac{\nu\theta}{\theta_n} + x_2 \sin \frac{\nu\theta}{\theta_n} ,$$

$$z_2(\nu) = -x_1 \cos \frac{\nu\theta}{\theta_n} + x_2 \cos \frac{\nu\theta}{\theta_n},$$

$$z_k(\nu) = x_k \qquad , \qquad (k \ge 3) .$$

Denote the tangent vector to the curve $z(\nu)$ by $\dot{\underset{\sim}{z}}(\nu) = \left(\frac{dz_i(\nu)}{d\nu}\right)$.

An easy calculation shows that

$$\dot{z}_1(\nu) = \theta\theta_n^{-1} z_2(\nu)$$

$$\dot{z}_2(\nu) = -\theta\theta_n^{-1} z_1(\nu)$$

$$\dot{z}_k(\nu) = 0 \qquad (k > 2) .$$

Let us denote the Euclidean magnitude of a vector $\underset{\sim}{w} = (w_i)$ by $|\underset{\sim}{w}| = (\Sigma_{i=1}^n w_i^2)^{1/2}$; then we see from above that

$$(2.4.15) \qquad |\dot{\underset{\sim}{z}}(\nu)| \le \theta$$

We can estimate $\|\Delta_\theta v\|_{L_2(A)}$ as follows:

$$\int_A (\Delta_\theta v)^2 dA = \theta^{-2} \int_A \left(\int_0^1 \frac{d}{d\nu} \tau_{\nu\theta} v \, d\nu \right)^2 dA$$

$$= \theta^{-2} \int_A \left(\int_0^1 \tau_{\nu\theta} \underset{\sim}{\nabla} v \cdot \dot{\underset{\sim}{z}}(\nu) \, d\nu \right)^2 dA$$

$$\leq \theta^{-2} \int_A \left(\int_0^1 |\tau_{\nu\theta} \underset{\sim}{\nabla} v| \, |\dot{\underset{\sim}{z}}(\nu)| \, d\nu \right)^2 dA$$

$$\leq \theta^{-2} \int_A \left(\int_0^1 |\tau_{\nu\theta} \underset{\sim}{\nabla} v| \, \theta \, d\nu \right)^2 dA$$

$$\leq \int_A \int_0^1 |\tau_{\nu\theta} \underset{\sim}{\nabla} v|^2 \, d\nu \, dA$$

$$= \int_0^1 \int_A |\tau_{\nu\theta} \underset{\sim}{\nabla} v|^2 \, dA \, d\nu$$

$$= \int_0^1 \int_A |\underset{\sim}{\nabla} v|^2 \, dA \, d\nu$$

$$= \|\underset{\sim}{\nabla} v\|^2_{L_2(A)}$$

$$\leq \|v\|^2_{H^1(A)} .$$

The C. B. S. (Cauchy-Bunyakovski-Schwarz) inequality, (2.4.15), Fubini's theorem, and the formula

$$\int_A |\tau_{\nu\theta} \underset{\sim}{\nabla} w|^2 = \int_A |\underset{\sim}{\nabla} w|^2 dA \qquad (w \in H^1(A))$$

are used to obtain the above estimates. It follows that

$$\text{(2.4.16)} \qquad \int_A p \, \angle_\theta v \, dA \leq \|p\|_{L_2(A)} \|\Delta_\theta v\|_{L_2(A)}$$

$$\leq \|p\|_{L_2(A)} \|v\|_{H^1(A)} .$$

From an inspection of the expression for $F(\theta, u, v)$ (2.4.8), we see that

$$(2.4.17) \qquad \left\| \frac{1}{\theta} F(\theta, u, v) \right\|_{L_2(A)} \leq c_{11} \| u \|_{H^1(A)} \| v \|_{H^1(A)}$$

$$\leq c_{11} \| p \|_{L_{2(A)}} \| v \|_{H^1(A)} ,$$

again by (2.4.4). The lemma follows immediately from (2.4.14), (2.4.16) and (2.4.17).

Lemma 2.4.3.

(i) $\theta_n^{-1} \frac{\partial x_i}{\partial \theta_k} < c_4 ,$

(ii) $\frac{\partial x_i}{\partial \theta_n} < c_4 ,$

(iii) $\frac{\partial \theta_k}{\partial x_i} < c_5 \theta_n^{-1} \qquad (k < n) .$

From the definition of θ_i ,

$$x_{k+1} = x_k \tan \theta_k \qquad (k < n) ,$$

and

$$\theta_n^2 = x_n^2 + x_{n-1}^2 + , \ldots , + x_1^2$$

$$(2.4.18) \qquad = x_n^2 + x_n^2 \cot^2 \theta_{n-1} + x_n^2 \cot^2 \theta_{n-1} \cot^2 \theta_{n-2} +$$

$$\ldots x_n^2 \cot^2 \theta_{n-1} \cot^2 \theta_{n-2} \ldots \cot^2 \theta_1$$

$$= x_n^2 \, w^2 (\theta_k) . \qquad (k < n)$$

Recall that A' is the intersection of A with a cone lying in the interior of the first quadrant. This implies that $w(\theta_k)$ is bounded away from zero (since $\cot \theta_k$, $k < n$, is positive) and $\csc \theta_k$ is bounded from above (since A' is bounded away from the hyperplanes $x_k = 0$). Solving for the x_i's we find from (2.4.18) that

$$x_n = \theta_n w(\theta_k)^{-1} \qquad (k < n)$$

$$x_{n-1} = \theta_n w(\theta_k)^{-1} \cot \theta_{n-1}$$

. . . . . .

$$x_1 = \theta_n w(\theta_k)^{-1} \cot \theta_{n-1} \ldots \cot \theta_1 .$$

Parts (i) and (ii) of the lemma follow immediately.

Part (iii) is obtained from the formula

$$\frac{\partial \theta_k}{\partial x_i} = \frac{\partial}{\partial x_i} \operatorname{Arctan} \frac{x_{k+1}}{x_k}$$

$$= \begin{cases} x_{k+1} (x_k^2 + x_{k+1}^2)^{-1} & i = k \\ -x_k (x_k^2 + x_{k+1}^2)^{-1} & i = k+1 \\ 0 & i \neq k, k+1, n . \end{cases}$$

It then follows that

$$\left| \frac{\partial \theta_k}{\partial x_{k+1}} \right| = \left| x_k (x_k^2 + x_{k+1}^2)^{-1/2} \right| \left| (x_k^2 + x_{k+1})^{-1/2} \theta_n \right| \theta_n^{-1}$$

$$\leq c_5 \theta_n^{-1}$$

and a similar estimate holds for $\frac{\partial \theta_k}{\partial x_k}$.

Lemma 2.4.4 .

$$-u_{\theta\theta} = \{(a'_{nn} u_{\theta_n})_{\theta_n} - (a'_{nn})_{\theta_n} u_{\theta_n}\} (a'_{nn})^{-1}$$

Proof. First, since $\frac{\partial \theta_n}{\partial x_i} = x_i \theta_n^{-1}$, $\nabla\theta_n \cdot \nabla\theta_n = 1$. By the strict ellipticity of Λ_0 , we know that

$$a_{ij} \frac{\partial \theta_n}{\partial x_i} \frac{\partial \theta_n}{\partial x_j} = a'_{nn}$$

is bounded away from zero. For $\phi \in C_0^1(A)$,

$$\int_A \{(a'_{nn} u_{\theta_n})_{\theta_n} - (a'_{nn})_{\theta_n} u_{\theta_n}\} a'^{-1}_{nn} \phi \, dA$$

$$= \int_A \{-a'_{nn} u_{\theta_n} (a'^{-1}_{nn} \phi)_{\theta_n} - (a'_{nn})_{\theta_n} u_{\theta_n} a'^{-1}_{nn} \phi\} dA$$

$$= \int_A \{-a'_{nn} u_{\theta_n} (a'_{nn})^{-1} \phi_{\theta_n} + a'_{nn} u_{\theta_n} (a'_{nn})_{\theta_n} a'^{-2}_{nn} \phi$$

$$- (a'_{nn})_{\theta_n} u_{\theta_n} a'^{-1}_{nn} \phi\} dA$$

$$= \int_A u_{\theta_n} \phi_{\theta_n} \, dA$$

and, by definition of generalized derivatives, $u_{\theta_n \theta_n}$ exists and is equal to

$$-[(a_{nn} u_{\theta_n})_{\theta_n} - (a'_{nn})_{\theta_n} u_{\theta_n}] a'^{-1}_{nn} \in L_2(A) .$$

We see from Corollary 1 above that $\Lambda_t(t)$ considered as an operator on $H^2(A)$, has an inverse $\Lambda_0^{-1}(t)$, mapping $H^2(A)$ into $L_2(A)$.

It will be convenient to introduce the notation

$$\Delta \Lambda_0(t) = \Lambda_0(t+\Delta t) - \Lambda_0(t)$$

$$\Delta \Lambda_0^{-1}(t) = \Lambda_0^{-1}(t+\Delta t) - \Lambda_0(t)$$

$$\Delta w(t) = w(t+\Delta t) - w(t)$$

where $w(t)$ is a vector-valued function.

Lemma 2.4.5. (Lady.zhenskaya). For $v \in L_2(A)$,

$$\|\Delta \Lambda_0^{-1}(t) v\|_{H^2(A)} \leq c_{12} \Delta t \|v\|_{L_2(A)}. \tag{2.4.20}$$

Further, if $\frac{dp(t)}{dt}$ exists, and if $u(t) = \Lambda_0^{-1}(t) p(t)$, then $\frac{du(t)}{dt}$ exists and

$$\frac{du}{dt} = -\Lambda_0(t)\left({}_0(t) + \Lambda_0^{-1}(t) p(t) + \Lambda_0^{-1}(t) \frac{dp}{dt}\right).$$

Proof. It follows from the boundedness of second derivatives of a_{ij} that, if $w \in H^2(A)$,

$$\|\Delta \Lambda_0 w\|_{L_2(A)} = \|(\Delta a_{ij} w_{x_i})_{x_j}\|_{L_2(A)} \tag{2.4.21}$$

$$\leq c_{14} \Delta t \|w\|_{H^2(A)}.$$

Also, denoting the identity map on $L_2(A)$ by I,

$$\Delta \Lambda_0^{-1}(t) = \Lambda_0^{-1}(t+\Delta t) - \Lambda_0^{-1}(t) \tag{2.4.22}$$

$$= \Lambda_0^{-1}(t)(\Lambda_0(t)\Lambda_0^{-1}(t+\Delta t) - I)$$

$$= \Lambda_0^{-1}(t)(-\Delta \Lambda_0(t))\Lambda_0^{-1}(t+\Delta t).$$

Using (2.4.22), (2.4.1), and (2.4.21), we obtain the following estimates for $v \in L_2(A)$:

$$\|\Delta\Lambda_0^{-1}(t)v\|_{H^2(A)} = \|-\Lambda_0^{-1}(t)(\Delta\Lambda_0(t))\,\Lambda_0^{-1}(t+\Delta t)v\|_{H^2(A)}$$

$$\leq \gamma\,\|(\Delta\Lambda_0(t))\Lambda_0^{-1}(t+\Delta t)v\|_{L_2(A)}$$

$$\leq \gamma c_{14}\Delta t\gamma\,\|v\|_{L_2(A)}.$$

Inequality (2.4.20) is now proved.

Using (2.4.22), we obtain the formulas

$$\Delta u = \Lambda_0^{-1}(t+\Delta t)p(t+\Delta t) - \Lambda_0^{-1}(t)p(t)$$

$$= \Delta\Lambda_0^{-1}p(t+\Delta t) + \Lambda_0^{-1}(t)\Delta p(t)$$

$$= \Lambda_0^{-1}(t)(-\Delta\Lambda_0(t))\Lambda_0^{-1}(t+\Delta t)p(t+\Delta t) + \Lambda_0^{-1}(t)\Delta p(t)$$

Add

$$\Lambda_0^{-1}(t+\Delta t)\Delta\Lambda(t)\,\Lambda_0^{-1}(t+\Delta t)p(t+\Delta t) - \Lambda_0^{-1}(t)\Delta p(t)$$

to both sides of the last relation and divide by Δt; the result is

$$\frac{\Delta u}{\Delta t} + \Lambda_0^{-1}(t+\Delta t)\frac{\Delta\Lambda_0(t)}{\Delta t}\Lambda_0^{-1}(t+\Delta t)p(t+\Delta t) - \Lambda_0^{-1}(t)\frac{\Delta p(t)}{\Delta t}$$

$$= \Delta\Lambda_0^{-1}\frac{\Delta\Lambda_0}{\Delta t}\Lambda_0^{-1}(t+\Delta t)p(t+\Delta t).$$

By (2.4.20) and (2.4.21), $\Delta\Lambda_0^{-1}$ tends to zero as $\Delta t \to 0$ and $\frac{\Delta\Lambda_0}{\Delta t}$ is bounded. Hence the right side of the above relation vanishes in the limit as $\Delta t \to 0$. It follows that

$$\frac{du}{dt} = -\Lambda_0^{-1}(t)\,(\Lambda_0(t))_t\,\Lambda_0^{-1}(t)\,p(t) + \Lambda_0^{-1}(t)\,\frac{dp}{dt} \quad .$$

CHAPTER III

EXISTENCE AND UNIQUENESS OF A WEAK SOLUTION TO THE MIXED PROBLEM

The goal of this chapter is to establish the existence and uniqueness of a solution to the mixed problem. In Section 3.1 an energy inequality is proved by integrating the expression $b_r u_{x_r} \wedge_2 u$ over a "light cone" and applying the divergence theorem. In section 3.2 we use the methods of Ladyzhenskaya to prove that a weak solution to (1.1.1) may be approximated by functions in $H^2(Q(0,T))$. Application of the energy inequality to a sequence of approximating functions yields the existence of a weak solution. It is shown that the weak solution satisfies the energy inequality, and the energy inequality is used to prove the uniqueness of a solution.

3.1 An Energy Inequality. For convenience, we adopt the notation

$$e(u) = u^2 + u_t^2 + a_{ij} u_{x_i} u_{x_j}$$

Lemma 3.1.1. If q is a quadratic polynomial, with bounded coefficients, in u and its first derivatives, then $|q| \leq c_1 e(u)$ where c_1 is a constant depending on the bound c_2 of the coefficients in q.

Proof. Let $w u_{x_r} u_{x_s}$ be any term of q. Then

$$w u_{x_s} u_{x_r} \leq 2^{-1} c_2 (u_{x_s}^2 + u_{x_r}^2)$$

$$\leq 2^{-1} c_2 (u^2 + u_t^2 + u_{x_i} u_{x_i})$$

$$\leq 2^{-1} c_2 (u^2 + u_t^2 + a_{ij} u_{x_i} u_{x_j}).$$

Strict ellipticity of Λ_0 is used to obtain the last inequality. The lemma follows immediately.

Let $C(\delta; T, x')$, with $\delta > 0$, be the cone

$$t - T + (\Sigma_1^n (x_i - x_i')^2)^{1/2} c_M^{-1} < \delta.$$

Denote $Q(0, T) \cap C(\delta; T, x')$ and $A(t) \cap C(\delta; T, x')$ by $Q'(0, T)$ and $A'(t)$, respectively. For each $\mu > 0$, let G_μ be the subset of R^{n+1} whose distance to $B(0, T)$ is no greater than μ. We assume μ, δ, x' are choosen so that $G_\mu \subset C(\delta; T, x')$. Since G_μ is a compact set containing $B(0, T)$ in its interior, we may assume that on $Q(0, T)$, the vector $\underset{\sim}{b}$ has the properties given in Lemma 2.1.1. In particular, $b_i = 0$ on the lateral surface of $C(\delta; T, x')$.

Theorem 3.1.2. If $u \in H^2(Q(0, T))$, and if

$$\Lambda_2 u = p$$

then

$$(3.1.1) \qquad \int_{A'(t)} e(u)\, dA \leq c_3 \int_{A'(0)} e(u)\, dA + c_4 \int_{Q'(0, T)} p^2\, dQ.$$

The constants c_3, c_4 depend on μ, δ, T, x', and the maximum of $\{b_r, (b_r)_{x_s}, a_{ij}, (a_{ij})_{x_s}, a_i, a\}$ in $Q(0, T)$.

Corollary (The Energy Inequality). If $p \in L_2(Q(0, T))$ and if

$$\int_{A(0)} e(u)\, dA < \infty ,$$

then

$$(3.1.2) \qquad \int_{A(t)} e(u)\, dA \leq c_3 \int_{A(0)} e(u)\, dA + c_4 \int_{Q(0,T)} p^2\, dQ .$$

Remark. Note that $C(\delta; T, x')$ is required to contain $B(0, T)$ in its interior. For the case in which $A(t)$ does not depend on time, this theorem can be proved without this requirement; the theorem is then known as the domain of dependence theorem.

Proof. The proof of this theorem can be sketched as follows. We first write $b_r u_{x_r} \Lambda_2 u$ as a divergence expression plus a quadratic expression in u and its first derivatives. (The terms of $b_r u_{x_r}$ are the multipliers of the (a, b, c)-method.) Then the divergence theorem is applied (in the form of Lemmas 2.3.2 and 2.3.3), and the surface integrals estimated. The theorem will then follow by a standard argument.

For $u \in H^2(Q(0, T))$, the following identities hold:

$$(3.1.3) \qquad (b_r u_{x_r}) u_{tt} = b_r ((u_{x_r} u_t)_t - u_{tx_r} u_t)$$

$$= (b_r u_{x_r} u_t)_t - \frac{1}{2} (b_r u_t^2)_{x_r} + q_1 ,$$

where

$$q_1 = -(b_r)_t u_{x_r} u_t + \frac{1}{2} (b_r)_{x_r} u_t^2 ;$$

and

$$(3.1.4)\qquad b_r u_{x_r} \wedge_0 u = b_r u_{x_r} ((a_{ij} u_{x_i})_{x_j} - u)$$

$$= b_r ((a_{ij} u_{x_i} u_{x_r})_{x_j} - a_{ij} u_{x_i} u_{x_r x_j}) - b_r u_{x_r} u$$

$$= (a_{ij} u_{x_i} u_{x_r} b_r)_{x_j} - \frac{1}{2}(b_r a_{ij} u_{x_i} u_{x_j})_{x_r} - \frac{1}{2}(b_r u^2)_{x_r} + q_2,$$

where

$$q_2 = -(b_r)_{x_j} u_{x_r} a_{ij} u_{x_i} + \frac{1}{2} b_r (a_{ij})_{x_r} u_{x_i} u_{x_j}$$

$$+ \frac{1}{2}(b_r)_{x_r} a_{ij} u_{x_i} u_{x_j} + \frac{1}{2}(b_r)_{x_r} u^2 .$$

If (3.1.4) is subtracted from (3.1.3), and if we set $q_3 = q_1 - q_2$, then

$$(3.1.5)\qquad b_r u_{x_r} (u_{tt} - \wedge_0 u) = (b_r u_{x_r} u_t)_t - (a_{ij} u_{x_i} u_{x_r} b_r)_{x_j}$$

$$- \frac{1}{2} b_r (u_t^2 - a_{ij} u_{x_i} u_{x_j} - u^2)_{x_r} + q_3 .$$

By hypothesis,

$$u_{tt} - \wedge_0 u = p - \wedge_1 u .$$

If we multiply the above equation by $b_r u_{x_r}$, make use of (3.1.5), and integrate over $Q'(0, t)$, the result is

$$(3.1.6)\qquad \int_{Q'(0,t)} \{[(b_r u_{x_r} u_t)_t - (a_{ij} u_{x_i} b_r u_{x_r})_{x_j}] -$$

$$- \frac{1}{2}[b_r (u_t^2 - a_{ij} u_{x_i} u_{x_j} - u^2)]_{x_r}\} dQ$$

$$= \int_{Q'(0,t)} [b_r u_{x_r} (p - a_i u_{x_i} - (a-1) u) - q_3] dQ$$

$$\leq \int_{Q'(0,t)} (\frac{1}{2} p^2 + q_4) dQ ,$$

where

$$q_4 = \frac{1}{2}(b_r u_{x_r})^2 + (a_i u_{x_i} - (a-1)u)^2 - q_3 .$$

Denote the lateral surface of the cone $C(\delta; T, x')$ between the hyperplanes $t = 0$ and $t = \tau$ by $\partial C(\tau)$ and the unit normal to $\partial C(\tau)$ by $\underset{\sim}{n} = (n_s)$. If we apply Lemma 2.3.2, to the last square bracket on the left side of (3.1.6) and Lemma 2.3.3, with $v_0 = u_t$, $v_j = -a_{ij} u_{x_i}$, to the first square bracket, we obtain the inequality

$$\begin{aligned}(3.1.7)\quad & \int_{A'(t)} (b_r u_{x_r} u_t - \frac{1}{2} b_0 (u_t^2 - a_{ij} u_{x_i} u_{x_j} - u^2))\, dA \\ & - \int_{A'(0)} \{b_r u_{x_r} u_t - \frac{1}{2} b_0 (u_t^2 - a_{ij} u_{x_i} u_{x_j} - u^2)\}\, dA \\ & + \int_{\partial C(t)} \{b_r u_{x_r} (u_t n_0 - a_{ij} u_{x_i} n_j) - \frac{1}{2} b_r (u_t^2 - a_{ij} u_{x_i} u_{x_j} - u^2) n_r\}\, d\partial C \\ & \leq \int_{Q'(0,t)} (\frac{1}{2} p^2 + q_4)\, dQ .\end{aligned}$$

Let I_1 and I_2 denote the integrands of the integrals over $A'(t)$ and $\partial C(t)$, respectively. Recall to mind the constants α and β introduced in (2.1.3). We now prove the inequalities,

$$\begin{aligned}(3.1.8)\quad & I_2 \geq 0 \\ & c_5 e(u) \leq I_1 \leq c_6 e(u),\end{aligned}$$

where

$$c_5 = \frac{\alpha\beta}{2} > 0 \quad \text{and} \quad c_6 = \frac{2-\alpha}{2} > 0 .$$

On $\partial C(t)$, the components of $\underset{\sim}{n}$, the unit exterior normal, are

$$n_0 = (c_M^2 + 1)^{-1/2} ,$$

$$n_i = n_0 (x_i - x_i') (\Sigma_{j=1}^{n} (x_j - x_j)^2)^{-1/2} c_M^{-1} .$$

From the strict ellipticity of Λ_0, it follows that

$$a_{ij} \frac{n_i}{n_0} \frac{n_j}{n_0} \leq c_M^2 \frac{n_i}{n_0} \frac{n_i}{n_0}$$

$$= 1 .$$

Also, recall that $\underset{\sim}{b}$ has been extended in such a way that $b_i = 0$ on $\partial C(t)$ and $b_0 > 0$. Thus I_2 satisfies the following relations on $\partial C(t)$:

$$I_2 = b_0 \left(\frac{1}{2} (u_t^2 + a_{ij} u_{x_i} u_{x_j}) n_0 - a_{ij} u_{x_i} u_t n_j + \frac{1}{2} u^2 n_0 \right)$$

$$= \frac{1}{2} b_0 n_0 \left(u_t^2 + a_{ij} u_{x_i} u_{x_j} - 2u_t a_{ij} u_{x_i} \frac{n_j}{n_0} + u^2 \right)$$

$$\geq \frac{1}{2} b_0 n_0 \left(u_t^2 a_{ij} \frac{n_i}{n_0} \frac{n_j}{n_0} + a_{ij} u_{x_i} u_{x_j} - 2u_t a_{ij} u_{x_i} \frac{n_j}{n_0} + u^2 \right)$$

$$= \frac{1}{2} b_0 n_0 \left(a_{ij} \left(u_t \frac{n_i}{n_0} - u_{x_i} \right) \left(u_t \frac{n_j}{n_0} - u_{x_j} \right) + u^2 \right)$$

$$\geq 0 .$$

We next estimate I_1. Using the CBS inequality, (2.1.5, (iv)), Minkowski's inequality, and strict ellipticity of Λ_0, we conclude that

$$| b_i b_0^{-1} u_{x_i} u_t | \leq | b_0^{-1} (b_j b_j)^{1/2} (u_{x_i} u_{x_i})^{1/2} u_t |$$

$$\leq (1-\alpha) (u_{x_i} u_{x_i})^{1/2} | u_t |$$

$$\leq (1-\alpha) \frac{e(u)}{2}$$

(The time-like character of B is used in the inequality $(b_i b_i)^{1/2} b_0^{-1} \leq (1-\alpha)$).

We then find that

$$I_1 = b_0 [\, e(u) + 2\, b_i\, b_0^{-1}\, u_{x_i}\, u_t \,] 2^{-1}$$

$$\geq b_0 [\, e(u) - 2 (\tfrac{1-\alpha}{2})\, e(u) \,] 2^{-1}$$

$$\geq \frac{\beta \alpha}{2}\, e(u)\,,$$

where the last inequality follows from (2.1.5, (iii)). Thus $c_5\, e(u) \leq I_1$, where $c_5 = \alpha\beta/2$. Similarily,

$$I_1 \leq b_0 [\, e(u) + 2\, \frac{1-\alpha}{2}\, e(u) \,] 2^{-1}$$

$$\leq (2-\alpha)\, b_0\, e(u)\, 2^{-1}\,.$$

Thus $I_1 \leq c_6\, e(u)$, where we may choose $c_6 = \frac{2-\alpha}{2}$ because $b_0 \leq 1$. The proof of inequalities (3.1.8) is complete.

Now, (3.1.7) can be written as

$$\int_{A'(t)} I_1\, dA + \int_{\partial C(t)} I_2\, d\partial C = \int_{A'(0)} I_1\, dA + \int_{Q'(0,\,t)} (\tfrac{1}{2} p^2 + q_4)\, dQ\,.$$

Therefore, by (3.1.8),

$$c_5 \int_{A'(t)} e(u)\, dA \leq c_6 \int_{A'(0)} e(u)\, dA + \int_{Q'(0,\,t)} (\tfrac{1}{2} p^2 + q_4)\, dQ\,;$$

or, by Lemma 3.1.1,

$$(3.1.9) \quad \int_{A'(t)} e(u)\, dA \leq c_7 \int_{A'(0)} e(u)\, dA + c_8 \int_{Q'(0,\,t)} e(u)\, dQ + c_9 \int_{Q'(0,\,t)} p^2\, dQ\,.$$

Set

$$q(t) = \int_{Q'(0,\,t)} e(u)\, dQ\,,$$

$$m(t) = \int_{Q'(0,\,t)} p^2\, dQ\,,$$

and denote differentiation with respect to t by a prime.

Observing that

$$q'(t) = \int_{A'(t)} e(u)\, dA\,,$$

we see from (3.1.9) that

$$q'(t) - c_8\, q(t) \leq c_7\, q'(0) + c_9\, m(t)\,. \tag{3.1.10}$$

But

$$[\,q(t)\exp(-c_8 t)\,]' \equiv \exp(-c_8 t)[\,q'(t) - c_8\, q(t)\,]\,;$$

consequently,

$$[\,q(t)\exp(-c_8 t)\,]' \leq \exp(-c_8 t)[\,c_7\, q'(0) + c_9\, m(t)\,]\,.$$

This inequality implies

$$\int_0^T [q(t)\exp(-c_8 t)]'\, dt \equiv \exp(-c_8 T)\, q(T)$$

$$\leq -\frac{c_7}{c_8}(\exp(-c_8 T) - 1)q'(0) + c_9 \int_0^T \exp(-c_8 t)m(t)dt\,,$$

or

$$q(T) \leq \frac{c_7}{c_8}\,(\exp(-c_8 T) - 1)\, q'(0) + c_9 \exp(c_8 T) \int_0^T \exp(-c_8 t)\, m(t)\, dt\,.$$

If the last inequality is introduced into (3.1.10), the desired inequality (3.1.2) follows immediately, since

$$\int_0^T \exp(-c_8 t)\, m(t)\, dt \leq T m(T) .$$

Proof of Corollary. Since the constants c_3 and c_4 in Theorem (3.1.2) depend upon $(b_r)_{x_i}$ (via the constant in Lemma 3.1.1), they depend upon δ and μ (recall that each b_i has support on $G_\mu \cap Q(0, T)$). These constants depend upon δ only in so far as δ must be choosen relative to μ as to satisfy the hypothesis of Theorem 3.1.2. But if $\delta' \geq \delta$, then, under the hypothesis of the theorem,

$$C(\delta'; T, x') \supset C(\delta; T, x') \supset G_\mu \cap Q(0, T)$$

and since b_i 0 exterior to $C(\delta; T, x')$, the same $\underset{\sim}{b}$ may be used in proving the Theorem relative to $C(\delta'; T, x')$ as was just used relative to $C(\delta; T, x')$. Thus the constants c_3 and c_4 are independent of δ' if $\delta' > \delta$, and the Corollary follows from the limit relation

$$\lim_{\delta \to \infty} c(\delta; T, x') \quad A(T) = \lim_{t \to \infty} A'(T)$$

$$= A(T)$$

(similarly for $A(0)$).

3.2 Existence and Uniqueness. Let A denote the exterior of a sphere in R^n and, for $t_2 > t_1 > 0$, $Q^*(t_1, t_2) = [t_1, t_2] \times A$.

Lemma 3.2.1. (Ladyzhenskaya). If, for some η to be specified below $(0 < \eta < 1)$, $p \in L_2(Q^*(T-\eta, T))$, and if

$$\int_{Q^*(T-\eta, T)} p \wedge_2 u \, dQ = 0 \tag{3.2.1}$$

for each $u \in W_2^2(Q^*(T-\eta, T))$ for which

$$u|_{(T-\eta, T) \times \partial A} = u|_{t=T-\eta} = u_t|_{t=T-\eta} = 0,$$

then $p \equiv 0$.

Proof. Throughout the proof, t will be restricted to $[T-\eta, T]$. Given p, we shall construct a u satisfying the hypothesis and show that the only way (3.2.1) can be satisfied is for p to vanish identically.

On $Q^*(T-\eta, T)$, define p_1, v, and u by the equations

$$p_1(t_1, x) = \int_t^T p(t_1, x) \, dt_1,$$

$$\wedge_0(t) v = -p_1(t), \qquad v = 0 \text{ on } \partial A,$$

$$u(t,x) = \begin{cases} 0 & T-\eta < t < \tau \\ \int_\tau^t \left(\int_\tau^{t_1} v(t_2, x) \, dt_2 \right) dt_1 & \tau \leq t \leq T. \end{cases}$$

On the basis of the results of Section 2.4 and the defining equations (3.2.3), the following relations can be established:

$$\int_A v \Lambda_0 v \, dA = \int_A (a_{ij} v_{x_i} v_{x_j} + v^2) \, dA,$$

$$\int_A v ((a_{ij})_t v_{x_i})_{x_j} dA = \int_A (a_{ij})_t v_{x_i} v_{x_j} \, dA,$$

(3.2.3)

$$u_{tt} = v,$$

$$p = \frac{d}{dt} (\Lambda_0 v),$$

$$p_1(T, x) = v(T, x) = u(\tau, x) = u_t(\tau, x) = 0 .$$

In the following calculations the several relations (3.2.3) and integration by parts are used repeatedly. The quantities $c_{10}, \ldots, c_{16}$ are constants depending on the bounds of the coefficients of Λ_2 and their derivatives; $q_5, \ldots, q_8$ are quadratic expressions in v, v_{x_i}, u, u_{x_r}, $u_{x_r x_s}$, and $u_{t x_r x_i}$ with coefficients depending on $c_{10}, \ldots, c_{16}$.

The terms in (3.2.1) can be transformed by integration by parts in the following manner:

$$\int_{Q^*(\tau, T)} p(\Lambda_0 - \Lambda_1) u \, dQ = \int_A p_1 (\Lambda_0 - \Lambda_1) u \, dA \Big]_\tau^T + \int_{Q^*(\tau, T)} p_1 (\Lambda_0 u - \Lambda_1 u)_t \, dQ$$

$$= - \int_{Q^*(\tau, T)} (\Lambda_0 u_{tt}) (\Lambda_0 u_t + \Lambda_{0t} u - (\Lambda_1 u)_t) \, dQ$$

$$= - \int_{Q^*(\tau, T)} \{ \frac{1}{2} (\Lambda_0 u_t)_t^2 - (\Lambda_{0t} u_t)(\Lambda_0 u_t) +$$

(3.2.4)

$$+ (\Lambda_0 u_{tt})(\Lambda_{0t} u - (\Lambda_1 u)_t) \} \, dQ$$

$$= - \frac{1}{2} \int_A (\Lambda_0 u_t)^2 \Big|_{t=T} dA + \int_{Q^*(\tau, T)} \{ (\Lambda_{0t} u_t)(\Lambda_0 u_t)$$

$$- (\Lambda_0 u_{tt})(\Lambda_{0t} u - (\Lambda_1 u)_t \} \, dQ ;$$

$$\int_{Q^*(\tau, T)} p u_{tt}\, dQ = \int_{Q^*(\tau, T)} (-\Lambda_0 v)_t v\, dQ$$

(3.2.5)

$$= \int_{Q^*(\tau, T)} \{\tfrac{1}{2} (v\Lambda_0 v)_t + \tfrac{1}{2} (v\Lambda_{0t} v)\}\, dQ$$

$$= \tfrac{1}{2} \int_A v\Lambda_0 v\big|_{t=\tau} dA - \tfrac{1}{2} \int_{Q^*(\tau, T)} (a_{ij})_t v_{x_i} v_{x_j}\, dQ .$$

If (3.2.4) and (3.2.5) are substituted into (3.2.1) and the result is rearranged, the equation

$$\tfrac{1}{2} \int_A (v\Lambda_0 v)\, dA\big|_{t=T} + \tfrac{1}{2} \int_A (\Lambda_0 u_t)^2\big|_{t=T} dA$$

(3.2.6)

$$= \int_{Q^*(t, T)} q_5\, dQ - \int_{Q^*(\tau, T)} (\Lambda_0 u_{tt})(\Lambda_0 u_t - (\Lambda_1 u)_t)\, dQ$$

is obtained.

The last integral on the right can be rewritten in the form

$$\int_{Q^*(\tau, T)} (\Lambda_0 u_{tt})(\Lambda_{0t} u - (\Lambda_1 u)_t)\, dQ$$

$$= \int_{Q^*(\tau, T)} \{(\Lambda_0 u_t)(\Lambda_{0t} u - (\Lambda_1 u)_t)\}_t dQ - \int_{Q^*(\tau, T)} \{(\Lambda_0 u_t)(\Lambda_{0t} u_t + \Lambda_{0tt} u - (\Lambda_1 u)_{tt})$$

(3.2.7)

$$+ (\Lambda_{0t} u_t)(\Lambda_{0t} u - \Lambda_1 u)_t)\}\, dQ$$

$$= \int_A (\Lambda_0 u_t)(\Lambda_{0t} u)\big|_{t=T} dA - \int_A \Lambda_0 u_t (\Lambda_1 u)_t\big|_{t=T} dA + \int_{Q^*(\tau, T)} q_6\, dQ .$$

The first integral on the right can be estimated as follows:

$$-\int_A (\Lambda_0 u_t)(\Lambda_{0t} u)\,|_{t=T}\,dA = -\int_A \frac{1}{2}(\Lambda_0 u_t)\Big(2\Lambda_{0t}\int_\tau^T u_z(z,x)\,dz\Big)\Big|_{t=T} dA$$

$$\le \frac{1}{2}\Big(\int_A (\Lambda_0 u_t)^2\,|_{t=T}\,dA\Big)^{1/2}\Big(4\int_A \Big(\Lambda_{0t}\int_\tau^T u_z\,dz\Big)^2 dA\Big)^{1/2}$$

$$\le \frac{1}{8}\int_A (\Lambda_0 u_t)^2\,|_{t=T}\,dA$$

(3.2.8)

$$+2\int_A \Big[c_{10}\int_\tau^T \Big(\sum_{i,j}^n u_{zx_i x_j} + \sum_i^n u_{zx_i}\Big)\,dz\Big]^2 dA$$

$$\le \frac{1}{8}\int_A (\Lambda_0 u_t)^2\,|_{t=T}\,dA$$

$$+2c_{10}^2\int_A (T-\tau)\int_\tau^T \Big(\sum_{i,j} u_{zx_i x_j} + \sum_{i=1}^n u_{zx_i}\Big)^2 dz\,dA$$

$$\le \frac{1}{8}\int_A (\Lambda_0 u_t)^2\,|_{t=T}\,dA + \int_{Q^*(\tau,T)} q_6\,dQ.$$

The second integral on the right of (3.2.7) can similarly be estimated.

$$\int_A (\Lambda_0 u_t)(\Lambda_1 u)_t\,|_{t=T}\,dA = \int_A \Big(\frac{1}{2}\Lambda_0 u_t\Big)\Big(2\int_\tau^T (\Lambda_1 u)_{tt}\,dt\Big)\Big|_{t=T} dA$$

(3.2.9)

$$\le \frac{1}{2}\Big(\int_A (\Lambda_0 u_t)^2\,|_{t=T}\,dA\Big)^{1/2}\Big[4\int_A \Big(\int_\tau^T (\Lambda_1 u)_{tt}\,dt\Big)^2 dA\Big]$$

$$\le \frac{1}{8}\int_A (\Lambda_0 u_t)^2\,|_{t=T}\,dA + \int_{Q^*(\tau,T)} q_7\,dQ\ .$$

Combining (3.2.6), (3.2.7), (3.2.8), and (3.2.9), we obtain the inequality

(3.2.10)

$$\int_A v\Lambda_0 v|_{t=\tau}\,dA + \int_A (\Lambda_0 u_t)\,|_{t=T}\,dA \le \int_{Q^*(\tau,T)} q_8\,dQ\ .$$

By an argument similar to that of Lemma 3.1.1, it can be shown that

$$q_8 \leq c_{11}(u^2 + \Sigma_{r=0}^{n} u_{x_r}^2 + \Sigma_{i=1}^{n} u_{tx_i}^2 + \Sigma_{i,j=1}^{n} (u_{x_i x_j}^2 + u_{tx_i x_j}^2) + v^2 + a_{ij} v_{x_i} v_{x_j}) , \tag{3.2.11}$$

where $v = u_{tt}$.

The $L_2(Q^*(\tau, T))$ norms of u and all its derivatives appearing in (3.2.12) will now be shown to be dominated by the $L_2(Q^*(\tau, T))$ norm of $\Lambda_0 u_t$. Firstly, Corollary I of Theorem 2.4.1 is used in the following estimate for $u_{tx_i x_j}$:

$$\| u_{tx_i x_j} \|^2_{L_2(Q^*(\tau, T))} = \int_\tau^T \int_A u_{tx_i x_j}^2 \, dA \, dt$$

$$\leq \int_\tau^T \| u_t \|^2_{H^2(A)} \, dt$$

$$\leq \gamma^2 \int_\tau^T \| \Lambda_0(t) u_t(t) \|^2_{L_2(A)} \, dt$$

$$\leq \gamma^2 \| \Lambda_0 u_t \|^2_{L_2(Q(\tau, T))} .$$

Similarily,

$$\int_{Q^*(\tau, T)} u^2 dQ = \int_{Q^*(\tau, T)} (\int_\tau^t u_t(t_1, x) \, dt_1)^2 \, dQ$$

$$\leq \int_{Q^*(\tau, T)} (T-\tau) \int_\tau^t u_t^2(t_1, x) \, dt_1 \, dQ$$

$$\leq \eta \int_{Q^*(\tau, T)} (\int_\tau^T u_t^2(t_1, x) \, dt_1) \, dQ$$

$$\leq \eta^2 \| u_t^2 \|_{L_2(Q^*(\tau, T))}$$

$$\leq \eta^2 \gamma^2 \| \Lambda_0 u_t \|^2_{L_2(Q^*(\tau, T))} .$$

The last inequality is obtained by repeating the steps in estimating $u_{tx_i x_j}$ and recalling that, by hypothesis, $\eta < 1$. By analoguous arguments, it can be shown that $\| u_{tx_i} \|^2_{L_2(Q^*(\tau, T)}$, $\| u_{x_i} \|^2_{L_2(Q^*(\tau, T))}$ and $\| u_{x_i x_j} \|^2_{L_2(Q^*(\tau, T))}$ are also dominated by $\gamma^2 \| \Lambda_0 u_t \|^2_{L_2(Q^*(\tau, T))}$. Making use of these inequalities in (3.2.1), we obtain the following estimate:

$$\int_A (a_{ij} v_{x_i} v_{x_j} + v^2) \Big|_{t=\tau} dA + \int_A (\Lambda_0 u_t)^2 \Big|_{t=T} dA$$

(3.2.12)

$$\leq c_{12} \int_{Q^*(\tau, T)} \{(a_{ij} v_{x_i} v_{x_j} + v^2) + (\Lambda_0 u_t)^2\} dQ$$

Define $w(t) = u_t(t) - u(T)$. Then,

$$\int_{Q^*(\tau, T)} (\Lambda_0 u_t)^2 dQ = \int_\tau^T \| \Lambda_0 (t) u_t (t) \|^2_{L_2(A)} dt$$

$$= \int_\tau^T \| \Lambda_0 (t) (w(\tau) - w(t)) \|^2_{L_2(A)} dt$$

(3.2.13)

$$\leq \int_\tau^T 2 (\| \Lambda_0 (t) w(\tau) \|^2_{L_2(A)} + \| \Lambda_0 (t) w(t) \|^2_{L_2(A)}) dt$$

$$\leq 2 \int_\tau^T (c_{13} \| w(\tau) \|^2_{H^2(A)} + \| \Lambda_0 (t) w(t) \|^2_{L_2(A)}) dt$$

$$\leq 2 \int_\tau^T (\gamma^2 c_{13} \| \Lambda_0 (\tau) w(\tau) \|^2_{L_2(A)}$$

$$+ \| \Lambda_0 (t) w(t) \|^2_{L_2(A)}) dt .$$

The last two inequalities follow from the boundedness of the coefficients of Λ_0 and from Corollary 1 of Theorem 2.4.1. The estimate (3.2.12) can now be written as

$$\int_A \{a_{ij} v_{x_i} v_{x_j} + v^2 + (\Lambda_0 w)^2\}\Big|_{t=\tau} dA$$

$$\leq c_{12} \int_{Q^*(\tau, T)} \{a_{ij} v_{x_i} v_{x_j} + v^2 + 2(\Lambda_0(t) w(t))^2$$

$$+ 2\gamma_0^2 c_{13} (\Lambda_0(\tau) w(\tau))^2\} \, dQ \tag{3.2.14}$$

$$\leq c_{12} \int_{Q^*(\tau, T)} \{a_{ij} v_{x_i} v_{x_j} + v^2 + (\Lambda_0 w)^2\} dQ$$

$$+ c_{14}(T-\tau) \int_A (\Lambda_0 w)^2 \Big|_{t=\tau} dA .$$

But the constant c_{14} and η may be choosen so that $\eta < (2c_{14})^{-1} < 1$. We do this. Thus, $(T-\tau)c_{14} < 2^{-1}$, and (3.2.14) become

$$\int_A (a_{ij} v_{x_i} v_{x_j} + v^2 + (\Lambda_0 w)^2\Big|_{t=\tau} dA$$

$$\leq c_{15} \int_{Q(\tau, T)} (a_{ij} v_{x_i} v_{x_j} + v^2 + (\Lambda_0 w)^2) \, dQ . \tag{3.2.15}$$

Define

$$m(\tau) = \int_{Q^*(\tau, T)} (a_{ij} v_{x_i} v_{x_j} + v^2 + (\Lambda_0 w)^2) \, dQ ,$$

and denote differentiation with respect to τ by a prime. Then (3.2.15) becomes

$$-m'(\tau) \leq c_{15} m(\tau) . \tag{3.2.16}$$

If we multiply (3.2.16) by $\exp(c_{15}\tau)$, we obtain the differential inequality

$$0 \leq (m(\tau)\exp(c_{15}\tau))' .$$

Integrating the above inequality from t to T, and noting that $m'(T) = 0$, we find

$$0 \leq -m'(t)\exp(c_{15}t) .$$

Since $m'(t) \geq 0$, we conclude that $m'(t) = 0$ for all $t \in [T-\eta, T]$. It follows that $v \equiv 0$ on $[T-\eta, T]$, and hence p_1 and p are also zero on $[T-\eta, T]$.

We now extend the above lemma to a region with a time-dependent boundary. Let $Q^*(T-\eta, T)$ and A be as in Lemma 3.2.1. We assume moreover that $B(T-\eta, T)$ is contained in the interior of $Q^*(T-\eta, T)$. Let

$$S = \{u \mid u \in W_2^2(Q(T-\eta, T)), u|_{B(T-\eta, T)} = u|_{t=T-\eta} = u_t|_{t=T-\eta} = 0\},$$

and let N be the image of S under Λ_2.

Lemma 3.2.2. If $p \in L_2(Q(T-\eta, T))$ is orthogonal to N (i.e., if

$$\int_{Q(T-\eta, T)} p\Lambda_2 u \, dQ = 0$$

for each $u \in S$), then $p = 0$.

Proof. By a well-known theorem of functional analysis, this lemma is equivalent to the statement that N is dense in $L_2(Q(T-\eta, T))$.

For $k = 1, 2, \ldots$ let

$$Q^k(T-\eta, T) = \{(t, x) \mid (t, x) \in Q(T-\eta, T) \text{ and } \operatorname{dist}[(t, x), B] > \frac{1}{k}\},$$

and let $L_2^k(Q(T-\eta, T))$ denote the functions in $L_2(Q(T-\eta, T))$ that have support in $Q^k(T-\eta, T)$. It is well-known that $\bigcup_{k=1}^{\infty} L_2^k(Q(T-\eta, T))$ is dense in $L_2(Q(T-\eta, T))$. If we prove N is dense in each $L^k(Q(T-\eta, T))$, then N will be dense in $\bigcup^{\infty} L_2^k(Q(T-\eta, T))$ and hence in $L_2(Q(T-\eta, T))$. The lemma will then be proved.

We prove that N is dense in $L_2^k(Q(T-\eta, T))$ by proving that if

$$\int_{Q(T-\eta, T)} z/\backslash_2 u \, dQ = 0 \tag{3.2.17}$$

for all $u \in S$ and for all $z \in L_2^k(Q(T-\eta, T))$, then $z = 0$.

Let z^* denote the extension of z to $L_2(Q^*(T-\eta, T))$ that is zero on $Q^*(T-\eta, T) - Q^k(T-\eta, T)$. Let h_k be a function in $C^2(Q^*(T-\eta, T))$ that satisfies the conditions

$$h_k(t, x) = 0 \qquad (t, x) \in Q^*(T-\eta, T) - Q(T-\eta, T),$$

$$h_k(t, x) = 1 \qquad (t, x) \in Q^k(T-\eta, T).$$

For all $u^* \in W_2^2(Q^*(T-\eta, T))$ that satisfy the hypothesis of Lemma 3.2.1, the following identity holds

$$\int_{Q^*(T-\eta, T)} z^* /\backslash_2 u^* \, dQ \equiv \int_{Q^*(T-\eta, T)} z^* /\backslash_2 (h_k u^*) \, dQ + \int_{Q^*(T-\eta, T)} z^* /\backslash_2 (1-h_k) u^* \, dQ \tag{3.2.18}$$

The second integral on the right vanishes because the intersection of the supports of z^* and $1 - h_k$ is void. Since h_k has support on $Q(T-\eta, T)$ and since $u^*|_{t=T-\eta} = u_t|_{T-\eta} = 0$, $h^k u^*$ may be considered as an element

in S; the first integral on the right of (3.2.18) vanishes by (3.2.17). In other words,

$$\int_{Q^*(T-\eta,\, T)} z^* \Lambda_2 u^* \, dQ = 0,$$

and, by Lemma 3.2.1, $z^* = 0$. It follows that $z = 0$, and the lemma is proved.

In the next lemma we show that a weak solution to (1.1.1) may be approximated by functions in $H^2(Q(0, T))$. The form of the proof is due to O. Ladyzhenskaya.

We define

$$H^1(A(0)) = \{u \mid u \in W_2^1(A(0)),\ u|_{\partial A(0)} = 0\},$$

$$M = L_2(Q(0, T)) \times H^1(A(0)) \times L_2(A(0)),$$

and the operator Λ_3 mapping $H^2(Q(0, T))$ into M by

$$\Lambda_3 : u \to (\Lambda_2 u,\ u(0),\ u_t(0)).$$

Lemma 3.2.3. The range of Λ_3 is dense in M.

Proof. The lemma is clearly equivalent to the following one: if $(p, f, g) \in M$ is orthogonal to the range of Λ_3, that is, if

$$\int_{Q(0, T)} p \Lambda_2 u \, dQ + \int_{A(0)} (fu + f_{x_i} u_{x_i} + g u_t) \, dA = 0 \tag{3.2.19}$$

for all $u \in H^2(Q(0, T))$, then

$$(p, f, g) = (0, 0, 0)$$

Let η and S be defined as in Lemma 3.3.2. Let $v \in S$ and extend v to $H^2(Q(0, T))$ by defining v to be zero for $0 < t < T-\eta$. With this choice of u in (3.2.19),

$$\int_{Q(T-\eta, T)} p \Lambda_2 v \, dQ = \int_{Q(0, T)} p \Lambda_2 v \, dQ$$

$$= 0 .$$

By Lemma 3.2.2, $p = 0$ in $L_2(Q(T-\eta, T))$. Let t_1 be the greatest lower bound in $[0, T]$ of the set of all $t \in [0, T]$ such that

$$\int_{Q(t_1, T)} p^2 \, dQ = 0 .$$

We have just shown that $t_1 \leq T-\eta$. If $t_1 > 0$, we can similarily show that, for some $\eta > 0$,

$$\int_{Q(t_1-\eta, t_1)} p^2 \, dQ = 0 .$$

This contradicts the definition of t_1.

Therefore $t_1 = 0$, and (3.2.19) now becomes

$$\int_{A(0)} (f u + f_{x_i} u_{x_i} + g u_t) \, dA = 0 . \tag{3.2.20}$$

Let U_ϵ be an n-dimensional sphere in the interior of $A(0)$, and choose t_2 small enough that $[0, t_2] \times U_\epsilon$ lies in the interior of $Q(0, T)$. Let $h(x) \in C^2(A(0))$ have support in U_ϵ, and let $m(t) \in C^2(R^1)$ satisfy the conditions

$$m(t) = 1 \qquad (0 \leq t \leq \frac{1}{2} t_2),$$

$$m(t) = 0 \qquad (t \geq t_2).$$

Set

$$z(t,x) = m(t)\,h(x) \qquad (t,x) \in [0, t_2] \times U_\epsilon,$$

$$z(t,x) = 0 \qquad (t,x) \in [0, t_2] \times U_\epsilon.$$

Substituting $z(t,x)$ into (3.2.20), we obtain the relation

$$\int_{A(0)} (hf + h_{x_i} f_{x_i})\, dA = 0$$

for $h(x)$ in $C^2(U_\epsilon)$. Consequently, $f = 0$ in $W_2^1(U_\epsilon)$. Since U_ϵ can be choosen to lie in the support of f, we conclude that the support of f is empty, and $f = 0$ in $W_2^1(A(0))$.

The relation (3.2.20) now becomes

$$\int_{A(0)} g\, u_t\, dA = 0. \tag{3.2.21}$$

We define

$$z^*(t,x) = t\,z(t,x).$$

Then $z^* \in H^2(Q(0,T))$,

$$z^*(0,x) = 0,$$

and

$$z_t^*(0,x) = h(x).$$

Substituting z^* into (3.2.23) yields the identity

$$\int_A g\, h(x)\, dA = 0$$

for each h in $C^2(A(0))$ that has support in U_ϵ, and as above, it follows that $g = 0$ in $L_2(A(0))$. Hence

$$(p, f, g,) = (0, 0, 0) .$$

We are now ready to prove our main result.

Theorem 3.2.4. There exists a unique weak solution to (1.1.1) with $(p, f, g) \in M$. Further, this solution satisfies the inequality (3.1.1) and the energy inequality (3.1.2).

Given $(p, f, g) \in M$, by Lemma 3.2.3, there exist sequences $\{p^k, f^k, g^k\}$ and $\{u^k\}$, such that

$$\Lambda_2 u^k = p^k ,$$

$$u^k|_{t=0} = f^k, \quad u^k_t|_{t=0} = g^k$$

and such that

$$(p^k, f^k, g^k) \to (p, f, g)$$

in M. It follows from the energy inequality that

$$\int_{A(t)} (u^k - u^m)^2 + (u^k_t - u^m_t)^2 + a_{ij}(u^k_{x_i} - u^m_{x_i})(u^k_{x_j} - u^m_{x_j})\, dA \tag{3.2.22}$$

tends to zero uniformly in t, $0 \le t \le T$. Thus $\{u^k\}$ converges in $H^1(A(t))$

for each t, and the limiting values of u^k in $H^1(A(t))$ satisfy the inequalities (3.1.1) and (3.1.2). If we integrate (3.2.22) from 0 to T, we see that u^k converges in $H^1(Q(0, T))$. Denote the limit element by u.

We show that u is a weak solution to (1.1.1). Suppose $\phi \in C^2(Q(0, T))$, $\phi|_{t=T} = \phi_t|_{t=T} = \phi|_{B(0, T)} = 0$. We apply Lemma 2.3.1 to the vector $(v_r^k)=(\phi u_t^k, \ldots, \phi a_{ij} u_{x_i}^k, \ldots)$ and conclude that

$$\lim_{k\to\infty} \int_{Q(0, T)} (v_s^k)_{x_s} = \lim_{k\to\infty} \int_{Q(0, T)} \{\phi_t u_t^k - a_{ij} u_{x_i}^k \phi_{x_j} + \phi(u_{tt}^k - (a_{ij} u_{x_i}^k)_{x_j})\} dQ$$

$$= \lim_{k\to 0} \int_{A(0)} \phi_t u_t^k(0)\, dA . \tag{3.2.23}$$

Since

$$u_{tt}^k - (a_{ij} u_{x_i}^k)_{x_j} = p^k - a_i u_{x_i}^k - a u^k$$

and

$$u_t^k|_{t=0} \to g(x),$$

(3.2.23) becomes, in the limit as $k \to \infty$,

$$\int_{Q(0, T)} \{\phi_t u_t - a_{ij} u_{x_i} \phi_{x_j} + \phi(p - a_i u_{x_i} - a)\} dQ = \int_{A(0)} \phi_t g\, dA .$$

It follows that u is a weak solution to (1.1.1).

Suppose u^1 and u^2 are distinct weak solutions to (1.1.1), then $u^1 - u^2$ is a weak solution to (1.1.1) with $p = f = g = 0$. By (3.1.2), $u_1 - u_2 = 0$, and therefore $u_1 \equiv u_2$. We conclude that the solution to (1.1.1) is unique.

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