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ASYMPTOTIC APPROXIMATIONS WITH ERROR BOUNDS FOR ZEROS OF AIRY AND CYLINDRICAL FUNCTIONS

by
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ABSTRACT

ASYMPTOTIC APPROXIMATIONS WITH ERROR BOUNDS FOR ZEROS OF AIRY AND CYLINDRICAL FUNCTIONS

by

Herbert W. Hethcote

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The asymptotic behavior of the zeros of any function can be deduced from the asymptotic behavior of the function itself. Theorems are established which show how error bounds for the asymptotic approximation of both real and complex zeros depend on error bounds for the asymptotic approximation of the functions. Four applications are given.

If a_n is the nth negative zero of the Airy function $Ai(z)$, then

$$a_n = -[3\pi(4n - 1)/8]^{2/3} + \sigma_n$$

where $|\sigma_n| \leq 0.0163/(n - 0.263)^{4/3}$ for $n \geq 1$. If $j_{\nu,1}$ is the first positive zero of the Bessel function $J_\nu(x)$, then

$$j_{\nu,1} = \nu - 1.85576\,\nu^{1/3} + \alpha\nu^{-1/3}$$

where

$$0.500 \leq \alpha \leq 1.357 \quad \text{for} \quad \nu \geq 1$$

and

$$0.935 \leq \alpha \leq 1.105 \quad \text{for} \quad \nu \geq 10 .$$

For $n \geq 0.314 - \mathrm{Re}(\nu)/2 + 1.38\,|4\nu^2 - 1|$, there exist large complex zeros $j_{\nu,n}$ of $J_\nu(z)$ with ν complex such that

$$|j_{\nu,n} - (n\pi + \nu\pi/2 - \pi/4)| \leq \frac{0.188|4\nu^2 - 1|}{n - 0.314 + \mathrm{Re}(\nu)/2} .$$

The above asymptotic approximation is the one-term McMahon approximation. An error bound for the two-term McMahon approximation has also been found.

The zeros $\nu_n(\beta)$ of the Hankel function $H_\nu^1(\beta)$ as a function of the order ν are residue poles of the Watson transformation in the problem of scalar diffraction by a sphere. An asymptotic approximation theorem is proved and used to obtain an error bound for the approximation of $H_\nu^1(\beta)$ by the Airy function Ai(z). If numerically estimated values for some constants are used in the above error bound, then it is shown that

$$\nu_1(\beta) = \beta + (\frac{\beta}{2})^{1/3} a_1 e^{-2\pi i/3} + (\frac{2}{\beta})^{1/3} \frac{a_1^2 e^{-4\pi i/3}}{60} + \alpha(\beta)$$

where $|\alpha(\beta)| \leq 1.71/\beta^{2/3}$ for $\beta \geq 20$.

The theorems and methods of this thesis could also be applied to find error bounds for the asymptotic approximations of real or complex zeros of other functions such as Legendre functions, Coulomb wave functions, confluent hypergeometric functions, and orthogonal polynomials.

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INTRODUCTION

In 1886, Stieltjes [32] applied repeated integration by parts to obtain the result

$$\int_0^\infty \frac{e^{-t}dt}{t+x} = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{x^k} + (-1)^n n! \int_0^\infty \frac{e^{-t}\,dt}{(t+x)^{n+1}}$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{x^k} + O(x^{-n-1}) \qquad \text{for} \quad x > 0 \quad .$$

Although the series with $n = \infty$ is clearly divergent, it is an asymptotic power series in Poincaré's sense [29]. In this case,

$$\left| \int_0^\infty \frac{e^{-t}dt}{t+x} - \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{x^k} \right| \leq \left| (-1)^n n! \int_0^\infty \frac{e^{-t}dt}{(t+x)^{n+1}} \right|$$

$$\leq \frac{n!}{x^{n+1}} \int_0^\infty e^{-t}\,dt = \frac{n!}{x^{n+1}} \quad .$$

Thus we have an error bound $n!/x^{n+1}$ on the difference between the integral $\int_0^\infty \frac{e^{-t}dt}{t+x}$ and the partial sum of the asymptotic expansion $\sum_{k=1}^{n} \frac{(-1)^k(k-1)!}{x^k}$. An approximate value of the integral with an error bound can now be computed for any positive value of x.

Error bounds for asymptotic expansions have not often been found, especially if the procedure used for approximating an integral is Watson's lemma, the method of Laplace, the method of stationary phase, steepest descents, or the saddle point method. The problem of obtaining error bounds for asymptotic approximations seems to have been somewhat neglected since Poincaré first used the big O and little o notation with regard to asymptotics.

If two differential equations are similar for large values of the independent variable or a parameter, then it can sometimes be shown that the solution of one equation is an asymptotic approximation of a solution of the other. If this method of comparing differential equations is used, then error bounds have not often been found for the difference between the function and its asymptotic approximation.

The usual computational procedure in using an asymptotic expansion to approximate a function for specific values of the variables is to truncate the asymptotic expansion at its smallest term. However, this procedure can lead to an incorrect approximation, since the difference between the function and this truncated asymptotic expansion is not, in general, related to the magnitude of the smallest term. For an example, see Olver [28, § 2].

For real functions, some error bounds on asymptotic expansions have been found [36, Ch. 7-8]. Often the error in truncating the asymptotic expansion is less than the modulus of the first nonvanishing omitted term, but this must be proved in each individual

case. Recently, Olver [24], [26], [27], [28] has found error bounds for the difference between real or complex solutions of differential equations and the partial sums of their asymptotic expansions.

We contribute to the study of zeros of special functions by establishing a method for obtaining error bounds for asymptotic approximations of real or complex zeros and by applying the method to various asymptotic approximations of real and complex zeros of Airy and cylindrical functions. Before giving a summary of our methods and results, we will review some previous results concerning the asymptotic expansions of zeros.

The zeros of Bessel functions have been studied extensively. In Watson's Theory of Bessel Functions [36, Ch. 15] it is shown that Bessel functions have an infinite number of zeros, that these zeros are real and simple except possibly at zero, that certain zeros interlace, and that the zeros are in certain intervals. Watson also presents some asymptotic expansions of zeros which he says are useful for calculations. In Higher Transcendental Functions, Vol. II [8, § 7.9], results are presented concerning the simplicity of zeros, the interlacing of zeros, and the reality or complexity of zeros. The Handbook of Mathematical Functions [1, § 9.5] gives many asymptotic expansions of zeros of Bessel functions including a new uniform asymptotic expansion. Graphs showing approximate locations of complex zeros are given and asymptotic expansions of zeros of cross products of Bessel functions are presented.

Tricomi [33] pointed out that the asymptotic behavior of the zeros of any function could be deduced from the asymptotic behavior of the function itself, and he applied this idea to functions such as Bessel functions, Legendre polynomials, confluent hypergeometric functions, orthogonal polynomials, and others. Where Tricomi used the asymptotic expansion of a function and Taylor's Theorem in his proof, Gatteschi used known error bounds on the truncated asymptotic expansion and the intermediate value form of the Taylor series remainder to find error bounds on the asymptotic expansions of zeros of the functions. He applied his idea to zeros of Legendre polynomials, spherical and ultraspherical polynomials, and Bessel functions [11], [12], [14].

It is known that for $\nu \geq -1$, there are an infinite number of real positive simple zeros of $J_\nu(z)$. Using a method due to Stokes, McMahon found the asymptotic expansion for the nth positive zero $j_{\nu,n}$ of $J_\nu(z)$ for fixed ν to be

$$j_{\nu,n} \sim (n + \frac{\nu}{2} - \frac{1}{4})\pi - \frac{4\nu^2 - 1}{8\pi(n + \frac{\nu}{2} - \frac{1}{4})} - \frac{4(4\nu-1)(28\nu^2 - 31)}{3[8\pi(n + \frac{\nu}{2} - \frac{1}{4})]^3} - \cdots .$$

Similar expansions exist for large zeros of $J'_\nu(z)$, $Y'_\nu(z)$, $Y_\nu(z)$ and cross products of Bessel functions [36, p. 507]. Gatteschi found error bounds for the two term McMahon asymptotic approximations of zeros of $J_\nu(x)$, $J'_\nu(x)$, and

$J_0(kx)Y_0(x) - J_0(x)Y_0(kx)$ for $\nu \geq 0$ and real x [11], [12], [13], [14].

To the author's knowledge, no one has previously found error bounds for asymptotic expansions of complex zeros of special functions. Here we present two theorems for finding error bounds on asymptotic approximations of both real and complex zeros of functions. The theorems are rigorous statements of the intuitive idea that if two functions are close together, then their zeros will be close together. The approximation theorem for real zeros is:

Theorem 1.1.1. In the interval $[b - \rho, b + \rho]$, suppose $f(x) = g(x) + \epsilon(x)$, where $f(x)$ is continuous, $g(x)$ is differentiable, $g(b) = 0$, $m = \min|g'(x)| > 0$, and $E = \max|\epsilon(x)| < \min\{|g(b-\rho)|, |g(b+\rho)|\}$. Then there exists a zero c of $f(x)$ in the interval such that $|c - b| \leq E/m$.

This theorem and the one below yield error bounds on asymptotic approximations of zeros, since, in the applications, E approaches zero asymptotically as some parameter or variable approaches infinity so that the zeros of $f(x)$ approach the zeros of $g(x)$ asymptotically. The theorem for complex zeros is:

Theorem 2.1.1. Suppose $f(z) = g(z) + \epsilon(z)$, where $f(z)$ and $g(z)$ are holomorphic for $|z - b| \leq \rho$, b being a simple zero and the only zero of $g(z)$ in the disk. Suppose $m > E$,

where $m = \min_{\theta} |\, g(b + \rho e^{i\theta})|$, $E = \max_{\theta} |\, \epsilon(b + \rho e^{i\theta})|$, and suppose $M' = \max_{\theta} |g'(b + \rho e^{i\theta})|$. Then there exists a simple zero c of $f(z)$ and c is the only zero of $f(z)$ in the above disk; moreover, $|c - b| \leq \frac{E\rho^2 M'}{m(m-E)}$.

The two comparison functions we shall use for $g(z)$ are $\cos(z + \psi)$ and $\mathrm{Ai}(z)$. Other comparison functions are discussed in § 1.4 and § 4.3.

We use these theorems to study the asymptotic approximations of zeros of Airy and cylindrical functions. The only real zeros of $\mathrm{Ai}(z)$ occur on the negative real axis and a_n denotes the nth negative zero. In § 1.2 we show that $a_n = -[3\pi(4n - 1)/8]^{2/3} + \sigma_n$ where $|\sigma_n| \leq 0.163/(n - 0.263)^{4/3}$ for $n \geq 1$. Throughout this thesis, three or more significant figures are given for numbers. The last significant figure is the result of rounding to the nearest digit except for numbers in inequalities, which are rounded to obtain the weakest inequality. The second application is to the first positive zero $j_{\nu,1}$ of the Bessel function $J_\nu(x)$ in case the order ν is large. In § 1.3 we show that $j_{\nu,1} = \nu - 1.85576\nu^{1/3} + \alpha\nu^{-1/3}$ where $0.500 \leq \alpha \leq 1.357$ for $\nu \geq 1$ and $0.935 \leq \alpha \leq 1.105$ for $\nu \geq 10$. Similar expressions for the next positive zeros $j_{\nu,n}$ of $J_\nu(x)$ can be found. The methods of Chapter 4 could be used to

find error bounds on the above asymptotic approximation for complex values of ν.

The third application of the theorems is to the McMahon asymptotic approximation of complex zeros of $J_\nu(z)$ for complex values of ν. In Chapter 2, we show that there exist simple zeros $j_{\nu,n}$ of $J_\nu(z)$ with ν complex such that $n \geq 0.314 - \mathrm{Re}(\nu)/2 + 1.38\,|4\nu^2 - 1|$ implies

$$|j_{\nu,n} - (n + \nu/2 - 1/4)\pi| \leq \frac{0.188\,|4\nu^2 - 1|}{n - 0.314 + \mathrm{Re}(\nu)/2} \quad ;$$

and

$$|j_{\nu,n} - [(n + \frac{\nu}{2} - \frac{1}{4}) - \frac{4\nu^2 - 1}{8\pi(n + \nu/2 - 1/4}]| \leq \frac{0.0246\,|4\nu^2-1|\,|4\nu^2-9|}{(n - 1/2 + \mathrm{Re}(\nu)/2)^2}$$

$$+ \frac{0.0288\,|4\nu^2 - 1|}{(n - 1/2 + \mathrm{Re}(\nu)/2)^2} + \frac{0.00016\,|4\nu^2-1|^3}{(n - 1/4 + \mathrm{Re}(\nu)/2)^3}$$

if $n \geq 1/2 - \mathrm{Re}(\nu)/2 + \max\{3.2,\ 0.76\,|4\nu^2 - 1|\}$.

The last application of the theorems is to the complex zeros of the Hankel function $H_\nu^1(\beta)$ as a function of the order ν for real positive β. We will call these zeros Watson poles since they were first used as residue poles in the Watson transformation [21, pp. 31-32]. The Watson pole asymptotic approximation is:

$$\nu_n(\beta) = \beta + (\frac{\beta}{2})^{1/3} a_n e^{-2\pi i/3} + (\frac{2}{\beta})^{1/3} \frac{a_n^2 e^{-4\pi i/3}}{60} + O(1/\beta) \quad ,$$

where a_n is the nth negative zero of the Airy function $Ai(z)$. In order to find error bounds on the Watson pole asymptotic approximation, we need to know error bounds on the asymptotic approximation of $H_\nu^1(\beta)$ in terms of the Airy function $Ai(u)$. In Chapter 3, we use Olver's technique of bounding the successive approximation solution of the integral equation for the error term to establish an approximation theorem with error bounds for analytic functions in terms of the Airy function $Ai(u)$. We apply this theorem to $H_\nu^1(\beta)$ and also find bounds on some constants occurring in the bound on the error term.

In Chapter 4 we use our approximation theorem for complex zeros together with the approximation with error bounds for $H_\nu^1(\beta)$ in terms of $Ai(u)$ to find an expression with several constants, which bounds the error in the Watson pole asymptotic approximation. Using numerically computed values of these constants, it is shown that

$$\nu_1(\beta) = \beta + (\frac{\beta}{2})^{1/3} a_1 e^{-2\pi i/3} + (\frac{2}{\beta})^{1/3} \frac{a_1^2 e^{-4\pi i/3}}{60} + \alpha(\beta)$$

where $|\alpha(\beta)| \leq 1.71/\beta^{2/3} \leq 0.232$ for $\beta \geq 20$.

In addition to their importance in theoretical expressions, asymptotic approximations of zeros are often used for the actual

computation of zeros. J. C. P. Miller [20] used the asymptotic expansions of zeros of Airy functions in compiling part of the tables of zeros of Airy functions, but he does not discuss error bounds for the asymptotic approximations. In the case of Bessel functions, the expansion of the small zeros for large ν and McMahon's asymptotic expansion as given earlier are used to compute zeros in Zeros of Bessel Functions and Associated Values [23, §2-4]. A uniform asymptotic expansion of zeros, which comes from the uniform asymptotic expansion of Bessel functions, is also used to compute zeros [23]. Error bounds for this approximation could be found by using the methods of Chapter 4.

In Zeros of Bessel Functions and Associated Values [23], no mention is made of Gatteschi's results concerning error bounds for McMahon's approximation for zeros nor is the error in using any truncated asymptotic expansions discussed. One assumes that the usual computational procedure of truncating at the smallest term was followed. In Error Bounds for Asymptotic Expansions, Olver [28, §1.3] presents strong arguments for finding error bounds on differences between functions and the partial sums of their asymptotic expansions, but he seems to have ignored the problem of error bounds on the asymptotic expansions of zeros in [23].

We have only applied the theorems for finding error bounds for approximations of zeros to real and complex zeros of Airy and cylindrical functions. However, given any asymptotic approximation of a function with known error bounds, an asymptotic approximation of the zeros of the function by the zeros of the comparison function with error bounds can be found by using our theorems. Our theorems allow us to quantify Tricomi's claim that the asymptotic behavior of the zeros of any function can be deduced from the asymptotic behavior of the function itself by showing how error bounds for the asymptotic approximation of zeros depend on error bounds for the asymptotic approximation of the functions.

Although the cosine and Airy function $Ai(z)$ have been used as comparison functions in the examples given here, any comparison function with known or tabulated zeros could be used. Other possible comparison functions include the Airy function $Bi(z)$, Bessel functions, the derivatives of Airy and Bessel functions, and the reciprocal of the gamma function. In §1.4 and §4.3, it is pointed out that error bounds can be found for the asymptotic approximations of zeros of Airy functions, Bessel functions, cross products of Bessel functions, Legendre functions, Coulomb wave functions, confluent hypergeometric functions, parabolic cylinder functions, orthogonal polynomials, and the derivatives of all of these functions.

CHAPTER 1

THE ASYMPTOTIC APPROXIMATION OF REAL ZEROS

In this chapter we first establish an approximation theorem with error bounds relating the real zeros of a function to the zeros of an approximating or comparison function. Corollaries are given where the comparison functions are the cosine and the Airy function Ai(x). In § 1.2 we use the corollary with the cosine as the comparison function to study the zeros of the Airy function Ai(z). We show that $a_n = -[3\pi(4n - 1)/8]^{2/3} + \sigma_n$ where a_n is the nth negative zero of Ai(z) and $|\sigma_n| \leq 0.0163/(n - 0.263)^{4/3}$ for $n \geq 1$.

In § 1.3 the corollary with Ai(x) as the comparison function is used to study the asymptotic approximation of the first positive zero $j_{\nu,1}$ of $J_\nu(x)$ for large real ν. We show that $j_{\nu,1} = \nu - (\frac{\nu}{2})^{1/3} a_1 + \alpha\nu^{-1/3}$ where

$$0.500 \leq \alpha \leq 1.357 \quad \text{for} \quad \nu \geq 1$$

and

$$0.935 \leq \alpha \leq 1.105 \quad \text{for} \quad \nu \geq 10 \quad .$$

Other possible applications of the theorem with various comparison functions are given in § 1.4.

§1.1. An Approximation Theorem for Real Zeros.

In this section we establish a theorem for the approximation of real zeros of functions by the zeros of a comparison function. Then we give corollaries with the cosine and the Airy function Ai(x) as comparison functions.

Theorem 1.1.1. In the interval $[b - \rho, b + \rho]$, suppose $f(x) = g(x) + \epsilon(x)$ where $f(x)$ is continuous, $g(x)$ is differentiable, $g(b) = 0$, $m = \min|g'(x)| > 0$, and $E = \max|\epsilon(x)| < \min\{|g(b - \rho)|, |g(b + \rho)|\}$. Then there exists a zero c of $f(x)$ in the interval such that $|c - b| \leq E/m$.

Proof. Since $\min|g'(x)| > 0$, then b is a simple zero and the only zero of $g(x)$ in the interval. Hence $g(b - \rho)$ and $g(b + \rho)$ are of opposite signs. Assume $g(b - \rho) > 0$ and $g(b + \rho) < 0$. Since $\max|\epsilon(x)| < |g(b - \rho)|$, then $f(b - \rho) = g(b - \rho) + \epsilon(b - \rho) > 0$; and since $\max|\epsilon(x)| < |g(b + \rho)|$, then $f(b + \rho) = g(b + \rho) + \epsilon(b + \rho) < 0$. Since $f(b - \rho) > 0$ and $f(b + \rho) < 0$, there exists at least one zero c of $f(x)$ in the interval. A similar reasoning implies the existence of a zero if $g(b - \rho) < 0$ and $g(b + \rho) > 0$. By the Mean Value Theorem, for some $\theta \in [b - \rho, b + \rho]$,

$|(g(c) - g(b))/(c - b)| = |g'(\theta)| \geq \min|g'(x)| = m$. Thus

$|c - b| \leq |g(c) - 0|/m = |f(c) - \epsilon(c)|/m = |\epsilon(c)|/m \leq E/m$.

Corollary 1.1.1. In the interval $[n\pi - \pi/2 - \psi - \rho,\ n\pi - \pi/2 - \psi + \rho]$, where $\rho < \pi/2$, suppose $f(x) = \cos(x + \psi) + \epsilon(x)$, $f(x)$ is continuous, and $E = \max|\epsilon(x)| < \sin\rho$. Then there exists a zero c of $f(x)$ in the interval such that $|c - (n\pi - \pi/2 - \psi)| \leq E/\cos\rho$.

Proof. If $g(x) = \cos(x + \psi)$ and $b = n\pi - \pi/2 - \psi$ in Theorem 1.1.1, then $m = \min|-\sin(x + \psi)| = \min_{[-\rho, \rho]}|\cos y| = \cos\rho > 0$, and $E = \max|\epsilon(x)| < \min\{|\cos(n\pi - \pi/2 - \rho)|,\ |\cos(n\pi - \pi/2 + \rho)|\} = \sin\rho$.

Corollary 1.1.2. In the interval $[a_n - \rho,\ a_n + \rho]$ where a_n is the nth negative zero of $Ai(x)$ and ρ is small enough so that $m = \min|Ai'(x)| > 0$, suppose $f(x) = Ai(x) + \epsilon(x)$, $f(x)$ is continuous, and $E = \max|\epsilon(x)| < \min\{|Ai(a_n - \rho)|,\ |Ai(a_n + \rho)|\}$. Then there exists a zero c of $f(x)$ in the interval such that $|c - a_n| \leq E/m$.

Proof. Let $g(x) = Ai(x)$ and $b = a_n$ in Theorem 1.1.1.

§ 1.2. The Asymptotic Approximation of the Zeros of the Airy Function Ai(z).

The only zeros of $Ai(z)$ occur on the negative real axis [1, p. 450] and a_n denotes the nth negative zero. Here we use Corollary 1.1.1 to show that $a_n = -[3\pi(4n-1)/8]^{2/3} + \sigma_n$ where $|\sigma_n| \leq 0.0163/(n-0.263)^{4/3}$ for $n \geq 1$. Hence a_n is approximated by the first term of its asymptotic expansion:

$$a_n \sim -\lambda^{2/3}(1 + 5/48\lambda^2 - 5/36\lambda^4 + \ldots) \quad ,$$

where $\lambda = 3\pi(4n-1)/8$ [20, § 4.7]. The bound on $|\sigma_n|$ is an asymptotic error bound since it is of the same order of magnitude as the second term in the asymptotic expansion. The method of this section could also be applied to the similar asymptotic expansions for the zeros of $Ai'(z)$, $Bi(z)$, and $Bi'(z)$ [1, p. 450].

If $\zeta = 2x^{3/2}/3$, then $Ai(-x) = (x^{1/2}/3)[J_{1/3}(\zeta) + J_{-1/3}(\zeta)]$ [1, 10.4.15]. If we use the asymptotic expansions of $J_{1/3}(\zeta)$ and $J_{-1/3}(\zeta)$, then

$$Ai(-x) = (1/\pi^{1/2}x^{1/4})(\cos(\zeta - \pi/4)[1 - \frac{385}{10368\zeta^2} + \ldots]$$

$$+ \sin(\zeta - \pi/4)[\frac{5}{72\zeta} + \ldots]) \quad .$$

The error in truncating each of the above series is less than the next term in absolute value [36, § 7.32]. In this application of Corollary 1.1.1, if we let ζ be the independent variable,

$f(\zeta) = x^{1/4}\pi^{1/2}\mathrm{Ai}(-x)$, and $\psi = -\pi/4$; then

$$|\epsilon(\zeta)| \leq \frac{385}{10368\zeta^2} + \frac{5}{72\zeta} .$$

If $\zeta \geq 2.3$, then $|\epsilon(\zeta)| \leq 0.0373$. If $\rho = 0.04$, then $E = \max|\epsilon(\zeta)| \leq 0.0373 < 0.040 = \sin\rho$. By Corollary 1.1.1 there exists a zero c_n of $f(\zeta)$ such that

$$c_n \in [n\pi - \pi/4 - 0.04,\ n\pi - \pi/4 + 0.04]$$

and

$$|c_n - (n\pi - \pi/4)| \leq E/\cos\rho \leq 0.086/(n\pi - \pi/4 - 0.04)$$

if $n \geq 1$. The assumption that $\zeta \geq 2.3$ is justified since c_1 belongs to the interval $[2.32,\ 2.40]$.

If the zeros of $\mathrm{Ai}(z)$ and $f(\zeta)$ are compared, then $c_n = 2(-a_n)^{3/2}/3$ and $|(-a_n)^{3/2} - 3\pi(4n - 1)/8| \leq 0.041/(n - 0.263)$ for $n \geq 1$. If we use Taylor's formula [1, 3.6.1], then we obtain the theorem below.

Theorem 1.2.1. There exist zeros a_n of $\mathrm{Ai}(z)$ such that

$$a_n = -[3\pi(4n - 1)/8]^{2/3} + \sigma_n$$

where $|\sigma_n| \leq 0.0163/(n - 0.263)^{4/3}$ for $n \geq 1$.

For $n \geq N > 1$, a slightly better bound on the absolute error σ_n could be found; however, the above error bound has the advantage that it applies to all of the zeros of $\mathrm{Ai}(z)$.

For example, $a_1 = -2.320 + \sigma_1$ where $|\sigma_1| \leq 0.025$. The number 0.025 seems to be a good bound since if we use the tabulated value of a_1 [1, p. 478], then the absolute error is 0.01811. The percentage error is less than 1.07 per cent. The percentage error for $n = 10$ is less than 0.0063 per cent.

§ 1.3. The Asymptotic Approximation of the First Positive Zero of the Bessel Function of the First Kind

In this section we use Corollary 1.1.2 to study the asymptotic expansion of the first positive zero $j_{\nu,1}$ of $J_\nu(x)$ for large real ν. The asymptotic expansion of $j_{\nu,1}$ as given by Airey [3], Watson [36, p.516], and Tricomi [34] is:

$$j_{\nu,1} = \nu + 1.85576\nu^{1/3} + 1.03315\nu^{-1/3} - 0.00397\nu^{-1} - 0.0908\nu^{-5/3} + 0.043\nu^{-7/3} + \ldots .$$

We show that

$$(1.3.1) \qquad j_{\nu,1} = \nu - (\nu/2)^{1/3} a_1 + \alpha\nu^{-1/3} = \nu + 1.85576\nu^{1/3} + \alpha\nu^{-1/3}$$

where

$$0.500 \leq \alpha \leq 1.357 \quad \text{for} \quad \nu \geq 1$$

and

$$0.935 \leq \alpha \leq 1.105 \quad \text{for} \quad \nu \geq 10 \quad .$$

The method of this section could also be used to study the asymptotic expansions for the next positive zeros $j_{\nu,n}$ of $J_\nu(x)$ and the

first positive zeros of $Y_\nu(x)$, $J_\nu'(x)$ and $Y_\nu'(x)$ [1, p. 371].

First we will find error bounds on the uniform asymptotic approximation of the Bessel function of the first kind in terms of the Airy function $Ai(x)$ [1, 9.3.35]. If $C_\nu(z)$ denotes any solution of Bessel's differential equation, then $z^{1/2}C_\nu(\nu z)$ satisfies

$$w''(z) - \left(\frac{1 - z^2}{z^2}\nu^2 - \frac{1}{4z^2}\right)w = 0 \qquad [1, 9.1.49] \quad .$$

If we use Olver's transformations [22, § 4],

$$(1.3.2) \qquad \zeta\left(\frac{d\zeta}{dz}\right)^2 = \frac{1 - z^2}{z^2}, \quad z(0) = 1, \quad W = \left(\frac{dz}{d\zeta}\right)^{-1/2}w, \quad x = \nu^{2/3}\zeta \quad ,$$

then $\left(\frac{1 - z^2}{\zeta}\right)^{1/4} C_\nu(\nu z)$ satisfies

$$(1.3.3) \qquad W''(x) - [x + f(x)]W = 0$$

where $f(x) = \nu^{-4/3}F(\nu^{-2/3}x)$ and $F(\zeta) = \frac{5}{16\zeta^2} + \frac{\zeta}{4}\frac{z^2(z^2 + 4)}{(z^2 - 1)^3}$.

Here we will assume z is real and $\nu > 0$.

To the differential equation (1.3.3) we now apply the approximation theorem of Olver [26, § 3] to obtain:

$$w_1(x) = Ai(x) + \epsilon_1(x)$$

and

$$w_2(x) = Bi(x) + \epsilon_2(x) \qquad ,$$

where

$$|\epsilon_1(x)| \le (1/\lambda_1)[e^{\lambda_1 F_1(x)} - 1]E^{-1}(x)M(x) \quad ,$$

$$|\epsilon_2(x)| \le (\lambda_2/\lambda_1)[e^{\lambda_1 F_2(x)} - 1]E(x)M(x) \quad ,$$

$$\lambda_1 = 1.430, \quad \lambda_2 = 1.315 \quad ,$$

$$F_1(x) = \int_x^\infty |t^{-1/2} f(t)|\, dt \quad , \quad \text{and} \quad F_2(x) = \int_a^x |t^{-1/2} f(t)|\, dt \quad .$$

Since the coefficient of $w'(x)$ in (1.3.3) is zero, the Wronskian of any two solutions of 1.3.3 is a constant. If $a = \infty$, then the Wronskian $W[w_1(x), w_2(x)] = W[Ai(x), Bi(x)] \neq 0$ so that $w_1(x)$ and $w_2(x)$ are linearly independent.

Since $(\frac{1 - z^2}{\zeta})^{1/4} J_\nu(\nu z)$ satisfies the differential equation (1.3.3), it is a linear combination of w_1 and w_2:

$$(\frac{1 - z^2}{\zeta})^{1/4} J_\nu(\nu z) = c_1(\nu) w_1(x) + c_2(\nu) w_2(x) \quad .$$

If z approaches zero, then ζ approaches infinity and x approaches infinity so that

$$(\frac{1 - z^2}{\zeta})^{1/4} J_\nu(\nu z) \to (\frac{1 - z^2}{\zeta})^{1/4} \frac{(\nu z/2)^\nu}{\Gamma(\nu + 1)} \to 0 \quad ,$$

and

$$w_1(x) = Ai(x) + w_1(x) \to \frac{1}{2\pi^{1/2} \zeta^{1/4} \nu^{1/6} e^{2\nu\zeta^{3/2}/3}} + 0 \to 0$$

[1, 9.1.7, 10.4.59]. Since $Bi(x) \to 2E(x)M(x)/5^{1/2}$ [26, 8.09], then for some small positive δ

$$w_2(x) = Bi(x) + \epsilon_2(x)$$

$$\geq 2E(x)M(x)(1 - \delta)/5^{1/2} - (\lambda_2/\lambda_1)[e^{\lambda_1 F_2(x)} - 1]E(x)M(x)$$

$$= E(x)M(x)\{2(1 - \delta)/5^{1/2} - (\lambda_2/\lambda_1)[e^{\lambda_1 F_2(x)} - 1]\}$$

$$\to \infty \quad \text{as} \quad x \to \infty$$

if we choose a large enough so that $2(1 - \delta)/5^{1/2} > (\lambda_2/\lambda_1)[e^{\lambda_1 F_2(x)} - 1]$. It is always possible to satisfy the above inequality since $F_2(x)$ can be made arbitrarily small by taking a large enough. Therefore, $c_2(\nu) = 0$.

From the previous paragraph we have $(\frac{1 - z^2}{\zeta})^{1/4} J_\nu(\nu z) - c_1(\nu)w_1(x) = 0$ for $a < \zeta < \infty$. Now $w_1(x)$ is independent of a and $(\frac{1 - z^2}{\zeta})^{1/4} J_\nu(\nu z) - c_1(\nu)w_1(x)$ is a solution of the differential equation (1.3.3) which has continuous coefficients for $-\infty < \zeta < \infty$. Therefore, $(\frac{1 - z^2}{\zeta})^{1/4} J_\nu(\nu z) - c_1(\nu)w_1(x)$ is the unique solution 0 throughout $-\infty < \zeta < \infty$ [4, p. 27]. If we use [1, 9.1.7 and 9.3.38], then

$$c_1(\nu) = \lim_{z\to 0} \left(\frac{1-z^2}{\zeta}\right)^{1/4} \frac{\nu^\nu z^\nu 2\pi^{1/2}\zeta^{1/4}\nu^{1/6} e^{\nu\left[\ell n \frac{1+(1-z^2)^{1/2}}{z} - (1-z^2)^{1/2}\right]}}{2^\nu \Gamma(\nu+1)}$$

$$= \frac{\nu^{\nu+1/6} 2\pi^{1/2}}{e^\nu \Gamma(\nu+1)} \quad .$$

Thus

$$(1.3.4) \qquad J_\nu(\nu z) = \left(\frac{\zeta}{1-z^2}\right)^{1/4} \frac{\nu^{\nu+1/6} 2\pi^{1/2}}{e^\nu \Gamma(\nu+1)} [Ai(\nu^{2/3}\zeta) + \epsilon_1(\nu^{2/3}\zeta)]$$

for $-\infty < \zeta < \infty$ where

$$(1.3.5) \qquad |\epsilon_1(\nu^{2/3}\zeta)| \le \frac{1}{1.430}[e^{1.430 F_1(\nu^{2/3}\zeta)} - 1]E^{-1}(\nu^{2/3}\zeta)M(\nu^{2/3}\zeta)$$

and

$$F_1(\nu^{2/3}\zeta) = \int_{\nu^{2/3}\zeta}^{\infty} |t^{-1/2} f(t)|\, dt$$

$$= \frac{1}{\nu}\int_{\zeta}^{\infty} |s^{-1/2} F(s)|\, ds$$

$$\le V_{-\infty,\infty}\left\{\int_0^x t^{-1/2} f(t)\, dt\right\}/\nu$$

$$\le 0.2103/\nu \qquad [25, \text{ p. } 9] \quad .$$

Let us now use (1.3.4) in Corollary 1.1.2 with $\nu^{2/3}\zeta$ as the independent variable,

$$f(\nu^{2/3}\zeta) = \left(\frac{1 - z^2}{\zeta}\right)^{1/4} \frac{e^{\nu}\Gamma(\nu+1)}{\nu^{\nu+1/6}2\pi^{1/2}} J_{\nu}(\nu z) \quad ,$$

$a_1 = -2.33811$ [1, p. 478], and $\rho \leq 0.25$. Now $\nu^{2/3}\zeta \leq -2.33811 + 0.25 = -2.088$ implies $E^{-1}(\nu^{2/3}\zeta) = 1$ and $M(\nu^{2/3}\zeta) \leq 0.47$ [26, § 2]. The inequality (1.3.5) becomes $|\epsilon_1(\nu^{2/3}\zeta)| \leq 0.116$ if $\nu \geq 1$. From tables of Airy functions [20], $\min\{|Ai(a_1 + \rho)|, |Ai(a_1 - \rho)|\} \geq 0.117$ and $m = \min|Ai'(x)| \geq 0.67600$ if $\rho = 0.17$. If we use $\rho = 0.17$ in Corollary 1.1.2, then $\nu^{2/3}\zeta_1 = a_1 + \delta$ where $|\delta| \leq 0.329[e^{0.302/\nu} - 1]/m \leq 0.21/\nu$ if $\nu \geq 1$.

From (1.3.4) $j_{\nu,1} = \nu z_1$, where $z_1 = 1 - 2^{-1/3}\zeta_1 + \zeta_1^2 z''(\theta)/2$ from (1.3.1) and Taylor's formula [1, 3.6.1]. If we use $\zeta_1 = \dfrac{a_1 + \delta}{\nu^{2/3}}$, then

$$j_{\nu,1} = \nu - (\nu/2)^{1/3}a_1 - (\nu/2)^{1/3}\delta + (a_1 + \delta)^2 z''(\theta)/2$$

$$= \nu - 1.85576\nu^{1/3} + \alpha\nu^{-1/3} \quad .$$

If $\nu \geq 1$, then $|\nu^{2/3}2^{-1/3}\delta| \leq 0.167$, $4.70 \leq (a_1 + \delta)^2 \leq 6.30$, and $-2.508 \leq \theta \leq 0$. From Appendix I, $0.284 \leq z''(\theta) \leq 0.378$ and therefore $0.500 = -0.167 + 0.667 \leq \alpha \leq 0.167 + 1.19 = 1.357$. If we combine this result with a similar result for $\nu \geq 10$, then we

obtain the theorem below.

Theorem 1.3.1. If $j_{\nu,1}$ denotes the first positive zero of $J_\nu(x)$ for real ν, then

$$j_{\nu,1} = \nu - (\nu/2)^{1/3} a_1 + \alpha\nu^{-1/3} = \nu + 1.85576\nu^{1/3} + \alpha\nu^{-1/3}$$

where $0.500 \leq \alpha \leq 1.357$ for $\nu \geq 1$ and

$0.935 \leq \alpha \leq 1.105$ for $\nu \geq 10$.

For example, $14.433 \leq j_{10,1} \leq 14.511$ or $j_{10,1} = 14.472 \pm 0.039$. The percentage error is less than 0.27 per cent. The number 0.039 seems to be a reasonable bound since if we use the tabulated value of $j_{10,1}$ [23, p. 14], then the absolute error is 0.0035.

§ 1.4. Other Possible Applications

Because $\sin(x + \phi) = \cos(x + \phi - \pi/2)$, Corollary 1.1.1 can be used with $\psi = \phi - \pi/2$ if the comparison function in Theorem 1.1.1 is the sine. Since Airy functions, Bessel functions, cross products of Bessel functions, Legendre functions, Coulomb Wave functions, confluent hypergeometric functions, and the derivatives of all of these functions have asymptotic approximations in terms of cosines or sines, Corollary 1.1.1 could be used to find error bounds on the asymptotic approximations of the zeros of these functions. An analysis similar to the one used in Chapter 2 could be used to study the McMahon asymptotic expansions

of zeros of $J_\nu(x)$, $Y_\nu(x)$, $J'_\nu(x)$, $Y'_\nu(x)$, $J_\nu(x)Y_\nu(\lambda x) - J_\nu(\lambda x)Y_\nu(x)$, and $J'_\nu(x)Y'_\nu(\lambda x) - J'_\nu(\lambda x)Y'_\nu(x)$ for real ν and x [1, § 9.5].

Corollary 1.1.2 could be used to study the uniform asymptotic expansion of zeros of $J_\nu(x)$, $J'_\nu(x)$, $Y_\nu(x)$, and $Y'_\nu(x)$ [1, pp. 371-372]. Since parabolic cylinder functions have asymptotic expansions in terms of Airy functions [1, § 19.7], Corollary 1.1.1 could be used to find error bounds on the asymptotic approximations of their zeros. For example, if c is a zero of Whittaker's function $V(a,x)$, then for $a < 0$, $c = 2|a|^{1/2}\theta$ where $(1/4)(\arccos\theta - \theta(1-\theta^2)^{1/2}) \sim (-a_n)^{3/2}/6|a|$ [1, § 19.26].

Other possible comparison functions which could be used in Theorem 1.1.1 include $Ai'(x)$, $Bi(x)$, $Bi'(x)$, $J_n(x)$, $J_n'(x)$, $Y_n(x)$, $Y_n'(x)$, and many others. For example, if $j_{b-1,r}$ is the rth positive zero of $J_{b-1}(x)$, then the rth positive zero x_r of Kummer's function $M(a,b,x)$ has the asymptotic approximation $x_r \sim j^2_{b-1,r}(2b - 4a)$ [1, § 13.7]. Since $J_\nu(x) \sim (x/2)^\nu/\Gamma(\nu+1)$ for positive x and large negative ν, the zeros ν_n of $J_\nu(x)$ as a function of ν for fixed x are asymptotically near the negative integers which are the zeros of $1/\Gamma(\nu+1)$ [7]. The asymptotic distribution of zeros of

orthogonal polynomials near the middle of the interval of orthogonality depends on the zeros of trigonometric functions and near the endpoints of the interval depends on the zeros of Bessel functions [9, p. 204]. Hence these zeros could be approximated asymptotically by using Theorem 1.1.1 or its corollaries; see [1, § 22.16].

CHAPTER 2

THE McMAHON ASYMPTOTIC APPROXIMATION OF COMPLEX ZEROS OF THE BESSEL FUNCTION OF THE FIRST KIND

In this chapter we prove an approximation theorem with error bounds relating the complex zeros of a holomorphic function to the zeros of a comparison function. Corollaries with the cosine and the Airy function $Ai(z)$ as comparison functions are given. The corollary with the cosine as the comparison function is used to study the McMahon asymptotic expansion of the complex zeros $j_{\nu,n}$ of $J_\nu(z)$ for complex ν and large n. The McMahon expansion

$$j_{\nu,n} \sim (n + \nu/2 - 1/4)\pi - \frac{4\nu^2 - 1}{8\pi(n + \nu/2 - 1/4)} - \frac{4(4\nu^2-1)(28\nu^2-31)}{384\pi^3(n + \nu/2 - 1/4)^3} - \cdots$$

was found by McMahon using a method due to Stokes [36, p. 507].

We will show that $J_\nu(z)$ with ν complex has simple zeros $j_{\nu,n}$ such that $n \geq 0.314 - \mathrm{Re}(\nu)/2 + 1.38|4\nu^2 - 1|$ implies

$$|j_{\nu,n} - (n + \nu/2 - 1/4)\pi| \leq \frac{0.188|4\nu^2 - 1|}{n - 0.314 + \mathrm{Re}(\nu)/2},$$

and

$n \geq 1/2 - \mathrm{Re}(\nu)/2 + \max\{3.2,\ 0.76|4\nu^2 - 1|\}$ implies

$$|j_{\nu,n} - (n\pi + \nu\pi/2 - \pi/4 - \frac{4\nu^2 - 1}{8\pi(n + \nu/2 - 1/4)})| \leq 0.0246\frac{|4\nu^2-1|\,|4\nu^2-9|}{(n - 1/2 + \mathrm{Re}(\nu)/2)^2}$$

$$+\frac{0.0288|4\nu^2-1|}{(n - 1/2 + \mathrm{Re}(\nu)/2)^2} + \frac{0.00016|4\nu^2 - 1|^3}{(n - 1/4 + \mathrm{Re}(\nu)/2)^3} .$$

We call these the one- and two-term McMahon approximations of $j_{\nu,n}$. The error bounds are asymptotic error bounds since they are of the same order of magnitude as the next term in the asymptotic expansion of McMahon. The method used here could also be applied to the similar expansions for large zeros of $J'_\nu(z)$, $Y_\nu(z)$, $Y'_\nu(z)$, and cross products of Bessel functions [1, § 9.5].

Our proof also establishes the existence of infinitely many zeros of $J_\nu(z)$ for complex ν and that these zeros are simple. These conclusions are not new, as Watson [36, Ch.15] shows that there are exactly m zeros of $z^{-\nu}J_\nu(z)$ between the imaginary axis and the line $\mathrm{Re}(z) = (m + \mathrm{Re}(\nu)/2 + 1/4)\pi$ for m sufficiently large. He also shows that these zeros are real and simple except possibly at $z = 0$.

For real ν, the McMahon asymptotic expansion for $j_{\nu,m}$ has been studied by Luigi Gatteschi. For real x and $0 < n < 1$, he used known error bounds on the expansion of $J_n(x)$ in terms of

trigonometric functions to find error bounds for the two-term McMahon asymptotic approximation of $j_{n,m}$ [11]. Gatteschi [13] has also found inequalities satisfied by the zeros of any function $f(x)$ such that $f(x) = (1 + \delta(x))\sin x + \epsilon(x)\cos x - r(x)$ for $\delta(x) > -1$ where $\alpha < x < \beta$. As an application, he proved that if $n \geq 0$ and $j_{n,m} \geq (2n + 1)(2n + 3)/\pi$, then

$$\left| j_{n,m} + \frac{(n,1)}{2X_{nm}} - X_{nm} \right| \leq \frac{A\left|(n,1)\right|^3 + B\left|(n,1)(n,2)\right| + C\left|(n,3)\right| + 2(n,1)^2}{\left\{(2m + n - 1)\pi\right\}^3}$$

where $X_{nm} = (m + n/2 - 1/4)\pi$, $(n,p) = \dfrac{\Gamma(n + p + 1/2)}{p!\,\Gamma(n - p + 1/2)}$, $A = 0.65$, $B = 1.29$, $C = 3.35$ for $n \leq 7/2$; and $A = 0.573$, $B = 1.146$, $C = 11.9$ for $n \geq 7/2$. Gatteschi also studied the McMahon asymptotic expansions for the real zeros of $J_n'(x)$ and $J_0(kz)Y_0(z) - J_0(z)Y_0(kz)$ [12], [14].

The corollary with the Airy function $Ai(z)$ as the comparison function is used in § 4.1 to find error bounds on the Watson pole asymptotic approximation. Other possible applications of Theorem 2.1.1 and its corollaries are presented in § 4.3.

§2.1. An Approximation Theorem for Complex Zeros

In this section we use Rouché's Theorem and a special form of the inverse function theorem to establish an approximation theorem with error bounds for complex zeros of holomorphic functions in terms of the zeros of a holomorphic comparison function. Then we give two corollaries with the cosine and the Airy function $Ai(z)$ as comparison functions.

The special form of the inverse function theorem [16, p. 265] which will be used in the proof of Theorem 2.1.1 is given below as a lemma.

Lemma 2.1.1. Suppose that $h(z)$ is holomorphic in $|z| < R$, $h(0) = 0$, $h'(0) \neq 0$, and $h(z) \neq 0$ for $0 < |z| < r \leq R$. Let C be the circle $|z| = \rho$, $\rho < r$. Then

$$d(w) = \frac{1}{2\pi i} \int_C \frac{t\,h'(t)dt}{h(t) - w}$$

defines a holomorphic function of w, at least for $|w| < m = \min_{\theta} |h(\rho e^{i\theta})|$. For such values of w, $z = d(w)$ is the only solution of $h(z) = w$ which tends to zero with w.

Our approximation theorem for complex zeros is given below.

Theorem 2.1.1. Suppose $f(z) = g(z) + \epsilon(z)$, where $f(z)$ and $g(z)$ are holomorphic for $|z - b| \leq \rho$, b being a simple zero and the only zero of $g(z)$ in the disk. Suppose $m > E$, where $m = \min_{\theta} |g(b + \rho e^{i\theta})|$, $E = \max_{\theta} |\epsilon(b + \rho e^{i\theta})|$; and suppose $M' = \max_{\theta} |g'(b + \rho e^{i\theta})|$. Then there exists a simple zero c of $f(z)$ and c is the only zero of $f(z)$ in the above disk; moreover,

$$|c - b| \leq \frac{E\rho^2 M'}{m(m-E)} .$$

Proof. The hypothesis $m > E$ implies $|g(z)| > |\epsilon(z)|$ on the circle with center b and radius ρ. By Rouché's Theorem there exists a simple zero c of $f(z)$, and c is the only zero of $f(z)$ in the above disk.

If we take $h(z)$ in Lemma 2.1.1 to be $g(b + z)$, then $h(0) = g(b) = 0$ and $h'(0) = g'(b) \neq 0$. There exists some $R > \rho$ such that $h(z) = g(b + z)$ is holomorphic for $|z| < R$ and $h(z) = g(b + z) \neq 0$ for $0 < |z| < r = R$. In Lemma 2.1.1 let C be the circle $|z - b| = \rho < r = R$. Then

$$d(w) = \frac{1}{2\pi i} \int_C \frac{tg'(b + t)dt}{g(b+t) - w}$$

is the inverse of $w = g(b + z)$ for $|w| < m = \min_{\theta} |g(b + \rho e^{i\theta})|$.

If $a = c - b$, then $w(a) = g(c) = f(c) - \epsilon(c) = -\epsilon(c)$, and

$|w(a)| = |\epsilon(c)| < E < m$ so that

$$c - b = a = d(w(a)) = \frac{1}{2\pi i} \int_C \frac{t g'(b + t)dt}{g(b + t) - w(a)} .$$

The series

$$\frac{1}{g(b + t) - w(a)} = \frac{1}{g(b + t)} + \frac{w(a)}{[g(b + t)]^2} + \ldots + \frac{w(a)^k}{[g(b+t)]^{k+1}} + \ldots$$

is uniformly convergent with respect to t and $w(a)$ if $t \in C$ and $|w(a)| < m(1 - \delta)$ where $\delta > 0$. If we multiply by the bounded function $tg'(b + t)$ and integrate term by term, then

$$c - b = \sum_{k=0}^{\infty} w^k(a) \frac{1}{2\pi i} \int_C \frac{t g'(b + t)dt}{[g(b+t)]^{k+1}}$$

$$= \frac{1}{2\pi i} \int_C \frac{t g'(b + t)dt}{g(b + t)} + \sum_{k=1}^{\infty} w(a)^k \frac{1}{2\pi i} \int_C \frac{t g'(b + t)dt}{[g(b + t)]^{k+1}} .$$

Since $\frac{g'(b + t)}{g(b + t)}$ has a pole of order 1 at $t = 0$, $\frac{tg'(b + t)}{g(b + t)}$ is holomorphic for $|z - b| \leq R$ and the first term above vanishes. Hence

$$|c - b| \leq \sum_{k=1}^{\infty} |w(a)|^k \frac{1}{2\pi} \int_C \frac{|tg'(b + t)||dt|}{|g(b + t)|^{k+1}}$$

$$\leq \sum_{k=1}^{\infty} E^k \frac{\rho^2 M'}{m^{k+1}}$$

$$\leq \frac{E\rho^2 M'}{m(m-E)} .$$

Corollary 2.1.1. If $f(z) = \cos(z + \psi) + \epsilon(z)$ is holomorphic for $|z - b| \leq \rho \leq \pi/2$, where $b = n\pi - \pi/2 - \psi$ for some integer n, and $E = \max_{\theta} |\epsilon(b + \rho e^{i\theta})| < \sin \rho$, then there exists a simple zero c of $f(z)$ and c is the only zero of $f(z)$ in the above disk; moreover,

$$|c - b| \leq \frac{E\rho^2 \cosh \rho}{\sin \rho (\sin \rho - E)} \quad .$$

Proof. In Theorem 2.1.1, let $g(z) = \cos(z + \psi)$ and $b = n\pi - \pi/2 - \psi$ so that

$$m = \min_{\theta} |g(b + \rho e^{i\theta})| = \min_{\theta} |\cos(n\pi - \pi/2 + \rho e^{i\theta})|$$

$$= \min_{\theta} |\sin(\rho e^{i\theta})| = \sin \rho \quad ,$$

and

$$M' = \max_{\theta} |g'(b + \rho e^{i\theta})| = \max_{\theta} |\sin(n\pi - \pi/2 + \rho e^{i\theta})|$$

$$= \max_{\theta} |\cos(\rho e^{i\theta})| = \cosh \rho \quad .$$

Corollary 2.1.2. Suppose $f(z) = \mathrm{Ai}(z) + \epsilon(z)$ is holomorphic for $|z - a_n| \leq \rho$, where a_n is the nth negative zero of $\mathrm{Ai}(z)$ and ρ is small enough so that a_n is the only zero of $\mathrm{Ai}(z)$ in the disk. Suppose $m > E$, where $m = \min_{\theta} |\mathrm{Ai}(a_n + \rho e^{i\theta})|$ and $E = \max_{\theta} |\epsilon(a_n + \rho e^{i\theta})|$; and suppose $M' = \max_{\theta} |\mathrm{Ai}'(a_n + \rho e^{i\theta})|$. Then there exists a simple zero c of $f(z)$ and c is the only zero of $f(z)$ in the above disk;

moreover, $|c - b| \leq \frac{E\rho^2 M'}{m(m-E)}$.

Proof. In Theorem 2.1.1, let $g(z) = Ai(z)$ and $b = a_n$.

§ 2.2. The One-Term McMahon Asymptotic Approximation

In this section we use Corollary 2.1.1 to show that for $n \geq 0.314 - Re(\nu)/2 + 1.38|4\nu^2 - 1|$, there exist simple zeros $j_{\nu,n}$ of $J_\nu(z)$ with ν complex such that

$$|j_{\nu,n} - (n + \nu/2 - 1/4)\pi| \leq \frac{0.188|4\nu^2 - 1|}{n - 0.314 + Re(\nu)/2} .$$

The asymptotic approximation of the Bessel function of the first kind in terms of a cosine is

$(\pi z/2)^{1/2} J_\nu(z) = \cos(z - \nu\pi/2 - \pi/4) + \epsilon(z)$ where

$$|\epsilon(z)| \leq [\,|e^{iz - i\nu\pi/2}| + \pi|e^{-iz + i\nu\pi/2}|/2\,]\frac{|4\nu^2 - 1|}{8|z|}\exp\left(\frac{\pi|4\nu^2 - 1|}{8|z|}\right)$$

for $|\arg z| \leq \pi/2$ [28, p. 179]. In this application of Corollary 2.1.1, let $f(z) = (\pi z/2)^{1/2} J_\nu(z)$ and $\psi = -\nu\pi/2 - \pi/4$. If $Re(z) \geq 0$ and $z \neq 0$, then $f(z)$ is holomorphic.

If we choose $\rho = \pi/4$, then $|z - b| \leq \pi/4$ implies $|z| \geq |b| - \pi/4 \geq Re(b) - \pi/4 \geq n\pi - \pi/2 + \pi Re(\nu)/2$. Thus for $|z - b| \leq \pi/4$,

$$E = \max_{\theta} |\epsilon(b + \pi e^{i\theta}/4)| \leq \frac{1.79|4\nu^2 - 1|}{8(n - 1/2 + \mathrm{Re}(\nu)/2)} \exp\left[\frac{|4\nu^2 - 1|}{8(n - 1/2 + \mathrm{Re}(\nu)/2)}\right] .$$

Let us find how large n must be to imply $E \leq \sin \rho/2 = 0.354$. Now $y \leq k/(1+k)$ implies $y \exp y \leq y/(1 - y) \leq k$. If we let $y = \frac{|4\nu^2 - 1|}{8(n - 1/2 + \mathrm{Re}(\nu)/2)}$ and $k = 0.354/1.79 = 0.198$, then $n \geq 1/2 - \mathrm{Re}(\nu)/2 + 0.76|4\nu^2 - 1|$ implies $y \leq 0.165$, which in turn implies $E \leq \sin \rho/2 < \sin \rho$. Also $\mathrm{Re}(z) \geq \mathrm{Re}(b) - \pi/4 \geq 0.76|4\nu^2 - 1| \geq 0$ so that our assumption that $\mathrm{Re}(z) \geq 0$ is justified.

If we apply Corollary 2.1.1, $f(z)$ has a simple zero c, which is the only zero of $f(z)$ in the disk. Moreover, for $n \geq 1/2 - \mathrm{Re}(\nu)/2 + 0.76|4\nu^2 - 1|$,

$$|c - (n + \nu/2 - 1/4)\pi| \leq \frac{E(\pi/4)^2 \cosh(\pi/4)}{\sin(\pi/4)[\sin(\pi/4) - E]} .$$

Since $\mathrm{Re}(z) \geq 0$ and $z \neq 0$, c is a zero of $J_\nu(z)$, which we will denote $j_{\nu,n}$. Hence we have proved the theorem below.

Theorem 2.2.1. For $n \geq 1/2 - \mathrm{Re}(\nu)/2 + 0.76|4\nu^2 - 1|$, there exist simple zeros $j_{\nu,n}$ of $J_\nu(z)$ with ν complex such that

$$|j_{\nu,n} - (n + \nu/2 - 1/4)\pi| \leq \frac{0.865|4\nu^2 - 1|}{n - 1/2 + \mathrm{Re}(\nu)/2} .$$

Obviously, this approximation is most accurate for large n and small ν.

If we choose $\rho = 0.2$ instead of $\rho = \pi/4$, then we obtain the theorem below.

Theorem 2.2.2. For $n \geq 0.314 - \mathrm{Re}(\nu)/2 + 1.38|4\nu^2 - 1|$, there exist simple zeros $j_{\nu,n}$ of $J_\nu(z)$ with ν complex such that

$$|j_{\nu,n} - (n + \nu/2 - 1/4)\pi| \leq \frac{0.188|4\nu^2 - 1|}{n - 0.314 + \mathrm{Re}(\nu)/2} .$$

For example, if $\nu = i$, then $b = n\pi + i\pi/2 - \pi/4$ and $n \geq 0.314 - \mathrm{Re}(i)/2 + 1.38(5) = 7.20$ implies $|j_{i,n} - (n + i/2 - 1/4)\pi| \leq 0.94/(n - 0.314)$. Thus $|j_{i,8} - (8 + i/2 - 1/4)\pi| \leq 0.137$. In this case the percentage error is less than 0.57 per cent.

§2.3. The Two-Term McMahon Asymptotic Approximation

In this section we use Corollary 2.1.1 to show that for $n \geq 1/2 - \mathrm{Re}(\nu)/2 + \max\{3.2,\ 0.76|4\nu^2 - 1|\}$, there exist simple zeros $j_{\nu,n}$ of $J_\nu(z)$ with ν complex such that

$$|j_{\nu,n} - (n\pi + \nu\pi/2 - \pi/4 - \frac{4\nu^2 - 1}{8\pi(n + \nu/2 - 1/4)})|$$

$$\leq 0.0246 \frac{|4\nu^2-1|\ |4\nu^2-9|}{(n - 1/2 + \mathrm{Re}(\nu)/2)^2} + \frac{0.0288|4\nu^2-1|}{(n - 1/2 + \mathrm{Re}(\nu)/2)^2}$$

$$+ \frac{0.00016|4\nu^2 - 1|^3}{(n - 1/4 + \mathrm{Re}(\nu)/2)^3} .$$

Let us consider the asymptotic approximation for the Bessel function of the first kind:

$$(\pi z/2)^{1/2} J_\nu(z) = \cos(z - \nu\pi/2 - \pi/4) - \sin(z - \nu\pi/2 - \pi/4)(4\nu^2 - 1)/8z + \epsilon_2(z)$$

where

$$|\epsilon_2(z)| \leq \{|e^{iz - i\nu\pi/2}| + 2|e^{-iz + i\nu\pi/2}|\} \frac{|4\nu^2 - 1||4\nu^2 - 9|}{8^2|z|^2} \exp\left(\frac{\pi|4\nu^2 - 1|}{8|z|}\right)$$

for $|\arg z| \leq \pi/2$ [28, p. 179]. If we use the identity $1/w = 1/v - (w - v)/wv$, then

$$(\pi z/2)^{1/2} J_\nu(z) = \cos(z - \nu\pi/2 - \pi/4) - \sin(z - \nu\pi/2 - \pi/4)\frac{(4\nu^2 - 1)}{8\pi(n + \nu/2 - 1/4)} + \eta(z)$$

where

$$|\eta(z)| \leq |\epsilon_2(z)| + \frac{|z - \pi(n + \nu/2 - 1/4)||4\nu^2 - 1||\sin(z - \nu\pi/2 - \pi/4)|}{|\pi(n + \nu/2 - 1/4)|\, 8|z|}.$$

Using a trigonometric identity, we find

$$\frac{(\pi z/2)^{1/2} J_\nu(z)}{\left[1 + \frac{(4\nu^2 - 1)^2}{[8\pi(n + \nu/2 - 1/4)]^2}\right]^{1/2}} = \cos\left(z - \nu\pi/2 - \pi/4 + \arctan\frac{4\nu^2 - 1}{8\pi(n + \nu/2 - 1/4)}\right) + \epsilon(z)$$

where $|\epsilon(z)| \leq |n(z)|$. In this application of Corollary 2.1.1, let $f(z)$ be the left hand side of the above equation and

$$\psi = -\nu\pi/2 - \pi/4 + \arctan\frac{4\nu^2 - 1}{8\pi(n + \nu/2 - 1/4)}.$$

In § 2.2 we established that $n \geq 1/2 - \mathrm{Re}(\nu)/2 + 0.76|4\nu^2 - 1|$ implies

$$y/\pi = \frac{4\nu^2 - 1}{8\pi(n + \nu/2 - 1/4)} \leq 0.0525.$$

By Taylor's Theorem [2, p. 101], if $f(u)$ is analytic inside a circle of radius R and C is a circle about the origin of radius $r < R$, then

$$f(u) = f(0) + u f'(0)/1! + \ldots + u^{n-1} f^{(n-1)}(0)/(n-1)! + u^n f_n(u)$$

where

$$|f_n(u)| = \left|\frac{1}{2\pi i}\int_C \frac{f(z)dz}{z^n(z-u)}\right| \leq \frac{\max_\theta |f(r e^{i\theta})|}{r^{n-1}(r - |u|)}$$

if $|u| < r$.

From [1, 4.4.39] we have

$$|\arctan(r e^{i\theta})|^2 \leq \frac{1}{4}\arctan^2\left(\frac{2r\cos\theta}{1 - r^2}\right) + \frac{1}{16}\ln^2\left[\left(\frac{r+1}{r-1}\right)^2\right]$$

$$\leq 1/4 + \ln^2 81/16 = 1.46 \quad \text{if} \quad r = 0.8 < 1 = R \quad .$$

Hence

$$|\arctan(y/\pi)| \leq 1.62|y/\pi| \leq 0.085$$

and

$$|\arctan(y/\pi) - y/\pi| \leq 2.53|y/\pi|^3 \quad .$$

With $\rho = \pi/4$, $|z - b| \leq \pi/4$ implies

$|z| \geq |b| - \pi/4 \geq n\pi - \pi/2 + \pi \operatorname{Re}(\nu)/2 - 0.085$. We now assume that n is large enough so that $\pi(n - 1/2 + \operatorname{Re}(\nu)/2) - 0.085 \geq 10$, which implies $|z| \geq 10$. Thus

$$\frac{|4\nu^2 - 1|}{8|z|} \leq \frac{|4\nu^2 - 1|}{8\pi(n - 1/2 + \operatorname{Re}(\nu)/2)} + \frac{0.085|4\nu^2 - 1|}{80\pi(n - 1/2 + \operatorname{Re}(\nu)/2)} \leq 0.053.$$

Now for $|z - b| \leq \pi/4$,

$$|\epsilon(z)| \leq |\eta(z)| \leq |\epsilon_2(z)| + \frac{|z - \pi(n + \nu/2 - 1/4)||4\nu^2 - 1||\sin(z - \nu\pi/2 - \pi/4)|}{8\pi|n + \nu/2 - 1/4||z|}$$

$$\leq 0.075.$$

Hence $E \leq 0.075 < 0.707 = \sin(\pi/4)$. If we combine our two assumptions, then we have the single condition

$$n \geq 1/2 - \operatorname{Re}(\nu)/2 + \max\{3.2,\ 0.76|4\nu^2 - 1|\} \quad .$$

If we use Corollary 2.1.1, then the above condition implies that

$$|j_{\nu,n} - [n\pi + \nu\pi/2 - \pi/4 - \arctan(\frac{4\nu^2 - 1}{8\pi(n + \nu/2 - 1/4)})]|$$

$$\leq \frac{E(\pi/4)^2 \cosh(\pi/4)}{\sin(\pi/4)[\sin(\pi/4) - 0.075]} \leq 1.84E \quad .$$

We now use the earlier result that $|\arctan(y/\pi) - (y/\pi)| \leq 2.53|(y/\pi)|^3$ to obtain

$$|j_{\nu,n} - [n\pi + \nu\pi/2 - \pi/4 - \frac{4\nu^2 - 1}{8\pi(n + \nu/2 - 1/4)}]| \leq 1.84E$$

$$+ 2.53 \frac{|4\nu^2 - 1|^3}{(8\pi)^3 |n + \nu/2 - 1/4|^3} \quad .$$

The theorem below follows from this inequality.

Theorem 2.3.1. For $n \geq 1/2 - \mathrm{Re}(\nu)/2 + \max\{3.2, 0.76|4\nu^2 - 1|\}$, there exist simple zeros $j_{\nu,n}$ of $J_\nu(z)$ with ν complex such that

$$|j_{\nu,n} - (n\pi + \nu\pi/2 - \pi/4 - \frac{4\nu^2 - 1}{8\pi(n + \nu/2 - 1/4)})|$$

$$\leq \frac{0.0246|4\nu^2 - 1||4\nu^2 - 9|}{(n - 1/2 + \mathrm{Re}(\nu)/2)^2} + \frac{0.0288|4\nu^2 - 1|}{(n - 1/2 + \mathrm{Re}(\nu)/2)^2}$$

$$+ \frac{0.00016|4\nu^2 - 1|^3}{(n - 1/4 + \mathrm{Re}(\nu)/2)^3} \quad .$$

For example, if we let $\nu = i$, then

$n \geq 1/2 + \max\{3.2, 0.76(5)\} = 4.3$ implies

$$|j_{i,n} - (n\pi + i\pi/2 - \pi/4 + \frac{5}{8\pi(n + i/2 - 1/4)})| \leq \frac{1.78}{(n - 1/2)^2} \quad .$$

The percentage error when $n = 5$ is less than 0.56 per cent.

CHAPTER 3

THE UNIFORM ASYMPTOTIC APPROXIMATION OF CYLINDRICAL FUNCTIONS

In this chapter we establish a theorem which gives error bounds for the asymptotic approximation of an analytic solution of a differential equation by the Airy function Ai(z). Then we apply this theorem to obtain the following uniform asymptotic approximation for the Hankel function of the first kind,

$$H_\nu^1(\nu z) = \left(\frac{\zeta}{1-z^2}\right)^{1/4} \frac{2^{3/2}}{\nu^{1/3}\, e^{\pi i/3}}\left[\mathrm{Ai}(\nu^{2/3}\zeta e^{2\pi i/3}) + \epsilon(\nu^{2/3}\zeta e^{2\pi i/3}, \nu)\right]$$

for $0 \leq \arg \nu \leq \pi/2$, where a bound on $|\epsilon_2(\nu^{2/3}\zeta e^{2\pi i/3}, \nu)|$ is given in Theorem 3.2.1. This asymptotic approximation with error bounds is used in Chapter 4 to study the Watson pole asymptotic approximation.

§ 3.1. An Approximation Theorem for Analytic Functions

In this section we prove an approximation theorem for analytic functions with a technique used by Olver [24], [26], [27]. Olver's technique consists of bounding the successive approximation solution of the integral equation for the error. The error bound in

our theorem is better than twice as good as the error bound given in Olver's Theorem 3 [27, § 5].

First we present some standard notation concerning Airy functions of a complex variable [27, § 5]. Let $S_1 = \{ z : |\arg z| \leq \pi/3 \}$, $S_2 = \{ z : -\pi \leq \arg z \leq -\pi/3 \}$, and $S_3 = \{ z : \pi/3 \leq \arg z \leq \pi \}$. Let $E(z) = |\exp(2z^{3/2}/3)|$ and $E^{-1}(z) = 1/E(z)$, where $z^{1/2}$ has its principal value. For z belonging to S_1 or S_3 define

$$E(z)|\mathrm{Ai}(z)| = M_2(z) \sin \chi_2(z) \quad ,$$

$$E^{-1}(z)|\mathrm{Ai}(z\, e^{-2\pi i/3})| = M_2(z) \cos \chi_2(z) \quad ,$$

$$E(z)|\mathrm{Ai}'(z)| = N_2(z) \sin \psi_2(z) \quad ,$$

and

$$E^{-1}(z)|\mathrm{Ai}'(z\, e^{-2\pi i/3})| = N_2(z) \cos \psi_2(z) \quad ,$$

where $M_2(z)$, $N_2(z)$, $\chi_2(z)$, and $\psi_2(z)$ are real valued non-negative functions. Define the real valued non-negative functions $M_3(z)$, $N_3(z)$, $\chi_3(z)$, and $\psi_3(z)$ in a similar manner when z belongs to S_1 or S_2.

From the above definitions,

$$M_2^2(z) = E^2(z)|\mathrm{Ai}^2(z)| + E^{-2}(z)|\mathrm{Ai}^2(z\, e^{-2\pi i/3})|$$

and

$$N_2^2(z) = E^2(z)|\mathrm{Ai}'(z)|^2 + E^{-2}(z)|\mathrm{Ai}'(ze^{-2\pi i/3})|^2$$

if z belongs to S_1 or S_3. Define

$$\nu_1 = \max_{z \in S_1 \cup S_3} \{\pi |z|^{1/2} M_2^2(z)\}$$

and

$$\nu_2 = \max_{z \in S_1 \cup S_3} \{\pi E(z) M_2(z) |z^{1/2} Ai(z)|\} \quad .$$

In §3.3 we will show that

$$\nu_1 = \max_{z \in S_1 \cup S_2} \{\pi |z|^{1/2} M_3^2(z)\}$$

and

$$\nu_2 = \max_{z \in S_1 \cup S_2} \{\pi E(z) M_3(z) |z^{1/2} Ai(z)|\} \quad .$$

Let us consider the differential equation

$$w''(z) - [z + f(z)]w(z) = 0 \quad , \tag{3.1.1}$$

where z is a complex variable.

Theorem 3.1.1. Let $f(z)$ in 3.1.1 be analytic in the simply connected domain D and let $H(b)$ be those points in D which can be joined to a fixed point b in D by a path Γ satisfying the following conditions:

1) Γ lies in D .

2) Γ is a finite number of Jordan arcs each with a parametric equation of the form $t = t(\tau)$ where τ is a real parameter, $t''(\tau)$ is continuous, and $t'(\tau)$ does

not vanish. If b is the point at infinity on a straight line L, then Γ coincides with L for all sufficiently large t.

3) As t traverses Γ from a to b, then $\mathrm{Re}(2t^{3/2}/3)$ is monotonic non-decreasing.

Then the differential equation 3.1.1 has an analytic solution $w(z)$ such that

$$w(z) = Ai(z) + \epsilon(z) \quad \text{and} \quad w'(z) = Ai'(z) + \eta(z) \quad ,$$

where

$$|\epsilon(z)| \leq (\nu_2/\nu_1)[e^{2\nu_1 F(z)} - 1]E^{-1}(z)M_j(z)$$

and

$$|\eta(z)| \leq (\nu_2/\nu_1)[e^{2\nu_1 F(z)} - 1]E^{-1}(z)N_j(z)$$

if $z \in H(b)$. Here $j = 2$ if $z \in S_1 \cup S_3$, $j = 3$ if $z \in S_1 \cup S_2$, and

$$F(z) = \int_z^b |t^{-1/2} f(t)|\,|dt| \quad .$$

Proof. The differential equation for the error term $\epsilon(z)$ is easily found to be

$$\epsilon''(z) - z\,\epsilon(z) = f(z)[Ai(z) + \epsilon(z)] \quad .$$

The equivalent integral equation with $\epsilon(b) = \epsilon'(b) = 0$ is

$$\epsilon(z) = 2\pi e^{\pm \pi i/6} \int [Ai(z)Ai(t e^{\pm 2\pi i/3}) - Ai(z e^{\pm 2\pi i/3})Ai(t)] f(t)[Ai(t) + \epsilon(t)]dt$$

where we use the minus signs in $S_1 \cup S_3$ and the plus signs in $S_1 \cup S_2$. All integrals in this proof are along the path Γ from z to b.

If we define a kernel by

$$K(z,t) = 2\pi e^{-\pi i/6}[Ai(z)Ai(t e^{-2\pi i/3}) - Ai(z e^{-2\pi i/3})Ai(t)] \quad ,$$

then the integral equation in $S_1 \cup S_3$ becomes

$$\epsilon(z) = \int K(z,t)f(t)Ai(t)dt + \int K(z,t)f(t)\epsilon(t)dt \quad .$$

We now solve this integral equation by successive approximations. Let

$$y_0(z) = \int K(z,t)f(t)Ai(t)dt$$

and

$$y_n(z) = \int K(z,t)f(t)y_{n-1}(t)dt \quad \text{for} \quad n = 1, 2, \ldots \ .$$

From hypothesis 3), $E(t) \geq E(z)$ so that

$$|K(z,t)| = 2\pi\,|\mathrm{Ai}(z)\mathrm{Ai}(te^{-2\pi i/3}) - \mathrm{Ai}(ze^{-2\pi i/3})\mathrm{Ai}(t)|$$

$$\leq 2\pi\,[E^{-1}(z)M_2(z)E(t)M_2(t)\sin\chi_2(z)\cos\chi_2(t)$$

$$+ E(z)M_2(z)E^{-1}(t)M_2(t)\sin\chi_2(t)\cos\chi_2(z)]$$

$$\leq 2\pi E^{-1}(z)E(t)M_2(z)M_2(t)\sin[\chi_2(z)+\chi_2(t)]$$

$$\leq 2\pi E^{-1}(z)E(t)M_2(z)M_2(t) \quad .$$

Hence

$$|y_0(z)| \leq \int |K(z,t)f(t)\mathrm{Ai}(t)|\,|dt|$$

$$\leq \int 2\pi E^{-1}(z)E(t)M_2(z)M_2(t)\,|f(t)\mathrm{Ai}(t)|\,|dt|$$

$$\leq 2E^{-1}(z)M_2(z)\,\nu_2 \int |t^{-1/2}f(t)|\,|dt|$$

$$= 2\nu_2 E^{-1}(z)M_2(z)F(z) \quad .$$

It is easy to show by induction that for $n \geq 1$,

$$|y_n(z)| \le \int |K(z,t)f(t)y_{n-1}(t)|\,|dt|$$

$$\le 2\pi E^{-1}(z)M_2(z)\int E(t)M_2(t)|f(t)|\nu_2 E^{-1}(t)M_2(t)2^n\nu_1^{n-1}F^n(t)/n!\,|dt|$$

$$\le \nu_2 E^{-1}(z)M_2(z)\int 2^{n+1}\nu_1^n|t^{-1/2}f(t)|\,F^n(t)/n!\,|dt|$$

$$\le \nu_2 E^{-1}(z)M_2(z)2^{n+1}\nu_1^n F^{n+1}/(n+1)! \quad .$$

If $\epsilon(z) = \sum_{n=0}^{\infty} y_n(z)$, then

$$|\epsilon(z)| \le \sum_{n=0}^{\infty} |y_n(z)|$$

$$\le \sum_{n=0}^{\infty} \nu_2 E^{-1}(z)M_2(z)\ 2^{n+1}\nu_1^n F^{n+1}(z)/(n+1)!$$

$$\le \frac{\nu_2}{\nu_1} E^{-1}(z)M_2(z)[e^{2\nu_1 F(z)} - 1] \quad .$$

If $E^{-1}(z)$, $M_2(z)$, and $F(z)$ are bounded for $z \in H(b)$, then the above series converges uniformly to $\epsilon(z)$. In order to show that $\epsilon(z)$ satisfies the integral equation, we consider

$K(z,t)f(t)\epsilon(t) = \lim_{N\to\infty} E_N(z,t)$, where $E_N(z,t) = \sum_{n=0}^{N} K(z,t)f(t)y_n(t)$.

Now

$$|E_N(z,t)| \le \sum_{n=0}^{\infty} |K(z,t)f(t)y_n(t)|$$

$$\le \sum_{n=0}^{\infty} 2\pi E^{-1}(z)M_2(z)M_2(t)|f(t)|\nu_2 M_2(t)2^{n+1}\nu_1^n F^{n+1}(t)/(n+1)!$$

$$\le \sum_{n=0}^{\infty} \nu_2 E^{-1}(z)M_2(z)|t^{-1/2}f(t)|2^{n+2}\nu_1^{n+1}F^{n+1}(t)/(n+1)!$$

$$\le 2\nu_2 E^{-1}(z)M_2(z)|t^{-1/2}f(t)|[e^{2\nu_1 F(t)} - 1] \quad .$$

The integral

$$\int 2\nu_2 E^{-1}(z)M_2(z)|t^{-1/2}f(t)|[e^{2\nu_1 F(t)} - 1]|dt|$$

$$= 2\nu_2 E^{-1}(z)M_2(z)[e^{2\nu_1 F(z)}/2\nu_1 - F(z)]$$

is bounded. By the Lebesgue dominated convergence theorem [31, p. 200], we conclude

$$\int K(z,t)f(t)\epsilon(t)dt = \lim_{N\to\infty} \int E_N(z,t)dt$$

$$= \sum_{n=0}^{\infty} \int K(z,t)f(t)y_n(t)dt$$

$$= \sum_{n=0}^{\infty} y_{n+1}(t)$$

$$= \epsilon(z) - \int K(z,t)f(t)Ai(t)dt \quad .$$

Hence $\epsilon(z)$ satisfies the integral equation.

The bound given for $|\eta(z)|$ follows from term-by-term differentiation of the series for $\epsilon(z)$ [2, p. 139] and the inequality

$$|Ai'(z)Ai(te^{-2\pi i/3}) - e^{-2\pi i/3}Ai'(ze^{-2\pi i/3})Ai(t)|$$

$$\leq E^{-1}(z)E(t)N_2(z)M_2(t) \quad .$$

A similar proof holds for $z \in S_1 \cup S_2$.

§ 3.2. The Uniform Asymptotic Approximation of Bessel Functions

In this section Theorem 3.1.1 is used to find error bounds on the uniform asymptotic approximation of $J_\nu(\nu z)$, $H_\nu^1(\nu z)$, and $H_\nu^2(\nu z)$ for large complex ν. These uniform asymptotic expansions of Bessel functions in terms of Airy functions were first found by Olver [22, § 4-5].

Let $C_\nu(x)$ denote any non-trivial solution of Bessel's differential equation $x^2C''(x) + xC'(x) + (x^2 - \nu^2)C = 0$. If we reduce this differential equation to normal form, then $z^{1/2}C_\nu(\nu z)$ satisfies $w''(z) - [\frac{1-z^2}{z^2}\nu^2 - \frac{1}{4z^2}]w(z) = 0$. We now use Olver's transformations [22, § 4]

$$(3.2.1) \qquad \zeta(\frac{d\zeta}{dz})^2 = \frac{1 - z^2}{z^2} \quad \text{and} \quad W = (\frac{dz}{d\zeta})^{-1/2} w \quad .$$

Then $(\frac{1 - z^2}{\zeta})^{1/4}C_\nu(\nu z)$ satisfies $W''(\zeta) - [\nu^2\zeta + F(\zeta)]W(\zeta) = 0$,

where $F(\zeta) = \frac{5}{16\zeta^2} + \frac{\zeta}{4}\,\frac{z^2(z^2+4)}{(z^2-1)^3}$. If we let $x = \nu^{2/3}\zeta$, then our differential equation becomes

$$W''(x) - [x + f(x,\nu)]W(x) = 0 \quad , \tag{3.2.2}$$

where $f(x,\nu) = \nu^{-4/3}F(\nu^{-2/3}x)$.

The function $F(\zeta)$ is analytic in D, which is the entire ζ plane with cuts along the rays $e^{\pm\pi i/3}$ from $(3\pi/2)^{2/3}$ to ∞ [22, § 4]. Thus $f(x,\nu)$ is an analytic function of x in the simply connected domain D_x which is the entire x plane with cuts along the rays $\nu^{2/3}e^{\pm\pi i/3}$ from $(3\pi|\nu|/2)^{2/3}$ to ∞. If $b = \nu^{2/3}\infty$ in Theorem 3.1.1 with $0 \le \arg \nu \le \pi/2$, then $H(b)$ includes all points in the sector such that $|\arg(-x)| < \pi/3$. If x_0 is such that $|\arg(-x_0)| < \pi/3$, then a suitable path Γ would be a straight line from x_0 to the origin and then a straight line to $b = \nu^{2/3}\infty$. If we use (3.2.2) as the differential equation in Theorem 3.1.1, then for $0 \le \arg \nu \le \pi/2$,

$w_1(x,\nu) = Ai(x) + \epsilon_1(x,\nu)$ and $\frac{\partial w_1(x,\nu)}{\partial x} = Ai'(x) + \eta_1(x,\nu)$

where

$$|\epsilon_1(x,\nu)| \le (\nu_2/\nu_1)[e^{2\nu_1 F_1(x,\nu)} - 1]E^{-1}(x)M_j(x) \quad ,$$

$$|\eta_1(x,\nu)| \le (\nu_2/\nu_1)[e^{2\nu_1 F_1(x,\nu)} - 1]E^{-1}(x)N_j(x) \quad ,$$

and

$$F_1(x,\nu) = \int_x^b |t^{-1/2} f(t,\nu)|\,|dt| \quad .$$

In order to obtain some linearly independent solutions of the differential equation (3.2.2), we let $u = xe^{2\pi i/3}$ and $v = xe^{-2\pi i/3}$. Then

$$W''(u) - [u + e^{-4\pi i/3} f(ue^{-2\pi i/3}, \nu)]W(u) = 0 \tag{3.2.3}$$

and

$$W''(v) - [v + e^{4\pi i/3} f(ve^{2\pi i/3}, \nu)]W(v) = 0 \quad . \tag{3.2.4}$$

If we apply Theorem 3.1.1 to the differential equation (3.2.3) and return to x as the independent variable, then for $0 \le \arg \nu \le \pi/2$, $w_2(x,\nu) = Ai(xe^{2\pi i/3}) + \epsilon_2(xe^{2\pi i/3}, \nu)$ and $\frac{\partial w_2(x,\nu)}{\partial x} = e^{2\pi i/3}[Ai'(xe^{2\pi i/3}) + \eta_2(xe^{2\pi i/3}, \nu)]$ where

$$|\epsilon_2(xe^{2\pi i/3}, \nu)| \le (\nu_2/\nu_1)[e^{2\nu_1 F_2(xe^{2\pi i/3}, \nu)} - 1]E^{-1}(xe^{2\pi i/3})M_j(xe^{2\pi i/3}) \ ,$$

$$|\eta_2(xe^{2\pi i/3}, \nu)| \le (\nu_2/\nu_1)[e^{2\nu_1 F_2(xe^{2\pi i/3}, \nu)} - 1]E^{-1}(xe^{2\pi i/3})N_j(xe^{2\pi i/3}) \ ,$$

and

$$F_2(xe^{2\pi i/3}, \nu) = \int_{xe^{2\pi i/3}}^b |t^{-1/2} f(te^{-2\pi i/3}, \nu)|\,|dt| \quad .$$

If we apply Theorem 3.1.1 to (3.2.4), then for $0 \le \arg \nu \le \pi/2$,

$$w_3(x,\nu) = Ai(xe^{-2\pi i/3}) + \epsilon_3(xe^{-2\pi i/3}, \nu)$$

and

$$\frac{\partial w_3(x,\nu)}{\partial x} = e^{-2\pi i/3}[Ai'(xe^{-2\pi i/3}) + \eta_3(xe^{-2\pi i/3},\nu)]$$

where

$$|\epsilon_3(xe^{-2\pi i/3},\nu)| \le (\nu_2/\nu_1)[e^{2\nu_1 F_3(xe^{-2\pi i/3},\nu)} - 1]E^{-1}(xe^{-2\pi i/3})M_j(xe^{-2\pi i/3}),$$

$$|\eta_3(xe^{-2\pi i/3},\nu)| \le (\nu_2/\nu_1)[e^{2\nu_1 F_3(xe^{-2\pi i/3},\nu)} - 1]E^{-1}(xe^{-2\pi i/3})N_j(xe^{-2\pi i/3}),$$

and

$$F_3(xe^{-2\pi i/3},\nu) = \int_{xe^{-2\pi i/3}}^{b}{}_3 |t^{-1/2} f(te^{2\pi i/3},\nu)|\,|dt| \quad .$$

Before proceeding, we outline the next steps. We will choose the paths Γ_1 and Γ_2 so that they depend on a parameter a. After showing that $w_1(x,\nu)$ and $w_2(x,\nu)$ are linearly independent, we will have

$$\left(\frac{1-z^2}{\zeta}\right)^{1/4} H_\nu^1(\nu z) = c_1(\nu)w_1(x,\nu) + c_2(\nu)w_2(x,\nu)$$

since $\left(\frac{1-z^2}{\zeta}\right)^{1/4} H_\nu^1(\nu z)$ satisfies the differential equation (3.2.2). As z approaches $i\infty$, we will show that $\left(\frac{1-z^2}{\zeta}\right)^{1/4} H_\nu^1(\nu z)$ and $w_2(x,\nu)$ approach zero while $w_1(x,\nu)$ approaches infinity for a sufficiently large. Thus we will see

that $c_1(\nu) = 0$ and $\left(\frac{1 - z^2}{\zeta}\right)^{1/4} H_\nu^1(\nu z) = c_2(\nu) w_2(x, \nu)$.

If we let $\zeta_2 = \zeta e^{2\pi i/3}$ and we let Γ_2 be the straight line from $\nu^{2/3}\zeta_2$ to $b = \nu^{2/3}\infty$ where $\zeta_2 > a$, a being a parameter, then

$$F_2(x e^{2\pi i/3}, \nu) = \int_{\nu^{2/3}\zeta_2}^{\nu^{2/3}\infty} |t^{-1/2} f(t e^{-2\pi i/3}, \nu)| \, |dt|$$

$$= \frac{1}{|\nu|} \int_{\zeta_2}^{\infty} |s^{-1/2} F(s e^{-2\pi i/3})| \, |ds| \quad .$$

If Γ_1 is the straight line from $\nu^{2/3}\zeta_2 e^{2\pi i/3}$ to $b = \nu^{2/3} a e^{2\pi i/3}$ for $\zeta_2 > a$, then

$$F_1(x, \nu) = \int_{\nu^{2/3}\zeta_2 e^{2\pi i/3}}^{\nu^{2/3} a e^{2\pi i/3}} |t^{-1/2} f(t, \nu)| \, |dt|$$

$$= \frac{1}{|\nu|} \int_{a}^{\zeta_2} |t^{-1/2} F(t e^{2\pi i/3})| \, |dt| \quad .$$

Since $f(\zeta, \nu) = O(\zeta^{-2})$, $F_1(x, \nu)$ and $F_2(x e^{2\pi i/3}, \nu)$ can be made arbitrarily small by choosing a sufficiently large. Since the coefficient of $w'(x)$ in (3.2.2) is zero, the

Wronskian of any two solutions is a constant [17, p. 356]. If $a = \zeta_2 = \infty$, then $x = \nu^{2/3} e^{-2\pi i/3} \infty$ and the Wronskian

$$W[w_1(x,\nu),\ w_2(x,\nu)] = W[Ai(x),\ Ai(x e^{2\pi i/3})] = e^{-\pi i/6}/2\pi \neq 0$$

so that $w_1(x,\nu)$ and $w_2(x,\nu)$ are linearly independent for $0 \leq \arg \nu \leq \pi/2$ [1, 10.4.11].

Since $(\frac{1 - z^2}{\zeta})^{1/4} H_\nu^1(\nu z)$ satisfies the differential equation (3.2.2), it must be a linear combination of $w_1(x,\nu)$ and $w_2(x,\nu)$:

$$(\frac{1 - z^2}{\zeta})^{1/4} H_\nu^1(\nu z) = c_1(\nu)\, w_1(x,\nu) + c_2(\nu)\, w_2(x,\nu) \quad .$$

If $z \to i\infty$, then $\zeta \to \infty e^{-2\pi i/3}$ and $\zeta_2 \to \infty$ so that

$$(\frac{1 - z^2}{\zeta})^{1/4} H_\nu^1(\nu z) \to (\frac{1 - z^2}{\zeta})^{1/4} (\frac{2}{\pi\nu z})^{1/2} e^{i(\nu z - \nu\pi/2 - \pi/4)} \to 0 \quad ,$$

and

$$w_2(x,\nu) \to \frac{e^{-2\nu\zeta_2^{3/2}/3}}{2\pi^{1/2}\nu^{1/6}\zeta^{1/4}e^{\pi i/6}} + 0 \to 0$$

[1, 9.2.3 and 10.4.59]. For some positive δ

$$|w_1(x,\nu)| \geq |Ai(x)| - |\epsilon_1(x,\nu)|$$

$$\geq \frac{e^{2\nu\zeta_2^{3/2}/3}}{\pi^{1/2}|\nu|^{1/6}|\zeta|^{1/4}}\left[\frac{|1-\delta|}{2} - \frac{\nu_2}{\nu_1^{1/2}}\left[e^{2\nu_1 F_1(x,\nu)} - 1\right]\right]$$

$$\to \infty \qquad \text{as} \quad \zeta_2 \to \infty$$

if we choose a large enough so that $\frac{|1-\delta|}{2} > \frac{\nu_2}{\nu_1^{1/2}}\left[e^{2\nu_1 F_1(x,\nu)} - 1\right]$.

Therefore, $c_1(\nu) = 0$. Using $z = 2(-\zeta)^{3/2}/3 + \pi/2 + O(|\zeta|^{-3/2})$ [22, 4.14], we find

$$c_2(\nu) = \lim_{z\to i\infty} \frac{H_\nu^1(\nu z)}{w_2(x,\nu)}$$

$$= \lim_{z\to i\infty} \frac{(1-z^2)^{1/4}\, 2^{3/2} e^{i\nu z - i\nu\pi/2 - \pi/4}\, 2\pi^{1/2}\nu^{1/6}\zeta^{1/4} e^{\pi i/6}}{\zeta^{1/4}(\pi\nu z)^{1/2} e^{2\nu\zeta^{3/2}/3}}$$

$$= \frac{2^{3/2} e^{-\pi i/3}}{\nu^{1/3}} \quad .$$

Now $\left(\frac{1-z^2}{\zeta}\right)^{1/4} H_\nu^1(\nu z) - c_2(\nu) w_2(x,\nu)$ is analytic throughout D_x for $0 \leq \arg\nu \leq \pi/2$ and vanishes on the arc $a < \zeta_2 < \infty$ so that it must vanish everywhere in D_x. Thus we have proved the theorem below.

Theorem 3.2.1. If $0 \le \arg \nu \le \pi/2$ and

$|\arg(-\nu^{2/3}\zeta e^{2\pi i/3})| < \pi/3$, then

$$H_\nu^1(\nu z) = \left(\frac{\zeta}{1-z^2}\right)^{1/4} \frac{2^{3/2}}{\nu^{1/3} e^{\pi i/3}} [Ai(\nu^{2/3}\zeta e^{2\pi i/3}) + \epsilon_2(\nu^{2/3}\zeta e^{2\pi i/3}, \nu)]$$

where

$$\zeta\left(\frac{d\zeta}{dz}\right)^2 = \frac{1-z^2}{z^2} \quad , \qquad z(0) = 1 \quad ,$$

$$|\epsilon_2(\nu^{2/3}\zeta e^{2\pi i/3}, \nu)| \le (\nu_2/\nu_1)[e^{2\nu_1 F_2(\nu^{2/3}\zeta e^{2\pi i/3}, \nu)} - 1] \cdot$$

$$\cdot E^{-1}(\nu^{2/3}\zeta e^{2\pi i/3}) M_j \psi^{2/3}\zeta e^{2\pi i/3}) \quad ,$$

$$F_2(\nu^{2/3}\zeta e^{2\pi i/3}, \nu) = \frac{1}{|\nu|} \int_{\zeta e^{2\pi i/3}}^{\infty} |s^{-1/2} F(s e^{-2\pi i/3})| \, |ds| \quad ,$$

$$F(\zeta) = \frac{5}{16\zeta^2} + \frac{\zeta}{4} \frac{z^2(z^2+4)}{(z^2-1)^3} \quad ,$$

and ν_1, ν_2, E^{-1}, and M_j are defined in §3.1.1.

If we let $\phi(\zeta) = \left(\frac{4\zeta}{1-z^2}\right)^{1/4}$, then

$$H_\nu^{1'}(\nu z) = \left(\frac{1-z^2}{\zeta}\right)^{1/4} \frac{-2^{3/2} e^{\pi i/3}}{z\nu^{2/3}} \left\{ Ai'(\nu^{2/3}\zeta e^{2\pi i/3}) \right.$$

$$\left. + \eta_2(\nu^{2/3}\zeta e^{2\pi i/3}, \nu) + \frac{e^{-\pi i/3}\phi'(\zeta)}{\nu^{2/3}\phi(\zeta)} [Ai(\nu^{2/3}\zeta e^{2\pi i/3}) + \epsilon_2(\nu^{2/3}\zeta e^{2\pi i/3}, \nu)] \right\} .$$

An analysis similar to the proof of Theorem 3.2.1 shows that $\left(\frac{1-z^2}{\zeta}\right)^{1/4} J_\nu(\nu z)$ is a multiple of $w_1(x,\nu)$. In this case Γ_1 is the straight line from $\nu^{2/3}\zeta$ for $\zeta > a$ to $b = \nu^{2/3}\infty$ and Γ_2 is the straight line from $\nu^{2/3}\zeta e^{2\pi i/3}$ to $b = \nu^{2/3} a e^{2\pi i/3}$ for $\zeta > a$. The result is

$$J_\nu(\nu z) = \left(\frac{\zeta}{1-z^2}\right)^{1/4} \frac{\nu^{\nu+1/6} 2\pi^{1/2}}{e^\nu \Gamma(\nu+1)} [Ai(\nu^{2/3}\zeta) + \epsilon_1(\nu^{2/3}\zeta, \nu)] \quad .$$

It can also be shown that

$$H_\nu^2(\nu z) = \left(\frac{\zeta}{1-z^2}\right)^{1/4} \frac{2^{3/2} e^{\pi i/3}}{\nu^{1/3}} [Ai(\nu^{2/3}\zeta e^{-2\pi i/3}) + \epsilon_3(\nu^{2/3}\zeta e^{-2\pi i/3}, \nu)] \quad .$$

§ 3.3. Bounds for the Constants ν_1 and ν_2

In this section we use the definitions of § 3.1 to establish that $\nu_1 \geq \nu_2$,

$$\nu_1 = \max_{z \in S_1 \cup S_3} \{\pi |z|^{1/2} M_2^2(z)\} = \max_{z \in S_1} \{\pi |z|^{1/2} M_2^2(z)\} = \max_{z \in S_1 \cup S_2} \{\pi |z|^{1/2} M_3^2(z)\},$$

and

$$\nu_2 = \max_{z \in S_1 \cup S_3} \{\pi E(z) M_2(z) |z^{1/2} Ai(z)|\} = \max_{z \in S_1 \cup S_2} \{\pi E(z) M_3(z) |z^{1/2} Ai(z)|\} .$$

By using some numerical calculations and error bounds for asymptotic expansions, we show that $\nu_1 \leq 1.288$.

Since

$$\pi |z|^{1/2} M_2^2(z) = \pi |z|^{1/2} M_2(z)[E^2(z)|Ai^2(z)| + E^{-2}(z)|Ai^2(z e^{-\pi i/3})|]^{1/2}$$

$$\geq \pi E(z) M_2(z) |z^{1/2} Ai(z)| \quad ,$$

then

$$\nu_1 = \max_{z \in S_1 \cup S_3} \{\pi |z|^{1/2} M_2^2(z)\} \geq \max_{z \in S_1 \cup S_3} \{\pi E(z) M_2(z) |z^{1/2} Ai(z)|\} = \nu_2 \quad .$$

If we use the definitions of § 3.1, then

$$E(\bar{z}) = |\exp \tfrac{2}{3}(r e^{-i\theta})^{3/2}| = |\exp \tfrac{2}{3} r^{3/2} \cos(3\theta/2)|$$

$$= |\exp \tfrac{2}{3}(r e^{i\theta})^{3/2}| = |\exp(\tfrac{2}{3} z^{3/2})| = E(z) \quad ,$$

and

$$M_2^2(\bar{z}) = E^2(\bar{z})|Ai^2(\bar{z})| + E^{-2}(\bar{z})|Ai^2(\bar{z} e^{-2\pi i/3})|$$

$$= E^2(z)|\overline{Ai^2(z)}| + E^{-2}(z)|\overline{Ai^2(z e^{2\pi i/3})}|$$

$$= M_3^2(z) \quad .$$

If $z \in S_3$, then $M_2^2(z)$ can be shown to be equal to $M_3^2(z e^{-2\pi i/3})$ which is equal to $M_2^2(\overline{z e^{-2\pi i/3}})$ where $\overline{z e^{-2\pi i/3}} \in S_1$. Thus $\nu_1 = \max_{z \in S_1} \{\pi |z|^{1/2} M_2^2(z)\}$. Since

$$\max_{z \in S_1 \cup S_2} \{\pi |z|^{1/2} M_2^2(z)\} = \max_{z \in S_1} \{\pi |z|^{1/2} M_2^2(z)\}$$ also, then

$$\nu_1 = \max_{z \in S_1 \cup S_3} \{\pi |z|^{1/2} M_2^2(z)\} = \max_{z \in S_1} \{\pi |z|^{1/2} M_2^2(z)\}$$

$$= \max_{z \in S_1 \cup S_2} \{\pi |z|^{1/2} M_3^2(z)\} .$$

Since $E(z) = E(\bar{z})$, $M_2(z) = M_3(\bar{z})$, and $|Ai(z)| = |Ai(\bar{z})|$; then

$$\nu_2 = \max_{z \in S_1 \cup S_3} \{\pi E(z) M_2(z) |z^{1/2} Ai(z)|\}$$

$$= \max_{z \in S_1 \cup S_2} \{\pi E(z) M_3(z) |z^{1/2} Ai(z)|\}$$

We will now use some numerical calculations and some error bounds for asymptotic expansions to show that $1.25 \leq \nu_1 \leq 1.288$ and $0.87 \leq \nu_2/\nu_1$. Now $Ai(z) = z^{i/2} K_{1/3}(\zeta)/\pi 3^{1/2}$ where $\zeta = 2z^{3/2}/3$ [1, 10.4.14] and $K_{1/3}(\zeta) = \pi^{1/2} e^{-\zeta} [1 + \gamma_1]/(2\zeta)^{1/2}$ for $|\arg \zeta| \leq \pi/2$

where $|\gamma_1| \leq \frac{5}{24|z|^{3/2}} \exp(\frac{5}{24|z|^{3/2}})$ [28, p. 180]. Therefore,

$$Ai(z) = e^{-2z^{3/2}/3}[1 + \gamma_1]/2\pi^{1/2}z^{1/4}$$

and

$$\pi^{1/2}|z|^{1/4}|e^{2z^{3/2}/3}|\,|Ai(z)| = |1 + \gamma_1|/2$$

for $|\arg z| \leq \pi/3$. Thus

$$\pi|z|^{1/2}E^2(z)|Ai^2(z)| \leq [1 + \frac{5}{24|z|^{3/2}} \exp(\frac{5}{24|z|^{3/2}})]^2/4$$

for $|\arg z| \leq \pi/3$.

If $0 \leq \arg z \leq \pi/3$ and $z_1 = z e^{-2\pi i/3}$, then

$-2\pi/3 \leq \arg z_1 \leq -\pi/3$ and

$\pi|z_1|^{1/2}E^2(z_1)|Ai^2(z_1)| \leq |1 + \gamma_2|^2/4$ where

$$|\gamma_2| \leq \frac{5\pi}{48|z|^{3/2}} \exp(\frac{5\pi}{48|z|^{3/2}}) \quad [28, p. 180].$$

Therefore,

$$\pi|z|^{1/2}E^{-2}(z)|Ai^2(z e^{-2\pi i/3})|$$

$$\leq [1 + \frac{5\pi}{48|z|^{3/2}} \exp(\frac{5\pi}{48|z|^{3/2}})]^2/4$$

for $0 \leq \arg z \leq \pi/3$.

If $-\pi/3 \le \arg z \le 0$ and $z_1 = -z e^{-2\pi i/3} = z e^{\pi i/3}$,

then $0 \le \arg z_1 \le \pi/3$. Now

$$Ai(-z_1) = [e^{i(\xi_1 - \pi/4)}(1 + \epsilon_1) + e^{-i(\xi_1 - \pi/4)}(1 + \epsilon_2)]/2\pi^{1/2} z_1^{1/4}$$

where

$$|\epsilon_1| \le \frac{5}{24|z_1|^{3/2}} \exp\left(\frac{5}{24|z_1|^{3/2}}\right) \quad \text{and}$$

$$|\epsilon_2| \le \frac{5\pi}{48|z_1|^{3/2}} \exp\left(\frac{5\pi}{48|z_1|^{3/2}}\right) \quad [1, 10.4.15], [28, p. 177].$$

Thus

$$\pi |z|^{1/2} E^{-2}(z) |Ai^2(z e^{-\pi i/3})|$$

$$\le \left[1 + \frac{5}{48|z|^{3/2}} \exp\left(\frac{5}{24|z|^{3/2}}\right) + \frac{5\pi}{96|z|^{3/2}} \exp\left(\frac{5\pi}{48|z|^{3/2}}\right)\right]^2$$

for $-\pi/3 \le \arg z \le 0$.

If we combine the previous results, then

$$\pi |z|^{1/2} M_2^2(z) \le \left[1 + \frac{5}{24|z|^{3/2}} \exp\left(\frac{5}{24|z|^{3/2}}\right)\right]^2/4$$

$$+ \left[1 + \frac{5\pi}{48|z|^{3/2}} \exp\left(\frac{5\pi}{48|z|^{3/2}}\right)\right]^2/4$$

for $0 \le \arg z \le \pi/3$; and if $-\pi/3 \le \arg z \le 0$, then

$$\pi|z|^{1/2}M_2^2(z) \le \left[1 + \frac{5}{24|z|^{3/2}} \exp\left(\frac{5}{24|z|^{3/2}}\right)\right]^2/4$$

$$+ \left[1 + \frac{5}{48|z|^{3/2}} \exp\left(\frac{5}{24|z|^{3/2}}\right) + \frac{5\pi}{96|z|^{3/2}} \exp\left(\frac{5\pi}{48|z|^{3/2}}\right)\right]^2$$

If we let $|z| \ge 3$ in these bounds, then $\pi|z|^{1/2}M_2^2(z) \le 0.55$ for $0 \le \arg z \le \pi/3$, and $\pi|z|^{1/2}M_2^2(z) \le 1.288$ for $-\pi/3 \le \arg z \le 0$. The maximum value of $\pi|z|^{1/2}M_2^2(z)$ when evaluated on a 540 point grid over $|z| \le 3$ in S_1 is 1.195. If we combine the bounds on $\pi|z|^{1/2}M_2^2(z)$ for $|z| \le 3$ and $|z| \ge 3$, then $\pi|z|^{1/2}M_2^2(z) \le 1.288$ in S_1, and consequently $\nu_1 \le 1.288$.

In [27, § 5] it is remarked that $\nu_1 \ge 5/4$ and $\nu_2 \ge (5/4)^{1/2}$. These two results are easily verified by using the definition of ν_1 and ν_2 and the result of letting z approach $\infty e^{-\pi i/3}$ in the previous asymptotic expansions. If we combine all of our bounds, then $1.288 \ge \nu_1 \ge 1.25$ and $1 \ge \nu_2/\nu_1 \ge 0.87$.

CHAPTER 4

THE ASYMPTOTIC APPROXIMATION OF WATSON POLES

In this chapter the asymptotic approximation of $H_\nu^1(\beta)$ in terms of the Airy function $Ai(u)$ from Chapter 3 is used in Corollary 2.1.2 to obtain asymptotic approximations for the zeros of $H_\nu^1(\beta)$ as a function of the order ν. These zeros are called Watson poles since they were first used as residue poles in the Watson transformation [21, pp. 32-33]. One of the Watson pole asymptotic approximations for ν_n, the nth zero of $H_\nu^1(\beta)$ considered as a function of the order ν, is

$$\nu_n(\beta) = \beta + (\frac{\beta}{2})^{1/3} a_n e^{-2\pi i/3} + (\frac{2}{\beta})^{1/3} \frac{a_n^2 e^{-4\pi i/3}}{60} + \alpha_n(\beta) \quad .$$

In § 4.1 we find a bound on $\alpha_n(\beta)$ in terms of several constants. These constants cannot be evaluated using analytic methods. Consequently, values of these constants have been numerically computed. The methods of computation and the results are given in Appendices II, III, and IV. In § 4.2 we use these computed values to obtain

$$|\alpha_1(\beta)| \leq 1.71/\beta^{2/3} \leq 0.232 \quad \text{for} \quad \beta \geq 20 \quad .$$

We also find that

$$\nu_1(\beta) = \beta + (\frac{\beta}{2})^{1/3} a_1 e^{-2\pi i/3} + \theta_1(\beta)$$

where

$$|\theta_1(\beta)| \leq 1.775/\beta^{1/3} \quad \text{for} \quad \beta \geq 20 \quad ,$$

and

$$\nu_1(\beta) = \beta + \psi_1(\beta) \quad \text{where} \quad |\psi_1(\beta)| \leq 2.096\beta^{1/3} \quad \text{for} \quad \beta \geq 20.$$

We give a brief history of Watson poles below.

The problem of scalar diffraction of high frequency waves by a totally reflecting sphere is of physical interest [21]. If λ is the wavelength of the incident waves, a is the radius of the sphere, and $k = 2\pi/\lambda$, then that the waves are of high frequency relative to the size of the sphere is expressed by the conditions $ka \gg 1$ and $(ka)^{1/3} \gg 1$. Such a problem might arise if the wave function is the velocity potential of sound waves and it is scattered by an accoustically soft sphere or if the Schrödinger wave function is diffracted by a hard core potential in quantum mechanics [21, pp. 23-24].

For high frequencies a large number of terms of the partial wave series solution is needed to obtain a good approximation to the total field [21, p. 24]. In order to avoid this difficulty, Watson [35], in1918, transformed the partial wave series

$$\psi(r,\theta) = \frac{1}{2}\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}[h_{\ell}^{2}(\rho) + S_{\ell}(\beta)h_{\ell}^{1}(\rho)]P_{\ell}(\cos\theta) \quad ,$$

where $\beta = ka$, $\rho = kr$, $\psi(a,\theta) = 0$, h_{ℓ}^{j} are the spherical Hankel functions, and $S_{\ell}(\beta) = -h_{\ell}^{2}(\beta)/h_{\ell}^{1}(\beta)$, into an integral

$$\psi(r,\theta) = \int_{C}[f(\nu,\beta,\rho)P_{\nu-1/2}(\cos\theta)e^{-i\nu\pi/2}\nu/\cos\pi\nu]d\nu \quad ,$$

where

$$f(\nu,\beta,\rho) = [(\pi/2\rho)^{1/2}e^{-\pi i/4}/2][H_{\nu}^{2}(\rho) - \frac{H_{\nu}^{2}(\beta)}{H_{\nu}^{1}(\beta)}H_{\nu}^{1}(\rho)] \quad .$$

After using a symmetry argument to change the contour C, Watson changed the integral into the residue series

$$\psi(r,\theta) = \sum_{m=0}^{\infty}(-1)^{m}2\pi e^{i\pi/4}(\pi/2\rho)^{1/2}\sum_{n=1}^{\infty}\nu_{n}\exp[i\pi\nu_{n}(2m+1/2)] \cdot$$

$$\cdot \frac{H_{\nu_n}^{1}(\rho)H_{\nu_n}^{1}(\beta)P_{\nu_n-1/2}(\cos\theta)}{[\frac{\partial}{\partial\nu}H_{\nu}^{1}(\beta)]_{\nu=\nu_n}} \quad .$$

In the above integral, the poles of the integrand are the poles of $S(\nu,\beta) = -H_{\nu}^{2}(\beta)/H_{\nu}^{1}(\beta)$, which in turn are the zeros $\nu_{n}(\beta)$ of $H_{\nu}^{1}(\beta)$ as a function of the order ν. Since $H_{-\nu}^{1}(\beta) = e^{i\nu\pi}H_{\nu}^{1}(\beta)$, the zeros ν_{n} are symmetrically distributed with respect to the origin. However, the zeros contained inside

the curve C are those in the upper half plane. There are an infinite number of these zeros and they are in the first quadrant close to the curve $\mathrm{Re}[(\nu^2 - \beta^2)^{1/2} - \nu \operatorname{arccosh}(\nu/\beta)] = 0$ [21, p. 32]. This curve cuts the real axis at $\nu = \beta$, at an angle of $\pi/3$. We will study ν_n near the origin as the first terms are the dominant terms of the residue series. These same poles have also been used by Regge and others in recent quantum mechanical studies [30].

Because of the importance of these poles in the residue series, numerous studies have been made concerning them. Perhaps the first asymptotic approximation of the zeros of $H_\nu^1(\beta)$ as a function of the order ν was found by van der Pol and Bremmer [5] using the Debye expansion for the Bessel function.

Franz [10] used the Sommerfeld integral representation for the Hankel function and obtained an asymptotic approximation by using the saddlepoint method. From this approximation, he found essentially the following Watson pole asymptotic expansion

$$\nu_n(\beta) = \beta + (\beta/2)^{1/3} a_n e^{-2\pi i/3} + (\frac{2}{\beta})^{1/3} \frac{a_n^2 e^{-4\pi i/3}}{60} + O(\beta^{-1}) \quad ,$$

where a_n is the nth negative zero of the Airy function Ai(x).

Next Magnus and Kotin [19] did an extensive theoretical study of these zeros and obtained results such as the simplicity

of the zeros, the existence of infinitely many zeros, and the asymptotic behavior of the zeros for $|\nu|$ large. Later, Keller, Rubinow, and Goldstein [18] obtained asymptotic expansions of $\nu_n(\beta)$ for large and small values of β and also for large values of n. Their expansion for large β is essentially the Watson pole asymptotic expansion given above. In addition, they computed approximate values for certain zeros for specific values of β. However, neither they nor anyone else has obtained a priori error bounds for these approximations to the zeros.

Cochran [6] also considers the zeros of $H_\nu^1(\beta)$ as a function of ν. Among other things, he uses Olver's uniform asymptotic expansion [22, § 4] for $H_\nu^1(\nu z)$ and Taylor's theorem to obtain the Watson pole asymptotic expansion given above. He states that the method can be extended to find as many terms in the Watson pole asymptotic expansion as desired.

It is desirable to determine a priori error bounds for the Watson pole asymptotic expansion given above in order to obtain error bounds on Watson's approximation to the total field in the scattering problem. In the shadow region $a/r > \sin\theta$, various approximations yield

$$\psi(r,\theta) \sim \frac{e^{-\pi i/6}}{(2\pi)^{1/2}} \left(\frac{\beta}{2}\right)^{1/3} \left(\frac{a}{r^2-a^2}\right)^{1/4} \frac{e^{ik(r^2-a^2)^{1/2}}}{(kr\sin\theta)^{1/2}} \cdot$$

$$\cdot \sum_{m=0}^{M} (-1)^m \sum_{n=1}^{N} \frac{\left[e^{i\nu_n \gamma_m + i\pi/4} + e^{i\nu_n \delta_m - i\pi/4}\right]}{[Ai'(a_n)]^2} \quad ,$$

where $\gamma_m = \arcsin(a/r) - \theta + 2m\pi$ and

$\delta_m = \arcsin(a/r) + \theta + 2m\pi$ [21, p. 38] .

In particular, ν_n is approximated six times in obtaining the above shadow region expansion. The approximations used are $\nu_n \sim \beta$ and $\nu_n \sim \beta + (\beta/2)^{1/3} a_n e^{-2\pi i/3}$. Obtaining a priori bounds on these approximations would be an important first step in determining the accuracy of the shadow region approximation given above.

§ 4.1. The Watson Pole Asymptotic Approximation

In this section Corollary 2.1.2 is applied to the asymptotic approximation $H_\nu^1(\beta)$ from Theorem 3.2.1 to obtain asymptotic approximations for the zeros $\nu_n(\beta)$ of $H_\nu^1(\beta)$ as a function of the order ν (these zeros are called Watson poles). The principal Watson pole asymptotic approximation derived here is

$$\nu_n(\beta) = \beta + \left(\frac{\beta}{2}\right)^{1/3} a_n e^{-2\pi i/3} + \left(\frac{2}{\beta}\right)^{1/3} \frac{a_n^2 e^{-4\pi i/3}}{60} + \alpha_n(\beta) \quad ,$$

where a bound on $\alpha_n(\beta)$ in terms of several constants is given in Theorem 4.1.1.

We choose β (real and positive) and u (complex) as the two independent variables; however, we shall also use the other variables ν, z, and ζ which depend on u and β according to the equations $u = \nu^{2/3} e^{2\pi i/3} \zeta$, $\beta = \nu z$, $\zeta (\frac{d\zeta}{dz})^2 = \frac{1 - z^2}{z^2}$, and $z(0) = 1$. If $\rho < |a_n| 3^{1/2}/2$, where a_n is the nth negative zero of the Airy function Ai(u), then the entire disk $|u - a_n| \leq \rho$ is contained in the sector such that $|\arg(-u)| < \pi/3$. If we assume $0 \leq \arg \nu \leq \pi/2$, then from Theorem 3.2.1 we have

$$(4.1.1) \qquad \frac{\nu^{1/3} e^{\pi i/3}}{2^{3/2}} \left(\frac{1 - z^2}{\zeta}\right)^{1/4} H_\nu^1(\beta) = \mathrm{Ai}(u) + \epsilon_2(u, \beta) \quad ,$$

where

$$|\epsilon_2(u, \beta)| \leq (\nu_2/\nu_1)[e^{2\nu_1 F_2(u,\beta)} - 1] E^{-1}(u) M_j(u)$$

and

$$F_2(u, \beta) = \frac{|z|}{\beta} \int_{\zeta e^{2\pi i/3}}^{\infty} |s^{-1/2} F(s e^{-2\pi i/3})| \, |ds| \quad .$$

In our application of Corollary 2.1.2, we take $f(u, \beta) = \left(\frac{1 - z^2}{\zeta}\right)^{1/4} \frac{e^{\pi i/3} \nu^{1/3}}{2^{3/2}} H_\nu^1(\beta)$. It is clear

that the continuous dependence of f and ϵ on the parameter β does not affect the application of the corollary. We need to show that $|u - a_n| \leq \rho$ implies $f(u, \beta)$ is holomorphic. If $|u - a_n| \leq \rho$ and $\rho < 3^{1/2}|a_n|/2$, then $|\arg(-u)| < \pi/3$. We now show that the rays $e^{\pm \pi i/3}$ are not in that part of the ζ plane where $|\arg(-u)| = |\arg(-\frac{\beta^{2/3} e^{2\pi i/3} \zeta}{z^{2/3}})| < \pi/3$.

If $\zeta = r e^{\pi i/3}$, then $z = Re^{-\pi i/2}$ [22, § 4], and $\arg(\frac{-\beta^{2/3} e^{2\pi i/3} r e^{\pi i/3}}{R^{2/3} e^{-\pi i/3}}) = \pi/3$. If $\zeta = r e^{-\pi i/3}$, then $z = Re^{\pi i/2}$ and $\arg(\frac{-\beta^{2/3} e^{2\pi i/3} r e^{-\pi i/3}}{R^{2/3} e^{\pi i/3}}) = \pi$. Thus z is a holomorphic function of ζ in D, the ζ plane cut from $(3\pi/2)^{2/3}$ to ∞ along the rays $e^{\pm \pi i/3}$ [22, § 4].

Since z is never zero or negative, $z^{-2/3}$ is also a holomorphic function of z and, consequently, of ζ. Thus $u = \nu^{2/3} e^{2\pi i/3} \zeta = \beta^{2/3} e^{2\pi i/3} \zeta / z^{2/3}$ is a holomorphic function of ζ. Since $\frac{du}{d\zeta} = \frac{\beta^{2/3} e^{2\pi i/3}}{z^{2/3}} (1 + \frac{2\zeta^{3/2}}{3\sqrt{1-z^2}}) \neq 0$, the inverse function theorem implies that ζ is a holomorphic function of u. Therefore, z and ν are also holomorphic functions of u. Now $(1 - z^2)/\zeta$ is never zero or negative in D so that $((1 - z^2)/\zeta)^{1/4}$ is a holomorphic function of ζ and, consequently, of u. Moreover, $H_\nu^1(\beta)$ is an entire function and $\nu^{1/3}$ is

holomorphic for $0 \leq \arg \nu \leq \pi/2$. Thus $f(u, \beta)$ is a holomorphic function of u for $|u - a_n| \leq \rho < 3^{1/2}|a_n|/2$.

Since $E(\beta) = \max_{|u-a_n| \leq \rho} |\epsilon_2(u, \beta)|$ is a decreasing function of β, we choose β large enough so that $m > E(\beta)$. Then by Corollary 2.1.2, there exists a simple zero c of $f(u, \beta)$ such that $|c - a_n| \leq \rho$ and $|c - a_n| \leq \frac{E(\beta)\rho^2 M'}{m(m - E(\beta))}$. Since the expression $(\frac{1 - z^2}{\zeta})^{1/4} \frac{e^{\pi i/3} \nu^{1/3}}{2^{3/2}}$ is never zero for $\zeta \in D$, c must be the desired zero of $H_\nu^1(\beta)$ as a function of u. If we let $c = \nu^{2/3} e^{2\pi i/3} b$, then b is a zero of $H_\nu^1(\beta)$ as a function of ζ and

$$b = (a_n + \delta)\, e^{-2\pi i/3} z^{2/3} / \beta^{2/3} \tag{4.1.2}$$

where $|\delta| \leq \rho$ and

$$|\delta| \leq \frac{E(\beta)\, \rho^2 M'}{m(m - E(\beta))} \,. \tag{4.1.3}$$

We now obtain an expression for the zero of $H_\nu^1(\beta)$ as a function of order by using $\nu_n(\beta) = \beta / z(b)$.

From [22, § 4], $z(\zeta) = 1 + 2^{-1/3}\zeta + \frac{3}{10} 2^{-2/3} \zeta^2 + \frac{1}{700} \zeta^3 + \ldots$.

From this we find that

$$\frac{1}{z(\zeta)} = 1 + 2^{-1/3}\zeta + \frac{7}{10} 2^{-2/3} \zeta^2 + \frac{139}{700} \zeta^3 + \ldots + d_{n-1} \zeta^{n-1} + \zeta^n R_n(\zeta) \,, \tag{4.1.4}$$

where

$$(4.1.5) \qquad R_n(\zeta) = \frac{1}{2\pi i} \int_C \frac{dw}{z(w)w^n(w - \zeta)} \quad ,$$

C being a circle around the origin with radius $R < (3\pi/2)^{2/3}$;

and

$$(4.1.6) \qquad z^{2/3}(\zeta) = 1 - \frac{2^{2/3}}{3}\zeta + \frac{2}{45}2^{1/3}\zeta^2 + \ldots + b_{n-1}\zeta^{n-1} + \zeta^n S_n(\zeta) \quad ,$$

where

$$(4.1.7) \qquad S_n(\zeta) = \frac{1}{2\pi i} \int_C \frac{z^{2/3}(w)dw}{w^n(w - \zeta)} \quad ,$$

with C as above [2, p. 101]. If we use $\nu_n = \beta/z(b)$ and (4.1.4), then

$$\nu_n(\beta) = \beta + 2^{-1/3}\beta b + \frac{7}{10}2^{-2/3}\beta b^2 + \frac{139}{700}\beta b^3 + \beta b^4 R_4(b) \quad .$$

If we use (4.1.2) to replace one b in the second through fourth terms in the expression above, then

$$\nu_n(\beta) = \beta + \frac{\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}z^{2/3}}{2^{1/3}} + \frac{7\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}z^{2/3}b}{10(2^{2/3})}$$

$$+ \frac{139\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}z^{2/3}b^2}{700} + \beta b^4 R_4(b) \quad .$$

If we now use (4.1.6) for $z^{2/3}(b)$, then

$$\nu_n(\beta) = \beta + \frac{\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}}{2^{1/3}} - \frac{2^{1/3}\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b}{3}$$

$$+ \frac{2\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b^2}{45} + \frac{\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b^3 S_3(b)}{2^{1/3}}$$

$$+ \frac{7\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b}{10(2^{2/3})} - \frac{7\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b^2}{30}$$

$$+ \frac{7\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b^3 S_2(b)}{10(2^{2/3})} + \frac{139\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b^2}{700}$$

$$+ \frac{139\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b^3 S_1(b)}{700} + \beta b^4 R_4(b)$$

$$= \beta + \frac{\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}}{2^{1/3}} + \frac{2^{1/3}\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b}{60}$$

$$+ \frac{61\,\beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b^2}{6300}$$

$$+ \beta^{1/3}(a_n + \delta)e^{-2\pi i/3}b^3\left(\frac{S_3}{2^{1/3}} + \frac{7S_2}{10(2^{2/3})} + \frac{139S_1}{700}\right)$$

$$+ \beta b^4 R_4(b) \quad .$$

If we use (4.1.2) again to replace one b in the third and fourth terms and use (4.1.6) for $z^{2/3}$ again, then

$$(4.1.8)\quad \nu_n(\beta) = \beta + \frac{\beta^{1/3}(a_n+\delta)e^{-2\pi i/3}}{2^{1/3}} + \frac{2^{1/3}(a_n+\delta)^2 e^{-4\pi i/3}}{60\,\beta^{1/3}}$$

$$- \frac{(a_n+\delta)^2 e^{-4\pi i/3} b}{700\,\beta^{1/3}} + \frac{(a_n+\delta)^2 e^{-4\pi i/3} b^2\, 2^{1/3} S_2}{\beta^{1/3}}\left(\frac{1}{60} + \frac{61\,S_1}{6300}\right)$$

$$+ \beta^{1/3}(a_n+\delta)e^{-2\pi i/3} b^3\left(\frac{S_3}{2^{1/3}} + \frac{7S_2}{10(2^{2/3})} + \frac{139\,S_1}{700}\right)$$

$$+ \beta\, b^4 R_4(b) \quad .$$

Using (4.1.8) we obtain the lemma below.

<u>Lemma</u> 4.1.1. Suppose $0 \le \arg\nu \le \pi/2$, $M' = \max_\theta |\operatorname{Ai}'(a_n + \rho e^{i\theta})|$, and $\rho < 3^{1/2}|a_n|/2$ where a_n is the nth negative zero of the Airy function $\operatorname{Ai}(u)$. Suppose β is large enough so that

$$m = \min_\theta |\operatorname{Ai}(a_n + \rho e^{i\theta})| > E(\beta) = \max_\theta |\epsilon_2(a_n + \rho e^{i\theta}, \beta)|$$

where $|\epsilon_2(u,\beta)| \le (\nu_2/\nu_1)[e^{2\nu_1 F_2(u,\beta)} - 1]E^{-1}(u)M_j(u)$ and

$$F_2(u,\beta) = \frac{|z|}{\beta}\int_{\zeta e^{2\pi i/3}}^{\infty} |s^{-1/2}F(s e^{-2\pi i/3})|\,|ds| \quad .$$

Then there exists a simple zero ν_n of $H_\nu^1(\beta)$ as a function of the order ν such that

$$(4.1.9)\qquad \nu_n(\beta) = \beta + (\frac{\beta}{2})^{1/3} a_n e^{-2\pi i/3} + (\frac{2}{\beta})^{1/3} \frac{a_n^2 e^{-4\pi i/3}}{60} + \alpha_n(\beta)$$

where

$$(4.1.10)\qquad |\alpha_n(\beta)| \le |\frac{\delta\beta^{1/3}}{2^{1/3}}| + |\frac{2^{1/3}(2a_n\delta + \delta^2)}{60\,\beta^{1/3}}| + |\frac{(a_n+\delta)^2 b}{700\,\beta^{1/3}}|$$

$$+ \frac{|(a_n+\delta)|^2|b|^2}{\beta^{1/3}}(\frac{2^{1/3}|S_2|}{60} + \frac{61|S_1|}{6300})$$

$$+ \beta^{1/3}|a_n+\delta|\,|b|^3(\frac{|S_3|}{2^{1/3}} + \frac{7|S_2|}{10(2^{2/3})} + \frac{139|S_1|}{700})$$

$$+ \beta|b|^4|R_4(b)| \quad .$$

We now define some quantities which are useful in simplifying the hypotheses of Lemma 4.1.1. Let

$$(4.1.11)\qquad E^{-1} = \max_\theta |\exp(\frac{-2}{3}(a_n + \rho e^{i\theta})^{3/2})| \quad ,$$

$$(4.1.12)\qquad M_j = \max_\theta |M_j(a_n + \rho e^{i\theta})| \quad ,$$

$$(4.1.13)\qquad P_r = \max_{|\zeta|=r} |z(\zeta)| \quad ,$$

$$(4.1.14)\qquad Q_r = \min_{|\zeta|=r} |z(\zeta)| \quad ,$$

and

$$(4.1.15)\qquad I_r = \max_{|\zeta|\le r} \int_{\zeta e^{2\pi i/3}}^{\infty} |s^{-1/2} F(s e^{-2\pi i/3})|\,|ds| \quad .$$

Assume $|\zeta/z^{2/3}| \le (|a_n| + \rho)/\beta_0^{2/3}$ implies $|\zeta| \le r$.

If $\beta \ge \beta_0$, then $|\zeta| \le r$ and

$$(4.1.16)\qquad E(\beta) \le E(\beta_0) \le \max_{|\zeta|=r} |\epsilon_2(u,\beta_0)| \le [e^{2\nu_1 P_r I_r/\beta_0} - 1]E^{-1}M_j$$

since $\nu_2 \le \nu_1$ from §3.3. Thus $E(\beta) < m$ if $\beta \ge \beta_0$ where β_0 and its corresponding r are chosen so that $[e^{2\nu_1 P_r I_r/\beta_0} - 1]E^{-1}M_j < m$. Now

$$E(\beta) \le [e^{2\nu_1 P_r I_r/\beta} - 1]E^{-1}M_j$$

$$\le \frac{2\nu_1 P_r I_r E^{-1} M_j}{\beta(1 - 2\nu_1 P_r I_r/\beta_0)} \quad ,$$

so that

$$(4.1.17)\qquad |\delta| \le \frac{E(\beta)\rho^2 M'}{m(m - E(\beta))} \le \frac{2\nu_1 P_r I_r E^{-1} M_j \rho^2 M'}{\beta m(1 - 2\nu_1 P_r I_r/\beta_0)(m - [e^{2\nu_1 P_r I_r/\beta_0} - 1]E^{-1}M_j)}$$

$$= \frac{D}{\beta} \quad .$$

Bounds on $|S_n|$ and $|R_n|$ in (4.1.10) are found by using (4.1.7) and (4.1.5) to obtain

$$(4.1.18)\qquad |S_n(b)| \le \left|\frac{1}{2\pi i}\int_C \frac{z^{2/3}(w)dw}{w^n(w-b)}\right|$$

$$\le \frac{P_R^{2/3}}{R^{n-1}(R-r)} \quad ,$$

where $r < R < (3\pi/2)^{2/3}$, and

$$(4.1.19)\qquad |R_n(b)| \le \left|\frac{1}{2\pi i}\int_C \frac{dw}{z(w)w^n(w-b)}\right|$$

$$\le \frac{1}{Q_R R^{n-1}(R-r)}$$

where $r < R < (3\pi/2)^{2/3}$.

Using the definitions above and Lemma 4.1.1, we obtain the theorem below.

Theorem 4.1.1. Suppose $0 \le \arg \nu \le \pi/2$, $\rho < 3^{1/2}|a_n|/2$ and $M' = \max_\theta |Ai'(a_n + \rho e^{i\theta})|$. Suppose $\beta \ge \beta_0$ where $|\zeta/z^{2/3}| \le (|a_n| + \rho)/\beta_0^{2/3}$ implies $|\zeta| \le r$, and $[e^{2\nu_1 P_r I_r/\beta_0} - 1]E^{-1}M_j < m = \min_\theta |Ai(a_n + \rho e^{i\theta})|$. Then there exists a simple zero ν_n of $H_\nu^1(\beta)$ as a function of the order ν such that

$$(4.1.20)\qquad \nu_n(\beta) = \beta + \left(\frac{\beta}{2}\right)^{1/3} a_n e^{-2\pi i/3} + \left(\frac{2}{\beta}\right)^{1/3} \frac{a_n^2 e^{-4\pi i/3}}{60} + \alpha_n(\beta)$$

where

$$(4.1.21)\qquad |\alpha_n(\beta)| \le \frac{D}{2^{1/3}\beta^{2/3}} + \frac{2^{1/3}(2|a_n| + \rho)D}{60\beta^{4/3}} + \frac{(|a_n| + \rho)^3 P_r^{2/3}}{700\beta}$$

$$+ \frac{(|a_n| + \rho)^4 P_r^{4/3}}{\beta^{5/3}}\left(\frac{2^{1/3}|S_2|}{60} + \frac{61|S_1|}{6300}\right)$$

$$+ \frac{(|a_n| + \rho)^4 P_r^{2}}{\beta^{5/3}}\left(\frac{|S_3|}{2^{1/3}} + \frac{7|S_2|}{10(2^{2/3})} + \frac{139|S_1|}{700}\right)$$

$$+ \frac{(|a_n| + \rho)^4 P_r^{8/3}}{\beta^{5/3}}|R_4| \quad .$$

We remark that the bound on $\alpha_n(\beta)$ given by (4.1.21) is an asymptotic bound of the order $\beta^{-2/3}$; however, it is not of the order β^{-1} of the next term. An asymptotic error bound which is of the order of the next term is given in the corollary below.

Corollary 4.1.1. If the hypotheses of Theorem 4.1.1 are satisfied, then there exists a simple zero ν_n of $H_\nu^1(\beta)$ as a function of the order ν such that

$$(4.1.22)\qquad \nu_n = \beta + (\beta/2)^{1/3} a_n e^{-2\pi i/3} + \theta_n(\beta)$$

where

$$(4.1.23)\qquad |\theta_n(\beta)| \le \frac{1}{\beta^{1/3}}\Big[\frac{2^{1/3}|a_n|^2}{60} + \frac{D}{2^{1/3}\beta_0^{1/3}} + \frac{2^{1/3}(2|a_n| + \rho)D}{60\beta_0}$$

$$+ \frac{(|a_n| + \rho)^3 P_r^{2/3}}{700\,\beta_0^{2/3}} + \frac{(|a_n| + \rho)^4 P_r^{4/3}}{\beta_0^{4/3}}\Big(\frac{2^{1/3}|S_2|}{60} + \frac{61|S_1|}{6300}\Big)$$

$$+ \frac{(|a_n| + \rho)^4 P_r^2}{\beta_0^{4/3}}\Big(\frac{|S_3|}{2^{1/3}} + \frac{7|S_2|}{10(2^{2/3})} + \frac{139|S_1|}{700}\Big)$$

$$+ \frac{(|a_n| + \rho)^4 P_r^{8/3}}{\beta_0^{4/3}}|R_4|\Big] .$$

The proof of the theorem below is similar to the proof of Theorem 4.1.1.

Theorem 4.1.2. If the hypotheses of Theorem 4.1.1 are satisfied, then there exists a simple zero ν_n of $H_\nu^1(\beta)$ as a function of order ν such that

$$\nu_n(\beta) = \beta + \Big(\frac{\beta}{2}\Big)^{1/3} a_n e^{-2\pi i/3} + \Big(\frac{2}{\beta}\Big)^{1/3}\frac{a_n^2 e^{-4\pi i/3}}{60} + \alpha_n(\beta)$$

where

$$|\alpha_n(\beta)| \le \frac{D}{2^{1/3}\beta^{2/3}} + \frac{2^{1/3}(2|a_n| + \rho)D}{60\beta^{4/3}} + \frac{2^{1/3}(|a_n| + \rho)^3 P_r^{2/3}|S_1|}{60\,\beta^{4/3}}$$

$$+ \frac{(|a_n| + \rho)^3 P_r^{4/3}}{\beta}\Big(\frac{|S_2|}{2^{1/3}} + \frac{7|S_1|}{10(2^{2/3})}\Big) + \frac{(|a_n| + \rho)^3 P_r^2 |R_3|}{\beta} .$$

Even though the bound on $\alpha_n(\beta)$ given here is simpler, the bound in Theorem 4.1.1 is better because more cancellation occurs among the terms before they become error terms. Other expressions which bound $\alpha_n(\beta)$ can be found; however, (4.1.21) turns out to be a reasonable compromise between smaller but more complicated bounds and simpler but larger bounds.

§ 4.2. The First Watson Pole Approximation

In this section we use computed values of the constants occurring in Theorem 4.1.1 to find error bounds for the first Watson pole asymptotic approximations. Some of the principal results are:

$$\nu_1(\beta) = \beta + (\frac{\beta}{2})^{1/3} a_1 e^{-2\pi i/3} + (\frac{2}{\beta})^{1/3} \frac{a_1^2 e^{-4\pi i/3}}{60} + \alpha_1(\beta) \quad ,$$

where $|\alpha_1(\beta)| \leq 1.71/\beta^{2/3} \leq 0.232$ for $\beta \geq 20$,

$$\nu_1(\beta) = \beta + (\beta/2)^{1/3} a_1 e^{-2\pi i/3} + \theta_1(\beta) \quad ,$$

where $|\theta_1(\beta)| \leq 1.775/\beta^{1/3}$ for $\beta \geq 20$, and

$$\nu_1(\beta) = \beta + \psi_1(\beta)$$

where $|\psi_1(\beta)| \leq 2.096\beta^{1/3}$ for $\beta \geq 20$. We emphasize that each of the above bounds is not rigorous in the sense that each is based on various numerically computed constants.

The first zero of $Ai(u)$ is $a_1 = -2.33810$ [1, p. 478]. If we choose $\rho = 0.1$, then $0.1 \le \rho < 3^{1/2}|a_1|/2 = 2.02$. We use these values in Theorem 4.1.1 to obtain the theorem below.

Theorem 4.2.1. Suppose $0 \le \arg\nu \le \pi/2$ and suppose $\beta \ge \beta_0$ where $|\xi/z^{2/3}| \le 2.438/\beta_0^{2/3}$ implies $|\xi| \le r$ and $[e^{2\nu_1 P_r I_r/\beta_0} - 1]E^{-1}M_j < m = \min_\theta |Ai(a_1 + 0.1\, e^{i\theta})|$.

Then there exists a simple zero ν_1 of $H_\nu^1(\beta)$ as a function of the order ν such that

$$\nu_1(\beta) = \beta + (\frac{\beta}{2})^{1/3} a_1 e^{-2\pi i/3} + (\frac{2}{\beta})^{1/3} \frac{a_1^2 e^{-4\pi i/3}}{60} + \alpha_1(\beta) \tag{4.2.1}$$

$$= \beta + 1.856\, \beta^{1/3} e^{\pi i/3} + 1.145\, e^{2\pi i/3}/\beta^{1/3} + \alpha_1(\beta) \quad ,$$

where

$$|\alpha_1(\beta)| \le \frac{0.794\, D}{\beta^{2/3}} + \frac{0.101\, D}{\beta^{4/3}} + 0.0207\, \frac{P_r^{2/3}}{\beta} \tag{4.2.2}$$

$$+ \frac{35.4 P_r^{4/3}}{\beta^{5/3}} (0.0210|S_2| + 0.0097|S_1|)$$

$$+ \frac{35.4\, P_r^2}{\beta^{5/3}} (0.794|S_3| + 0.44|S_2| + 0.199|S_1|)$$

$$+ \frac{35.4\, P_r^{8/3}}{\beta^{5/3}} |R_4| \quad .$$

The corollary below is similar to Corollary 4.1.1.

Corollary 4.2.1. If the hypotheses of Theorem 4.2.1 are satisfied, then there exists a simple zero ν_1 of $H_\nu^1(\beta)$ as a function of the order ν such that

$$\nu_1(\beta) = \beta + (\beta/2)^{1/3} a_1 e^{-2\pi i/3} + \theta_1(\beta) \tag{4.2.3}$$

$$= \beta + 1.856\beta^{1/3} e^{\pi i/3} + \theta_1(\beta)$$

where

$$|\theta_1(\beta)| \leq 1.145/\beta^{1/3} + |\alpha_1(\beta)| \quad , \tag{4.2.4}$$

and

$$\nu_1(\beta) = \beta + \psi_1(\beta) \tag{4.2.5}$$

where

$$|\psi_1(\beta)| \leq 1.856\,\beta^{1/3} + 1.145/\beta^{1/3} + |\alpha_1(\beta)| \quad . \tag{4.2.6}$$

In order to use Theorem 4.2.1 and Corollary 4.2.1 to obtain numerical error bounds on the asymptotic approximations of the first Watson pole, we need to use some numerically computed quantities. In Appendix II, the values of $m = \min_\theta |\mathrm{Ai}(a_1 + \rho e^{i\theta})|$, $M' = \max_\theta |\mathrm{Ai}'(a_1 + \rho e^{i\theta})|$, $E^{-1} = \max_\theta |\exp(-\frac{2}{3}(a_1 + \rho e^{i\theta})^{3/2})|$, and $M_j = \max_\theta |M_j(a_1 + \rho e^{i\theta})|$ as defined in § 4.1 are tabulated for various values of ρ. Appendix III contains a table of values

of $P_r = \max_{|\zeta|=r} |z(\zeta)|$, $Q_r = \min_{|\zeta|=r} |z(\zeta)|$, $\min_{|\zeta|=r} |\zeta/z^{2/3}|$,

and $\max_{|\zeta|=r} |F(\zeta)|$ for various values of r. Appendix IV contains values of I_r defined by (4.1.15). In § 3.3 it was shown that $\nu_1 \leq 1.29$.

Assume $\beta_0 = 20$. From Appendix III,

$|\zeta/z^{2/3}| \leq 2.438/(20)^{2/3} = 0.33$ implies $|\zeta| \leq 0.4 = r$.

From Appendices III and IV, $P_{0.4} = 1.35$ and $I_{0.4} \leq 0.1302$.

From Appendix II, $m = 0.070$, $M' = 0.710$, and

$E^{-1}M_j = 0.292$ for $\rho = 0.1$. Thus

$E(\beta_0) \leq [e^{(2.58)(1.35)(0.1302)/20} - 1](0.292) = 0.006 < m = 0.070$,

and $D = 0.214$ if we use (4.1.17). From (4.1.19) with

$R = 2.6$ and $Q_R = 0.045$ from Appendix III,

$$|R_6| \leq \frac{1}{(0.045)(2.6)^5(2.6 - 0.4)} = 0.084 \quad .$$

Since $\frac{1}{z(\zeta)} = 1 + 2^{-1/3}\zeta + \frac{7}{10}2^{-2/3}\zeta^2 + \frac{139}{700}\zeta^3 + 0.161\zeta^4 + 0.080\zeta^5 + \ldots$,

then $|R_4| \leq 0.161 + 0.4(0.080) + (0.4)^2(R_6) \leq 0.206$. From

(4.1.18) with $R = 2.6$ and $P_R = 4.247$ from Appendix III,

$|S_4| \leq 0.068$. Since $z^{2/3}(\zeta) = 1 - \frac{2^{2/3}}{3}\zeta - \frac{2}{45}2^{1/3}\zeta^2 + \frac{836}{14175}\zeta^3 + \ldots$,

then

$$|S_3| \leq 836/14175 + 0.4|S_4| \leq 0.087 \quad ,$$

$$|S_2| \leq 2^{4/3}/45 + 0.4|S_3| \leq 0.074 \quad ,$$

and

$$|S_1| \leq 2^{2/3}/3 + 0.4|S_2| \leq 0.56 \quad .$$

If we use the above values of the constants in Theorem 4.2.1, then

$$|\alpha_1(\beta)| \leq 1.71/\beta^{2/3} \leq 0.232 \quad \text{for} \quad \beta \geq 20 \quad .$$

In Corollary 4.2.1,

$$|\theta_1(\beta)| \leq 1.775/\beta^{1/3} \quad \text{for} \quad \beta \geq 20$$

and

$$|\psi_1(\beta)| \leq 2.096\beta^{1/3} \quad \text{for} \quad \beta \geq 20 \quad .$$

One way to judge our error bounds is to find what percentage of the approximating value they represent. For example, if we use (4.2.1), then

$$\nu_1(20) = 23.925 + i\,7.060 + \alpha_1(20)$$

where $|\alpha_1(20)| \leq 0.232$. The relative error in this case is less than 1 per cent.

§ 4.3. Other Possible Applications

Many of the applications discussed in § 1.4 for real zeros are also possible for complex zeros. Corollary 2.1.1 could be used to study the other McMahon expansions for complex zeros of $J_\nu'(z)$, $Y_\nu(z)$, and $Y_\nu'(z)$ as given in [1, § 9.5]. Asymptotic approximations of zeros of cross products such as

$J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)$ and $J'_\nu(z)Y'_\nu(\lambda z) - J'_\nu(\lambda z)Y'_\nu(z)$ could also be studied [1, § 9.5]. Other functions which have complex zeros near the cosine zeros include Legendre, Coulomb Wave, and confluent hypergeometric functions.

Corollary 2.1.2 could be used to find error bounds on asymptotic approximations of complex zeros as a function of argument or order of all Bessel and Hankel functions. For example, the error bound on the asymptotic approximation of the first zero $j_{\nu,1}$ of $J_\nu(z)$ for complex ν could be found. The asymptotic approximation of $j_{\nu,1}$ was studied in § 1.3 for real ν. The uniform asymptotic expansion of zeros of $J_\nu(z)$ for large orders ν, which was mentioned in § 1.4, could also be studied.

Corollary 2.1.2 could also be used to find the zero approximations of $\frac{d}{d\beta} H^1_\nu(\beta)$ and $\frac{d}{d\beta} H^1_\nu(\beta) + iZ H^1_\nu(\beta)$ where Z is a constant [18], [6]. These zeros are poles in the problem of scattering by a cylinder. The zeros of Weber parabolic cylinder functions of large order could also be approximated by expression involving the zeros of Airy functions [1, § 19.26].

Other comparison functions in addition to the cosine and the Airy function $Ai(z)$ could be used in Theorem 2.1.1. For example, other Airy functions such as $Bi(z)$ and the derivatives $Ai'(z)$ or $Bi'(z)$ could be used. If the Bessel function of the first kind is used as a comparison function, then Jacobi, ultraspherical,

Legendre, and Laguerre polynomials have zeros which are asymptotic to functions involving zeros of $J_\nu(z)$ [1, § 22.16]. Another possible application of the theorem might be to the rth positive zero x_r of $M(a,b,x)$, Kummer's confluent hypergeometric function, which has the approximation $x_r \sim j^2_{b-1,r}/(2b - 4a)$, where $j_{b-1,r}$ is the rth positive zero of $J_{b-1}(x)$ [1, § 13.7].

Only some of the possible applications of Theorem 2.1.1 and its corollaries have been presented. The theorem is applicable to all cases concerning approximation of zeros of a function by zeros of a comparison function. If we are given an asymptotic approximation of a function by a comparison function with known error bounds, then we can find an asymptotic approximation of the zeros of the function by the zeros of the comparison function with error bounds.

APPENDIX I

We will show that $z''(\xi)$ is a non-decreasing function of ξ for $\xi \leq 0$ or $z \geq 1$, where $z(0) = 1$ and $z'(\xi) = -z(-\xi)^{1/2}/(z^2 - 1)^{1/2}$ from (1.3.1). We find that $z''(\xi) = z[(z^2 - 1)^{3/2} - 2(-\xi)^{3/2}]/2(-\xi)^{1/2}(z^2 - 1)^2$ and $z'''(\xi) = z[(z^2 - 1)^3 + 6(-\xi)^{3/2}(z^2 - 1)^{3/2} - 4(-\xi)^3(1 + 3z^2)]/4(-\xi)^{3/2}(z^2 - 1)^{7/2}$. Since $(-\xi)^{3/2} = 3[(z^2 - 1)^{1/2} - \text{arcsec } z]/2$ for $\xi \leq 0$ [1, 9.3.39], then $z'''(\xi) = z[(z^2 - 1)^3 - 18(z^2 - 1)(z^2 + 1) + 9(z^2 - 1)^{1/2}(5z^2 + 3)\text{arcsec } z - 9(3z^2 + 1)\text{arcsec}^2 z]/4(-\xi)^{3/2}(z^2 - 1)^{7/2}$. If $x = (z^2 - 1)^{1/2}$ and $g_1(z)$ is the expression contained in the brackets in the above expression for $z'''(\xi)$, then arcsec z = arctan x [15, p. 295] and $g_1(x) = x^6 - 18x^4 - 36x^2 + (45x^3 + 72x)\arctan x - (27x^2 + 36)\arctan^2 x$. In the next paragraph we show that $g_1(x) \geq 0$ for $x \geq 0$, which implies $g_1(z) \geq 0$ for $z \geq 1$ so that $z'''(\xi) \geq 0$ for $\xi \leq 0$. Thus $z''(\xi)$ is a non-decreasing function for $\xi \leq 0$ so that $z''(b) \leq z''(\theta) \leq z''(0)$ for $b \leq \theta \leq 0$. The values of $z''(\xi)$ used in § 1.3 are $z''(0) = 0.378$, $z''(-0.50) = 0.366$, and $z''(-2.508) = 0.284$.

If the argument : $g_n'(x) \geq 0$ for $x \geq 0$ and $g_n(0) \geq 0$ imply $g_n(x) \geq 0$ for $x \geq 0$, which in turn implies $g_{n-1}'(x) \geq 0$ for $x \geq 0$, is applied for $n = 10, 9, \ldots, 2, 1$, where $g_n(x)$ and $g_{n-1}'(x)$ are given below, then $g_1(x) \geq 0$ for $x \geq 0$.

Now

$$g_1'(x) = 3x[2x^6 - 22x^4 - 33x^2 + (45x^3 + 51x)\arctan x - (18x^2 + 18)\arctan^2 x]/(1 + x^2)$$

$$= 3x\, g_2(x)/(1 + x^2) \quad ,$$

$$g_2'(x) = [12x^7 - 76x^5 - 109x^3 - 15x + (135x^4 + 150x^2 + 15)\arctan x$$

$$- (36x^3 + 36x)\arctan^2 x]/(1 + x^2)$$

$$= g_3(x)/(1 + x^2) \quad ,$$

$$g_3'(x) = 4[21x^6 - 95x^4 - 48x^2 + (135x^4 + 57x)\arctan x - (27x^2 + 9)\arctan^2 x]$$

$$= 4g_4(x) \quad ,$$

$$g_4'(x) = [126x^7 - 254x^5 - 341x^3 - 39x + (405x^4 + 408x^2 + 39)\arctan x$$

$$- (54x^3 + 54x)\arctan^2 x]/(1 + x^2)$$

$$= g_5(x)/(1 + x^2) \quad ,$$

$$g_5'(x) = 2[441x^8 - 194x^6 - 944x^4 - 327x^2 + (810x^5 + 1164x^3 + 354x)\arctan x$$

$$- (81x^4 + 108x^2 + 27)\arctan^2 x]/(1 + x^2) \quad ,$$

$$= 2g_6(x)/(1 + x^2) \quad ,$$

$$g_6'(x) = 2[1764x^7 - 582x^5 - 1483x^3 - 150x + (2025x^4 + 1665x^2 + 150)\arctan x$$

$$- (162x^3 + 108x)\arctan^2 x]$$

$$= 2g_7(x) \quad ,$$

$$g_7'(x) = 2[6174x^8 + 4719x^6 - 2667x^4 - 1467x^2 + (4050x$$

$$+ (4050x^5 + 5553x^3 + 1557x)\arctan x$$

$$- (243x^4 + 297x^2 + 54)\arctan^2 x]/(1 + x^2)$$

$$= 2g_8(x)/(1 + x^2) \quad ,$$

$$g_8'(x) = [49392x^9 + 77706x^7 + 21696x^5 - 8049x^3$$

$$- 1377x + (20250x^6 + 36423x^4 + 17622x^2 + 1449)\arctan x$$

$$- (972x^5 + 1566x^3 + 594x)\arctan^2 x]/(1 + x^2)$$

$$= g_9(x)/(1 + x^2) \quad ,$$

$$g_9'(x) = 2[222,264x^8 + 271,871x^6 + 64,365x^4 - 3987x^2 + 36$$

$$+ (60,750x^5 + 71,874x^3 + 17,028x)\arctan x$$

$$- (2430x^4 + 2349x^2 + 297)\arctan^2 x]$$

$$= 2g_{10}(x) \quad ,$$

and

$$g'_{10}(x) = [3556,\ 224x^9 + 6619,\ 876x^7 + 3700,\ 072x^5$$

$$+ 642,720x^3 + 18,108x + (607,500x^6$$

$$+ 1029,024x^4 + 455,904x^2 + 32,868)\arctan x$$

$$- (19,440x^5 + 28,836x^3 + 9396x)\arctan^2 x]/(1 + x^2)$$

$$\geq 0 \quad \text{for} \quad x \geq 0 \quad ,$$

since $x > \arctan x$ for $x \geq 0$.

APPENDIX II

From [1, § 10.4], $Ai(z) = c_1 f(z) - c_2 g(z)$ where

$c_1 = 1/3^{2/3}\Gamma(2/3)$, $c_2 = 1/3^{1/3}\Gamma(1/3)$, $f(z) = \sum_0^\infty 3^k(\frac{1}{3})_k \frac{z^{3k}}{(3k)!}$,

$g(z) = \sum_0^\infty 3^k(\frac{2}{3})_k \frac{z^{3k+1}}{(3k+1)!}$, $(\alpha + 1/3)_0 = 1$, and

$3^k(\alpha + 1/3)_k = (3\alpha + 1)(3\alpha + 4)\ldots(3\alpha + 3k - 2)$.

This power series expansion was used to compute the values of $m = \min_\theta |Ai(a_1 + \rho e^{i\theta})|$, $M' = \max_\theta |Ai'(a_1 + \rho e^{i\theta})|$, and $M_j = \max_{|z-a_1| \le \rho} M_j(z)$ for various values of ρ.

A bound on the remainder of the truncated power series was found and used to determine that ten terms of the power series yield at least three decimal accuracy on all of the computed quantities. Symmetry properties from § 1.3 were used to reduce the amount of computation. The numbers in the table below are lower bounds on m and upper bounds on M', E^{-1}, M_j, and $E^{-1}M_j$, where $E^{-1} = \max_\theta |\exp(-\frac{2}{3}(a_1 + \rho e^{i\theta})^{3/2})|$.

ρ	m	M'	E^{-1}	M_j	$E^{-1}M_j$
0.1	0.070	0.710	1.166	0.25	0.292
0.2	0.138	0.735	1.359	0.28	0.381
0.3	0.202	0.778	1.584	0.32	0.507
0.4	0.262	0.839	1.850	0.36	0.666
0.5	0.313	0.924	2.161	0.40	0.865
0.8	0.410	1.34	3.50	0.53	1.86

APPENDIX III

For various values of r, the values of

$$Q_r = \min_{|\zeta|=r} |z(\zeta)|, \quad P_r = \max_{|\zeta|=r} |z(\zeta)|, \quad \min_{|\zeta|=r} |\zeta/z^{2/3}|, \quad \text{and}$$

$\max_{|\zeta|=r} |F(\zeta)|$ were computed where $\zeta\left(\frac{d\zeta}{dz}\right)^2 = \frac{1 - z^2}{z^2}$,

$z(0) = 1$, and $F(\zeta) = \frac{5}{16\zeta^2} + \frac{z^2(z^2 + 4)}{(z^2 - 1)^3}$.

Given ζ, the equation $\alpha - \tanh \alpha = 2\zeta^{3/2}/3$ was solved for α to four decimal places by the method of false position and $z = \operatorname{sech} \alpha$ was found. Graphs of these transformations are given in [22, § 4]. The values of Q_r were assumed at ζ positive whereas the values of P_r and $\min_{|\zeta|=r} |\zeta/z^{2/3}|$ were assumed at ζ negative. The values in the table below are lower bounds on Q_r and $\min_{|\zeta|=r} |\zeta/z^{2/3}|$ and upper bounds on P_r and $\max_{|\zeta|=r} |F(\zeta)|$.

r	Q_r	P_r	$\min_{\lvert\zeta\rvert=r} \lvert\zeta/z^{2/3}\rvert$	$\max_{\lvert\zeta\rvert=r} \lvert F(\zeta)\rvert$
0.1	0.922	1.081	0.095	0.021
0.2	0.848	1.166	0.181	0.024
0.3	0.779	1.255	0.258	0.027
0.4	0.712	1.348	0.330	0.030
0.5	0.650	1.444	0.392	0.033
0.75	0.510	1.700	0.526	0.041
1.0	0.392	1.979	0.635	0.051
2.0	0.111	3.330	0.901	0.26
2.5	0.053	4.082		
2.6	0.045	4.247		

APPENDIX IV

Here we find bounds on $I_r = \max_{|\zeta| \le r} \int_{\zeta e^{2\pi i/3}}^{\infty} |s^{-1/2} F(s e^{-2\pi i/3})| \, |ds|$

for various values of r. Now

$$I_r \le \max_{|\zeta| \le r} \int_{\zeta e^{2\pi i/3}}^{0} |s^{-1/2} F(s e^{-2\pi i/3})| \, |ds| + \int_{0}^{\infty} |s^{-1/2} F(s e^{-2\pi i/3})| \, ds .$$

If $r = 1$, then

$$\max_{|\zeta| \le 1} \int_{\zeta e^{2\pi i/3}}^{0} |s^{-1/2} F(s e^{-2\pi i/3})| \, |ds|$$

$$\le \max_{|t|=0.25} |F(t)| \, 2t^{1/2} \Big]_0^{0.25} + \max_{|t|=0.50} |F(t)| \, 2t^{1/2} \Big]_{0.25}^{0.5}$$

$$+ \max_{|t|=0.75} |F(t)| \, 2t^{1/2} \Big]_{0.5}^{0.75} + \max_{|t|=1.0} |F(t)| \, 2t^{1/2} \Big]_{0.75}^{1.0} .$$

If $r = 0.5$, then only the first two terms on the right hand side of the inequality are needed. If $r = 0.25$, then only the first term is needed. If we use the values of $\max_{|\zeta|=r} |F(\zeta)|$ from Appendix III, then

$$\max_{|\zeta|\le r} \int_{\zeta e^{2\pi i/3}}^{0} |s^{-1/2}F(s e^{-2\pi i/3})|\,|ds| \le \begin{cases} 0.0662 & \text{if } r = 1.0 \\ 0.040 & \text{if } r = 0.5 \\ 0.026 & \text{if } r = 0.25 \end{cases}.$$

Since the integrand $|s^{-1/2}F(s e^{-2\pi i/3})| = \left|\frac{5}{16s^{5/2}} + \frac{s^{1/2}}{4}\,\frac{z^2(z^2+4)}{(z^2-1)^3}\right|$ is decreasing on $[0,\infty)$, we can bound the integral $\int_0^\infty |s^{-1/2}F(s e^{-2\pi i/3})|\,|ds|$ by an upper Riemann sum. As our partition, we choose $\{0, 0.1, 0.2, \ldots, 1.8, 1.9, 2.0, 3.0, \ldots 28, 29, 30, 40, 50, \ldots\}$. Thus

$$\int_0^\infty |s^{-1/2}F(s e^{-2\pi i/3})|\,ds \le \max_{0\le s\le 0.1} |F(s e^{-2\pi i/3})|\,2s^{1/2}\Big]_0^{0.1}$$

$$+ \sum_{n=1}^{19} 0.1 \max_{0.1\,n\le s\le 0.1(n+1)} |s^{-1/2}F(s e^{-2\pi i/3})|$$

$$+ \sum_{n=0}^{29} \max_{n\le s\le n+1} |s^{-1/2}F(s e^{-2\pi i/3})|$$

$$+ \sum_{n=0}^{\infty} 10 \max_{30+10\,n\le s\le 40+10\,n} |s^{-1/2}F(s e^{-2\pi i/3})|.$$

The values of the integrand $\left|\frac{5}{16s^{5/2}} + \frac{s^{1/2}z^2(z^2+4)}{4(z^2-1)^3}\right|$ were computed by using $\alpha - \tanh\alpha = -2s^{3/2}/3$ and $z = \operatorname{sech}\alpha$

as in Appendix III to find $z(s\,e^{-2\pi i/3})$. If we use these values, then

$$\int_0^{\infty} |s^{-1/2} F(s\,e^{-2\pi i/3})|\,ds \le 0.01065 + 0.03575 + 0.04255 + 0.00125$$
$$= 0.09020 \quad .$$

If we combine our two integral estimates, then

$$I_r = \max_{|\zeta| \le r} \int_{\zeta\, e^{2\pi i/3}}^{\infty} |s^{-1/2} F(s\,e^{-2\pi i/3})|\,|ds| \le \begin{cases} 0.1564 & \text{if } r = 1.0 \\ 0.1302 & \text{if } r = 0.5 \\ 0.1162 & \text{if } r = 0.25 \end{cases} .$$

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