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EXTREMAL RINGS AND

QUASICONFORMAL MAPPINGS IN 3-SPACE

by

Glen Douglas Anderson

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of the requirements for the degree of
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TABLE OF CONTENTS

	Page
ACKNOWLEDGMENTS	ii
I. INTRODUCTION	1
II. SYMMETRIZATION OF SPACE RINGS	9
1. Introduction	9
2. Notation	10
3. Admissible functions	10
4. Simple admissible functions	11
5. Special admissible functions	14
STEINER SYMMETRIZATION	
6. Steiner symmetrization of open and closed sets	18
7. Steiner symmetrization of rings	21
8. Surface area under Steiner symmetrization	23
9. Steiner symmetrization of functions	25
10. Equimeasurability	29
11. Proof of Theorem 2.1	32
SCHWARZ SYMMETRIZATION	
12. Schwarz symmetrization of open and closed sets	37
13. Schwarz symmetrization of rings	39
14. Surface area under Schwarz symmetrization	40
15. Schwarz symmetrization of functions	41
16. Equimeasurability	43
17. Proof of Theorem 2.2	45
18. Extremal space rings	47

	Page
III. ASYMPTOTIC BEHAVIOR OF EXTREMAL RINGS	54
1. Introduction	54
2. Proof of Theorem 3.1	55
3. Dilatations of a homeomorphism	66
4. Behavior of mod $R_{3,d}$ for small a	70
5. Behavior of mod $R_{3,s}$ for small a	75
6. Behavior of mod $R_{3,d}$ as a tends to 1	80
7. Behavior of mod $R_{3,s}$ as a tends to 1	86
IV. THE COEFFICIENTS OF ELLIPSOIDS	94
1. Introduction	94
2. Hyperbolic density	96
3. The prolate ellipsoid	97
4. The outer coefficient of the oblate ellipsoid	111
5. The inner coefficient of the oblate ellipsoid	117
6. The general ellipsoid	124
BIBLIOGRAPHY	129

Chapter I

INTRODUCTION

A famous theorem due to Riemann [1, p. 172] states that any simply-connected plane domain ([1]) whose boundary consists of more than one point can be mapped conformally onto the unit disk. Thus any two such domains may be mapped conformally onto each other, and we say that they are conformally equivalent. In fact, there are exactly three conformal equivalence classes of simply-connected plane domains--these have the unit disk, the entire plane, and the extended plane as canonical members.

For domains of higher connectivity the number of conformal equivalence classes is much greater. We focus our attention on doubly-connected domains, which are also called rings. That is, a plane ring is a finite ([1]) plane domain whose complement with respect to the extended plane consists of two components. It is well known that any such plane ring R_2 can be mapped conformally onto a circular annulus $0 \leq r_1 < |z| < r_2 \leq \infty$, and $0 < r_1 < r_2 < \infty$ if and only if neither of the two components of the complement reduces to a point.

([1]) **A plane domain is an open, connected set in the extended plane. A** domain is said to be finite if it does not contain the point at infinity. A domain will be called simply-connected if its complement is connected.

The ratio r_2/r_1 is unique ; that is, it is independent of the auxiliary conformal mapping [1, pp. 199-204] . Thus for each real number there is a class of conformally equivalent plane rings, and the ratio r_2/r_1 is a conformal invariant attached to the class of which R_2 is a member. We then define the modulus of the plane ring R_2 as

$$\text{mod } R_2 = \log r_2/r_1 . \tag{1.1}$$

The modulus of a plane ring has the characterization

(Cf. [4, p.225] or [12])

$$\inf_u \int_{R_2} |\nabla u|^2 \, d\sigma = \frac{2\pi}{\text{mod } R_2} , \tag{1.2}$$

where u ranges over all real-valued functions which are continuously differentiable in R_2 and have boundary values 0 and 1 on the respective boundary components. If neither component of the complement of R_2 is degenerate, then by Dirichlet's principle equality holds in (1.2) if

$$u(z) = \frac{\log |w(z)/r_1|}{\log r_2/r_1} ,$$

where $w(z)$ is a conformal mapping of R_2 onto $r_1 < |w| < r_2$. This alternative definition (1.2) of mod R_2 does not depend upon conformal

mappings, and hence suggests how to define the modulus of a ring in 3-space, where the number of domains conformally equivalent to the unit ball is very limited.

In 3-space the counterpart of the extended plane is the Möbius space $\bar{E}^3$, which consists of Euclidean 3-space E^3 plus the point at infinity. A space ring R_3 is a domain in E^3 whose complement with respect to the Möbius space $\bar{E}^3$ consists of two components, one of which is bounded.

The Riemann theorem fails very badly in 3-space. In fact, by Liouville's theorem [19], the only conformal mappings there are the Möbius transformations--i.e., compositions of similarity transformations and inversions. Hence the only domains in 3-space which are conformally equivalent to the unit ball are half spaces and balls. It follows, too, that the only space rings conformally equivalent to a spherical annulus are those bounded by two spheres or by a plane and a sphere.

For this reason it is unsatisfactory to seek a definition of mod R_3 corresponding to (1.1). Hence we generalize (1.2), using Loewner's definition [26] of conformal capacity,

$$\text{cap } R_3 = \inf_u \int_{R_3} |\nabla u|^3 \, d\omega, \tag{1.3}$$

where the infimum is taken over all real-valued functions u which are continuously differentiable in R_3 and have boundary values 0 on ∂C_0 and 1 on ∂C_1. Here C_0, C_1 are the components of the complement of R_3 and C_0 is bounded.

Next we define the modulus of R_3 as

$$\text{mod } R_3 = \left[\frac{4\pi}{\text{cap } R_3} \right]^{1/2}. \tag{1.4}$$

Then mod R_3 is a conformal invariant, i.e., invariant with respect to Möbius transformations, and the modulus of the spherical annulus $r_1 < |x| < r_2$ turns out to be $\log r_2/r_1$. For a proof of this see [14].

If the modulus of a ring R' maximizes the moduli of all rings R in a large class to which R' belongs, we will refer to R' as an extremal ring. For example, suppose $R = (R \cup C_0) - C_0$ is any space ring such that C_0 and $R \cup C_0$ have the same volume as the balls $|x| \leq r_1$ and $|x| < r_2$, respectively, and let R' be the spherical annulus $r_1 < |x| < r_2$. By means of point symmetrization Gehring [14] proved that mod $R \leq$ mod R'. In the same paper he solved other extremal problems by means of spherical symmetrization.

It appears that symmetrization is a particularly effective tool for solving such problems. In Chapter II of this paper we introduce Steiner and Schwarz symmetrizations of space rings and prove a pair of inequalities of the type

$$\text{mod } R \leq \text{mod } R' ,$$

where R' is a space ring obtained from the space ring R by means of one of these symmetrizations.

This work leads naturally to two extremal rings, which we label $R_{3,s}$ and $R_{3,d}$. For fixed a, $0 < a < 1$, the ring $R_{3,s}$ consists of the unit ball minus the slit $-a \leq x_1 \leq a$, $x_2 = x_3 = 0$, while $R_{3,d}$ is obtained from the unit ball by omission of a closed central disk of radius a (See Figure 1, page 51).

These rings have the following extremal properties (Cf. § 2.18). Let R be a space ring consisting of the unit ball minus a continuum C. If the diameter of C is at least $2a$, then $\text{mod } R \leq \text{mod } R_{3,s}$. If the projection of C on some diametral plane is at least πa^2 in area, then $\text{mod } R \leq \text{mod } R_{3,d}$.

Schwarz symmetrization also has the following interesting consequence (Cf. § 2.18). Suppose R_3 is a 3-space ring whose complementary components C_0, C_1 lie in a plane Π, say $x_3 = 0$, and that $R_2 = \Pi \cap R_3$ is a plane ring with the same complementary components. Suppose the areas of C_0 and $(R_2 \cup C_0)$ are πr_1^2 and πr_2^2, respectively. Then $\text{mod } R_3 \leq \text{mod } R_3'$, where R_3' is the space ring obtained from R_3 by replacing C_0 by the disk $|x| \leq r_1$, $x_3 = 0$ and C_1 by $|x| \geq r_2$, $x_3 = 0$.

In Chapter III we show that $\operatorname{mod} R_{3,d} \leq \operatorname{mod} R_2 \leq \operatorname{mod} R_{3,s}$, where now R_2 denotes the plane ring consisting of the unit disk minus the central slit $-a \leq x_1 \leq a$, $x_2 = 0$. In addition we investigate the asymptotic behavior of the moduli of these extremal rings as a tends to 0 and to 1. We discover that $\operatorname{mod} R_{3,s}$ and $\operatorname{mod} R_{3,d}$ behave essentially like the modulus of the spherical annulus $a < |x| < 1$ as a tends to 0. But as a tends to 1 the asymptotic behavior of these three rings differs markedly one from another. In particular, $\operatorname{mod} R_{3,d}$ tends to 0 much faster than $\operatorname{mod} R_{3,s}$.

Closely related to the theory of rings is that of quasiconformal mappings. Suppose f is a homeomorphism of Ω onto Ω', where Ω and Ω' represent domains in E^3. If R is a bounded space ring with $\bar{R} \subset \Omega$ then $R' = f[R]$ is a bounded space ring with $\bar{R}' \subset \Omega'$. Then the <u>inner</u> and <u>outer dilatations</u> of f are defined (Cf. § 3.3) as

$$K_I(f) = \sup_R \frac{\operatorname{mod} R}{\operatorname{mod} R'}, \quad K_O(f) = \sup_R \frac{\operatorname{mod} R'}{\operatorname{mod} R},$$

where the suprema are taken over all bounded rings R with $\bar{R} \subset \Omega$ for which $\operatorname{mod} R$ and $\operatorname{mod} R'$ are not both infinite. Clearly these dilatations are both 1 if f is a Möbius transformation; the converse is also true [13, Theorem 16]. If both are bounded by $K < \infty$, then f is called a K-<u>quasiconformal mapping.</u> A <u>quasiconformal mapping</u>

is one which is K-quasiconformal for some K (Cf. § 4.1). Plane quasiconformal mappings may be defined similarly [16].

Historically, plane quasiconformal mappings were first treated (under another name) in 1928 by H. Grötzsch in the solution of the following extremal problem : Of all mappings of one rectangle onto another with vertices corresponding, to find the mapping most nearly conformal, that is, with smallest dilatation. He determined that the extremal function was the natural affine mapping. Though the paper itself lay hidden in an obscure journal for many years, the class of functions which he considered later became an important analytic tool.

Quasiconformal mappings "are intimately and essentially, even surprisingly, connected with the most classical problems in the theory of analytic functions" , as Ahlfors [3] eloquently puts it. For example, Teichmüller's study [37] of Riemann surfaces established a direct connection between quasiconformal mappings and quadratic differentials. These mappings are also encountered in the theory of partial differential equations as solutions of Beltrami systems.

Suppose Ω is a domain in E^3 which can be mapped quasiconformally onto the unit ball B . We set

$$K_I(\Omega) = \inf_f K_I(f) , \quad K_O(\Omega) = \inf_f K_O(f) ,$$

where the infima are taken over all homeomorphisms f of Ω onto B (Cf. § 4.1) . These numbers are called the inner and outer coefficients of quasiconformality of Ω , respectively.

A quasiconformal mapping f of Ω onto B is called extremal for the inner coefficient if $K_I(f) = K_I(\Omega)$, and similarly for the outer coefficient. In the plane, a large class of extremal quasiconformal mappings have a particularly simple form ; locally they are the composition of a conformal mapping, an affine mapping, and a second conformal mapping. It is natural to ask if one of the extremal mappings of an ellipsoid onto a ball is the natural affine mapping.

In Chapter IV we show that the affine mapping of the prolate ellipsoid is not extremal for either coefficient. We also prove that for every ellipsoid the affine mapping is not extremal for the outer coefficient. In fact, for each of these cases we show that even the orders of the dilatations of the affine mappings are larger than the orders of the corresponding coefficients of the ellipsoids. The problem for the inner coefficient of the oblate ellipsoid appears to be much more difficult. For in this instance we prove that the inner dilatation of the affine mapping has the same order as that of the inner coefficient of the ellipsoid.

Chapter II

SYMMETRIZATION OF SPACE RINGS

1. Introduction.

Symmetrization is a geometric operation invented by Jacob Steiner and developed by Pólya and Szegö [30] . The purpose of this chapter is to obtain two upper bounds

$$\text{mod } R \leq \text{mod } R' \tag{2.1}$$

for the modulus of a bounded space ring R , where R' is a space ring obtained from R by means of symmetrization.

Two well-known kinds of symmetrization in the plane are the Steiner [21] and point [7] symmetrizations. Analogues of these in 3-space -- known as the Steiner and Schwarz symmetrizations [30] , respectively -- are obtained by performing the corresponding plane symmetrization in each plane normal to a fixed line, say the x_3 axis.

Two other symmetrizations in 3-space -- spherical [30] and point -- were studied in [14] by Gehring, who proved that (2.1) is valid for each of these methods. We shall show that Gehring's proofs may be modified to yield (2.1) also for the Steiner and Schwarz symmetrizations in 3-space.

In the final section of the chapter, we consider the application of

Schwarz symmetrization to a space ring R_3 of the following type : Both components of the complement of R_3 lie in a plane Π, and $R_2 = \Pi \cap R_3$ is a plane ring with the same complementary components. In conclusion, we derive the extremal properties of the two extremal rings $R_{3,s}$ and $R_{3,d}$ resulting from these symmetrizations.

2. Notation.

Throughout this paper, points in the Möbius space $\bar{E}^3$ will be designated by the small letters x and y, or by the capital letters P and Q. The rectangular coordinates of x will be denoted by (x_1, x_2, x_3), while P_1 and P_2 will stand for different points. Points will be regarded as vectors, and the norm of x will be indicated by $|x|$.

Given a set E, we let ∂E designate its boundary, $C(E)$ its complement, $\bar{E}$ its closure, and Int E its interior. For $k = 1, 2, 3$, the k-dimensional Lebesgue measure is denoted by m_k.

3. Admissible functions.

In the definition (1.3) of conformal capacity of a space ring R the infimum was taken over all real-valued functions u which are continuously differentiable in R and satisfy the proper boundary conditions. We now relax the differentiability requirement and consider

a larger class of competing functions.

A real-valued function $u = u(x)$ is said to be ACL or absolutely continuous on lines in a domain Ω if, given any ball B with $B \subset \Omega$, u is absolutely continuous in the usual sense on almost all line segments in B which are parallel to the coordinate axes. If u is continuous and ACL in a space ring R, and if u has boundary values 0 on ∂C_0 and 1 on ∂C_1, then u is called an admissible function for the ring R.

By Lemma 1 in [14],

$$\operatorname{cap} R \leq \inf_u \int_R |\nabla u|^3 \, d\omega ,$$

where the infimum is taken over all admissible u. The opposite inequality being obvious, we conclude that in the definition (1.3) of cap R the infimum may be taken over all admissible functions for R.

4. Simple admissible functions.

As in [14] we find it desirable to consider admissible functions whose level surfaces are polyhedral.

Suppose $v = v(x)$ is an admissible function for a space ring R. Let $0 < \varepsilon < 1/2$ and, for $x \in R$, define

$$v_1 = v_1(x) = \begin{cases} 0 & \text{if } v < \varepsilon , \\ \dfrac{v - \varepsilon}{1 - 2\varepsilon} & \text{if } \varepsilon \le v \le 1 - \varepsilon , \\ 1 & \text{if } 1 - \varepsilon < v , \end{cases}$$

and extend v_1 to be 0 on C_0 and 1 on C_1. Then the set of x where $0 < v_1 < 1$ is a **bounded** subset of R and lies at a positive distance d from $C(R)$. Let B be the ball $|y| < c$, $c < d$, and define

$$v_2(x) = \frac{1}{m_3(B)} \int_B v_1(x+y)\, d\omega .$$

This function v_2 is continuously differentiable in R and has boundary values 0 on ∂C_0 and 1 on ∂C_1. The set of x where $0 < v_2 < 1$ lies at a positive distance b from $C(R)$. Moreover (See [14], §§ 5, 7),

$$\text{cap } R = \inf_{v_2} \int_R |\nabla v_2|^3 \, d\omega , \tag{2.2}$$

where the infimum is taken over all such functions v_2.

Next consider a partition of E^3 into congruent tetrahedra $\{T\}$ of diameter $\delta < b$, with b as given in the last paragraph. Then define a new function $u = u(x)$ so that u is a linear function of the

coordinate variables in each tetrahedron and so that $u = v_2$ on the vertices of each tetrahedron. Then u is called a simple admissible function for R. A simple admissible function satisfies the definition of § 2.3 and is thus an admissible function. Furthermore,

$$\lim_{\delta \to 0} \int_R |\nabla u|^3 \, d\omega = \int_R |\nabla v_2|^3 \, d\omega$$

so that with (2.2),

$$\text{cap } R = \inf_u \int_R |\nabla u|^3 \, d\omega , \tag{2.3}$$

where the infimum is taken over all simple admissible functions u for R ([14], § 7).

Obviously $0 \leq u \leq 1$ for all x. For fixed a, $0 < a < 1$, let F and Σ be the sets of x on which $u \leq a$ and $u = a$, respectively. Then F is a closed polyhedron, that is, the union of a finite number of closed (possibly degenerate) tetrahedra. It is clear that $\partial F \subset \Sigma$; furthermore,

LEMMA 2.1. Suppose a is different from the finite set of values assumed by v_2, and hence by u, on the vertices of $\{T\}$. Then $\partial F = \Sigma$.

Proof: It is sufficient to show that $\Sigma \subset \partial F$. Fix a point $P \in \Sigma$. Then $P \in \bar{T}$, for some tetrahedron T, and by hypothesis P is not a vertex of T.

Let P_i, $i=1,2,3,4$ be the vertices of T. Then for each i, $u(P_i) \neq a$. There must be one vertex P_i' for which $u(P_i') > a$. Otherwise $u(P_i) < a$ for all i and by the linearity of u, $u < a$ at each point of $\bar{T}$, contrary to the assumption that $u(P) = a$.

Let I be the segment joining P and P_i'. Then $I \subset \bar{T}$ and u varies linearly on I. Hence $u > a$ at each point of I except at P, and P is therefore a limit point of the set of x where $u > a$. But since also $P \in F$, we must have $P \in \partial F$, and this completes the proof of the lemma.

5. Special admissible functions.

Let R be any bounded ring and v a simple admissible function for R. Fix a, $0 < a < 1$: in this section we shall always assume a to be different from the finite set of values assumed by v on the vertices of $\{T\}$. Let Σ be the polyhedral surface consisting of the set of x where $v = a$.

In the proofs for Steiner symmetrization, certain technical difficulties arise in case a line parallel to the x_3 axis meets a level surface Σ of v in an infinite set. The same is true for Schwarz

symmetrization in case the intersection of Σ with some plane normal to the x_3 axis is too large.

To avoid these troubles, we shall determine a function $u = v \circ f^{-1}$, where f is a suitable rotation, and show that in definition (1.3) we need consider only functions of this type. The rotation f will be chosen so that u is admissible for R and so that each (polyhedral) level surface $f[\Sigma]$ of u has no faces parallel to a given line.

In the Steiner case the line is the x_3 axis, and then each surface $f[\Sigma]$ meets each line parallel to the x_3 axis in a finite set. For the Schwarz case the line need only lie in the x_1x_2 plane, and then each plane normal to the x_3 axis meets each $f[\Sigma]$ in the union of a finite number of closed linear segments. It is obviously sufficient to consider the x_3 axis, since the proof would be the same for any other line.

Now suppose Σ meets one of the finite number of tetrahedra $T_j \in \{T\}$ in which v is nonconstant. By the linearity of v in T_j, $\Sigma \cap T_j$ is contained in a plane $c_{1j}x_1 + c_{2j}x_2 + c_{3j}x_3 = a$, where the c_{ij} are independent of the particular value of a. Therefore there is a finite collection of unit vectors n_j associated with the tetrahedra $\{T\}$ such that each plane face of each level surface $v = a$ has one of the vectors n_j as its normal.

We wish to determine a rotation f so that none of the vectors

$f[n_j]$ is normal to the x_3 axis. For this it is sufficient to require that the projection of each vector $f[n_j]$ on the x_2x_3 plane be non-degenerate and form a positive acute angle with the x_2 axis.

Let $f_3(x) = x'$ denote a rotation through an angle φ_3 about the x_3 axis. Suppose the projection of some n_j on the x_2x_3 plane is the zero vector. This will not be true of any of the vectors $f_3[n_j]$ if $0 < \varphi_3 < \Phi_3 \leq \frac{\pi}{2}$, where Φ_3 denotes the smallest positive angle which the x_1 axis makes with the (nonzero) projection of any n_j on the x_1x_2 plane.

A second rotation $f_1(x') = x''$ through an angle φ_1, $0 < \varphi_1 < \Phi_1 \leq \frac{\pi}{2}$, about the x_1 axis will ensure that the projection of each vector $f_1 \circ f_3[n_j]$ on the x_2x_3 plane also forms a positive acute angle with the x_2 axis. We may take Φ_1 to be the least positive angle between the x_2 axis and the projection of any vector $f_3[n_j]$ on the x_2x_3 plane. Hence none of the vectors $f_1 \circ f_3[n_j]$ will be normal to the x_3 axis.

Since the point set E on which $0 < v < 1$ lies at a positive distance from $C(R)$, we may assume that the angles φ_1 and φ_3 are so small that if $x \in E$ then $x'' \in R$. Now we let $f = f_1 \circ f_3$ and define a function $u = u(x)$ by the condition $u(x) = v(f^{-1}(x))$. It is clear that u is an admissible function for R. The level surfaces of u are polyhedral but have no faces parallel to the x_3 axis, because for fixed a,

$$\{x \mid u(x) = a\} = \{x \mid v(f^{-1}(x)) = a\} = \{f(x) \mid v(x) = a\} = f[\{x \mid v(x) = a\}] .$$

As usual, we assume that a is not one of the values acquired by v on the vertices of $\{T\}$. In the applications we shall refer to such a function as a special admissible function for R.

We prove

LEMMA 2.2. The conformal capacity of a space ring R is

$$\operatorname{cap} R = \inf_{u} \int_{R} |\nabla u|^3 \, d\omega,$$

where the infimum is taken over all special admissible functions for R.

Proof: Let v be a simple admissible function for R, and let $u = v \circ f^{-1}$ be a special admissible function for R obtained from v as described above. It is easy to see that the Jacobian of the rotation f is 1 and that $|\nabla u(x)| = |\nabla v(f^{-1}(x))|$. Hence

$$\int_{R} |\nabla u|^3 \, d\omega = \int_{R} |\nabla v|^3 \, d\omega . \tag{2.4}$$

Taking the infimum on the left in (2.4) over all special admissible u and then on the right over all simple admissible v we have, by virtue of (2.3),

$$\inf_u \int_R |\nabla u|^3 \, d\omega \leq \inf_v \int_R |\nabla v|^3 \, d\omega = \text{cap } R .$$

But the opposite inequality follows from the fact that each special admissible function for R is also admissible for R.

STEINER SYMMETRIZATION

This method of symmetrization is discussed in [30] and is a generalization to 3-space of Steiner symmetrization in the plane [21].

6. Steiner symmetrization of open and closed sets.

Let G be a bounded open set in E^3. We define a second set G^*, called the Steiner symmetrization of G with respect to the x_1x_2 plane, as follows: Let $L = L(x_1, x_2)$ denote the line in E^3 through $(x_1, x_2, 0)$ that is parallel to the x_3 axis. Then $L \cap G^* = \phi$ if and only if $L \cap G = \phi$. If $L \cap G \neq \phi$ and $m_1(L \cap G) = \ell > 0$, then $L \cap G^*$ is an open segment of length ℓ which is bisected by the x_1x_2 plane.

If F is a bounded closed set in E^3, we define F^* as above except in the second case, where we take $L \cap F^*$ to be a closed segment of length ℓ which is bisected by the x_1x_2 plane. If $\ell = 0$, then $L \cap F^*$ is the single point $(x_1, x_2, 0)$.

LEMMA 2.3. If E^* is the Steiner symmetrization of a bounded closed or open set E, then E^* is closed or open, respectively.

Proof: Suppose F is closed, and let $\{P_n\}$ be a sequence of points in F^* with limit point P_o. We will show that P_o is in F^*. In rectangular coordinates let $P_n = (x_{1n}, x_{2n}, x_{3n})$, $P_o = (x_{1o}, x_{2o}, x_{3o})$. Obviously $x_{in} \to x_{io}$, $i = 1, 2, 3$. Since $P_n \in F^*$, all points x on $L(x_{1n}, x_{2n})$ with $|x_3| \leq |x_{3n}|$ are in F^*. Hence $\ell(x_{1n}, x_{2n}) \geq |2x_{3n}|$, where $\ell(x_{1n}, x_{2n})$ denotes the length of $F \cap L(x_{1n}, x_{2n})$. If we show that $\ell(x_1, x_2)$ is upper semicontinuous (usc) we will have $\ell(x_{1o}, x_{2o}) \geq \limsup_{n \to \infty} \ell(x_{1n}, x_{2n}) \geq \lim_{n \to \infty} |2x_{3n}| = |2x_{3o}|$, and hence $P_o \in F^*$.

To show that $\ell(x_1, x_2)$ is usc, let E_n, E_o be the orthogonal projections of $L_n \cap F$, $L_o \cap F$, respectively, on the x_3 axis. Here $L_n = L(x_{1n}, x_{2n})$ and $L_o = L(x_{1o}, x_{2o})$. Then $E_o \supset \limsup_{n \to \infty} E_n$, for if $P = (0, 0, x_3) \in E_n$ for infinitely many n, then $P_n = (x_{1n}, x_{2n}, x_3) \in F \cap L_n$ for infinitely many n. But $P_n \to (x_{1o}, x_{2o}, x_3) \in L_o$, and since F is closed it follows that $(x_{1o}, x_{2o}, x_3) \in F \cap L_o$. This implies that $P = (0, 0, x_3) \in E_o$.

Since F is bounded, $m_1[\bigcup_n E_n] < \infty$, and therefore by [32, Theorem 9.3, p.17], $\ell(x_{1o}, x_{2o}) = m_1(E_o) \geq m_1(\limsup_{n \to \infty} E_n)$ $\geq \limsup_{n \to \infty} m_1(E_n) = \limsup_{n \to \infty} \ell(x_{1n}, x_{2n})$. That is, $\ell(x_1, x_2)$ is usc, and F^* is closed.

Next suppose G is open, and let H denote a closed right cylinder of height $2h$ which is symmetric in the x_1x_2 plane, has its bases parallel to this plane, and contains G. Then $H - G$ is closed, and the previous argument shows that $2h - \ell(x_1, x_2)$ is usc, where $\ell(x_1, x_2)$ denotes the length of $L \cap G$, $L = L(x_1, x_2)$.

Let $\{P_n\}$ be a sequence of points in $H^* - G^*$ with limit point P_o. Using the same notation as before, it follows that all points x on $L(x_{1n}, x_{2n})$ with $|x_{3n}| \leq |x_3| \leq h$ are in $H^* - G^*$. Hence by upper semicontinuity $2h - \ell(x_{1o}, x_{2o}) \geq 2h - |2x_{3o}|$, and $H^* - G^*$ is closed. This means that G^* is open in $H^* = H$ and hence open.

LEMMA 2.4. If E^* is the Steiner symmetrization of a bounded connected (closed or open) set E, then E^* is connected.

Proof: Let E_o be the orthogonal projection of E on the x_1x_2 plane. Then E_o is connected, for if $E_o = E_1 \cup E_2$ sep, let $H_i = \bigcup(L \cap E_i)$, $i = 1, 2$, where the union is taken over all L such that $L \cap E_i \neq \phi$. Then E meets each of the sets H_i, which are separated since $\infty \notin L$, and hence E is not connected ([36], Theorem 16, p. 18).

Now $E^* = \bigcup(L \cap E^*)$, where the union is taken over all L such that $L \cap E_o \neq \phi$. But since each $L \cap E^*$ is connected, and since each nonempty set $L \cap E^*$ meets the connected set E_o, E^*

is connected ([29], Corollary 2, p. 74) .

LEMMA 2.5. If E^* is the Steiner symmetrization of a bounded (closed or open) set E, then $C(E^*)$ is connected.

Proof: For suppose $x = (x_1, x_2, p) \in C(E^*)$. Then $(x_1, x_2, x_3) \in C(E^*)$ for $|x_3| \geq |p|$. But $\{(x_1, x_2, x_3) | x_3 \geq |p|\}$ and $\{(x_1, x_2, x_3) | x_3 \leq -|p|\}$ are connected sets, and each meets $U=\{x \mid |x|>r\}$, where r is chosen so large that $U \cap E^* = \phi$. Thus $C(E^*)$ is connected.

7. Steiner symmetrization of rings.

Now let R be any bounded ring in E^3. Then $R \cup C_0 = C(C_1)$ is open, C_0 is closed, and we define the Steiner symmetrization ([2]) R^* of the ring R as

$$R^* = (R \cup C_0)^* - C_0^* .$$

LEMMA 2.6. If R is a bounded ring, then R^* is a ring.

Proof: By Lemmas 2.3 and 2.4, C_0^* is closed and connected .

([2]) Although the Steiner symmetrization of R as a ring and the Steiner symmetrization of R as an open set (Cf. § 2.6) are not the same, for convenience of notation we shall let R^* denote the former.

By Lemma 2.3, $(R \cup C_0)^*$ is open, so that $C_1^* = C(R \cup C_0)^*$ is closed. Also by Lemma 2.5, C_1^* is connected.

We must show that R^* is connected. First let D_1 be a component of $R^* \cap \{x_3 > 0\}$ and let $D_2 = R^* \cap \{x_3 > 0\} - D_1$. Suppose $D_2 \neq \phi$. Then the projections of D_1, D_2 on the x_1x_2 plane must meet in some point $(x_1, x_2, 0)$ since otherwise R would be disconnected, by the argument in Lemma 2.4. Let $P_i \in L \cap D_i$, $i = 1,2$, where $L = L(x_1, x_2)$. Then $L \cap R^* \cap \{x_3 > 0\}$ is a connected set which joins P_1 and P_2 and hence D_1 and D_2. But this is impossible since D_1 and D_2 were separated sets. Hence $R^* \cap \{x_3 > 0\}$ is connected. Similarly $R^* \cap \{x_3 < 0\}$ is connected. But both $R^* \cap \{x_3 > 0\}$ and $R^* \cap \{x_3 < 0\}$ have limit points in the connected set $R^* \cap \{x_3 = 0\}$, and R^* is connected.

Letting R denote any bounded ring and R^* its Steiner ring symmetrization, it is now our aim to prove that R^* satisfies the following extremal property.

THEOREM 2.1. $\quad \operatorname{mod} R \leq \operatorname{mod} R^*$.

Before proving this theorem we must first study some geometrical properties of Steiner-symmetrized sets.

8. Surface area under Steiner symmetrization.

Suppose F is a compact set and $\Sigma = \partial F$ its boundary. By the Steiner symmetrization ([3]) of Σ we shall mean the set $\Sigma^* = \partial F^*$, where F^* is the Steiner symmetrization of F. The following result is well known, being attributed to Steiner (Cf. [30], p. 182).

LEMMA 2.7. *If F is a closed polyhedron, if none of the faces of $\Sigma = \partial F$ is parallel to the x_3 axis, and if F^* is the Steiner symmetrization of F, then $\Sigma^* = \partial F^*$ is a polyhedral surface whose area does not exceed that of Σ.*

Proof: As in § 2.6, we let $L = L(x_1, x_2)$ be the line in E^3 through $(x_1, x_2, 0)$ which is parallel to the x_3 axis. If $L \cap F \neq \phi$, L meets Σ in a finite number of points (x_1, x_2, x_{3i}), $i = 1, 2, \ldots, 2m$, $m = m(x_1, x_2)$, where $x_{3i} \geq x_{3j}$ if $i < j$. (If there is a double point in $L \cap F$ -- that is, a point on the intersection of two plane faces of Σ -- then this is to be counted twice.) The set $L \cap \Sigma$ is the boundary of $L \cap F$ as a linear point set, and $L \cap F$ has the same length as $L \cap F^*$. To $L \cap F$ there corresponds a line segment $L \cap F^*$ bisected by the $x_1 x_2$ plane, the endpoints of which have the coordinates

([3]) Although the Steiner symmetrization of Σ as the boundary of F and the Steiner symmetrization of Σ as a closed set (Cf. § 2.6) are not the same, for convenience of notation we shall let Σ^* denote the former.

$(x_1, x_2, \pm x_3)$, where

$$2x_3 = \sum_{i=1}^{2m} (-1)^{i-1} x_{3i} . \tag{2.5}$$

On Σ , x_3 is a piecewise linear function of x_1, x_2 , and hence absolutely continuous in the sense of Tonelli. By (2.5) the same is true of the x_3 coordinate as a function of x_1, x_2 on the Steiner-symmetrized surface Σ^* . Hence we may apply the standard integration formulas [32, p. 181] for surface area . Now let D be the projection of Σ and Σ^* on the $x_1 x_2$ plane. Then using the triangle inequality and the fact that $m \geq 1$, we have

$$\begin{aligned}
\int_{\Sigma^*} d\sigma^* &= 2 \int_D \left[1 + \left(\frac{\partial x_3}{\partial x_1}\right)^2 + \left(\frac{\partial x_3}{\partial x_2}\right)^2 \right]^{1/2} dx_1 dx_2 \\
&= 2 \int_D \left[1 + \frac{1}{4} \left(\sum_{i=1}^{2m} (-1)^{i-1} \frac{\partial x_{3i}}{\partial x_1} \right)^2 + \frac{1}{4} \left(\sum_{i=1}^{2m} (-1) \frac{\partial x_{3i}}{\partial x_2} \right)^2 \right]^{1/2} dx_1 dx_2 \\
&= \int_D \left| \left(2 , \sum_{i=1}^{2m} (-1)^{i-1} \frac{\partial x_{3i}}{\partial x_1} , \sum_{i=1}^{2m} (-1)^{i-1} \frac{\partial x_{3i}}{\partial x_2} \right) \right| dx_1 dx_2 \\
&\leq \int_D \sum_{i=1}^{2m} \left| \left(1, (-1)^{i-1} \frac{\partial x_{3i}}{\partial x_1} , (-1)^{i-1} \frac{\partial x_{3i}}{\partial x_2} \right) \right| dx_1 dx_2 \\
&= \int_D \sum_{i=1}^{2m} \left[1 + \left(\frac{\partial x_{3i}}{\partial x_1}\right)^2 + \left(\frac{\partial x_{3i}}{\partial x_2}\right)^2 \right]^{1/2} dx_1 dx_2 = \int_\Sigma d\sigma .
\end{aligned}$$

For application in the proof of Theorem 2.1 we observe that in Lemma 2.7 we have actually shown the following. Given any rectangle E in the x_1x_2 plane, the area of the part of Σ^* which projects into E is not greater than the area of the part of Σ which projects into E.

9. Steiner symmetrization of functions.

Let $u = u(x)$ be a continuous real-valued function defined in E^3 which is bounded above, and let the set of x where $u < A = \sup u$ be bounded. For $a < A$ let G_a and F_a be the open and closed sets where $u < a$ and $u \leq a$, respectively, and let G_a^* and F_a^* be the Steiner symmetrizations of these sets.

We then define $u^* = u^*(x)$, the Steiner symmetrization of u, as follows: Fix a point x. If $x \in F_a^*$ for some $a < A$, we set

$$u^*(x) = \inf \{a \mid x \in F_a^*\}.$$

Otherwise we let $u^*(x) = A$.

LEMMA 2.8. *For each* $a < A = \sup u$, $L \cap G_a^*$ *and* $L \cap F_a^*$ *are the sets in* L *where* $u^* < a$ *and* $u^* \leq a$, *respectively.*

Proof: Suppose $x \in L \cap F_a^*$. Then by definition $u^*(x) = \inf\{b \mid x \in L \cap F_b^*\}$. But since $x \in L \cap F_a^*$ this infimum must be $\leq a$. Hence $u^*(x) \leq a$.

To show the opposite inclusion, fix $x \in L$ so that $u^*(x) \leq a$. Then $u^*(x) = \inf\{b \mid x \in L \cap F_b^*\} \leq a$. This means that $x \in L \cap F_b^*$ for all $b > a$, or that $x \in L \cap \bigcap_{b>a} F_b^*$. But by hypothesis on u all the sets $L \cap F_b^*$ are bounded, so that $L \cap \bigcap_{b>a} F_b^* = L \cap F_a^*$, and $x \in L \cap F_a^*$.

The proof for G_a is similar.

COROLLARY 2.1. The functions u and u^* assume exactly the same values $a < A = \sup u$ on each fixed line L.

Proof: It follows from Lemma 2.8 that $L \cap (F_a^* - G_a^*)$ is the set of points on L where $u^* = a$. But it is already known that $L \cap (F_a - G_a)$ is the set of points on L where $u = a$. Since these sets are simultaneously empty or nonempty the result follows.

Lemma 2.8 shows that G_a^* and F_a^* are the sets of x where $u^* < a$ and $u^* \leq a$, respectively. But since G_a^* is open and F_a^* closed by Lemma 2.3, it follows that u^* is continuous. In fact we can show that u^* satisfies a Lipschitz condition whenever u does. (Cf. Theorem 4.5 of [21] and Lemma 4 of [14].)

LEMMA 2.9. If u^* is the Steiner symmetrization of u and if

$$|u(P_1) - u(P_2)| \leq M|P_1 - P_2|$$

for all points P_1 and P_2, then

$$|u^*(Q_1) - u^*(Q_2)| \leq M|Q_1 - Q_2| \tag{2.6}$$

for all points Q_1 and Q_2.

Proof: Fix two points Q_1 and Q_2 with $u^*(Q_1) \leq u^*(Q_2)$ and let $a_1 = u^*(Q_1)$ and $d = |Q_1 - Q_2|$. For (2.6) it is sufficient to prove that

$$u^*(Q_2) \leq a_2 = a_1 + Md. \tag{2.7}$$

Let (x_{11}, x_{21}) and (x_{12}, x_{22}) be the (x_1, x_2) coordinates of Q_1, Q_2, respectively. Let E_1 be the set of points on $L_1 = L(x_{11}, x_{21})$ where $u \leq a_1$, and let E_2 be the set of points on $L_2 = L(x_{12}, x_{22})$ whose distance from E_1 does not exceed d. By Corollary 2.1, E_1 is nonempty, and since the distance from L_1 to L_2 is no greater than d it is clear that E_2 is nonempty. E_2 is closed, and for each $P_2 \in E_2$ there exists $P_1 \in E_1$ such that $|P_1 - P_2| \leq d$. Thus

$$u(P_2) \leq u(P_1) + Md \leq a_2 ,$$

and $u \leq a_2$ at every point of E_2 .

Next let E_1^* and E_2^* be the Steiner symmetrizations of E_1 and E_2 , respectively. Then by Lemma 2.8, E_1^* is the set of points on L_1 where $u^* \leq a_1$. Moreover, since $E_2 \subset \{x \mid u \leq a_2\}$ it follows by Lemma 2.8 that $E_2^* \subset \{x \mid u^* \leq a_2\}$, or $u^* \leq a_2$ in E_2^* . Hence $Q_1 \in E_1^*$ and to obtain (2.7) it is sufficient to show that $Q_2 \in E_2^*$.

Let H be the orthogonal projection of E_1 on L_2 , and let H^* be the orthogonal projection of E_1^* on L_2 . Since E_2 is the set of points of L_2 which lie within distance d of E_1 , it is the set of points of L_2 which lie within distance c of H , where c is given by the Pythagorean theorem as $c^2 = d^2 - (x_{12}-x_{11})^2 - (x_{22}-x_{21})^2$. Now H^* is equal to the Steiner symmetrization of H , and the length of these two sets are therefore equal. Thus

$$m_1(E_2^*) = m_1(E_2) \geq m_1(H) + 2c = m_1(H^*) + 2c .$$

That is, E_2^* includes all points on L_2 within distance c from H^* , and hence all points on L_2 within distance d from E_1^* . Thus $Q_2 \in E_2^*$ and (2.7) is true.

10. Equimeasurability.

Let u be one of the special admissible functions for R defined in § 2.5, and let u^* be the Steiner symmetrization of u. Let $0 = a_1 < a_2 < \ldots < a_n = 1$ be the finite set of values assumed by u on the vertices of the tetrahedra $\{f[T]\}$, where f is the rotation in § 2.5. Let D_i and D_i^* be the sets of points x where $a_i < u < a_{i+1}$ and $a_i < u^* < a_{i+1}$, respectively.

Fix i; let $\rho^* = \rho^*(x)$ be a real-valued function defined on D_i^*, such that ρ^* is nonnegative, continuous, and symmetric with respect to the x_1x_2 plane. That is, the value of ρ^* is the same at the two points $(x_1, x_2, \pm x_3)$. Next, for each $a_i < a < a_{i+1}$ let F and F^* denote the sets on which $u \leq a$ and $u^* \leq a$; G and G^* the sets on which $u < a$ and $u^* < a$; and Σ and Σ^* the sets on which $u = a$ and $u^* = a$, respectively. Then by Lemma 2.1 we have $\Sigma = \partial F$. By definition, $\Sigma^* = \partial F^*$.

Suppose $L \cap \Sigma \neq \phi$. Then because u is a special admissible function for R, $L \cap \Sigma$ is a finite set. Further, $L \cap \Sigma^*$ consists of two points (perhaps coincident), symmetric with respect to the x_1x_2 plane, because

$$m_1(L \cap \Sigma^*) = m_1(L \cap (F^* - G^*)) = m_1(L \cap (F - G)) = m_1(L \cap \Sigma) = 0 .$$

For this reason ρ^* assumes exactly one value on each nonempty intersection $L \cap \Sigma^*$. We define a function $\rho = \rho(x)$ on D_i by requiring ρ to take this value at each point of the corresponding intersection $L \cap \Sigma$.

LEMMA 2.10. If ρ, ρ^* are functions as defined above, then ρ is continuous in D_i and

$$\int_{D_i} \rho^p \, d\omega = \int_{D_i^*} \rho^{*p} \, d\omega \tag{2.8}$$

for all $p > 0$.

Proof: For suppose ρ is not continuous in D_i. Then we can find $P_o \in D_i$ and a sequence $\{P_n\}$ in D_i such that $P_n \longrightarrow P_o$ but $\lim_{n\to\infty} \rho(P_n) = c \neq \rho(P_o)$. In rectangular coordinates let $P_n = (x_{1n}, x_{2n}, x_{3n})$, $P_o = (x_{1o}, x_{2o}, x_{3o})$. Then $x_{in} \longrightarrow x_{io}$, $i = 1, 2, 3$. We let $L_n = L(x_{1n}, x_{2n})$ and $L_o = L(x_{1o}, x_{2o})$.

By Corollary 2.1 there exist points Q_n in $L_n \cap D_i^*$ such that $u^*(Q_n) = u(P_n)$. By the Bolzano-Weierstrass theorem we can pick a subsequence $\{Q_{n_k}\}$ such that $Q_{n_k} \longrightarrow Q_o$. But $x_{in_k} \longrightarrow x_{io}$, so that $Q_o \in L_o$. Then by the continuity of u and u^* we have

$$u(P_o) = \lim_{n\to\infty} u(P_n) = \lim_{n_k\to\infty} u(P_{n_k}) = \lim_{n_k\to\infty} u^*(Q_{n_k}) = u^*(Q_o).$$

Hence $Q_o \in D_i^*$ and

$$\rho(P_o) = \rho^*(Q_o) = \lim_{n_k \to \infty} \rho^*(Q_{n_k}) = \lim_{n_k \to \infty} \rho(P_{n_k}) = c .$$

But this contradicts the earlier assumption that $\rho(P_o) \neq c$. Hence ρ is continuous in D_i.

To prove (2.8) let $b > 0$, and let G and G^* denote the subsets of D_i and D_i^* where $\rho < b$ and $\rho^* < b$, respectively. Given $L = L(x_1, x_2)$, let E be the set of values assumed by u^* on $L \cap G^*$. Then G and G^* are open because ρ and ρ^* are continuous, and hence $L \cap G^*$ is a linear open set. Thus $L \cap G^*$ is the countable union of open intervals I_n. If I_n' is the set of values assumed by u^* on I_n, then I_n' must be an interval because u^* is continuous. Thus E is the countable union of disjoint (possibly degenerate) linear intervals J_m.

We will show that, because u^* is the Steiner symmetrization of u, the sets of points on L for which $u \in J_m$ and $u^* \in J_m$ have equal length, where J_m is one of the intervals in E. Suppose $J_m = (c, d)$. Then $\{x | u(x) \in J_m\} = G_d - F_c$, $\{x | u^*(x) \in J_m\} = G_d^* - F_c^*$ (Cf. § 2.9), and because all sets involved are bounded,

$$m_1\{x|u \in J_m\} = m_1[L \cap (G_d - F_c)] = m_1[L \cap G_d - L \cap F_c]$$

$$= m_1(L \cap G_d) - m_1(L \cap F_c) = m_1(L \cap G_d^*) - m_1(L \cap F_c^*)$$

$$= m_1[L \cap (G_d^* - F_c^*)] = m_1\{x|u^* \in J_m\} .$$

The other cases $J_m = [c,d]$, $J_m = (c,d]$, $J_m = [c,d)$ are similar.

Now by Corollary 2.1, $L \cap G$ and $L \cap G^*$ are just the sets in L where u and u^*, respectively, take on values in E and, summing over the disjoint linear intervals J_m, we conclude that $L \cap G$ and $L \cap G^*$ have equal length.

Now fix $b_1 < b_2$ and let H and H^* be the subsets of D_i and D_i^* where $b_1 \leq \rho < b_2$ and $b_1 \leq \rho^* < b_2$, respectively. Then by the previous argument $L \cap H$ and $L \cap H^*$ have equal length for each L. Thus, using Fubini's theorem [32, p. 76], integration with respect to x_1 and x_2 shows that $m_3(H) = m_3(H^*)$. Hence ρ and ρ^* are equimeasurable and (2.8) follows from [20, p. 277].

11. Proof of Theorem 2.1.

Let R be a bounded space ring and R^* its Steiner symmetrization. We want to prove that $\operatorname{mod} R \leq \operatorname{mod} R^*$ or, equivalently, that

(2.9) $$\text{cap } R^* \leq \text{cap } R .$$

Let u be a special admissible function for R. Since $\nabla u = 0$ in all but a finite set of tetrahedra, u satisfies a uniform Lipschitz condition; therefore by Lemma 2.9 this must be true also of u^* ([4]), so that u^* is ACL. Now $R^* = (R \cup C_0)^* - C_0^*$, where $(R \cup C_0)^*$ and C_0^* are the Steiner symmetrizations of $R \cup C_0$ and C_0. Since u is 0 on C_0 and 1 on C_1, C_0 is contained in the set of x where $u \leq 0$ while $R \cup C_0$ contains the set of x where $u < 1$. This, together with the fact that $0 \leq u^* \leq 1$, implies that u^* is 0 on C_0^* and 1 on C_1^*. We conclude that u^* is admissible for the ring R^*.

To prove (2.9) it will be sufficient to show that

(2.10) $$\int_{R^*} |\nabla u^*|^3 \, d\omega \leq \int_R |\nabla u|^3 \, d\omega .$$

For then we obtain

$$\text{cap } R^* \leq \int_R |\nabla u|^3 \, d\omega ,$$

and taking the infimum over all special admissible functions u yields (2.9).

([4]) This condition follows automatically, of course, from the fact that u^* is piecewise linear (Cf. (2.5)). However, for other kinds of symmetrization, including the Schwarz case, this stronger argument is needed.

As in § 2.10 let $0 = a_1 < a_2 < \ldots < a_n = 1$ be the finite set of values assumed by u on the vertices of the tetrahedra $\{f[T]\}$, and let D_i and D_i^* be the sets on which $a_i < u < a_{i+1}$ and $a_i < u^* < a_{i+1}$, respectively. If the set of x where $u^* = a_i$ has positive m_3-measure, then almost all of its points are points of linear density in the directions of the coordinate axes (Cf. [32], p. 298). Since ∇u^* will vanish at almost all such density points ([4]) we conclude that the integral of $|\nabla u^*|^3$ over this set will vanish. Hence to prove (2.10) it is sufficient to show that

$$\int_{D_i^*} |\nabla u^*|^3 \, d\omega \leq \int_{D_i} |\nabla u|^3 \, d\omega \tag{2.11}$$

for $i = 1, \ldots, n-1$.

Let ρ^* be a real-valued function defined on D_i^* as in § 2.10, and let ρ be the corresponding function defined on D_i. We will show that

$$\int_{D_i^*} \rho^* \, |\nabla u^*| \, d\omega \leq \int_{D_i} \rho \, |\nabla u| \, d\omega . \tag{2.12}$$

If $a_i < a < a_{i+1}$, then by Corollary 2.1 Σ and Σ^* have the same projection on the x_1x_2 plane. Let the x_1x_2 plane be divided into congruent rectangles E_k of diameter δ and let Σ_k and Σ_k^* denote the portions of Σ and Σ^*, respectively, whose projections

on the x_1x_2 plane are contained in E_k. Then Lemma 2.7 shows that

$$\int_{\Sigma_k^*} d\sigma^* \leq \int_{\Sigma_k} d\sigma$$

for each k. Since ρ and ρ^* are continuous and equal on corresponding intersections $L \cap \Sigma$ and $L \cap \Sigma^*$ for $a_i < a < a_{i+1}$,

$$(2.13) \qquad \int_{\Sigma^*} \rho^* d\sigma^* = \lim_{\delta \to 0} \sum_k \int_{\Sigma_k^*} \rho^* d\sigma^* = \lim_{\delta \to 0} \sum_k (\max_{\Sigma_k^*} \rho^*) \int_{\Sigma_k^*} d\sigma^*$$

$$\leq \lim_{\delta \to 0} \sum_k (\max_{\Sigma_k} \rho) \int_{\Sigma_k} d\sigma = \lim_{\delta \to 0} \sum_k \int_{\Sigma_k} \rho \, d\sigma = \int_{\Sigma} \rho \, d\sigma .$$

Appealing now to a partial area formula due to Federer [9, Theorem 3.1] we see that ([4])

$$(2.14) \qquad \int_{D_i^*} \rho^* |\nabla u^*| \, d\omega = \int_{a_i}^{a_{i+1}} \left[\int_{\Sigma^*} \rho^* d\sigma^* \right] da$$

and

$$(2.15) \qquad \int_{D_i} \rho \, |\nabla u| \, d\omega = \int_{a_i}^{a_{i+1}} \left[\int_{\Sigma} \rho \, d\sigma \right] da .$$

But (2.13) is valid for $a_i < a < a_{i+1}$. Combining (2.13), (2.14), (2.15) therefore yields (2.12) as desired.

Now $|\nabla u^*|^2$ is bounded, measurable, and symmetric in the x_1x_2 plane. Then by a theorem of Vitali-Carathéodory ([32], p. 75) and a theorem of Baire on the approximation of lower semicontinuous functions by continuous functions ([23], p. 150), we can find a sequence of functions ρ_n^* which are nonnegative and continuous in D_i^*, symmetric with respect to the x_1x_2 plane, and which converge boundedly to $|\nabla u^*|^2$ almost everywhere in D_i^*. From (2.12) we see that

$$\int_{D_i^*} |\nabla u^*|^3 \, d\omega = \lim_{n\to\infty} \int_{D_i^*} \rho_n^* |\nabla u^*| d\omega \leq \liminf_{n\to\infty} \int_{D_i} \rho_n |\nabla u| \, d\omega . \tag{2.16}$$

But by Hölder's inequality and Lemma 2.10 we have

$$\int_{D_i} \rho_n |\nabla u| d\omega \leq \left[\int_{D_i} \rho_n^{3/2} d\omega\right]^{2/3} \left[\int_{D_i} |\nabla u|^3 d\omega\right]^{1/3}$$

$$= \left[\int_{D_i^*} \rho_n^{*\,3/2} d\omega\right]^{2/3} \left[\int_{D_i} |\nabla u|^3 d\omega\right]^{1/3} . \tag{2.17}$$

Then

$$\liminf_{n\to\infty} \int_{D_i} \rho_n |\nabla u| d\omega \le [\lim_{n\to\infty} \int_{D_i^*} \rho_n^{*3/2} d\omega]^{2/3} [\int_{D_i} |\nabla u|^3 d\omega]^{1/3}$$

(2.18)

$$= [\int_{D_i^*} |\nabla u^*|^3 d\omega]^{2/3} [\int_{D_i} |\nabla u|^3 d\omega]^{1/3} .$$

Therefore by (2.16) and (2.18),

$$\int_{D_i^*} |\nabla u^*|^3 d\omega \le [\int_{D_i^*} |\nabla u^*|^3 d\omega]^{2/3} [\int_{D_i} |\nabla u|^3 d\omega]^{1/3},$$

which is equivalent to (2.11). This completes the proof of Theorem 2.1.

SCHWARZ SYMMETRIZATION

This method of symmetrization is discussed in [30] and [33]. It is a limiting case of the spherical symmetrization discussed in [30] and employed in [14].

12. Schwarz symmetrization of open and closed sets.

Let G be a bounded open set in E^3. We define a second set G^{**}, called the Schwarz symmetrization of G with respect to the x_3 axis, as follows: Let $\Pi = \Pi(x_3)$ denote the plane through $(0, 0, x_3)$ that is normal to the x_3 axis. Then $\Pi \cap G^{**} = \phi$ if and

only if $\Pi \cap G = \phi$. If $\Pi \cap G \neq \phi$ and $m_2(\Pi \cap G) = A > 0$, take $\Pi \cap G^{**}$ to be an open disk of area A with center on the x_3 axis.

If F is a bounded closed set in E^3 , we define F^{**} as above except in the second case, where we take $\Pi \cap F^{**}$ to be a closed disk of area A with center on the x_3 axis . If $A = 0$, then $\Pi \cap F^{**}$ is the single point $(0, 0, x_3)$.

The proofs of some of the lemmas for Schwarz symmetrization are similar to those of the corresponding lemmas for Steiner symmetrization, with the plane Π playing the role formerly served by the line L . When this is the case, the lemmas will be stated without proof.

LEMMA 2.11. If E^{**} is the Schwarz symmetrization of a bounded closed or open set E , then E^{**} is closed or open, respectively.

LEMMA 2.12. If E^{**} is the Schwarz symmetrization of a bounded connected (closed or open) set E , then E^{**} is connected.

LEMMA 2.13. If E^{**} is the Schwarz symmetrization of a bounded (closed or open) set E , then $C(E^{**})$ is connected.

13. Schwarz symmetrization of rings.

Now let R be a bounded ring in E^3, and let C_0 and C_1 be the two components of $C(R)$. Then $R \cup C_0 = C(C_1)$ is open, C_0 is closed, and we define the Schwarz symmetrization ([5]) of R with respect to the x_3 axis as

$$R^{**} = (R \cup C_0)^{**} - C_0^{**} .$$

LEMMA 2.14. If R is a bounded ring, then R^{**} is a ring.

The proof of this lemma rests in part upon Lemmas 2.11 to 2.13 and is analogous to the proof of Lemma 2.6.

It is now our aim to prove that the ring R^{**} satisfies the following extremal property.

THEOREM 2.2. $$\text{mod } R \leq \text{mod } R^{**} .$$

Before coming to the proof of this theorem, we first study some geometrical properties of Schwarz-symmetrized sets.

([5]) Although the Schwarz symmetrization of R as a ring and the Schwarz symmetrization of R as an open set (Cf. § 2 12) are not the same, for convenience of notation we shall let R^{**} denote the former.

14. Surface area under Schwarz symmetrization.

Suppose F is a compact set and $\Sigma = \partial F$ its boundary. By the Schwarz symmetrization ([6]) of Σ we shall mean the set $\Sigma^{**} = \partial F^{**}$, where F^{**} is the Schwarz symmetrization of F.

As indicated in § 2.5, each surface Σ in the proof of Theorem 2.2 is polyhedral, and none of the faces of Σ is normal to the x_3 axis. For such a surface we can show that the surface area is not increased by Schwarz symmetrization. This result was proved by H.A. Schwarz in [34, pp. 327-35]. Alternatively, a proof could be given along the same lines as that for spherical symmetrization in [14].

LEMMA 2.15. *If F is a closed polyhedron, if none of the faces of $\Sigma = \partial F$ is normal to the x_3 axis, and if F^{**} is the Schwarz symmetrization of F, then $\Sigma^{**} = \partial F^{**}$ is a surface of revolution whose area does not exceed that of Σ.*

We remark that the same proof shows that the area of the part of Σ between two planes $x_3 = p_1$ and $x_3 = p_2$, $p_1 < p_2$, is not increased by Schwarz symmetrization.

([6]) Although the Schwarz symmetrization of Σ as a boundary and the Schwarz symmetrization of Σ as a closed set are not the same (Cf. § 2.12), for convenience we shall let Σ^{**} denote the former.

15. Schwarz symmetrization of functions.

Let $u = u(x)$ be a continuous real-valued function defined in E^3 which is bounded above, and let the set of x where $u < A = \sup u$ be bounded. For $a < A$ let G_a and F_a be the open and closed sets of x on which $u < a$ and $u \leq a$, respectively, and let G_a^{**} and F_a^{**} be the Schwarz symmetrizations of these sets.

We then define $u^{**} = u^{**}(x)$, the Schwarz symmetrization of u, as follows: Fix a point x. If $x \in F_a^{**}$ for some $a < A$, we set

$$u^{**}(x) = \inf \{a \mid x \in F_a^{**}\} .$$

Otherwise we let $u^{**}(x) = A$. Then we have

LEMMA 2.16. For each $a < A = \sup u$, $\Pi \cap G_a^{**}$ and $\Pi \cap F_a^{**}$ are the sets on Π where $u^{**} < a$ and $u^{**} \leq a$, respectively.

We remark that, as in the Steiner case, this shows that u^{**} is continuous. Moreover, we show that u^{**} satisfies a Lipschitz condition whenever u does.

LEMMA 2.17. If u^{**} is the Schwarz symmetrization of u and if

$$|u(P_1) - u(P_2)| \leq M |P_1 - P_2|$$

<u>for all points</u> P_1 <u>and</u> P_2 , <u>then</u>

$$|u^{**}(Q_1) - u^{**}(Q_2)| \leq M|Q_1 - Q_2| \tag{2.19}$$

<u>for all points</u> Q_1 <u>and</u> Q_2 .

<u>Proof:</u> Fix two points Q_1 and Q_2 with $u^{**}(Q_1) \leq u^{**}(Q_2)$, and let $a_1 = u^{**}(Q_1)$ and $d = |Q_1 - Q_2|$. For (2.19) it is sufficient to prove that

$$u^{**}(Q_2) \leq a_2 = a_1 + Md . \tag{2.20}$$

Let x_{31} and x_{32} be the x_3 coordinates of Q_1 and Q_2 , respectively. Let E_1 be the set of points on $\Pi_1 = \Pi(x_{31})$ where $u \leq a_1$, and let E_2 be the set of points on $\Pi_2 = \Pi(x_{32})$ whose distance from E_1 does not exceed d . As in the proof of Lemma 2.9 , it then follows that $u \leq a_2$ at every point of E_2 .

Next let E_1^{**} and E_2^{**} be the Schwarz symmetrizations of E_1 and E_2 , respectively. Then E_1^{**} is, by Lemma 2.16 , the set of points on Π_1 where $u^{**} \leq a_1$. Moreover, since $E_2 \subset \{x | u \leq a_2\}$, it follows that $E_2^{**} \subset \{x | u^{**} \leq a_2\}$. Hence $Q_1 \in E_1^{**}$, and to obtain (2.20) it is sufficient to show that $Q_2 \in E_2^{**}$.

Let H be the orthogonal projection of E_1 on Π_2 , and let H^{**}

be the orthogonal projection of E_1^{**} on Π_2. Since E_2 is the set of points of Π_2 that lie within distance d of E_1, it is the set of points of Π_2 that lie within distance c of H, where $c^2 = d^2 - (x_{32} - x_{31})^2$. Now H^{**} is equal to the Schwarz symmetrization of H, and the areas of these two sets are therefore equal. But then by the Brunn- Minkowski inequality ([35], p. 84),

$$r_2 \geq r_1 + c ,$$

where r_1 and r_2 are the radii of E_1^{**} and E_2^{**}, respectively. That is, E_2^{**} includes all points within distance c from H, and hence within distance d from E_1^{**}. Thus $Q_2 \in E_2^{**}$ and (2.19) is true.

16. Equimeasurability.

Let u be a special admissible function for R, and for fixed a, $0 < a < 1$, let Σ be the set of x on which $u = a$. It a is not one of the finite set of exceptional values $0 = a_1 < a_2 < \ldots < a_n = 1$ assumed by u on the vertices of certain tetrahedra $\{T\}$ (Cf. § 2.5), then Σ is a polyhedral surface having no faces parallel to the x_1x_2 plane.

Now let u^{**} be the Schwarz symmetrization of u, and let

D_i and D_i^{**} be the sets of x where $a_i < u < a_{i+1}$ and $a_i < u^{**} < a_{i+1}$, respectively.

Fix i; let $\rho^{**} = \rho^{**}(x)$ be a real-valued function defined on D_i^{**}, such that ρ^{**} is nonnegative, continuous, and symmetric with respect to the x_3 axis. That is, the value of ρ^{**} at a point x depends only on r and x_3, where (r, φ, x_3) are cylindrical coordinates for x. Next, for each $a_i < a < a_{i+1}$ let F and F^{**} denote the sets of x on which $u \leq a$ and $u^{**} \leq a$; G and G^{**} the sets on which $u < a$ and $u^{**} < a$; and Σ and Σ^{**} the sets on which $u = a$ and $u^{**} = a$, respectively. Then F is a closed polyhedron and by Lemma 2.2 $\Sigma = \partial F$. By definition, $\Sigma^{**} = \partial F^{**}$.

Since each polyhedral surface Σ has no faces normal to the x_3 axis, each nonempty intersection $\Pi \cap \Sigma$ consists of the union of a finite number of closed linear segments. Also $\Pi \cap \Sigma^{**}$ consists of a circle (perhaps degenerate) with center on the x_3 axis. Hence ρ^{**} assumes exactly one value on each nonempty intersection $\Pi \cap \Sigma^{**}$. Define a function $\rho = \rho(x)$ on D_i by requiring ρ to take this value at each point of the corresponding intersection $\Pi \cap \Sigma$.

LEMMA 2.18. If ρ, ρ^{**} are functions as defined above, then ρ is continuous in D_i and

$$\int_{D_i} \rho^p \, d\omega = \int_{D_i^{**}} \rho^{**p} \, d\omega$$

for all $p > 0$.

17. Proof of Theorem 2.2.

Let R be a bounded space ring and R^{**} its Schwarz symmetrization with respect to the x_3 axis. We want to prove that mod $R \leq$ mod R^{**} or, equivalently, that

(2.21) $$\text{cap } R^{**} \leq \text{cap } R .$$

Let u be a special admissible function for R, such that no level surface Σ of u has a plane face normal to the x_3 axis. Since $\nabla u = 0$ in all but a finite set of tetrahedra, u satisfies a uniform Lipschitz condition. By Lemma 2.17 this must therefore be true also of u^{**}, so that u^{**} is ACL. As in the Steiner case, we see that u^{**} is 0 on C_0^{**} and 1 on C_1^{**}, where C_0^{**} and C_1^{**} are the components of $C(R^{**})$. Hence u^{**} is admissible for the ring R^{**}.

Let $0 = a_1 < a_2 < \ldots < a_n = 1$ be the finite set of exceptional values for u (Cf. § 2.16), and let D_i and D_i^{**} be the sets of x on which $a_i < u < a_{i+1}$ and $a_i < u^{**} < a_{i+1}$, respectively. As in the

proof of Theorem 2.1 , (2.21) will follow if we show that

$$\int_{D_i^{**}} |\nabla u^{**}|^3 \, d\omega \leq \int_{D_i} |\nabla u|^3 \, d\omega \tag{2.22}$$

for $i = 1, \ldots, n-1$.

Fix i ; let ρ^{**} be a real-valued function defined on D_i^{**} as in § 2.16 , and let ρ be the corresponding function defined on D_i . We will show that

$$\int_{D_i^{**}} \rho^{**} |\nabla u^{**}| \, d\omega \leq \int_{D_i} \rho \, |\nabla u| \, d\omega \ . \tag{2.23}$$

Let the x_3 axis be partitioned into intervals I_k of equal length δ , and let Σ_k and Σ_k^{**} denote the portions of Σ and Σ^{**} , respectively, whose projections on the x_3 axis are contained in I_k . Then by Lemma 2.15 ,

$$\int_{\Sigma_k^{**}} d\sigma^{**} \leq \int_{\Sigma_k} d\sigma$$

for each k . Since ρ and ρ^{**} are continuous (See Lemma 2.18) and equal on corresponding intersections $\Pi \cap \Sigma$ and $\Pi \cap \Sigma^{**}$ for $a_i < a < a_{i+1}$,

$$\int_{\Sigma^{**}} \rho^{**}\, d\sigma^{**} = \lim_{\delta \to 0} \sum_k \int_{\Sigma_k^{**}} \rho^{**}\, d\sigma^{**} = \lim_{\delta \to 0} \sum_k (\max_{\Sigma_k^{**}} \rho^{**}) \int_{\Sigma_k^{**}} d\sigma^{**}$$

(2.24)

$$\leq \lim_{\delta \to 0} \sum_k (\max_{\Sigma_k} \rho) \int_{\Sigma_k} d\sigma = \lim_{\delta \to 0} \sum_k \int_{\Sigma_k} \rho\, d\sigma = \int_{\Sigma} \rho\, d\sigma \ .$$

Then making use of (2.24) and Federer's theorem [9] we achieve (2.23) .

Now $|\nabla u^{**}|^2$ is bounded, measurable, and symmetric with respect to the x_3 axis. We can find a sequence of functions ρ_n^{**} which are nonnegative and continuous in D_i^{**} and symmetric in the x_3 axis, and which converge boundedly to $|\nabla u^{**}|^2$ almost everywhere in D_i^{**}. By applying Hölder's inequality, Lemma 2.18 , and a limiting process to this sequence of functions exactly as in the proof of Theorem 2.1 , we are able to achieve (2.22) and thus to complete the proof of Theorem 2.2 .

18. Extremal space rings.

An important type of ring R_3 in 3-space is that for which both components of $C(R_3)$ lie in a plane Π , say $x_3 = 0$, and for which the configuration $R_2 = \Pi \cap R_3$ is a plane ring.

If R_2 is a plane ring and C **is the bounded component of** $C(R_2)$,

we say that R_2 is metrically equivalent to the circular annulus $r_1 < |x_1 + ix_2| < r_2$ if the areas of C and $R \cup C$ are πr_1^2 and πr_2^2, respectively.

The simplest form of plane symmetrization is point symmetrization, which replaces the ring R_2 by the circular annulus to which it is metrically equivalent. The effect of point symmetrizing a plane ring R_2 is to replace it by a plane ring R_2' such that $\operatorname{mod} R_2 \leq \operatorname{mod} R_2'$ (Cf. [7]). But this also replaces R_3 by a new space ring R_3', where $C(R_3') = C(R_2')$, and we shall show that the plane symmetrization increases the space modulus also. In the proof of this result we shall need the following

LEMMA 2.19. Let R_3 be an unbounded space ring with nondegenerate boundary components C_0 and C_1. Given $\varepsilon > 0$, there exists a bounded ring R which separates the components of $C(R_3)$ and for which

$$\operatorname{mod} R_3 < (1 + \varepsilon)^{1/2} \operatorname{mod} R . \tag{2.25}$$

Proof: Since the boundary components of R_3 are nondegenerate, it follows by [26] that $\operatorname{cap} R_3 > 0$. Therefore, there exists a simple admissible function for R_3 such that

$$(1+\varepsilon)\ \text{cap}\ R_3 > \int_{R_3} |\nabla u|^3\, d\omega\ .$$

Now let E_i be the component of the set of x where $u = i$, $i = 0, 1$, which contains C_i. Since $u(x) = 1$ for $|x|$ sufficiently large, $C(E_1)$ must be bounded.

Next, we see that E_0 and E_1 are disjoint continua. Hence by Lemma 3.5 of [15] there exists a ring R, with C_0' and C_1' as the components of $C(R)$, such that $\partial C_i' \subset E_i \subset C_i'$ for $i = 0, 1$. Because $R \subset C(E_1)$, R is bounded. Since $C_i \subset E_i \subset C_i'$, R separates the components of $C(R_3)$. Finally, since $\partial C_i' \subset E_i$, $u = 0$ on $\partial C_0'$ and 1 on $\partial C_1'$. Thus u is an admissible function for R and hence

$$\text{cap}\ R \leq \int_R |\nabla u|^3\, d\omega \leq \int_{R_3} |\nabla u|^3\, d\omega \leq (1+\varepsilon)\ \text{cap}\ R_3\ ,$$

from which (2.25) follows.

COROLLARY 2.2. Let R_3 be a 3-space ring such that both components of $C(R_3)$ lie in a plane Π and determine a plane ring R_2 there. Let R_3' be the 3-space ring obtained by point symmetrizing the plane ring R_2. Then

$$\text{mod } R_3 \leq \text{mod } R_3' .$$

Proof: For convenience let Π be the plane $x_3 = 0$. We may assume that $\text{mod } R_3 < \infty$, for otherwise [26] shows that both rings have a degenerate boundary component, $\text{mod } R_3' = \infty$, and there is nothing to prove.

Given $\varepsilon > 0$, by Lemma 2.19 there exists a bounded ring R which separates the components of $C(R_3)$ and for which (2.25) holds.

Now, since R is a bounded ring, we have by Theorem 2.2,

$$\text{mod } R \leq \text{mod } R^{**} ; \tag{2.26}$$

where R^{**} is the Schwarz symmetrization of R with respect to the x_3 axis. But it is evident that R^{**} separates the boundary components of R_3' so that Lemma 2 of [14] yields

$$\text{mod } R^{**} \leq \text{mod } R_3' . \tag{2.27}$$

Combining (2.25), (2.26), and (2.27), we have

$$(1 + \varepsilon)^{-1/2} \text{mod } R_3 < \text{mod } R \leq \text{mod } R^{**} \leq \text{mod } R_3' ,$$

and the corollary follows when we let ε approach zero.

Now fix a, $0 < a < 1$. In the remainder of this section and throughout Chapter III we shall let R_2 denote the plane ring which consists of the unit disk minus the symmetric slit $-a \leq x_1 \leq a$, $x_2 = 0$. By $R_{3,s}$ and $R_{3,d}$ we shall mean the space rings obtained from R_2 by rotation about the x_1 and x_2 axes, respectively (See Figure 1). The ring $R_{3,s} = R_{3,s}(a)$ consists of the unit ball minus the slit $-a \leq x_1 \leq a$, $x_2 = x_3 = 0$, while the ring $R_{3,d} = R_{3,d}(a)$ consists of the unit ball minus the closed central disk $|x| \leq a$, $x_2 = 0$.

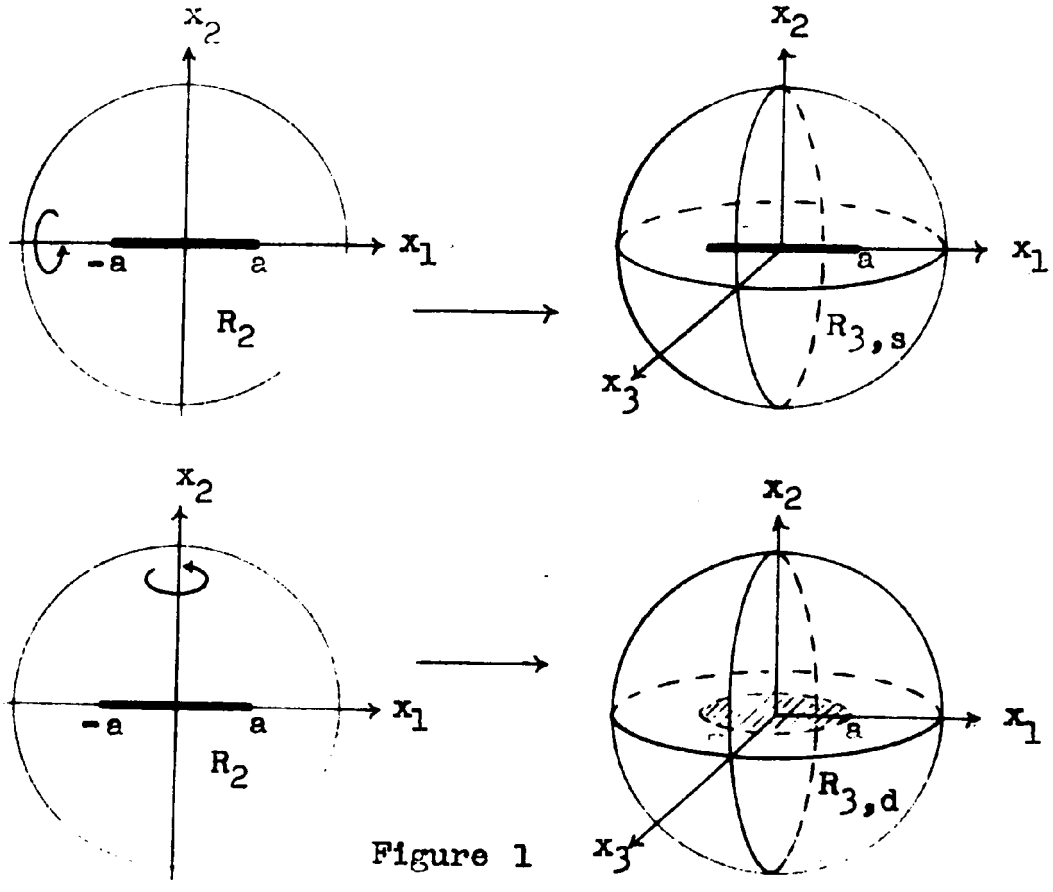


Figure 1

The ring $R_{3,d}$ is extremal in the following sense .

COROLLARY 2.3. Let R be any space ring consisting of the unit ball minus a continuum C , and suppose that the projection of C on some diametral plane Π is πa^2 in area. Then

$$\text{mod } R \leq \text{mod } R_{3,d}(a) .$$

Proof: For convenience let Π be the plane $x_3 = 0$. Suppose R^* is the ring obtained from R by Steiner symmetrization with respect to the x_1x_2 plane. Then it follows from Theorem 2.1 that mod $R \leq$ mod R^* . Now replace the continuum C by its projection on the x_1x_2 plane. This yields a new ring R' , and by the monotoneity of the space modulus ([14], Lemma 2) mod $R^* \leq$ mod R' . Finally, Schwarz symmetrization with respect to the x_3 axis replaces R' by a ring $R_{3,d}(a)$ consisting of the unit ball minus a closed disk $|x| \leq a$, $x_3 = 0$, for some a , $0 < a < 1$. By Theorem 2.2 we have mod R' $\leq$ mod $R_{3,d}(a)$, which proves the corollary.

The ring $R_{3,s}$ has the following extremal property.

COROLLARY 2.4. Let R be any space ring consisting of

the unit ball minus a continuum whose diameter is $2a$, $0 < a < 1$. Then

$$\text{mod } R \leq \text{mod } R_{3,s}(a) .$$

Proof: Let P_1, P_2 be points of C such that $|P_1 - P_2| \geq 2a$. For convenience we may assume that P_1P_2 is parallel to the x_1 axis. Then Schwarz symmetrization of R with respect to the x_1 axis yields a new ring R' and, by Theorem 2.2, $\text{mod } R \leq \text{mod } R'$. This statement follows also from Gehring's result on spherical symmetrization [14, Theorem 1] or the work of Šabat in [33].

But R' is a ring consisting of the unit ball minus a continuum C' which contains a segment $p \leq x_1 \leq p + 2a$, $x_2 = x_3 = 0$, and if C' is replaced by this segment a new ring R'' results such that $\text{mod } R' \leq \text{mod } R''$.

Finally, Steiner symmetrization of R'' with respect to the x_2x_3 plane yields $R_{3,s}(a)$ and, by Theorem 3.1, $\text{mod } R'' \leq \text{mod } R_{3,s}(a)$.

Because of the extremal properties of $R_{3,s}$ and $R_{3,d}$ indicated in Corollaries 2.3 and 2.4, these rings deserve to be studied further. In the next chapter we shall derive the asymptotic behavior of their moduli as a tends to 0 and to 1.

Chapter III

ASYMPTOTIC BEHAVIOR OF EXTREMAL RINGS

1. Introduction.

In this chapter we shall investigate the behavior of the extremal rings $R_{3,d}$ and $R_{3,s}$ introduced in Chapter II (Cf. § 2.18). Our first goal is the following result :

THEOREM 3.1. *Let R_2 be the plane ring consisting of the unit disk minus the slit $-a \leq x_1 \leq a$, $x_2 = 0$, and let $R_{3,d}$ and $R_{3,s}$ be the space rings obtained by rotating R_2 about the x_2 and x_1 axes, respectively. Then*

$$\text{mod } R_{3,d} \leq \text{mod } R_2 \leq \text{mod } R_{3,s} . \tag{3.1}$$

After this we determine the behavior of mod $R_{3,d}$ and mod $R_{3,s}$ as a tends to 0 and to 1. We show that (mod $R_{3,d}$ - log $1/a$) and (mod $R_{3,s}$ - log $1/a$) are monotone decreasing functions of a in the interval $0 < a < 1$, and we provide bounds for the limits of these differences as a tends to 0.

As a approaches 1, however, mod $R_{3,d}$ behaves like

$d_1(1-a)^{1/2}$, while mod $R_{3,s}$ is asymptotic to $d_2[\log \frac{1+a}{1-a}]^{-1/2}$; d_1 and d_2 are bounded. Thus as a tends to 1 , mod $R_{3,d}$ approaches zero much faster than mod $R_{3,s}$, though these moduli are of the same order as a tends to 0 .

2. Proof of Theorem 3.1.

Our proof requires the following result from [11] .

LEMMA 3.1. Suppose that $y = f(x)$ is a continuously differentiable homeomorphism of $r_1 < |x| < r_2$ onto a space ring R , that $f(x)$ has a nonvanishing Jacobian, and that $f(x)$ maps each radius of $r_1 < |x| < r_2$ onto a curve that is normal to the image of each surface $|x| = r$. Then

$$(3.2) \qquad \int_{r_1}^{r_2} D_1(f,r)\frac{dr}{r} \le \text{mod } R \le \int_{r_1}^{r_2} D_2(f,r)\frac{dr}{r} ,$$

where, for $r_1 < r < r_2$,

$$D_1(f,r) = \min_{|x|=r} [\frac{N(x)^3}{J(x)}]^{1/2}, \quad D_2(f,r) = \max_{|x|=r} [\frac{N(x)^3}{J(x)}]^{1/2} .$$

Here $J(x)$ is the absolute value of the Jacobian, and $N(x)$ is the stretching normal to $|x| = r$, that is ,

$$N(x) = \lim_{p \to 0} \frac{|f(x + px) - f(x)|}{|px|} \qquad (p \ \underline{\text{real}}) .$$

We shall obtain space mappings f_d and f_s of a spherical annulus onto $R_{3,d}$ and $R_{3,s}$, respectively, and show that these mappings satisfy the hypotheses of Lemma 3.1. After this we prove that $D_2(f_d, r) \leq 1 \leq D_1(f_s, r)$ and finally apply (3.2) to achieve (3.1).

The plane annulus $A < |x_1 + ix_2| < 1$ is mapped conformally (See [6, p.28] or [27, pp. 280-95]) onto R_2 by the Jacobian elliptic sine function

$$y_1 + iy_2 = f(x_1 + ix_2) = k^{1/2} sn(i(u + iv), k) , \tag{3.3}$$

where

$$u + iv = \frac{2K}{\pi} \log \frac{x_1 + ix_2}{A} - iK , \quad A = \exp\left(\frac{-\pi K'}{4K}\right) , \quad k = a^2 , \tag{3.4}$$

$\log (x_1 + ix_2)$ denotes the principal branch of the logarithm, and

$$K = K(k) = \int_0^1 [(1 - t^2)(1 - k^2 t^2)]^{-1/2} \, dt , \tag{3.5}$$

$$K' = K(k') , \quad k' = (1 - k^2)^{1/2} .$$

For our purpose it will be convenient to employ Jacobi's imaginary transformation $sn(iz, k) = i\, tn(z, k')$ ([5, p. 37], [6, #125.02], [40, p. 505]) to rewrite (3.3) as

$$y_1 + iy_2 = f(x_1 + ix_2) = i\, k^{1/2}\, tn(u + iv, k') . \tag{3.6}$$

We define the space mapping $y = f_d(x)$ from $A < |x| < 1$ onto $R_{3,d}$ by rotating $A < |x_1+ix_2| < 1$ and R_2 about the x_2 and y_2 axes (See Figure 2). That is, we let

$$t + iy_2 = f(s + ix_2) , \quad \varphi = \theta , \tag{3.7}$$

where (s, θ) and (t, φ) are polar coordinates in the x_1x_3 and y_1y_3 planes, respectively.

From (3.7) and the fact that f is a plane conformal mapping it follows that $y = f_d(x)$ has continuous first partials, and similarly for the inverse $x = f_d^{-1}(y)$. Obviously the mapping is one-to-one. Thus $y = f_d(x)$ is a continuously differentiable homeomorphism.

To determine the Jacobian $J(x)$ of f_d we note that the stretching in each direction in a plane $\theta = \theta_o$ is $|f'(s+ix_2)|$, while normal to this plane it is t/s. Thus $J(x) = \frac{t}{s}\,|f'(s+ix_2)|^2$, where $x = (s, \theta, x_2)$ in cylindrical coordinates. From the conformality of

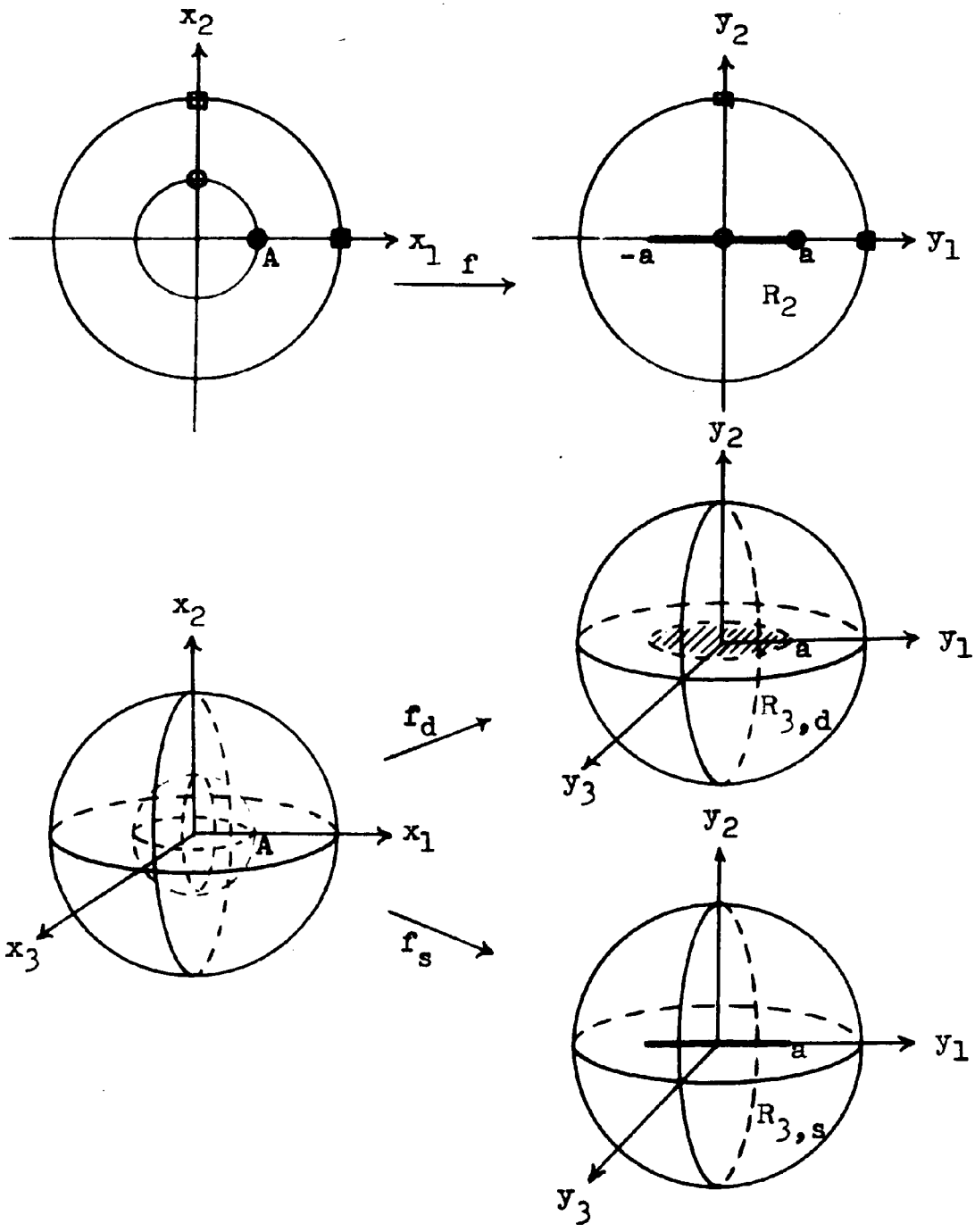


Figure 2

f it follows that $J(x) \neq 0$ whenever $s \neq 0$. But when s tends to 0 the ratio t/s tends to $|f'|$, and $J(x) \neq 0$ for all x.

Now let I be a fixed radius of $A < |x| < 1$, and let γ be the curve in $R_{3,d}$ onto which I is mapped by f_d. From the symmetry with respect to the axes of rotation and the fact that f maps each plane ring $\theta = \theta_o$, $A < |x| < 1$ conformally into the plane $\varphi = \theta_o$, it follows that γ is normal to the image of each sphere $|x| = r$, $A < r < 1$.

Thus the mapping $y = f_d(x)$ satisfies all of the hypotheses of Lemma 3.1, so that we are justified in using (3.2) to find bounds for mod $R_{3,d}$.

Now $N(x) = |f'(s + ix_2)|$, so that

$$\frac{N(x)^3}{J(x)} = \frac{s}{t} |f'(s + ix_2)|$$

for each x. Thus

$$D_1(f_d, r) = \min_{|x_1 + ix_2| = r} \left| \frac{x_1}{y_1} f'(x_1 + ix_2) \right|^{1/2},$$

(3.8)

$$D_2(f_d, r) = \max_{|x_1 + ix_2| = r} \left| \frac{x_1}{y_1} f'(x_1 + ix_2) \right|^{1/2}.$$

Now by (3.6) and the differentiation formula

$\frac{d}{dz}$ (tn z) = (dn z)/(cn^2 z) (See [5, p. 9], [6, #731.10], [40, pp. 492, 3]), we have

$$|f'| = \frac{2K}{\pi r} k^{1/2} \left| \frac{dn(u + iv, k')}{cn^2(u + iv, k')} \right| ,$$

while the addition theorems ([5, p. 38], [6, #125.01]) give

$$(3.9) \qquad |f'| = \frac{2K}{\pi r} k^{1/2} \frac{[d^2C^2D^2 + k'^4 s^2 c^2 S^2]^{1/2} (1 - S^2 d^2)}{c^2C^2 + s^2 d^2 S^2 D^2} ,$$

and

$$y_1 = -k^{1/2} \operatorname{Im} \frac{sD - icdSC}{cC + isdSD} = k^{1/2} \frac{Sd(c^2C^2 + s^2D^2)}{c^2C^2 + s^2d^2S^2D^2} ,$$

$$(3.10) \qquad y_2 = k^{1/2} \operatorname{Re} \frac{sD - icdSC}{cC + isdSD} = k^{1/2} \frac{scCD(1 - S^2d^2)}{c^2C^2 + s^2d^2S^2D^2} .$$

Here we have used the notation

$$s = sn\left(\frac{2K}{\pi} \log \frac{r}{A}, k'\right), \quad c = cn\left(\frac{2K}{\pi} \log \frac{r}{A}, k'\right), \quad d = dn\left(\frac{2K}{\pi} \log \frac{r}{A}, k'\right) ,$$

$$(3.11) \qquad S = sn\left(\frac{2K}{\pi} \psi, k\right) , \quad C = cn\left(\frac{2K}{\pi} \psi, k\right) , \quad D = dn\left(\frac{2K}{\pi} \psi, k\right) ,$$

where $(r, \frac{\pi}{2} - \psi)$ are polar coordinates in the $x_1 x_2$ plane.

But using the identities $C^2 = 1 - S^2$, $D^2 = 1 - k^2S^2$, $c^2 + s^2 = 1$, $c^2 + k^2s^2 = d^2$ ([5, p. 9] , [6, #121.00], [40, p. 493]) we achieve

$$c^2C^2 + s^2D^2 = c^2(1 - S^2) + s^2(1 - k^2S^2)$$

$$= (s^2 + c^2) - S^2(c^2 + k^2s^2) = 1 - S^2d^2 . \tag{3.12}$$

Hence from (3.9), (3.10), (3.12) , and the fact that $x_1 = r \sin\psi$, $x_2 = r \cos\psi$, there results

$$\left| \frac{x_1}{y_1} f' \right| = \frac{2K}{\pi} \left| \sin\psi \right| \left[\frac{C^2D^2}{S^2} + k'^4 \frac{s^2c^2}{d^2} \right]^{1/2} ,$$

$$\left| \frac{x_2}{y_2} f' \right| = \frac{2K}{\pi} \left| \cos\psi \right| \left[k'^4 \frac{S^2}{C^2D^2} + \frac{d^2}{s^2c^2} \right]^{1/2} . \tag{3.13}$$

We wish to show that $D_2(f_d, r) \leq 1$, $A < r < 1$. For this it is sufficient to prove that $\sup_r D_2(f_d, r) \leq 1$. According to Landen's Transformation ([5, p. 72], [6, #163.01], [40, p. 507]) ,

$$\frac{sc}{d} = (1+k)^{-1} \operatorname{sn}\left[(1+k) \frac{2K}{\pi} \log \frac{r}{A} , \frac{1-k}{1+k} \right] . \tag{3.14}$$

Since $\operatorname{sn}(u,k)$ is maximum when $u = K$ (Cf. [6, #121.02, #122.02] , [40, p. 499]) and because $K\left(\frac{1-k}{1+k}\right) = (1+k)\frac{K'}{2}$

([5, p. 72] , [6, #164.02], [40, p. 507]) we see that sc/d has its maximum value $(1+k)^{-1}$ when $\log r/A = \pi K'/4K = \log 1/A$, that is, when $r = 1$. Thus

$$\sup_r D_2(f_d, r)^2 = \sup_\psi \frac{2K}{\pi} (\sin \psi) \left[\frac{C^2D^2}{S^2} + (1-k)^2\right]^{1/2} . \tag{3.15}$$

Now by use of the identities $C^2 = 1 - S^2$, $D^2 = 1 - k^2S^2$ ([6, #121.00] , [40, p. 493]) we see that

$$C^2D^2 + (1 - k)^2S^2 = (1 - kS^2)^2 ,$$

so that (3.15) reduces to

$$\sup_r D_2(f_d, r)^2 = \sup_\psi \frac{2K}{\pi} \frac{(\sin \psi)}{S} (1 - kS^2) . \tag{3.16}$$

Suppose now we have proved

LEMMA 3.2. <u>For each</u> ψ ,

$$1 - k \leq \frac{\sin \psi}{\operatorname{sn}\left(\frac{2K}{\pi} \psi , k\right)} \left[1 - k \operatorname{sn}^2\left(\frac{2K}{\pi} \psi , k\right)\right] \leq \frac{\pi}{2K} . \tag{3.17}$$

Then it will follow from (3.16) and the second half of (3.17)

that $D_2(f_d, r) \leq 1$ for $A < r < 1$, and by Lemma 3.1 we have

$$\text{(3.18)} \qquad \text{mod } R_{3,d} \leq \int_A^1 D_2(f_d, r) \frac{dr}{r} \leq \log \frac{1}{A} = \text{mod } R_2 .$$

The first half of (3.17) will be needed in § 3.4. We postpone the proof of this lemma and consider next the space mapping $y = f_s(x)$ from $A < |x| < 1$ onto $R_{3,s}$ obtained from the plane mapping (3.3) by rotating $A < |x_1 + ix_2| < 1$ and R_2 about the x_1 and y_1 axes (See Figure 2). A proof similar to that given for f_d shows that f_s satisfies the hypotheses of Lemma 3.1 and that

$$\text{(3.19)} \qquad D_1(f_s, r) = \min_{|x_1+ix_2|=r} \left| \frac{x_2}{y_2} f'(x_1 + ix_2) \right|^{1/2},$$

$$D_2(f_s, r) = \max_{|x_1+ix_2|=r} \left| \frac{x_2}{y_2} f'(x_1+ix_2) \right|^{1/2} .$$

Now $f(x_1+ix_2)$ maps the half ring $A < |x_1+ix_2| < 1$, $x_2 > 0$ conformally onto the semidisk $|y_1 + iy_2| < 1$, $y_2 > 0$. Then by the reflection principle, f maps the half ring $A < |x_1 + ix_2| < 1/A$, $x_2 > 0$ conformally onto the upper half plane $y_2 > 0$. The inverse mapping $x_1 + ix_2 = f^{-1}(y_1 + iy_2)$ therefore maps the half plane $y_2 > 0$ conformally into the half plane $x_2 > 0$. Hence (Cf. [11, p. 149]) by

the half plane form of Schwarz's Lemma ([7]),

$$|f'(x_1 + ix_2)|^{-1} \leq \frac{x_2}{y_2} ,$$

or

$$\left| \frac{x_2}{y_2} f'(x_1 + ix_2) \right| \geq 1 \tag{3.20}$$

for $A < |x_1 + ix_2| < 1/A$, $x_2 > 0$. But then $D_1(f_s, r) \geq 1$ by (3.19). Therefore Lemma 3.1 yields

$$\operatorname{mod} R_{3,s} \geq \int_A^1 D_1(f_s, r) \frac{dr}{r} \geq \log 1/A = \operatorname{mod} R_2 .$$

But this inequality together with (3.18) gives (3.1).

To prove Lemma 3.2 we make use of the Gauss Transformation ([5, pp. 72, 3], [6, #164.01-.02]) to write

$$\frac{2K}{\pi} \frac{1 - k\,\mathrm{sn}^2\left(\frac{2K}{\pi}\psi, k\right)}{\mathrm{sn}\left(\frac{2K}{\pi}\psi, k\right)} = \frac{2K_1}{\pi} \frac{\mathrm{dn}\left(\frac{2K_1}{\pi}\psi, k_1\right)}{\mathrm{sn}\left(\frac{2K_1}{\pi}\psi, k_1\right)} , \tag{3.21}$$

where $k_1 = \frac{2k^{1/2}}{1+k}$, $k_1' = \frac{1-k}{1+k}$, and $K_1 = K(k_1) = (1+k)K$.

Thus it is sufficient to show that

([7]) Here (3.20) is a special case of a general inequality of type (4.9), which follows from the theory of hyperbolic measure.

$$\frac{2K}{\pi} \leq \frac{sn(\frac{2K}{\pi}\psi\ ,\ k)}{dn(\frac{2K}{\pi}\psi\ ,\ k)} \csc\psi \leq \frac{1}{k'}$$

for all ψ ; here we have used (3.21) and then replaced k_1 by k .

Since $sn(u+K) = (cn\ u)/(dn\ u)$ and $dn(u+K) = k'/(dn\ u)$ ([6, #122.03], [40, p. 500]) , substitution of $\psi + \frac{\pi}{2}$ for ψ reduces this inequality to the equivalent condition ,

$$(3.22) \qquad \frac{2k'K}{\pi} \leq \frac{cn(\frac{2K}{\pi}\psi\ ,\ k)}{\cos\psi} \leq 1$$

for all ψ .

The infinite product expansion for $cn\ u$ ([6, #909.01], [8, p. 74]) shows , if $0 \leq k < 1$, that

$$\frac{cn\ (\frac{2K}{\pi}\psi\ ,\ k)}{\cos\psi} = \prod_{n=1}^{\infty} A_n(\psi)\ ,$$

where ([8])

$$A_n(\psi) = [\frac{1 - q^{2n-1}}{1 + q^{2n}}]^2 \cdot \frac{1 + 2q^{2n}\cos 2\psi + q^{4n}}{1 - 2q^{2n-1}\cos 2\psi + q^{4n-2}} \ .$$

([8]) Here and elsewhere, unless otherwise specified, q represents Jacobi's Nome $q = \exp(-\pi K'/K)$. As k increases from 0 to 1 , q does likewise.

Since $0 \leq q < 1$, obviously $A_n(\psi) \leq A_n(0) = 1$ for all ψ, and the second half of (3.22) follows immediately.

But we observe also that $A_n(\psi) \geq A_n(\frac{\pi}{2})$ for all ψ, so that the function $(\mathrm{cn}(\frac{2K}{\pi}\psi, k))/(\cos\psi)$ in (3.22) has itssmallest value when $\psi = \frac{\pi}{2}$. We show that this minimum is $2k'K/\pi$. By the special values $\mathrm{sn}\,K = 1$, $\mathrm{cn}\,K = 0$, $\mathrm{dn}\,K = k'$ ([6, #122.02], [40, p. 499]), the derivative $\frac{d}{du}\,\mathrm{cn}\,u = -\mathrm{sn}\,u\,\mathrm{dn}\,u$ ([6, #731.02], [40, pp. 492, 3]), and L'Hôpital's Rule, it is easy to see that

$$\lim_{\psi \to \frac{\pi}{2}} \frac{\mathrm{cn}(\frac{2K}{\pi}\psi, k)}{\cos\psi} = \frac{2K}{\pi} \lim_{\psi \to \frac{\pi}{2}} \frac{\mathrm{sn}(\frac{2K}{\pi}\psi, k)\,\mathrm{dn}(\frac{2K}{\pi}\psi, k)}{\sin\psi} = \frac{2k'K}{\pi}.$$

This completes the proof of (3.22) and hence of the lemma.

3. Dilatations of a homeomorphism.

In order to obtain asymptotic formulas for the moduli of $R_{3,d}$ and $R_{3,s}$ we need additional tools.

First, we introduce the concept of the modulus of a family of arcs in 3-space. A set γ in E^3 is said to be an arc if it is homeomorphic to a linear interval. If E_1 and E_2 are two sets which meet the closure of a set E, then an arc γ is said to join E_1 and E_2 in E if $\gamma \subset E$ and $\bar{\gamma} \cap E_1 \neq \phi$, $\bar{\gamma} \cap E_2 \neq \phi$.

Now let Γ be a family of locally rectifiable arcs in E^3. We denote by $F(\Gamma)$ the family of density functions ρ which are non-negative and Borel measurable in E^3 and for which

$$\int_{\gamma} \rho \, ds \geq 1 \tag{3.23}$$

for each arc $\gamma \in \Gamma$. Then following Väisälä [39] we define the modulus $M(\Gamma)$ of the arc family Γ as

$$M(\Gamma) = \inf_{\rho} \int_{E^3} \rho^3 \, d\omega, \tag{3.24}$$

where the infimum is taken over all functions $\rho \in F(\Gamma)$. (See also [10] and [15].) Non-locally rectifiable arcs may be included in the family simply by letting $F(\Gamma)$ be the family of functions ρ for which (3.23) holds whenever $\gamma \in \Gamma$ is locally rectifiable.

We can now state the following lemma [11, Theorem 1] concerning the modulus of an arc family.

LEMMA 3.3. *Suppose* R *is a space ring and that* Γ *is the family of arcs which join the components of* ∂R *in* R. *Then*

$$\operatorname{cap} R = M(\Gamma). \tag{3.25}$$

Next suppose Ω and Ω' are finite domains in E^3 and that f is a homeomorphism of Ω onto Ω'. If R is a bounded space ring with $\bar{R} \subset \Omega$, then $R' = f[R]$ is a bounded space ring with $\bar{R}' \subset \Omega'$. The inner dilatation $K_I(f)$ and the outer dilatation $K_0(f)$ of the homeomorphism f are defined as

$$(3.26) \qquad K_I(f) = \sup_R \frac{\operatorname{mod} R}{\operatorname{mod} R'} , \quad K_0(f) = \sup_R \frac{\operatorname{mod} R'}{\operatorname{mod} R} ,$$

where the suprema are taken over all bounded rings R with $\bar{R} \subset \Omega$ for which $\operatorname{mod} R$ and $\operatorname{mod} R'$ are not both infinite. Then

$$K(f) = \max (K_I(f) , K_0(f))$$

is called the maximal dilatation of f([9]).

We remark that if f is a homeomorphism of a ring R onto a ring R', then it follows from Lemma 2.19 and (3.26) that

$$(3.27) \qquad K_I(f) \geq \frac{\operatorname{mod} R}{\operatorname{mod} R'} , \quad K_0(f) \geq \frac{\operatorname{mod} R'}{\operatorname{mod} R} .$$

([9]) Although $K = K(k)$ and $K = K(f)$ are used to denote an elliptic integral and the maximal dilatation of a homeomorphism, respectively, the context will always make clear the meaning of K.

It is important in our work to have available the so-called analytic definition of the dilatations of a homeomorphism f. Suppose that f is a homeomorphism of a finite domain Ω in E^3. For each point $x \in \Omega$ we let

$$L(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}, \quad \ell(x) = \liminf_{y \to x} \frac{|f(y) - f(x)|}{|y - x|},$$

(3.28)

$$J(x) = \limsup_{r \to 0} \frac{m_3(B')}{m_3(B)},$$

where $B = B^3(x, r)$ is the ball with center x and radius r, and $B' = f[B]$. At a point of differentiability $L(x)$ and $\ell(x)$ are just the maximum and minimum stretchings under f, and $J(x)$ is the absolute value of the Jacobian.

LEMMA 3.4. Suppose that f is a homeomorphism of a finite domain $\Omega \subset E^3$. If f is differentiable with $J > 0$ a.e. in Ω and if f is ACL in Ω, then

$$K_I(f)^2 = \operatorname{ess\,sup}_{x \in \Omega} \frac{J(x)}{\ell(x)^3}, \quad K_O(f)^2 = \operatorname{ess\,sup}_{x \in \Omega} \frac{L(x)^3}{J(x)}.$$

This is lemma 1.1 of [15]

4. Behavior of $\mod R_{3,d}$ for small a.

Let us turn now to the study of the asymptotic behavior of these extremal rings as a tends to 0 and to 1.

THEOREM 3.2. As a function of a, $(\mod R_{3,d}(a) - \log 1/a)$ is monotone increasing in the interval $0 < a < 1$, and

$$\lim_{a \to 0} (\mod R_{3,d}(a) - \log 1/a) = c_1 . \tag{3.29}$$

where $\frac{3}{2}\log 2 - \frac{\pi}{4} \leq c_1 \leq \log 2$.

Proof: For let $0 < a' < a < 1$. If R is the image of $R_{3,d}(a')$ under the (conformal) mapping $y = (a/a')x$, then $\mod R_{3,d}(a') = \mod R$. Since $R_{3,d}(a)$ and $1 < |x| < a/a'$ are two disjoint rings separating the boundary components of R, we see from Lemma 2 of [14] that

$$\mod R_{3,d}(a) + \log a/a' \leq \mod R = \mod R_{3,d}(a') ,$$

from which the monotoneity follows (Cf. Lemma 6 in [14]).

To obtain the upper bound we make use of Theorem 3.1, (3.4), and the limit $\lim_{k \to 0} k^2 \exp(\pi K'/K) = 16$ (Cf. [6, #112.04, #901.00], [8, p. 88]) to conclude that

$$\lim_{a \to 0} (\operatorname{mod} R_{3,d} - \log 1/a) \leq \lim_{a \to 0} (\operatorname{mod} R_2 - \log 1/a)$$

$$(3.30) \qquad = \lim_{k \to 0} \left[\frac{\pi K'}{4K} + \log k^{1/2} \right] = \log 2 .$$

To obtain the lower bound we shall apply the left hand side of (3.2) to $\operatorname{mod} R_{3,d}$. Thus by (3.8) and (3.13) we must determine

$$(3.31) \qquad D_1(f_d, r)^2 = \min_{|x_1+ix_2|=r} \frac{2K}{\pi} |\sin \psi| \left[\frac{C^2 D^2}{S^2} + k'^4 \frac{s^2 c^2}{d^2} \right]^{1/2} ,$$

where S, C, D, s, c, d have the meanings assigned in (3.11). We assert that the minimum is achieved, for each r, when $\psi = \frac{\pi}{2}$. To see this, we use (3.14), the identity $sn^2 z + cn^2 z = 1$ ([6, #121.00], [40, p. 493]), and the relation $C^2 D^2 + (1-k)^2 S^2 = (1-kS^2)^2$ proved in § 3.2 after (3.15), to rewrite (3.31) as

$$(3.32) \qquad D_1(f_d, r)^2 = \frac{2K}{\pi} \min_{\psi} |\sin \psi| \left[\frac{(1-kS^2)^2}{S^2} - (1-k)^2 \, cn^2(z, k_1') \right]^{1/2} ,$$

where $z = (1+k) \frac{2K}{\pi} \log \frac{r}{A}$ and $k_1' = \frac{1-k}{1+k}$.

Then because $\sin \psi$ is maximum when $\psi = \frac{\pi}{2}$, it is clear from (3.32) that it will be sufficient to show that $g(\psi) = (\sin \psi)(1-kS^2)/S$ is minimum when $\psi = \frac{\pi}{2}$. But since $g(\frac{\pi}{2}) = 1-k$ ([6, #122.02],

[40, p. 499]) , it follows from the first half of Lemma 3.2 that

$$g(\psi) = \frac{(\sin \psi)}{S} (1 - kS^2) \geq 1 - k = g(\frac{\pi}{2}) ,$$

and we conclude that the minimum in (3.32) occurs when $\psi = \frac{\pi}{2}$.

Hence by (3.31) ,

$$D_1(f_d, r)^2 = \frac{2K}{\pi} k'^2 \frac{sc}{d} ,$$

and by Lemma 3.1 this means that

$$\text{(3.33)} \qquad \text{mod } R_{3,d} \geq k' (\frac{2K}{\pi})^{1/2} \int_A^1 (\frac{sc}{d})^{1/2} \frac{dr}{r} .$$

Later, evaluating the integral in (3.33) , we will prove

LEMMA 3.5. <u>For each</u> a , $0 < a < 1$,

$$\text{(3.34)} \qquad \text{mod } R_{3,d}(a) \geq \frac{1}{2} (\frac{\pi}{2K})^{1/2} [K([\frac{1+k'}{2}]^{1/2}) - K([\frac{1-k'}{2}]^{1/2})] ,$$

<u>where</u> $k = a^2$, $k' = (1-a^4)^{1/2}$, <u>and</u> $K = K(k)$ <u>denotes the elliptic integral</u> (3.5) .

Using Lemma 3.5 and the fact that $K(0) = \frac{\pi}{2}$, we now have

$$\lim_{a \to 0} (\mathrm{mod}\, R_{3,d} - \log 1/a) \geq \frac{1}{2} \lim_{k \to 0} [\log k + (\frac{\pi}{2K})^{1/2} K([\frac{1+k'}{2}]^{1/2})] - \frac{\pi}{4}$$

(3.35)

$$= \frac{1}{2} \lim_{k \to 0} [\log k + K([\frac{1+k'}{2}]^{1/2})] - \frac{\pi}{4} .$$

Now the limit $\lim_{k \to 0} (K' - \log 4/k) = 0$ ([6, #112.01], [40, p. 521]) gives

$$\lim_{k \to 0} [\log k + K([\frac{1+k'}{2}]^{1/2})] = \lim_{k \to 0} [\log k + \log 4[\frac{2}{1-k'}]^{1/2}]$$

$$= \lim_{k \to 0} \log 4[2(1+k')]^{1/2} = \log 8 .$$

This result, together with (3.35) , shows that

$$\lim_{a \to 0} (\mathrm{mod}\, R_{3,d} - \log 1/a) \geq \frac{3}{2} \log 2 - \frac{\pi}{4} ,$$

which is the lower bound in the theorem.

To prove Lemma 3.5 we need to evaluate the integral in (3.33) . By the half angle formulas ([6, #124.02], [18, p. 120]) , and the identity $dn^2 z + k^2 sn^2 z = 1$ ([6, #121.00], [40, p. 493]) we may write

$$(3.36)\qquad \frac{sc}{d} = \frac{sn(\frac{4K}{\pi}\log\frac{r}{A}, k')}{1+dn(\frac{4K}{\pi}\log\frac{r}{A}, k')} = \frac{1 - dn(\frac{4K}{\pi}\log\frac{r}{A}, k')}{k'^2\, sn(\frac{4K}{\pi}\log\frac{r}{A}, k')} .$$

Now making the change of variables

$$(3.37)\qquad sn(\frac{4K}{\pi}\log\frac{r}{A}, k') = \frac{t^2}{2 - t^2}$$

we have

$$(3.38)\qquad \left[\frac{1 - dn(\frac{4K}{\pi}\log\frac{r}{A}, k')}{k'^2\, sn(\frac{4K}{\pi}\log\frac{r}{A}, k')}\right]^{1/2} = \frac{[1 - \frac{1-k'}{2}t^2]^{1/2} - [1 - \frac{1+k'}{2}t^2]^{1/2}}{k't}$$

and

$$(3.39)\qquad \frac{dr}{r} = \frac{\pi}{4K}[(1-t^2)(1 - \frac{1-k'}{2}t^2)(1 - \frac{1+k'}{2}t^2)]^{-1/2}\, t\, dt .$$

When $r = A, 1$ then $t = 0, 1$, respectively, and by means of (3.36), (3.38), and (3.39) we may reduce (3.33) to

$$\text{mod } R_{3,d} \geq \frac{1}{2}(\frac{\pi}{2K})^{1/2}\left[\int_0^1 [(1-t^2)(1 - \frac{1+k'}{2}t^2)]^{-1/2}\, dt - \int_0^1 [(1-t^2)(1 - \frac{1-k'}{2}t^2)]^{-1/2}\, dt\right] .$$

Finally, consulting the definition (3.5) of the elliptic integral $K = K(k)$ we arrive at (3.34), and the lemma is proved.

5. Behavior of $\operatorname{mod} R_{3,s}$ for small a.

THEOREM 3.3. As a function of a, $(\operatorname{mod} R_{3,s}(a) - \log 1/a)$ is monotone decreasing in the interval $0 < a < 1$, and

$$\lim_{a \to 0} (\operatorname{mod} R_{3,s}(a) - \log 1/a) = c_2 , \tag{3.40}$$

where $\log 2 \leq c_2 \leq \frac{3}{2} \log 2 + \frac{\pi}{4}$.

Proof: The argument for monotoneity is the same as that given in Theorem 3.2.

To obtain the lower bound we make use of Theorem 3.1 as in (3.30) to conclude

$$\lim_{a \to 0} (\operatorname{mod} R_{3,s} - \log 1/a) \geq \lim_{a \to 0} (\operatorname{mod} R_2 - \log 1/a)$$

$$= \log 2 .$$

To obtain the upper bound we shall apply the right hand side of (3.2) to $\operatorname{mod} R_{3,s}$. Thus by (3.19) and (3.13) we must determine

$$D_2(f_s, r)^2 = \max_{|x_1+ix_2|=r} \frac{2K}{\pi} (\cos \psi)\left[k'^4 \frac{S^2}{C^2 D^2} + \frac{d^2}{s^2 c^2}\right]^{1/2} , \tag{3.41}$$

where S, C, D, s, c, d are the functions defined in (3.11).

We assert that this maximum is achieved, for each fixed r, when $\psi = 0$. Because of the special values $\operatorname{sn} 0 = 0$, $\operatorname{cn} 0 = \operatorname{dn} 0 = 1$ ([5, p. 9], [6, #122.01]), this is equivalent to the assertion that

$$(\cos^2\psi)\left[k'^4 \frac{S^2}{C^2D^2} + \frac{d^2}{s^2c^2}\right] \leq \frac{d^2}{s^2c^2}$$

for all ψ.

By a simple rearrangement and use of the identity $\cos^2\psi = 1 - \sin^2\psi$, this becomes

$$k'^4 \frac{s^2c^2}{d^2} \leq \frac{C^2D^2}{S^2} \tan^2\psi \quad . \tag{3.42}$$

We shall later prove

LEMMA 3.6. For each ψ,

$$\frac{2k'K}{\pi} \leq \frac{\operatorname{cn}\left(\frac{2K}{\pi}\psi, k\right)}{\operatorname{sn}\left(\frac{2K}{\pi}\psi, k\right)} \tan\psi \leq \frac{\pi}{2K} \quad . \tag{3.43}$$

Since $D = \operatorname{dn}\left(\frac{2K}{\pi}\psi, k\right) \geq k'$ ([6, #121.02], [40, pp. 493, 9]), it follows from (3.43) that

(3.44) $$\frac{CD}{S} \tan \psi \geq \frac{2k'^2 K}{\pi} .$$

Then, comparing (3.44) with (3.42), we see that to show (3.41) maximum when $\psi = 0$ it is sufficient to prove that

(3.45) $$\frac{sc}{d} \leq \frac{2K}{\pi} .$$

We proved earlier, after (3.14), that $sc/d \leq (1+k)^{-1}$. Thus (3.45) is implied by

(3.46) $$1 \leq (1+k) \frac{2K}{\pi} .$$

The latter inequality, however, is trivial, since $k \geq 0$ and $K \geq \frac{\pi}{2}$.

We conclude that (3.42) is valid, and that the maximum in (3.41) occurs for $\psi = 0$. Hence

$$D_2(f_s, r)^2 = \frac{2K}{\pi} \frac{d}{sc} .$$

By virtue of Lemma 3.1, this means that

(3.47) $$\operatorname{mod} R_{3,s} \leq \left(\frac{2K}{\pi}\right)^{1/2} \int_A^1 \left(\frac{d}{sc}\right)^{1/2} \frac{dr}{r} .$$

At the end of this section we will evaluate the integral in (3.47), thus proving

LEMMA 3.7. For each a, $0 < a < 1$,

$$(3.48) \qquad \operatorname{mod} R_{3,s}(a) \leq \frac{1}{2}\left(\frac{\pi}{2K}\right)^{1/2}\left[K\left(\left[\frac{1+k'}{2}\right]^{1/2}\right)+K\left(\left[\frac{1-k'}{2}\right]^{1/2}\right)\right],$$

where $k = a^2$, $k' = (1-a^4)^{1/2}$, and $K = K(k)$ denotes the elliptic integral (3.5).

Then, by (3.48) and the same proof given in § 3.4 after the statement of Lemma 3.5, we achieve

$$(3.49) \qquad \lim_{a \to 0} (\operatorname{mod} R_{3,s} - \log 1/a) \leq \frac{3}{2}\log 2 + \frac{\pi}{4},$$

which is the upper bound in the theorem.

We remark that the bounds in (3.40) were obtained by Gehring ([11, § 9] and [14, § 21]), the upper bound by a quite different method. Our method has the advantage, however, of yielding not only the limit (3.49), but also an upper bound for $\operatorname{mod} R_{3,s}(a)$ in terms of a (See Lemma 3.7).

Proof of Lemma 3.6. By the infinite product expansion for $\operatorname{tn} u = (\operatorname{sn} u)/(\operatorname{cn} u)$ ([6, # 909.03] [8, p. 74]), $0 \leq k < 1$,

$$(3.50)\qquad \frac{\operatorname{tn}\left(\frac{2K}{\pi}\psi, k\right)}{\tan\psi} = \frac{2K}{\pi}\prod_{n=1}^{\infty} A_n(\psi),$$

where ([8])

$$A_n(\psi) = \left[\frac{1+q^{2n}}{1-q^{2n}}\right]^2 \frac{1 - 2q^{2n}\cos 2\psi + q^{4n}}{1 + 2q^{2n}\cos 2\psi + q^{4n}} .$$

Since $0 \le q < 1$, we see that $1 = A_n(0) \le A_n(\psi)$ for all ψ. Therefore

$$\frac{\operatorname{tn}\left(\frac{2K}{\pi}\psi, k\right)}{\tan\psi} \ge \frac{2K}{\pi} ;$$

this is the second half of (3.43).

Moreover, $A_n(\psi) \le A_n\left(\frac{\pi}{2}\right)$ for all ψ. Using the special value $\operatorname{sn}K = 1$ ([6, #122.02], [40, p. 499]) and the limit at the end of § 3.2 we obtain

$$\lim_{\psi \to \frac{\pi}{2}} \frac{\operatorname{tn}\left(\frac{2K}{\pi}\psi, k\right)}{\tan\psi} = \lim_{\psi \to \frac{\pi}{2}} \frac{\cos\psi}{\operatorname{cn}\left(\frac{2K}{\pi}\psi, k\right)} = \frac{\pi}{2k'K} .$$

This establishes the first half of (3.43).

Proof of Lemma 3.7. We need to evaluate the integral in (3.47). Making the change of variables (3.37) we have

$$(3.51)\qquad \left[\frac{1 + \operatorname{dn}\left(\frac{4K}{\pi}\log\frac{r}{A}, k'\right)}{\operatorname{sn}\left(\frac{4K}{\pi}\log\frac{r}{A}, k'\right)}\right]^{1/2} = \frac{\left[1 - \frac{1-k'}{2}t^2\right]^{1/2} + \left[1 - \frac{1+k'}{2}t^2\right]^{1/2}}{t} .$$

Then , proceeding exactly as in the proof of Lemma 3.5, we arrive at (3.48) by means of (3.47), (3.36), (3.51), and (3.39) .

6. Behavior of $\mod R_{3,d}$ as a tends to 1 .

THEOREM 3.4.

$$(3.52) \qquad \limsup_{a \to 1} \frac{\mod R_{3,d}(a)}{[1-a]^{1/2}} \leq \frac{\pi}{2} = 1.57 \ldots$$

and

$$(3.53) \qquad \liminf_{a \to 1} \frac{\mod R_{3,d}(a)}{[1-a]^{1/2}} \geq \frac{3}{6}^{1/2} = 0.288 \ldots .$$

Proof: To obtain the upper bound (3.52) let Γ_1 be the family of arcs in $R_{3,d}$ joining the components of $\partial R_{3,d}$, and let Γ be the subfamily of circular arcs γ which are normal to the boundary components. Since $\Gamma_1 \supset \Gamma$ we have $F(\Gamma_1) \subset F(\Gamma)$ and, by (3.24), $M(\Gamma_1) \geq M(\Gamma)$. We will prove that

$$(3.54) \qquad M(\Gamma) \geq \frac{32}{\pi} \frac{a^2}{1-a^2} .$$

Then, using Lemma 3.3 , together with (3.24) and (3.54) , we will have

$$\text{cap } R_{3,d} = M(\Gamma_1) \geq M(\Gamma) \geq \frac{32}{\pi} \frac{a^2}{1-a^2} \quad .$$

By definition (1.4) of the modulus, this is equivalent to

$$(3.55) \qquad \text{mod}^2 R_{3,d} = \frac{4\pi}{\text{cap } R_{3,d}} \leq \frac{\pi^2}{8} \frac{1-a^2}{a^2} \quad .$$

Then (3.52) follows directly from (3.55) .

To prove (3.54) let Ω be the intersection of the plane quadrant $x_1 > 0$, $x_2 > 0$ with the union of the curves in Γ considered as point sets, and let Ω' be the image of Ω under

$$(3.56) \qquad y_1 + iy_2 = f(x_1+ix_2) = \frac{1 + (x_1+ix_2)}{1 - (x_1+ix_2)} \quad .$$

Then Ω' is the quarter annulus $1 < |y_1+iy_2| < b = \frac{1+a}{1-a}$, $y_1>0$, $y_2>0$ (See Figure 3) .

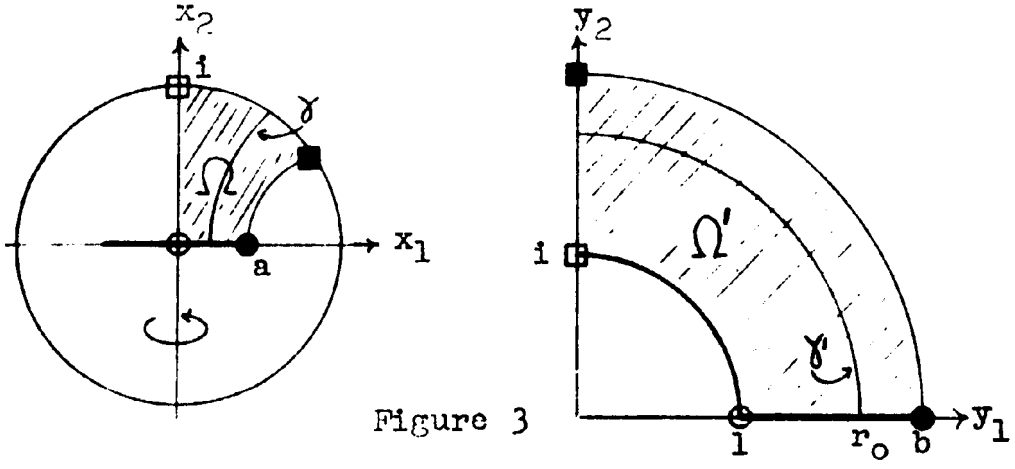


Figure 3

If g is the inverse of f, then

$$x_1 + ix_2 = g(y_1+iy_2) = \frac{(y_1+iy_2) - 1}{(y_1+iy_2) + 1} \tag{3.57}$$

and it is easy to see that

$$|g'(y_1+iy_2)| = \frac{2}{|y_1+iy_2+1|^2}\ , \quad x_1 = \frac{|y_1+iy_2|^2 - 1}{|y_1+iy_2+1|^2}\ . \tag{3.58}$$

Let $\rho \in F(\Gamma)$. Then for each $\gamma \in \Gamma$ in Ω we have, by virtue of (3.23),

$$1 \leq [\int_{\gamma} \rho\, ds\ ]^3 = [\int_{\gamma} \rho' |g'|\, ds']^3 \tag{3.59}$$

where $\rho = \rho(x_1+ix_2)$, $\rho' = \rho'(y_1+iy_2) = \rho\ (g(y_1+iy_2))$; and $\gamma' = f[\gamma]$, a quarter circle $r = r_o$, $0 < \varphi < \frac{\pi}{2}$, for some fixed r_o, $1 < r_o < b$ (See Figure 3). Then an application of Hölder's inequality to (3.59) yields

$$1 \leq [\int_{\gamma'} \rho'^3 x_1 |g'|^2 ds'] [\int_{\gamma'} |g'|^{1/2} x_1^{-1/2} ds']^2\ . \tag{3.60}$$

Using (3.58) we reduce (3.60) to

$$\int_{\gamma'} \rho'^3 x_1 |g'|^2 ds' \geq \left[\int_{\gamma'} \left(\frac{2}{|y_1+iy_2|^2 - 1}\right)^{1/2} ds'\right]^{-2}$$

(3.61)

$$= \left[\int_0^{\frac{\pi}{2}} \left(\frac{2}{r^2-1}\right)^{1/2} r\, d\varphi\right]^{-2} = \frac{2}{\pi^2}\,\frac{r^2-1}{r^2},$$

where (r, φ) are polar coordinates in the y_1+iy_2 plane.

Next we see that

$$\text{(3.62)} \qquad \int_{E^3} \rho^3\, d\omega \geq 2\int_0^{2\pi} \left(\int_\Omega \rho^3 x_1\, d\sigma\right) d\theta = 4\pi \int_{\Omega'} \rho'^3 x_1 |g'|^2\, d\sigma .$$

But by (3.61)

$$\int_{\Omega'} \rho'^3 x_1 |g'|^2\, d\sigma' = \int_1^b \left(\int_{\gamma'} \rho'^3 x_1 |g'|^2\, ds'\right) dr$$

$$\geq \frac{2}{\pi^2}\int_1^b (1 - r^{-2})\, dr$$

$$= \frac{2}{\pi^2}(b + b^{-1} - 2) = \frac{8}{\pi^2}\,\frac{a^2}{1-a^2} .$$

Hence (3.62) gives the estimate

$$\int_{E^3} \rho^3\, d\omega \geq \frac{32}{\pi}\,\frac{a^2}{1-a^2} ,$$

from which (3.54) then follows by taking the infimum over all $\rho \in F(\Gamma)$.

Next we obtain the lower bound (3.53). By (1.4) and Lemma 3.3,

$$\text{(3.63)} \qquad \operatorname{mod}^2 R_{3,d} = \frac{4\pi}{M(\Gamma)} ,$$

where Γ is the family of arcs which join the components of $\partial R_{3,d}$ in $R_{3,d}$. Then if $\rho \in F(\Gamma)$,

$$M(\Gamma) \leq \int_{E^3} \rho^3 \, d\omega$$

yields by (3.24) an upper bound for $M(\Gamma)$, and hence by (3.63) a lower bound for $\operatorname{mod} R_{3,d}$.

Let (r, θ, x_2) be cylindrical coordinates in 3-space, $r \geq 0$, and define

$$\text{(3.64)} \qquad \rho(x) = \begin{cases} \dfrac{2}{1-r} & \text{if } 0 \leq |x_2| \leq 1-r, \quad 0 \leq r \leq a, \\ \dfrac{2}{1-a} & \text{if } 0 \leq |x_2| \leq 1-r, \quad a \leq r \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

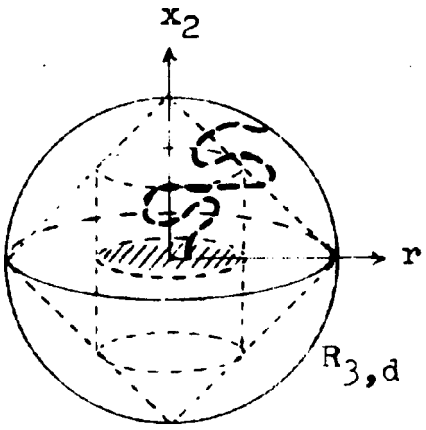


Figure 4

If $\gamma \in \Gamma$ is an arc joining the components of $\partial R_{3,d}$ (See Figure 4) then $ds^2 = r^2\, d\theta^2 + dr^2 + dx_2^2 \geq dr^2 + dx_2^2$ so that $ds \geq \frac{1}{2}(|dr| + |dx_2|)$. Now let r_1 be the smallest value of r on γ and let r_2 be the value of r where γ first meets the surface $r + |x_2| = 1$ as we move away from the central disk in $R_{3,d}$. Then $r_1 \leq r_2$ and $\rho(x) \geq 2/(1-r_1)$ **on the part of** γ **where** $r + |x_2| \leq 1$. **Thus**

$$\int_\gamma \rho\, ds \geq \frac{2}{1-r_1} \int_\gamma ds \geq \frac{1}{1-r_1}\left[\int_\gamma |dr| + \int_\gamma |dx_2|\right]$$

$$\geq \frac{1}{1-r_1}[(r_2-r_1) + (1-r_2)] = 1,$$

and $\rho \in F(\Gamma)$.

Then

$$\int_{E_3} \rho(x)^3 \, d\omega = \int_{|x|<1} \rho(x)^3 \, r \, dr \, d\theta \, dx_2$$

$$(3.65) \qquad = 32\pi \int_{r=0}^{a} \int_{x_2=0}^{1-r} \frac{dx_2 \, r \, dr}{(1-r)^3} + \frac{32\pi}{(1-a)^3} \int_{r=a}^{1} \int_{x_2=0}^{1-r} dx_2 \, r \, dr$$

$$= 32\pi \left[\log (1-a) + \frac{a}{1-a} \right] + \frac{16\pi(1+2a)}{3(1-a)} < \frac{48\pi}{1-a} \, .$$

Thus by (3.65) and Lemma 3.3 ,

$$\mathrm{mod}^2 R_{3,d} > \frac{1-a}{12}$$

and finally

$$\liminf_{a \to 1} \frac{\mathrm{mod}\, R_{3,d}}{[1-a]^{1/2}} \geq \left(\frac{1}{12} \right)^{1/2} = \frac{3}{6}^{1/2} \, .$$

7. Behavior of $\mathrm{mod}\, R_{3,s}$ as a tends to 1 .

THEOREM 3.5. *As a function of* a , $\left[\log \frac{1+a}{1-a} \right]^{1/2} \mathrm{mod}\, R_{3,s}(a)$ *is monotone increasing in the interval* $0 < a < 1$, *and*

$$(3.66) \qquad \lim_{a \to 1} \left[\log \frac{1+a}{1-a} \right]^{1/2} \mathrm{mod}\, R_{3,s}(a) = d \, ,$$

where $1/2 \leq d \leq q$,

$$q = \int_0^{\pi/2} (\sin t)^{-1/2}\, dt = 2.62 \ldots . \tag{3.67}$$

Proof: We first prove the asserted monotoneity. There exists a Möbius transformation which carries $R_{3,s}(a)$ conformally onto the space ring $R(b)$ consisting of the half space $x_2 > 0$ minus a closed slit from 1 to b along the x_2 axis (See Figure 5), with

$$b = \left[\frac{1+a}{1-a}\right]^2 . \tag{3.68}$$

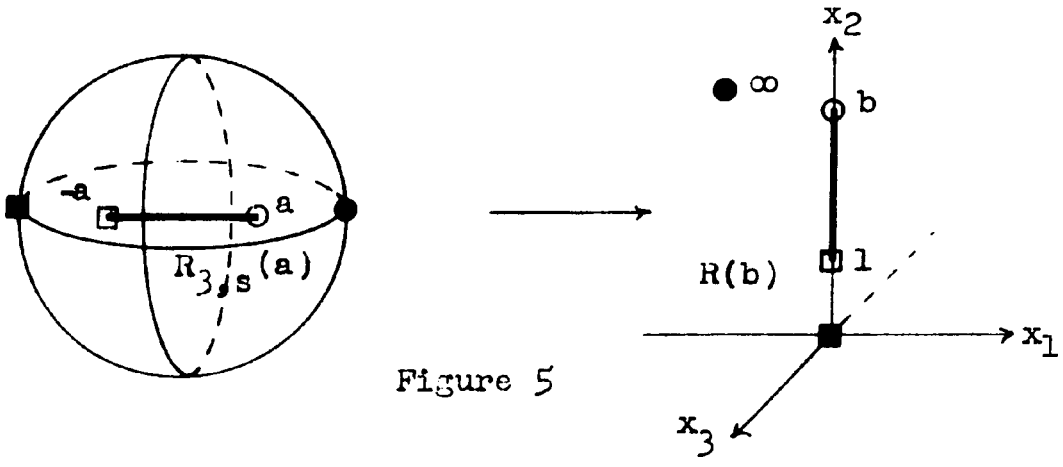


Figure 5

The mapping

$$g(x) = x|x|^{p-1}, \quad p = \frac{\log b'}{\log b} \tag{3.69}$$

is a homeomorphism of $\bar{E}^3$ onto itself carrying $R(b)$ onto $R(b')$. Assuming for definiteness that $1 < b' < b$, we have $0 < p < 1$.

We now wish to determine $K_0(g)$ so that we may apply (3.27) to arrive at monotoneity. Given a point x, consider a plane containing the segment $1 \leq x_2 \leq b$, $x_1 = x_3 = 0$ and the radius vector to x. Let (r, φ) be polar coordinates in this plane. In the radial direction at x the stretching factor is then $\left| \frac{\partial g}{\partial r} \right| = p\, r^{p-1}$. In this plane the stretching factor perpendicular to the radius vector at x is $\left| \frac{1}{r} \frac{\partial g}{\partial \varphi} \right| = r^{p-1}$; this must also be the stretching normal to the plane at x. Thus we conclude from (3.28) that $J(x) = p\,|x|^{3(p-1)}$ and $L(x) = |x|^{p-1}$. Hence by Lemma 3.4 it follows that

$$K_0(g) = p^{-1/2}. \tag{3.70}$$

Now

$$K_0(g) \geq \frac{\operatorname{mod} R(b')}{\operatorname{mod} R(b)}, \tag{3.71}$$

by (3.27). Therefore (3.69), (3.70), and (3.71) yield

$$\left[\frac{\log b}{\log b'} \right]^{1/2} \geq \frac{\operatorname{mod} R(b')}{\operatorname{mod} R(b)}.$$

Because we chose $b > b'$ this means that $(\log b)^{1/2} \operatorname{mod} R(b)$

is a monotone increasing function of b . Since by (3.68) a and b increase together, this shows that $[\log \frac{1+a}{1-a}]^{1/2} \bmod R_{3,s}$ increases monotonically with a as asserted.

To obtain the upper bound in the theorem let E be the slit $1 \leq x_2 \leq b$, $x_1 = x_3 = 0$ and Γ' the family of arcs γ' in $1 < |x| < b$, $x_2 > 0$ which join E to the plane annulus $1 < |x_1+ix_3| < b$, $x_2 = 0$ (See Figure 6) . Then Lemma 3.8 of [15] gives

$$M(\Gamma') = \frac{2\pi}{q^2} \log b \,, \tag{3.72}$$

where q is as in (3.67) .

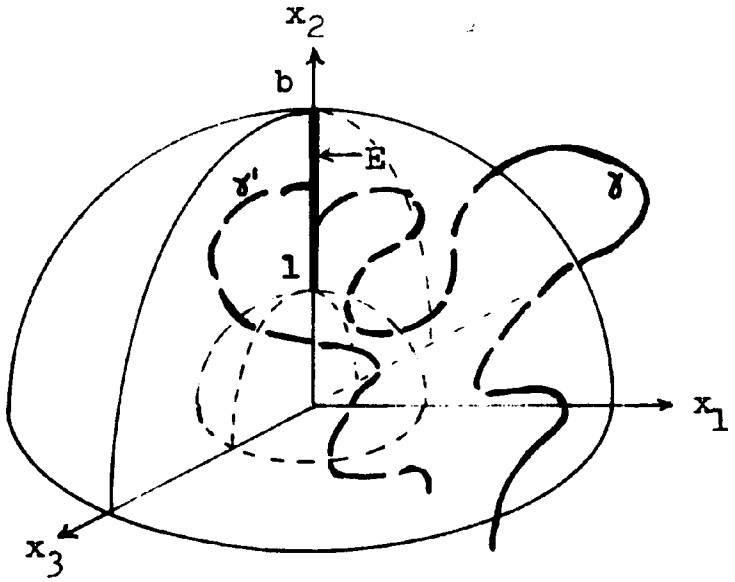


Figure 6

Now let Γ be the family of arcs γ joining the boundary components of $R(b)$. It follows from Lemma 3.3 that

$$\text{cap } R(b) = M(\Gamma) . \tag{3.73}$$

But since $\Gamma \supset \Gamma'$ we have $F(\Gamma) \subset F(\Gamma')$ and, by (3.24), $M(\Gamma) \geq M(\Gamma')$. Making use of (3.72) and (3.73), we have

$$\text{cap } R(b) = M(\Gamma) \geq M(\Gamma') = \frac{2\pi}{q^2} \log b . \tag{3.74}$$

Of course $\text{cap } R_{3,s} = \text{cap } R(b)$, since conformal capacity is invariant with respect to Möbius transformations. Then from (3.74), (3.68), and the definition (1.4) of the modulus of a space ring one arrives at

$$\text{mod}^2 R_{3,s} = \frac{4\pi}{\text{cap } R_{3,s}} \leq \frac{2q^2}{\log b} = \frac{q^2}{\log \frac{1+a}{1-a}} . \tag{3.75}$$

From (3.75) and the monotoneity it follows that

$$0 < \lim_{a \to 1} \left[\log \frac{1+a}{1-a} \right]^{1/2} \text{mod } R_{3,s} \leq q .$$

To evaluate the integral q in (3.67) we make the change of

variables

$$sn^2(u, (\tfrac{1}{2})^{1/2}) = \frac{2 \sin t}{1+ \sin t} .$$

Then

$$du = (2 \sin t)^{-1/2} \, dt$$

and $u = 0$, $K((\frac{1}{2})^{1/2})$ when $t = 0$, $\frac{\pi}{2}$, respectively. Hence

$$q = \int_0^{\pi/2} (\sin t)^{-1/2} dt = 2^{1/2} \int_0^{K((\frac{1}{2})^{1/2})} du = 2^{1/2} K((\tfrac{1}{2})^{1/2}) .$$

Now use of the tables [6, p. 323] or [24, p. 250] gives $q = 2.62\ldots$.

To obtain the lower bound we use the same method as for (3.53) in Theorem 3.4. Let (r, θ, x_1) be cylindrical coordinates in 3-space, $r \geq 0$, and define

$$(3.76) \quad \rho(x) = \begin{cases} \frac{2}{1-|x_1|} & \text{if } r \leq 1 - |x_1| , \ 0 \leq |x_1| \leq a , \\ \frac{2}{1-a} & \text{if } r \leq 1 - |x_1| , \ a \leq |x_1| \leq 1 , \\ 0 & \text{elsewhere.} \end{cases}$$

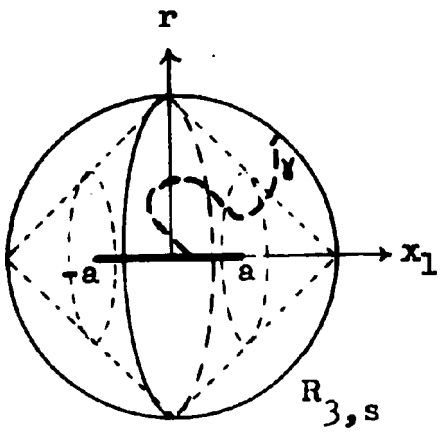


Figure 7

If $\gamma \in \Gamma$ is an arc joining the components of $\partial R_{3,s}$ (See Figure 7) it follows as in the proof of Theorem 3.4 that

$$\int_{\gamma} \rho \, ds \geq 1$$

and $\rho \in F(\Gamma)$.

Then

$$\int_{E^3} \rho(x)^3 \, d\omega = \int_{|x|<1} \rho(x)^3 \, r \, dr \, d\theta \, dx_1$$

$$= 32\pi \int_{x_1=0}^{a} \int_{r=0}^{1-x_1} \frac{r \, dr \, dx}{(1-x_1)^3} 1 + \frac{32\pi}{(1-a)^3} \int_{x_1=a}^{1} \int_{r=0}^{1-x_1} r \, dr \, dx_1 \tag{3.77}$$

$$= 16\pi \log \frac{1}{1-a} + \frac{16\pi}{3} < 16\pi \left[\log \frac{1+a}{1-a} + \frac{1}{3} \right].$$

Thus by (1.4), Lemma 3.3, and (3.77) we have

$$\mathrm{mod}^2 R_{3,s} = \frac{4\pi}{M(\Gamma)} > \frac{1}{4[\log \frac{1+a}{1-a} + \frac{1}{3}]}$$

and

$$\lim_{a \to 1} \log \frac{1+a}{1-a} \ \mathrm{mod}^2 R_{3,s} \geq \frac{1}{4} .$$

as asserted.

COROLLARY 3.1.

$$\lim_{a \to 1} \frac{\mathrm{mod}\ R_{3,d}(a)}{\mathrm{mod}\ R_{3,s}(a)} = 0 .$$

Proof: By (3.52) and (3.66) we have

$$0 \leq \limsup_{a \to 1} \frac{\mathrm{mod}\ R_{3,d}}{\mathrm{mod}\ R_{3,s}} \leq 2\pi \lim_{a \to 1} [\,(1-a) \log \frac{1+a}{1-a}\,]^{1/2} = 0 .$$

Chapter IV

THE COEFFICIENTS OF ELLIPSOIDS

1. Introduction.

In chapter III we introduced the inner dilatation $K_I(f)$ and the outer dilatation $K_0(f)$ of a homeomorphism f. It is obvious from the definition that

$$K_I(f) = K_0(f^{-1}), \quad K_0(f) = K_I(f^{-1}) \tag{4.1}$$

for each mapping f. Moreover, it follows from (4.1) and Theorem 5 of [13] that

$$K_I(f) \leq K_0(f)^2, \quad K_0(f) \leq K_I(f)^2, \tag{4.2}$$

so that these dilatations are simultaneously finite or infinite. In the former case, f is said to be quasiconformal; f is said to be K-quasiconformal if the maximal dilatation $K(f)$ (Cf. § 3.3) is bounded by K, $1 \leq K < \infty$.

Suppose Ω is a domain in E^3 which is homeomorphic to the unit ball. We set

$$K_I(\Omega) = \inf_f K_I(f), \qquad K_0(\Omega) = \inf_f K_0(f),$$

where the infima are taken over all homeomorphisms f of Ω onto the unit ball. These numbers $K_I(\Omega)$, $K_0(\Omega)$ are called the inner and outer coefficients of quasiconformality of Ω, respectively (Cf. [15]).

From (4.2) it follows that

$$K_I(\Omega) \leq K_0(\Omega)^2, \quad K_0(\Omega) \leq K_I(\Omega)^2.$$

Thus these coefficients are simultaneously finite or infinite. In the former case, Ω is said to be quasiconformally equivalent to a ball. A mapping f of a domain Ω onto the unit ball is said to be extremal for the inner or outer coefficient if $K_I(f) = K_I(\Omega)$ or $K_0(f) = K_0(\Omega)$, respectively.

The problem of extremal plane quasiconformal mappings was first treated by Grötzsch [17], who proved that the extremal quasiconformal mapping of one rectangle onto another with vertices corresponding in the same order is the affine mapping. As observed by Ahlfors [2], this actually shows that the extremal quasiconformal mapping of one quadrilateral ([10]) onto another is the composition of

([10]) A quadrilateral is a Jordan domain with four distinguished boundary points called vertices. Any such domain can be mapped conformally onto some rectangle with vertices corresponding.

a conformal mapping, an affine mapping, and a second conformal mapping. The work of Teichmüller [37] shows that there is a large class of extremal quasiconformal mappings which are given locally as the composition of such a sequence of mappings.

In view of this it is natural to ask if one of the extremal quasiconformal mappings of an ellipsoid $x_1^2/a^2 + x_2^2/b^2 + x_3^2/c^2 < 1$ onto a ball is the affine mapping $(y_1, y_2, y_3) = (x_1/a, x_2/b, x_3/c)$. We prove that for each coefficient of a prolate ellipsoid the affine mapping is not extremal, and that the same is true for the outer coefficient of all ellipsoids.

2 Hyperbolic density.

In our consideration of the prolate ellipsoid it will be convenient to make use of the hyperbolic density (See [28], pp. 45-51). Let Ω be a simply-connected plane domain, with more than one boundary point in the ζ plane, and let $\omega(\zeta)$ be a conformal mapping of Ω onto the upper half plane $\operatorname{Im} \omega > 0$. The element of length $d\lambda(\zeta)$ of the hyperbolic metric is

$$d\lambda(\zeta) = \frac{|d\omega|}{2 \operatorname{Im} \omega}, \tag{4.3}$$

and

$$(4.4) \qquad \rho(\zeta) = \rho(\zeta, \Omega) = \frac{d\lambda(\zeta)}{|d\zeta|}$$

is called the hyperbolic density for Ω.

The density $\rho(\zeta, \Omega)$ is a monotone function of Ω: that is, if $\Omega_2 \subset \Omega_1$, then $\rho(\zeta, \Omega_2) \geq \rho(\zeta, \Omega_1)$ for all $\zeta \in \Omega_2$

The function ρ satisfies the differential equation

$$(4.5) \qquad \Delta \log \rho = 4\rho^2,$$

where Δ is the Laplacian differential operator

If $\zeta = \zeta(z)$ maps Ω_z conformally onto Ω_ζ, then $\omega(\zeta(z))$ maps Ω_z conformally onto $\operatorname{Im} \omega > 0$, and it follows from the chain rule for derivatives that ρ satisfies the transformation law

$$(4.6) \qquad \rho(z, \Omega_z) = \rho(\zeta, \Omega_\zeta) \left|\frac{d\zeta}{dz}\right|$$

3. The prolate ellipsoid

We consider first the prolate ellipsoid We prove

THEOREM 4.1. Let E_p be any prolate ellipsoid Then the affine mapping of E_p onto the unit ball is not extremal for either

the inner or the outer coefficient. In fact, both dilatations of a prolate ellipsoid are bounded.

Proof: According to [25, p. 177] or [27, pp. 295, 6], the plane mapping

$$(4.7) \qquad y_1+iy_2 = f(x_1+ix_2) = k^{1/2}\, sn\left(\frac{2K}{\pi}\zeta, k\right)$$

maps the ellipse $x_1^2/a^2 + x_2^2/b^2 < 1$ conformally onto the unit disk, where $x_1+ix_2 = \sin\zeta$ and $\zeta = \alpha+i\beta$, K and K' are the elliptic integrals defined in (3.5), sn denotes the Jacobian elliptic sine function, and

$$(4.8) \qquad a = \cosh t, \quad b = \sinh t, \quad t = \frac{\pi K'}{4K}.$$

If the x_1+ix_2 and y_1+iy_2 planes are rotated about the real axes, there results a space mapping $y = f_p(x)$ of the prolate ellipsoid $E_p: x_1^2/a^2 + x_2^2/b^2 + x_3^2/b^2 < 1$ onto the unit ball. In the next section we will consider the analogous mapping $y = f_o(x)$ obtained from f by rotation about the imaginary axes in both planes; f_o maps the oblate ellipsoid $E_o: x_1^2/a^2 + x_2^2/b^2 + x_3^2/a^2 < 1$ onto the unit ball. (See Figure 8).

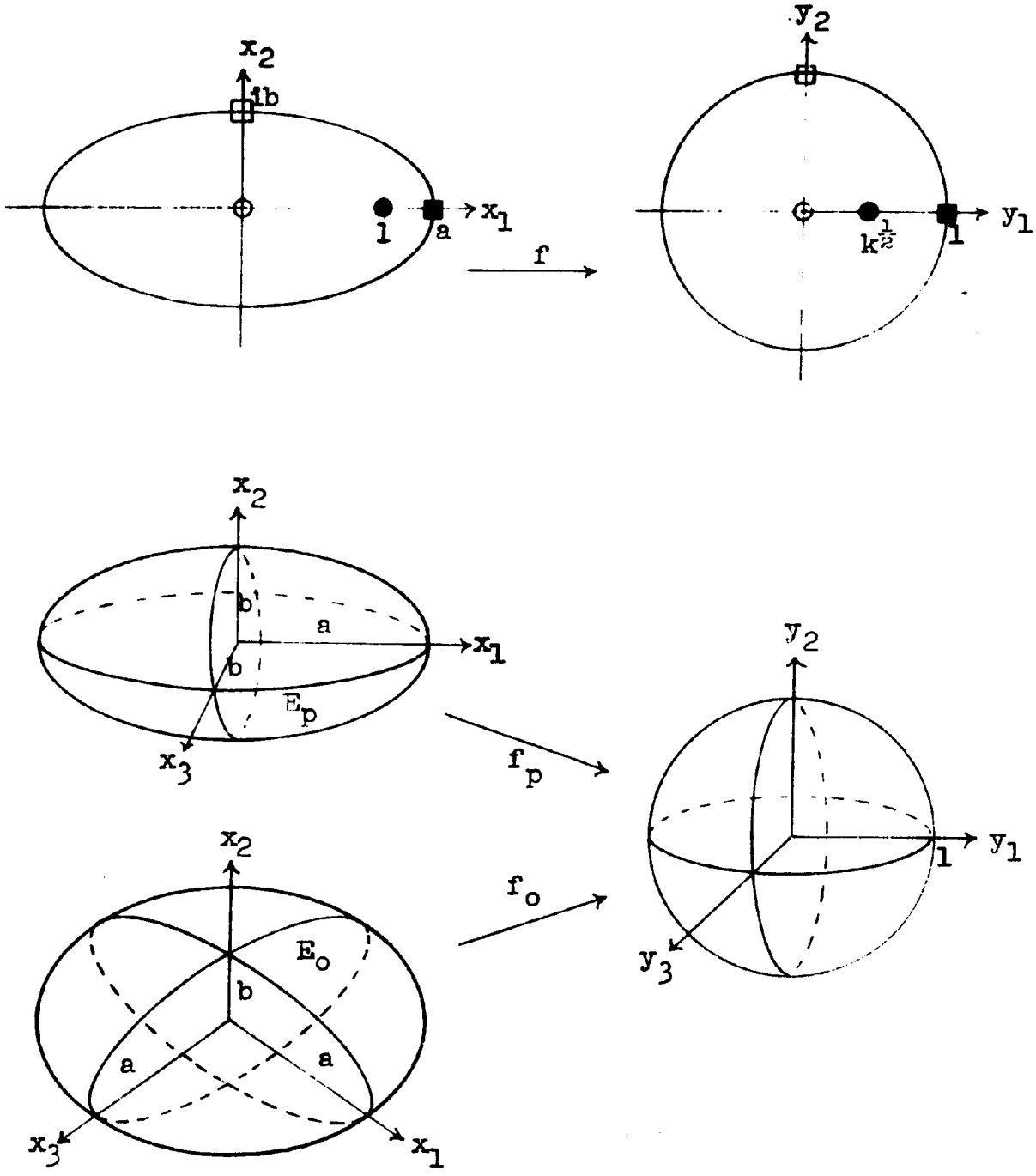


Figure 8

Now if ρ_1 is the hyperbolic density of the upper half plane $\Omega_1 : x_2 > 0$, $x_3 = 0$, then by (4.3) and (4.4),

$$\rho_1 = \rho(x_1 + ix_2, \Omega_1) = \frac{1}{2x_2} .$$

Next let Ω_2 denote the semi-ellipse $x_1^2/(\cosh 2t)^2 + x_2^2/(\sinh 2t)^2 < 1$, $x_2 > 0$, $x_3 = 0$ (See Figure 9). By virtue of the reflection principle, $f(x_1+ix_2)$ maps Ω_2 conformally onto the upper half plane $y_2 > 0$, $y_3 = 0$. Hence by (4.6) the hyperbolic density of Ω_2 is

$$\rho_2 = \rho(x_1+ix_2, \Omega_2) = \frac{|f'(x_1+ix_2)|}{2y_2} .$$

Since $\Omega_2 \subset \Omega_1$ it follows then that everywhere in Ω_2,

$$(4.9) \qquad \frac{\rho_2}{\rho_1} = \frac{x_2}{y_2} |f'| \geq 1 .$$

We remark that (4.9) is a form of Schwarz's Lemma (Cf. (3.20)).

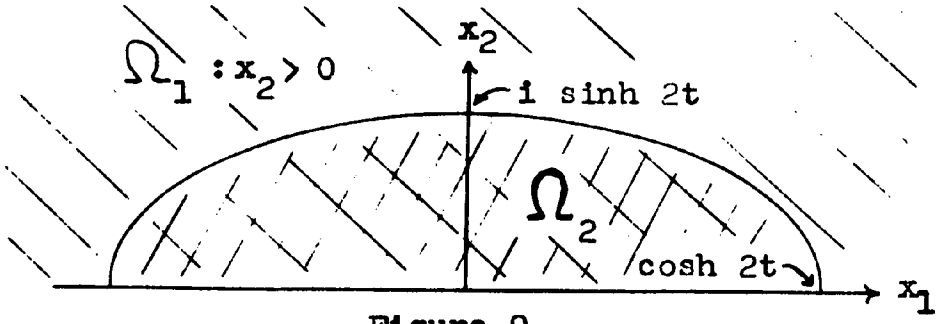


Figure 9

We are now able to determine the dilatations of f_p. By the axial symmetry of the mapping it is sufficient to consider a point x in the x_1+ix_2 plane. In every direction in this plane the stretching factor is $|f'(x_1+ix_2)|$, while normal to the plane it is y_2/x_2. We note that y_2/x_2 tends to $|f'|$ as x_2 tends to 0 and so y_2/x_2 has a well-defined positive limit at each point of the axis of rotation such that $-\cosh 2t < x_1 < \cosh 2t$, and $(x_2/y_2)|f'| = 1$ there.

By means of (3.28) and (4.9) we therefore see that $L(x)=|f'|$, $\ell(x) = y_2/x_2$, $J(x) = (y_2/x_2)\,|f'|^2$, and Lemma 3.4 gives

$$K_I(f_p) = \sup\left(\frac{x_2}{y_2}|f'|\right), \qquad K_O(f_p) = \sup\left(\frac{x_2}{y_2}|f'|\right)^{1/2}, \tag{4.10}$$

where the suprema are taken over all points $(x_1, x_2, 0)$ in the ellipse $x_1^2/a^2 + x_2^2/b^2 < 1$, $x_3 = 0$, with a, b as in (4.8).

From (4.9) and the differential equation (4.5) it follows that

$$\Delta \log u = \Delta \log \frac{\rho_2}{\rho_1} = 4(\rho_2^2 - \rho_1^2) \geq 0 \tag{4.11}$$

in Ω_2, where for convenience we employ the notation

$$u = \frac{x_2}{y_2}\,|f'|\,. \tag{4.12}$$

Because of (4.11) and [38, Theorem II.18], log u is subharmonic in Ω_2. Consequently ([22, p. 75], [38, Theorem II.12]) the suprema in (4.10) must occur on the boundary of the ellipse $x_1^2/a^2 + x_2^2/b^2 < 1$, $x_3 = 0$. In fact, by the symmetry of the mapping f, it is clear that this maximum will occur on the upper half of the boundary, namely on the arc γ: $x_1^2/a^2 + x_2^2/b^2 = 1$, $x_2 \geq 0$, $x_3 = 0$ (See Figure 10).

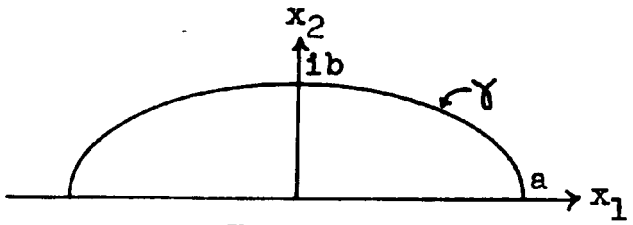


Figure 10

Since $\beta = t = \pi K'/4K$ on γ, where $x_1 + ix_2 = \sin(\alpha + i\beta)$, $f(x_1+ix_2) = k^{1/2}\ \mathrm{sn}(\frac{2K}{\pi}\alpha + i\frac{K'}{2}, k)$ there. By the addition theorem for sn ([5, p. 38], [6, #125.01]) we have

$$f(x_1 + ix_2) = k^{1/2}\,\frac{Sd + i\,CDsc}{1 - s^2D^2} \tag{4.13}$$

on γ, where, in a notation similar to (3.11),

$$s = \mathrm{sn}(\frac{K'}{2}, k'), \quad c = \mathrm{cn}(\frac{K'}{2}, k'), \quad d = \mathrm{dn}(\frac{K'}{2}, k'),$$

$$S = \mathrm{sn}(\frac{2K}{\pi}\alpha, k), \quad C = \mathrm{cn}(\frac{2K}{\pi}\alpha, k), \quad D = \mathrm{dn}(\frac{2K}{\pi}\alpha, k). \tag{4.14}$$

But the special values ([6, #122.10], [18, p. 120])

$$(4.15) \qquad s = (1+k)^{-1/2}, \quad c = k^{1/2}(1+k)^{-1/2}, \quad d = k^{1/2},$$

and the identity $k^2S^2 = 1 - D^2$ ([6, #121.00], [40, p. 493]) reduce (4.13) to

$$(4.16) \qquad f(x_1+ix_2) = k\,\frac{S + i\,\frac{CD}{1+k}}{1 - \frac{D^2}{1+k}} = \frac{(1+k)S + iCD}{1+kS^2}.$$

Next, an application of the Gauss Transformation ([5 pp. 72 3], [6, #164.01-.02]) to (4.16) gives

$$(4.17) \qquad f(x_1+ix_2) = sn\left(\frac{2K_1}{\pi}\alpha, k_1\right) + i\,cn\left(\frac{2K_1}{\pi}\alpha, k_1\right)$$

or γ, where as usual $k_1 = \frac{2k^{1/2}}{1+k}$, $k_1' = \frac{1-k}{1+k}$, and

$$(4.18) \qquad K_1 = K(k_1) = (1+k)K, \quad K_1' = K(k_1') = (1+k)\frac{K'}{2}$$

Then by the derivatives for the elliptic functions ([6, #731.01-.02], [40, pp. 492, 3]) and (4.18) we have

$$f' = \frac{2K_1}{\pi}\,\frac{dn\left(\frac{2K_1}{\pi}\alpha, k_1\right)\left[cn\left(\frac{2K_1}{\pi}\alpha, k_1\right) - i\,sn\left(\frac{2K_1}{\pi}\alpha, k_1\right)\right]}{\cos(\alpha + it)}$$

By using the identity $sn^2 z + cn^2 z = 1$ ([6, #121.00], [40, p. 493]) we achieve simply

$$(4.19) \qquad |f'| = \frac{2K_1}{\pi} \frac{dn(\frac{2K_1}{\pi}\alpha, k_1)}{[\cos^2\alpha + \sinh^2 t]^{1/2}} .$$

But on γ (See Figure 10), $x_2 = \text{Im} \sin(\alpha + it) = \cos\alpha \sinh t$, and by (4.17) $v_2 = cn(\frac{2K_1}{\pi}\alpha, k_1)$ there. Consequently, everywhere on γ,

$$(4.20) \qquad u = \frac{x_2}{v_2}|f'| = \frac{2K_1}{\pi} \frac{dn(\frac{2K_1}{\pi}\alpha, k_1)}{cn(\frac{2K_1}{\pi}\alpha, k_1)} \frac{\cos\alpha \sinh t}{[\cos^2\alpha + \sinh^2 t]^{1/2}} .$$

By the special values $cn\, 0 = dn\, 0 = 1$ ([6, #122.01], [40, p. 493]) we see that (4.20) reduces to $(2K_1/\pi)\tanh t$ when $\alpha = 0$. We will show that this is the largest value of u on γ (See Figure 10). But this is obviously equivalent to

$$\frac{dn^2(\frac{2K_1}{\pi}\alpha, k_1)}{cn^2(\frac{2K_1}{\pi}\alpha, k_1)} \cdot \frac{\cos^2\alpha \cosh^2 t}{\cos^2\alpha + \sinh^2 t} \leq 1 ,$$

for all k_1, $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, where $t = \frac{\pi K'}{4K}$. But recognizing that $\frac{\pi K'}{4K} = \frac{\pi K_1'}{2K_1}$ by (4.18), and replacing k_1 by k, we see that this inequality is equivalent to

LEMMA 4.1. For all k, $0 \le k < 1$, and all α, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$,

$$\frac{dn^2\left(\frac{2K}{\pi}\alpha, k\right)}{cn^2\left(\frac{2K}{\pi}\alpha, k\right)} \quad \frac{\cos^2\alpha \cosh^2\frac{\pi K'}{2K}}{\cos^2\alpha + \sinh^2\frac{\pi K'}{2K}} \le 1 . \tag{4.21}$$

Assuming Lemma 4.1, whose proof we shall give later, we have by (4.10), (4.18), and (4.20)

$$K_I(f_p) = (1+k)\frac{2K}{\pi}\tanh\frac{\pi K'}{4K} = \frac{2K_1}{\pi}\tanh\frac{\pi K_1'}{2K_1} ,$$

$$K_O(f_p) = \left[(1+k)\frac{2K}{\pi}\tanh\frac{\pi K'}{4K}\right]^{1/2} = \left[\frac{2K_1}{\pi}\tanh\frac{\pi K_1'}{2K_1}\right]^{1/2} . \tag{4.22}$$

That these dilatations are bounded is a consequence of the limit

$$\lim_{k_1 \to 1}\frac{2K_1}{\pi}\tanh\frac{\pi K_1'}{2K_1} = \left[\lim_{k_1 \to 1} K_1'\right]\left[\lim_{k_1 \to 1}\frac{\tanh\frac{\pi K_1'}{2K_1}}{\frac{\pi K_1'}{2K_1}}\right] = \frac{\pi}{2} .$$

We wish to compare the dilatations (4.22) with those of the natural affine mapping g_p of E_p onto the unit ball. Here

$$g_p(x) = (x_1/a, x_2/b, x_3/b) .$$

Since $a > b$, we have $L(x) = 1/b$ $\ell(x) = 1/a$, $J(x) = 1/ab^2$ for the mapping g_p and by Lemma 3.4, (4.8), and (4.18),

$$K_1(g_p) = \frac{a}{b} \coth \frac{\pi K'}{4K} \coth \frac{\pi K_1'}{2K_1},$$

(4.23)

$$K_0(g_p) = \left(\frac{a}{b}\right)^{1/2} \left[\coth \frac{\pi K}{4K}\right]^{1/2} \left[\coth \frac{\pi K_1'}{2K_1}\right]^{1/2}.$$

Comparing (4.22) and (4.23) we see that to show the affine mapping not extremal for either coefficient it is sufficient to prove

$$\frac{2K_1}{\pi} \tanh^2 \frac{\pi K_1'}{2K_1} \leq 1 \tag{4.24}$$

for all $0 \leq k_1 < 1$, with strict inequality if $0 < k_1 < 1$. In fact, we can show

LEMMA 4.2. *For each* k, $0 \leq k < 1$,

$$\left[\frac{2k'K}{\pi}\right]^{1/2} \leq \tanh \frac{\pi K'}{2K} \leq \left[\frac{\pi}{2K}\right]^{1/2} \tag{4.25}$$

with strict inequality if $0 < k < 1$.

The second half of (4.25) is equivalent to (4.24), while the first half will be needed in § 4.4.

Assuming these two lemmas, the proof of Theorem 4.1 is complete. We note that as k tends to 0 the ratio of the axes of the prolate ellipsoid E_p tends to $\coth \infty = 1$; that is, the prolate ellipsoid degenerates to a ball. Our work here then shows that except in the trivial case each dilatation of f_p is smaller than the corresponding dilatation of the affine mapping g_p, so that the affine mapping of the prolate ellipsoid onto a ball is not extremal for either coefficient.

Proof of Lemma 4.1. By the identities $C^2 = 1 - S^2$, $D^2 = 1 - k^2 S^2$ ([6, #121.00], [40, p. 493]) (4.21) is reduced to showing that

$$(4.26) \qquad \begin{aligned} & S^2(\cos^2\alpha + \sinh^2 2t - k^2 \cos^2\alpha \cosh^2 2t) \\ & \leq \cos^2\alpha + \sinh^2 2t - \cos^2\alpha \cosh^2 2t \end{aligned}$$

Here t and S have the values assigned in (4.8) and (4.14), respectively, so that $2t = \pi K'/2K$. Employing the identities $\sin^2\alpha + \cos^2\alpha = 1$ and $1 + \sinh^2 2t = \cosh^2 2t$, we may rewrite (4.26) as

$$S^2(\sin^2\alpha \sinh^2 2t + k^2 \cos^2\alpha \cosh^2 2t) \leq \sin^2\alpha \sinh^2 2t .$$

A second use of the identity $C^2 = 1 - S^2$ and a simple rearrangement reduces this to the factored form

$$k'^2 \coth^2 2t \leq \frac{C^2}{S^2} \tan^2 \alpha \tag{4.27}$$

Now for Lemma 4.1 to be true it is necessary and sufficient that (4.27) hold for the most unfavorable α, $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$. But by Lemma 3.6,

$$\frac{C}{S} \tan \alpha \geq \frac{2k'K}{\pi}$$

for all α, with equality when $\alpha = \frac{\pi}{2}$. Hence, because of (4.27), (4.21) is equivalent to the assertion that

$$\coth \frac{\pi K'}{2K} \leq \frac{2K}{\pi} \tag{4.28}$$

for each k, $0 \leq k < 1$.

Now we let q denote Jacobi's Nome $\exp(-\pi K'/K)$. If $0 \leq k < 1$ then $0 \leq q < 1$, and it is easy to see that

$$\coth \frac{\pi K'}{2K} = \frac{1+q}{1-q} = 1 + 2\sum_{n=1}^{\infty} q^n \tag{4.29}$$

Next, for the same values of k and q, it follows from the Fourier series for dn u ([6, #908.03], [18, p. 286]) that $2K/\pi$ has the following expansion as a function of q:

$$\frac{2K}{\pi} = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \tag{4.30}$$

Comparing (4.29) with (4.30) we see that (4 28) is equivalent to the inequality

$$\sum_{n=1}^{\infty} q^n \leq 2\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \tag{4.31}$$

for $0 \leq q < 1$. But for these values of q and for $n \geq 1$ it is trivial that

$$q^n \leq 2\,\frac{q^n}{1+q^{2n}} ,$$

and (4.31) follows. This establishes (4.28) and hence Lemma 4.1.

Proof of Lemma 4.2. In our proof we shall need to use both (4.29) and

$$\tanh \frac{\pi K'}{2K} = \frac{1-q}{1+q} = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^n \tag{4 32}$$

Next we appeal to [6 #1050.01, #1051.01] or [31, pp. 314, 315, 343] for the series expansions

(4.33) $$\left[\frac{2K}{\pi}\right]^{1/2} = 1+2\sum_{n=1}^{\infty} q^{n^2}$$

and

(4.34) $$\left[\frac{2k'K}{\pi}\right]^{1/2} = 1+2\sum_{n=1}^{\infty} (-1)^n q^{n^2}$$

Each of these expansions is valid for the range of k and q under consideration

By combining (4 29) and (4 33) we see that the second half of Lemma 4 2 is equivalent to the inequality

(4 35) $$\sum_{n=1}^{\infty} q^{n^2} \leq \sum_{n=1}^{\infty} q^{n},$$

$0 \leq q < 1$, with strict inequality if $0 < q < 1$ But for these values of q $q^{n^2} \leq q^n$ for each $n \geq 1$, and (4 35) follows Equality can occur in (4 35) only in the degenerate case $q = 0$, that is, when $k=0$

By combining (4 32) and (4 34) we see that the other half of the Lemma is equivalent to the inequality

(4 36) $$\sum_{n=1}^{\infty} (-1)^n q^n \geq \sum_{n=1}^{\infty} (-1)^n q^{n^2}$$

for each $0 \leq q < 1$ with strict inequality if $0 < q < 1$ But an elementary induction argument shows that

$$q^{2n} - q^{2n-1} \geq q^{(2n)^2} - q^{(2n-1)^2}$$

for $n \geq 1$ if $0 \leq q < 1$. There is strict inequality unless $q = 0$. This establishes (4.36) and therefore completes the proof of Lemma 4.2.

4. The outer coefficient of the oblate ellipsoid.

THEOREM 4.2. Let E_o be any oblate ellipsoid. Then the affine mapping of E_o onto the unit ball is not extremal for the outer coefficient.

Let E_o denote the ellipsoid $x_1^2/a^2 + x_2^2/b^2 + x_3^2/a^2 < 1$, where $a > b$ have the values assigned in (4.8). To prove this theorem for E_o it would be natural to try a method analogous to that used in the previous section. That is, let f_o denote the mapping of E_o onto the ball B obtained from the function f of (4.7) by rotating both planes about the imaginary axes (See Figure 8). We show, however, that this method is unsuccessful, and afterward prove the theorem in another way.

By the axial symmetry of $y = f_o(x)$ it is again sufficient to consider a point in the x_1+ix_2 plane to determine the dilatations of f_o. As in the previous problem the stretchings here in the directions of the x_1, x_2, x_3 axes are $|f'|$, $|f'|$, y_1/x_1, respectively. As

before, y_1/x_1 has a well-defined limit at all points of the axis of rotation inside E_o. There are now two cases to be considered.

(i) If $|f'| \geq y_1/x_1$ at the point $(x_1, x_2, 0)$, then $L(x) = |f'|$, $\ell(x) = y_1/x_1$, and $J(x) = (y_1/x_1)|f'|^2$, so that

$$\frac{J(x)}{\ell(x)^3} = \left[\frac{x_1}{y_1}|f'|\right]^2 , \quad \frac{L(x)^3}{J(x)} = \frac{x_1}{y_1}|f'| .$$

(ii) If $|f'| \leq y_1/x_1$, then $L(x) = y_1/x_1$, $\ell(x) = |f'|$, and

$$\frac{J(x)}{\ell(x)^3} = \left[\frac{x_1}{y_1}|f'|\right]^{-1} , \quad \frac{L(x)^3}{J(x)} = \left[\frac{x_1}{y_1}|f'|\right]^{-2} .$$

By Lemma 3.4 we must therefore have

$$K_1(f_o) = \sup \max \left(\frac{x_1}{y_1}|f'| , \left[\frac{x_1}{y_1}|f'|\right]^{-1/2} \right)$$

(4.37)

$$K_0(f_o) = \sup \max \left(\left[\frac{x_1}{y_1}|f'|\right]^{1/2} , \left[\frac{x_1}{y_1}|f'|\right]^{-1} \right) ,$$

where the suprema are taken over all points in the ellipse $x_1^2/a^2 + x_2^2/b^2 < 1$, $x_3 = 0$.

At the point $x_1 = \cosh t$, $x_2 = x_3 = 0$ we have $\alpha = \frac{\pi}{2}$, $\beta = t$. (Recall that $x_1 + ix_2 = \sin(\alpha + i\beta)$). Thus by (4.19), (4.18), and the fact that $y_1 = 1$ here (Cf. (4.17)) we have

(4.38)
$$\frac{x_1}{y_1}\,|f'| = \frac{2K_1}{\pi}\,k_1'\,\coth\frac{\pi K_1'}{2K_1}$$

at this point. Hence by (4.37) and (4.38) we know that

(4.39)
$$K_I(f_o) \geq \left[\frac{2K_1}{\pi}\,k_1'\,\coth\frac{\pi K_1'}{2K_1}\right]^{-1/2}$$

$$K_0(f_o) \geq \left[\frac{2K_1}{\pi}\,k_1'\,\coth\frac{\pi K_1'}{2K_1}\right]^{-1} .$$

Next consider the affine mapping

$$g_o(x) = (x_1/a,\ x_2/b,\ x_3/a)$$

of the oblate ellipsoid E_o onto the unit ball. At any point the stretchings in the directions of the coordinate axes are $1/a$, $1/b$, $1/a$, respectively. Since $a > b$ we have $L(x) = 1/b$, $\ell(x) = 1/a$, $J(x) = 1/a^2b$. Thus by (4.8),

$$\frac{J(x)}{\ell(x)^3} = \frac{a}{b} = \coth t\,, \quad \frac{L(x)^3}{J(x)} = \frac{a^2}{b^2} = \coth^2 t\,.$$

By reason of Lemma 3.4 and (4.18) this means that

$$K_I(g_o) = \left(\frac{a}{b}\right)^{1/2} = \left[\coth \frac{\pi K_1'}{2K_1}\right]^{1/2},$$

(4.40)

$$K_0(g_o) = \frac{a}{b} = \coth \frac{\pi K_1'}{2K_1} .$$

In Lemma 4.2 we proved that

$$\tanh \frac{\pi K_1'}{2K_1} > \left[\frac{2k_1' K_1}{\pi}\right]^{1/2}$$

for each k_1, $0 < k_1 < 1$. But then from (4.39) and (4.40) we see that each dilatation of f_o exceeds the corresponding dilatation of g_o.

Proof of Theorem 4.2. The proof we present is based on the work of Gehring and V[illegible] [15]. We define a radial mapping $y = h(x)$ from E_o to the unit ball as follows. If $P \in E_o$, $P \neq 0$, then P has a unique representation of the form $P = sQ$, where $Q \in \partial E_o$ and $0 < s < 1$. For each such $P \in E_o$ we define

$$h(P) = s^p h(Q), \quad h(Q) = \frac{Q}{|Q|}, \tag{4.41}$$

where p is some fixed positive number, and we let $h(0) = 0$.

Next for each $Q \in \partial E_o$ let $\psi(Q)$ denote the acute angle which the tangent plane at Q makes with the ray from 0 to Q, and let $\psi = \min \psi(Q)$ for all $Q \in \partial E_o$.

To determine ψ it is sufficient, by symmetry, to consider points in the plane $x_3 = 0$, that is, on the boundary of the ellipse $x_1^2/a^2 + x_2^2/b^2 < 1$, $x_3 = 0$. At a point $Q = (x_1, x_2, 0) \in \partial E_o$ the slope of the radius vector is x_2/x_1, while the slope of the tangent to the ellipse is $-(b^2x_1)/(a^2x_2)$. Thus

$$\tan \psi(Q) = \frac{x_2/x_1 + (b^2x_1)/(a^2x_2)}{1-b^2/a^2} = \frac{b^2}{1-b^2/a^2}\,\frac{1}{x_1x_2} \quad . \tag{4.42}$$

On the boundary of the ellipse we have, when $x_1^2 = a^2/2$,

$$x_1^2x_2^2 = \frac{b^2}{a^2}x_1^2(a^2-x_1^2) = a^2b^2/4 \ .$$

On the other hand, replacing the geometric by the arithmetic mean gives

$$2x_1x_2 = 2\left(\frac{b}{a}\right)^{1/2} x_1 \left(\frac{a}{b}\right)^{1/2} x_2 \le \frac{b}{a}x_1^2 + \frac{a}{b}x_2^2 = ab$$

there and we conclude that the maximum value of x_1x_2 on the boundary is $ab/2$. From this and (4.42) it follows that

$$\psi = \arctan\frac{2ab}{a^2-b^2} = \arcsin\frac{2ab}{a^2+b^2} \ .$$

Now take the exponent p in (4.41) to be

$$(4.43) \qquad p = \sin\psi = \frac{2ab}{a^2+b^2} \quad .$$

Then by Theorem 5.1 of [15] it follows that

$$(4.44) \qquad K_I(h)^2 \leq 2^{-1/2} \cot\frac{\psi}{2} \csc\frac{\psi}{2} \ ,$$

$$K_0(h)^2 \leq 2^{1/2} \cot\frac{\psi}{2} \cos\frac{\psi}{2} \ .$$

Then from (4.43), (4.44), and the half angle formulas from elementary trigonometry we have

$$(4.45) \qquad K_I(h)^2 \leq \frac{a^2}{b^2}\left[\frac{a^2+b^2}{2a^2}\right]^{1/2}, \quad K_0(h)^2 \leq \frac{a}{b}\left[\frac{2a^2}{a^2+b^2}\right]^{1/2} .$$

Now consider again the affine mapping g_o of E_o onto the unit ball. We proved earlier (Cf. (4.40)) that

$$(4.46) \qquad K_I(g_o) = \left(\frac{a}{b}\right)^{1/2}, \qquad K_0(g_o) = \frac{a}{b} \quad .$$

But comparison of (4.45) and (4.46) shows that $K_0(g_o) > K_0(h)$ because $a > b$. That is, the affine mapping is not extremal for the outer coefficient. The method gives no information about the inner coefficient, since the upper bound for $K_I(h)$ in (4.45) is larger than $K_I(g_o)$.

5. The inner coefficient of the oblate ellipsoid.

In Theorem 4.1 we found that for the prolate ellipsoid $E_p : x_1^2/a^2 + x_2^2/b^2 + x_3^2/b^2 < 1$ the affine mapping g_p has the dilatations

$$K_I(g_p) = \frac{a}{b}, \qquad K_0(g_p) = (\frac{a}{b})^{1/2},$$

while the mapping f_p showed that

$$K_I(E_p) = 0(1), \quad K_0(E_p) = 0(1)$$

as $\frac{a}{b}$ tends to ∞. Thus for large $\frac{a}{b}$ the affine mapping is very far from being extremal.

The situation is similar for the outer coefficient of the oblate ellipsoid $E_o : x_1^2/a^2 + x_2^2/b^2 + x_3^2/a^2 < 1$. For in Theorem 4.2 we saw that the affine mapping g_o has dilatations

$$K_I(g_o) = (\frac{a}{b})^{1/2}, \quad K_0(g_o) = \frac{a}{b},$$

and we produced a mapping h of E_o onto the unit ball such that

$$K_0(h) \leq (\frac{a}{b})^{1/2} [\frac{2a^2}{a^2+b^2}]^{1/4} = 0((\frac{a}{b})^{1/2})$$

as $\frac{a}{b}$ tends to ∞.

The problem of determining whether the affine mapping of an oblate ellipsoid is extremal for the inner coefficient appears to be much more difficult. For in this case Theorem 4.3 below shows that the inner dilatation of the affine map has the same order as $K_I(E_o)$ as $\frac{a}{b}$ tends to ∞.

THEOREM 4.3. Let E_o be the oblate ellipsoid $x_1^2/a^2 + x_2^2/b^2 + x_3^2/a^2 < 1$, $a > b$. Then $K_I(E_o)/(a/b)^{1/2}$ is monotonically decreasing as a/b tends to ∞, and

$$\lim_{a/b \to \infty} \frac{K_I(E_o)}{(a/b)^{1/2}} = A ,$$

where $0.47 \leq A \leq 1$.

Proof: We recall from (4.40) that

$$K_I(g_o) = \left(\frac{a}{b}\right)^{1/2} ,$$

where g_o is the affine mapping of E_o onto the unit ball. This proves that $K_I(E_o) \leq (a/b)^{1/2}$, so that $A \leq 1$.

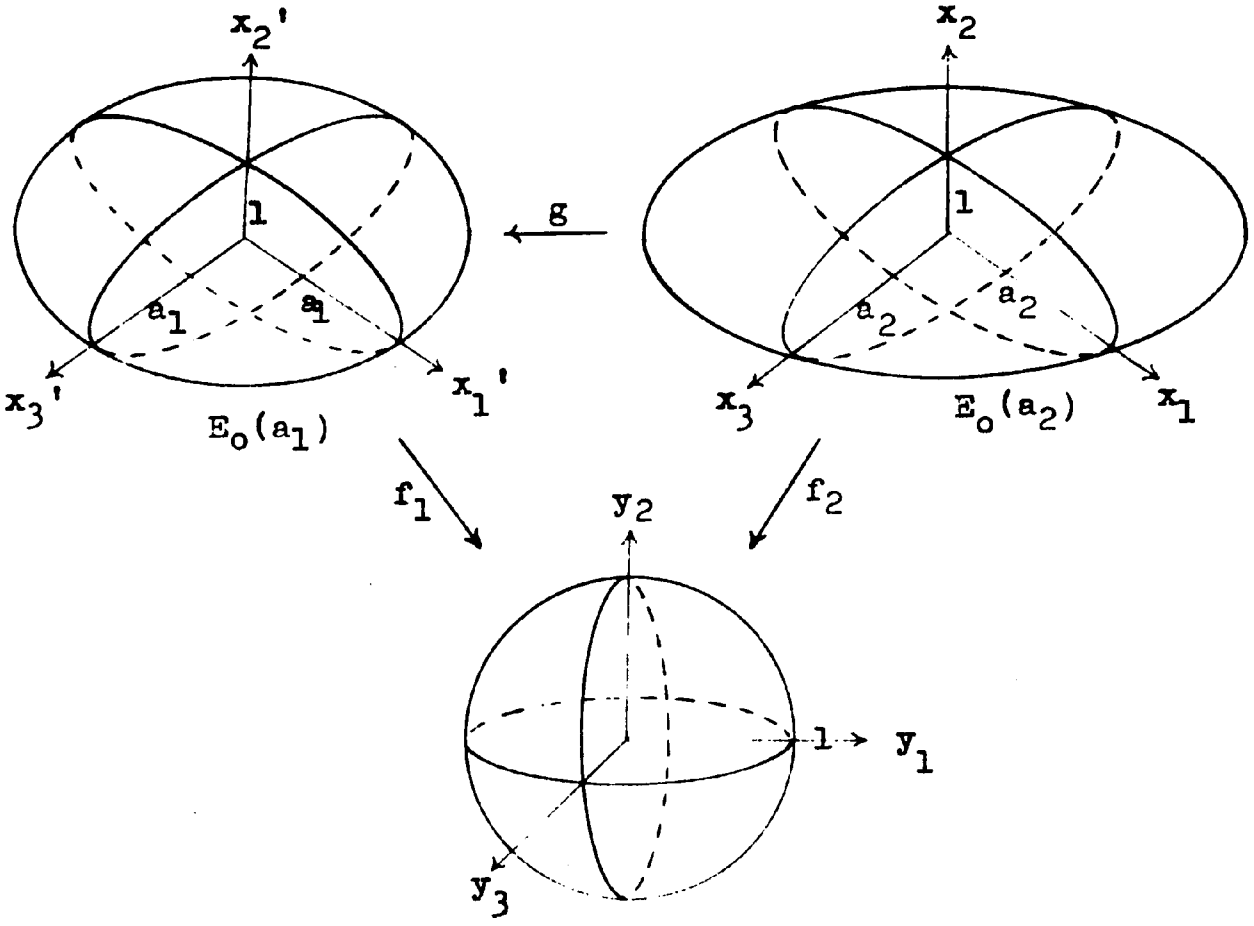


Figure 11

For convenience we now assume, as we obviously may, that $b = 1$. To prove the monotoneity, let f_1 be any quasi-conformal mapping of $E_o = E_o(a_1)$ onto the unit ball, and let

$$x' = g(x) = (a_1 x_1/a_2 , \ x_2 , \ a_1 x_3/a_2)$$

be the natural affine mapping of $E_o(a_2)$ onto $E_o(a_1)$, where $a_2 > a_1$ (See Figure 11) . Then $y = f_2(x)$ is a quasiconformal mapping of $E_o(a_2)$ onto the unit ball, where $f_2 = f_1 \circ g$. It is easy to see by Lemma 3.4 that

$$K_I(g) = (a_2/a_1)^{1/2} ,$$

and by definition (3.26) there results —

$$(4.47) \qquad K_I(f_2) \leq K_I(f_1) \, K_I(g) = (\frac{a_2}{a_1})^{1/2} \, K_I(f_1) .$$

Taking the infimum first over all f_2 on the left and then over all f_1 on the right in (4.47) we achieve

$$K_I(E_o(a_2)) \leq (\frac{a_2}{a_1})^{1/2} K_I(E_o(a_1)) ,$$

from which the asserted monotoneity follows.

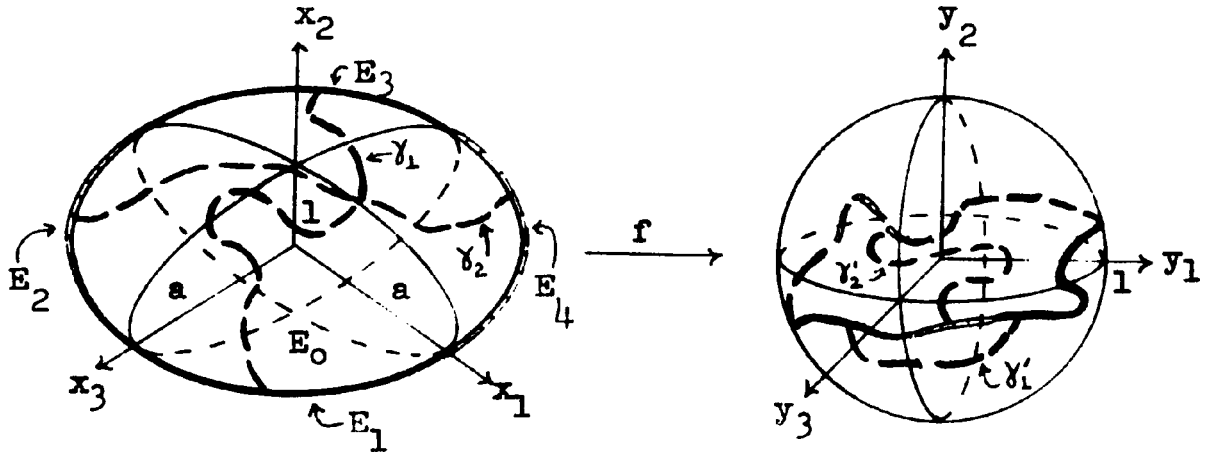


Figure 12

We now obtain the lower bound in the theorem. Our method is similar to an unpublished argument of Gehring and Väisälä. Let f be any quasiconformal mapping of E_o onto the unit ball. Let E_i denote the arc of the circle $x_1^2 + x_3^2 = a^2$, $x_2 = 0$ which is contained in the closed i-th quadrant of $x_2 = 0$, $i = 1, 2, 3, 4$, and let Γ_1 and Γ_2 be the families of arcs which join E_1 to E_3 and E_2 to E_4, respectively, in E_o. Let $\Gamma_i' = f[\Gamma_i]$ (See Figure 12).

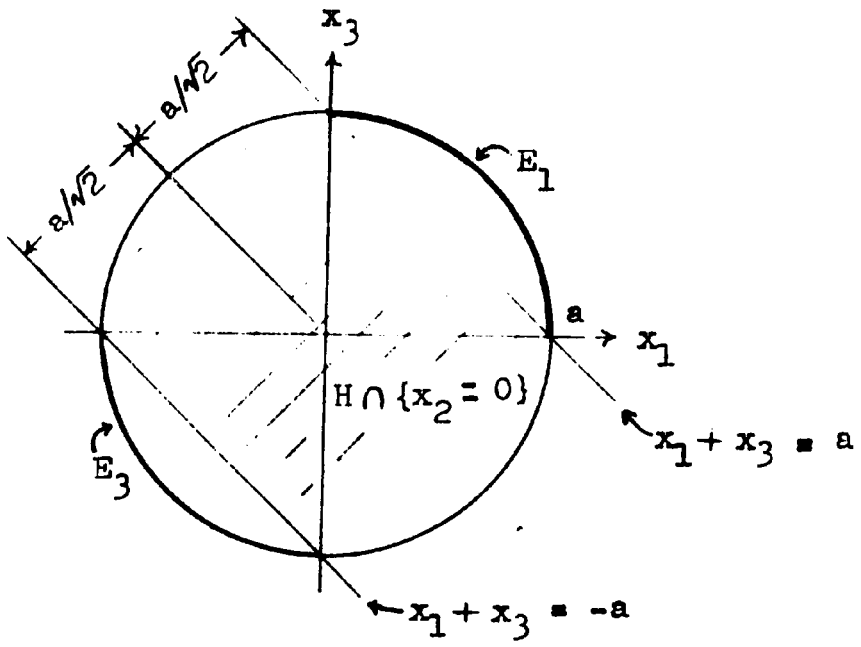


Figure 13

Now let H be the portion of E_o which is included between the two planes $x_1 + x_3 = \pm a$ (See Figure 13), and let ρ be the density function which is $1/(\sqrt{2}\,a)$ in H and 0 in C(H). Then

for each $\gamma_1 \in \Gamma_1$,

$$\int_{\gamma_1} \rho \, ds \geq 1$$

and $\rho \in F(\Gamma_1)$. Moreover, an elementary integration shows that

$$m_3(H) = \frac{2\pi}{a} \int_0^{a/\sqrt{2}} (a^2 - u^2) \, du = \frac{5\sqrt{2}\,\pi}{6} a^2 ,$$

where $u = (x_1 + x_3) / 2^{1/2}$. Thus

$$(4.48) \qquad M(\Gamma_1) = M(\Gamma_2) \leq \int_H \rho^3 \, d\omega = \frac{m_3(H)}{2\sqrt{2}\, a^3} = \frac{5\pi}{12a} .$$

Next, by Corollary 3.3 of [15] we have either

$$(4.49) \qquad M(\Gamma_1') \geq v(1) \text{ or } M(\Gamma_2') \geq v(1) ,$$

where for $s > 0$, $v(s)$ denotes the modulus of the family of arcs which join the segment $-1 \leq x_1 \leq 0$, $x_2 = x_3 = 0$ to the ray $s \leq x_1 \leq \infty$, $x_2 = x_3 = 0$ in the half space $x_2 > 0$. By a simple estimate based on the upper bound (3.48) it is easy to show that

$$(4.50) \qquad v(1) \geq \frac{2\pi}{(4.6)^2} .$$

Hence by (4.48), (4.49), (4.50), and Lemma 1.2 of [15]

we have

$$K_I(f)^2 \geq \max_i \frac{M(\Gamma_i')}{M(\Gamma_i)} \geq \frac{4.8}{(4.6)^2} a , \quad i = 1, 2,$$

so that

$$K_I(E_o) = \inf_f K_I(f) \geq (0.47)\, a^{1/2} .$$

Remark: Let Γ denote the family of curves in E_o which join the axis $-1 \leq x_2 \leq 1$, $x_1 = x_3 = 0$ to the circle $x_1^2 + x_3^2 = a^2$, $x_2 = 0$, and let $\Gamma' = f[\Gamma]$, where f is any quasiconformal mapping of E_o onto the unit ball. In (3.14) of [15] Gehring and Väisälä conjectured that $M(\Gamma') \geq (\frac{\pi}{q})^2$, where q is the value of the elliptic integral in (3.67). If this conjecture were true, then an argument similar to that given above shows that the lower bound for A in the theorem could be improved to $A \geq 0.63$.

6. The general ellipsoid.

In conclusion we consider a general ellipsoid, that is, an ellipsoid which is neither prolate nor oblate. We prove

THEOREM 4.4. Let E be a general ellipsoid. Then the affine mapping of E onto the unit ball is not extremal for the outer

coefficient.

Proof: Let $1 < b < a < \infty$ and let E denote the general ellipsoid $x_1^2/a^2 + x_2^2 + x_3^2/b^2 < 1$. The affine mapping

$$y = g(x) = (x_1/a, x_2, x_3/b)$$

of E onto the unit ball may be regarded as the composition of two affine mappings -- an affine mapping

$$x' = g_1(x) = (bx_1/a, x_2, x_3)$$

of E onto the oblate ellipsoid $E_o : x_1'^2/b^2 + x_2'^2 + x_3'^2/b^2 < 1$, followed by a second affine mapping

$$y = g_2(x') = (x_1'/b, x_2', x_3'/b)$$

of E_o onto the unit ball (See Figure 14). For this composition of mappings we shall show that

(4.51) $$K_0(g) = K_0(g_1)\, K_0(g_2) .$$

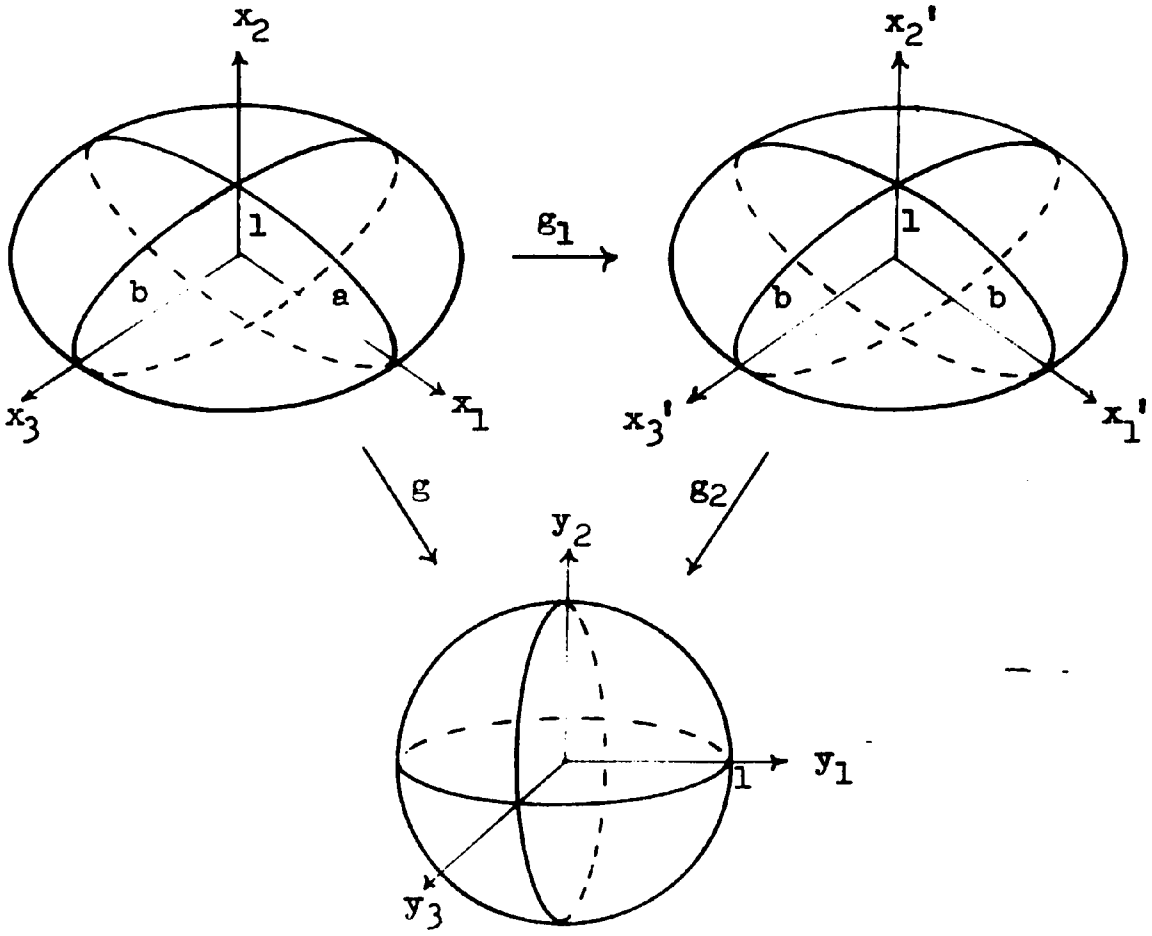


Figure 14

For g the stretchings in the x_1, x_2, x_3 directions are $1/a$, 1, $1/b$, respectively, and we obtain

$$K_0(g)^2 = ab . \tag{4.52}$$

Next, g_1 has stretchings b/a, 1, 1 in the directions of the coordinate axes, and since $b/a < 1$ we achieve

(4.53) $$K_0(g_1)^2 = a/b \ .$$

In (4.52) and (4.53) we have used Lemma 3.4.

Moreover, as we proved earlier (Cf. (4.40)),

(4.54) $$K_0(g_2)^2 = b^2 \ .$$

Then (4.51) follows from (4.52), (4.53), and (4.54).

Now let f_2 be the quasiconformal mapping of the oblate ellipsoid E_o given in § 4.4. Then the composition of g_1 with f_2 is a new quasiconformal mapping h of E onto the unit ball. Moreover, by definition (3.26) we have

(4.55) $$K_0(h) \leq K_0(g_1)\, K_0(f_2) \ .$$

But in Theorem 4.2 we proved

(4.56) $$K_0(f_2) < K_0(g_2) \ .$$

Finally, (4.51), (4.54), and (4.56) together show that

$$K_0(h) \leq K_0(g_1)\, K_0(f_2) < K_0(g_1)\, K_0(g_2) = K_0(g) \ ,$$

so that the affine mapping of E onto the unit ball is not extremal for the outer coefficient.

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