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THE DISTRIBUTION MODULO 1 OF OSCILLATING FUNCTIONS

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CHAPTER 1

NOTATION AND DEFINITIONS

In what follows,

$[x]$ denotes the integral part of a real number x: $[x]$ is the largest integer not exceeding x.

$\langle x \rangle$ denotes the fractional part of x: $\langle x \rangle = x - [x]$.

$\{n \mid P(n)\}$ denotes the set of integers n such that the proposition $P(n)$ about n holds.

$\#\{n \mid P(n)\}$ denotes the number of elements in the set $\{n \mid P(n)\}$.

$e(x)$ is an abbreviation for $e^{2\pi i x}$.

$\sum_{k=A}^{B} a_k$, if A and B are not integers, denotes $\sum_{k=[A]+1}^{[B]} a_k$.

Δa_k denotes $a_{k+1} - a_k$.

$\sum_{n} f(n)$ is an abbreviation for $\sum_{n=n_0}^{\infty} f(n)$ where n_0 is sufficiently large that $f(n)$ is defined for $n \geq n_0$.

$J_n(x)$ denotes the Bessel function of the first kind of order n.

$0, o$: $f(x) = 0(g(x))$ as $x \to a$ if $\frac{|f(x)|}{g(x)}$ is bounded in a neighborhood of a. $f(x) = o(g(x))$ as $x \to a$ if $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$. When not otherwise specified, a will be ∞.

$\mathrm{Re}(z)$ denotes the real part of the complex number z.

c denotes an unspecified constant, perhaps different on different occurrences.

Uniform distribution modulo 1:

Given a sequence of real numbers $\{a_n\}$ with fractional parts $\langle a_n \rangle$,

let

$$N_n(a, b) = \#\{m \leq n \mid a \leq \langle a_m \rangle < b\} .$$

If each subinterval of $[0, 1]$ gets its expected number of points; that is, if

$$\lim_{n \to \infty} \frac{1}{n} N_n(a, b) = b - a$$

for all a and b such that $0 \leq a \leq b \leq 1$, then $\{a_n\}$ is said to

be uniformly distributed modulo 1. We note that to show that a sequence is uniformly distributed (mod 1), it suffices to verify that

$$\lim_{n \to \infty} \frac{1}{n} N_n(0, \alpha) = \alpha$$

for all α, $0 \leq \alpha \leq 1$, since

$$N_n(a, b) = N_n(0, b) - N_n(0, a) .$$

Distribution function:

If

$$\omega(\alpha) = \lim_{n \to \infty} \frac{1}{n} N_n(0, \alpha)$$

exists and is continuous for all α, $0 \leq \alpha \leq 1$, then we call $\omega(\alpha)$ the distribution function for the sequence $\{a_n\}$.

Throughout, when we say that a function is increasing (decreasing), we mean that it is strictly monotone increasing (decreasing).

INTRODUCTION

The study of uniformly distributed sequences began with Hermann Weyl's 1916 monograph [12] in which he established Weyl's criterion, basic to most subsequent work:

$\{a_n\}$ is uniformly distributed (mod 1)

if and only if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e(ma_k) = 0$$

for each positive integer m, where

$$e(ma_k) = e^{2\pi i m a_k}.$$

Since that time, the theory of exponential sums of the type $\sum_{k=1}^{n} e(a_k)$ has been considerably advanced, notably by Vinogradov (see, for example, [10]) because of its application to the problems of Waring (representing an integer as the sum of kth powers) and Goldbach (representing an integer as the sum of primes). In another direction (the one in which this thesis lies), uniformly distributed sequences have continued to be studied for their own sake. A summary of results up to 1935 can be found in Koksma's book [5] and work from then until 1961 is summarized in a paper of Cigler

and Helmberg [2].

We give two results which will be referred to later. The first is Fejér's Theorem:

> If $f(x)$ is continuous and increasing such that
>
> $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to \infty} f(x)/x = 0$, and
>
> $\lim_{x \to \infty} xf'(x) = \infty$, then $\{f(n)\}$ is uniformly
>
> distributed (mod 1).

The proof is adapted from Pólya and Szegö [8] and is given here because it illustrates some of the methods we will use.

Let $F(t)$ be the function inverse to $f(x)$. Then the three displayed hypotheses of the theorem are equivalent to

$$\lim_{t \to \infty} F(t) = \infty, \quad \lim_{t \to \infty} F(t)/t = \infty, \quad \text{and} \quad \lim_{t \to \infty} \frac{F'(t)}{F(t)} = 0.$$

Let

$$N(n, \alpha) = \#\{m \leq n \mid \langle f(m) \rangle < \alpha \} \ ;$$

to show that the sequence $\{f(n)\}$ is uniformly distributed (mod 1), we need only show, as noted above, that

$$\lim_{n \to \infty} \frac{1}{n} N(n, \alpha) = \alpha \tag{1}$$

for all α, $0 \leq \alpha \leq 1$.

There is no loss of generality in assuming that $f(x) \geq 0$, for discarding the finite range of values of x for which $f(x)$ is negative does not affect the distribution of $\{f(n)\}$. Thus we can write

$$N(n, \alpha) = \sum_{k=0}^{T} N(n, k, \alpha) \tag{2}$$

where

$$N(n, k, \alpha) = \#\{m \leq n \mid k \leq f(m) < k + \alpha\}$$

and T is the integral part of the largest value that $f(x)$ can assume for $x \leq n$, namely $T = [f(n)]$. Let us fix n and assume that $f(n)$ is an integer, so that

$$T = f(n) \text{ and } n = F(T) ;$$

the error introduced by this assumption can be shown to be negligible. We omit this verification as obscuring the outline of the proof.

Since $F(t)$ increases,

$$N(n, k, \alpha) = \#\{m \leq n \mid F(k) \leq m < F(k + \alpha)\}$$

$$= [F(k + \alpha)] - [F(k)]$$

$$= F(k + \alpha) - F(k) + 0(1),$$

where $0(1)$ is a function of k which remains bounded as $k \to \infty$. In this case,

$$|0(1)| = |\langle F(k + \alpha)\rangle - \langle F(k)\rangle| \leq 1.$$

Since $F'(t)$ increases, it follows that $F(k + \alpha) - F(k)$ increases with increasing k, so

$$(3) \qquad \sum_{k=1}^{T} (F(k + \alpha) - F(k)) = \int_{1}^{T} (F(x + \alpha) - F(x))\, dx + 0(F(T + \alpha) - F(T)).$$

Noting that

$$\int_{1}^{T} F(x + \alpha)\, dx = \int_{1+\alpha}^{T+\alpha} F(x)\, dx$$

and that by the mean-value theorem,

$$F(x + \alpha) - F(x) = 0(F'(x)),$$

from (3) we have

$$(4)\qquad \sum_{k=1}^{T} (F(k+\alpha) - F(k)) = \int_{1}^{1+\alpha} F(x)\,dx + \int_{T}^{T+\alpha} F(x)\,dx + 0(F'(T)).$$

Now

$$\int_{1}^{1+\alpha} F(x)\,dx = 0(1)$$

and in the interval $[T, T+\alpha]$, $F(x)$ differs from $F(T)$ by an amount which is $0(F'(T))$, so from (4) we have

$$(5)\qquad \sum_{k=1}^{T} (F(k+\alpha) - F(k)) = 0(1) + \alpha F(T) + 0(F'(T)) .$$

From this and (2) we see that

$$(6)\qquad N(n,\alpha) = \alpha F(T) + 0(F'(T)) + 0(T)$$

$$= \alpha n + 0(F'(T)) + 0(f(n)) .$$

Thus, to show that (1) holds, we need only show that the error terms in (6) are negligible. But

$$\frac{F'(T)}{n} = \frac{F'(T)}{F(T)} \to 0 \quad \text{as} \quad T \to \infty.$$

By assumption $f(n)/n \to 0$ as $n \to \infty$ and so from (6)

$$\lim_{n \to \infty} \frac{1}{n} N(n,\alpha) = \alpha$$

and the theorem is proved.

The second result is the following:

$\{n\theta\}$ is uniformly distributed (mod 1) if and only if θ is irrational.

This follows from Weyl's criterion. We see that for $m\theta \notin \mathbb{Z}$,

$$\sum_{k=1}^{n} e(mk\theta) = \sum_{k=1}^{n} (e^{2\pi i m\theta})^k = e^{2\pi i m\theta} \frac{1 - e^{2\pi i mn\theta}}{1 - e^{2\pi i m\theta}} ,$$

so

$$(7) \qquad \left| \sum_{k=1}^{n} e(mk\theta) \right| \leq \frac{2}{|1 - e^{2\pi i m\theta}|} = \frac{1}{|\sin \pi m\theta|} .$$

If θ is irrational, then for any integer m, $m\theta$ is not an integer. From (7),

$$\left| \frac{1}{n} \sum_{k=1}^{n} e(mk\theta) \right| \to 0$$

as $n \to \infty$ for each m. If $\theta = p/q$, then for $m = q$, $e(mk\theta) = e(kp) = 1$ for each k, and

$$\sum_{k=1}^{n} e(mk\theta) = n,$$

so Weyl's criterion shows that $\{n\theta\}$ is not uniformly distributed (mod 1).

This was the first of the metric theorems on uniform distribution, that is, theorems saying that a sequence $\{s_n(x)\}$ is uniformly distributed (mod 1) for almost all x in a certain range. In the line of descent which led to this thesis, the next result is that of Koksma, who proved in 1935 that $\{x^n\}$ is uniformly distributed (mod 1) for almost all $x > 1$. (Although the theorem is true for almost all x, no example of a number x for which it holds is yet known. Nor is any $x > 1$ known for which $\{x^n\}$ is not uniformly distributed (mod 1), except for the trivial case when x is an integer.) This theorem naturally suggests investigating the distribution of $\{z^n\}$ in the unit square, where z is a complex number. This was done in 1953 by LeVeque [7]. In the course of showing that $\{z^n\}$ is indeed uniformly distributed in the unit square for almost all z, there arose the question of the distribution of $\{r^n \cos n\theta\}$ or, more generally, $\{f(n)P(g(n))\}$ for periodic P. This was the first consideration of the distribution of oscillating functions; the present work is the second.

In the general case, LeVeque obtained results when at least one of the sequences $\{f(n)\}$, $\{g(n)\}$ is lacunary-that is, when there is a positive number δ, independent of n, such that $f(n+1)/f(n) \geq 1 + \delta$ (or $g(n+1)/g(n) \geq 1 + \delta$) for $n = 1, 2, \ldots$.

Thus the rate of growth of lacunary sequences is at least exponential, since $f(n) \geq (1+\delta)^n f(0)$. When neither f nor g is lacunary, a result was obtained only for the special case $P(x) = \cos x$, namely

> $\{n^j \cos(n^k y)\}$ is uniformly distributed (mod 1) for almost all y when j is an integer and $j \geq k \geq 1$.

In this paper, we extend and generalize this result. In case $k = 1$, we remove the restriction that the multiplier of the cosine increase as rapidly as n: we prove in Theorem 1 that if P is a periodic function satisfying certain smoothness conditions, then $\{f(n) P(ny)\}$ is uniformly distributed (mod 1) for almost all y for any sufficiently smooth function f which increases, however slowly, as long as $f(x) = o(x^{1/2})$.

Theorem 2 gives a partial result on the distribution of $\{f(n) P(g(n))\}$ where $g(n)$, in contrast to the factor n^k in LeVeque's result, increases less rapidly than n. Each of the functions f and g is required to satisfy the requirements of Fejér's Theorem (Theorem 2 being based on it) so that, from the conditions $\lim_{x \to \infty} f(x)/x = \lim_{x \to \infty} g(x)/x = 0$, the functions increase less rapidly than x; and from the conditions $\lim_{x \to \infty} xf'(x) = \lim_{x \to \infty} xg'(x) = \infty$, the functions increase

more rapidly than log x. In addition, the functions must satisfy other conditions. As a supplement to this result, we show that for smooth functions g, it is necessary to have some increasing multiplier of the periodic function in order to have uniform distribution (mod 1). In Theorem 3, we show that $\{P(g(n))\}$ is not uniformly distributed for a large class of functions with rates of growth varying from barely increasing to exponential.

Then we attempt to improve on the result for the distribution of $\{f(n) \cos n^k y\}$ when k is a positive integer. We show in Theorem 4 that the sequence is uniformly distributed (mod 1) for almost all y when f(x) is a smooth function whose rate of growth lies between log x and x^9. The restriction log x is inherent in the theorem we apply to get the result; the restriction x^9 is probably irrelevant, and arises from the method used.

We also remove the requirement in LeVeque's result that j be an integer. Combining this with Theorem 4, we see that $\{n^j \cos n^k y\}$ is uniformly distributed (mod 1) for almost all y, for all j except possibly those such that $9 < j < k$.

From Theorem 1, in the special case $P(x) = \cos x$, and Theorem 4 in the special case $k = 1$, we have that $\{f(n) \cos ny\}$ is uniformly distributed (mod 1) for almost all y when f is a smooth function, increasing without bound, such that $f(x) = o(x^{1/2})$ (Theorem 1) or f(x) has rate of growth between log x and x^9

(Theorem 4). In Theorem 5, we show that $\{f(n) \cos ny\}$ is uniformly distributed (mod 1) when f' is positive and bounded away from 0. Theorems 1, 4, and 5 thus show that $\{f(n) \cos ny\}$ is uniformly distributed (mod 1) for almost all y when f is a smooth function increasing without bound, no matter at what rate.

In proving Theorems 4 and 5, we use a recently established theorem due to Davenport, Erdös, and LeVeque [3] which is the most powerful tool presently available establishing uniform distribution almost everywhere. We quote it here for reference:

Theorem A: Given a sequence of bounded and integrable functions $\{s_n(x)\}$ on $a \leq x \leq b$, let m be a positive integer,

$$S(N, x) = \frac{1}{n} \sum_{n=1}^{N} e(ms_n(x)) ,$$

and

$$I(N) = \int_a^b |S(N, x)|^2 \, dx .$$

If

$$\sum_N \frac{I(N)}{N}$$

converges for each m, then $\{s_n(x)\}$ is uniformly distributed (mod 1) for almost all x.

CHAPTER II

The Distribution of $\{f(n)P(ny)\}$

We will show that $\{f(n)P(ny)\}$ is uniformly distributed (mod 1) for almost all y when P is periodic and f smoothly increases to infinity. More precisely, we suppose that P has the following properties:

(A)
P is bounded and periodic with period p,

P is piecewise monotone (that is, made up of a finite number of monotone components), and differentiable on each monotone component;

at each interior point in a monotone component, some derivative $P^{(n)}$ of P is not zero. At the left-(right-) hand endpoint of each component, some right-(left-) hand derivative of P is not zero.

The third condition insures that P is never too close to being a constant-if it were, the sequence would behave like $\{cf(n)\}$, and it can be shown that functions which increase slowly (so that $\lim_{x \to \infty} xf'(x) = 0$) are not uniformly distributed (mod 1). We also suppose that f has the properties

(B)
f increases without bound,

f' decreases, and

$f(x) = o(x^{1/2})$.

Finally, we assume that

(C) y/p is not a rational number.

This excludes only countably many y (namely, rational multiples of p), so almost all y are still included.

The plan of the proof is as follows. We can fairly easily get information on the distribution of $\{BP(ny)\}$ when B is a constant, and we find that the larger B is, the closer the distribution is to being uniform. We then approximate $f(n)P(ny)$ by a step function so closely that it and $\{f(n)P(ny)\}$ have the same limiting distribution.

The proof is in a sequence of lemmas. Lemma 1 is a well-known fact, included for completeness. Lemma 2 gives a criterion for a sequence to be uniformly distributed (mod 1) if a nearby sequence is. Lemmas 3-6 prepare for Lemma 7, which asserts that the step function approximating $f(x)P(xy)$ is uniformly distributed (mod 1). Lemma 8, a consequence of Lemmas 2 and 7, gives a complicated condition for f to satisfy which suffices for $\{f(n)P(ny)\}$ to be uniformly distributed (mod 1) for almost all y. Lemma 9 shows that a function f satisfies the condition of Lemma 8 if it has the properties listed in (B). Lemmas 8 and 9 yield Theorem 1, which asserts that if (A)-(C) hold, then $\{f(n)P(ny)\}$ is uniformly distributed (mod 1) for almost all y.

Lemma 1:

If $\{a_k\}$ and $\{d_k\}$ are sequences such that $\lim_{k \to \infty} d_k = 0$ and $\Delta a_k = a_{k+1} - a_k$ is positive and bounded away from zero, then

$$\frac{1}{a_n} \sum_{k=1}^{n-1} (\Delta a_k)\, d_k \to 0$$

as $n \to \infty$.

Proof:

For any integer m, $1 \leq m \leq n-1$,

$$\sum_{k=1}^{n-1} (\Delta a_k) d_k = \sum_{k=1}^{m-1} (\Delta a_k) d_k + \sum_{k=m}^{n-1} (\Delta a_k)\, d_k$$

$$\leq (\max_{1 \leq t < m} d_t) \sum_{k=1}^{m-1} (\Delta a_k) + (\max_{m \leq t < n} d_t) \sum_{k=m}^{n-1} (\Delta a_k).$$

If we put

$$D_{mn} = \max_{m \leq t < n} d_t ,$$

then

$$\sum_{k=1}^{n-1} (\Delta a_k) d_k \leq c(a_m - a_1) + D_{mn}(a_n - a_m) \leq c a_m + D_{mn} a_n .$$

(Here, as throughout, c denotes an unspecified constant, whose value may be different on different occurrences.) Hence

$$\frac{1}{a_n} \sum_{k=1}^{n-1} (\Delta a_k) d_k \leq c \frac{a_m}{a_n} + D_{mn} .$$

If we choose $m = m(n)$ to be the largest integer such that $a_m \leq a_n^{1/2}$, then $\lim_{n \to \infty} m(n) = \infty$. (If not, there would exist an integer m_0 such that $a_{m_0+1} > a_n^{1/2}$ for all n. This is impossible, for the relation $\Delta a_k \geq c > 0$ implies that $\lim_{k \to \infty} a_k = \infty$). Thus $D_{mn} \to 0$ as $n \to \infty$, so

$$\frac{1}{a_n} \sum_{k=1}^{n-1} (\Delta a_k) d_k \leq \frac{c}{a_n^{1/2}} + D_{mn} = o(1).$$

Lemma 2:

Suppose $\{x_k\}$ and $\{y_k\}$ are sequences such that

$$\lim_{k \to \infty} (x_k - y_k) = 0 .$$

Then if either sequence is uniformly distributed (mod 1), so is the other.

Proof:

Let m be a positive integer and put

$$S_x(N) = \sum_{k=1}^{N} e(mx_k) \quad \text{and} \quad S_y(N) = \sum_{k=1}^{N} e(my_k),$$

where $e(mx_k) = e^{2\pi i m x_k}$. Thus

$$(1) \qquad S_x(N) - S_y(N) = \sum_{k=1}^{N} e(mx_k)(1 - e(my_k - mx_k)).$$

Let us put

$$(2) \qquad R_m(N) = \sum_{k=1}^{N} |1 - e(my_k - mx_k)|,$$

so that from (1),

$$|S_x(N) - S_y(N)| \leq R_m(N).$$

If we suppose that $\{x_k\}$ is uniformly distributed (mod 1), then the necessity of Weyl's Criterion for uniform distribution (mod 1) (see the Introduction) yields

$$\lim_{N \to \infty} \frac{S_x(N)}{N} = 0 \quad \text{for each} \quad m.$$

To show that $\{y_k\}$ is uniformly distributed (mod 1), it is enough, from the sufficiency of Weyl's Criterion, to show that

(3) $$\lim_{N \to \infty} \frac{1}{N} R_m(N) = 0 \quad \text{for each} \quad m.$$

But since for any θ,

$$|1 - e(m\theta)|^2 = 2(1 - \cos 2\pi m\theta) = 4 \sin^2 \pi m\theta,$$

we have from (2),

$$R_m(N) = \sum_{k=1}^{N} 2|\sin \pi m (y_k - x_k)| \leq 2\pi m \sum_{k=1}^{N} |y_k - x_k| .$$

Thus

$$\frac{1}{N} R_m(N) \leq \frac{2\pi m}{N} \sum_{k=1}^{N} |y_k - x_k| ,$$

and from Lemma 1 (with $a_k = k$),

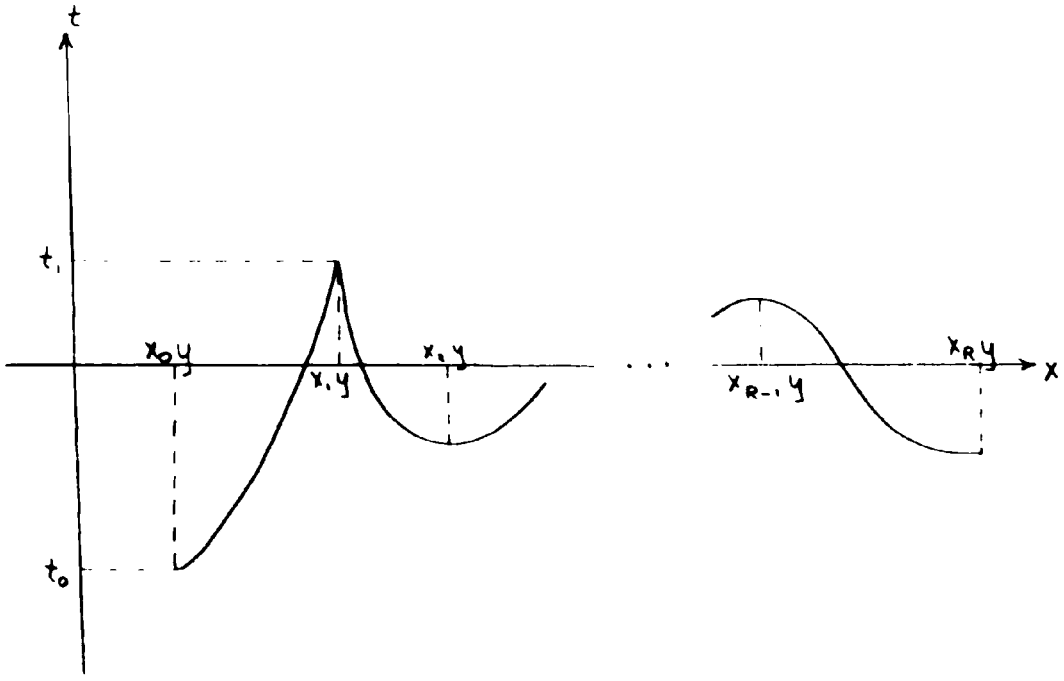
$$\frac{1}{N} R_m(N) \to 0 \quad \text{as} \quad N \to \infty,$$

(3) holds, and the lemma is proved.

We decompose a period of P into its strictly monotone components $P_1, P_2, \ldots, P_R$ (see Figure 1). Since P' has only finitely many zeros in each period, there are only finitely many of the P_i. We define $x_0, \ldots, x_R$ and $t_0, \ldots, t_R$, as shown in

the figure, by letting $x_{i-1}y$ and $x_i y$ be the endpoints of the monotone component P_i and

$$t_i = P(x_i y), \quad i = 0, 1, \ldots, R.$$



A Period of $P(x)$

Figure 1

For $0 \leq \alpha \leq 1$, let

$$N(B, n, \alpha) = \#\{m \leq n \mid \langle BP(my) \rangle < \alpha\}. \tag{4}$$

Let $\omega(B, \alpha)$ denote the distribution function of $\{BP(ny)\}$:

$$\omega(B, \alpha) = \lim_{n \to \infty} \frac{1}{n} N(B, n, \alpha) ,$$

if the limit exists. We later show that it does exist by computing it.

Let

$$N_i(B, n, \alpha) = \#\{m \leq n \mid \langle BP_i(my) \rangle < \alpha\} . \tag{5}$$

(Since P_i is defined only on the intervals $(x_{i-1}y + kp, x_i y + kp)$, $k = 0, 1, \ldots,$ (5) could more precisely be written as

$$N_i(B, n, \alpha) = \#\{m \leq n \mid m \in (x_{i-1}y + kp, x_i y + kp)$$

$$\text{for some } k \text{ and } \langle BP_i(my) \rangle < \alpha\} .)$$

Let

$$\omega_i(B, \alpha) = \lim_{n \to \infty} \frac{1}{n} N_i(B, n, \alpha) , \tag{6}$$

again if the limit exists, for $i = 1, 2, \ldots, R$. We compute the limit in Lemma 3. From (6),

$$\omega(B, \alpha) = \sum_{i=1}^{R} \omega_i(B, \alpha) , \tag{7}$$

if all the limits in (6) exist. We will consider $\omega_1(B, \alpha)$; each of the other terms in (7) may be treated similarly. We suppose that

P_1 is (strictly) increasing on $(x_0 y, x_1 y)$. At the end of Lemma 4, we will show that the assumption that P_1 decreases would not change the final result.

Lemma 3:

With the above notation,

$$\omega_1(B,\alpha) = \sum_{i=Bt_0}^{Bt_1-1} \frac{1}{p}\left(P_1^{-1}\left(\frac{i+\alpha}{B}\right) - P_1^{-1}\left(\frac{i}{B}\right)\right) + \Gamma(B),$$

where $\lim_{B\to\infty} \Gamma(B) = 0$, for all α.

(As noted in the section of notation, if the limits on a sum are not integers, then $\Sigma_A^B = \Sigma_{[A]+1}^{[B]}$.)

Proof:

Suppose for the moment that Bt_0 and Bt_1 are integers. In that case,

$$N_1(B,n\alpha) = \#\{m \le n \mid \langle BP_1(my)\rangle < \alpha\}$$

$$= \sum_{i=Bt_0}^{Bt_1-1} \#\{m \le n \mid i \le BP_1(my) < i+\alpha\}. \tag{8}$$

Since

$$P_1(my) = P_1\left(p\left[\frac{my}{p}\right] + p\left\langle\frac{my}{p}\right\rangle\right) = P_1\left(p\left\langle\frac{my}{p}\right\rangle\right)$$

and P_1^{-1} exists and is an increasing function, (8) becomes

$$N_1(B,n,\alpha) = \sum_{i=Bt_0}^{Bt_1-1} \#\{m \le n \mid \frac{i}{B} \le P_1(p\langle\frac{my}{p}\rangle) < \frac{i+\alpha}{B}\}$$

(9)

$$= \sum_{i=Bt_0}^{Bt_1-1} \#\{m \le n \mid \frac{1}{p} P_1^{-1}(\frac{i}{B}) \le \langle\frac{my}{p}\rangle < \frac{1}{p} P_1^{-1}(\frac{i+\alpha}{B})\}.$$

Assumption (C) asserts that y/p is irrational, so (as noted in the Introduction), $\{ny/p\}$ is uniformly distributed (mod 1). Recall that for a uniformly distributed sequence $\{a_k\}$, and $0 \le a < b \le 1$,

$$\lim_{n\to\infty} \frac{1}{n} \#\{m \le n \mid a \le \langle a_m\rangle < b\} = b - a;$$

hence dividing by n in (9) and taking the limit as $n \to \infty$, we get

$$\omega_1(B,\alpha) = \lim_{n\to\infty} \frac{1}{n} N_1(B,n,\alpha)$$

(10)

$$= \sum_{i=Bt_0}^{Bt_1-1} \frac{1}{p}\left(P_1^{-1}(\frac{i+\alpha}{B}) - P_1^{-1}(\frac{i}{B})\right).$$

Hence if Bt_0 and Bt_1 are integers, the conclusion of the lemma holds with the error term $\Gamma(B)$ equal to zero. In case Bt_0 and Bt_1 are not integers, we consider the sum in (8) between the limits $[Bt_0] + 1$ and $[Bt_1] - 1$. This introduces two errors,

which give rise to $\Gamma(B)$:

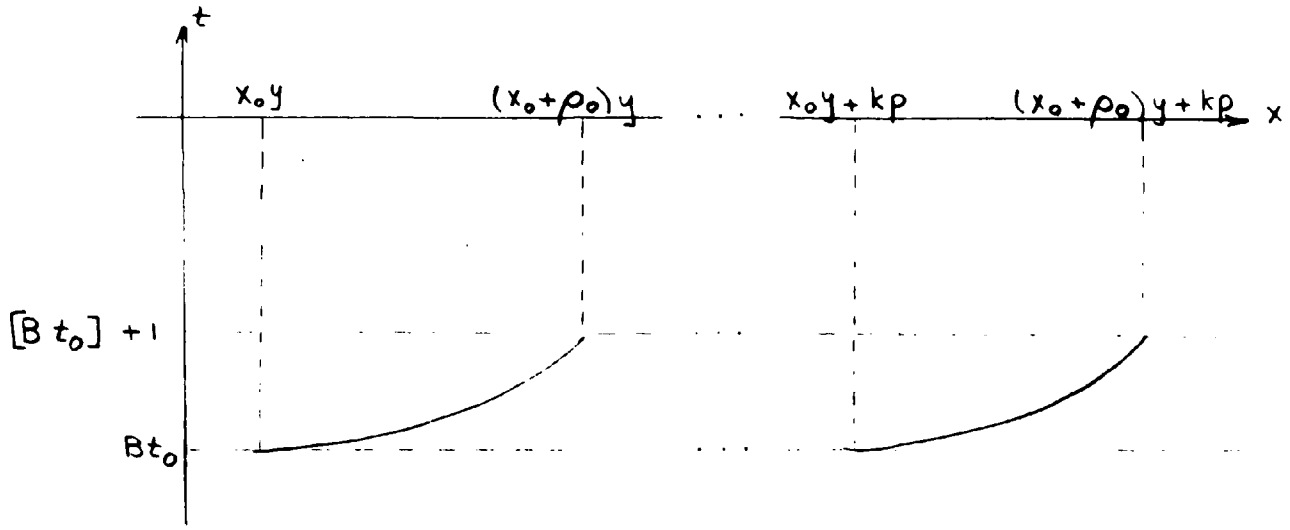
$$(11) \qquad N_1(B, n, \alpha) = \sum_{i=[Bt_0]+1}^{[Bt_1]-1} \#\{m \leq n \mid i \leq BP_1(my) < i + \alpha\}$$

$$+ E_0(B, n, \alpha) + E_1(B, n, \alpha),$$

where $E_0(B, n, \alpha)$ is the error committed by ignoring in the sum the portion of the graph of $BP_1(x)$ lying between Bt_0 and $[Bt_0] + \alpha$. $E_1(B, n, \alpha)$ is the similar error near Bt_1. We will first consider $E_0(B, n, \alpha)$.

Define $\rho_0 = \rho_0(B)$, as shown in Figure 2, to be the number such that $BP_1((x_0 + \rho_0)y) = [Bt_0] + 1$. As B increases, the common length of the intervals $(x_0 y + kp, (x_0 + \rho_0)y + kp)$, $k = 0, 1, \ldots,$ decreases, and hence these intervals may be expected to contain fewer integers. To show that the number of integers is negligible, we first show that

$$(12) \qquad \lim_{B \to \infty} \rho_0(B) = 0.$$



$BP_1(x)$ between Bt_0 and $[Bt_0] + 1$

Figure 2

Because we have assumed in (A) that at each point of each component some derivative of P is not zero, there is a smallest integer r, $r \geq 1$, so that $P_1^{(r)}(x_0 y) \neq 0$. Hence by Taylor's Theorem,

$$P_1(x_0 y + \rho_0 y) - P_1(x_0 y) = \frac{(\rho_0 y)^r}{r!} P_1^{(r)}(x_0 y) + c\rho_0^{r+1}.$$

But from Figure 2,

$$P_1(x_0 y) = \frac{Bt_0}{B} \quad \text{and} \quad P_1(x_0 y + \rho_0 y) = \frac{1}{B}([Bt_0] + 1),$$

so

$$[Bt_0] + 1 - Bt_0 = B(P_1(x_0y + \rho_0 y) - P_1(x_0 y))$$

$$= c\, B \rho_0^r (1 + \rho_0).$$

The left-hand side of this equation is bounded above by 1, so

$$c\, B \rho_0^r (1 + \rho_0) \leq 1,$$

whence

$$\rho_0 \leq \frac{1}{c\, B^{1/r}}$$

and (12) holds.

We now get an upper bound for $E_0(B, n, \alpha)$ in terms of ρ_0. We note that E_0 is zero if $[Bt_0] + \alpha < Bt_0$, for then no integer m satisfies

$$Bt_0 \leq BP_1(my) < [Bt_0] + \alpha.$$

In the contrary case, the error is

$$\#\{m \leq n \mid Bt_0 \leq BP_1(my) < [Bt_0] + \alpha\},$$

and so an upper bound for $E_0(B, n, \alpha)$ is given by this expression with $\alpha = 1$:

$$E_0(B, n, \alpha) \leq E_0(B, n) = \#\{m \leq n \mid Bt_0 \leq BP_1(my) \leq [Bt_0] + 1\} .$$

From Figure 2, we see that

$$E_0(B, n) = \#\{m \leq n \mid x_0 y + tp \leq my \leq (x_0 + \rho_0)y + tp$$

$$\text{for some integer } t\} .$$

If we set

$$z_k = x_o + \frac{kp}{y} , \quad k = 0, 1, \ldots ,$$

then

$$E_0(B, z_n y) = \#\{m \leq z_n y \mid z_t y \leq my \leq z_t y + \rho_0 y \text{ for some } t\}$$

or

$$(13) \quad E_0(B, z_n y) = \#\{m \leq z_n y \mid z_t \leq m \leq z_t + \rho_0 \text{ for some } t\} .$$

Note that t can assume only the values $0, 1, \ldots, n$. Since $\lim_{B \to \infty} \rho_0(B) = 0$ and we are interested only in large values of B, we may assume that $y\rho_0 < 1$, so that each of the intervals in Figure 2 has length less than 1. We also suppose that $\rho_0 < 1$. In (13), we considered the positive integers up to $z_n y$ to see how

many of them fell in the specified intervals; if instead we look at the intervals and see how many of them contain an integer, we see that since each interval has length smaller than 1 and can contain at most one integer,

$$(14) \qquad E_0(B, z_n y) = \#\{t \leq n \mid z_t \leq m \leq z_t + \rho_0 \text{ for some integer } m\}.$$

Now $\{kp/y\}$ is uniformly distributed (mod 1) and $\{x_0 + kp/y\}$, being only a translation of it, is also uniformly distributed (mod 1). Thus from (14),

$$E_0(B, z_n y) = \#\{t \leq n \mid \langle z_t \rangle \geq 1 - \rho_0\}$$

and

$$\frac{1}{n} E_0(B, z_n y) \to \rho_0 \quad \text{as} \quad n \to \infty.$$

Hence

$$\lim_{n \to \infty} \frac{E_0(B, z_n y)}{z_n y} = \lim_{n \to \infty} \frac{n}{z_n y} \cdot \frac{E_0(B, z_n y)}{n},$$

and since

$$\frac{n}{z_n y} = \frac{n}{x_0 y + np} \to \frac{1}{p} \quad \text{as} \quad n \to \infty,$$

it follows that

$$\lim_{n \to \infty} \frac{E_0(B, z_n y)}{z_n y} = \frac{\rho_0}{p} .$$

Since the numbers z_k are uniformly spaced along the axis, it follows that $E_0(B, n)/n$ also approaches ρ_0/p: let k be the integer such that $z_{k-1} y \leq n < z_k y$; then

$$\lim_{n \to \infty} \left| \frac{E_0(B, z_k y)}{z_k y} - \frac{E_0(B, n)}{n} \right| \leq \lim_{n \to \infty} \frac{|E_0(B, z_k y) - E_0(B, n)|}{z_{k-1} y}$$

$$\leq \lim_{n \to \infty} \frac{z_k y - n}{z_{k-1} y} \leq \lim_{n \to \infty} \frac{y(z_k - z_{k-1})}{y z_{k-1}}$$

$$= \lim_{n \to \infty} \frac{p}{x_0 y + np} = 0.$$

In the same way, if we define $\rho_1 = \rho_1(B)$ to be such that

$$P_1((x_1 - \rho_1)y) = [Bt_1],$$

then we can show that

$$\lim_{B \to \infty} \rho_1(B) = 0 \tag{15}$$

and

$$\lim_{n \to \infty} \frac{E_1(B, n)}{n} = \frac{1}{p} \rho_1 .$$

It thus follows from (11) that Lemma 3 holds with

$$0 \leq \Gamma(B) \leq \frac{1}{p} (\rho_0 + \rho_1).$$

From (12) and (15) we see that

$$\lim_{B \to \infty} \Gamma(B) = 0,$$

completing the proof.

Lemma 4:

(16) $$\lim_{B \to \infty} \omega_1(B, \alpha) = \frac{\alpha}{p} (x_1 y - x_0 y) .$$

Proof:

P_1 is differentiable in the interior of the interval in which it is defined, so for $[Bt_0] + 1 \leq t \leq [Bt_1] - 1$, $P_1^{-1}(t)$ is differentiable. From Lemma 3 and the mean-value theorem we have

(17) $$\omega_1(B, \alpha) = \sum_{i = Bt_0}^{Bt_1 - 1} \frac{1}{p} \frac{\alpha}{B} \frac{d}{dt} P_1^{-1}(t) \Big|_{t = \xi_i} + \Gamma(B),$$

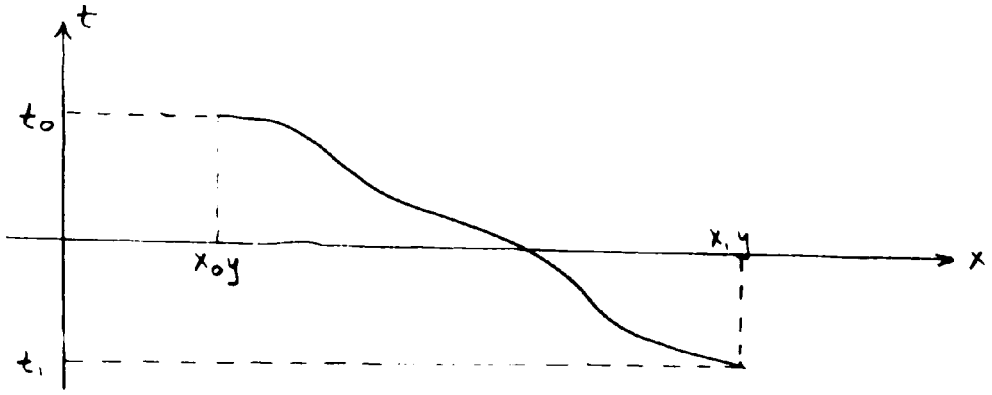
where $i/B \leq \xi i \leq (I + \alpha)/B$. But (17) is an approximating sum for a Riemann integral:

$$\lim_{B\to\infty} \omega_1(B,\alpha) = \frac{\alpha}{p} \int_{t_0}^{t_1} \left(\frac{d}{dt} P_1^{-1}(t)\right) dt + \lim_{B\to\infty} \Gamma(B)$$

$$= \frac{\alpha}{p} \left(P_1^{-1}(t_1) - P_1^{-1}(t_0)\right)$$

$$= \frac{\alpha}{p} (x_1 y - x_0 y),$$

so the lemma is proved, under the assumption made just before Lemma 3 that P_1 is an increasing function. If we had supposed P_1 to be decreasing, then P_1^{-1} would also have been decreasing and (9) would be



Decreasing $P_1(x)$

Figure 4

$$(18)\quad N_1(B,n,\alpha) = \sum_{i=Bt_1}^{Bt_0-1} \#\left\{m \leq n \,\middle|\, \frac{1}{p} P_1^{-1}\left(\frac{i+\alpha}{B}\right) < \left\langle \frac{my}{p} \right\rangle \leq \frac{1}{p} P_1^{-1}\left(\frac{i}{B}\right)\right\}.$$

The errors E_0 and E_1 would be handled similarly, and the conclusion of Lemma 3 would be

$$\omega_1(B,\alpha) = \sum_{i=Bt_1}^{Bt_0-1} \frac{1}{p}\left(P_1^{-1}\left(\frac{i}{B}\right) - P_1^{-1}\left(\frac{i+\alpha}{B}\right)\right) + \Gamma(B),$$

where $\Gamma(B) \to 0$ as $B \to \infty$. We would proceed in Lemma 4 to get the analogue of (18),

$$\lim_{B\to\infty} \omega_1(B,\alpha) = -\frac{\alpha}{p}\int_{t_1}^{t_0}\left(\frac{d}{dt}\,P_1^{-1}(t)\right)dt$$

$$= -\frac{\alpha}{p}\left(P_1^{-1}(t_0) - P_1^{-1}(t_1)\right)$$

$$= \frac{\alpha}{p}\left(x_1 y - x_0 y\right),$$

which is the same as the conclusion of Lemma 4.

In the same way as we got (16), we can show that

$$\lim_{B\to\infty} \omega_i(B,\alpha) = \frac{\alpha}{p}\left(x_i y - x_{i-1} y\right) \tag{19}$$

for $i = 2, 3, \ldots, R$, irrespective of whether P_i increases or decreases.

From (19) we get

Lemma 5:

(20) $$\lim_{B \to \infty} \omega(B, \alpha) = \alpha.$$

Proof:

Since

$$\omega(B, \alpha) = \sum_{i=1}^{R} \omega_i(B, \alpha) ,$$

we have from (19) that

$$\lim_{B \to \infty} \omega(B, \alpha) = \frac{\alpha}{p} \sum_{i=1}^{R} (x_i y - x_{i-1} y)$$

$$= \frac{\alpha}{p} (x_R y - x_0 y)$$

and $x_R y - x_0 y = p.$

Before going to the next lemma, we introduce the idea of the discrepancy of a sequence. The discrepancy is a measure of how closely the distribution of the first n terms of a sequence approaches the distribution of the entire sequence. Let $\{a_k\}$ be a sequence with distribution function $\omega(\alpha)$, so

$$\omega(\alpha) = \lim_{n \to \infty} \frac{1}{n} \#\{m \leq n \mid \langle a_m \rangle < \alpha\} .$$

The discrepancy, $D(n)$, is defined by

$$D(n) = \sup_{0 \leq \alpha \leq 1} D(n, \alpha) \tag{21}$$

where

$$D(n, \alpha) = \frac{1}{n} \#\{m \leq n \mid \langle a_m \rangle < \alpha\} - \omega(\alpha). \tag{22}$$

Note that $\lim_{n \to \infty} D(n) = 0$.

For later purposes, we need an upper bound for the discrepancy of the sequence $\{BP(ny)\}$. This can be made to depend on the discrepancy of the sequence $\{ny/p\}$ for which estimates are known. Such an estimate is given in the next lemma.

Lemma 6:

If $N_j(B, n, \alpha)$ is defined as in (5) and $\omega_j(B, \alpha)$ as in (6), then

$$N_j(B, n, \alpha) = n\omega_j(B, \alpha) + O(Bn^{\delta}) - n\Gamma_1(B), \tag{23}$$

for any $\delta > 0$ for almost all y. $\Gamma_1(B) \to 0$ as $B \to \infty$.

Proof:

We consider the case $j = 1$; the others can be handled similarly. For a uniformly distributed sequence (mod 1), $\{n\theta\}$, from (21) and (22) we have

$$\#\{m \le n \mid a \le \langle m\theta \rangle < b\}$$

$$= \#\{m \le n \mid 0 \le \langle m\theta \rangle < b\} - \#\{m \le n \mid 0 \le \langle m\theta \rangle < a\}$$

$$= nb + nD(n, b) - na - nD(n, a)$$

$$= n(b - a) + c_1 nD(n),$$

where $|c_1| < 2$. Applying this to each term of the sum in (11), we have

$$\#\{m \le n \mid i \le BP_1(my) < i + \alpha\}$$

$$= \#\{m \le n \mid \frac{1}{p} P_1^{-1}(\frac{i}{B}) \le \left\langle \frac{my}{p} \right\rangle < \frac{1}{p} P_1^{-1}(\frac{i+\alpha}{B})\} \tag{24}$$

$$= \frac{n}{p} (P_1^{-1}(\frac{i+\alpha}{B}) - P_1^{-1}(\frac{i}{B})) + c_1 nD(n).$$

We now get an estimate for $D(n)$, the discrepancy of $\{ny/p\}$, valid for almost all y.

As in [5, page 27], let E be the upper limit of numbers $\Omega > 0$ for which the irrational number θ admits the approximation $1/t^{\Omega}$ (that is, for which the inequality

$$|\theta - \frac{p}{q}| < \frac{1}{q^{\Omega}}$$

has infinitely many solutions in integers p and q). Let $\eta = E - 1$. If $\eta < \infty$, then θ is said to be of type $I\eta$. A well-known theorem of

Khintchine [4 , or 5, page 47] asserts that the inequality

$$\left| \theta - \frac{p}{q} \right| < \frac{1}{qf(q)}$$

has infinitely many or finitely many solutions for almost all θ according as $\int^{\infty} dx/f(x)$ diverges or converges. Since $\int^{\infty} dx/x = \infty$, almost all θ admit the approximation $1/t^2$. Since $\int^{\infty} dx/x^{1+\epsilon} < \infty$ for all $\epsilon > 0$, almost all θ do not admit the approximation $1/t^{2+\epsilon}$, for $\epsilon > 0$. Almost all irrational numbers are therefore of type I1. It is known [5, page 103] that for all θ of type I1,

$$D(n) = O(n^{-1+\delta}) \text{ for } \delta > 0.$$

Applying this and summing the right-most member of (24) over i from $[Bt_0] + 1$ to $[Bt_1] - 1$, we get from Lemma 3,

$$n\omega_1(B, \alpha) - n \Pi(B) + O(Bn^{\delta}).$$

The left-most member of (24), summed over i, is, from (10),

$$N_1(B, n, \alpha) - E_0(B, n, \alpha) - E_1(B, n, \alpha).$$

Since $0 \leq E_0(B, n, \alpha) + E_1(B, n, \alpha) \leq n \Pi(B)$, we have

$$N_1(B, n, \alpha) = n\omega_1(B, \alpha) - n \Pi_1(B) + O(Bn^{\delta})$$

where $\Gamma_1(B) \to 0$ as $B \to \infty$. This gives Lemma 6.

As an immediate corollary of Lemma 6, we have, summing (23) over $j = 1, 2, \ldots, R$,

$$N(B, n, \alpha) = n\omega(B, \alpha) - n\,\Gamma_2(B) + O(Bn^{\delta}) \tag{25}$$

for almost all y. $\Gamma_2(B) \to 0$ as $B \to \infty$. The y for which (25) does not hold are those such that y/p is a rational number or is closely approximable by rational numbers (that is, not of type II).

We now define a step function which closely approximates $f(x)P(xy)$ and show that it is uniformly distributed (mod 1).

Lemma 7:

Let $\{a_k\}$ be a sequence of numbers such that

$$a_1 < a_2 < \cdots, \quad \lim_{k \to \infty} \Delta a_k = \infty, \quad \text{and} \quad \lim_{k \to \infty} \frac{\Delta a_k}{a_k} = 0.$$

Let f be an increasing function with $\lim_{x \to \infty} f(x) = \infty$. Suppose that for $\delta > 0$ sufficiently small, $\lim_{n \to \infty} nf(a_n)/a_n^{1-\delta} = 0$. Define g by

$$g(x) = f(a_k)\,P(xy) \quad \text{when} \quad a_k \leq x < a_{k+1},$$

$k = 1, 2, \ldots$. Then $\{g(n)\}$ is uniformly distributed (mod 1) for almost all y.

Proof:

Put

$$N'(n, \alpha) = \#\{m \leq n \mid \langle g(m) \rangle < \alpha\} .$$

We will show that $N'(a_n, \alpha)/a_n \to \alpha$ as $n \to \infty$, and then that $N'(n, \alpha)/n$ also approaches α as $n \to \infty$. Now

$$N'(a_n, \alpha) = \sum_{k=1}^{n-1} \#\{a_k \leq m < a_{k+1} \mid \langle g(m) \rangle < \alpha\}$$

$$= \sum_{k=1}^{n-1} \#\{a_k \leq m < a_{k+1} \mid \langle f(a_k) P(my) \rangle < \alpha\} \tag{26}$$

$$= \sum_{k=1}^{n-1} M(a_k, a_{k+1}, \alpha), \text{ say.}$$

Note that from (4)

$$M(a_k, a_{k+1}, \alpha) = N(f(a_k), a_{k+1}, \alpha) - N(f(a_k), a_k, \alpha),$$

so from (25)

$$M(a_k, a_{k+1}, \alpha) = a_{k+1}\omega(f(a_k), \alpha) + O(f(a_k) a_{k+1}^{\delta}) + a_{k+1}\Gamma_2(f(a_k))$$

$$- a_k \omega(f(a_k), \alpha) + O(f(a_k) a_k^{\delta}) - a_k \Gamma_2(f(a_k))$$

$$= \Delta a_k \, \omega(f(a_k), \alpha) + O(f(a_k) a_{k+1}^{\delta}) + \Delta a_k \, \Gamma_2(f(a_k)).$$

We know from Lemma 5 that as $k \to \infty$,

$$\omega(f(a_k),\ \alpha) = \alpha + o(1).$$

Also, as $k \to \infty$, $\Gamma_2(f(a_k)) = o(1)$. Substituting in (2 6),

$$N'(a_n, \alpha) = \sum_{k=1}^{n-1} \left(\Delta a_k(\alpha + o(1)) + O(f(a_k)a_{k+1}^{\delta}) \right)$$

$$= \sum_{k=1}^{n-1} \Delta a_k(\alpha + o(1)) + O(nf(a_n)\, a_n^{\delta})$$

$$= a_n\alpha - a_1\alpha + \sum_{k=1}^{n-1} (\Delta a_k)\, o(1) + O(nf(a_n)\, a_n^{\delta}),$$

so

$$(27) \qquad \frac{1}{a_n} N'(a_n, \alpha) = \alpha + O(\frac{1}{a_n}) + \frac{1}{a_n} \sum_{k=1}^{n-1} (\Delta a_k)\, o\,(1) + O\left(\frac{nf(a_n)}{a_n^{1-\delta}}\right) .$$

From Lemma 1,

$$\frac{1}{a_n} \sum_{k=1}^{n-1} (\Delta a_k)\, o(1) = o\,(1),$$

and we have assumed that

$$\frac{nf(a_n)}{a_n^{1-\delta}} = o(1).$$

The error terms in (27) are all $o(1)$, so

$$\frac{1}{a_n} N'(a_n, \alpha) = \alpha + o(1).$$

To complete the proof, we need only show that if k is defined by $a_k \leq n < a_{k+1}$, then

$$\lim_{n \to \infty} \left| \frac{1}{n} N'(n, \alpha) - \frac{1}{a_k} N'(a_k, \alpha) \right| = 0. \tag{28}$$

But

$$\left| \frac{1}{n} N'(n, \alpha) - \frac{1}{a_k} N'(a_k, \alpha) \right| \leq \frac{1}{a_k} \left| N'(n, \alpha) - N'(a_k, \alpha) \right|$$

$$\leq \frac{1}{a_k} (n - a_k) \leq \frac{\Delta a_k}{a_k},$$

and we assumed that $\lim_{k \to \infty} \Delta a_k / a_k = 0$. Thus (28) holds and the lemma is proved.

We combine Lemmas 7 and 2 to get a set of conditions sufficient for the uniform distribution (mod 1) of $\{f(n) P(ny)\}$ for almost all y.

Lemma 8:

Let $\{a_k\}$ be a sequence of numbers such that

$$a_1 < a_2 < \cdots, \quad \lim_{k \to \infty} \Delta a_k = \infty, \quad \text{and} \quad \lim_{k \to \infty} \frac{\Delta a_k}{a_k} = 0.$$

Suppose f is an increasing function such that $\lim_{x \to \infty} f(x) = \infty$, $\lim_{k \to \infty} k f(a_k)/a_k^{1-\delta} = 0$ for $\delta > 0$ sufficiently small, and $\lim_{k \to \infty} (f(a_{k+1}) - f(a_k)) = 0$. If P has the properties (A) listed on p.15 then $\{f(n)\ P(ny)\}$ is uniformly distributed (mod 1) for almost all y.

Proof:

Define g by

$$g(x) = f(a_k)\ P(xy) \quad \text{when} \quad a_k \leq x < a_{k+1}.$$

Then $\{g(n)\}$ is uniformly distributed (mod 1) for almost all y by Lemma 7. We will apply Lemma 2. Let

$$h(x) = f(x)\ P(xy)\ ;$$

then since P is bounded,

$$|h(n) - g(n)| \leq c\ |f(n) - f(a_k)|$$

$$\leq c(f(a_{k+1}) - f(a_k)).$$

Thus

$$\lim_{n \to \infty} |h(n) - g(n)| = 0,$$

and $\{h(n)\}$ is uniformly distributed (mod 1) for almost all y.

We now show that to each smooth, slowly increasing function f, there corresponds a sequence $\{a_k\}$ which satisfies conditions very similar to those of Lemma 8.

Lemma 9:

Suppose that f increases, f' decreases, $\lim_{x \to \infty} f(x) = \infty$, and $\lim_{x \to \infty} f(x)/x^{1/2} = 0$. Then there exists an increasing function $a(x)$ such that

$$\text{(i)} \qquad \lim_{x \to \infty} a'(x) = \infty,$$

$$\text{(ii)} \qquad \lim_{x \to \infty} a''(x) = 0,$$

$$\text{(iii)} \qquad \lim_{x \to \infty} \frac{a'(x)}{x} = 0,$$

$$\text{(iv)} \qquad \lim_{x \to \infty} \frac{d}{dx} f(a(x)) = 0, \text{ and}$$

$$\text{(v)} \qquad \lim_{x \to \infty} \frac{xf(a(x))}{a^{1-\delta}(x)} = 0,$$

for $\delta > 0$ sufficiently small.

Proof:

The appropriate function is

$$a(x) = x^{2-\gamma},$$

where γ will be chosen later. That (i)-(iii) hold is immediate.

To verify (iv), note that

$$\frac{d}{dx} f(a(x)) = a'(x)\, f'(a(x)). \tag{29}$$

Since f' decreases, and we may assume $f(0) \geq 0$,

$$\frac{f(x)}{x} \geq \frac{f(x) - f(0)}{x - 0} \geq f'(x)$$

so from (29),

$$\frac{d}{dx} f(a(x)) \leq \frac{(2-\gamma)\, x^{1-\gamma} f(x^{2-\gamma})}{x^{2-\gamma}} = \frac{c\, f(x^{2-\gamma})}{x} .$$

If we put $y = x^{2-\gamma}$, then

$$\frac{d}{dx} f(a(x)) \leq \frac{c\, f(y)}{y^{1/(2-\gamma)}} ,$$

which tends to zero as x (and therefore y) increases without bound, since $\lim_{x \to \infty} f(x)/x^{1/2} = 0$.

It remains to verify (v). We have

$$\frac{x f(a(x))}{(a(x))^{1-\delta}} = \frac{x f(x^{2-\gamma})}{x^{(2-\gamma)(1-\delta)}} = \frac{f(x^{2-\gamma})}{x^{(2-\gamma)(1-\delta)-1}} .$$

We choose

$$\gamma = \frac{\delta}{1 - \delta}$$

so $2 - \gamma = (2 - 3\delta)/(1 - \delta)$

and

$$\frac{xf(a(x))}{(a(x))^{1-\delta}} = \frac{f(x^{(2 - 3\delta)/(1 - \delta)})}{x^{1 - 3\delta}} .$$

If $y = x^{(2 - 3\delta)/(1-\delta)}$, then $x = y^{(1 - \delta)/(2 - 3\delta)}$, and

$$\frac{xf(a(x))}{(a(x))^{1-\delta}} = \frac{f(y)}{y^{(1 - 3\delta)(1 - \delta)/(2 - 3\delta)}} .$$

For $\delta > 0$ sufficiently small, the exponent $(1 - 3\delta)(1 - \delta)/(2 - 3\delta)$ is arbitrarily close to and greater than $1/2$. Since we assumed that $f(x)/x^{1/2} = o(1)$, it follows that

$$\frac{xf(a(x))}{(a(x))^{1-\delta}} \to 0 \text{ as } x \to \infty ,$$

and all parts of the lemma are verified.

We can at last state the main theorem.

Theorem 1:

Suppose that f increases without bound, f' decreases, and $f(x) = o(x^{1/2})$. Suppose that P is bounded, periodic, and piecewise monotone, and at each point in each monotone component, some derivative of P is not zero. Then $\{f(n)\ P(ny)\}$ is uniformly

distributed (mod 1) for almost all y.

Proof:

We will apply Lemmas 8 and 9. Take the function $a(x) = x^{2-\gamma}$ of Lemma 9 and choose $a_k = a(k)$, $k = 1, 2, \ldots$. To prove the theorem, we need only show that the hypotheses of Lemma 8 are satisfied, namely

$$\text{(a)} \qquad a_k < a_{k+1},$$

$$\text{(b)} \qquad \lim_{k \to \infty} \Delta a_k = \infty,$$

$$\text{(c)} \qquad \lim_{k \to \infty} \frac{\Delta a_k}{a_k} = 0,$$

$$\text{(d)} \qquad \lim_{k \to \infty} (f(a_{k+1}) - f(a_k)) = 0, \text{ and}$$

$$\text{(e)} \qquad \lim_{k \to \infty} \frac{k f(a_k)}{a_k^{1-\delta}} = 0,$$

for $\delta > 0$ sufficiently small.

(a) and (b) follow from (i) of Lemma 9. Since

$$\Delta a_k \leq \max_{k \leq x \leq k+1} a'(x)$$

$$= a'(k) + \max_{k \leq x \leq k+1} |a'(x) - a'(k)|$$

$$\leq a'(k) + \max_{k \leq x \leq k+1} |a''(x)|,$$

(c) follows from (ii) and (iii) of Lemma 9. Also

$$f(a_{k+1}) - f(a_k) \leq \Delta a_k f'(a(k))$$

$$\leq a'(k) f'(a(k)) + f'(a(k)) \max_{k \leq x \leq k+1} a''(x),$$

so (d) follows from (iv). Finally, (e) follows from (v), and the theorem is proved.

CHAPTER III

The Distribution of $\{f(n)\,P(g(n))\}$

In this chapter, we use the method of proof of Fejér's Theorem (see the Introduction) to get a partial result on the distribution of $\{f(n)\,P(g(n))\}$.

We will suppose that P is differentiable, bounded, periodic with period p, and that P' vanishes only finitely many times in each period. We also suppose that near each relative extremum of P, P'' is bounded away from zero. This will be made more precise later. After the proof is complete, we will note how some of these conditions could be relaxed.

We will impose conditions on f and g slightly stronger than those of Fejér's Theorem: we suppose that f increases without bound as $x \to \infty$, $\lim_{x\to\infty} f(x)/x = 0$, $\lim_{x\to\infty} xf'(x) = \infty$, and $f''(x) \leq 0$ for x sufficiently large. We suppose that g satisfies the same conditions. (The condition $f''(x) \leq 0$ is not part of Fejér's Theorem.) In addition, we suppose that $g'(x)/f'(x) = o(1)$ and $f(x)g'(x) = o(1)$.

If P, f, and g satisfy the conditions listed above, then $\{f(n)\,P(g(n))\}$ is uniformly distributed (mod 1). Note that, unlike Theorem 1, this is not a metric theorem.

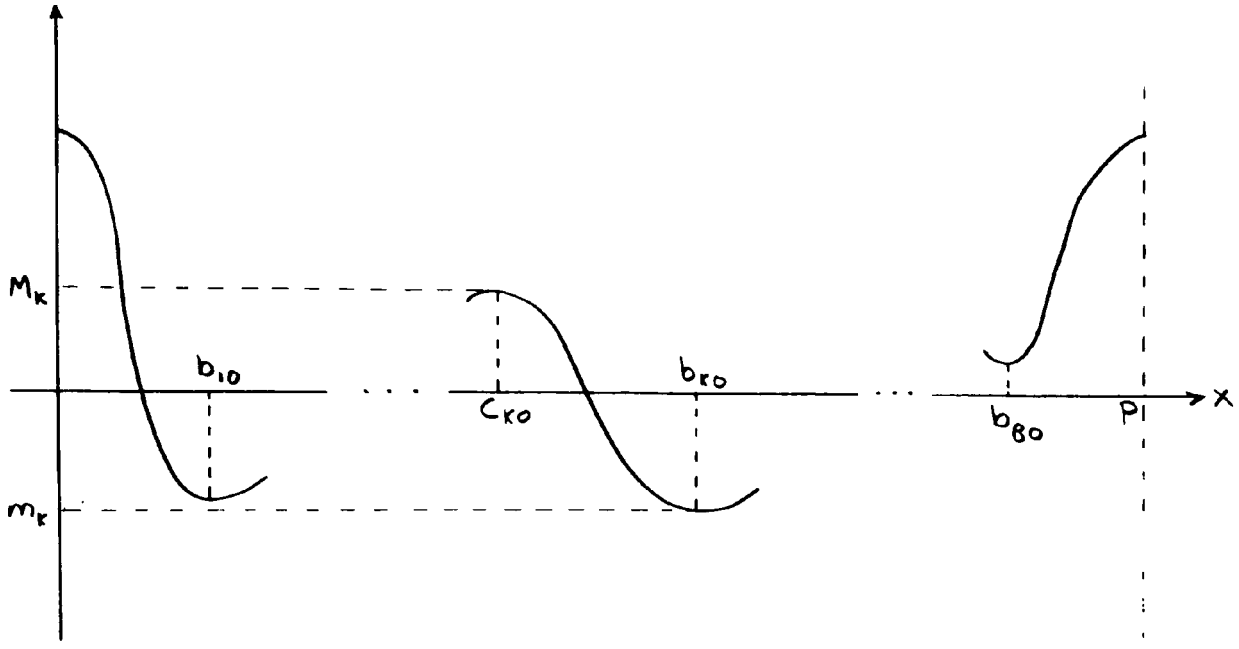
In spite of the number of conditions, there reamin functions f and g which satisfy them. For example,

$f(x) = x^{\alpha}$ and $g(x) = x^{\beta}$ with $0 < \beta < \alpha < \alpha + \beta < 1$,

$f(x) = x^{\alpha}$ and $g(x) = (\log x)^{\beta}$ with $0 < \alpha < 1$, $1 < \beta$, and

$f(x) = (\log x)^{\alpha}$ and $g(x) = (\log x)^{\beta}$ with $1 < \beta < \alpha$.

We first develop notation. As shown in Figure 1, let P have relative maxima at $0 = c_{10}, c_{20}, \ldots, c_{B0}$ with $P(c_{k0}) = M_k$ and relative minima at $b_{10}, b_{20}, \ldots, b_{B0}$ with $P(b_{k0}) = m_k$. B is the number of relative maxima in a period of P. There is no loss of generality in assuming that P has a relative maximum at zero, for if it does not, a translation will make it so. Such a translation does not affect the distribution of $\{f(n)\, P(g(n))\}$.



A period of $P(x)$

Figure 1

Let Ω_k be a positive number, no matter how small, $k = 1, 2, \ldots, B$. We suppose that P'' is bounded away from zero in an interval I_k, (see Figure 2), determined by Ω_k, around each relative maximum of P. Similarly, let ω_k be a positive number. We suppose that P'' is bounded away from zero in an interval i_k around each relative minimum of P (see Figure 2).

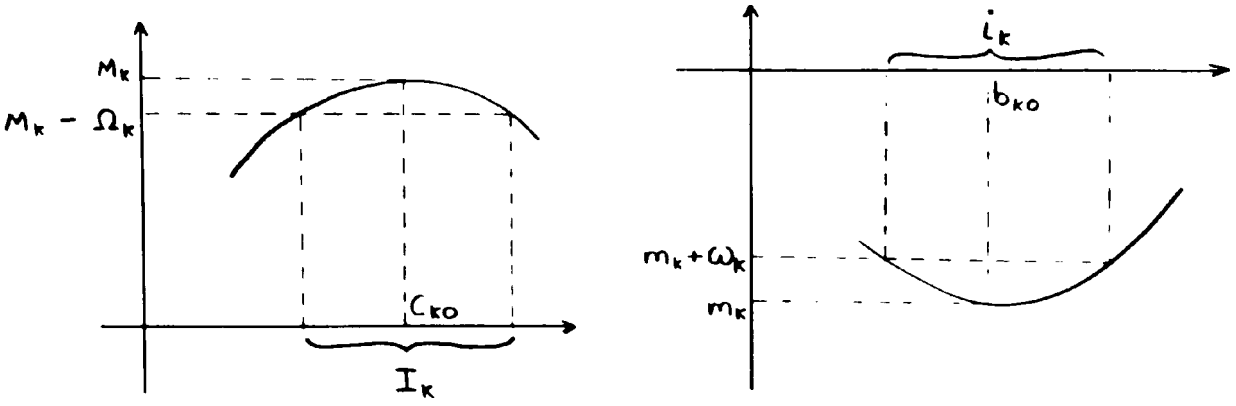


Figure 2

Let $G(t)$ be the function inverse to $g(x)$. Let

$$c_{kj} = G(c_{k0} + jp), \quad j = 1, 2, \ldots,$$

so that

$$P(g(c_{kj})) = P(c_{k0} + jp) = M_k ;$$

and let

$$b_{kj} = G(b_{k0} + jp), \quad j = 1, 2, \ldots,$$

so that

$$P(g(b_{kj})) = P(b_{ko} + jp) = m_k.$$

By assumption, $f''(x) \leq 0$ for x sufficiently large. However, there is no loss of generality in assuming that $f''(x) \leq 0$ for all x. The finite range of values of x excluded does not affect the distribution of $\{f(n) P(g(n))\}$. In the same way, we may assume that $f(x) \geq 0$ for all x, since it is eventually so. We then have

$$(1) \qquad \frac{f(x)}{x} \geq \frac{f(x) - f(0)}{x - 0} \geq f'(x),$$

so it follows from $f(x)/x = o(1)$ that $f'(x) = o(1)$.

Let

$$h(x) = f(x)\, P(g(x)).$$

Then

$$h'(x) = f'(x)\, P(g(x)) + f(x)\, P'(g(x))\, g'(x).$$

At c_{kj}, $P'(c_{kj}) = 0$ and

$$h'(c_{kj}) = M_k f'(c_{kj}).$$

If we suppose that $M_k > 0$, then the graph of h is rising at c_{kj}, although since $f'(x) = o(1)$, the slope decreases as j increases. Even though P decreases between c_{k0} and b_{k0}, f may be increasing so rapidly (when its argument is small) that $h'(x)$ does not vanish between $x = c_{kj}$ and $x = b_{kj}$. But since $f'(x) \to 0$ as $x \to \infty$, we might expect that for large j, h' will vanish between c_{kj} and b_{kj}, corresponding to the decrease of P from $P(c_{k0}) = M_k$ to $P(b_{k0}) = m_k$. This is the case, as we verify below. If we suppose that $M_k < 0$, then the graph of h is falling at c_{kj}. We expect that for large j, h' will vanish between $b_{k,j-1}$ and c_{kj}, corresponding to the increase of P from $P(b_{k,j-1}) = m_k$ to $P(c_{kj}) = M_k$. This also is the case.

Note that

$$\frac{h'(x)}{f'(x)} = P(g(x)) + \frac{f(x)\,g'(x)}{f'(x)}\,P'(g(x)) . \tag{2}$$

From (1),

$$0 \leq \frac{f'(x)}{f(x)\,g'(x)} \leq \frac{1}{xg'(x)} .$$

Since $xg'(x) \to \infty$ as $x \to \infty$, it follows that

$$\lim_{x \to \infty} \frac{f(x)\, g'(x)}{f'(x)} = \infty. \tag{3}$$

We want to find a point r_{kj}, near c_{kj}, such that $h'(r_{kj}) = 0$. Suppose that M_k is positive. For x slightly greater than c_{kj}, $P'(g(x))$ is negative. Since P is bounded, it follows from (3) that for x large enough, the rightmost member of (2) will be larger than $P(g(x))$. So, when j is sufficiently large--say $j \geq j_0$--we can associate with each zero $x = c_{kj}$ of $P'(g(x))$ a nearby point--say r_{kj}--such that $r_{kj} > c_{kj}$ and $h'(r_{kj}) = 0$. If M_k is negative, then for x slightly smaller than c_{kj}, $P'(g(x))$ is positive. In this case, the nearby point is to the left of c_{kj}. In case $M_k = 0$, then $h'(c_{kj}) = 0$, and we choose $r_{kj} = c_{kj}$. Since the next lemmas are concerned with estimating $|r_{kj} - c_{kj}|$, we disregard this case. There is no loss of generality in assuming that r_{kj} exists for all j, instead of for j sufficiently large, for discarding the finite portion of the graph of h where $j < j_0$ has no effect on the distribution of $\{h(n)\}$.

As above, with the zero of $P'(g(x))$ at $x = b_{kj}$, we can associate a nearby point s_{kj} such that $h'(s_{kj}) = 0$. In case m_k is negative, $b_{kj} < s_{kj}$; if m_k is positive, $s_{kj} < b_{kj}$; if $m_k = 0$, $b_{kj} = s_{kj}$.

Let

$$T_{kj} = h(r_{kj}) \quad \text{and} \quad S_{kj} = h(s_{kj}),$$

and put

$$U_{kj}(x) = \begin{cases} h(x) \text{ when } s_{kj} \leq x \leq r_{k+1,\,j} \\ 0 \text{ otherwise} \end{cases}$$

and

$$D_{kj}(x) = \begin{cases} h(x) \text{ when } r_{kj} \leq x \leq s_{kj} \\ 0 \text{ otherwise.} \end{cases}$$

The functions U_{kj} and D_{kj}, $k = 1, 2, \ldots, B$, $j = 0, 1, \ldots$, give a decomposition of h into monotone components.

We first prove three lemmas.

If a curve is too close to being horizontal, its distribution may not be uniform. Lemma 10 shows that with what we have assumed, the intervals from r_{kj} to c_{kj} and from s_{kj} to b_{kj}, in which the graph of h is nearly horizontal, are relatively short. These intervals are on only one side of the relative extremum; using Lemma 11, we show in Lemma 12 that the corresponding intervals on the other side of the extremum are also relatively short.

Lemma 10:

If $c_{kj} < r_{kj}$, then $r_{kj} - c_{kj} = \epsilon_k(j)/g'(r_{kj})$; if $r_{kj} < c_{kj}$, then $c_{kj} - r_{kj} = \epsilon_k(j)/g'(c_{kj})$, where $\lim \epsilon_k(j) = 0$, for $k = 1, 2, \ldots, B$.

Proof:

The two cases are similar. We treat the first. By assumption, $g''(x) \leq 0$ for x sufficiently large. As before, we may assume that $g''(x) \leq 0$ for all x. Thus

$$g(r_{kj}) - g(c_{kj}) \geq (r_{kj} - c_{kj})\, g'(r_{kj}) \geq 0,$$

whence the lemma will be proved if we show that

$$\lim_{j \to \infty} (g(r_{kj}) - g(c_{kj})) = 0. \tag{4}$$

Now $h'(r_{kj}) = 0$, so

$$f(r_{kj})\, g'(r_{kj})\, P'(g(r_{kj})) + f'(r_{kj})\, P(g(r_{kj})) = 0,$$

or

$$0 \leq - P'(g(r_{kj})) = P(g(r_{kj}))\, \frac{f'(r_{kj})}{f(r_{kj})\, g'(r_{kj})} .$$

But P is bounded, and from (3), $f'(x)/f(x)g'(x) = o(1)$, so that

$$\lim_{j \to \infty} P'(g(r_{kj})) = 0.$$

Since $P'(g(c_{kj})) = 0$, we have

$$\lim_{j \to \infty} (P'(g(r_{kj})) - P'(g(c_{kj}))) = 0. \tag{5}$$

We have assumed that in an interval about every relative extremum, P has a second derivative which is bounded away from zero. In particular, $P''(c_{k0}) \neq 0$. Since $P''(g(c_{kj})) = P''(g(G(c_{k0} + jp))) = P''(c_{k0} + jp) = P''(c_{k0})$, we have that $P''(g(c_{kj}))$ is a constant independent of j. Hence

$$P'(g(r_{kj})) - P'(g(c_{kj})) = (g(r_{kj}) - g(c_{kj}))\, P''(g(c_{kj})) + O((g(r_{kj}) - g(c_{kj}))^2) .$$

From this and (5), it follows that

$$g(r_{kj}) - g(c_{kj}) \to 0 \quad \text{as} \quad j \to \infty ,$$

so (4) holds and the lemma is proved.

In the same way, we can show that if $s_{kj} < b_{kj}$, then $b_{kj} - s_{kj} = \epsilon_k(j)/g'(b_{kj})$ and if $b_{kj} < s_{kj}$, then $s_{kj} - b_{kj} = \epsilon_k(j)/g'(s_{kj})$, for each k.

Lemma 11:

$$\lim_{j \to \infty} \frac{r_{kj}}{r_{k,j+1}} = 1 \quad \text{and} \quad \lim_{j \to \infty} \frac{s_{kj}}{s_{k,j+1}} = 1$$

for each k.

Proof:

We first note that the lemma is satisfied if we replace r with c:

$$\frac{c_{k,j+1} - c_{kj}}{c_{k,j+1}} = \frac{G(c_{k0} + p(j+1)) - G(c_{k0} + pj)}{G(c_{k0} + p(j+1))}$$

$$= \frac{pG'(c_{k0} + p\eta_j)}{G(c_{k0} + p(j+1))} ,$$

where $j \leq \eta_j \leq j+1$. Since $G'(t)$ is an increasing function of t,

$$\frac{c_{k,j+1} - c_{kj}}{c_{k,j+1}} \leq \frac{c\, G'(c_{k0} + p(j+1))}{G(c_{k0} + p(j+1))} .$$

But $\lim_{t \to \infty} G(t)/G'(t) = \infty$; thus $\lim_{j \to \infty} c_{kj}/c_{k,j+1} = 1$ for each k.

Regardless of the relative positions of r_{kj} and c_{kj}, we have

$$\frac{r_{k,j+1} - r_{kj}}{r_{k,j+1}} < \frac{c_{k,j+2} - c_{k,j-1}}{c_{kj}} .$$

But

$$\frac{c_{k,j+2} - c_{k,j-1}}{c_{kj}} \leq \frac{c_{k,j+2} - c_{k,j+1}}{c_{k,j+1}} \cdot \frac{c_{k,j+1}}{c_{kj}} + \frac{c_{k,j+1} - c_{kj}}{c_{kj}}$$

$$+ \frac{c_{kj} - c_{k,j-1}}{c_{kj}},$$

and each of these terms is $o(1)$. Thus $\lim_{j \to \infty} r_{kj}/r_{k,\,j+1} = 1$.

In the same way, we can show that

$$s_{k,\,j+1} - s_{kj} = o(s_{kj})$$

for each k, proving the lemma.

Let c'_{kj} be such that r_{kj} is between c_{kj} and c'_{kj} and $h(c_{kj}) = h(c'_{kj})$ (see Figure 3).

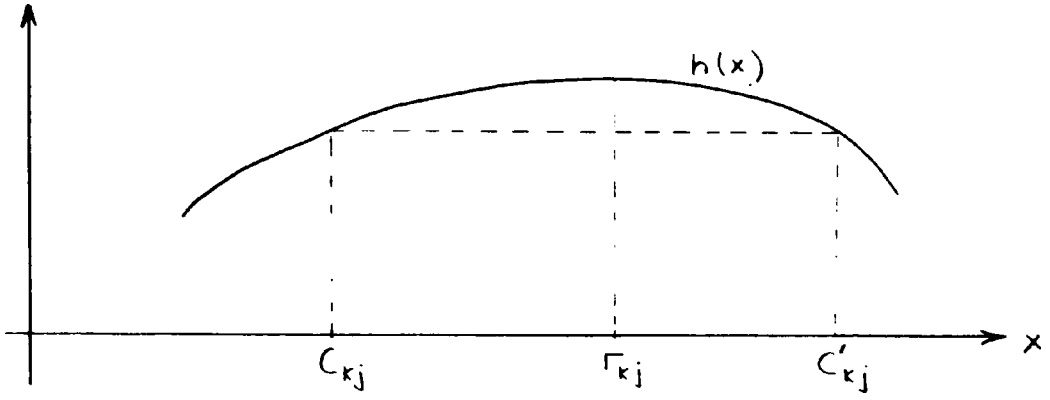


Figure 3

In Figure 3, we have assumed the relative maximum of h at r_{kj} to be positive. If we had assumed it to be negative, we would have

$c'_{kj} < r_{kj} < c_{kj}$.

Lemma 12:

If $c'_{kj} > c_{kj}$, then $c'_{kj} - c_{kj} = \epsilon_k(j)/g'(c'_{kj})$, and if $c_{kj} > c'_{kj}$, then $c_{kj} - c'_{kj} = \epsilon_k(j)/g'(c_{kj})$, where $\epsilon_k(j) \to 0$ as $j \to \infty$, for each k.

Proof:

We will suppose that M_k is positive; the other case is similar. Since $h(c_{kj}) = h(c'_{kj})$,

$$
\begin{aligned}
0 &= f(c'_{kj})\, P(g(c'_{kj})) - f(c_{kj})\, P(g(c_{kj})) \\
&= f(c'_{kj})\,(P(g(c'_{kj})) - P(g(c_{kj}))) \qquad (6) \\
&\quad + P(g(c_{kj}))\,(f(c'_{kj}) - f(c_{kj}))\,.
\end{aligned}
$$

From Taylor's Theorem and the fact that $P'(g(c_{kj})) = 0$, we have

$$
P(g(c'_{kj})) - P(g(c_{kj})) = \frac{1}{2}\,(g(c'_{kj}) - g(c_{kj}))^2\, P''(g(\xi_{kj}))\,,
$$

where $c_{kj} \leq \xi_{kj} \leq c'_{kj}$. Substituting this in (6), applying the mean value theorem, and using $P(g(c_{kj})) = M_k$, we have

$$
\begin{aligned}
M_k(c'_{kj} - c_{kj})\, f'(\psi_{kj}) = \frac{1}{2}\, f(c'_{kj})\,(-P''(g(\xi_{kj}))\, x \qquad (7) \\
x\,(g(c'_{kj}) - g(c_{kj}))^2\,,
\end{aligned}
$$

where $c_{kj} \leq \psi_{kj} \leq c'_{kj}$. From the assumption that in the interval I_k,

P'' is bounded away from zero, it follows that P'' is bounded away from zero in the interval I_{kj}, defined as the inverse image of $(T_{kj} - \Omega_k f(c_{kj}), T_{kj})$ about r_{kj}.

We would like to apply this in (7), but to do so, we must be sure that

$$h(r_{kj}) - h(c_{kj}) \leq \Omega_k f(c_{kj}) .$$

It is enough that

$$h(r_{kj}) - h(c_{kj}) = o(f(c_{kj})) ;$$

we verify this now:

$$\begin{aligned} h(r_{kj}) - h(c_{kj}) &= f(r_{kj}) P(g(r_{kj})) - f(c_{kj}) P(g(c_{kj})) \\ &= f(r_{kj}) (P(g(r_{kj})) - P(g(c_{kj}))) \\ &\quad + P(g(c_{kj})) (f(r_{kj}) - f(c_{kj})) . \end{aligned} \tag{8}$$

Since the first term in the rightmost member of (8) is negative and $P(g(c_{kj})) = M_k$,

$$h(r_{kj}) - h(c_{kj}) \leq M_k (r_{kj} - c_{kj}) f'(c_{kj}).$$

From (1)

$$h(r_{kj}) - h(c_{kj}) \leq M_k \frac{r_{kj} - c_{kj}}{c_{kj}} f(c_{kj}) .$$

But from Lemma 11,

$$\frac{r_{kj} - c_{kj}}{c_{kj}} \leq \frac{r_{kj} - r_{k,j-1}}{r_{k,j-1}} = \frac{r_{kj}}{r_{k,j-1}} \cdot \frac{r_{kj} - r_{k,j-1}}{r_{kj}} = o(1).$$

Thus we may apply the bound on P'' and the mean value theorem in (7) to get

$$M_k(c'_{kj} - c_{kj}) f'(\psi_{kj}) \geq c(c'_{kj} - c_{kj}) g'^2(c'_{kj}) f(c'_{kj}).$$

Thus

$$c'_{kj} - c_{kj} \leq \frac{cf'(\psi_{kj})}{f(c_{kj}) g'^2(c'_{kj})} \leq \frac{cf'(c_{kj})}{f(c'_{kj}) g'^2(c'_{kj})} .$$

Since $f'(c_{kj}) \leq f(c_{kj})/c_{kj}$,

$$(9) \qquad c'_{kj} - c_{kj} \leq c \frac{f(c_{kj})}{f(c'_{kj})} \cdot \frac{c'_{kj}}{c_{kj}} \cdot \frac{1}{c'_{kj} g'(c'_{kj})} \cdot \frac{1}{g'(c'_{kj})} .$$

We will show that in (9), the first two fractions approach 1 as $j \to \infty$. The third approaches zero as $j \to \infty$ and is the $\epsilon_k(j)$ of the lemma. It follows from Lemma 11 that $c'_{kj}/c_{kj} \to 1$ as $j \to \infty$:

regardless of the relative positions of c_{kj} and c'_{kj}, the interval with these points as endpoints is contained in the interval $(r_{k,\,j-1},\ r_{k,\,j+1})$. Since

$$\frac{r_{k,j+1} - r_{k,j-1}}{r_{k,\,j-1}} \leq \frac{r_{k,j+1} - r_{kj}}{r_{kj}} \cdot \frac{r_{kj}}{r_{k,j-1}} + \frac{r_{kj} - r_{k,j-1}}{r_{kj}} = o(1),$$

we have

$$0 \leq \frac{|c'_{kj} - c_{kj}|}{\max(c_{kj}, c'_{kj})} \leq \frac{r_{k,j+1} - r_{k,\,j-1}}{r_{k,j-1}} = o(1),$$

and the assertion is verified. The following calculation shows that $f(c_{kj})/f(c'_{kj}) \to 1$ as $j \to \infty$:

$$f(c'_{kj}) - f(c_{kj}) \leq (c'_{kj} - c_{kj})\, f'(c_{kj})$$

$$\leq \frac{c'_{kj} - c_{kj}}{c_{kj}}\, f(c_{kj}) = o(f(c_{kj})).$$

Finally, $c'_{kj} g'(c'_{kj}) \to \infty$ as $j \to \infty$, so the lemma follows from (9).

If we define b'_{kj} in the same manner as we defined c'_{kj}, we can show that if $b'_{kj} > b_{kj}$, then $b'_{kj} - b_{kj} = \epsilon_k(j)/g'(b'_{kj})$, and if $b_{kj} > b'_{kj}$ then $b_{kj} - b'_{kj} = \epsilon_k(j)/g'(b_{kj})$.

Theorem 2:

Suppose that P is everywhere differentiable, bounded, and

periodic with period p; that P' has only finitely many zeros in each period; and that in a neighborhood around each relative extremum, P'' is bounded away from zero. Suppose that f increases, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x)/x = 0$, $\lim_{x\to\infty} xf'(x) = \infty$, and $f''(x) \leq 0$ for x sufficiently large. Suppose that g satisfies the same conditions as f, and in addition that $g'(x)/f'(x) = o(1)$ and $f(x)g'(x) = o(1)$. Then $\{f(n)P(g(n))\}$ is uniformly distributed (mod 1).

Proof:

As before, let

$$h(x) = f(x)P(g(x))$$

and put

$$Z_n(\alpha) = \#\{m \leq n \mid \langle h(m)\rangle < \alpha\},$$

for $0 \leq \alpha \leq 1$. To show that $\{h(n)\}$ is uniformly distributed (mod 1), it suffices to show that for each such α,

$$\lim_{n\to\infty} \frac{1}{n} Z_n(\alpha) = \alpha.$$

For given n, there is a unique integer j such that $r_{1j} \leq n < r_{1,\,j+1}$. We note that

$$\left| \frac{1}{n} Z_n(\alpha) - \frac{1}{r_{1j}} Z_{r_{1j}}(\alpha) \right| \leq \frac{1}{r_{1j}} \left| Z_n(\alpha) - Z_{r_{1j}}(\alpha) \right|$$

$$= \frac{1}{r_{1j}} \#\{m \mid r_{1j} < m \leq n \text{ and } \langle h(m) \rangle < \alpha\}$$

$$\leq \frac{1}{r_{1j}} (n - [r_{1j}]) \leq \frac{r_{1,j+1} - r_{1j}}{r_{1j}} + O\left(\frac{1}{r_{1j}}\right) .$$

From Lemma 11, this quantity is $o(1)$, so that $Z_n(\alpha)/n$ and $Z_{r_{1j}}(\alpha)/r_{1j}$ approach the same limit. To prove the theorem, we thus need only show that

$$\lim_{n \to \infty} \frac{Z_{r_{1n}}(\alpha)}{r_{1n}} = \alpha . \tag{10}$$

The plan of the proof is as follows. We separate h into $2B$ components, namely

$$h_{k1}(x) = \sum_{j=0}^{\infty} U_{kj}(x)$$

and

$$h_{k2}(x) = \sum_{j=0}^{\infty} D_{kj}(x) ,$$

for $k = 1, 2, \ldots, B$.

We wish to count the number of integers m for which $\langle h(m) \rangle < \alpha$. If we disregard the ranges of x for which h_{k1}

or h_{k2} vanish, then we can write

$$Z_{r_{1n}}(\alpha) = \sum_{k=1}^{B} \#\{m \leq r_{1n} \mid \langle h_{k1}(m) \rangle < \alpha\}$$

$$+ \sum_{k=1}^{B} \#\{m \leq r_{1n} \mid \langle h_{k2}(m) \rangle < \alpha\} .$$

We can achieve this by temporarily agreeing that $<0> = 1$. Denote the first term in the right of the last equation by $V_{kn}(\alpha)$ and the second by $W_{kn}(\alpha)$. We can write

$$W_{kn}(\alpha) = \sum_{j=0}^{n-1} P_{kj}(\alpha)$$

and

$$V_{kn}(\alpha) = \sum_{j=0}^{n-1} Q_{kj}(\alpha) ,$$

where (setting $r_{B+1,j} = r_{1,\ j+1}$)

$$P_{kj}(\alpha) = \#\{m \mid r_{kj} \leq m < s_{kj} \text{ and } <D_{kj}> < \alpha\}$$

and

$$Q_{kj}(\alpha) = \#\{m \mid s_{kj} \leq m < r_{k+1,j} \text{ and } <U_{kj}(m)> < \alpha\} ,$$

where, since m is restricted to an interval in which D_{kj} (or U_{kj}) is not a constant, we need not require that $<0> = 1$.

If $d_{kj}(t)$ denotes the function inverse to the non-zero part of $D_{kj}(t)$, we will show that

$$P_{kj}(\alpha) = \sum_{r = S_{kj}}^{T_{kj} - 1} ([d_{kj}(r)] - [d_{kj}(r + \alpha)]) + E_{kj} , \tag{11}$$

where E_{kj} denotes a negligible error. Removing the greatest-integer brackets will introduce another negligible error, so

$$P_{kj}(\alpha) = \sum_{r = S_{kj}}^{T_{kj} - 1} (d_{kj}(r) - d_{kj}(r + \alpha)) + E_{kj} .$$

We then replace the sum with an integral, again with negligible error:

$$P_{kj}(\alpha) = \int_{S_{kj}}^{T_{kj} - 1} (d_{kj}(x) - d_{kj}(x + \alpha))\, dx + E_{kj} .$$

We next show that we can write

$$P_{kj}(\alpha) = - \int_{S_{kj}}^{T_{kj}} \alpha\, d'_{kj}(x)\, dx + E_{kj} ,$$

from which

$$\begin{aligned} P_{kj}(\alpha) &= \alpha(d_{kj}(S_{kj}) - d_{kj}(T_{kj})) + E_{kj} \\ &= \alpha(s_{kj} - r_{kj}) + E_{kj}. \end{aligned} \tag{12}$$

In a similar manner, we are able to show that

$$Q_{kj}(\alpha) = \alpha(r_{k+1,j} - s_{kj}) + E_{kj}.$$

Thus

$$\sum_{k=1}^{B} (P_{kj}(\alpha) + Q_{kj}(\alpha)) = \alpha(r_{1,j+1} - r_{1j}) + E_j$$

and

$$Z_{r_{1n}}(\alpha) = \sum_{j=0}^{n-1} \sum_{k=1}^{B} (P_{kj}(\alpha) + Q_{kj}(\alpha)) = \alpha(r_{1n} - r_{10}) + E_n ,$$

from which it will follow that

$$\lim_{n \to \infty} \frac{Z_{r_{1n}}(\alpha)}{r_{1n}} = \alpha ,$$

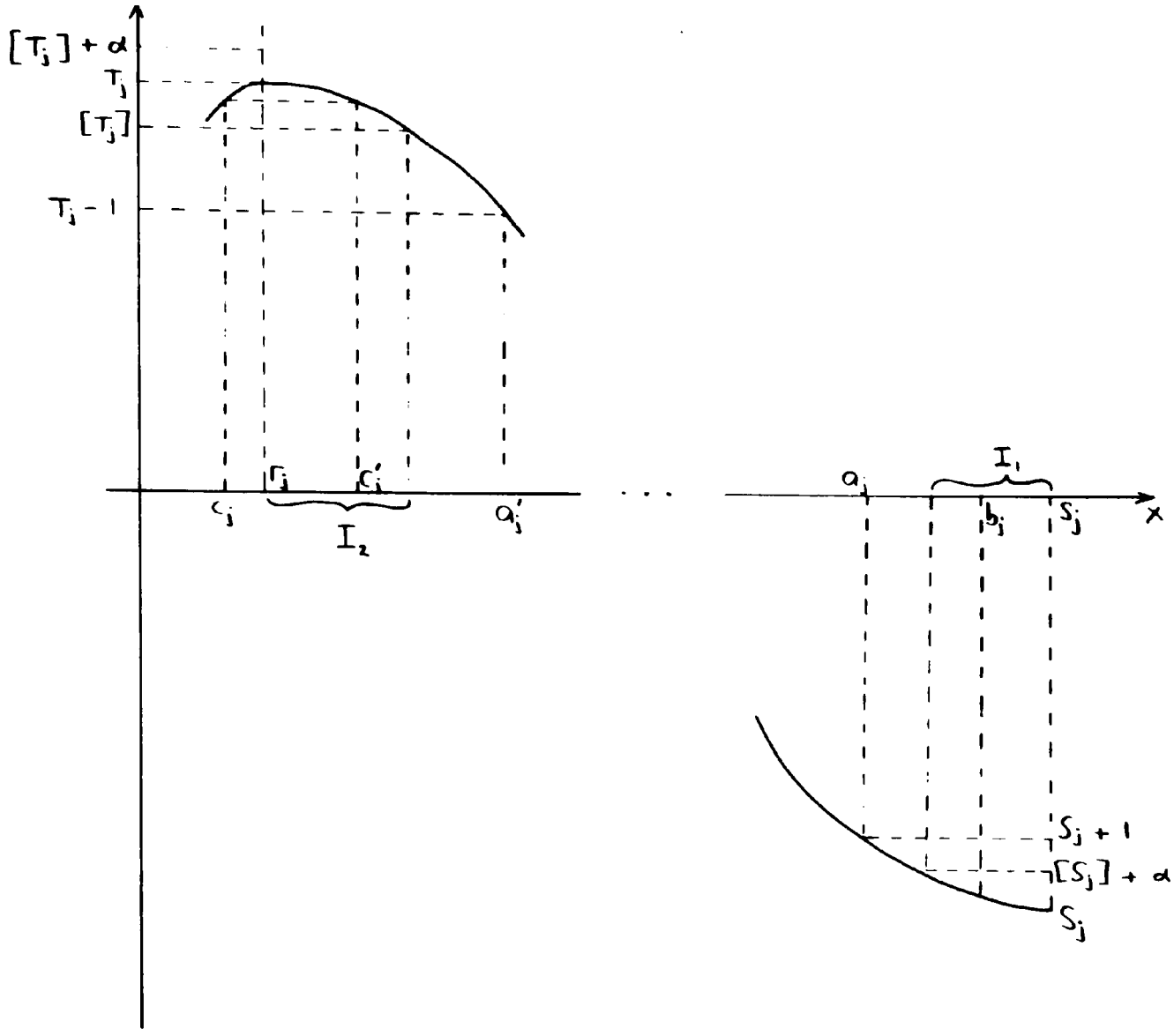
proving the theorem.

Thus, when we establish (12), we have enough. Since the treatment is the same regardless of the value of k, we will tkae $k = 1$ and drop the subscript, so

$$P_{kj}(\alpha) = P_j(\alpha), \quad D_{kj}(x) = D_j(x), \quad r_{kj} = r_j ,$$

and so on. We have

$$P_j(\alpha) = \sum_{i=-\infty}^{\infty} \#\{m \mid r_j \leq m < s_j \text{ and } i \leq D_j(m) < i + \alpha\} . \tag{13}$$



Endpoints of $D_j(x)$

Figure 4

(The sum is only formally infinite, since $S_j \leq D_j(t) \leq T_j$.) Let

$$A_j(x) = \#\{m \mid r_j \leq m < s_j \text{ and } D_j(m) < x\}. \tag{14}$$

From (13), we see that

$$P_j(\alpha) = \sum_{r=S_j}^{T_j-1} (A_j(r+\alpha) - A_j(r)) + E_{j1}(\alpha) + E_{j2}(\alpha) \tag{15}$$

(recall that $\Sigma_{S_j}^{T_j-1} = \Sigma_{[S_j]+1}^{[T_j]-1}$), where $E_{j1}(\alpha)$ and $E_{j2}(\alpha)$ are the errors made by ignoring those portions of $D_j(t)$ lying between S_j and $[S_j] + \alpha$, and between $[T_j]$ and T_j, respectively. In Figure 4, $E_{j1}(\alpha)$ is the number of integers in I_1 and $E_{j2}(\alpha)$ is the number of integers in I_2. An upper bound for the first error is $[s_j] - [a_j]$. If (as in Figure 4), b_j is between a_j and s_j, then $[s_j] - [a_j] = [s_j] - [b_j] + [b_j] - [a_j]$, with $[b_j] - [a_j] > 0$. If, on the other hand, b_j is to the left of a_j, then $[s_j] - [a_j] < [s_j] - [b_j]$. Thus we may assume that $a_j < b_j < s_j$, since this makes the estimate of the error which we are using larger, if anything. In case (as in Figure 4) m is negative, $b_j < s_j$. If m were positive, we would have $b_j' < s_j$. But since from Lemmas 10 and 12, both $|s_j - b_j|$ and $|s_j - b_j'|$ have the same upper bound, it is immaterial whether m is positive or not. Removing the greatest-integer brackets gives

$$E_{j1} \leq (s_j - b_j) + (b_j - a_j) + 2. \tag{16}$$

To show that this error is negligible, it suffices to show that

$$\frac{1}{r_n} \sum_{j=0}^{n-1} E_{j1} = o(1). \tag{17}$$

From Lemma 10, the first term in the right-hand side of (16) is bounded above in absolute value by $\epsilon(j)/g'(b_j)$ or $\epsilon(j)/g'(b'_j)$, whichever is larger; in either case, it is bounded by $\epsilon(j)/g'(r_{j+1})$. Also, (see Figure 5), since $h'(b_j) = mf'(b_j)$ (for $P'(g(b_j)) = 0$ and $P(g(b_j)) = m$),

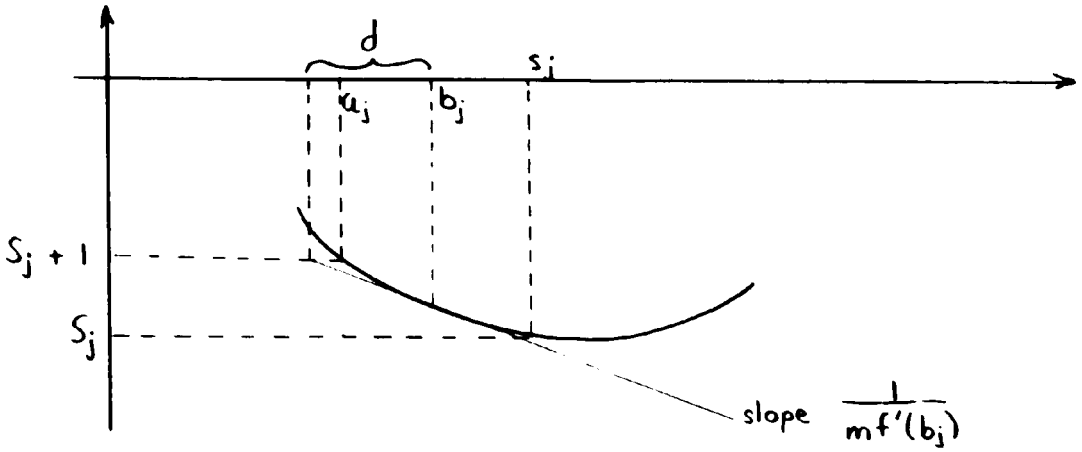


Figure 5

$$b_j - a_j \leq d = \frac{1}{|m| f'(b_j)} \leq \frac{c}{f'(r_{j+1})} .$$

Thus from (16),

$$\frac{1}{r_n}\sum_{j=0}^{n-1} E_{j1} \leq \frac{1}{r_n}\sum_{j=0}^{n-1}\left(\frac{\epsilon(j)}{g'(r_{j+1})} + \frac{c}{f'(r_{j+1})} + 2\right).$$

Replacing the sum by the corresponding integral introduces an error no larger than the maximum value of the integrand, so that

$$(18)\qquad \frac{1}{r_n}\sum_{j=0}^{n-1} E_{j1} \leq \frac{c}{r_n}\int_0^{n-1}\left(\frac{\epsilon(x)}{g'(G(r_0+(x+1)p))} + \frac{1}{f'(G(r_0+(x+1)p))} + 2\right)dx$$

$$+ \frac{c}{r_n}\left(\frac{1}{g'(r_n)} + \frac{1}{f'(r_n)} + 2\right).$$

In the second term in the right of (18), $1/r_n g'(r_n)$, $1/r_n f'(r_n)$, and $2/r_n$ are each $o(1)$. In the first term, let $u = G(r_0 + (x+1)p)$, so that $x = (g(u) - r_0 - p)/p$; then the first term becomes

$$\frac{c}{r_n}\int_{r_1}^{r_n}\left(\epsilon\left(\frac{g(u) - r_0 - p}{p}\right) + \frac{g'(u)}{f'(u)} + 2g'(u)\right) du .$$

Since each term in the integrand is $o(1)$, it follows from the continuous analogue of Lemma 1 that the entire term is $o(1)$. Hence

$$\frac{1}{r_n}\sum_{j=0}^{n-1} E_{j1} = o(1),$$

and E_{j1} is negligible.

We can show that E_{j2} is negligible in the same manner, except that since c_j' is between a_j and r_j, we must apply Lemma 12 instead of Lemma 10.

Letting E stand for any negligible error (in the sense that $\frac{1}{r_n} \sum_{j=0}^{n-1} E = o(1)$), we have from (15)

$$P_j(\alpha) = \sum_{r=S_j}^{T_j - 1} (A_j(r+\alpha) - A_j(r)) + E. \tag{19}$$

But from (14),

$$A_j(t) = \#\{m \mid r_j \leq m < s_j \text{ and } m > d_j(t)\},$$

so that

$$A_j(r+\alpha) - A_j(r) = \#\{m \mid d_j(r) \geq m > d_j(r+\alpha)\}$$

$$= [d_j(r)] - [d_j(r+\alpha)].$$

From (19),

$$P_j(\alpha) = \sum_{r=S_j}^{T_j - 1} (d_j(r) - d_j(r+\alpha)) + E_{j3}(\alpha) + E, \tag{20}$$

where $E_{j3}(\alpha)$ is the error arising from the removal of the greatest-integer brackets. It is thus bounded above by the number of terms to be summed:

$$E_{j3}(\alpha) \leq T_j - S_j = h(r_j) - h(s_j). \tag{21}$$

Since P is bounded,

$$E_{j3}(\alpha) \leq c(f(r_j) + f(s_j)) \leq c\, f(r_{j+1}) .$$

Hence

$$\frac{1}{r_n} \sum_{j=0}^{n-1} E_{j3}(\alpha) \leq \frac{c}{r_n} \sum_{j=0}^{n-1} f(r_j).$$

But

$$\frac{1}{r_n} \sum_{j=0}^{n-1} f(r_j) = \frac{1}{r_n} \int_0^{n-1} f(G(r_0 + (x+1)p))dx + O\left(\frac{f(r_n)}{r_n}\right)$$

$$= \frac{1}{r_n} \int_{r_1}^{r_n} \frac{1}{p} f(u)\, g'(u)\, du + O\left(\frac{f(r_n)}{r_n}\right) .$$

Since $f(u)\,g'(u)$ and $f(r_n)/r_n$ are both $o(1)$, it follows that $E_{j3}(\alpha)$ may be neglected.

Thus from (20),

$$P_j(\alpha) = \sum_{r=S_j}^{T_j - 1} (d_j(r) - d_j(r+\alpha)) + E.$$

Let the limits on the sum be denoted by $B_j = [S_j] + 1$ and $C_j = [T_j] - 1$. Then

$$P_j(\alpha) = \int_{B_j}^{C_j} (d_j(x) - d_j(x+\alpha))\, dx + E_{j4}(\alpha) + E, \tag{22}$$

where $|E_{j4}(\alpha)| \leq T_j - S_j$ and thus may be neglected just as $E_{j3}(\alpha)$ was. Since

$$\int_{B_j}^{C_j} d_j(x+\alpha)\,dx = \int_{B_j+\alpha}^{C_j+\alpha} d_j(x)\,dx ,$$

we have from (22),

$$P_j(\alpha) = \int_{B_j}^{C_j} d_j(x)\,dx - \int_{B_j+\alpha}^{C_j+\alpha} d_j(x)\,dx + E$$

$$= \int_{B_j}^{B_j+\alpha} d_j(x)dx - \int_{C_j}^{C_j+\alpha} d_j(x)\,dx + E \tag{23}$$

$$= I_{j1}(\alpha) - I_{j2}(\alpha) + E, \text{ say.}$$

By the mean-value theorem, it follows that

$$\alpha d_j(C_j + 1) \leq I_{j2}(\alpha) \leq \alpha d_j(C_j).$$

Since the distance from T_j to C_j is greater than that from T_j to $C_j + 1$, $I_{j2}(\alpha)$ differs from $\alpha d_j(T_j)$ by an amount no greater that $\alpha(d_j(C_j) - d_j(T_j)) = \alpha(d_j([T_j] - 1) - d_j(T_j))$; in the same way as $E_{j2}(\alpha)$ was shown to be negligible, this error can be shown to be negligible. Similarly,

$$I_{j1}(\alpha) = \alpha d_j(S_j) + E,$$

so from (23),

$$P_j(\alpha) = \alpha(d_j(S_j) - d_j(T_j)) + E = \alpha(s_j - r_j) + E,$$

so (12) is established, and with it, the theorem.

In the preceeding, we used the assumptions that P was everywhere differentiable, with each relative minimum followed by a relative maximum and vice versa, only for convenience of notation. After we had separated P into monotone components, we considered only one of them (for they are all similar), and addition gave the result. The assumption that P' vanished at the endpoints of each monotone component made the work if anything more difficult, since the more nearly horizontal P is, the more care is needed in estimation. Thus, Theorem 2 holds if P is piecewise monotone, and is differentiable in the interior of each monotone component. At the endpoints, P must have right- or left-hand derivatives (unless P is horizontal, in which case it must have a two-sided derivative). As before, we require that P be bounded and periodic, that P' vanishes only finitely many times in each period, and that P'' be bounded away from zero near points where P' is zero.

CHAPTER IV

The Distribution of $\{P(g(n))\}$

Suppose that P is a periodic function with a continuous distribution function (that is,

$$\lim_{n \to \infty} \frac{1}{n} \#\{m \leq n \mid \langle P(m) \rangle < \alpha\}$$

exists and is continuous for $0 \leq \alpha \leq 1$), and suppose that the latter function is not linear. As Salem [9 ; page 34] has observed, if $\{g(n)\}$ is uniformly distributed (mod 1), then $\{P(g(n))\}$ is not. In particular, from Fejér's Theorem (see the Introduction), $\{P(g(n))\}$ is not uniformly distributed (mod 1) when, roughly speaking, $g(x)$ is a smooth function whose rate of growth lies between $\log x$ and x. For more rapidly increasing functions, the non-uniform distribution (mod 1) of $\{P(n^{\alpha}y)\}$ $(\alpha \geq 1)$ and $\{P(y^n)\}$ for almost all y follows from the uniform distribution (mod 1) of $\{n^{\alpha}y\}$ $(\alpha \geq 1)$ and $\{y^n\}$ for almost all y.

It is to be expected that $\{P(g(n))\}$ will also be non-uniformly distributed (mod 1) for a slowly increasing function g. The following theorem shows that this is indeed sometimes the case.

Theorem 3:

Suppose P is periodic with period p, is bounded and has a first

derivative which has only finitely many zeros in each period. Suppose g is increasing, $\lim_{x \to \infty} g(x) = \infty$, and $\lim_{x \to \infty} xg'(x) = 0$. Then $\{P(g(n))\}$ is not uniformly distributed (mod 1).

Proof:

Like Theorem 2, this is not a metric theorem. Let

$$N(n, \alpha) = \#\{m \leq n \mid \langle P(g(m)) \rangle < \alpha\} .$$

We will show that $P(g(n))$ changes so slowly that the distribution of $\{P(g(m))\}$ for $m \leq n$ is always dominated by the value of $P(g(n))$; that is, if $\langle P(g(n)) \rangle = \alpha$, then almost all of the numbers $P(g(m))$, $m \leq n$, have fractional part less than α:

$$\#\{m \leq n \mid \langle P(g(m)) \rangle < \alpha\} = n - o(n) .$$

Thus the ratio $N(n, \alpha)/n$ continually oscillates from near zero to near one, never approaching a limit. In particular, in what follows we show that for proper choice of n and α (since to show non-uniform distribution (mod 1), we need only show that $\lim_{n \to \infty} N(n, \alpha)/n \neq \alpha$ for some particular α), we can make $N(n, \alpha)/n$ as close to 1 as we please. If we choose $\alpha < 1$, then the sequence cannot be uniformly distributed (mod 1).

Let G(t) denote the function inverse to g(x). Decompose P into its monotone components P_i, where P_i is defined and monotone on the interval (p_{i-1}, p_i), $i = 1, 2, \ldots, R-1$, $p_0 = 0$, and $p_R = p$, and is not defined elsewhere. Without loss of generality, we may assume that P_1 decreases on $(0, p_1)$. Put

$$N_1(n, \alpha) = \#\{m \leq n \mid \langle P_1(g(m)) \rangle < \alpha\}.$$

More precisely, since P_1 is not defined everywhere, we should write

$$N_1(n, \alpha) = \#\{m \leq n \mid g(m) \in (p_0 + kp, p_1 + kp) \text{ for some } k$$

$$\text{and } \langle P_1(g(m)) \rangle < \alpha\}.$$

Let

$$B = \begin{cases} P_1(p_1) & \text{if } P_1(p_1) \text{ is an integer} \\ [P_1(p_1)] + 1 & \text{otherwise} \end{cases}$$

and

$$T = [P_1(0)].$$

Fix α with $0 < \alpha < 1$, so that $[P_1(p_1)] + \alpha < P_1(p_1)$ if $P_1(p_1)$ is not an integer (if $P_1(p_1)$ is an integer no restriction is placed on α

so long as it is in $(0, 1)$). With α so chosen, the portion of the graph of P_1 below B contributes nothing to $N_1(n, \alpha)$, for all ordinates on that portion of the curve have fractional parts greater than α. Put

$$N_1(n, \alpha) = \sum_{k=B}^{T} N_{1k}(n, \alpha)$$

where

(1) $$N_{1k}(n, \alpha) = \#\{m \leq n \mid k \leq P_1(g(m)) < k + \alpha\}.$$

Let

$$\theta_k = P_1^{-1}(k), \qquad 0 \leq \theta_k \leq p_1$$

$$\phi_k = P_1^{-1}(k+\alpha), \quad 0 \leq \phi_k \leq p_1$$

for $k = B, B + 1, \ldots, T$ (see Figure 1), and set

$$R_{kr} = G(\theta_k + rp)$$

$$S_{kr} = G(\phi_k + rp)$$

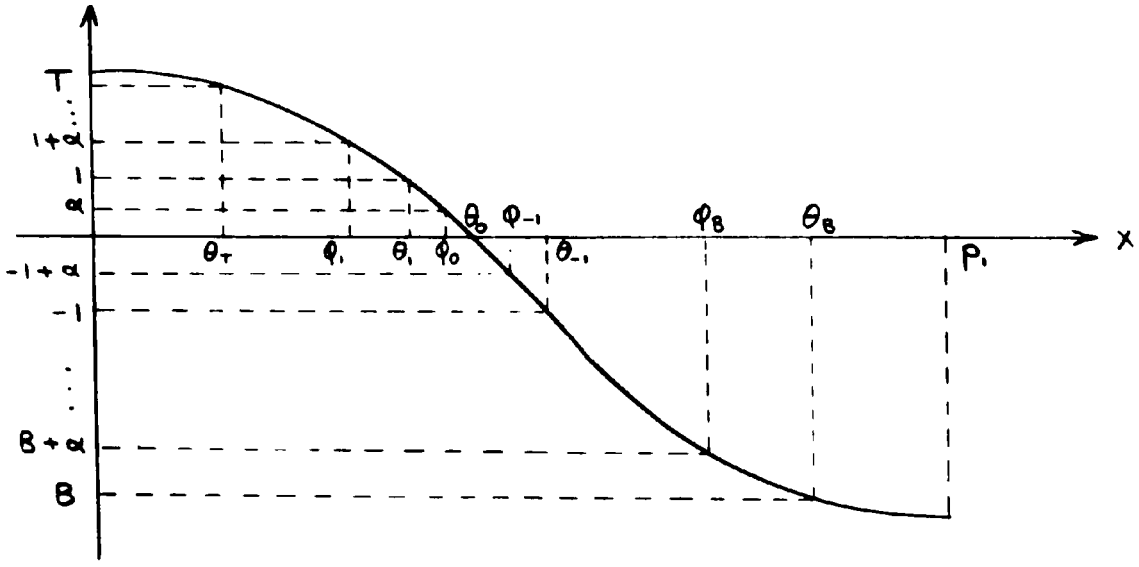
$r = 0, 1, \ldots,$ so that

$$P_1(g(R_{kr})) = k$$

and

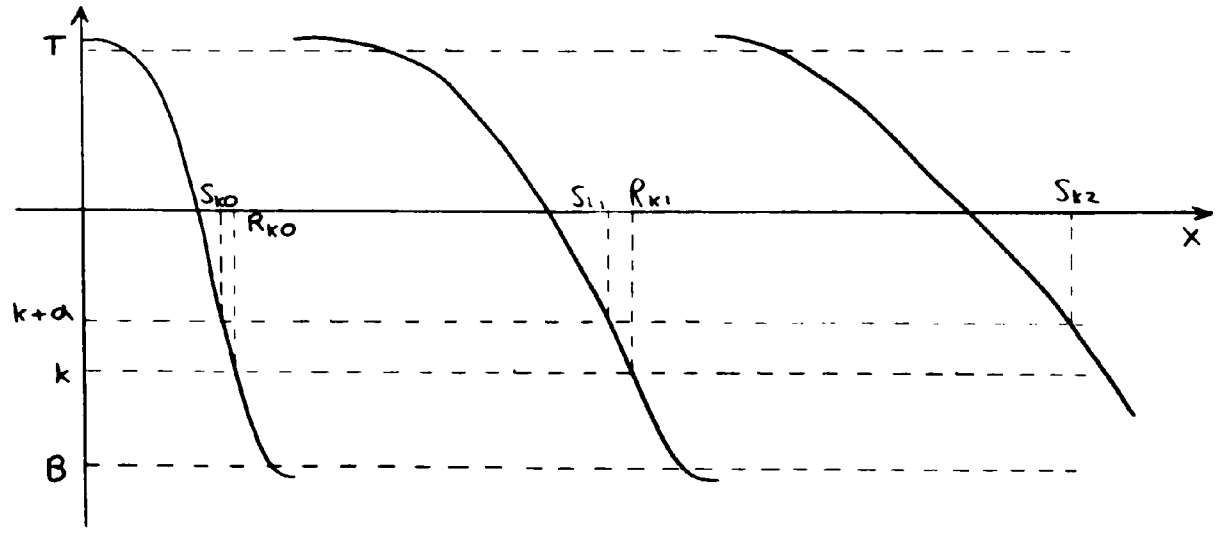
$$P_1(g(S_{kr})) = k + \alpha$$

in the rth "period" of $P_1(g(x))$, i.e., for $rp \leq g(x) < (r+1)p$ or, equivalently, for $G(rp) \leq x < G((r+1)p)$.



$P_1(x)$

Figure 1



$P_1(g(x))$

Figure 2

We show that because $G(t)$ increases more rapidly than e^t,

$$\lim_{a \to \infty} \frac{G(a - h)}{G(a)} = 0$$

for any $h > 0$. The hypotheses $\lim_{x \to \infty} g(x) = \infty$ and $\lim_{x \to \infty} xg'(x) = 0$ are equivalent to $\lim_{t \to \infty} G(t) = \infty$ and $\lim_{t \to \infty} G'(t)/G(t) = \infty$. If we put

$$H(t) = \log G(t),$$

then

$$\lim_{t \to \infty} H(t) = \infty \text{ and } \lim_{t \to \infty} H'(t) = \infty ;$$

also

$$\frac{G(a - h)}{G(a)} = \exp (H(a - h) - H(a)) .$$

An application of the mean-value theorem yields

$$0 \leq \lim_{a \to \infty} \frac{G(a - h)}{G(a)} \leq \lim_{a \to \infty} \exp \left(-h \min_{a - h \leq t \leq a} H'(t) \right).$$

Since the limit on the right is zero, the assertion is verified.

Since $g(x)$ is a function,

$$\phi_B - \theta_B < 0,$$

so by the preceeding remark,

$$(2) \qquad \lim_{r\to\infty} \frac{S_{Br}}{R_{Br}} = \lim_{r\to\infty} \frac{G(\phi_B + rp)}{G(\theta_B + rp)} = \lim_{r\to\infty} \frac{G(\phi_B + rp)}{G(\phi_B + rp - (\phi_B - \theta_B))} = 0$$

and

$$(3) \qquad \lim_{r\to\infty} \frac{R_{B,\,r-1}}{R_{Br}} = \lim_{r\to\infty} \frac{G(\theta_B + rp - p)}{G(\theta_B + rp)} = 0 .$$

We will show that

$$(4) \qquad \lim_{r\to\infty} \frac{N_{1B}(R_{Br},\alpha)}{R_{Br}} = 1.$$

This implies the theorem, for

$$\limsup_{n\to\infty} \frac{N(n,\alpha)}{n} \geq \limsup_{n\to\infty} \frac{N_{1B}(n,\alpha)}{n} = 1 > \alpha ,$$

which would contradict the uniform distribution (mod 1) of $\{P(g(n))\}$.

From (1),

$$(5) \qquad \begin{aligned} N_{1B}(R_{Br},\alpha) = {} & \#\{m \leq R_{B,\,r-1} \mid B \leq P_1(g(m)) < B + \alpha\} \\ & + \#\{R_{B,\,r-1} < m \leq R_{Br} \mid B \leq P_1(g(m)) < B + \alpha\} . \end{aligned}$$

We have $B \leq P_1(g(m)) < B + \alpha$ whenever

$$P_1^{-1}(B + \alpha) + tp < g(m) \leq P_1^{-1}(B) + tp$$

for some integer t. Hence, $B \leq P_1(g(m)) < B + \alpha$ whenever

$$\phi_B + tp < g(m) \leq \theta_B + tp ;$$

that is, when

$$G(\phi_B + tp) < m \leq G(\theta_B + tp) ,$$

and thus finally whenever

$$S_{Bt} < m \leq R_{Bt}$$

for some integer t. The second term on the right in (5) is thus

$$\begin{aligned} &\#\{R_{B,r-1} < m \leq R_{Br} \mid S_{Bt} < m \leq R_{Bt} \text{ for some } t\} \\ &= \#\{S_{Br} < m \leq R_{Br}\} = R_{Br} - S_{Br} + O(1). \end{aligned} \tag{6}$$

The first term on the right in (5) is bounded above by $R_{B,r-1}$, so from (6)

$$\lim_{r \to \infty} \frac{N_{1B}(R_{Br}, \alpha)}{R_{Br}} = \lim_{r \to \infty} \left(1 - \frac{S_{Br}}{R_{Br}} + O\left(\frac{R_{B,r-1}}{R_{Br}}\right) \right) .$$

From (2) and (3)

$$\lim_{r \to \infty} \frac{N_{1B}(R_{Br}, \alpha)}{R_{Br}} = 1,$$

so (4) holds and the theorem is proved.

CHAPTER V

The Distribution of $\{f(n) \cos n^{\beta} y\}$

LeVeque proved [7] that if β is a positive integer and a and α are positive numbers with $\alpha > \beta$, then $\{an^{\alpha} \cos n^{\beta} y\}$ is uniformly distributed (mod 1) for almost all y. We will show that $\{f(n) \cos n^{\beta} y\}$ is uniformly distributed (mod 1) for almost all y when f is one of a class of functions which includes $f(n) = an^{\alpha}$ with $0 \leq \alpha < \beta$ when $\alpha < 9$. Thus the distribution of $\{an^{\alpha} \cos n^{\beta} y\}$ is still not settled when $9 \leq \alpha < \beta$, though there seems no reason to doubt that it is uniform (mod 1) for almost all y.

The result is an application of Theorem A, cited in the Introduction. To apply the theorem, it is necessary to estimate the integral

$$E_{jk} = \int_0^{2\pi} e(mf(j) \cos j^{\beta} y - mf(k) \cos k^{\beta} y)\, dy ,$$

where m is a positive integer. The integrand can be expanded into a simple series of Bessel functions, which yields upon integration

$$E_{jk} = J_0(2\pi mf(j))\, J_0(2\pi mf(k)) + R_{jk} ,$$

where R_{jk} is an infinite sum of products of two Bessel functions. Lemma 14 is devoted to estimating R_{jk}. Lemma 13 provides an

inequality needed in the proof of the theorem.

Lemma 13:

If j and t are positive integers, then

$$\sum_{k=1}^{t} \frac{(j,k)^{\theta}}{k^{\theta}} \leq \begin{cases} \frac{c_{\epsilon}}{1-\theta} t^{1-\theta} j^{\epsilon} & \text{when } 0 < \theta < 1 \\ c_{\epsilon}(\log t + 1) j^{\epsilon} & \text{when } \theta = 1 \\ \frac{c_{\epsilon}}{\theta - 1} j^{\epsilon} & \text{when } \theta > 1 \end{cases}$$

for every $\epsilon > 0$. c_{ϵ} denotes a constant dependent on ϵ.

Proof:

Let

$$P(j,t) = \sum_{k=1}^{t} \frac{(j,k)^{\theta}}{k^{\theta}} .$$

If $d \mid j$, then $(j,k) = d$ for some of the values $k = id$, $i = 1, 2, \ldots, [t/d]$. Hence

$$(1) \quad P(j,t) = \sum_{d \mid j} \sum_{\substack{k \leq t \\ (k,j)=d}} \frac{d^{\theta}}{k^{\theta}} \leq \sum_{d \mid j} \sum_{i=1}^{[t/d]} \frac{d^{\theta}}{(id)^{\theta}} = \sum_{d \mid j} \sum_{i=1}^{[t/d]} \frac{1}{i^{\theta}} .$$

If $0 < \theta < 1$, then replacing the sum with its corresponding integral and noting that the error introduced is no larger than the maximum

value of the summand, we have

$$\sum_{i=1}^{[t/d]} \frac{1}{i^{\theta}} \leq \int_{1}^{[t/d]} \frac{dx}{x^{\theta}} + 1 \leq \frac{1}{1-\theta}([t/d]^{1-\theta} - 1) + 1,$$

so from (1) and the fact that the number of divisors of an integer j is $O(j^{\epsilon})$ for all $\epsilon > 0$,

$$(2) \qquad P(j,t) \leq \frac{c}{1-\theta} \sum_{d|j} [t/d]^{1-\theta} \leq \frac{ct^{1-\theta}}{1-\theta} \sum_{d|j} 1 \leq \frac{c_{\epsilon} t^{1-\theta} j^{\epsilon}}{1-\theta} .$$

If $\theta = 1$, then

$$\sum_{i=1}^{[t/d]} \frac{1}{i} \leq \log t/d + c,$$

so from (1)

$$(3) \qquad P(j,t) \leq \sum_{d|j} (\log t/d + c) \leq c(\log t + 1) \sum_{d|j} 1 \leq c_{\epsilon}(\log t + 1) j^{\epsilon}.$$

If $\theta > 1$, then

$$\sum_{i=1}^{[t/d]} \frac{1}{i^{\theta}} \leq \int_{1}^{[t/d]} \frac{dx}{x^{\theta}} + 1 = \frac{[t/d]^{1-\theta} - 1}{1-\theta} + 1 \leq \frac{1}{\theta - 1} + 1,$$

so from (1),

$$(4) \qquad P(j,t) \leq \frac{c}{\theta - 1} \sum_{d|j} 1 \leq \frac{c_{\epsilon} j^{\epsilon}}{\theta - 1} .$$

Equations (2)-(4) prove the lemma.

Lemma 14:

Let j, k, and β be positive integers with $k < j$ and put

$$j' = j^{\beta}/(j, k)^{\beta} \quad \text{and} \quad k' = k^{\beta}/(j, k)^{\beta}.$$

Let $\{b_n\}$ be an increasing sequence of positive numbers such that $b_n > 1$ and $\lim_{n \to \infty} b_n = \infty$, and let

$$S(j, k) = \sum_{t=1}^{\infty} | J_{tk'}(b_j) | \, | J_{tj'}(b_k) | ,$$

where $J_n(x)$ denotes the Bessel function of the first kind of order n and argument x. Then

$$S(j, k) \leq \frac{cb_j^{1/9} (\log b_j)^{2/3}}{k'} . \tag{5}$$

Proof:

From the Cauchy-Schwarz inequality,

$$S^2(j, k) \leq \sum_{t=1}^{\infty} (J_{tk'}(b_j))^2 \sum_{t=1}^{\infty} (J_{tj'}(b_k))^2$$

$$= S_1(j, k) \, S_1(k, j), \quad \text{say.}$$

We will show that, whether $k < j$ or not,

$$(6) \qquad S_1(j,k) \leq \frac{cb_j^{1/9}(\log b_j)^{2/3}}{k'} .$$

It follows from (6) and the facts that $b_j \geq b_k$ and $j' > k'$ that

$$S_1(k,j) \leq \frac{cb_k^{1/9}(\log b_k)^{2/3}}{j'} \leq \frac{cb_j^{1/9}(\log b_j)^{2/3}}{k'} ,$$

verifying (5).

We will truncate the series for $S_1(j,k)$ at b_j^2, separate the remainder into three parts, and estimate each part.

From the series expansion for $J_n(x)$ [11, page 22], namely

$$(7) \qquad J_n(x) = \frac{(x/2)^n}{n!}\left(1 - \frac{(x/2)^2}{1\cdot(n+1)} + \frac{(x/2)^4}{1\cdot 2\cdot(n+1)\cdot(n+2)} - \cdots\right),$$

we see that if $n \geq x^2$, the error made by neglecting all the terms after the first in the series in parentheses is at most $1/4$ in absolute value. Thus

$$(8) \qquad |J_n(x)| \leq c\frac{(x/2)^n}{n!} \quad \text{for} \quad n \geq x^2 .$$

If $t \geq b_j^2$, then $tk' \geq b_j^2$, so

$$|J_{tk'}(b_j)| \leq \frac{c(b_j/2)^{tk'}}{(tk')!} .$$

If we apply Stirling's formula, we have

$$|J_{tk'}(b_j)| < c\left(\frac{eb_j}{2tk'}\right)^{tk'} \frac{1}{(tk')^{1/2}} .$$

If we let

$$S_0(j,k) = \sum_{t=b_j^2}^{\infty} J_{tk'}^2(b_j) ,$$

then we have

$$S_0(j,k) \leq \frac{c}{k'} \sum_{t=b_j^2}^{\infty} \frac{1}{t} \left(\frac{eb_j}{2tk'}\right)^{2tk'} .$$

The sum converges, so

$$S_0(j,k) \leq \frac{c}{k'} . \tag{9}$$

We develop upper bounds for Bessel functions from the uniform asymptotic expansion

$$J_n(nx) \sim \left(\frac{4p}{1-x^2}\right)^{1/4} \left\{ \frac{Ai(n^{2/3}p)}{n^{1/3}} \sum_{k=0}^{\infty} \frac{a_k(p)}{n^{2k}} + \frac{Ai'(n^{2/3}p)}{n^{5/3}} \sum_{k=0}^{\infty} \frac{b_k(p)}{n^{2k}} \right\} , \tag{10}$$

valid for all $x > 0$ [1, equation 9.5.35]. Here p is given by

$$\frac{2}{3}(-p)^{3/2} = (x^2-1)^{1/2} - \operatorname{Arc\,cos}\frac{1}{x} \quad \text{if} \quad x \geq 1 , \tag{11}$$

$$(12) \qquad \frac{2}{3} p^{3/2} = \log \left(\frac{1 + (1-x^2)^{1/2}}{x} \right) - (1-x^2)^{1/2} \quad \text{if } x \leq 1$$

[1, equations 9.3.38, 9.3.39]; Ai is a function with the following properties [1, equations 10.4.59, 10.4.61]:

$$(13) \qquad |\mathrm{Ai}(x)| \leq c \min (1, |x|^{-1/4}) \text{ and}$$

$$(14) \qquad |\mathrm{Ai}'(x)| \leq c \max (1, |x|^{1/4}) .$$

Each of $a_0(p)$ and $b_0(p)$ is bounded above by 1 for all p.

From the uniform expansion (10), we are able to derive the bound

$$|J_n(nx)| < cn^{-1/2}$$

provided x is bounded away from 1. This is done in (15) and (16). If x is close to 1, we cannot do as well, but in (18) and (19) we show that

$$|J_n(nx)| < cn^{-1/3}$$

regardless of the value of x.

Suppose $x = 1 - \delta$, where $0 \leq \delta \leq 1$. From the dominant term of (10), and (13) and (14), we have

$$|J_n(nx)| \leq \frac{c\delta^{1/4}}{(2\delta-\delta^2)^{1/4}} \left(\frac{1}{n^{1/6}p^{1/4}} \cdot \frac{1}{n^{1/3}} + \frac{n^{1/6}p^{1/4}}{n^{5/3}} \right)$$

$$\leq \frac{c}{\delta^{1/4}n^{1/2}} \left(1 + \frac{p^{1/2}}{n}\right).$$

If we require that $\delta \leq 1/2$, then p is bounded and $p^{1/2}/n \leq c$. If $\delta \geq 1/2$, then it is known [11, page 243] that

$$|J_n(nx)| \leq cn^{-1/2},$$

so that in any event,

(15) $$|J_n(nx)| \leq \frac{c}{\delta^{1/4}n^{1/2}} \quad \text{when } x = 1 - \delta.$$

If $x = 1 + \delta$, $\delta > 0$, then $p \leq 0$, so that (10), and (13) and (14) give as before

$$|J_n(nx)| \leq \frac{c}{\delta^{1/4}n^{1/2}} \left(1 + \frac{(-p)^{1/2}}{n}\right).$$

Also as before, if $\delta \leq 1$, then $-p$ is bounded; if $\delta \geq 1$, then [11, page 244]

(16) $$|J_n(nx)| \leq \frac{c}{\delta^{1/4}n^{1/2}},$$

so (16) holds when $x = 1 + \delta$ for all $\delta > 0$.

We now suppose that $1 - \delta \leq x \leq 1$ We now show that

$$\log \frac{1+(1-x^2)^{1/2}}{x} - (1-x^2)^{1/2} \leq \frac{2}{3}\sqrt{2}\,\delta^{3/2} + O(\delta^2).$$

(Until the end of the proof of this lemma, O-symbols should be understood with the variable going to zero.) Since $(1+(1-x^2)^{1/2})/x$ is a decreasing function in the interval $(0, 1]$,

$$\log \frac{1+(1-x^2)^{1/2}}{x} \leq \log\,(1+(2\delta-\delta^2)^{1/2}) - \log\,(1-\delta) - (2\delta-\delta^2)^{1/2}$$

$$= (2\delta-\delta^2)^{1/2} - \frac{1}{2}(2\delta-\delta^2) + \frac{1}{3}(2\delta-\delta^2)^{3/2} + O(\delta^2)$$

$$+ \delta + O(\delta^2) - (2\delta-\delta^2)^{1/2}$$

$$= \frac{2}{3}\sqrt{2}\,\delta^{3/2} + O(\delta^2).$$

From (12) we see that

$$p \leq 2^{1/3}\delta + O(\delta^{4/3})$$

and hence

$$(17) \qquad \frac{4p}{1-x^2} \leq 4\left(\frac{2^{1/3}\delta + O(\delta^{4/3})}{2\delta-\delta^2}\right) = 2^{4/3} + O(\delta^{1/3}).$$

So from (10), (13), (14) and (17),

$$(18) \qquad |J_n(nx)| \leq c\left(\frac{1}{n^{1/3}} + \frac{n^{1/6}\, p^{1/4}}{n^{5/3}}\right)$$

$$\leq \frac{c}{n^{1/3}}\left(1 + \frac{\delta}{n^{7/6}}\right) \leq \frac{c}{n^{1/3}},$$

when $1 - \delta \leq x \leq 1$.

If $x = 1 + \delta$, then using the expansion [1, equation 4.4.41] valid for $|y| < 2$,

$$\operatorname{Arc\,cos}(1-y) = (2y)^{1/2}\left(1 + \sum_{k=1}^{\infty} \frac{1}{2^{3k}} \binom{2k}{k} \frac{y^k}{2k+1}\right),$$

we see that

$$(x^2-1)^{1/2} - \operatorname{Arc\,cos}\frac{1}{x} = (2\delta+\delta^2)^{1/2} - \operatorname{Arc\,cos}\frac{1}{1+\delta}$$

$$= (2\delta)^{1/2}(1+\delta/2)^{1/2} - \left(\frac{2\delta}{1+\delta}\right)^{1/2}\left(1 + \frac{1}{12}\,\frac{\delta}{1+\delta} + O(\delta^2)\right)$$

$$= (2\delta)^{1/2}(1+\delta/4+O(\delta^2)) - (2\delta)^{1/2}(1-\delta/2+O(\delta^2))\times$$

$$\times\left(1 + \frac{1}{12}\,\delta(1-\delta) + O(\delta^2)\right)$$

$$= (2\delta)^{1/2}(1+\delta/4+O(\delta^2)) - (2\delta)^{1/2}(1-5\delta/12+O(\delta^2))$$

$$= 2^{1/2}\left(\frac{2}{3}\,\delta^{3/2} + O(\delta^{5/2})\right).$$

From (11) we see that

$$-p = 2^{1/3}\delta + O(\delta^{5/3}),$$

and proceeding as in the case where $1 - \delta \leq x \leq 1$, we get

$$(19) \qquad |J_n(nx)| \leq \frac{c}{n^{1/3}} \quad \text{when} \quad 1 \leq x \leq 1 + \delta.$$

We now estimate $J^2_{tk'}(b_j)$ in the three cases

(i) $b_j/tk' \leq 1 - \delta$,

(ii) $1 - \delta \leq b_j/tk' \leq 1 + \delta$, and

(iii) $b_j/tk' \geq 1 + \delta$.

We may assume that δ is bounded away from 1. In case (i),

$$\frac{b_j}{(1-\delta)k'} \leq t \leq b_j^2 .$$

If the sums below have these limits, then from (15) we have

$$(20) \qquad \Sigma J^2_{tk'}(b_j) \leq c \Sigma \frac{1}{\delta^{1/2} tk'} \leq \frac{c \log b_j}{\delta^{1/2} k'} .$$

In case (ii),

$$\frac{b_j}{(1+\delta)k'} \leq t \leq \frac{b_j}{(1-\delta)k'} .$$

If the sums below have these limits, then from (18) and (19) we have

$$(21) \qquad \begin{aligned} \Sigma J^2_{tk'}(b_j) &\leq c \Sigma (k't)^{-2/3} \\ &\leq c \left(\frac{1+\delta}{b_j}\right)^{2/3} \frac{\delta b_j}{k'} \leq \frac{c \delta b_j^{1/3}}{k'} . \end{aligned}$$

In case (iii),

$$1 \leq t \leq \frac{b_j}{(1+\delta)k'} .$$

If the sums below have these limits, then from (16) we have

$$(22) \qquad \Sigma J^2_{tk'}(b_j) \leq c \Sigma \frac{1}{\delta^{1/2} tk'} \leq \frac{c \log b_j}{\delta^{1/2} k'} .$$

We choose δ so as to make the upper bounds in (20)-(22) of the same order of magnitude. Thus we want

$$\frac{\log b_j}{\delta^{1/2}} = \delta b_j^{1/3} ,$$

so we choose

$$\delta = b_j^{-2/9} (\log b_j)^{2/3} ,$$

from which

$$(23) \qquad \sum_{t=1}^{b_j^2} J_{tk'}^2(b_j) \leq \frac{c b_j^{1/9} (\log b_j)^{2/3}}{k'} .$$

From (9) and (23), it follows that (6) holds, and as was remarked, this is enough to prove the lemma.

Recall that $\Sigma_N = \Sigma_{N_0}^{\infty}$, where N_0 is sufficiently large to make the summand defined for all $N \geq N_0$.

Theorem 4:

If β is a positive integer and f is an increasing function for which $\lim_{x \to \infty} f(x) = \infty$,

$$\sum_N \frac{1}{N^3} \sum_{k<j\leq N} (f(j)f(k))^{-1/2} < \infty ,$$

and

$$\sum_N \frac{f^{1/9}(N) (\log f(N))^{2/3}}{N^{2-\epsilon}} < \infty \text{ for some } \epsilon > 0,$$

then $\{f(n) \cos n^{\beta} y\}$ is uniformly distributed (mod 1) for almost all y.

(In particular, the theorem holds for $f(x) = (\log x)^{\alpha}$ with $\alpha > 1$ and $f(x) = x^{\alpha}$ with $0 < \alpha < 9$.)

Proof:

The theorem is an application of Theorem A. Let

$$g(n, y) = f(n) \cos n^{\beta} y,$$

and let m be a positive integer. If we put

$$S(N, y) = \frac{1}{N} \sum_{n=1}^{N} e(mg(n, y))$$

and

$$I(N) = \int_0^{2\pi} |S(N, y)|^2 \, dy, \tag{24}$$

then, from Theorem A, to show that $\{g(n, y)\}$ is uniformly distributed (mod 1) for almost all y, it suffices to show that

$$\sum_{N} I(N)/N < \infty$$

for each m.

From (24) we have

$$I(N) = \frac{2\pi}{N} + \frac{1}{N^2} \sum_{k < j \leq N} 2 \operatorname{Re}(E_{jk}), \tag{25}$$

where

$$E_{jk} = \int_0^{2\pi} e(mg(j, y) - mg(k, y)) \, dy; \tag{26}$$

thus it suffices to show that

$$(27) \qquad \sum_{N} \frac{1}{N^3} \sum_{k < j \leq N} \operatorname{Re} E_{jk} < \infty$$

for each m.

If we expand (26), we have

$$\begin{aligned} \operatorname{Re} E_{jk} &= \int_0^{2\pi} \cos(2\pi m g(j, y)) \cos(2\pi m g(k, y))\, dy \\ &\quad + \int_0^{2\pi} \sin(2\pi m g(j, y)) \sin(2\pi m g(j, y))\, dy \qquad (28) \\ &= I_1(j, k) + I_2(j, k), \quad \text{say.} \end{aligned}$$

We consider $I_1(j, k)$ first. Let

$$b_n = 2\pi m f(n);$$

then from (28),

$$(29) \qquad I_1(j, k) = \int_0^{2\pi} \cos(b_j \cos j^{\beta} y) \cos(b_k \cos k^{\beta} y)\, dy.$$

With no loss of generality, we may assume that $b_n > 1$ for all n. Since $b_n \to \infty$ as $n \to \infty$, omitting the finite range of values for which $b_n \leq 1$ does not affect the distribution of $\{f(n) \cos n^{\beta} y\}$. We may thus apply Lemma 14 when needed, since $\{b_n\}$ satisfies its hypotheses. We expand the integrand in (29) in terms of Bessel functions. We know [11, page 22] that

$$(30) \qquad \cos(b_j \cos j^{\beta} y) = J_0(b_j) + 2 \sum_{r=1}^{\infty} (-1)^r J_{2r}(b_j) \cos 2rj^{\beta} y$$

From (8) we see that the series converges uniformly in y for $0 \leq y \leq 2\pi$. Thus we may substitute (30) in (29) and interchange the order of summation and intgration to get

$$I_1(j,k) = J_0(b_j) \int_0^{2\pi} \cos\,(b_k \cos\, k^\beta y)\, dy$$

$$+ 2 \sum_{r=1}^{\infty} (-1)^r J_{2r}(b_j) \int_0^{2\pi} \cos\, 2rj^\beta y \cos\,(b_k \cos\, k^\beta y) dy. \tag{31}$$

Applyi ng (30) with j replaced by k, the first term in the right of (31) is

$$2\pi J_0(b_j) J_0(b_k), \tag{32}$$

and the second, again interchanging the order of summation and integration is

$$4 \sum_{r=1}^{\infty} (-1)^r J_{2r}(b_j) \sum_{s=1}^{\infty} (-1)^s J_{2s}(b_k) \int_0^{2\pi} \cos\, 2rj^\beta y \cos\, 2sk^\beta y\, dy. \tag{33}$$

Now

$$\int_0^{2\pi} \cos\, 2rj^\beta y \cos\, 2sk^\beta y\, dy = \begin{cases} \pi & \text{if } rj^\beta = sk^\beta \\ 0 & \text{otherwise;} \end{cases}$$

if $j' = j^\beta/(j,k)^\beta$ and $k' = k^\beta/(j,k)^\beta$, then $rj^\beta = sk^\beta$ exactly when $r = tk'$ and $s = tj'$, $t = 1, 2, \ldots$, so (33) becomes

$$(34) \qquad 4\pi \sum_{t=1}^{\infty} (-1)^{t(k'+j')} J_{tk'}(b_j)\, J_{tj'}(b_k).$$

We note that the sum in (34) is bounded above in absolute value by $S(j,k)$, so from (31), (32), and (34) we have

$$(35) \qquad |I_1(j,k)| \leq 2\pi |J_0(b_j)|\,|J_0(b_k)| + 4\pi S(j,k).$$

$I_2(j,k)$ may be handled similarly. Writing

$$I_2(j,k) = \int_0^{2\pi} \sin(b_j \cos j^{\beta} y) \sin(b_k \cos k^{\beta} y)\, dy,$$

applying the relation [11, page 22]

$$\sin(b_j \cos j^{\beta} y) = 2 \sum_{r=0}^{\infty} (-1)^r J_{2r+1}(b_j) \cos(2r+1) j^{\beta} y,$$

and proceeding as for $I_1(j,k)$, we get

$$I_2(j,k) = 4\pi\, T(j,k),$$

where

$$T(j,k) = \begin{cases} \sum\limits_{\substack{t \geq 1 \\ t \text{ odd}}} (-1)^{\frac{1}{2}t(k'+j')} J_{tk'}(b_j) J_{tj'}(b_k) \\ \qquad \text{if } k' \text{ and } j' \text{ are odd} \\ 0 \qquad \text{otherwise.} \end{cases}$$

Thus

$$|I_2(j,k)| \leq S(j,k). \tag{36}$$

According to (27), (28), (35), and (36), to prove the theorem, it suffices to show that

$$\sum_N \frac{1}{N^3} \sum_{k<j\leq N} J_0(b_j)\, J_0(b_k) < \infty \tag{37}$$

and

$$\sum_N \frac{1}{N^3} \sum_{k<j\leq N} S(j,k) < \infty. \tag{38}$$

We know [11, page 199] that for all x,

$$|J_0(x)| \leq cx^{-1/2}.$$

Since $b_n = 2\pi mf(n)$, (37) holds if

$$\sum_N \frac{1}{N^3} \sum_{k<j\leq N} \frac{1}{(f(j)\,f(k))^{1/2}} < \infty.$$

But we have assumed this to be so. To establish (38), we may sum the inequality in (5) over j and k. Applying Lemma 13, we see that for every $\epsilon > 0$,

$$\sum_{k=1}^{j} \frac{b_j^{1/9}(\log b_j)^{2/3}}{k'} \leq b_j^{1/9}(\log b_j)^{2/3}\, c_\epsilon (\log j + 1) j^{\epsilon}$$

$$\leq c_\epsilon\, b_N^{1/9}(\log b_N)^{2/3}\, N^{\epsilon},$$

so

$$\sum_{k < j \leq N} S(j,k) \leq C_\epsilon\, f^{1/9}(N)\,(\log f(N))^{2/3}\, N^{1+\epsilon}$$

and

$$\sum_{N} \frac{1}{N^3} \sum_{k<j\leq N} S(j,k) \leq \sum_{N} \frac{c_\epsilon f^{1/9}(N)(\log f(N))^{2/3}}{N^{2-\epsilon}},$$

and we have assumed that the latter series converges. Thus the theorem is proved.

CHAPTER VI

The Distribution of $\{f(n) \cos ny\}$

In the special case $P(x) = \cos x$, Theorem 1 asserts that $\{f(n) \cos ny\}$ is uniformly distributed (mod 1) for almost all y when f increases without bound, f' decreases, and $f(x) = o(x^{1/2})$. Theorem 4, in the special case $\beta = 1$, asserts that $\{f(n) \cos ny\}$ is uniformly distributed (mod 1) when f is a smooth function with rate of growth between $\log x$ and x^9. We remove the restriction that $f(x)$ increase less rapidly than x^9 in the following theorem. As a result of Theorems 1, 4, and 5, we conclude that $\{f(x) \cos ny\}$ is uniformly distributed (mod 1) for almost all y when f is **a sufficiently smooth function which increases withqut bound, no matter at what rate.**

Theorem 5:

If f' is positive and bounded away from zero, then $\{f(n) \cos ny\}$ is uniformly distributed (mod 1) for almost all y.

Proof:

The proof is an application of Theorem A, combined with previous work of LeVeque. Note that in Theorem 1, we knew the y for which uniform distribution fails: those y such that $y/2\pi$ is rational or well approximable by rationals. In Theorem 5, we have no such information.

If we let m be a positive integer,

$$s_n(y) = f(n) \cos ny,$$

$$S(N, y) = \frac{1}{N} \sum_{n=1}^{N} e(m s_n(y)),$$

and

$$I(N) = \int_0^{2\pi} |S(N, y)|^2 \, dy, \tag{1}$$

then expanding (1) gives

$$I(N) = \frac{2\pi}{N} + \frac{1}{N^2} \sum_{k < j \leq N} 2 \operatorname{Re}(J_{jk}),$$

where $\operatorname{Re}(J_{jk})$ is the real part of the complex number J_{jk} and

$$J_{jk} = \int_0^{2\pi} e(ms_j(y) - ms_k(y)) \, dy. \tag{2}$$

From the conclusion of Theorem A, to prove the theorem, it suffices to show that

$$\sum_{N} \frac{1}{N^3} \sum_{k < j \leq N} \operatorname{Re}(J_{jk}) \tag{3}$$

converges for each m.

LeVeque has shown [7, page 371] that for an arbitrary double

sequence of positive numbers ω_{jk}, $j = 1, 2, \ldots$, $k = 1, 2, \ldots$,

$$\mathrm{Re}(J_{jk}) = O(\max(q_1, q_2, q_3)) ,$$

where

$$q_1 = \omega_{jk},$$

$$q_2 = \frac{\omega_{jk} \min\left((j-k) f(j),\ j(f(j) - f(k))\right)}{\left\{(j^2 f^2(j) - k^2 f^2(k))\ (j^2 - k^2)\right\}^{1/2}} , \text{ and} \tag{4}$$

$$q_3 = \frac{j}{\omega_{jk} \min\left((j-k) f(j),\ j(f(j) - f(k))\right)} .$$

We will choose ω_{jk} so that $q_1 = q_3 = \omega_{jk}$ and $q_2 = O(\omega_{jk})$, so that $\mathrm{Re}(J_{jk}) = O(\omega_{jk})$. We then show that

$$\sum_{k < j \leq N} \omega_{jk} = O(N^{3/2}) ,$$

which, according to (3), suffices to prove the theorem.

We will say that a pair (j, k) of integers with $k < j$ is in class C_1 if

$$(j - k) f(j) \leq j(f(j) - f(k)) , \tag{5}$$

and in class C_2 otherwise. In C_1, the minimum occurring in (4) is $f(j)\,(j-k)$. Since

(6) $$j^2 f^2(j) - k^2 f^2(k) \geq f^2(j)\,(j^2 - k^2),$$

we have

$$q_2 \leq \omega_{jk} \frac{f(j)\,(j-k)}{f(j)\,(j^2-k^2)} = \frac{\omega_{jk}}{j+k} = O(\omega_{jk}) .$$

If we choose

$$\omega_{jk} = \frac{j^{1/2}}{(j-k)^{1/2} f^{1/2}(j)} ,$$

then

$$q_3 = \frac{j}{\omega_{jk} f(j)\,(j-k)} = \omega_{jk} ,$$

so that q_1, q_2, and q_3 are all of order ω_{jk} and

$$\mathrm{Re}(J_{jk}) = O\left(\frac{j^{1/2}}{(j-k)^{1/2} f^{1/2}(j)}\right) .$$

Thus

$$\sum_{\substack{k<j\leq N \\ (j,k)\in C_1}} \mathrm{Re}(J_{jk}) \leq c \sum_{\substack{k<j\leq N \\ (j,k)\in C_1}} \frac{j^{1/2}}{f^{1/2}(j)} \frac{1}{(j-k)^{1/2}} .$$

From (5) we have $j/f(j) \leq k/f(k)$, so $(j/f(j))^{1/2}$ is a decreasing function of j and thus

$$(7)\qquad \sum_{\substack{k<j\le N\\ (j,k)\in C_1}} \mathrm{Re}(J_{jk}) \le c \sum_{\substack{k<j\le N\\ (j,k)\in C_1}} \frac{1}{(j-k)^{1/2}} \le c \sum_{j=2}^{N} \sum_{k=1}^{j-1} \frac{1}{(j-k)^{1/2}}$$

$$\le c \sum_{j=2}^{N} j^{1/2} \le cN^{3/2} .$$

On the other hand, if (j, k) is in class C_2, then the minimum in (4) is $j(f(j) - f(k))$ and it follows from (6) that

$$q_2 \le \frac{\omega_{jk}\, j\,(f(j) - f(k))}{f(j)\,(j^2 - k^2)} \le \frac{\omega_{jk}\, j\, f(j)\,(j - k)}{f(j)\,(j^2 - k^2)} = \omega_{jk}\, \frac{j}{j + k}$$

$$= O(\omega_{jk}).$$

If we choose

$$\omega_{jk} = (f(j) - f(k))^{-1/2} ,$$

we have

$$q_3 = \frac{j}{\omega_{jk}\, j(f(j) - f(k))} = \omega_{jk} ,$$

so for (j, k) in class C_2, each of q_1, q_2, and q_3 has order ω_{jk} and

$$\mathrm{Re}(J_{jk}) = O((f(j) - f(k))^{-1/2} .$$

Since $f'(x)$ is positive and bounded away from zero,

$$f(j) - f(k) > c(j - k),$$

so

$$(8) \qquad \sum_{\substack{k < j \le N \\ (j,k) \in C_2}} \mathrm{Re}(J_{jk}) \le c \sum_{j=1}^{N} \sum_{k=1}^{j-1} \frac{1}{(j-k)^{1/2}} \le cN^{3/2} .$$

From (7) and (8) we have

$$\sum_{k < j \le N} \mathrm{Re}(J_{jk}) \le cN^{3/2} ,$$

so the sum in (3) converges for each m and the theorem is proved.

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