THE STABILITY OF A STRATIFIED FLOW

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I. INTRODUCTION

For most flows whose stability has been investigated so far, instability occurs at rather large Reynolds numbers. The Orr-Sommerfeld equation[7,9] governing the stability of such flows must therefore be solved for large values of the Reynolds number, which occurs as a parameter of these equations. The asymptotic solutions of the Orr-Sommerfeld equation, appropriate for large Reynolds numbers, have singularities at the critical points where the wave velocity of the disturbance is equal to the velocity of the mean flow. Great care must therefore be exercised in the evaluation of these solutions as a critical point is crossed. These singularities of the solutions are, however, not inherent in the Orr-Sommerfeld equation, and are introduced entirely by the method of solution. For flows which can be expected to be unstable at low Reynolds numbers, the appropriate solutions can be expressed in ascending powers of the Reynolds number or of one of the coordinates. Since asymptotic solutions are not needed, the afore-mentioned singularities do not occur. The study of the stability of flows which can be expected to become unstable at low Reynolds numbers can therefore be carried out by conventional methods. In view of this, it is perhaps somewhat surprising that until recent years problems of hydrodynamic instability at low Reynolds numbers have been neglected by research workers. The study of these problems will lead to an understanding of many fascinating hydrodynamic phenomena.

The flow whose stability is studied here is a stratified flow of two fluids of equal viscosity but different densities. It is entirely

motivated by gravity and its velocity distribution is antisymmetric, with the lighter fluid flowing in a direction opposite to that of the heavier fluid, with a point of inflection of the velocity profile at the interface. Since a point of inflection is known to have a destabilizing effect, and since the work of Yih[11] and of Benjamin[1] has shown that a free surface also has a destabilizing effect, it can be expected to be unstable at low Reynolds numbers, especially if the slope of the mean flow is steep. The Orr-Sommerfeld equation will therefore be solved for small Reynolds numbers.

The present problem also has some bearing on many phenomena which occur in nature and in engineering practice. The intrusion of salt water into river estuaries and the intrusion of a cold front into warmer air are examples of stratified flows whose stability is of interest to the hydraulic engineer and the meteorologist. If the concept of eddy viscosity is introduced, the problem also has some bearing on the generation of waves by a turbulent wind. In chemical engineering, the type of flow investigated is encountered in extraction columns. The study of the present problem will provide some understanding of these phenomena encountered in nature or in industry.

II. THE PRIMARY FLOW

The present investigation concerns the stability of a steady laminar stratified flow of an incompressible viscous fluid between two parallel fixed planes. The spacing of the planes is denoted by 2b. The origin of the coordinate system is taken half-way between the planes, with the X axis parallel to the planes. The planes are inclined at an angle Θ with the horizontal. (See Figure 1.)

The fluid occupying the region $0 \leq Y \leq b$, with density ρ_1, flows up the inclined plane in the direction of negative X. The fluid occupying the region $-b \leq Y \leq 0$, with density ρ_2 greater than ρ_1, flows down the plane under the action of gravity. The viscosity μ of the two fluids are considered equal. Gravity is the sole motivating force for the flow, with the heavier fluid displacing the lighter fluid in a reservoir at $X = +\infty$. The volumetric discharge across the channel is zero.

The Navier-Stokes equations which govern the primary flow are

$$\begin{aligned} 0 &= - P_{1,X} + \rho_1 g \sin\Theta + \mu \bar{U}_{1,YY} , \\ 0 &= - P_{1,Y} - \rho_1 g \cos\Theta, \qquad \text{for} \qquad 0 \leq Y \leq b , \end{aligned} \tag{1}$$

and

$$\begin{aligned} 0 &= - P_{2,X} + \rho_2 g \sin\Theta + \mu \bar{U}_{2,YY} , \\ 0 &= - P_{2,Y} - \rho_2 g \cos\Theta, \qquad \text{for} \qquad -b \leq Y \leq 0 , \end{aligned} \tag{2}$$

where $\bar{U}_1$ and $\bar{U}_2$, the components of velocity of the two fluids in the X direction, are functions of Y only, and letter subscripts following commas denote partial differentiation with respect to that quantity. The components of velocity in the Y and Z directions are zero. Continuity is thus automatically satisfied.

These equations are easily solved for $\overline{U}_1$ and $\overline{U}_2$. It can be verified that

$$\overline{U}_1 = U_M \left[- \frac{Y}{b} + \left(\frac{Y}{b}\right)^2 \right], \quad 0 \leq Y \leq b \ , \tag{3}$$

and

$$\overline{U}_2 = - U_M \left[+ \frac{Y}{b} + \left(\frac{Y}{b}\right)^2 \right], \quad -b \leq Y \leq 0 \ , \tag{4}$$

where

$$U_M = \frac{(\rho_2 - \rho_1)}{4\mu} \ b^2 \ g \sin \theta \ . \tag{5}$$

is four times the maximum velocity. The boundary conditions corresponding to (3) and (4) are no slip at the fixed boundaries. The condition of zero volumetric discharge is also satisfied by (3) and (4).

III. FORMULATION OF THE STABILITY PROBLEM

A. Differential Equations

The basic equations to be satisfied are the Navier-Stokes equations and the continuity equations. As is customary in fluid stability problems, it is assumed that the velocity and pressure fields upon suffering a small disturbance are expressible in power series expansions in terms of an amplitude parameter ϵ, assumed small and constant.(6) Upon substituting these expansions into the field equations and setting coefficients of powers of ϵ to zero, it is noted that the ϵ^0 equations are those already described as governing the primary flow. The ϵ^1 equations are

$$u^*_{i,t} + \bar{U}_i u^*_{i,X} + \bar{U}_{i,Y} v^*_i = -\frac{1}{\rho_i} P^*_{i,X} + \frac{\mu}{\rho_i} \nabla^2 u^*_i , \tag{6}$$

$$v^*_{i,t} + \bar{U}_i v^*_{i,X} = -\frac{1}{\rho_i} P^*_{i,Y} + \frac{\mu}{\rho_i} \nabla^2 v^*_i , \tag{7}$$

$$u^*_{i,X} + v^*_{i,Y} = 0 , \tag{8}$$

where the disturbance is assumed to be a two-dimensional one. The convention has been adopted that capital letters refer to the primary flow and small letters with an asterisk in the upper right hand corner refer to the first-order disturbance flow. The equations hold in the upper region when $i = 1$ and in the lower region when $i = 2$.

A complete treatment of the stability problem requires consideration of higher powers of ϵ than the first, and also establishment of convergence and justification of the term-by-term differentiation of the

expansion in ϵ. However, as is customary, attention will be confined solely to the first order equations, with the assumption that for sufficiently small disturbances the linearized equations can be used without appreciable error.

To facilitate the solution of (6), (7), and (8), it is convenient to put them in dimensionless form. The following dimensionless parameters are introduced:

$$u_i = \frac{u_i^*}{U_M}\ ,\ v_i = \frac{v_i^*}{U_M}\ ,\ p_i = \frac{p_i^*}{\rho_i U_M^2}\ ,\ x = \frac{X}{b}\ ,\ y = \frac{Y}{b}\ ,$$
$$\tau = t\,\frac{U_M}{b}\ ,\ U_i = \frac{\overline{U}_i}{U_M}\ ,\ R_i = \frac{b\,U_M\,\rho_i}{\mu}\ ,\ r = \frac{\rho_2}{\rho_1}\ . \tag{9}$$

Formulating (6), (7), and (8) in terms of these parameters, the following dimensionless equations are obtained:

$$u_{i,\tau} + U_i u_{i,x} + U_{i,y} v_i = - p_{i,x} - \frac{1}{R_i}\ \nabla^2 u_i\ , \tag{10}$$

$$v_{i,\tau} + U_i v_{i,x} = - p_{i,y} + \frac{1}{R_i}\ \nabla^2 v_i\ , \tag{11}$$

$$u_{i,x} + v_{i,y} = 0\ , \tag{12}$$

in which R_i is the Reynolds number based on the half-spacing b and U_M, and represents the ratio of the inertia forces to the viscous forces. Since the flow is motivated by gravity, it is expected that the Reynolds number and the Froude number F, which represents the ratio of inertia forces to gravity forces, are not independent of one another. Such is indeed the case, for if F is defined by

$F^2 = \frac{U_M^2}{gb}$, then from the primary flow (5) provides the relationship that

$$F^2 = \frac{1}{4}\,(r-1)R_1 \sin\theta\ . \tag{13}$$

To reduce the number of field equations which are to be solved, it is convenient to introduce stream functions. To simplify solution of (10), (11), and (12), the solutions will be assumed to have exponential time factors. Further, to reduce the partial differential equations to ordinary differential equations, it is assumed that the disturbance may be resolved into Fourier components, which are thus periodic in x. A general disturbance then would consist of either a Fourier series or a Fourier integral of such components. To determine the stability of the flow it is sufficient to consider the effect of a single Fourier component of general period.

Only two-dimensional disturbances are considered, since the work of Squire[10], Yih[12], and others has shown that the stability or instability of a three-dimensional disturbance can be determined from that of a two-dimensional disturbance at a higher Reynolds number.

The velocity components and the pressure can thus be assumed to have the following forms:

$$\begin{aligned} u_1 &= f'(y)e^{i\alpha(x-c\tau)} , \\ v_1 &= -i\alpha f(y)e^{i\alpha(x-c\tau)} , \\ p_1 &= q_1(y)e^{i\alpha(x-c\tau)} , \end{aligned} \tag{14}$$

in $0 \leq y \leq 1$, and

$$\begin{aligned} u_2 &= h'(y)e^{i\alpha(x-c\tau)} , \\ v_2 &= -i\alpha h(y)e^{i\alpha(x-c\tau)} , \\ p_2 &= q_2(y)e^{i\alpha(x-c\tau)} , \end{aligned} \tag{15}$$

in $-1 \leq y \leq 0$, where primes denote differentiation with respect to y. The dimensionless wave number α is related to the wave length λ of the disturbance by $\alpha = \frac{2\pi b}{\lambda}$. The real part of c represents the dimensionless

disturbance celerity, and α times the imaginary part of c gives the dimensionless growth rate of the disturbance.

Upon solving (10), (11), (14) and (15) for the pressure terms, the following are obtained:

$$q_1 = \frac{1}{i\alpha R_1}(f''' - \alpha^2 f') + (c-U_1)f' + U_1' f ,$$
$$q_1' = \frac{1}{i\alpha R_1}(\alpha^2 f'' - \alpha^4 f) + \alpha^2(c-U_1)f , \tag{16}$$

in $0 \leq y \leq 1$, and

$$q_2 = \frac{1}{i\alpha R_2}(h''' - \alpha^2 h') + (c-U_2)h' + U_2' h ,$$
$$q_2' = \frac{1}{i\alpha R_2}(\alpha^2 h'' - \alpha^4 h) + \alpha^2(c-U_2)h , \tag{17}$$

in $-1 \leq y \leq 0$. The pressure terms can now be eliminated to obtain

$$f'''' + [-2\alpha^2 + i\alpha R_1(c-U_1)]\, f'' + [\alpha^4 + i\alpha R_1 U_1'' + i\alpha^3 R_1(U_1-c)]\, f = 0 \tag{18}$$

in $0 \leq y \leq 1$, and

$$h'''' + [-2\alpha^2 + i\alpha R_2(c-U_2)]\, h'' + [\alpha^4 + i\alpha R_2 U_2'' + i\alpha^3 R_2(U_2-c)]\, h = 0 \tag{19}$$

in $-1 \leq y \leq 0$. These equations are the Orr-Sommerfeld equations. From (3), (4), and (9) the functions U_1 and U_2 are:

$$U_1 = -y + y^2 ,$$
$$U_2 = -y - y^2 . \tag{20}$$

B. Boundary Conditions

Since the differential Equations (18) and (19) are two in number and each is of the fourth order, there must be eight boundary conditions imposed to specify the mathematical problem completely. Two conditions

are imposed at each of the fixed boundaries, and a total of four are imposed at the interface.

Since the fluids are considered viscous, there must be no slip at the fixed boundaries. Hence, in terms of the stream functions,

$$f(1) = 0,\ f'(1) = 0,\ h(-1) = 0,\ h'(-1) = 0\ . \tag{21}$$

The interface is assumed to be displaced from the x axis by a small amount which in dimensional form is denoted by $b\eta$. This introduces a further unknown into the problem, which, however, may be readily determined from the kinematical condition that the component of velocity at the interface must equal the time derivative of η. In what follows, quantities evaluated at the interface are expanded in a Taylors series about $y = 0$, and η is considered to be of the same order as the velocity disturbances. In the spirit of the previous linearization, disturbance terms are neglected if they are of an order greater than one.

At the interface, the velocity components must be continuous, hence

$$f(0) = h(0),\ f'(0) = h'(0)\ . \tag{22}$$

The kinematic condition at the interface is, in dimensionless terms,

$$v = \eta_{,\tau} \tag{23}$$

at $y = 0$. Then

$$\eta_{,\tau} = -i\alpha f(0)e^{i\alpha(x-c\tau)}$$

and upon solving for η ,

$$\eta = \frac{1}{c}\, f(0)e^{i\alpha(x-c\tau)}\ . \tag{24}$$

The shear stress must also be continuous across the interface. To the first order, the dimensionless shear stress at the interface is given by

$$\tau_{xy} = \frac{1}{R}\,(U' + u' + v_{,x} + U''\,\eta)\ . \tag{25}$$

Therefore the boundary condition imposed is, after some simplification,

$$4f(0) + cf''(0) - ch''(0) = 0 . \tag{26}$$

The dimensionless normal stress at the interface is

$$\tau_{yy} = - \frac{1}{\rho U_M^2} (P + P_{,Y} b\eta + 2\mu U_M v') - p, \tag{27}$$

where P is the pressure in the primary flow. The condition imposed is that the difference in normal stresses must be equal to the curvature of the interface times Γ, the surface tension. Since P must already be continuous, and since $P_{,Y} = -g \rho \cos \theta$, the boundary condition becomes

$$q_1(0) - rq_2(0) + \frac{1}{c} \left[\frac{(r-1)bg \cos \theta}{U_M^2} + \frac{\alpha^2}{W} \right] f(0) = 0, \tag{28}$$

where W is the Weber number, representing the ratio of inertia forces to surface tension, and is defined by

$$W = \frac{\rho_1 b U_M^2}{\Gamma} .$$

This can be written entirely in terms of the stream functions by using (16) and (17). The resulting equation is, after further simplification,

$$c[f'''(0) - h'''(0)] + (r-1)\, i\alpha R_1 c\, [f(0) - cf'(0)] + i\alpha[4 \cot \theta + \alpha^2 S] f(0) = 0. \tag{29}$$

The parameter S is defined by

$$S = \frac{R_1}{W} = \frac{\Gamma}{\mu U_M} , \tag{30}$$

and represents the ratio of surface tension to viscous forces.

IV. SOLUTION OF THE STABILITY PROBLEM

A. General

For the purpose of solving (18) and (19), much labor can be saved by employing the following notation:

$$\begin{aligned} \phi(y) &= \begin{cases} f(y) & \text{in } 0 \leq y \leq 1 , \\ h(y) & \text{in } -1 \leq y \leq 0 , \end{cases} \\ B &= \alpha^2 , \\ A &= \begin{cases} A_1 = i\alpha R_1 & \text{in } 0 \leq y \leq 1 , \\ A_2 = i\alpha R_2 & \text{in } -1 \leq y \leq 0 , \end{cases} \\ \delta &= \begin{cases} +1 & \text{in } 0 \leq y \leq 1 , \\ -1 & \text{in } -1 \leq y \leq 0 . \end{cases} \end{aligned} \tag{31}$$

Then (18) and (19) can both be solved simultaneously by solving

$$\phi'''' + [-2B + A(c+y - \delta y^2)]\, \phi'' + [B^2 + 2A\delta + AB\,(-c-y + \delta y^2)]\, \phi = 0 . \tag{32}$$

As discussed in the Introduction, instability can be expected to occur at low Reynolds numbers when the inclination θ approaches 90°. Two techniques at once are suggested as alternate methods of solution. The first, expansion of ϕ in ascending powers of the Reynolds number, is certainly the more typical of stability problems and is formulated in the following section. The alternate approach, expansion of ϕ in powers of the coordinate y, has, to the best of the author's knowledge, been employed successfully only by Benjamin[1] in spite of its early recognition by Kelvin[3] and others. Because of the nature of the problem at hand, this second method, employed in Section C, seems more useful in the region of interest, and conclusions are drawn on the basis of these calculations. Of course, the validity of approximations based on either method can be determined only a posteriori.

It may be noted that the mathematical problem posed does not lend itself well to any other of the usual procedures for the calculation of eigenvalues. The equations are not self-adjoint, so variational methods, such as those of Rayleigh-Ritz and Galerkin, can not be utilized. Also the problem is further complicated by the presence of the eigenvalue c in two of the boundary conditions. Therefore series expansions in terms of one of the parameters seems to be the most promising method of solution of the problem.

B. Expansion in Powers of the Reynolds Number

Following Yih[11] assume a solution of (32) of the form

$$\phi(y) = \sum_{n=0}^{\infty} A^n \, \phi^{(n)}(y) \, . \tag{33}$$

Since the coefficient of the highest derivative of ϕ is one, and the coefficients of the other terms are entire functions of A, B, and y, ϕ must be an entire function in the A-B plane.

Substituting (33) into (32), and setting coefficients of powers of A to zero gives

$$\phi^{(o)''''} - 2B\,\phi^{(o)''} + B^2\,\phi^{(o)} = 0 \, , \tag{34}$$

$$\phi^{(n)''''} - 2B\,\phi^{(n)''} + B^2\,\phi^{(n)} = (y^2\delta - y - c) \cdot (\phi^{(n-1)''} - B\,\phi^{(n-1)}) - 2\delta\phi^{(n-1)} \tag{35}$$

for $n \geq 1$. The solutions of (34) are, since $B = \alpha^2$,

$$\phi_1^{(o)} = e^{\alpha y}, \; \phi_2^{(o)} = e^{-\alpha y}, \; \phi_3^{(o)} = ye^{\alpha y}, \; \phi_4^{(o)} = ye^{-\alpha y}. \tag{36}$$

To solve (35), it is necessary to apply the operator $I = (D^2-B)^{-1}(D^2-B)^{-1}$ on functions of the form $y^m e^{\pm\alpha y}$. It can be verified that

$$I(e^{\pm\alpha y}y^m) = \frac{m!}{(m+4)!}\, y^{m+4}e^{\pm\alpha y}\ {}_1F_1(2,m+5, \mp 2\alpha y), \tag{37}$$

in which ${}_1F_1$ is the confluent hypergeometric function (or the Kummer function) defined by

$${}_1F_1(2,m+5, \mp 2\alpha y) = \sum_{n=0}^{\infty} \frac{(m+4)!(n+1)}{(m+n+4)!}\ (\mp 2\alpha y)^n. \tag{38}$$

On applying this to the determination of $\phi^{(1)}(y)$, the following is obtained:

$$\begin{aligned}
\phi_1^{(1)}(\alpha,y) &= \tfrac{1}{4!}\, e^{\alpha y}\, y^4\ {}_1F_1\ (2,5, -2\alpha y),\\
\phi_2^{(1)}(\alpha,y) &= \phi_1^{(1)}(-\alpha,y)\ ,\\
\phi_3^{(1)}(\alpha,y) &= [\tfrac{4}{6!}\,\alpha\delta y^6\ {}_1F_1(2,7, -2\alpha y) -\\
&\qquad \tfrac{\alpha}{5!}\, y^5\ {}_1F_1\ (2,6, -2\alpha y) -\\
&\qquad \tfrac{1}{4!}\,(c\alpha+\delta)\, y^4\ {}_1F_1\ (2,5, -2\alpha y)]\ e^{\alpha y}\ ,\\
\phi_4^{(1)}(\alpha,y) &= \phi_3^{(1)}(-\alpha,y)\ .
\end{aligned} \tag{39}$$

Further values of $\phi^{(n)}$ are obtainable in a similar manner.

The results can be more simply expressed in terms of a finite number of exponentials.[11] In any case, when the boundary conditions are finally applied and a determinant is obtained involving the solutions, an exact solution for the eigenvalue c is prohibited by the immense amount of labor involved in the calculations. It then is necessary to

terminate the solutions after a finite number of terms. The convergence of such a solution is not as easily recognizable as a solution obtained by a direct expansion in the coordinate y. A solution by expansion in powers of y is now presented.

C. Expansion in Powers of the Coordinate y

Following Benjamin[1], a solution of the form

$$\phi(y) = \sum_{n=0}^{\infty} a_n y^n \tag{40}$$

can be assumed. It is most convenient to expand around the origin since four of the boundary conditions must be satisfied there. Upon substituting (40) into (32) and setting coefficients of y^n to zero, the following recursion relation is obtained:

$$a_n = \frac{(n-4)!}{n!}\{(n-2)(n-3)(2B-Ac)a_{n-2} - (n-4)(n-3)Aa_{n-3} + [A\delta(n-4)(n-5)-B^2-2A\delta+ABc]\, a_{n-4} + ABa_{n-5} - AB\delta a_{n-6}\} . \tag{41}$$

The coefficients a_0, a_1, a_2, and a_3 may be chosen arbitrarily. They may thus be chosen to yield the following four solutions:

$$\phi_1(y) = 1 + \frac{(ABc-B^2-2A\delta)}{4!} y^4 + \frac{AB}{5!} y^5 + \frac{(2A^2c\delta-6AB\delta+3AB^2c-A^2Bc-2B^3)}{6!} y^6 + \frac{(6A^2\delta+5AB^2-4A^2Bc^2)}{7!} y^7 + \frac{(-20A^2-3B^4+6AB^3c-4A^2B^2c^2-20AB^2\delta+18A^2Bc\delta-2A^3c^2\delta+A^3Bc^3-4A^2B)}{8!} y^8 + \frac{(14AB^3+50A^2B\delta+9A^3Bc^2-22A^2B^2c-16A^3c\delta)}{9!} y^9 + \ldots , \tag{42}$$

$$\phi_2(y) = y + \frac{(ABc-B^2-2A\delta)}{5!}\,y^5 + \frac{2AB}{6!}\,y^6 + \frac{(2A^2c\delta-10AB\delta+3AB^2c-A^2Bc^2-2B^3)}{7!}\,y^7$$
$$+ \frac{(8A^2\delta+8AB^2-6A^2Bc)}{8!}\,y^8$$
$$+ \frac{(-36A^2-3B^4+6AB^3c-36AB^2\delta+30A^2Bc\delta-4A^2B^2c^2-2A^3c^2\delta-10A^2B+A^3Bc^3)}{9!}\,y^9$$
$$+ \frac{(20AB^3+122A^2B\delta+12A^3Bc^2-30A^2B^2c-20A^3c\delta)}{10!}\,y^{10} + \ldots, \qquad (43)$$

$$\phi_3(y) = y^2 + \frac{(4B-2Ac)}{4!}\,y^4 - \frac{2A}{5!}\,y^5 + \frac{(6B^2-6ABc+2A^2c^2)}{6!}\,y^6 + \frac{(8A^2c-10AB)}{7!}\,y^7$$
$$+ \frac{(8A^2+16AB\delta-20A^2c\delta+8B^3-12AB^2c+8A^2Bc^2-2A^3c^3)}{8!}\,y^8$$
$$+ \frac{(-36A^2\delta-28AB^2+44A^2Bc-18A^3c^2)}{9!}\,y^9$$
$$+ \frac{(10B^4-20AB^3c+80AB^2\delta+20A^2B^2c^2-164A^2Bc\delta+64A^2B+76A^3c^2\delta-10A^3Bc^3-}{10!}$$
$$\frac{56A^3c+2A^4c^4)}{10!}\,y^{10} + \ldots, \qquad (44)$$

$$\phi_4(y) = \frac{y^3}{3!} + \frac{(2B-Ac)}{5!}\,y^5 - \frac{2A}{6!}\,y^6 + \frac{(3B^2-3ABc+A^2c^2+4A\delta)}{7!}\,y^7$$
$$+ \frac{(6A^2c-8AB)}{8!}\,y^8$$
$$+ \frac{(24AB\delta-22A^2c\delta+10A^2+4B^3-6AB^2c+4A^2Bc^2-A^3c^3)}{9!}\,y^9$$
$$+ \frac{(-80A^2\delta-20AB^2+30A^2Bc-12A^3c^2)}{10!}\,y^{10}$$
$$+ \frac{(160A^2+5B^4-10AB^3c+60AB^2\delta+10A^2B^2c^2+62A^2B-142A^2Bc\delta+A^4c^4 -}{11!}$$
$$\frac{52A^3c-5A^3Bc^3+62A^3c^2\delta)}{11!}\,y^{11} \qquad (45)$$

$$+ \frac{(580A^3c\delta - 72A^3Bc^2 + 90A^2B^2c - 80A^3 - 40AB^3 - 64A^2B\delta + 20A^4c^3)}{12!} y^{12}$$

$$+ \ldots \quad . \qquad (45 \text{ cont'd})$$

The solutions must converge for all finite A, B, c, and y because of the absence of singularities in the differential equation. An inspection of (42), (43), (44) and (45) indicates that for the range of the parameters considered, the solutions converge rather rapidly. For convenience, the values of ϕ_1 and their derivatives are summarized in the Appendix. A further function ϕ_5, defined by

$$\phi_5(y) = c\phi_1(y) + \phi_2(y) - \delta\phi_3(y), \qquad (46)$$

ia also introduced. It is found convenient in later work to use the set of four solutions ϕ_2, ϕ_3, ϕ_4, and ϕ_5 .

The solutions of (18) and (19) are given by linear combinations of $f_1(y)$ and $h_1(y)$, these being the values of $\phi_1(y)$ when A and δ are assigned the values A_1, 1 and A_2, -1, respectively.

Satisfaction of boundary conditions (22) and (26) is insured by writing

$$f(y) = a_2f_2(y) + a_3f_3(y) + a_4f_4(y) + a_5f_5(y) , \qquad (47)$$

$$h(y) = a_2h_2(y) + a_3h_3(y) + a_5h_5(y) + a_6h_6(y) . \qquad (48)$$

To satisfy the remainder of the boundary conditions, it is necessary that

$$\begin{aligned}
0 &= a_2c(A_1 - A_2) + a_4i\alpha(4 \cot \theta + \alpha^2 S) - a_6 , \\
0 &= a_2f_2(1) + a_3f_3(1) + a_4f_4(1) + a_5f_5(1), \\
0 &= a_2f_2'(1) + a_3f_3'(1) + a_4f_4'(1) + a_5f_5'(1), \\
0 &= a_2h_2(-1) + a_3h_3(-1) + a_5h_5(-1) + a_6h_6(-1), \\
0 &= a_2h_2'(-1) + a_3h_3'(-1) + a_5h_5'(-1) + a_6h_6'(-1) .
\end{aligned} \qquad (49)$$

In general, this set of algebraic equations admits only zero values of the a_1. In order that a nontrivial solution exist, the determinant of the coefficients must vanish. Therefore for the existence of a nontrivial solution,

$$0 = \begin{vmatrix} c(A_1-A_2) & 0 & 1 & i\alpha(4\cot\theta + \alpha^2 S) & -1 \\ f_2(1) & f_3(1) & f_4(1) & f_5(1) & 0 \\ f_2'(1) & f_3'(1) & f_4'(1) & f_5'(1) & 0 \\ h_2(-1) & h_3(-1) & 0 & h_5(-1) & h_4(-1) \\ h_2'(-1) & h_3'(-1) & 0 & h_5'(-1) & h_4'(-1) \end{vmatrix} . \qquad (50)$$

This determinant may be expanded into a number of two-by-two determinants by a method such as Laplace's expansion.[2] This yields the result

$$\begin{aligned} 0 = {} & H_1F_4 + F_1H_4 - H_2F_6 - F_2H_6 + H_3F_5 + F_3H_5 + \\ & i\alpha(4\cot\theta + \alpha^2 S)(H_2F_3 - F_2H_3) + \\ & c(A_2-A_1)(F_1H_3 - H_1F_3) , \end{aligned} \qquad (51)$$

where the following notation has been introduced:

$$\begin{aligned} F_1 &= \begin{vmatrix} f_4(1) & f_5(1) \\ f_4'(1) & f_5'(1) \end{vmatrix} , & H_1 &= \begin{vmatrix} h_4(-1) & h_5(-1) \\ h_4'(-1) & h_5'(-1) \end{vmatrix} , \\ F_2 &= \begin{vmatrix} f_2(1) & f_4(1) \\ f_2'(1) & f_4'(1) \end{vmatrix} , & H_2 &= \begin{vmatrix} h_2(-1) & h_4(-1) \\ h_2'(-1) & h_4'(-1) \end{vmatrix} , \\ F_3 &= \begin{vmatrix} f_3(1) & f_4(1) \\ f_3'(1) & f_4'(1) \end{vmatrix} , & H_3 &= \begin{vmatrix} h_3(-1) & h_4(-1) \\ h_3'(-1) & h_4'(-1) \end{vmatrix} , \\ F_4 &= \begin{vmatrix} f_2(1) & f_3(1) \\ f_2'(1) & f_3'(1) \end{vmatrix} , & H_4 &= \begin{vmatrix} h_2(-1) & h_3(-1) \\ h_2'(-1) & h_3'(-1) \end{vmatrix} , \\ F_5 &= \begin{vmatrix} f_2(1) & f_5(1) \\ f_2'(1) & f_5'(1) \end{vmatrix} , & H_5 &= \begin{vmatrix} h_2(-1) & h_5(-1) \\ h_2'(-1) & h_5'(-1) \end{vmatrix} , \\ F_6 &= \begin{vmatrix} f_3(1) & f_5(1) \\ f_3'(1) & f_5'(1) \end{vmatrix} , & H_6 &= \begin{vmatrix} h_3(-1) & h_5(-1) \\ h_3'(-1) & h_5'(-1) \end{vmatrix} . \end{aligned} \qquad (52)$$

To complete the determination of the eigenvalue c it becomes necessary now to approximate the ϕ's, as obviously the infinite series form for the ϕ's must be terminated. It was decided to consider α and R to be of the same order, and to limit the calculation to fifth powers of α and R (and combinations of them). The odd power is introduced by the boundary condition on the normal stresses. No special assumption is made concerning c, although it is considered to be of the order of one. Computations based on such an approximation prove straight-forward but lengthy. To carry out the next-higher approximation (to the sixth power) would triple or quadruple the amount of work necessary, which is already considerable.

Treating α and R to be of the same order gives results which are valid in a small circle around the origin in the α - R plane. Calculations based on expansions to the first order in the Reynolds number as made in Section B would give results valid in a narrow strip along the α axis in the α - R plane. Since the results of most stability analyses (including this one as shall be seen) indicate the onset of instability with low wave number, it would seem that an approximation valid for a wider range of R is more desirable. This was the main factor in choosing an analysis based on expansion in terms of powers of the coordinate y.

It is desirable to state the range of α and R for which the approximation is valid. Certainly if α and R are both much less than one, the results are very good. For α and R of the order of one or slightly larger, it is felt that the conclusions are still qualitively true, since the denominators of the terms in the series increase in a factorial manner

while the numerators increase only gradually. However, the prohibitive amount of calculations necessary make any definite statements on the range of validity impossible.

Since the calculations are straight-forward, they are not included in the main body of the text. Results of the more important stages are presented in the Appendix. Since in several calculations it was necessary to take the difference between two small numbers of almost equal magnitude, it was considered useful to leave the numbers in fractional form. This also facilitated checking of the results.

When all calculations are carried out, an algebraic equation in integral powers of c is obtained. This is

$$
\begin{aligned}
0 = &- 320\, i\alpha \cot\theta + \frac{12}{7}(r+1)\, i\alpha R_1 - \frac{320}{3} i\alpha^3 \cot\theta + \frac{68}{21}(r-1)\,\alpha^2 R_1 \cot\theta \\
&- 80\, i\alpha^3 S - \frac{17}{1155}(r^2-1)\,\alpha^2 R_1^2 + \frac{86}{105}(r+1)\, i\alpha^3 R_1 + \frac{142}{10,395}(r^2+1)\, i\alpha^3 R_1^2 \cot\theta \\
&- \frac{8}{245} r\, i\alpha^3 R_1^2 \cot\theta + \frac{388}{315}(r-1)\,\alpha^4 R_1 \cot\theta - \frac{1856}{105} i\alpha^5 \cot\theta \\
&- \frac{80}{3} i\alpha^5 S + \frac{17}{21}\alpha^4 R_1 S\,(r-1) \\
&+ c\left\{-1920 - \frac{380}{7} i\alpha R_1 (r-1) - 1024\,\alpha^2 - \frac{80}{3}(r+1)\,\alpha^2 R_1 \cot\theta \right. \\
&\quad + \frac{1319}{2310}(r^2+1)\,\alpha^2 R_1^2 - \frac{62}{105}\alpha^2 R_1^2 r - \frac{608}{21}(r-1)\, i\alpha^3 R_1 - \frac{2048}{7}\alpha^4 \\
&\quad \left. - \frac{85}{378}(r^2-1)\, i\alpha^3 R_1^2 \cot\theta - \frac{928}{105}(r+1)\,\alpha^4 R_1 \cot\theta - \frac{20}{3}\alpha^4 R_1 S(r+1)\right\} \\
&+ c^2\left\{256\,(r+1)\, i\alpha R_1 - \frac{323}{63}(r^2-1)\,\alpha^2 R_1^2 + \frac{1024}{7} i\alpha^3 R_1 (r+1)\right. \\
&\quad \left. + \frac{6}{7}(r^2+1)\, i\alpha^3 R_1^2 \cot\theta + \frac{32}{15} i\alpha^3 R_1^2 r \cot\theta\right\} \\
&+ c^3\,\alpha^2 R_1^2 \left[(r^2+1)\,\frac{181}{14} + 29r\right] . \qquad (53)
\end{aligned}
$$

Of the three roots of this equation, only the one starting in positive powers of α and R is considered. The other two roots have lead terms of the type $\frac{1}{\alpha R}$. This does not imply a singularity in the problem. It is observed that in the differential equation and in the boundary condition [as written in the determinant (50)], c always is multiplied by αR and therefore ϕ does not have a singularity at $\alpha R = 0$. Neither would $\alpha R = 0$ imply an infinite celerity or growth rate for these disturbances, as αc_T remains finite as αR approaches zero. To the order of approximation considered these roots cannot be computed accurately. However, preliminary calculation of these roots shows a tendency for them to be more stable than the case considered.

The root of interest can be computed by assuming

$$c = c_0 + c_1\alpha + c_2R + c_3\alpha^2 + \ldots$$

and substituting this into (53). After this is done, upon separation into real and imaginary parts, the result is obtained

$$c_r = \frac{\alpha^2 R_1(r-1)}{1920} \left\{\frac{2,183}{64,680}(r+1)R_1 - \frac{61}{42}\alpha^2 S - \cot\theta\left[\frac{122}{21} - \frac{46}{35}\alpha^2\right]\right\} , \tag{54}$$

$$c_i = \frac{\alpha}{1920}\left\{\frac{(r+1)R_1}{7}(12 - 2/3\,\alpha^2) - 16\,\alpha^2 S\,(5 - \alpha^2) - \cot\theta\left[64\left(5 - \alpha^2 + \frac{\alpha^4}{21}\right) - \frac{443,683}{1,164,240}(r^2+1)\alpha^2 R_1^2 + \frac{2,787}{17,640}r\alpha^2 R_1^2\right] - \cot^2\theta\,[8/3\,(r+1)\alpha^2 R_1]\right\} . \tag{55}$$

The growth rate σ is given by

$$\sigma = \alpha c_i . \tag{56}$$

It can be argued[6] that the most likely wave number to occur is the one with the highest growth rate. This value of α can be determined by setting to zero the derivative of σ with respect to α. The relation obtained by this is

$$0 = \frac{\alpha}{1920}\left\{\frac{8(r+1)R_1}{7}\left(3 - \frac{1}{3}\alpha^2\right) - 32S\,\alpha^2\left(10 - 3\alpha^2\right) - \cot\theta\left[64\left(10 - 4\alpha^2 + \frac{6\alpha^4}{21}\right) - \frac{443,683}{291,060}\left(r^2+1\right)\alpha^2R_1^2 + \frac{2,787}{4,410}r\alpha^2R_1^2\right]\right.$$

$$\left. - \cot^2\theta\left[\frac{32}{3}\left(r+1\right)\alpha^2R_1\right]\right\} . \qquad (57)$$

For the case of vertical inclination, the possible roots of (57) are, besides $\alpha = 0$,

$$\alpha_{cr}^2 = \frac{1}{6}\left[10 + \beta \pm \left(100 - 88\,\beta + \beta^2\right)^{1/2}\right] , \qquad (58)$$

where

$$\beta = \frac{(r+1)R_1}{84\,S} . \qquad (59)$$

(In the case of $S = 0$, the roots reduce to 0, 3.)

For small β, (58) may be expanded using the binomial theorem to obtain

$$\alpha_{cr}^2 = \frac{10}{6}\left\{1 + .1\,\beta \pm \left(1 - .44\,\beta \mp .092\,\beta^2\right)\right\} , \qquad (60)$$

or, for the root of interest,

$$\alpha_{cr}^2 = \beta\left\{.9 + .15\beta\right\} . \qquad (61)$$

V. DISCUSSION OF RESULTS

A. The Special Case of Infinite Slope

An inspection of (55) and the corresponding Figures 2, 3, 4, and 5 reveals that for the case of vertical inclination and zero surface tension, every value of α makes c_i zero for zero Reynolds number. Hence the α axis is the neutral stability curve. (The analysis does not hold true for large values of α, however it can be argued on physical grounds that in the absence of surface tension there is no restoring force when the slope is infinite.) Surface tension has a stabilizing effect (Figures 2 and 5) and reduces the range of α for which instability occurs at zero Reynolds number to the point $\alpha = 0$, hence it can never completely prevent instability. From a physical point of view, the effect of surface tension decreases with curvature. From a mathematical point of view, S is always accompanied by a curvature term (α^2), so for very small wave numbers the stabilizing force would be small.

The shape of the growth rate curves of Figure 5 is very similar to the neutral stability curve shown by Yih.[11] It might be speculated that the disagreement between Yih's and Benjamin's[1] results could be due to the difference in the approximations which might yield different members of the same family of curves. Benjamin's result would be the true neutral stability curve, while Yih's would be a curve of constant (small) growth rate. Of course surface tension was not included in the work of Yih, but similar terms could be introduced by terminating the approximation at different places.

It is noted from (9) and (54) that the dimensional celerity of the disturbance is directly dependent on the difference in densities of

the two fluids and the velocity of the primary flow. A small Reynolds number implies either large viscosity or small difference in densities, hence the speed of propagation of the disturbance is small for the approximation considered. The celerity is also small for the case of small wave numbers. The disturbance anticipated by the analysis is then in effect almost a standing wave, in the sense that it does not propagate, at least not very quickly.

It is seen from (54) and Figure 6 that the surface tension could make the celerity negative, hence disturbances would travel uphill. This effect is perhaps unexpected, but not unreasonable. Such results have in fact been noted in experiments conducted by William M. Sangster[8] of the University of Iowa, for layers of different thicknesses.

If (54) is contrasted with the corresponding results of the stability analyses of Yih and Benjamin, it is seen that the celerities represented in their analyses have values of the same order as the primary flow for very small Reynolds and wave numbers while the celerities presented here are many orders smaller than the velocity of the primary flow. In this sense, the present problem is more stable than the case of no counterflow considered by Yih and Benjamin.

The direct dependence of the disturbance celerity on the difference in densities of the two fluids may seem surprising when compared with previous results[4] for gravity waves. The reason for this difference is that the present waves are viscosity waves rather than gravity waves.

The fact that the primary flow velocity is zero at the interface may explain some aspects of the disturbance celerity.

Although a free surface is generally regarded as a destabilizing agent, this effect is somewhat diminished in the present problem due to the counterflow. If the free surface were replaced by a rigid boundary, the problem would be the familiar one of plane Poiseuille flow, for which instability is known to occur for Reynolds numbers many orders higher than contemplated in the present analysis.(5) The present problem then in these aspects lies somewhere between the previously mentioned free surface flows and the Poiseuille flow.

Another interesting feature is that (24) shows that the disturbance amplitude is inversely proportional to c. For small c then, even if the y velocity component is small, the amplitude of the disturbance may be larger. This again is brought about by the fact that in the problem considered, the primary flow velocity at the interface is zero, so that the convective terms in the time derivative of η do not appear, as they do in Yih's and Benjamin's works.

Naturally the onset of instability at such low Reynolds numbers does not mean the onset of turbulence, but only the onset of waves at the free surface. The analysis performed here is, of course, based on the assumption that the two fluids do not mix across the interface, which would not be true in turbulent flow except under very special conditions.

B. Other Slopes

The results are not as informative for values of Θ other than 90°. As the inclination becomes even slightly less steep, the Θ terms in c_1 predominate and exert a strong stabilizing influence. For Θ more

than a degree or two away from vertical, the results probably are not sufficiently accurate to predict instability, and higher order approximations are necessitated. Neutral stability curves are shown in Figure 7 for various values of the parameter S for the case $\theta = 89.5°$. The curve for S = 0 bends towards the α axis and predicts instability again for all small Reynolds numbers. It is believed however that this is not a reliable result, and that if higher order terms were present the curve would be almost vertical for small α, and then would go in the positive R_1 direction, as do the curves for non-zero values of S. This same effect is shown in Figure 8. The results shown in Figure 9, with a non-zero value of S, are more reliable in regard to the shape of the curve, and show that for inclinations other than the vertical, critical Reynolds numbers do exist. It seems reasonable that the critical Reynolds number should occur at $\alpha = 0$. The greater surface tension effect and greater dissipation of the disturbance energy at higher wave numbers would both tend to stabilize the flow. Such tendencies were shown for the case of vertical inclination. A definite mathematical answer to this matter would require a higher order approximation.

It is interesting that since historically a point of inflexion in the velocity of the primary flow was known to be conducive to instability, the flow envisaged here is "less unstable" than that discussed by Benjamin and Yih, at least as far as the effect of slope is concerned. Benjamin's results show a lesser effect of terms involving θ. This may perhaps be explained physically again on the grounds that the velocity is zero at the interface, hence the energy which the primary flow is able to impart to the disturbance at this point is smaller than elsewhere in the flow.

VI. CONCLUSIONS

From the results presented, the following conclusions can be drawn:

a) For the case of vertical inclination, there are values of α for which the flow is unstable at all values of the Reynolds number. The celerity of such disturbances is small, but the effect on the interface can be large.

b) The stabilizing effect of surface tension is shown, although surface tension can never induce complete stability, since it has no effect at zero wave number.

c) For slopes other than the vertical, a critical Reynolds number exists. The value of the critical Reynolds number probably is the value of R_1 which occurs at $\alpha = 0$, although a more thorough investigation is needed in order that a definite statement can be made in this regard.

d) The presence of the interface is responsible for instability at low Reynolds numbers at steep slopes. However, the counterflow does contribute a stabilizing influence.

e) The critical wave number has been found and is given in Equation (61).

VII. SUGGESTIONS FOR FURTHER WORK

For results valid for higher Reynolds numbers or smaller slopes, higher approximations by the methods used in the present paper do not seem promising. The work involved becomes staggering and the approximations would have to be several orders higher to change the range of validity appreciably. It is known that for $\Theta = 0$ the flow is stable. It would be interesting to expand in negative powers of the Reynolds number to determine the effects of small slopes. Such a result would be of interest for flows at large Reynolds numbers.

The solution could be carried out for other discharges and differing viscosities. This would not greatly change the formulation of the problem, but would somewhat complicate the algebra involved in the calculation.

An interesting variation on this problem is the case of a continuously varying density distribution. In such a problem the boundary conditions become much simpler, because only the conditions of no slip need be imposed (at the rigid boundaries). The continuity of velocities and stresses at the interface which was represented in the boundary conditions in the present problem are now inherent in the stability equation, which, however, becomes much more complicated thereby.

APPENDICES

APPENDIX A

VALUES OF ϕ_1 AND THEIR FIRST THREE DERIVATIVES EVALUATED AT y = 0

n	$\phi_1^{(n)}(0)$	$\phi_2^{(n)}(0)$	$\phi_3^{(n)}(0)$	$\phi_4^{(n)}(0)$	$\phi_5^{(n)}(0)$
0	1	0	0	0	c
1	0	1	0	0	1
2	0	0	2	0	-2δ
3	0	0	0	1	0

APPENDIX B

VALUES OF ϕ_i AND THEIR FIRST DERIVATIVES EVALUATED AT $y = \delta$

$$\phi_1(\delta) = 1 - \frac{2\delta}{4!} A + \left(\frac{2c\delta}{6!} + \frac{28}{8!}\right)A^2 + \frac{c}{4!} AB - \frac{1}{4!} B^2 + \ldots$$

$$\phi_2(\delta) = \delta - \frac{2}{5!} A + \left(\frac{2c}{7!} + \frac{4\delta}{8!}\right)A^2 + \left(\frac{c\delta}{5!} + \frac{4}{7!}\right)AB - \frac{\delta}{5!} B^2 + \ldots$$

$$\phi_3(\delta) = 1 - \left(\frac{2c}{4!} + \frac{2\delta}{5!}\right)A + \frac{1}{3!} B + \left(\frac{2c^2}{6!} + \frac{44c\delta}{8!} + \frac{4}{8!}\right)A + \left(-\frac{c}{5!} - \frac{8\delta}{7!}\right)AB + \frac{1}{5!} B^2 + \ldots$$

$$\phi_4(\delta) = \frac{\delta}{3!} + \left(-\frac{c\delta}{5!} - \frac{10}{7!}\right)A + \frac{2\delta}{5!} B + \left(\frac{c^2\delta}{7!} + \frac{32c}{9!} + \frac{380\delta}{11!}\right)A^2 + \left(-\frac{3c\delta}{7!} - \frac{48}{9!}\right)AB + \frac{3\delta}{7!} B^2 + \ldots$$

$$\phi_5(\delta) = c - \frac{\delta}{3!} B + \left(\frac{c^2}{4!} + \frac{2c\delta}{5!} + \frac{12}{7!}\right)AB + \left(-\frac{c}{4!} - \frac{2\delta}{5!}\right)B^2 + \ldots$$

$$\phi_1'(\delta) = -\frac{2}{3!} B + \left(\frac{2c}{5!} + \frac{22\delta}{7!}\right)A^2 + \left(\frac{c\delta}{3!} - \frac{1}{5!}\right)AB - \frac{\delta}{3!} B^2 + \ldots$$

$$\phi_2'(\delta) = 1 - \frac{2\delta}{4!} A + \left(\frac{2c\delta}{6!} + \frac{28}{8!}\right)A^2 + \left(\frac{c}{4!} + \frac{2\delta}{6!}\right)AB - \frac{1}{4!} B^2 + \ldots$$

$$\phi_3'(\delta) = 2\delta + \left(-\frac{2c\delta}{3!} - \frac{2}{4!}\right)A + \frac{4\delta}{3!} B + \left(\frac{2c^2\delta}{5!} + \frac{36c}{7!} + \frac{28\delta}{8!}\right)A^2 + \left(-\frac{6c\delta}{5!} - \frac{54}{7!}\right)AB + \frac{6\delta}{5!} B^2 + \ldots$$

$$\phi_4'(\delta) = \frac{1}{2!} + \left(-\frac{c}{4!} - \frac{8\delta}{6!}\right)A + \frac{2}{4!} B + \left(\frac{c^2}{6!} + \frac{26c\delta}{8!} + \frac{26}{9!}\right)A^2 + \left(-\frac{3c}{6!} - \frac{5\delta}{7!}\right)AB + \frac{3}{6!} B^2 + \ldots$$

$$\phi_5'(\delta) = -1 - \frac{4}{3!} B + \left(\frac{c^2\delta}{3!} + \frac{2c}{4!} + \frac{68\delta}{7!}\right)AB + \left(-\frac{c\delta}{3!} - \frac{11}{5!}\right)B^2 + \ldots$$

APPENDIX C

VALUES OF THE DETERMINANTS F_i AND H_i

$$F_1 = \left(-\frac{c}{2!} - \frac{1}{3!}\right) + \left(\frac{c^2}{4!} + \frac{14c}{6!} + \frac{10}{7!}\right)A_1 + \left(-\frac{2c}{4!} - \frac{32}{6!}\right)B$$
$$+ \left(-\frac{c^3}{6!} - \frac{34c^2}{8!} - \frac{58c}{9!} - \frac{380}{11!}\right)A_1^2 + \left(\frac{8c^2}{6!} + \frac{29c}{7!} + \frac{240}{9!}\right)A_1B$$
$$+ \left(-\frac{8c}{6!} - \frac{24}{7!}\right)B^2 + \ldots$$

$$F_2 = \frac{2}{3!} + \left(-\frac{4c}{5!} - \frac{18}{7!}\right)A_1 + \frac{8}{5!}B + \left(\frac{6c^2}{7!} + \frac{106c}{9!} + \frac{632}{11!}\right)A_1^2 + \left(-\frac{32c}{7!} - \frac{336}{9!}\right)A_1B$$
$$+ \frac{32}{7!}B^2 + \ldots$$

$$F_3 = \frac{1}{3!} + \left(-\frac{8c}{6!} - \frac{8}{7!}\right)A_1 + \frac{16}{6!}B + \left(\frac{12c^2}{8!} + \frac{32c}{9!} + \frac{252}{11!}\right)A_1^2 + \left(-\frac{8c}{7!} - \frac{96}{9!}\right)A_1B$$
$$+ \frac{8}{7!}B^2 + \ldots$$

$$F_4 = 1 + \left(-\frac{6c}{4!} - \frac{2}{5!}\right)A_1 + \frac{1}{2!}B + \left(\frac{10c^2}{6!} + \frac{108c}{8!} + \frac{4}{8!}\right)A_1^2 + \left(-\frac{8c}{5!} - \frac{38}{7!}\right)A_1B$$
$$+ \frac{8}{5!}B^2 + \ldots$$

$$F_5 = (-1 - c) + \left(\frac{2c}{4!} + \frac{2}{5!}\right)A_1 + \left(-\frac{1}{2!}\right)B + \left(-\frac{2c^2}{6!} - \frac{44c}{8!} - \frac{4}{8!}\right)A_1^2$$
$$+ \left(\frac{2c^2}{4!} + \frac{40c}{6!} + \frac{38}{7!}\right)A_1B + \left(-\frac{2c}{4!} - \frac{8}{5!}\right)B^2 + \ldots$$

$$F_6 = (-1 - 2c) + \left(\frac{2c^2}{3!} + \frac{c}{3!} + \frac{2}{5!}\right)A_1 + \left(-\frac{4c}{3!} - \frac{1}{2!}\right)B +$$
$$+ \left(-\frac{2c^3}{5!} - \frac{50c^2}{7!} - \frac{9c}{7!} - \frac{4}{8!}\right)A_1^2 + \left(\frac{16c^2}{5!} + \frac{348c}{7!} + \frac{38}{7!}\right)A_1B$$
$$+ \left(-\frac{16c}{5!} - \frac{8}{5!}\right)B^2 + \ldots$$

$$H_1 = \left(-\frac{c}{2!} + \frac{1}{3!}\right) + \left(\frac{c^2}{4!} - \frac{14c}{6!} + \frac{10}{7!}\right)A_2 + \left(-\frac{2c}{4!} + \frac{32}{6!}\right)B$$
$$+ \left(-\frac{c^3}{6!} + \frac{34c^2}{8!} - \frac{58c}{9!} + \frac{380}{11!}\right)A_2^2 + \left(\frac{8c^2}{6!} - \frac{29c}{7!} + \frac{240}{9!}\right)A_2B$$
$$+ \left(-\frac{8c}{6!} + \frac{24}{7!}\right)B^2 + \ldots$$

$$H_2 = \left(-\frac{2}{3!}\right) + \left(\frac{4c}{5!} - \frac{18}{7!}\right)A_2 + \left(-\frac{8}{5!}\right)B + \left(-\frac{6c^2}{7!} + \frac{106c}{9!} - \frac{632}{11!}\right)A_2^2$$
$$+ \left(\frac{32c}{7!} - \frac{336}{9!}\right)A_2B + \left(-\frac{32}{7!}\right)B^2 + \ldots$$

$$H_3 = \frac{1}{3!} + \left(-\frac{8c}{6!} + \frac{8}{7!}\right)A_2 + \frac{16}{6!}B + \left(\frac{12c^2}{8!} - \frac{32c}{9!} + \frac{252}{11!}\right)A_2^2$$
$$+ \left(-\frac{8c}{7!} + \frac{96}{9!}\right)A_2B + \frac{8}{7!}B^2 + \ldots$$

$$H_4 = 1 + \left(-\frac{6c}{4!} + \frac{2}{5!}\right)A_2 + \frac{1}{2!}B + \left(\frac{10c^2}{6!} - \frac{108c}{8!} + \frac{4}{8!}\right)A_2^2$$
$$+ \left(-\frac{8c}{5!} + \frac{38}{7!}\right)A_2B + \frac{8}{5!}B^2 + \ldots$$

$$H_5 = (1 - c) + \left(-\frac{2c}{4!} + \frac{2}{5!}\right)A_2 + \frac{1}{2!}B + \left(\frac{2c^2}{6!} - \frac{44c}{8!} + \frac{4}{8!}\right)A_2^2$$
$$+ \left(\frac{2c^2}{4!} - \frac{40c}{6!} + \frac{38}{7!}\right)A_2B + \left(-\frac{2c}{4!} + \frac{8}{5!}\right)B^2 + \ldots$$

$$H_6 = (-1 + 2c) + \left(-\frac{2c^2}{3!} + \frac{c}{3!} - \frac{2}{5!}\right)A_2 + \left(\frac{4c}{3!} - \frac{1}{2!}\right)B + \left(\frac{2c^3}{5!} - \frac{50c^2}{7!} + \frac{9c}{7!} - \frac{4}{8!}\right)A_2^2$$
$$+ \left(-\frac{16c}{5!} + \frac{348c}{7!} - \frac{38}{7!}\right)A_2B + \left(\frac{16c}{5!} - \frac{8}{5!}\right)B^2 + \ldots$$

APPENDIX D

PRODUCTS OF THE F_i AND H_i

$$H_1F_4 = (-\frac{c}{2!} + \frac{1}{3!}) + (\frac{3c^2}{4!} - \frac{4c}{5!} - \frac{2}{6!})A_1 + (\frac{c^2}{4!} - \frac{14c}{6!} + \frac{10}{7!})A_2$$

$$+ (-\frac{2c}{3!} + \frac{92}{6!})B + (-\frac{5c^3}{6!} + \frac{354c^2}{9!} + \frac{2c}{7!} + \frac{6}{9!})A_1^2$$

$$+ (-\frac{c^3}{6!} + \frac{34c^2}{8!} - \frac{58c}{9!} + \frac{380}{11!})A_2^2 + (-\frac{6c^3}{4!4!} + \frac{3c^2}{6!}$$

$$- \frac{204c}{5!7!} - \frac{1}{3!7!})A_1A_2 + (\frac{39c^2}{6!} - \frac{86c}{7!} - \frac{1208}{5!7!})A_1B$$

$$+ (\frac{23c^2}{6!} - \frac{78c}{7!} + \frac{600}{9!})A_2B + (-\frac{62c}{6!} + \frac{192}{7!})B^2 + \ldots$$

$$F_1H_4 = (-\frac{c}{2!} - \frac{1}{3!}) + (\frac{c^2}{4!} + \frac{14c}{6!} + \frac{10}{7!})A_1 + (\frac{3c^2}{4!} + \frac{4c}{5!} - \frac{2}{6!})A_2$$

$$+ (-\frac{2c}{3!} - \frac{92}{6!})B + (-\frac{c^3}{6!} - \frac{34c^2}{8!} - \frac{58c}{9!} - \frac{380}{11!})A_1^2$$

$$+ (-\frac{5c^3}{6!} - \frac{354c}{9!} + \frac{2c}{7!} - \frac{6}{9!})A_2^2 + (-\frac{6c^3}{4!4!} - \frac{3c^2}{6!}$$

$$- \frac{204c}{5!7!} + \frac{1}{3!7!})A_1A_2 + (\frac{23c^2}{6!} + \frac{78c}{7!} + \frac{600}{9!})A_1B$$

$$+ (\frac{39c^2}{6!} + \frac{86c}{7!} - \frac{1208}{5!7!})A_2B + (-\frac{62c}{6!} - \frac{192}{7!})B^2 + \ldots$$

$$H_2F_6 = \left(\frac{4c}{3!} + \frac{2}{3!}\right) + \left(-\frac{80c^2}{6!} - \frac{40c}{6!} - \frac{4}{6!}\right)A_1 + \left(-\frac{8c^2}{5!} - \frac{132c}{7!} + \frac{18}{7!}\right)A_2$$

$$+ \left(\frac{256c}{6!} + \frac{28}{5!}\right)B + \left(\frac{4c^3}{6!} + \frac{100c^2}{3!7!} + \frac{3c}{7!} + \frac{1}{3!7!}\right)A_1^2$$

$$+ \left(\frac{12c^3}{7!} + \frac{220c^2}{9!} - \frac{10396c}{11!} + \frac{632}{11!}\right)A_2^2$$

$$+ \left(\frac{8c^3}{6!} + \frac{25c^2}{7!} - \frac{144c}{10!} - \frac{36}{5!7!}\right)A_1A_2$$

$$+ \left(-\frac{8c^2}{5!} - \frac{172c}{7!} - \frac{2192}{5!7!}\right)A_1B$$

$$+ \left(-\frac{176c^2}{7!} - \frac{6816c}{9!} + \frac{984}{9!}\right)A_2B$$

$$+ \left(\frac{512c}{7!} + \frac{312}{7!}\right)B^2 + \ldots$$

$$F_2H_6 = \left(\frac{4c}{3!} - \frac{2}{3!}\right) + \left(-\frac{8c^2}{5!} + \frac{132c}{7!} + \frac{18}{7!}\right)A_1 + \left(-\frac{80c^2}{6!} + \frac{40c}{6!} - \frac{4}{6!}\right)A_2$$

$$+ \left(\frac{256c}{6!} - \frac{28}{5!}\right)B + \left(\frac{12c^3}{7!} - \frac{220c^2}{9!} - \frac{10396c}{11!} - \frac{632}{11!}\right)A_1^2$$

$$+ \left(\frac{4c^3}{6!} - \frac{100c^2}{3!7!} + \frac{3c}{7!} - \frac{1}{3!7!}\right)A_2^2$$

$$+ \left(\frac{8c^3}{6!} - \frac{25c^2}{7!} - \frac{144c}{10!} + \frac{36}{5!7!}\right)A_1A_2$$

$$+ \left(-\frac{176c^2}{7!} + \frac{6816c}{9!} + \frac{984}{9!}\right)A_1B$$

$$+ \left(-\frac{8c^2}{5!} + \frac{172c}{7!} - \frac{2192}{5!7!}\right)A_2B$$

$$+ \left(\frac{512c}{7!} - \frac{312}{7!}\right)B^2 + \ldots$$

$$H_3F_5 = (-\frac{c}{3!} - \frac{1}{3!}) + (\frac{10c}{6!} + \frac{2}{6!})A_1 + (\frac{8c^2}{6!} + \frac{48c}{7!} - \frac{8}{6!})A_2$$

$$+ (-\frac{16c}{6!} - \frac{76}{6!})B + (-\frac{2c^2}{3!6!} - \frac{66c}{9!} - \frac{6}{9!})A_1^2$$

$$+ (-\frac{12c^3}{8!} - \frac{76c^2}{9!} + \frac{3268c}{11!} - \frac{252}{11!})A_2^2 + (-\frac{4c^2}{3!6!} - \frac{32c}{5!7!} + \frac{16}{5!7!})A_1A_2$$

$$+ (\frac{10c^2}{6!} + \frac{8c}{6!} + \frac{984}{5!7!})A_1B + (\frac{8c^2}{7!} + \frac{2496c}{9!} - \frac{384}{9!})A_2B$$

$$+ (-\frac{78c}{7!} - \frac{120}{7!})B^2 + \ldots$$

$$F_3H_5 = (-\frac{c}{3!} + \frac{1}{3!}) + (\frac{8c^2}{6!} - \frac{48c}{7!} - \frac{8}{7!})A_1 + (-\frac{10c}{6!} + \frac{2}{6!})A_2$$

$$+ (-\frac{16c}{6!} + \frac{76}{6!})B + (-\frac{12c^3}{8!} + \frac{76c^2}{9!} + \frac{3268c}{11!} + \frac{252}{11!})A_1^2$$

$$+ (\frac{2c^2}{3!6!} - \frac{66c}{9!} + \frac{6}{9!})A_2^2 + (\frac{4c^2}{3!6!} - \frac{32c}{5!7!} - \frac{16}{5!7!})A_1A_2$$

$$+ (\frac{8c^2}{7!} - \frac{2496c}{9!} - \frac{384}{9!})A_1B + (\frac{10c^2}{6!} - \frac{8c}{6!} + \frac{984}{5!7!})A_2B$$

$$+ (-\frac{78c}{7!} + \frac{120}{7!})B^2 + \ldots$$

$$H_2F_3 = -\frac{2}{3!3!} + (\frac{16c}{3!6!} + \frac{16}{3!7!})A_1 + (\frac{4c}{6!} - \frac{3}{7!})A_2 - \frac{80}{3!6!}\ B$$

$$+ (-\frac{4c^2}{8!} - \frac{64c}{3!9!} - \frac{84}{11!})A_1^2 + (-\frac{c^2}{7!} + \frac{106c}{3!9!} - \frac{632}{3!11!})A_2^2$$

$$+ (-\frac{32c^2}{5!6!} - \frac{8c}{5!7!} + \frac{144}{7!7!})A_1A_2 + (\frac{768c}{5!7!} + \frac{704}{10!})A_1B$$

$$+ (\frac{1088c}{5!7!} - \frac{848}{10!})A_2B - \frac{1856}{5!7!}\ B^2 + \ldots$$

$$F_2H_3 = \frac{2}{3!3!} + (-\frac{4c}{6!} - \frac{3}{7!})A_1 + (-\frac{16c}{3!6!} + \frac{16}{3!7!})A_2 + \frac{80}{3!6!}\ B$$

$$+ (\frac{c^2}{7!} + \frac{106c}{3!9!} + \frac{632}{3!11!})A_1^2 + (\frac{4c^2}{8!} - \frac{64c}{3!9!} + \frac{84}{11!})A_2^2$$

$$+ (\frac{32c^2}{5!6!} - \frac{8c}{5!7!} - \frac{144}{7!7!})A_1A_2 + (-\frac{1088c}{5!7!} - \frac{848}{10!})A_1B$$

$$+ (-\frac{768c}{5!7!} + \frac{704}{10!})A_2B + \frac{1856}{5!7!}\ B^2 + \ldots$$

$$F_1H_3 = (-\frac{2c}{4!} - \frac{20}{6!}) + (\frac{5c^2}{6!} + \frac{14c}{3!6!} + \frac{10}{3!7!})A_1 + (\frac{4c^2}{6!} + \frac{32c}{3!7!} - \frac{8}{3!7!})A_2$$

$$+ (-\frac{3c}{5!} - \frac{8}{6!})B + \ldots$$

$$H_1F_3 = (-\frac{2c}{4!} + \frac{20}{6!}) + (\frac{4c^2}{6!} - \frac{32c}{3!7!} - \frac{8}{3!7!})A_1$$

$$+ (\frac{5c^2}{6!} - \frac{14c}{3!6!} + \frac{10}{3!7!})A_2 + (-\frac{3c}{5!} + \frac{8}{6!})B + \ldots$$

APPENDIX E

SUMS OF THE PRODUCTS OF THE F_1 AND H_1

$$H_1F_4 + F_1H_4 = -c + (\frac{c^2}{3!} - \frac{10c}{6!} - \frac{4}{7!})A_1 + (\frac{c^2}{3!} + \frac{10c}{6!} - \frac{4}{7!})A_2$$

$$- \frac{4c}{3!} B + (- \frac{c^3}{5!} + \frac{48c^2}{9!} + \frac{86c}{9!} + \frac{280}{11!})A_1^2$$

$$+ (- \frac{c^3}{5!} - \frac{48c^2}{9!} + \frac{86c}{9!} - \frac{280}{11!})A_2^2 + (- \frac{c^3}{2!4!} - \frac{408c}{5!7!})A_1A_2$$

$$+ (\frac{62c^2}{6!} - \frac{8c}{7!} - \frac{1248}{10!})A_1B$$

$$+ (\frac{62c^2}{6!} + \frac{8c}{7!} - \frac{1248}{10!})A_2B - \frac{124c}{6!} B^2 + \ldots$$

$$- H_2F_6 - F_2H_6 = - \frac{8c}{3!} + (\frac{128c^2}{6!} + \frac{148c}{7!} + \frac{10}{7!})A_1$$

$$+ (\frac{128c^2}{6!} - \frac{148c}{7!} + \frac{10}{7!})A_2 - \frac{512c}{6!} B$$

$$+ (- \frac{40c^3}{7!} - \frac{980c^2}{9!} - \frac{13364c}{11!} - \frac{688}{11!})A_1^2$$

$$+ (- \frac{40c^3}{7!} + \frac{980c^2}{9!} - \frac{13364c}{11!} + \frac{688}{11!})A_2^2$$

$$+ (- \frac{16c^3}{6!} + \frac{288c}{10!})A_1A_2$$

$$+ (\frac{512c^2}{7!} + \frac{5568c}{9!} + \frac{3312}{10!})A_1B$$

$$+ (\frac{512c^2}{7!} - \frac{5568c}{9!} + \frac{3312}{10!})A_2B$$

$$- \frac{1024c}{7!} B^2 + \ldots$$

$$H_3F_5 + F_3H_5 = -\frac{2c}{3!} + (\frac{8c^2}{6!} + \frac{22c}{7!} + \frac{6}{7!})A_1 + (\frac{8c^2}{6!} - \frac{22c}{7!} + \frac{6}{7!})A_2$$

$$- \frac{32c}{6!} B + (- \frac{12c^3}{8!} - \frac{92c^2}{9!} - \frac{3992c}{11!} - \frac{408}{11!})A_1^2$$

$$+ (- \frac{12c^3}{8!} + \frac{92c^2}{9!} - \frac{3992c}{11!} + \frac{408}{11!})A_2^2 - \frac{64c}{5!7!} A_1A_2$$

$$+ (\frac{78c^2}{7!} + \frac{1536c}{9!} + \frac{2064}{10!})A_1B$$

$$+ (\frac{78c^2}{7!} - \frac{1536c}{9!} + \frac{2064}{10!})A_2B - \frac{156c}{7!} B^2 + \ldots$$

$$H_2F_3 - F_2H_3 = - \frac{4}{3!3!} + (\frac{40c}{3!6!} + \frac{34}{3!7!})A_1 + (\frac{40c}{3!6!} - \frac{34}{3!7!})A_2$$

$$- \frac{160}{3!6!} B + (- \frac{12c^2}{8!} - \frac{170c}{3!9!} - \frac{1136}{3!11!})A_1^2$$

$$+ (- \frac{12c^2}{8!} + \frac{170c}{3!9!} - \frac{1136}{3!11!})A_2^2$$

$$+ (- \frac{64c^2}{5!6!} + \frac{288}{7!7!})A_1A_2 + (\frac{1856c}{5!7!} + \frac{1552}{10!})A_1B$$

$$+ (\frac{1856c}{5!7!} - \frac{1552}{10!})A_2B - \frac{3712}{5!7!} B^2 + \ldots$$

$$F_1H_3 - H_1F_3 = - \frac{2}{3!3!} + (\frac{c^2}{6!} + \frac{130c}{3!7!} + \frac{3}{7!})A_1$$

$$+ (- \frac{c^2}{6!} + \frac{130c}{3!7!} - \frac{3}{7!})A_2 - \frac{16}{6!} B + \ldots$$

FIGURES 1 THROUGH 10

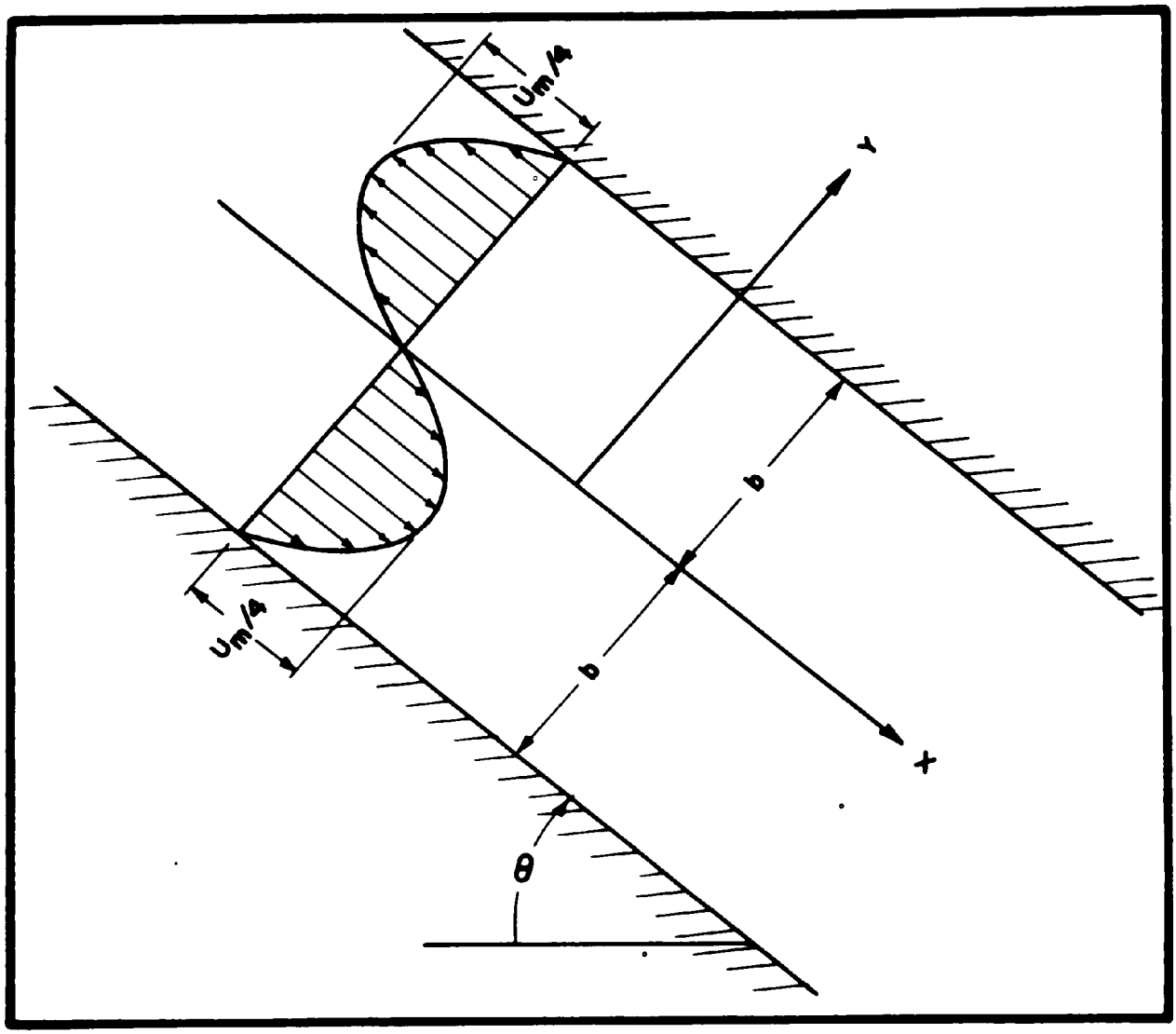


Figure 1. Diagram of the Primary Flow, Showing the Coordinate Axes and the Velocity Profile.

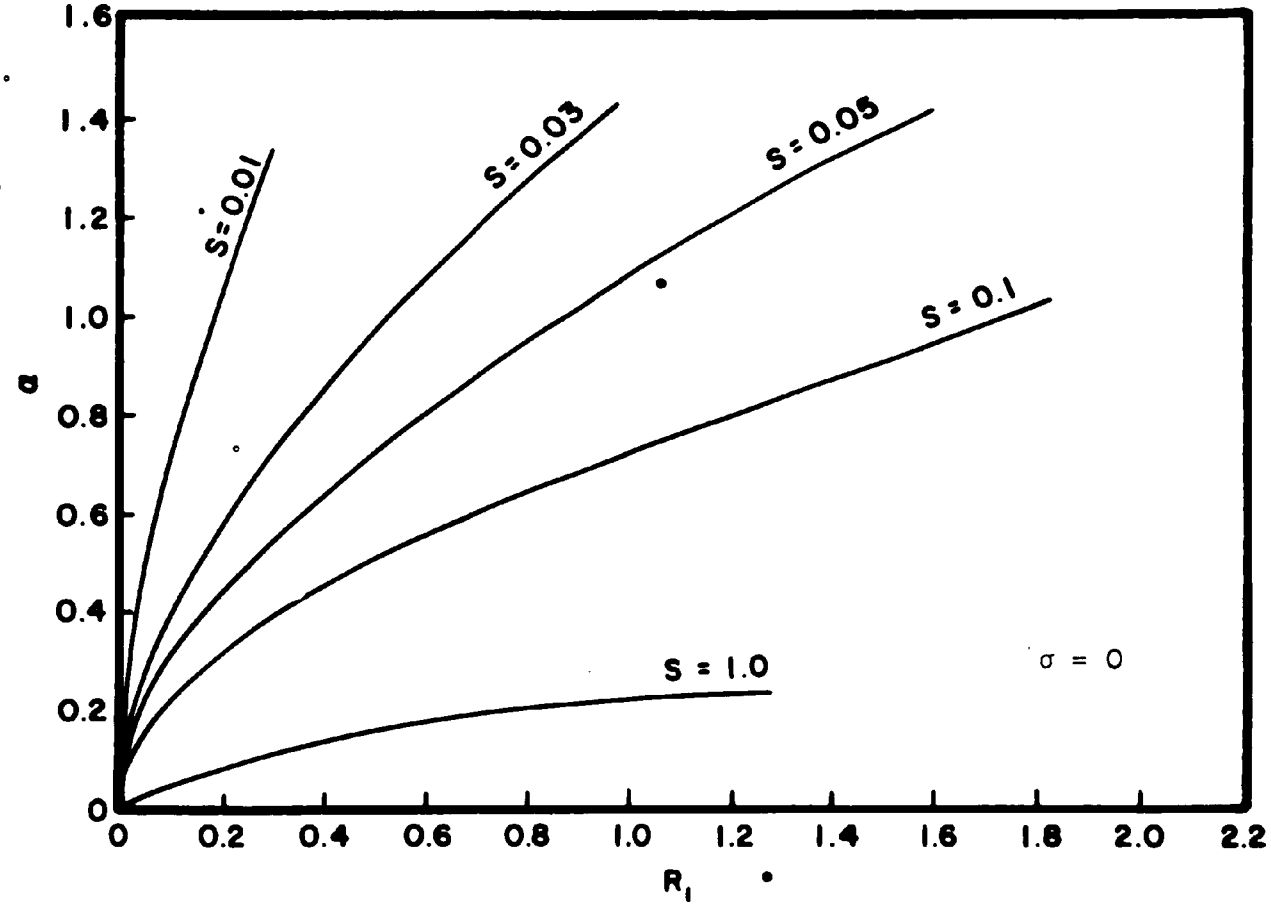


Figure 2. Curves of Neutral Stability for Flow Between Vertical Walls with Various Values of the Parameter S. r = 1.2. (For a Given S the Stable Region is Above the Curve and the Unstable Region is Below the Curve.)

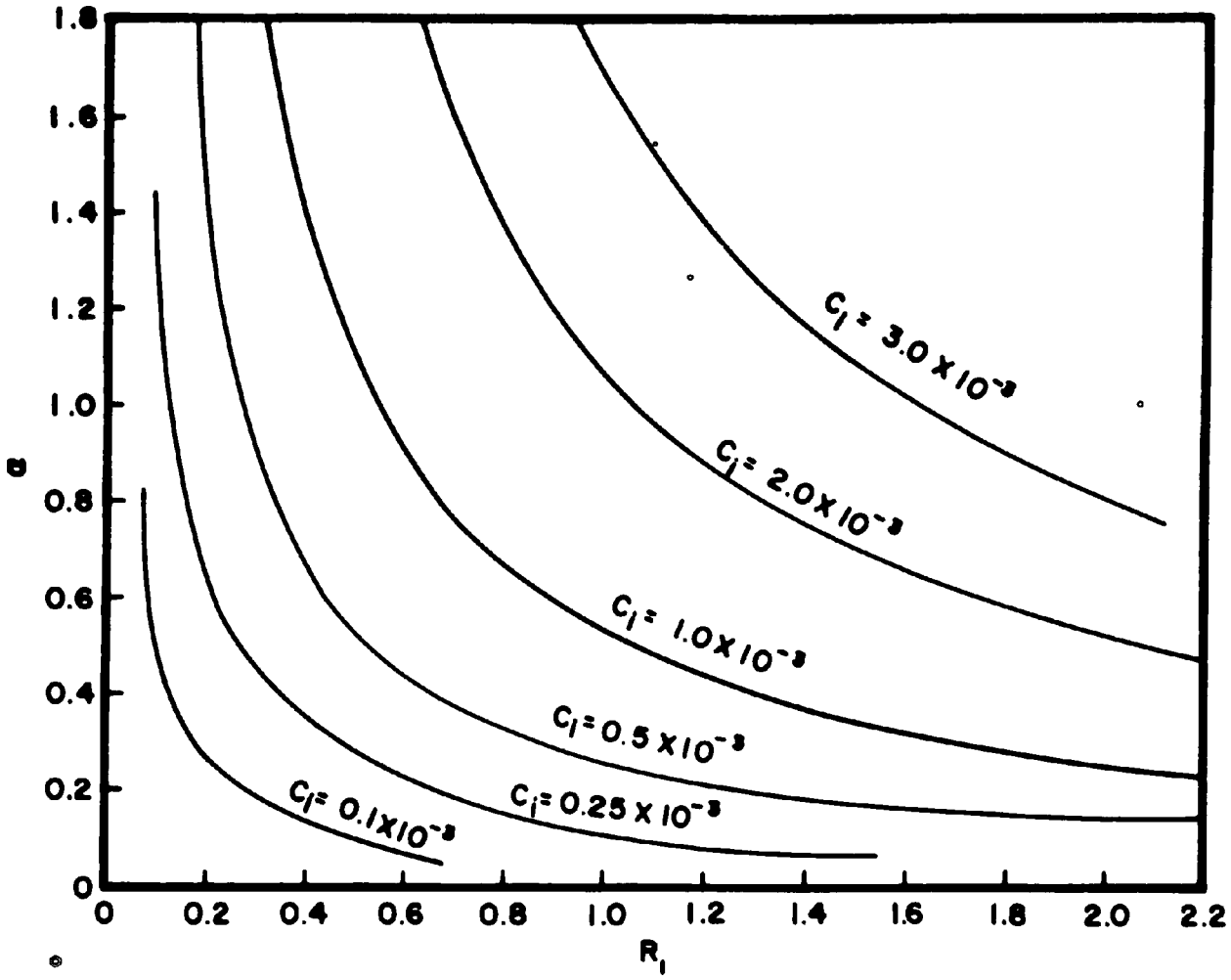


Figure 3. Curves of Constant c_1 for S = 0, $\theta = 90°$, r = 1.2.

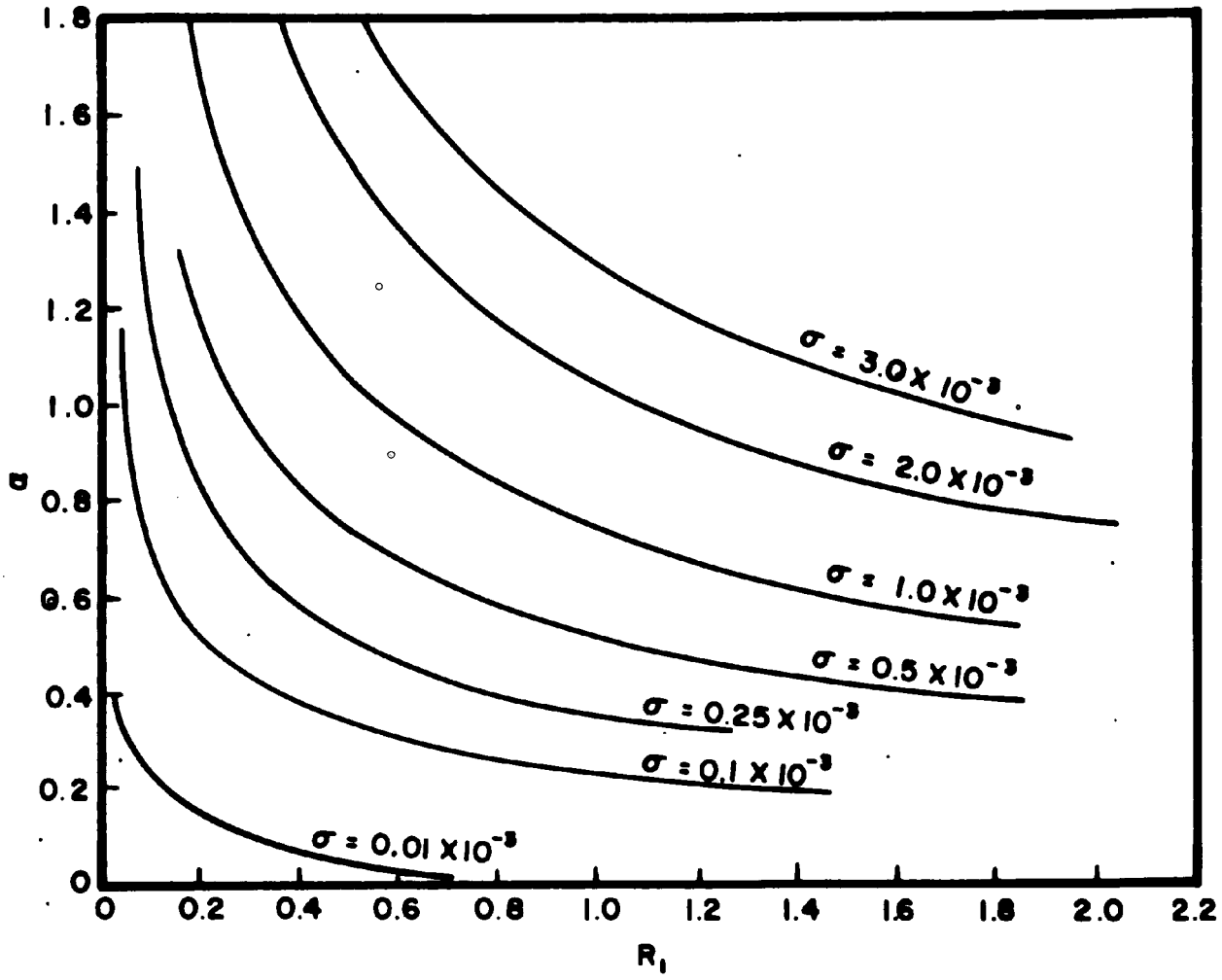


Figure 4. The Rate of Amplification of Waves of Various Wave Numbers for the Vertical Case. r = 1.2, S = 0.

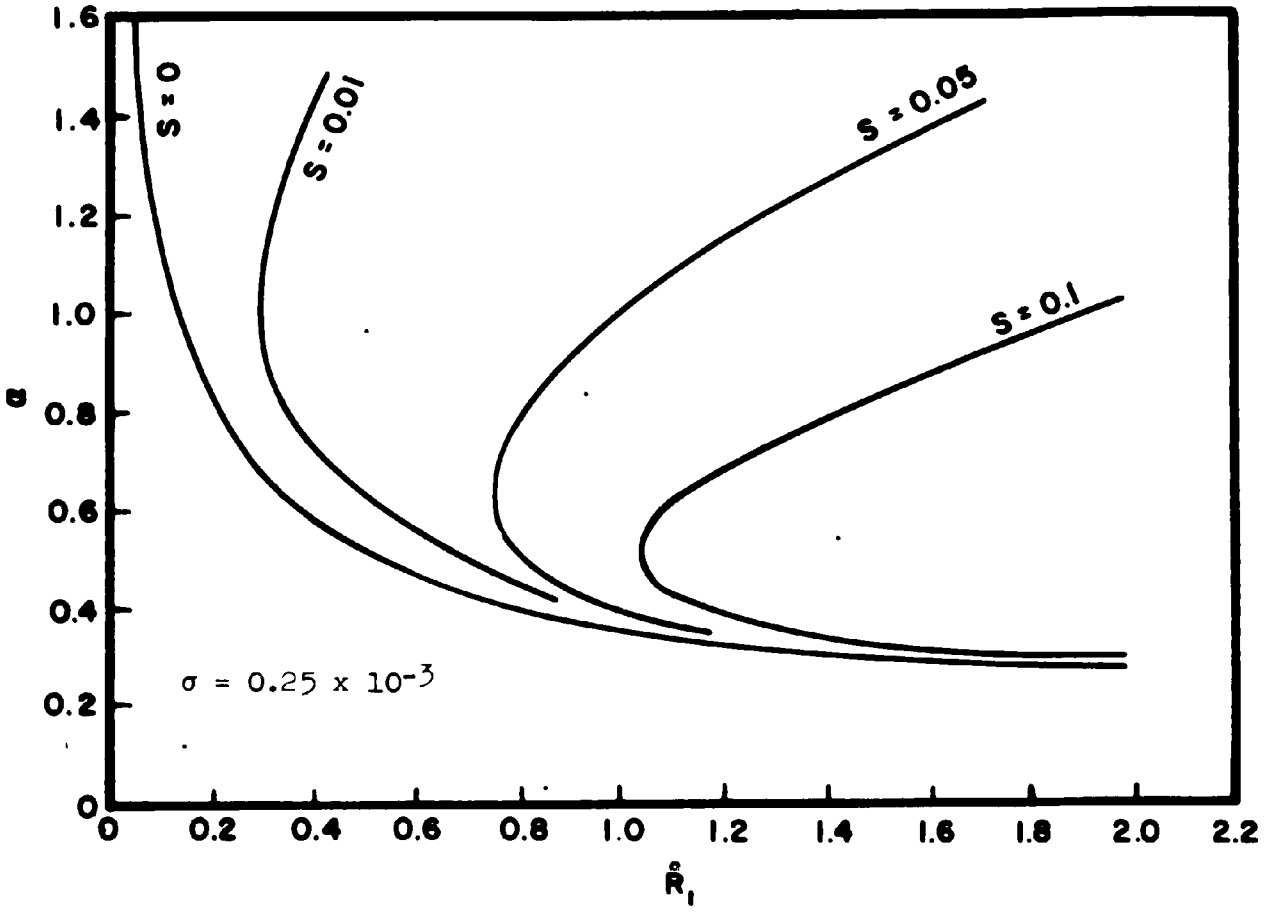


Figure 5. The Effect of Surface Tension on Growth Rate for the Case $\sigma = 0.25 \times 10^{-3}$, $r = 1.2$, $\theta = 90°$.

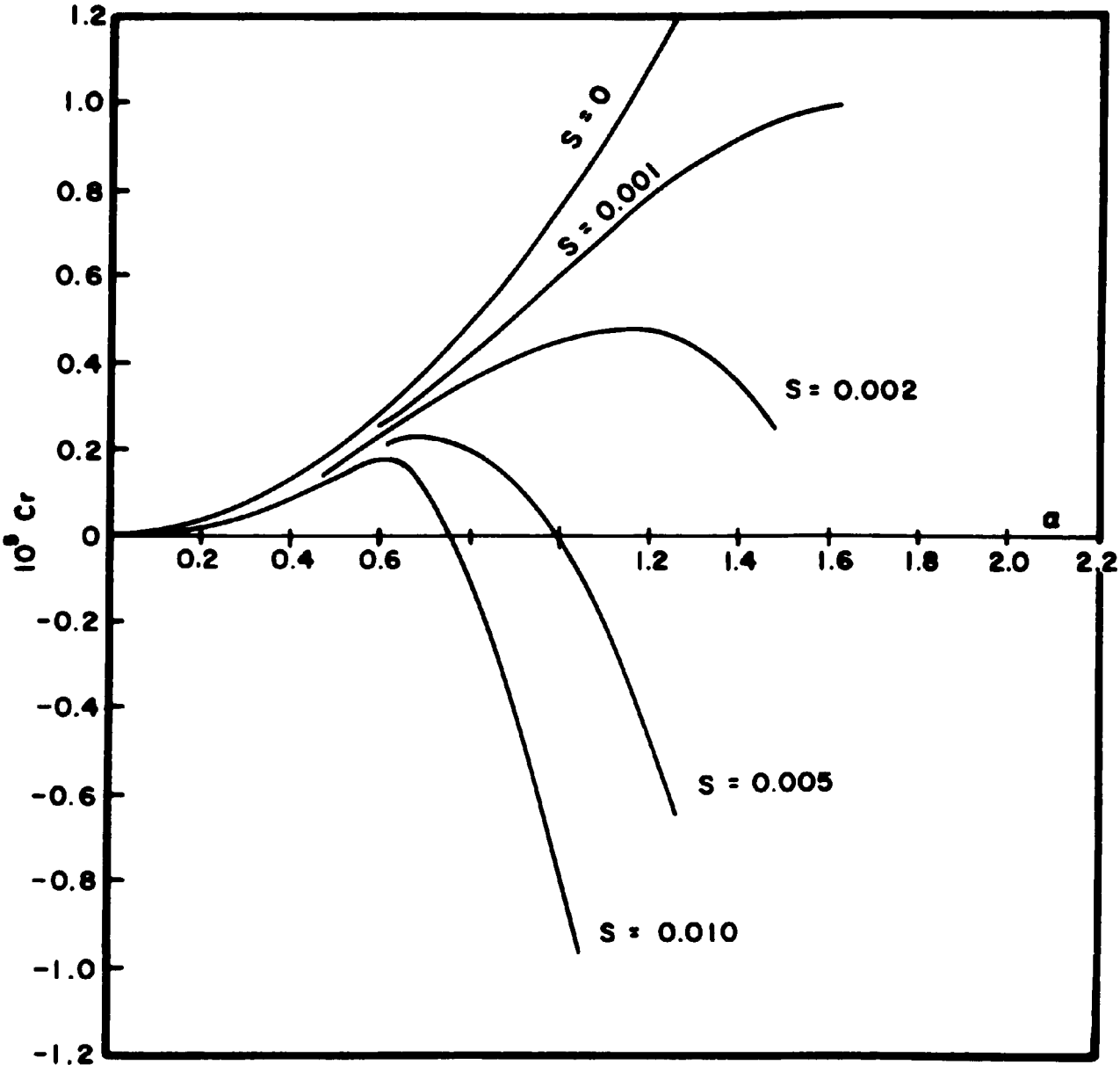


Figure 6. The Effect of Surface Tension on the Celerity of the Disturbance for the Vertical Case, $R_1 = 1$, $r = 1.2$.

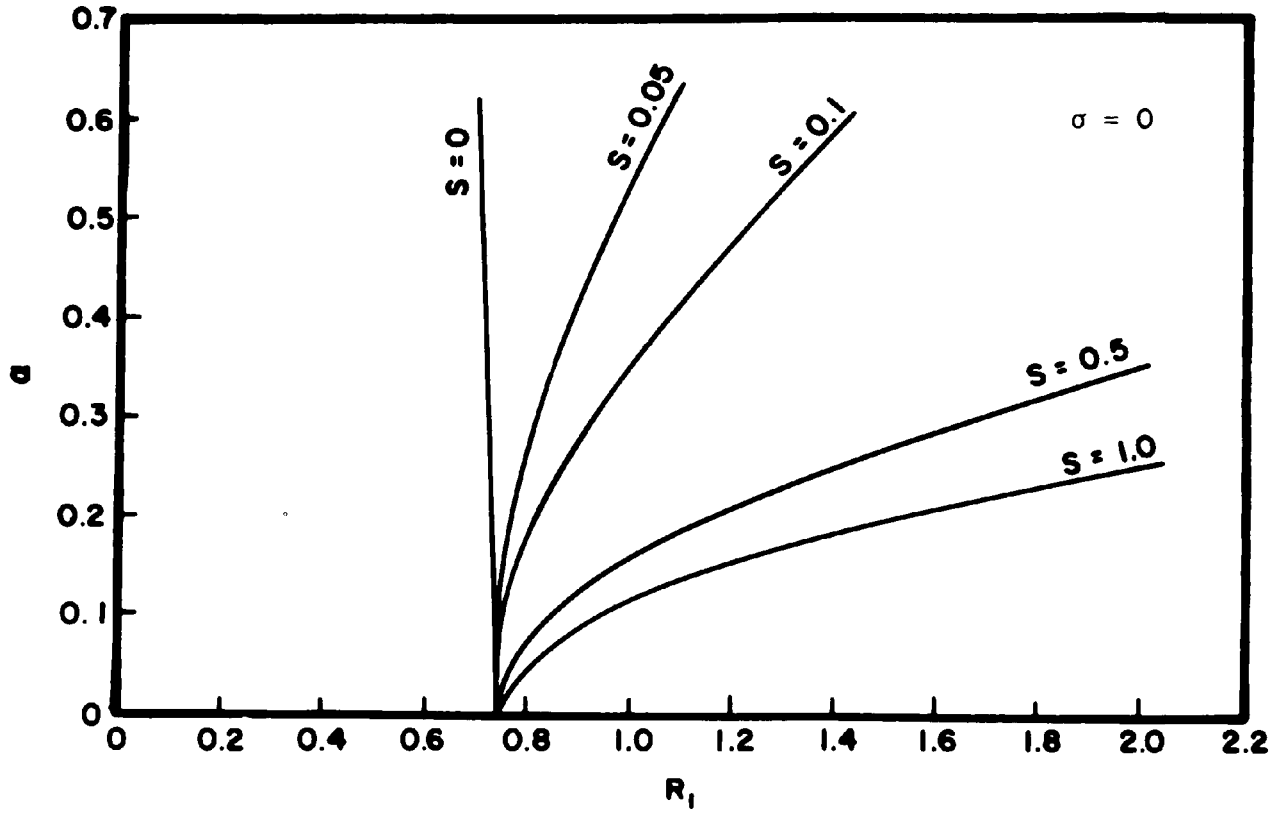


Figure 7. Neutral Stability Curves for the Case $\theta = 89.5°$, $r = 1.2$, for Various Values of the Parameter S.

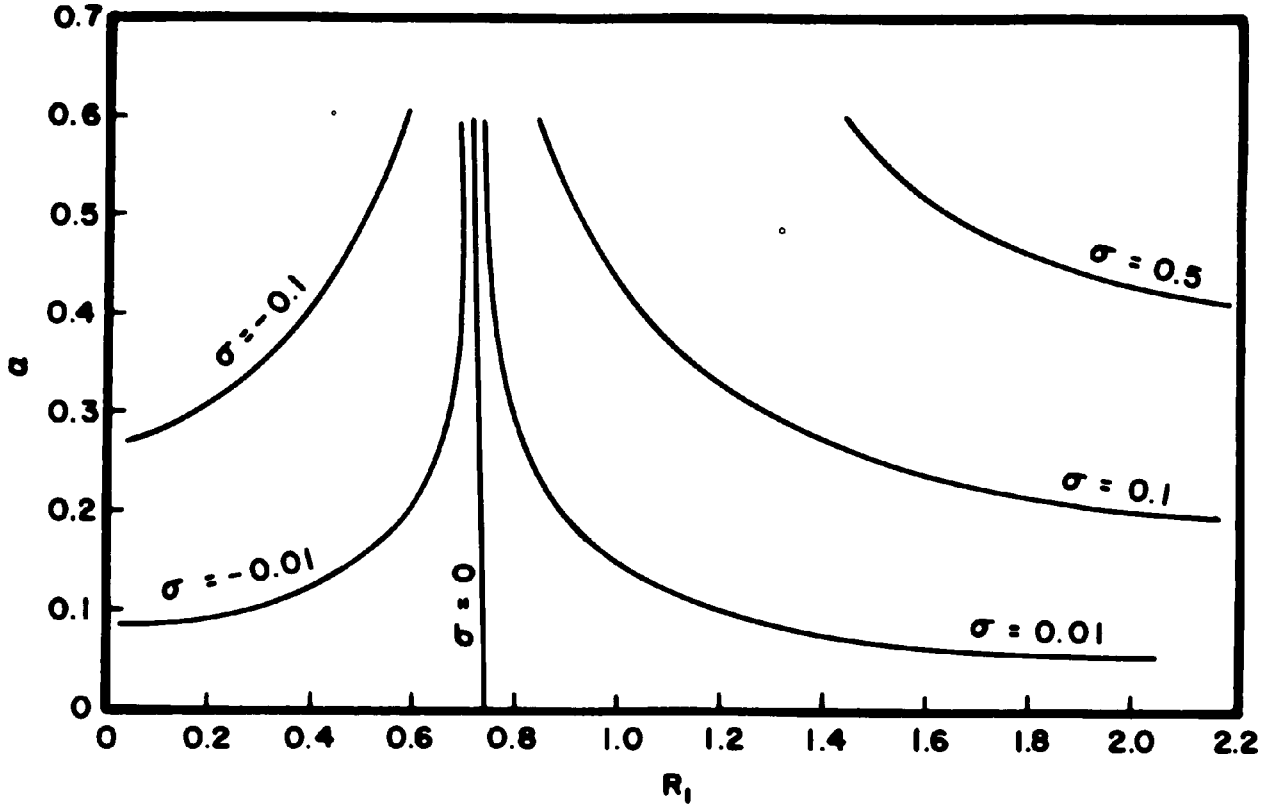


Figure 8. The Rate of Amplification of Waves of Various Wave Numbers for the Case $\theta = 89.5°$, $S = 0$, $r = 1.2$.

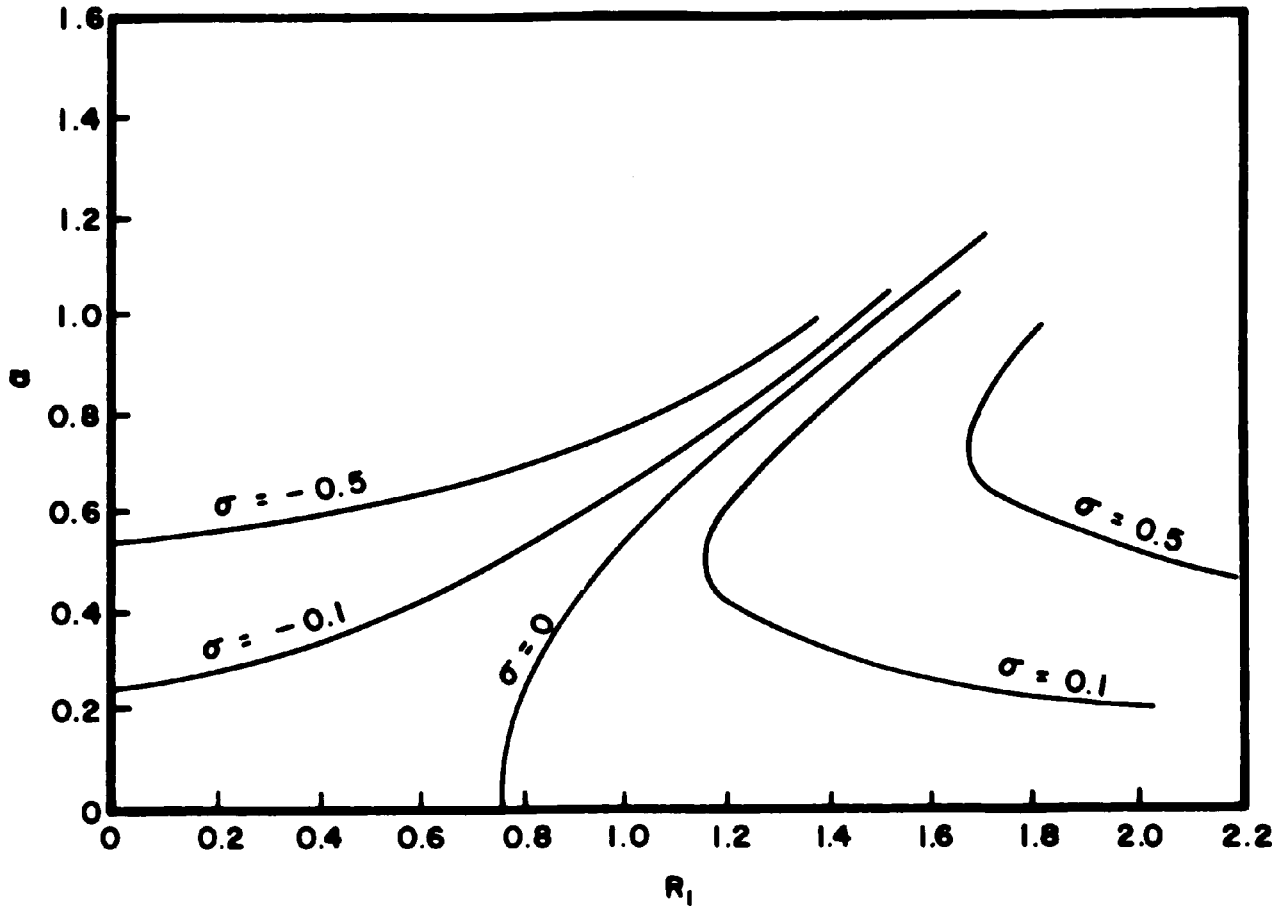


Figure 9. The Rate of Amplification of Waves of Various Wave Numbers for the Case $\theta = 89.5°$, $S = 0.05$, $r = 1.2$.

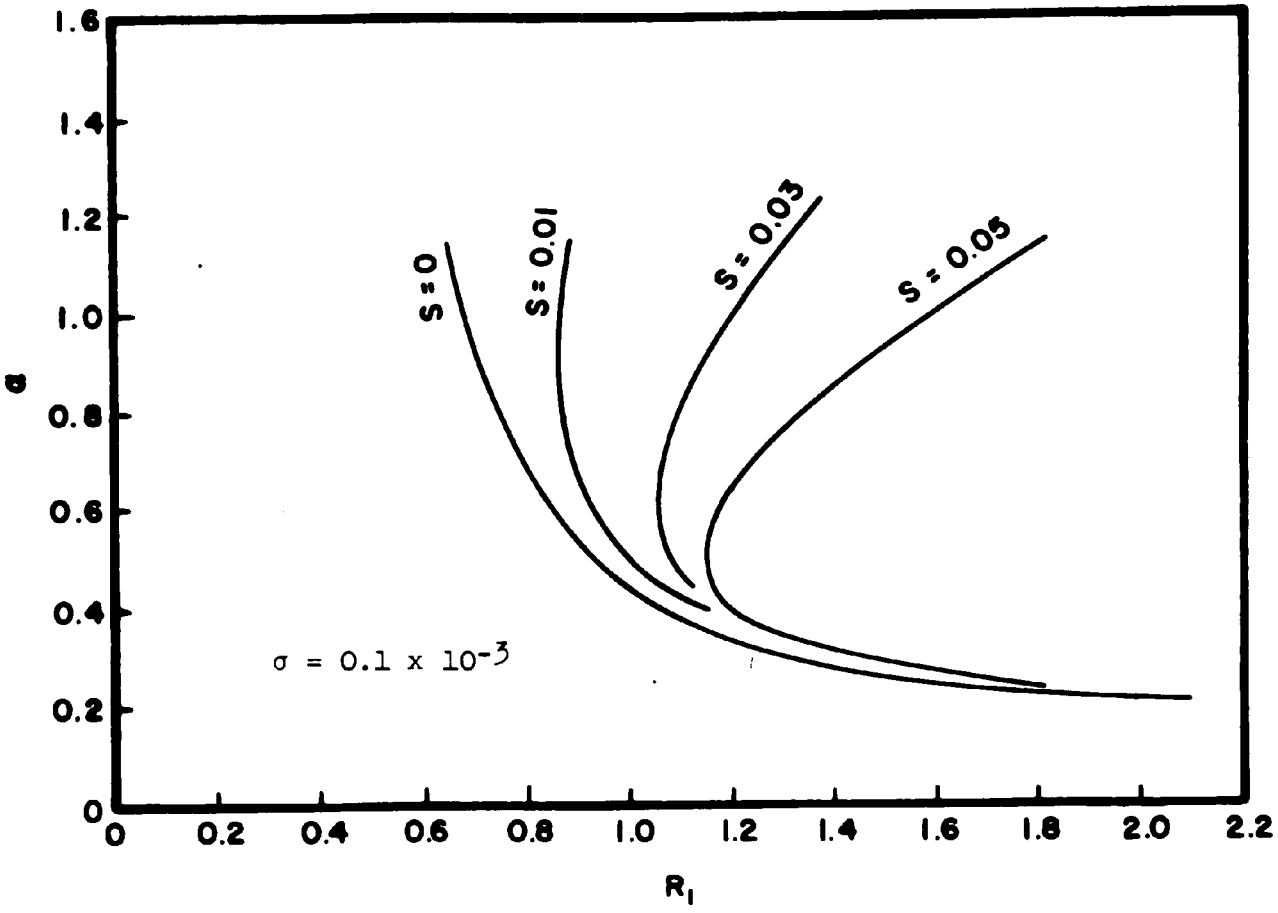


Figure 10. The Effect of Surface Tension on the Growth Rate for the Case $\sigma = 0.1 \times 10^{-3}$, $\theta = 89.5°$, $r = 1.2$.

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