

**EXTENSIONS OF THE LAW OF COMPARATIVE JUDGMENT
TO DISCRIMINABLE AND MULTIDIMENSIONAL STIMULI**

by

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CHAPTER I

INTRODUCTION

Recent American psychology has been characterized by extensive developments in measurement theory. The emergence of new models and techniques has made measurement feasible in many areas of psychology where it was previously not used. However, it is becoming more and more necessary for psychologists to examine in detail the assumptions and consequences of their measurement techniques. Any given method is somewhat limited in its generality of application, carrying with it the limitation to work with certain kinds of data, or types of responses, or classes of subjects. Hence, psychological studies have tended to become less comparable because of the great variety of methods used. What is needed is some way of regaining comparability without sacrificing the greater scope achieved with the availability of diverse methods.

A basic and important step toward establishing such comparability is to classify measurement methods according to a set of systematic criteria. Coombs (8) has presented such a system, based on the fundamental notion that the datum of any measurement process is a relation on a pair of points -- a distance measure in psychological space. Coombs' basic classification consists of three dichotomous statements regarding possible relations in data:

1. The relation is either a proximity relation ("0", stating that the members of the pair are near each other), or an order relation (">", stating that one member of the pair has more of some attribute);
2. The relation in 1 above is either on a pair of points, or on a pair of distances between pairs of points;
3. The members of a pair of points are either drawn from two different sets (a person and a stimulus, for example), or from the same set (two stimuli, for example).

When these three dichotomies occur in all combinations they produce the eightfold classification illustrated in Table 1.

TABLE 1
COOMBS' CLASSIFICATION OF MEASUREMENT RELATIONS

	Pair of points	Pair of distances between pairs of points
Members of the pair drawn from different sets	IIa IIb 0 >	Ia Ib 0 >
Members of the pair drawn from the same set	IIIa IIIb 0 >	IVa IVb 0 >

In addition to these three basic dichotomies, two further subdivisions are mentioned:

4. Measurement models may be either deterministic or probabilistic;

5. Measurement models may be unidimensional or multidimensional.

Such a classification allows one to express, in a simple manner, the differences and similarities among existing models, and to determine which methods are potentially comparable, and in what respects. Furthermore, new methods, or particular extensions of existing methods, are suggested by lacunae in the classification scheme.

It was precisely such considerations that led to the present study. Coombs' "method of similarities" (6) and Thurstone's "Law of Comparative Judgment" (LCJ) (44) both deal with order relations on points from the same set. However, they differ in every other respect; Coombs' is a deterministic multidimensional model of relations on distances between pairs, while Thurstone's is a probabilistic unidimensional model of relations on pairs of points. The methods are potentially comparable because they both deal with stimulus scaling -- they scale the distance relations of the items judged, without scaling the judges. (This is a characteristic of all models in quadrants III and IV of Coombs' classification.) Yet, they are not directly comparable because the method of similarities is applicable to relatively discriminable stimuli varying in one or more dimensions, while the LCJ is restricted to relatively indiscriminable stimuli varying in one dimension.

The goals of the theoretical portion of this thesis are an extension of Thurstone's LCJ to relatively discriminable stimuli, and the development of a multidimensional analogue of the LCJ. In the empirical portion, the scaling procedures developed in this study are evaluated and compared with those proposed by Torgerson (45),

Thurstone (44), and Coombs (6), as techniques for mapping psychological color space.

The plan of the study is as follows: Chapter II contains a historical review of multidimensional stimulus scaling, which sets the stage for the present developments. Chapter III is a presentation of Thurstone's LCJ. Chapter IV presents a new theory of the way in which pair comparison judgments are made, and shows that one consequence of this theory is a practical method of scaling discriminable stimuli by the LCJ. Chapter V develops a scaling model (based upon judgments of whether stimulus pairs are "same" or "different"), which is a multi-dimensional analogue of the LCJ. Chapter VI is an empirical study of the relation derived in Chapter IV. Chapter VII presents a comparison of the same-different procedure suggested in Chapter V, with the scaling methods of Torgerson, Thurstone, and Coombs, in a one-dimensional example. Chapter VIII describes a two-dimensional application of the same-different technique. Chapter IX concludes with a discussion of the results and some conjectures regarding further implications of these developments.

CHAPTER II

A REVIEW OF MULTIDIMENSIONAL STIMULUS SCALING

Methods of multidimensional stimulus scaling must have two special characteristics that distinguish them from the more conventional unidimensional scaling techniques. Firstly, the operations carried out by the experimenter and by the subject must be such that they impose no irrelevant constraints upon either the number or the character of the dimensions underlying the psychological space of the stimuli in question. Secondly, the method of analyzing the data must be one that allows the determination of the configuration of the space -- the number of dimensions and the positions of the stimuli on these dimensions.

There seem to be three distinguishable stages in the development of such methods. The first stage, historically, was the informal or intuitive development of multidimensional representations of sensory domains, particularly color and odor. This stage witnessed the introduction of most of the specific techniques of collecting data that have been utilized in later, more formal, methods. The second stage, which still represents the main stream of development today, involves the determination of a metric Euclidean space from symmetrical interpoint distances by a procedure due to Young and Householder (49). The third stage involves a nonmetric approach to determining the dimensions of a Euclidean space. This whole area of measurement is still evolving

rapidly and many issues and problems remain; one section of this review is devoted to these issues. The last section discusses where the present study fits into the classification presented.

Developments to 1930 -- Intuitive Methods

According to Boring (4) the color solid (color pyramid) was developed at least as early as 1772, when Lambert suggested a pyramid with white at its apex and the colors around the base. By 1810 an artist (Runge) had described a color solid very much like the modern double pyramid structure. These color solids, and others developed since, were not, apparently, the product of any formal multidimensional analysis. Rather, the relevant attributes, or dimensions, were determined by assuming that the psychological dimensions were directly related to simple physical variables or physiological modalities, or by intuitive and introspective evidence regarding the nature of the sensory experience. Detailed spacing of the stimuli along dimensions so determined has often been carried out by conventional unidimensional techniques.

Historically, the next developments came in the content area of odor. It seems that there is a positive correlation between the complexity or difficulty of a problem and the sophistication of the methods used to attack it. Color -- as a problem in multidimensional scaling -- was, or at least apparently was, too simple. The dimensions seemed to be obvious, and therefore the detailed scaling of the space could be carried out one dimension at a time with existing methods. Odor, on the other hand, presented a much more difficult problem. There were no known stimulus properties to which one might look for clues as to the dimensions involved. No general agreement has even

yet been reached as to the fundamental attributes or dimensions of odor (23).

Therefore, the studies of Henning in 1915 (20) on odor were not prejudiced by specific preconceptions of the nature of odor space. In order to develop an odor solid he used several multidimensional methods of collecting data -- that is, methods that placed no constraint upon the number and nature of the psychological dimensions. Apparently Henning invented these methods, and therefore deserves credit for taking the first steps toward developing multidimensional scaling methods.

One of his methods consisted of determining the chemical similarity of odors; this method is, of course, not a psychological one unless the special assumption is made that psychological dimensions and chemical dimensions are simply related. Two other methods of his were genuinely psychological, however, both involving determination of the relative similarity of odors, one by direct judgment, and the other by inference. The way in which Henning used these methods indicates that he was aware of the fact that distances in a multidimensional space must be symmetrical, or absolute, distances. In one of these methods the subject was given a large number of odors and instructed to smell a particular one. The subject's task was then to pick out the odor most similar to the first one, next to find a third odor which was most similar to the second, and so on, through long chains of odors. Data obtained in this manner thus told which stimuli were closest to which other stimuli. In the other method the subject's task was to assign a name to each odor as it was presented (that is, to identify the stimulus). Henning analyzed these data in terms of how often a

given stimulus was confused with each other stimulus, the inference being that a stimulus would most often be confused with stimuli that were close to it in psychological space. Unfortunately, Henning did not develop any formal method of analyzing the data he had thus collected on the similarity of stimuli. His famous "hollow triangular prism" of odors was apparently arrived at by an intuitive process taking into account all that he knew about odor. We do know that he considered and rejected other possible spatial representations of odors, but always within a 3-dimensional Euclidean framework.

Henning (20) also described a taste solid in the same book. This he represented as an equilateral tetrahedron with sweet, sour, bitter, and saline at its four corners. In both taste and odor the assumption was made that simple stimulus characteristics, or dimensions, would necessarily correspond to simple psychological dimensions. This is a particular illustration of a constraint of the nature of the dimensions of psychological space.

Odor psychology continued to be productive of multidimensional methods. Several American psychologists, in devising empirical tests of Henning's assertion about the prism, developed other methods. Most important is Macdonald's "An experimental study of Henning's system of olfactory qualities" (27) in which he reported two new methods which he presumably developed himself. The first of these was an incomplete method of triads.¹ Three stimuli were presented at a time, with two of the stimuli always being representatives of corners of Henning's

¹Actually, it is a partial use of what Torgerson (45) calls the complete method of triads and what Coombs (6) calls the method of propellers.

prism. The subject judged which of the two corners the third stimulus most nearly resembled. The second method was a graphical one. Bottles containing odors representing the four corners from a square surface of the prism were placed at the corners of a 15-inch square divided up into 3-inch squares. The subject placed the test odor at its proper relative position on the square. Although only odors which Henning had predicted would lie on this face were used, the subjects were permitted to reject any stimulus if they believed it did not belong. Macdonald developed formal, quantitative methods of analysis for the data that he collected in this manner, but he did not carry out a multidimensional analysis. Instead, he used his methods as checks on the dimensions that Henning had proposed. Like those working earlier in the field of color, he was trapped by his "knowledge" of the dimensions of the space. Macdonald did point out that the hollow triangular prism was inadequate to explain much of the data that he collected, and he suggested that "We might have a hollow hypersolid with solid tetrahedrons as its sides. There is no reason why mental continua should occur only under Euclidean [i.e., 3-dimensional] limitations." (27, p.551)

The last methodological development of this period came in a suggestion by Beebe-Center (51), that the judgment of whether a group of stimuli are "same" or "different" places no constraint upon the dimensions of difference which may be used by the subject. Again, the context in which this suggestion was made indicates an awareness that distances in a multidimensional space are symmetrical and therefore cannot be determined by judgments which order the stimuli along some dimension.

The Metric Euclidean Analysis of Interpoint Distances

Around 1930 two developments in psychological measurement were made which, while not directly concerned with multidimensional stimulus scaling, were to prove important in stimulating further developments in that area. The first of these was Thurstone's 1927 development of the Law of Comparative Judgment (LCJ) (44), which provided an explicit rationale for unidimensional stimulus scaling without regard for any underlying physical dimension. The second was the development of factor analytic techniques for determining the configuration of a set of points in a multidimensional Euclidean space, given the scalar products of vectors from the origin to each pair of points.

In 1937 Harold Gulliksen and M. W. Richardson interested two mathematicians in the problem of determining the configuration of a set of points whose mutual distances are known. Young and Householder (49) solved the basic problem and published a fundamental paper that provides the basis for a whole series of studies in this area.

Young and Householder's very elegant paper shows how: 1. to determine if a set of numbers, which allegedly represent interpoint distances, can be the mutual distances of a set of real points in a Euclidean space; 2. to find the least number of dimensions in which the points can be imbedded; and 3. to determine the configuration of the points in the space -- that is, their projections on the orthogonal Euclidean reference axes. These steps are carried out by forming a square matrix, B_1 , whose elements, b_{jk} , may be considered to be the scalar product of vectors from point 1 to points j and k . (The

elements b_{jk} may be expressed entirely in terms of the interpoint distances d_{ij} , d_{ik} , d_{jk} , by the cosine law.) Then, 1. If the matrix is positive semi-definite (that is, has only non-negative latent roots), the set of numbers may be considered to be distances between a set of points in Euclidean space; 2. The rank of the matrix is the dimensionality of the Euclidean space; 3. The matrix may be factored to obtain a matrix, A_1 , such that $B_1 = A_1 A_1'$. A_1 is an $(n-1) \times r$ (where r is the rank of B_1 and n is the number of stimuli) matrix of projections of points on r orthogonal axes with origin at the i^{th} point.

With these problems solved, Richardson (36) proceeded to apply the technique to judgments about a two-dimensional plane of psychological color space. Since Richardson published only an abstract, it is not clear how he dealt with some of the details of the scaling process. He used the triadic method, which requires the subject to say which pair is most alike and which pair is least alike in each set of three stimuli. These judgments may be regarded as three pair comparisons, where the elements compared are the distances between stimuli. Thurstone's LCJ. Case V (44), may then be applied to determine the psychological scale values of the distances between stimuli, measured on an interval scale.

A second theoretical paper by Young and Householder (50) makes three specific suggestions regarding the scaling procedures used to arrive at the interpoint distances: 1. They suggest that a complete method of pair comparisons of distances be used, rather than the incomplete triadic method proposed by Richardson. Richardson's method provides comparisons only for distances having one terminus in common. 2. They point out that the zero point necessary to convert the interval

scale of distances to a ratio scale can be determined by actually including zero distances (between a stimulus and itself) in the set to be compared. The LCJ, which Richardson used, provides only an interval scale, but his abstract did not indicate how the ratio scale values were determined. 3. They point out that the LCJ assumption of normal distribution of the discriminial process is necessarily violated when absolute distances are the stimuli to be judged, since these are never negative. They suggest the $\sqrt{}$ function as a possible alternative to the normal. (In Appendix 1 of the present study it is proved that the distribution is actually the square root of non-central chi-square, if the stimuli are assumed to be multivariate normal.)

A whole series of studies, beginning with Klingberg (24) in 1941, has added new methods of collecting data, developed new procedures for determining the interpoint distances from the data, and suggested improvements in the technique for determining the configuration of the space. All of these studies follow the same pattern. 1. Judgments are obtained which can be regarded as determined by a symmetric distance between stimulus pairs. 2. The judgments are converted to interpoint distances measured on a ratio scale. and 3. The Young and Householder procedure is applied to determine the configuration of the space. These studies will not be discussed in detail here, but the most important developments will be indicated. Messick (31) summarizes most of these papers and provides a comparison of the various procedures and the relative advantages and disadvantages of each.

Klingberg (24) was the first to publish a complete description of a multidimensional analysis. He devised two new methods of

collecting data. One involved application of the method of rank order to distances. In the other method he obtained ratings of the probability of war between pairs of sovereign states. For purposes of multidimensional analysis, he identified small probability as implying short distance and large probability as implying large distance. This is a special case of a number of procedures that might be called "restricted similarity" scaling. Presumably, such an analysis could be carried out using "trade distance," "similarity of cultural patterns," or any other subpart of similarity in general. Klingberg also devised a method for estimating the additive constant necessary to convert the interval scale of distances, obtained from the LCI scaling of similarity judgments, to a ratio scale. The method that he used restricted the analysis to three dimensions, and has not been used further.

Torgerson (45, 46) developed a better estimation procedure for determining the additive constant, based on the assumption that some set of three or more stimuli probably lie on, or near, a straight line. However, his estimate is not an unbiased one. Messick and Abelson (32) described an iterative procedure for determining the additive constant that minimizes the dimensionality of the Euclidean space. Rowan (38) has criticized this procedure because some additive constant can always be found which will imbed any set of numbers in a Euclidean space. This negates the possibility of testing whether the space is Euclidean. In reply, Messick (31) pointed out that it is still possible to determine if the points can be represented in a Euclidean space of dimensionality significantly smaller than the number of points.

Torgerson (45) also improved upon the matrix procedure suggested by Young and Householder. The method that they suggested would yield the same results no matter what point might be chosen as origin for infallible data; however, in the case of fallible data the configuration would differ slightly for each choice of origin. Torgerson's solution is to compute the scalar products with the centroid of the points as origin, thereby achieving a unique solution.

Torgerson (45), Messick (29), Gulliksen (16), Wilson (48), and Shepard (39,40) have described several new methods of collecting data on symmetric distances. Torgerson described the complete method of triads which allows independent judgment of each pair of distances in a triad. Messick extended the method of successive intervals to judgments of distances. Gulliksen described a graphic method that provides enough information for multidimensional scaling from a single series of judgments by one subject. Wilson and Shepard have dealt with the frequencies with which responses relevant to one stimulus are given for other stimuli and have converted these "confusion matrices" to distances between stimuli in such a way as to avoid the additive constant problem. Rowan (38) and Wilson (47) have proposed analogous distance transformations for data collected by the method of triads.

Non-metric Multidimensional Stimulus Scaling

A completely different approach to multidimensional stimulus scaling has been developed by Hays (18). His method proceeds from rank-orders of similarity of pairs of points, to a configuration consisting of orthogonal Euclidean dimensions on which the orders of

projections of stimuli are determined. Hays' method is an extension of Coombs' unidimensional analysis of similarity data (6), and an analogue of Hays' and Bennett's non-metric multidimensional analysis of preference data (3, 17).

Hays' non-metric method of analysis, since it requires only rank orders of similarity (or distance), can be applied to any data that are suitable for the Young and Householder procedure. His method has the added advantage of being applicable to some data which are not complete enough for the Young and Householder procedure. For example, it becomes possible to analyze interpoint distances expressed on an interval scale (it is not necessary to determine the additive constant); or, one can analyze data from a single administration of the method of triads to a single subject. Of course, the lower level of information used is paralleled by the correspondingly lower level of results obtained. No rotational procedures are available, and no metric information is obtained regarding the stimulus projections. Coombs (9) has described an application of this technique to Rothkopf's (37) same-different data on Morse Code signals.

A Review of Major Issues in Multidimensional Stimulus Scaling

The requirement that distances be symmetrical

In spite of general agreement that only symmetrical distances between points are suitable for representation in multidimensional space (see, for example, Galanter [12, p. 18] , or Householder and Landahl [21, p. 75]), Gulliksen (15) has explicitly proposed that asymmetrical distances, determined by the application of the LCJ to judgments of "greater than," be analyzed in a multidimensional space.

A detailed analysis of this suggestion will be made here, because one of the central contentions of the present study is that several explicit modifications in the LCJ are necessary to make it applicable to the multidimensional case.

Gulliksen first suggests that data collected by the method of pair comparisons be converted to interpoint distances, by the LCJ. He points out that if these distances are between points on a one-dimensional continuum, then the three distances between pairs in each triple are such that two of them add up to the third. A test can be made to see if the distances satisfy this requirement. If this criterion is not satisfied, Gulliksen says, ". . . there are two alternative procedures open. One is to seek some distribution law, other than the normal curve, such that when it is used the internal consistency check will be satisfied. The other is to assume that one is not dealing with a one-dimensional system and to see if the set of judgments will be 'consistent' if one adopts procedures proper for a space of dimensionality greater than one." (15, p. 207). He then proceeds to an illustration of the method, using hypothetical data in which the judgments are pair comparisons of the relative seriousness of six criminal offenses, arrayed in two dimensions. The analysis is carried out in six steps. 1. Ask "Which offense is more serious?" for each possible pair of six offenses. Determine the percentage of judgments "i more serious than j." 2. Convert the obtained percentages into distances by some appropriate procedure. 3. Observe that the internal consistency check for one dimension is violated. 4. Apply the Young and Householder procedure to the interpoint distances to determine the dimensionality of the space. 5. Determine the configuration of the space.

In a plane, each point is specified by two numbers. 6. "From the six double coordinates and the appropriate operation of vector subtraction we can calculate each of the 15 experimental distances, or predict the percentage of persons who will judge one offense more serious than the other." (15, p. 209). As examples of applications of this method he says:

It has been applied by Richardson (13) [36] to a set of colors differing in gray-value and in saturation. A two-dimensional continuum was found adequate in this case. Klingberg (11) [24] applied the method to a study of the likelihood of war between various nations. [Footnote :] In these studies of Richardson and Klingberg a slight variant of the method of paired [pair] comparisons called the 'method of triads' was used. Since the bases of the two methods are similar no detailed reconsideration is necessary. The method of triads uses paired comparisons of distances between objects, rather than of objects. Otherwise the considerations applying to one method also apply to the other. (15, p. 209)

There seem to be three specific mistakes or misinterpretations contained in this description. 1. The judgment of the subject, that one stimulus has more of some psychological attribute than some other stimulus, necessarily implies an ordering of the stimuli on a single continuum. Householder and Landahl made this point in 1945 (21). 2. The distance derived from a judgment of this sort is a directed, or algebraic, distance; such a distance cannot be represented by an absolute, undirected, distance in multidimensional space. This consequence is most noticeable here in that it makes impossible the alleged prediction of percentages from coordinates described in step 6, above. If the distribution assumption of the method asserts that judgments divided 60-40 imply a certain distance between stimuli, then we can infer backward that the existence of such a distance in the

space implies or predicts a 60-40 division of the judgments. But, there is no way to predict the direction of the 60-40 split from an undirected distance. 3. The methods used by Richardson and Klingberg are fundamentally different from judgments of the type: "Which offense is more serious?" The only similarity lies in the use of the LCJ to convert inconsistency of judgment into differences in scale values.

In summary, it would seem that, contrary to Gulliksen's suggestions, it is never proper to carry out a multidimensional analysis with distances derived by the usual LCJ procedure applied to judgments of "Which stimulus has more of some attribute?"

Stevens (41), in a paragraph which contained a reference to Gulliksen's paper at the end, suggested that a multidimensional analysis could be based upon incompatible rank orders from different subjects. Such analysis might be feasible,¹ but Gulliksen dealt only with group data and never suggested the separation of individual rank orders.

Euclidean and non-Euclidean metric spaces

A metric space must have mutual distances which satisfy four criteria: 1. the distances must be representable by non-negative, real

¹This situation, with regard to multidimensional analysis of order data, is complicated by Coombs' observations on the "double Law of Comparative Judgment" (5), in which he points out that individual rank orders of preference for stimuli can be analyzed on an unfolded scale, rather than on a scale of preferability. This analysis yields a joint space (7) with both people and stimuli located in the configuration. Bennett (3) and Hays (17) extended this unfolding method to the case of r dimensions, thus providing a genuine multidimensional analysis of data which is conventionally analyzed on one dimension by the LCJ. Their method does not, however, involve a multidimensional analysis of distances derived from the LCJ.

numbers; 2. the distance between two points must be the same in both directions; 3. a distance must equal zero if and only if it is the distance from a point to itself; and 4. for any three points, the sum of any two of the distances must be greater than or equal to the third. A great many metric spaces exist, and at least one non-Euclidean metric space has been proposed as a possible model of psychological space. On the basis of a neural model which behaves in such a way that differences in different modalities (dimensions) are additive in producing the total difference, Householder and Landahl (21) suggest a metric in which the distance between two points is the sum of their differences in each dimension of the space. In a Euclidean space, the distance between two points is the length of the straight line joining them. In a Euclidean space, the metric distance is independent of the particular reference frame chosen to describe the space -- that is, interpoint distances are not changed by rotation or translation of the axes chosen to describe the space. In this sense, a Euclidean metric space can be described equally well by an infinite number of sets of r dimensions.¹ In the case of the additive metric, the reference frame is not arbitrary. With fixed points, the interpoint distances change markedly with rotation of the reference frame.

Attneave (1) described several experiments which he interpreted as being better represented by the additive metric than the Euclidean metric. Attneave did not carry out a multidimensional analysis; instead he assumed that the relevant psychological dimensions could

¹Dr. Richard Kao (personal communication) points out that there are several metric spaces of potential interest to psychologists, and that rotation is possible in some of these, while not in others.

be specified by the simple physical dimensions of the stimuli that he used. As Householder and Landahl point out (21, p. 77), no such restrictive assumption is necessary. An additional restriction in Attneave's experiment was that his stimuli were varied along one physical dimension at a time. Torgerson (45) points out that the conditions Attneave used maximized the likelihood of the subjects' responding to simple physical dimensions in an additive way. Attneave also incorrectly asserts that the Young and Householder procedure "begs the question" of the appropriate metric for psychological space, by assuming that it is Euclidean. As is pointed out above, the Young and Householder procedure provides a test of the adequacy of a Euclidean representation.

Leaving aside the question of the adequacy of Attneave's interpretations, there remain several important questions regarding the characteristics of multidimensional psychological space. 1. What is the appropriate metric? Attneave says that it is not Euclidean, while most of the other studies indicate that a Euclidean representation is satisfactory. 2. Can the reference frames of psychological space be rotated? 3. Is psychological space metric at all? Attneave reports that some of his data violate the triangular inequality. Menger (28) asserts that the assumption of a normal distribution of discriminial processes will lead to a non-metric space.

Complexity of the judgment required of the subject

The complexity of the task presented to the subject in the methods discussed here ranges from the dichotomous judgment of whether a pair of stimuli are "same" or "different", through the dichotomous

judgment of whether the distance between one pair of stimuli is "greater" or "less" than the distance between another pair of stimuli, all the way to the judgment of quantitative ratios of distances between three stimuli as in Gulliksen's graphical method. Some psychologists apparently do not think complex judgments present a problem (43), but Goodman (13) and Galanter (12) have argued that one should never impose a more complex task upon the subject than is necessary for obtaining the desired information. Several studies have compared the results of different methods (2, 29, 45), but none of these was explicitly designed to study the effects of complexity of the subject's task.

A related issue concerns the question of using the subject's responses as direct information about the scales being sought, versus using the inconsistency of the subject's judgments to infer the information about scale differences. The former procedure is more direct, and generally requires less time for data collection. On the other hand, it may require judgments beyond the capacity of the subject. The latter procedure involves easier judgments, but they must be repeated many times. The non-metric method of Hays' (18) provides a way out of this dilemma, since it requires much less total information for the analysis.

Distribution function of discriminial processes, and the additive constant problem

It was pointed out above that the assumption of normal distribution of the discriminial process is necessarily violated when absolute distances are the stimuli to be judged, since they can never be negative. The studies using the LCJ procedure for scaling distances

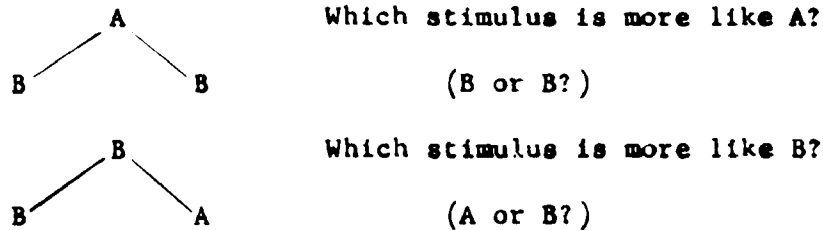
(29, 32, 45) have consistently avoided the simplest procedure for determining the additive constant -- namely, that of including experimental comparisons of zero distances. The assumption of normal distribution of discriminial processes is most seriously violated for small distances, since the absolute distances are curtailed at zero. It is suggested here that a desire to avoid the serious violation of the normal assumption led these investigators to develop very round-about methods for determining the true zero point on the scale of interpoint distance.

Torgerson has been most explicit about his reasons for omitting zero distance comparisons. He says:

It might be noted that the elements in the k th row and column [containing the proportion of times k is closer to k than to i] could be obtained experimentally. However, since the method would ordinarily be used in connection with supraliminal distances, the experimentally determined proportions would be either .00 or 1.00. As in paired comparisons, proportions of .00 or 1.00 cannot be utilized. It might also be mentioned that, in the experiments . . . it seemed advisable from the standpoint of subject rapport to stress the fact that no "joker" triads containing two or three identical stimuli would be presented. (45, p. 22)

As to his first reason, the method requires, not supraliminal distances, but differences between distances in the liminal range. These two conditions are not at all the same, and the latter condition suggests that, under some conditions at least, zero distances would not be detected 100% of the time. Torgerson, himself, found it necessary to impoverish his stimuli in order to make them partially indiscriminable. (45, p. 52) His second reason, the suggestion that triads containing two identical stimuli be excluded as unfair to the subjects, seems to impose unnecessary limitations on the data that might be collected by the

complete method of triads. There are actually two very different situations in which two stimuli are identical:



The first triad calls for an impossible discrimination, and it is clear that we might want to assure the subject ahead of time that he would not have to make such judgments. The second triad (which is the only one containing a zero distance) may actually call for a very easy discrimination. There is apparently no valid reason for not presenting such triads to subjects. Subjects in the present study (Chapter VII) did not regard such judgments as unfair; neither did they judge them with 100% accuracy, as Torgerson had suggested they might.

The Present Study

The present study develops a probabilistic multidimensional model of stimulus scaling based upon the relatively simple judgment of whether two stimuli are "same" or "different". It is analogous to the LCJ in assuming a normal distribution of the discriminial process of the stimulus elements (not the distances) in the space. It is also analogous in requiring only a dichotomous judgment on a pair of stimuli. A separate development seeks to extend probabilistic scaling techniques to relatively discriminable stimuli.

CHAPTER III

THURSTONE'S LAW OF COMPARATIVE JUDGMENT

The following two chapters present modifications of the LCJ, designed to extend its applicability to new kinds of material. Therefore, some space will be devoted here to Thurstone's original formulations of the law, in spite of its being so well known. In particular, the extensions of the next two chapters are extensions of Case V, so only that case is presented. Corresponding modifications of Cases I to IV would be relatively simple, so there is no loss of generality in limiting this treatment to Case V.

Case V, LCJ

A psychological attribute continuum in one dimension is postulated. To each physical stimulus there corresponds a normal distribution of values of the discriminial process on this continuum. (The discriminial process is the psychological process aroused when a stimulus is presented. The distribution of values occurs with multiple presentation.) The standard deviation (σ) of the discriminial process for each stimulus is the same for all stimuli. The scale value of the stimulus is defined as the mean of this distribution and is the value that one wishes to recover, or approximate, by a scaling technique. (Figure 1 illustrates the distributions of discriminial processes for two stimuli.) When two stimuli are presented to the subject, each gives rise to a single value, chosen at random from the corresponding

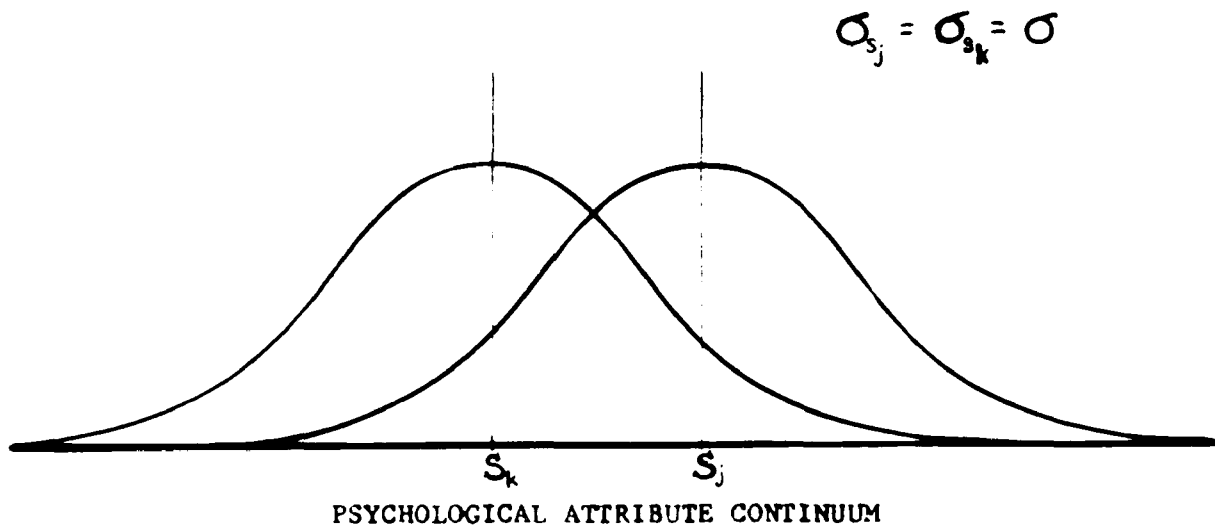


Fig. 1 - Distributions of discriminial processes for stimuli j and k .

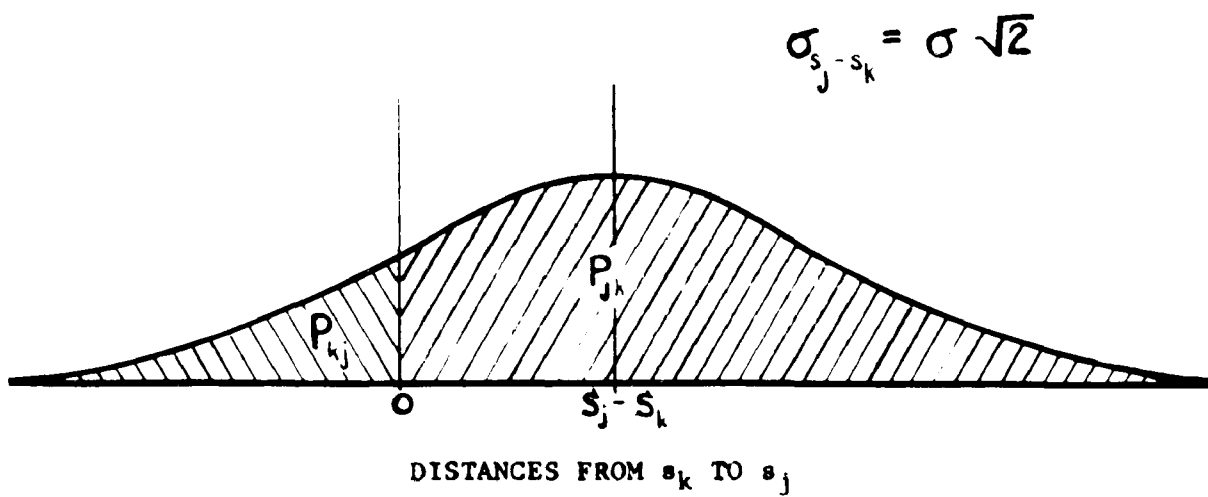


Fig. 2 - Distribution of differences in discriminial processes for pairwise random values of s_j and s_k .

distribution of discriminial processes. The stimulus corresponding to the larger sample value of the discriminial process is judged by the subject to have more of the attribute in question. This judgment is obtained on repeated independent presentations of each pair of stimuli. P_{jk} is the proportion of times stimulus j is judged "more than" stimulus k . The distances between scale values of the stimuli are inferred from these proportions: $S_j - S_k = x_{jk}$, where the unit of the scale is $\sigma\sqrt{2}$, and x_{jk} is the value of the abscissa of the normal curve, $N(0,1)$, below which P_{jk} of the area falls. (Figure 2 illustrates the distribution of differences in discriminial processes for two stimuli and the determination of their difference in scale values from P_{jk} .)

These interpoint distances are then used to determine the scale values on a psychological scale, at the level of an interval scale (with an arbitrary zero point). For n stimuli there are $(n-1)$ independent parameters required to locate the stimuli on the scale. (These may be regarded as the $n-1$ distances between adjacent stimuli, or, alternatively, as the $n-1$ distances from any one stimulus.) There are, however, $n(n-1)$ observational equations; therefore, the $(n-1)$ parameters are overdetermined. This overdetermination is customarily used to provide n estimates of the distance between each adjacent stimulus pair. The overdetermination (and the labor required of the subject) may be reduced by omitting certain comparisons.

One consequence of this method of scaling is that the distance between pairs of stimuli that are judged with perfect or nearly perfect consistency can not be directly estimated. This is so because the shape of the normal curve, assumed to underly the judgmental process, makes small differences in extreme proportions correspond to very large differences in scale

values. This means that to secure reliable estimates of large distances on this scale would require astronomical amounts of data. The usual practice is to consider that no information is obtained when the observed proportion is within, say, .10 or .05 of complete consistency. This means that the matrix of interpoint distances may be incomplete, but this is not serious because the scale values are overdetermined. The scale value for a stimulus is unobtainable only if the stimulus is relatively discriminable from every other stimulus.

In summary, the LCJ is a theory of perception which asserts that inconsistent judgments occur because the subject, in comparing stimuli, selects a single value of the discriminial process corresponding to each stimulus. The extent of the overlap of discriminial processes provides the measure of the distance between the means of the distributions. The subject is required to make a dichotomous judgment concerning the direction of difference between pairs of stimuli, and all degrees of difference are inferred from the proportion of times one stimulus is judged "more than" another.

CHAPTER IV

AN EXTENSION OF THE LAW OF COMPARATIVE JUDGMENT TO DISCRIMINABLE STIMULI

It has been commonly observed that stimuli that are relatively discriminable (seldom confused), when presented for unrestricted viewing under ideal conditions, become relatively indiscriminable when the viewing conditions are impoverished in some manner. Although this assertion is widely accepted, it cannot be predicted from the LCJ. Modification of one part of the perceptual model in Thurstone's LCJ makes this prediction possible. The modified model presented here predicts the quantitative extent of the shift in discriminability when impoverishment is carried out in tachistoscopic presentation of the stimulus pairs.

The assumption underlying this modification is that the subject, instead of picking one value from each discriminational process on which to base the judgment of "greater than," picks one value from each distribution in each small unit of time, t . Thus, if a total time T is allowed for observation of the stimuli, a sample of size $T/t = n$ will be taken from each distribution. Furthermore, it is postulated that the judgment of "greater than" will be made in terms of the mean (or median) of the accumulated values of each distribution. Now, since the standard deviation of the mean or median is an inverse linear function of the number of cases on which the mean or median is based, the standard deviation of the difference between means can be directly

manipulated by varying T . Since this standard deviation is the unit of measurement of the distance function, it is therefore possible to manipulate the distance (that is, the discriminability) between pairs of stimuli by varying the exposure time in a tachistoscopic arrangement. Quantitatively, this model predicts that the relation between distance and the square root of exposure time should be linear. From other considerations it seems reasonable to suppose that there is some maximum effectiveness of cumulated information; discriminability would not increase without limit. If we assert that the individual has a memory (storage) capacity M , then there is some T , $T_{\max} = Mt$, beyond which additional exposure will not increase the distance and discriminability. These relationships are all illustrated graphically in Figure 3.

The statement of Case V of the LCJ given in Chapter III may be modified in the following manner to include these developments (the underlined portions have been added to the original statement): . . . When two stimuli are presented to the subject for time T_{jk} , each gives rise to a single value chosen at random from the corresponding distribution of discriminial processes in each unit of time, t . The stimulus corresponding to the larger mean sample value of the discriminial process is judged by the subject to have more of the attribute in question. This judgment is obtained on repeated independent presentations of each pair of stimuli. P_{jk} is the proportion of times stimulus j is judged "more than" stimulus k . The distances between scale values of the stimuli are inferred from these proportions and times:

$$S_j - S_k = X_{jk} \sqrt{T_0 / T_{jk}} \quad \dots$$

In practice, application of this method might involve presentation of all stimulus pairs for some constant time, T_0 , say, $1/10$

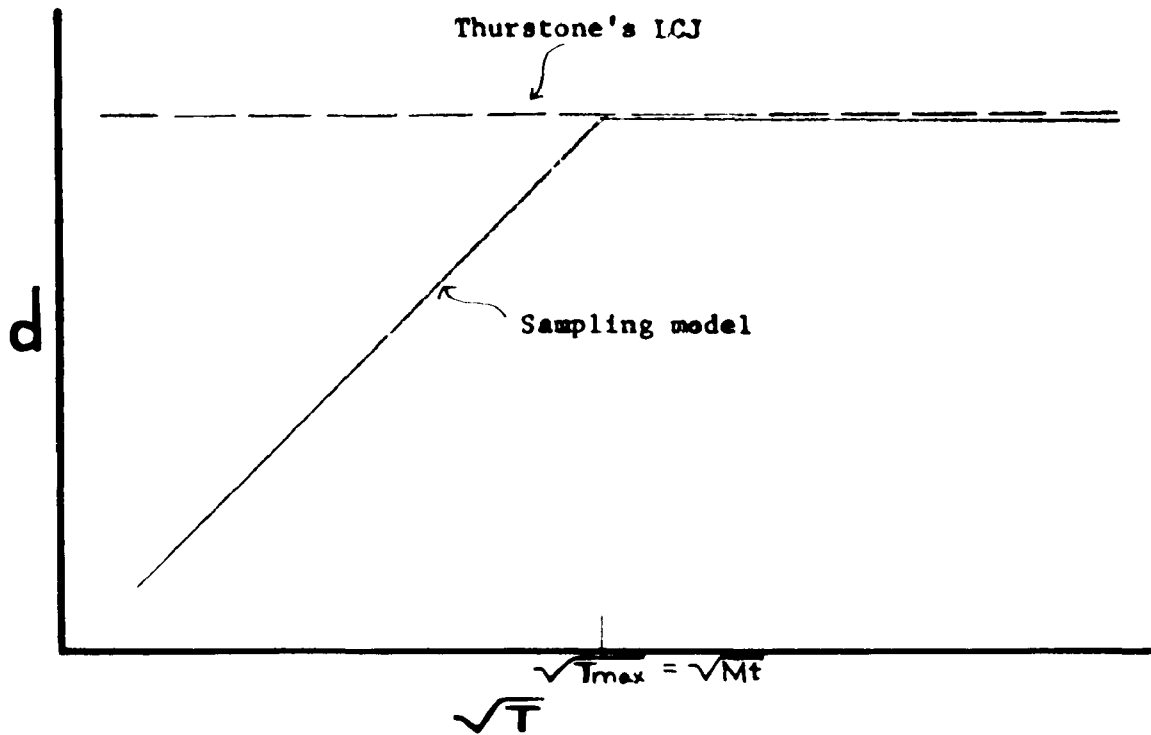


Fig. 3 - The predicted relationship between psychological distance, d , and the square root of exposure time, $\sqrt{T}$, for the two models.

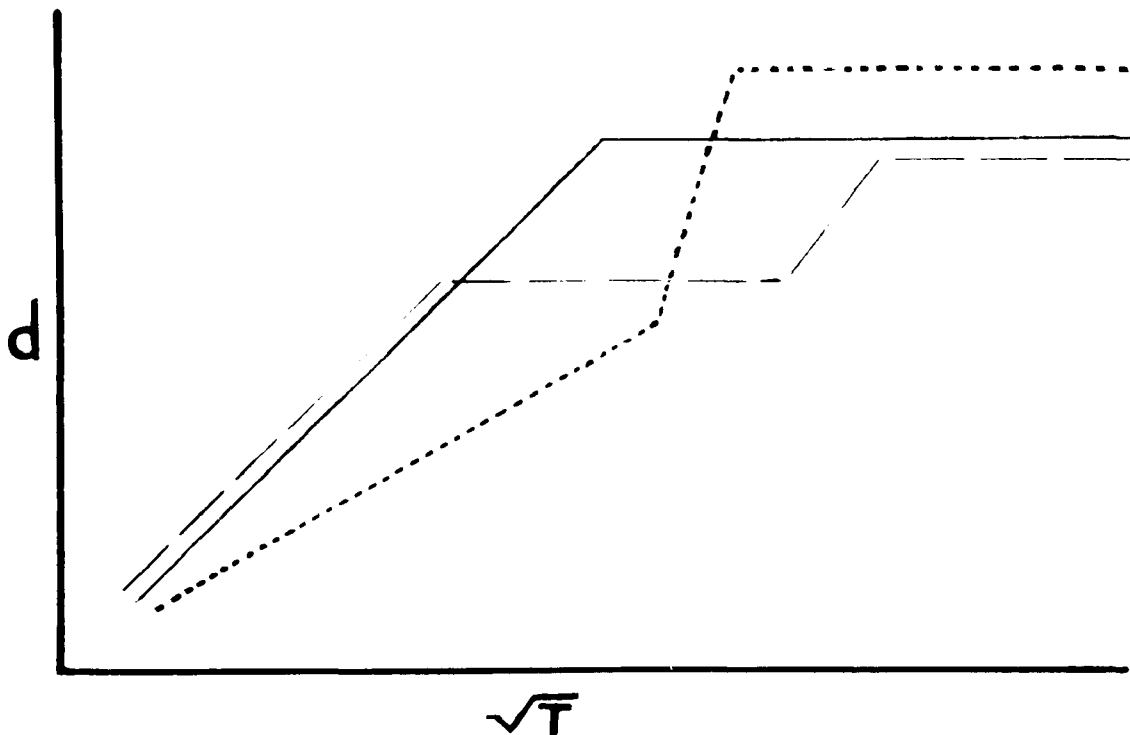


Fig. 4 - Possible departures from the predicted linear relationship between d and $\sqrt{T}$, due to qualitative changes with impoverishment.

second. Then those pairs that proved to be too discriminable at that speed could be suitably impoverished by presenting them at reduced times, T_{jk} . The revised observational equation above would then relate each observed proportion to the same unit in terms of the exposure times.

This model says that the change taking place in the perception of stimuli, as the stimuli are impoverished, is entirely quantitative -- the individual gets less information, but the information is always qualitatively the same. Suggestions have been made that with some kinds of stimuli, qualitative changes might take place. For example, if the response-time of "color receptors" in the eye is much longer than the response-time of "brightness receptors," then brightness differences might be perceived at fast speeds, and both brightness and color perceived at slower speeds. If such qualitative changes do take place, the function relating distance to the square root of exposure time would probably not be linear. The two broken curves in Figure 4 are intended to represent such situations. The dotted curve represents a situation in which the addition of another "modality" at a given exposure time causes the curve to become steeper. The dashed curve indicates what might happen if the first modality reached its maximum effectiveness before the second modality became effective at all. In this case the curve has a break in it.

The development described here is paralleled in two other studies, in different contexts. Green (14), in a paper on auditory discrimination, developed the identical relationship between exposure time and discriminability, including even the asymptote of the curve

due to the limited memory capacity. Landahl (25) considers the consequences of varying exposure time on the discrimination of a difference between two stimuli when there are random neural events at each synapse. His Figure 4 (25, p. 114) illustrates the relationship between the probability of being correct and exposure time. His predicted relationship is not exactly the same as the one described here, because of slightly different assumptions.

Further modifications of the LCJ will be presented in Chapter V, and an empirical test of the model described in this chapter will be presented in Chapter VI, with the brightness dimension of the psychological color space as the content area.

CHAPTER V

A MULTIDIMENSIONAL ANALOGUE OF THE LAW OF COMPARATIVE JUDGMENT

The present chapter is concerned with overcoming the unidimensional restriction inherent in Thurstone's scaling model. The resulting model is a multidimensional analogue of the LCJ. In terms of the classification provided in Chapter I, it is a probabilistic multidimensional model of a proximity relation on a pair of points from the same set (quadrant III b, see page 2). When this model is combined with the development of the previous chapter, it becomes possible to make the comparison, described in Chapter I, between Coombs' method of similarities and an extended form of Thurstone's LCJ.

A Multidimensional Model Based on Same-Different Judgments

A multidimensional analogue of the LCJ is presented here in a form parallelling the statement of Case V of the LCJ in Chapter III:

A psychological space of unknown dimensionality is postulated. To each physical stimulus there corresponds a multivariate normal distribution of values of the discriminial process in this space, uncorrelated over dimensions. The variability of every stimulus in every dimension is the same. The location of the stimulus is defined as the mean, or modal point, of this distribution and is the value that one wishes to recover or approximate by the scaling technique. When two stimuli are presented to the subject, each gives rise to a single point, chosen at random from the corresponding multivariate

distribution of the discriminial process. If the Euclidean distance between the two points is greater than some criterion distance, the stimuli are judged by the subject to be "different"; otherwise, they are judged to be the "same". The distance between the modal points of a pair of stimuli is inferred from the proportion of times the two stimuli are judged different in repeated presentations: $D_{jk} = \sqrt{\lambda_{jk}}$, where D_{jk} is the distance between the locations of stimuli j and k in psychological space, and λ_{jk} , (λ_{jk}), is the non-centrality parameter of the non-central chi-square distribution of distances between pairs.¹ From the interpoint distances thus determined, the Young and Householder procedure (49) can be used to determine the dimensionality of the space and the configuration of the stimuli in the space. As an alternative to this analysis, the proportions of judgments "different" can be more directly interpreted in terms of the relative distances between points (that is, the rank order of distances), in the manner described in Appendix 3, and these relative distances can be analyzed by Hays' nonmetric multidimensional unfolding technique (18). (Since the tables of non-central chi-square that would be necessary to carry out the former analysis do not exist, the latter method of analysis is the one applied to the same-different data described in Chapter VIII.)

In analogy with the LCJ, the transformation of the proportion of judgments "different" to a distance in the psychological space depends upon knowledge of the distribution function of distances between points sampled pairwise, at random, from the two distributions. It is proven in Appendix 1 that the distribution of such distances

¹This statement is proved in Appendix 1, and a routine procedure for calculating D_{jk} is given in Appendix 2.

squared is non-central chi-square with r degrees of freedom (where r is the dimensionality of the space), and with λ , the non-centrality parameter, equal to the square of the distance between the means of the multivariate normal distributions. Since this distribution is known, and since a parameter of the distribution is the square of the value we wish to recover with our scaling technique, we are now in a position to specify a procedure for determining interpoint distances from the proportion of judgments "different".

For n stimuli there are $n(n-1)$ interpoint distances to be determined; in addition r , the unknown dimensionality of the space, must be determined. However, there are only $n(n-1)$ observations available to determine these $n(n-1)+1$ unknowns. (There are also n observations of proportions for identical pairs from which the criterion distance may be determined.) Since there are more unknowns than observations, the method of analysis described here is an iterative one in which it is first assumed that r equals 1, and then the interpoint distances are determined and used to check the assumption of one dimension. If the interpoint distances so determined do not fit in a one-dimensional Euclidean space, we then assume that r equals 2, solve for the interpoint distances, and again use them to check the assumption of dimensionality. This process is repeated until the distances fit into the dimensionality assumed in determining the distances. Then the interpoint distances are used to determine the configuration of the space by the factor analytic procedure described by Young and Householder (49). A routine procedure for carrying out these steps is described in Appendix 2.

Hypothetical Example in One Dimension

The one-dimensional case is a special case which can be solved entirely in terms of the normal distribution, since the square of a normally distributed variable is chi-square with one degree of freedom. This special case will be illustrated here with hypothetical data, since it will serve to illustrate the principle upon which the judgments of same and different are made and since this is always the first step in the general solution.

We begin with the matrix of data given in Table 2. The entries are $P_{jk} = P_{kj}$ = the proportion of times stimuli j and k are judged to be different, including the case where $j = k$.

TABLE 2
PROPORTION OF TIMES STIMULI j AND k ARE JUDGED DIFFERENT
(HYPOTHETICAL DATA)

		k				
		A	B	C	D	E
j	A	.05				
	B	.23	.05			
	C	.60	.27	.05		
	D	.75	.29	.07	.05	
	E	.77	.33	.08	.052	.05

First consider the problem of determining the criterion distance, D_c . Figure 5 indicates, for the assumption of one dimension, the distribution of distances between pairs of points drawn at random from the same normal distribution. Note that the variance of this distribution is 1, since the variance of the stimulus distributions

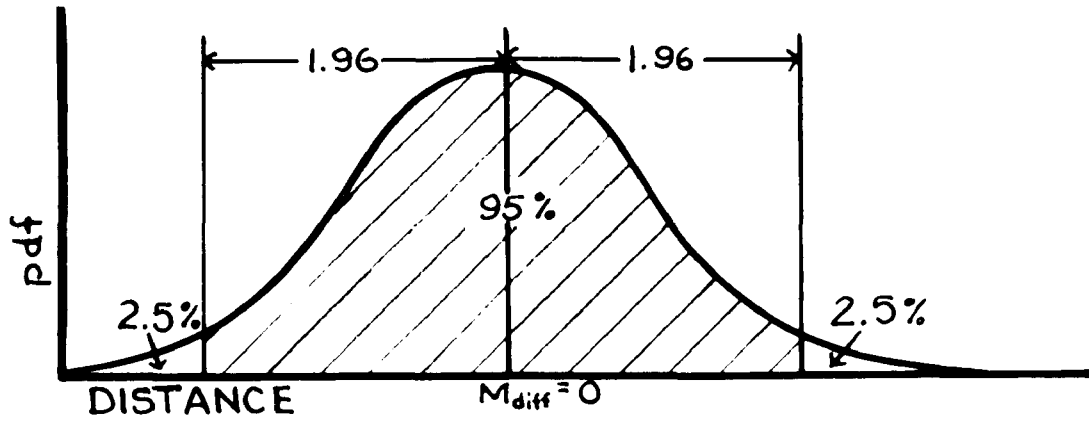


Fig. 5 - Distribution of differences between pairs from the same normal distribution.

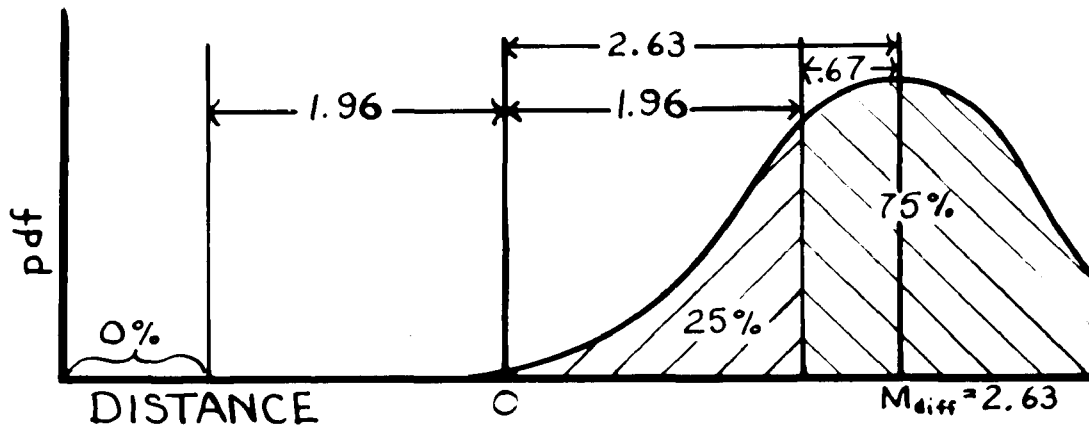


Fig. 6 - Distribution of differences between pairs from normal distributions corresponding to stimuli A and D.

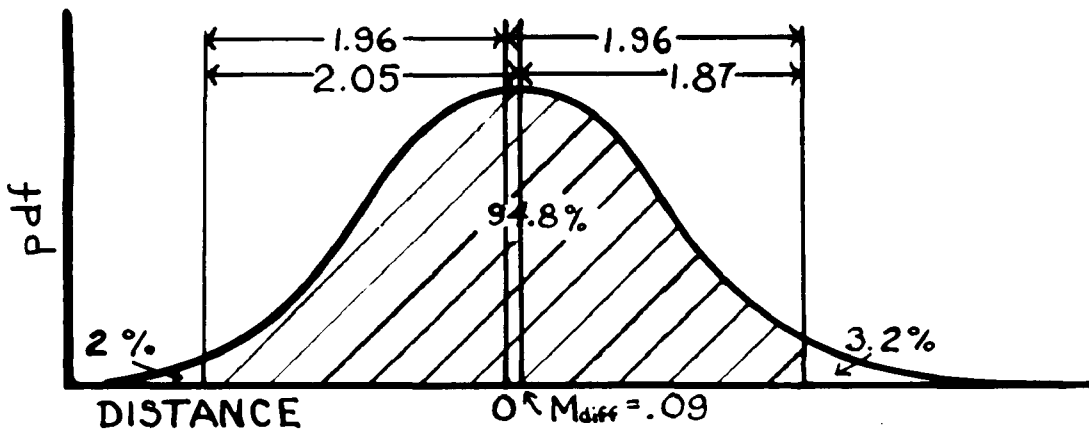


Fig. 7 - Distribution of differences between pairs from normal distributions corresponding to stimuli D and E.

has been arbitrarily set equal to $1/2$ in order to establish a convenient unit for the distance measure. Since 5% of this curve lies beyond 1.96, 1.96 must be the criterion distance, because 5% of identical pairs were judged to be different.^{1,2}

Figures 6 and 7 indicate the form of the solution of interpoint distances, given the criterion distance. Figure 6 shows the method of obtaining the distance between A and D, D_{AD} , given that the criterion distance is 1.96, and that 75% of the pairs A-D were judged different. Recall that the model states that all pairs farther apart than the criterion distance are judged "different"; all pairs closer together than the criterion distance are judged "same". The conversion of the P_{jk} matrix to the D_{jk} matrix may be stated as the problem of finding the mean of the normal curve, $N(D_{jk}, 1)$, which has $(1-P_{jk})$ of its area

¹The criterion distance is always to be determined from the proportion of times physically identical stimuli are judged "different." In these cases we know that the distance between the modal points of the corresponding distributions is zero (which means that the distribution of distances between random points is central chi-square), and therefore it is possible to determine the distance corresponding to a given proportion of "different" judgments. In general, we would expect the subject to vary the parameter of the criterion distance to suit various situations. It is presumed here that we can hold the D_c constant for a given experiment by holding constant the following parameters:

- P_s , the a priori subjective probability of the stimuli being the same;
- U_1 , the value to the subject of judging "same" when the stimuli are the same;
- U_2 , the value to the subject of judging "same" when the stimuli are different;
- U_3 , the value to the subject of judging "different" when the stimuli are different;
- U_4 , the value to the subject of judging "different" when the stimuli are the same.

²Note that in chi-square with 2 df, less than 5% of the curve would lie beyond $(1.96)^2$. Therefore, in the general procedure described in Appendix 2, the criterion distance must be redetermined for each assumed dimensionality.

in a band 1.96 on either side of zero. In practice this is found by moving an interval of width $2D_c$ along a table of the normal curve to find the segment with that width containing $(1-P_{jk})$ of the curve. Figure 7 illustrates the same procedure for determining the distance between D and E. In this case the distance between the means is small enough that a significant proportion of the normal curve with mean D_{DE} lies in both tails outside of the interval around the zero point.

Table 3 is the matrix of interpoint distances derived by this procedure. These hypothetical data have been set up so that the criteria of unidimensionality are exactly satisfied. In practice, the rank of the matrix of distances converted to scalar products is the criterion of dimensionality.¹

TABLE 3
DISTANCE BETWEEN j AND k (HYPOTHETICAL DATA)

		k				
		A	B	C	D	E
j	A	0				
	B	1.21	0			
	C	2.21	1.00	0		
	D	2.63	1.42	.42	0	
	E	2.72	1.51	.51	.09	0

¹With fallible data, the question of whether the criterion for a given dimension is approximately satisfied is precisely the same as the question of "When to stop factoring?" in conventional factor analysis. It is desired that the residuals remaining after extracting the r true dimensions should be insignificant and randomly distributed about zero.

Figure 8 shows the configuration of points satisfying these interpoint distances.



Fig. 8. - Scale values on one dimension, determined from Table 3. (Hypothetical data)

The general procedure described in Appendix 2 requires the following tables: 1. the unit normal table, for the solution when r is assumed to be 1; 2. tables of non-central chi-square with 2, 3, . . . , m degrees of freedom (where m is the maximum number of dimensions that might be of interest), for many relatively small values of the non-centrality parameter, and with each table containing many percentage points. Unfortunately, such tables are not available.¹ Patnaik (35) points out that non-central chi-square has been used almost exclusively to evaluate the power function of the chi-square test, and to develop the non-central F-distribution to evaluate the power function of the F-test. For these purposes, most interest has centered on a few percentage points of the curves (11). Patnaik also discusses various approximations to non-central chi-square. These approximations are not simple, and they are probably not accurate enough for our purposes.

Chapter VII presents a unidimensional analysis of same-different data collected on Munsell grey stimuli. Chapter VIII describes a multidimensional unfolding analysis of data collected on

¹If we assume that m equals nine, and that 30 values of the non-centrality parameter and 30 percentage points on each curve would be sufficient, then a table of 7,200 entries would be necessary. The development of such tables might be feasible if there were sufficient demand.

stimuli from the two-dimensional red plane of the Munsell system.

It should be apparent that the extension of the LCJ presented in Chapter IV is independent of the multidimensional model presented in this chapter. Each may be used separately, or they may be used in combination.

CHAPTER VI

EMPIRICAL STUDIES OF THE RELATIONSHIP BETWEEN EXPOSURE TIME AND DISCRIMINABILITY

The theoretical development of Chapter IV asserts that, at least over some part of the range, there will be a linear relationship between distance (as determined by the ICI) and the square root of exposure time of the stimulus pairs. The study reported in this chapter is a test of this assertion, using judgments of "brighter than" regarding Munsell grey papers.

Method

Two subjects, Ro and Ka, participated in the experiments described in this chapter and Chapter VII. Two different subjects, Ra and Vg, participated in the experiments described in Chapter VIII. All of the subjects were male college students: two undergraduate psychology majors, one architecture major, and one graduate history major. None had had extensive previous experience in this type of experiment. At the completion of the experiment all were paid on the basis of a standard hourly rate. The total time of participation was about 40 hours each for subjects Ro and Ka, and about 30 hours each for Ra and Vg, including practice sessions. Experimental sessions were from 1 to 2 hours long. In the 2-hour sessions a ten-minute break was taken half way through. In all sessions there were short breaks of a few minutes after each ten minutes of observation. No subject

participated in more than three sessions in a single day, and successive sessions were always separated by at least 2 hours. The author served as experimenter for all sessions. Data are presented and analyzed separately for each subject throughout the study.

The stimuli used in all cases were Munsell standard colored papers (33). 1-1/2 inch squares of these papers were cemented to 4" x 6" white cards. These 1-1/2 inch squares were presented to the subject through 1-inch circular holes in the frame at the rear of a tachistoscope; the subject was unable to see the edges of the colored papers or any part of the white cards on which they were mounted. The subjects were never permitted to see the cards outside of the tachistoscope. Questioning after completion of the experiments revealed that the subjects were not aware of the limited number of stimuli being presented. One subject, after viewing the same five Munsell grey values approximately 5,000 times each, guessed that he had been looking at "40 or 50 different stimuli."

The equipment used for presenting the stimuli was a two-channel Gerbrands tachistoscope, equipped with an interval timer so arranged that it could light the rear channel for a preset time whenever the subject pressed a telegraph key to indicate that he was ready. The stimuli were displayed in a frame at the back of the rear channel, where they could be rapidly changed by the experimenter. The side channel was normally lighted, except during the brief exposures of stimuli. Between exposures it contained only a white, diamond-shaped fixation point on a grey field located in the same visual plane as the stimuli, and centered in the visual display of the two or three stimuli. These arrangements are illustrated from the point of view

of the subjects in Figures 9 and 10.

The experiments were carried out in a small darkened room with the single window covered by a blackout shade. A curtain divided the experimenter's half of the room from the subject's half, and a single 7-watt bulb on the experimenter's side provided the illumination necessary for changing cards, setting the timer, and recording the responses.

The subjects were told that the experiment involved a comparison of several different methods of psychological measurement and that they would be required to make judgments ranging from very easy to practically impossible. They were also told that they were required to make a judgment on every presentation, even if they thought that they were only guessing, because such judgments were useful and more often correct than the subject believed them to be.

For the presentation of each stimulus set (either a pair or a triad) the experimenter placed the stimulus card under the spring clip at the rear of the tachistoscope and said "Ready". The subject then pressed a telegraph key to set the tachistoscope timer in operation whenever he was ready after the "ready" signal. In most cases the subject responded immediately, but in some cases he delayed in order to blink, adjust the eyeshade, or for some other reason. In any case, the stimuli were presented only when the subject decided he was prepared.

For the method of pair comparisons, used in this chapter and in Chapter VII, the instructions to the subject were:

Your task now is to determine which of the cards, the one on your right or the one on your left, is brighter or lighter. After you see the pair of cards call out 'right'

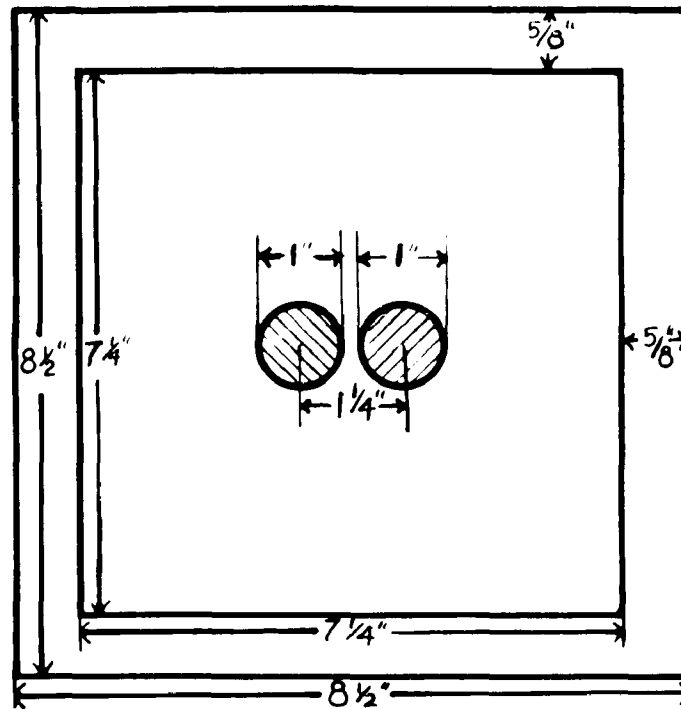


Fig. 9 - Subject's view of stimulus arrangement for all conditions involving pairs of stimuli. Viewing distance = 24 inches.

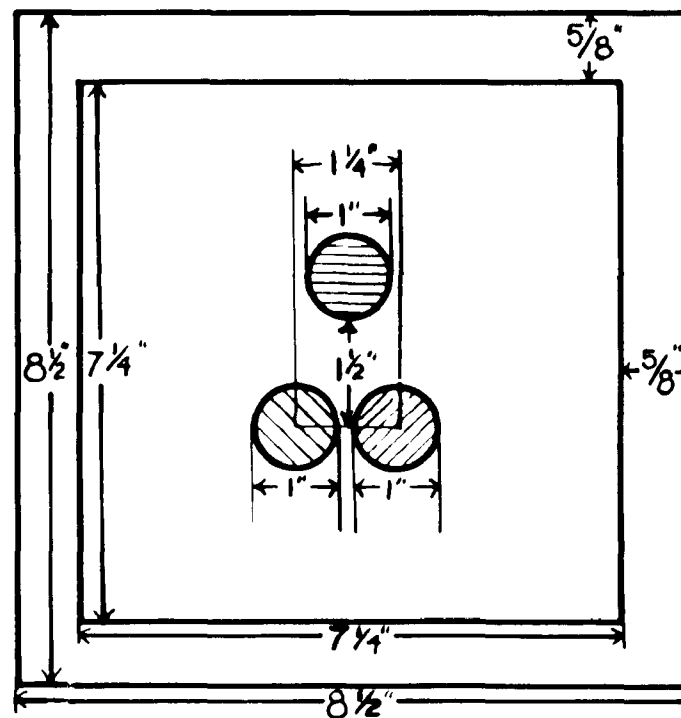


Fig. 10 - Subject's view of stimulus arrangement for all conditions involving three stimuli. Viewing distance = 24 inches.

if you think the card on the right is brighter; call out 'left' if you think the card on the left is brighter. Remember, you must respond each time, even if you think you are guessing.

Before each run of cards the subject was told: "Tell me which card is brighter, the one on your right or the one on your left."

For the experiment described in this chapter the stimuli were the Munsell grey values 6, 5, and 3.5. Six cards were made up with the three pairs and the "mirror images" of these three pairs (that is, with the stimuli reversed, right for left). These six cards were presented in a random sequence at a single timer setting for a total of 54 presentations. Then the timer was set at a different interval and another sequence of 54 presentations made. The particular timer intervals used were determined, as the experiment progressed, by the following criteria:

1. The intervals must not be so long that discriminations are made with 100% consistency;
2. The intervals must not be so short that the subject's responses show no better than chance accuracy;
3. Within this range, it is desired to have six or seven time intervals approximately equally spaced.

The intervals used were different for the two subjects and are given in Table 4 for Ro and in Table 6 for Ka.

Results

Table 4 presents the proportion of correct judgments as a function of exposure time, for the three pairs of stimuli, for subject Ro. Table 5 contains the distances determined from these proportions by Case V of the LCJ. The relation of the square root of exposure time to distance in Table 5 is presented graphically in Figure 11. The same

information for subject Ka is presented in Tables 6 and 7 and Figure 12.

The theory of the discrimination process developed in Chapter IV predicts a linear relationship between psychological distance, d , and the square root of exposure time, $\sqrt{T}$. The results clearly indicate that distance increases (discrimination becomes easier) as exposure time increases; all six best-fit-lines have positive slopes. The question of linearity is, of course, more difficult. The data points for large distances in Figures 11 and 12 are relatively unreliable in the vertical direction because they are distances determined by the "tails" of the normal curve, from probabilities in the neighborhood of .95. The data points for short time intervals are unreliable in the horizontal direction because the tachistoscope timer proved to be somewhat variable at very short time-interval settings. For subject Ro, two of the linear functions distinctly cross. In other words, Munsell greys 5 and 6 were more discriminable than greys 3.5 and 5 at brief exposure intervals, but with longer exposures the relationship was reversed. Such a reversal is contrary to the predictions of the theory. With the qualifications stated then, it may be said that these results are in close correspondence with the predicted linear function. There is no evidence in the present data for the predicted leveling off (deceleration) of the curve at an asymptote as the limit of memory capacity is approached.

TABLE 4
PROPORTION OF CORRECT JUDGMENTS "BRIGHTER" AS A
FUNCTION OF EXPOSURE TIME, SUBJECT R₀

Stimulus pair	Exposure time, in seconds						
	.0199	.0375	.0510	.0717	.0917	.1317	.1680
6,3.5	.670	.667	.804	.9009	.9700	.9455	1.000
6,5	.641	.726	.774	.769	.843	.852	.9109
5,3.5	.588	.588	.705	.752	.843	.9026	.9711

TABLE 5
INTERPOINT DISTANCES COMPUTED FROM THE PROPORTIONS
OF TABLE 4 BY CASE V OF THE LCJ, SUBJECT R₀

Exposure time, in seconds (The figures in parentheses are the square roots of the corresponding exposure times)							
Stimulus pair	.0199 (.141)	.0375 (.194)	.0510 (.226)	.0717 (.268)	.0917 (.303)	.1317 (.363)	.1680 (.410)
6,3.5	.440	.432	.856	1.287	1.880	1.602	--
6,5	.361	.600	.752	.736	1.007	1.045	1.346
5,3.5	.222	.222	.539	.681	1.007	1.295	1.897

TABLE 6

PROPORTION OF CORRECT JUDGMENTS "BRIGHTER" AS A
FUNCTION OF EXPOSURE TIME, SUBJECT Ka

Stimulus pair	Exposure time, in seconds					
	.0151	.0156	.0199	.0269	.0280	.0324
6,3.5	.506	.729	.819	.9598	.99310	.9665
6,5	.558	.667	.768	.868	.9110	.9611
5,3.5	.440	.597	.590	.664	.731	.708

TABLE 7

INTERPOINT DISTANCES COMPUTED FROM THE PROPORTIONS OF
TABLE 6 BY CASE V OF THE LCJ, SUBJECT Ka

Stimulus pair	Exposure time, in seconds (The figures in parentheses are the square roots of the corresponding exposure times)					
	.0151 (.123)	.0156 (.125)	.0199 (.141)	.0269 (.164)	.0280 (.167)	.0324 (.180)
6,3.5	.015	.610	.912	1.749	--	1.831
6,5	.146	.431	.732	1.117	1.347	1.764
5,3.5	-.151	.246	.227	.423	.616	.548

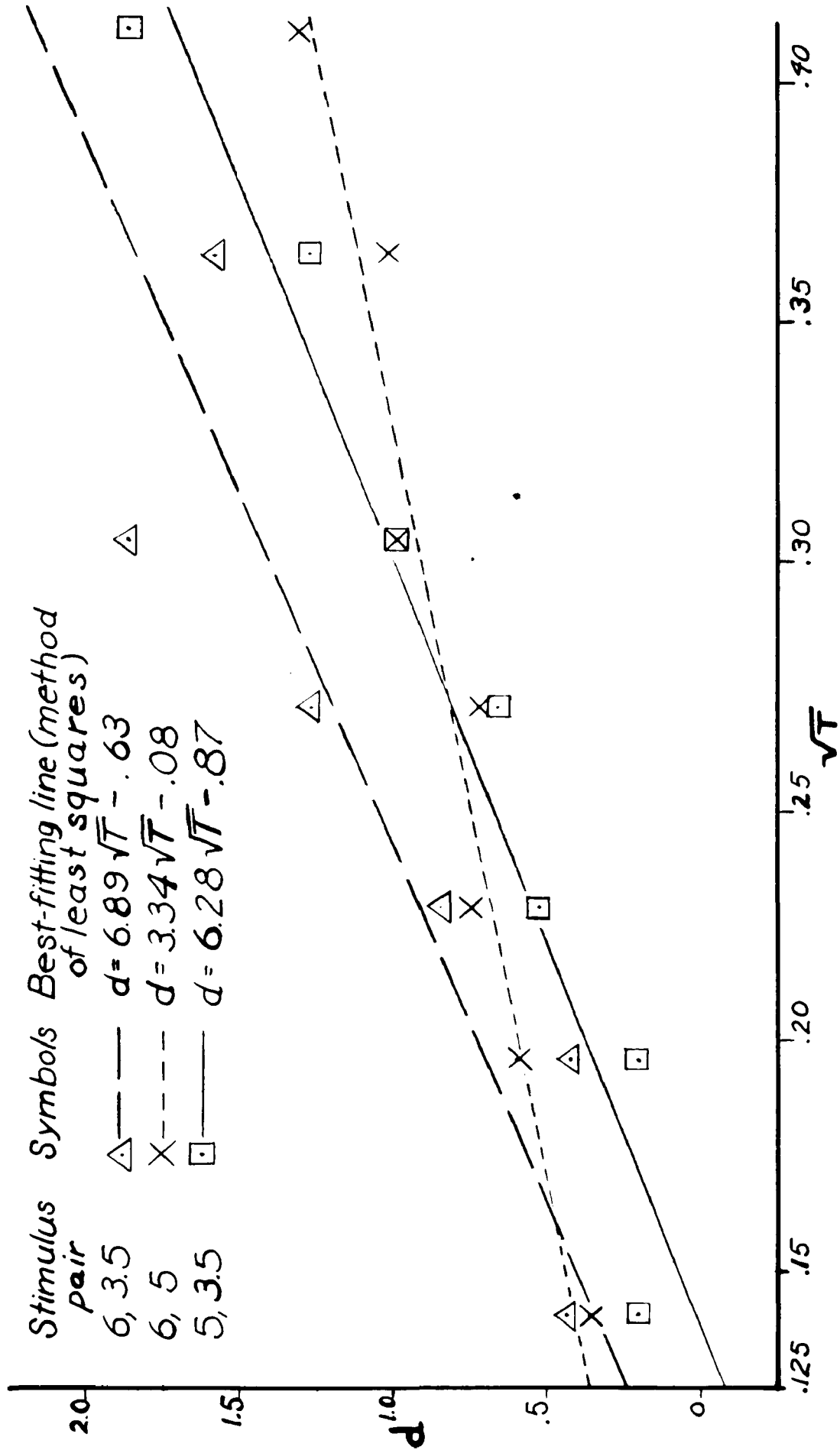


Fig. 11 - The relation between psychological distance, d , and the square root of exposure time, $\sqrt{T}$, subject Ro.

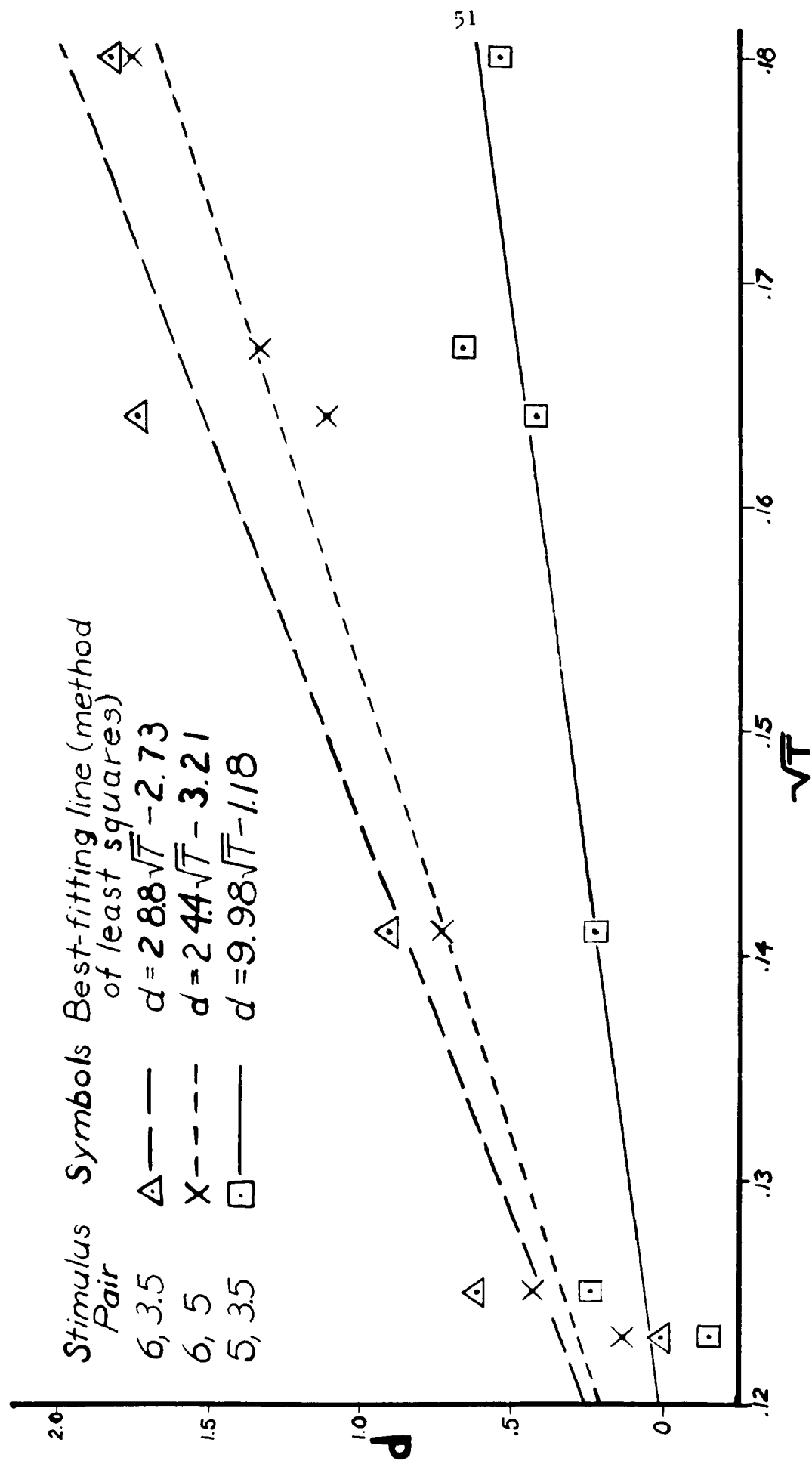


Fig. 12 - The relation between psychological distance, d , and the square root of exposure time, $\sqrt{T}$, subject Ka.

CHAPTER VII

COMPARISONS OF DIFFERENT SCALING TECHNIQUES IN ONE DIMENSION

The initial impetus to this study lay in the desire to compare the "incompatible" methods of stimulus scaling developed by Coombs and Thurstone. The developments of Chapters IV and V were designed to make this comparison possible, and this chapter is a report of experimental applications of these developments. The Munsell grey series was chosen as a content area because of its relative simplicity, and because previous work indicates that it represents a uni-dimensional psychological continuum (45). The following methods were compared as techniques for arriving at scale values for the five Munsell greys used: 1. Coombs' method of similarities, the special case of propellers (6); 2. Torgerson's complete method of triads (45); 3. The method of pair comparisons, analyzed by Thurstone's LCJ, Case V (44); 4. The same-different method described in Chapter V, analyzed by the procedure described in Appendix 2. Methods 2, 3, and 4 required that the stimuli be impoverished by being presented for brief time intervals in a tachistoscope. Data for the Coombs' analysis were independently collected, with the stimuli presented supraliminally.

Although there are some differences, the study by Beardslee (2) covered methods 1, 2, and 3 in a more intensive

manner¹, so the main interest here is centered on the comparison of those three methods with the new same-different method.

The experiments described here were carried out with the equipment and subjects described in Chapter VI. Five Munsell grey values were used: 7, 6, 5, 3.5, and 3. (7 is light and 3 is dark.) The instructions, conditions, and results for each method are given below.

Coombs' Method of Propellers

This procedure was given last in the experimental sequence for both subjects because it allowed the subjects a supraliminal view of the stimuli. The experimenter arranged the three stimuli as indicated in Figure 10, and then lighted the rear channel of the tachistoscope by operating the "manual" switch which bypasses the timer operation. The stimuli remained lighted until the subject had made his judgment. The instructions to the subject were:

Your task now is to determine whether the stimulus on your right or the stimulus on your left is more like (more similar to) the stimulus on the top. Which is more like the top, the one on the right or the one on the left? In this series I will turn the light on when the cards are ready. You will have as much time to respond as you want, and the light will not be turned off until you have responded. There is not necessarily any correct answer; what I am interested in is how the stimuli look to you. Take as long as you like; you should be fairly certain of your response before you give it. Call out 'right' or 'left' when you have reached your decision.

¹In the present study the methods of Coombs and Torgerson were applied to experimentally independent sets of data, while in the Beardslee study these two types of analysis were applied to a single set of data.

Before each run the subject was told: "Tell me which is more like the top, the one on the right or the one on the left." The cards were presented in random order, with each stimulus in turn serving as the hub of the propellor. Each propellor was replicated six times. In the very rare cases where there were inconsistencies, the judgments were replicated additional times at the end of the series.

Results -- Coombs' method of propellers

Table 8 gives the rank orders of similarity from each stimulus in turn (I scales), and the metric information derived from these ranks for subject Ro. Table 9 gives the same information for subject Ka. Figures 13 and 14 give possible configurations of points satisfying the relations indicated in Tables 8 and 9, for subjects Ro and Ka, respectively. Note that the metric information for the two subjects is incompatible; there is no single arrangement of stimuli that accounts for the judgments of both subjects.



Fig. 13 - Scale values compatible with the metric information from Coombs' method of propellers, subject Ro.



Fig. 14 - Scale values compatible with the metric information from Coombs' method of propellers, subject Ka.

TABLE 8

RANK ORDERS OF SIMILARITY FROM EACH STIMULUS (I SCALES)
AND METRIC INFORMATION CONTAINED THEREIN, SUBJECT Ro

I Scales					Metric Information Independent of Order Information
3	3.5	5	6	7	
3.5	3	5	6	7	$3-3.5 < 3.5-5$
5	6	3.5	7	3	$5-6 < 3.5-5 < 5-7 < 3-5$
6	5	7	3.5	3	$5-6 < 6-7 < 6-3.5$
7	6	5	3.5	3	

TABLE 9

RANK ORDERS OF SIMILARITY FROM EACH STIMULUS (I SCALES)
AND METRIC INFORMATION CONTAINED THEREIN, SUBJECT Ka

I Scales					Metric Information Independent of Order Information
3	3.5	5	6	7	
3.5	3	5	6	7	$3-3.5 < 3.5-5$
5	6	7	3.5	3	$5-7 < 3.5-5$
6	7	5	3.5	3	$6-7 < 5-6$
7	6	5	3.5	3	

Torgerson's Complete Method of Triads

The judgment in this procedure is the same as in Coombs' method of propellers, but the scaling is based upon the inconsistency of the judgment, rather than upon the direct interpretation of the subject's report, as in Coombs' analysis. Therefore, this method requires many replications of each triad, and relatively indiscriminable distances between pairs of stimuli. Each triad was replicated at least 80 times, and the desired degree of inconsistency was achieved by presenting the triads for the limited exposure times determined by preliminary experiments to provide satisfactory inconsistency. This time was .0510 seconds for subject Ro and .0269 seconds for subject Ka. The instructions to the subject were:

In this experiment you will see three stimuli, the usual two, plus one centered above the other two. Your task now is to determine whether the stimulus on the top is more like (more similar to) the one on the right or the one on the left. Which is more like the top, the one on the right or the one on the left? There is not necessarily any correct answer; what I am interested in is how the stimuli look to you. When I say 'Ready', press the key as soon as you are ready, and then report 'right' or 'left'. Remember that you must respond each time.

Before each individual run, the subject was told: "Tell me which is more like the top, the one on the right or the one on the left." The cards were presented in random order, with each stimulus serving as the center of the triad in turn.

Results -- Torgerson's complete method of triads

Table 10 gives the proportion of times each distance was judged greater than each other distance for subject Ro. Table 11 gives the differences between distances, determined from these proportions by Case V of the LCJ. Figure 15 gives the scale values determined from Table 11, with the additive constant determined from the average scale

TABLE 10

PROPORTION OF JUDGMENTS OF DISTANCE j LESS THAN DISTANCE k
(THAT IS, THE STIMULI IN j MORE SIMILAR THAN THE STIMULI
IN k), SUBJECT Ro

(Decimal points are omitted for convenience)

		k									
		7,6	6,5	5,3.5	3.5,3	7,5	6,3.5	5,3	7,3.5	6,3	7,3
j	7,7	488				699			9294		9647
	6,6	633	711				9166			9518	
	5,5		547	763		733		828			
	3.5,3.5			771	453		9778		9882		
	3,3				464			756		9513	9762
	7,6		547			711	893		9875	9213	9884
	6,5	453		738		659	9146	828		9036	
	5,3.5		262		256	368	750	483	9195		
	3.5,3			744			9405	821	9647	9397	9878
	7,5	289	341	632				618	9390		9268
	6,3.5	107	0854	250	0595				830	495	
	5,3		172	517	179	382				845	9286
	7,3.5	0125		0805	0353	0610	170				465
	6,3	0787	0964		0603		505	155			744
	7,3	0116			0122	0732		0714	535	256	

Note: The blank cells represent comparison of disjoint distances, data not obtainable by the method of triads.

TABLE 11

DIFFERENCES BETWEEN DISTANCES, DETERMINED FROM THE PROPORTIONS OF TABLE 10 BY CASE V OF THE LCJ, SUBJECT Ro. THE CELL ENTRIES ARE THE AMOUNT BY WHICH DISTANCE k EXCEEDS DISTANCE j

		k									
		7,6	6,5	5,3.5	3.5,3	7,5	6,3.5	5,3	7,3.5	6,3	7,3
j	7,7	-.030				.521			1.471		1.808
	6,6	.340	.555				1.383			1.663	
	5,5		.108	.714		.622		.946			
	3.5,3.5			.742	-.118		2.010		2.263		
	3,3				-.090			.694		1.658	1.980
	7,6		.108			.555	1.243		2.240	1.414	2.270
	6,5			.637		.410	1.369	.946		1.302	
	5,3.5						.690		1.402		
	3.5,3			.654			1.559	.916	1.808	1.553	2.250
	7,5			.337				.300	1.547		1.452
	6,3.5								.954		
	5,3			.043						1.015	1.465
	7,3.5										
6,3						.013				.654	
7,3								.088			

position of the 5 zero distances. Tables 12 and 13 and Figure 16 give the same information for subject Ka.

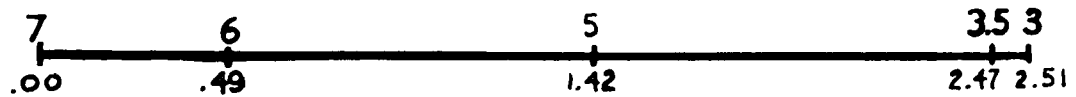


Fig. 15 - Scale values from Torgerson's complete method of triads, subject Ro.

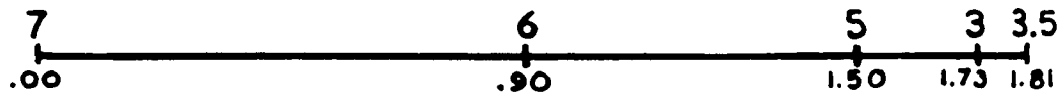


Fig. 16 - Scale values from Torgerson's complete method of triads, subject Ka.

It should be noted that the distances in Tables 11 and 13 contain persistent violations of the triangular equality required for a unidimensional scale. Consider the comparisons of the distances with zero distances (between a stimulus and itself) in Tables 11 and 13. The amount by which a distance exceeds a zero distance (obtained from the average of the two estimates in the tables) is, directly, a measure of the distance. For the two subjects, 23 of the 26 possible tests of the triangular equality show violations in the direction indicating that the space is non-metric. For example, for both subjects:

$$\overline{5-6} + \overline{6-7} < \overline{5-7}$$

$$\overline{3-3.5} + \overline{3.5-5} + \overline{5-6} + \overline{6-7} < \overline{3-7}$$

Torgerson (45) noted the same result in his study. This condition violates one of the requirements for a metric space.

The Method of Pair Comparisons, Analyzed by Case V, LCJ

The instructions for this method are given in the previous chapter. Each pair was replicated at least 100 times. To achieve the desired degree of inconsistency, the pairs were presented for the limited exposure time determined by preliminary experiments to provide

TABLE 12

PROPORTION OF JUDGMENTS OF DISTANCE j LESS THAN DISTANCE k
(THAT IS, THE STIMULI IN j MORE SIMILAR THAN THE STIMULI
IN k), SUBJECT Ka

(Decimal points are omitted for convenience)

		k									
		7,6	6,5	5,3.5	3.5,3	7,5	6,3.5	5,3	7,3.5	6,3	7,3
j	7,7	731				788			808		811
	6,6	642	352				545			519	
	5,5		792	500		809		451			
	3.5,3.5			600	537		836		889		
	3,3				426			648		704	870
	7,6		389			804	491		769	444	827
	6,5	611		306		804	667	245		574	
	5,3.5		694		370	780	796	620	889		
	3.5,3			630			811	673	870	722	855
	7,5	196	196	220				191	673		585
	6,3.5	509	333	204	189				811	500	
	5,3		755	380	327	809				704	778
	7,3.5	231		111	130	327	189				462
	6,3	556	426		278		500	296			778
	7,3	173			145	415		222	538	222	

Note: The blank cells represent comparisons of disjoint distance, data not obtainable by the method of triads.

TABLE 13

DIFFERENCES BETWEEN DISTANCES, DETERMINED FROM THE PROPORTIONS OF TABLE 12 BY CASE V OF THE LCJ, SUBJECT K_a. THE CELL ENTRIES ARE THE AMOUNT BY WHICH DISTANCE k EXCEEDS DISTANCE j

	7,6	6,5	5,3.5	3.5,3	7,5	6,3.5	5,3	7,3.5	6,3	7,3
7,7	.616				.800			.870		.882
6,6	.364	-.380				.113			.048	
5,5		.813	.000		.874		-.133			
3.5,3.5			.253	.093		.978		1.221		
3,3				-.186			.380		.536	1.127
7,6					.856			.768		.942
6,5	.282				.856	.432			.187	
5,3.5		.507			.772	.823	.305	1.221		
3.5,3			.332			.882	.449	1.127	.589	1.058
7,5								.449		.215
6,3.5	.023							.882	.000	
5,3		.690			.874				.536	.765
7,3.5										
6,3	.141									.765
7,3								.095		

satisfactory inconsistency. The time was .0510 seconds for subject Ro and .0199 seconds for subject Ka.

Results -- method of pair comparisons

Table 14 gives the proportion of judgments "brighter" for each pair for subject Ro. Table 15 gives the difference in scale value for each pair, derived from Table 14 by Case V of the LCJ. Figure 17 represents the scale values, determined from the distances of Table 15 by the usual LCJ procedure. Tables 16 and 17 and Figure 18 provide the same information for subject Ka.



Fig. 17 - Scale values from pair comparisons, analyzed by Thurstone's LCJ, Case V, subject Ro.

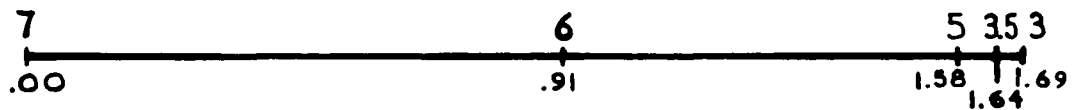


Fig. 18 - Scale values from pair comparisons, analyzed by Thurstone's LCJ, Case V, subject Ka.

Same-Different Method

This method also requires that the stimuli be relatively indistinguishable, and the same techniques were used to find a suitable exposure time. The time chosen was .0407 seconds for subject Ro and .0199 seconds for subject Ka. Each pair was replicated at least 100 times. The instructions to the subject were:

In this portion of the experiment you will be observing two stimuli. Your task is to tell me whether the two stimuli are the "same" or "different". Of all the pairs that you see, 1/3 will actually be physically the same and 2/3 will be different in varying degrees -- some very

TABLE 14

PROPORTION OF JUDGMENTS OF j BRIGHTER THAN k, SUBJECT Ro

		k				
		7	6	5	3.5	3
	7		.819	.9709	1.000	1.000
	6			.774	.804	.9706
j	5				.705	.769
	3.5					.480
	3					

TABLE 15

DISTANCE FROM k TO j, DERIVED FROM THE PROPORTIONS OF
TABLE 14 BY CASE V OF THE LCJ, SUBJECT Ro

		k				
		7	6	5	3.5	3
	7		.912	1.894	--	--
	6			.752	.856	1.890
j	5				.539	.736
	3.5					-.050
	3					

TABLE 16

PROPORTION OF JUDGMENTS OF j BRIGHTER THAN k, SUBJECT Ka

		k				
		7	6	5	3.5	3
j	7		.858	.9583	.900	.9587
	6			.768	.819	.758
	5				.590	.533
	3.5					.517
	3					

TABLE 17

DISTANCE FROM k TO j, DERIVED FROM THE PROPORTIONS OF
TABLE 16 BY CASE V OF THE LCJ, SUBJECT Ka

		k				
		7	6	5	3.5	3
j	7		1.073	1.731	1.282	1.736
	6			.732	.911	.700
	5				.227	.134
	3.5					.043
	3					

close and some very different. There are four possible kinds of judgments that we can diagram in the following way:

Proportion	Physical situation the pairs are:	Your judgment the pairs are:	
1/3	same	same	(correct)
		different	(wrong)
2/3	different	same	(wrong)
		different	(correct)

Note that there are two ways of being correct -- by saying 'same' for pairs that are actually the same and by saying 'different' for pairs that are actually different. There are also two ways of being wrong -- by saying 'different' for pairs actually the same and by saying 'same' for pairs that are actually different. I want you to regard all of these possible outcomes as equally weighted; that is, one kind of error is just as serious as the other kind, and one way of being correct is just as desirable as the other. Another way of saying the same thing would be to assume that I had money to pay you for each response. If I said that I would pay you 10¢ for each correct response, and if you had to pay me 10¢ for each wrong response, then all four responses would be 'equally important' or 'equally weighted'.

Before each run the subject was told: "Tell me whether the stimuli are 'same' or 'different'."

Results -- same-different method

Table 18 gives the proportion of judgments "different" for each pair of stimuli, for subject Ro. Table 19 gives the interpoint distances derived from Table 18 by the method described in Appendix 2. Note that the proportion of times a stimulus is judged the same as itself differs for the five stimuli. Therefore the criterion distance was computed separately for each pair, as described in Appendix 2. Figure 19 gives the scale values determined from Table 19 by the usual LCJ procedure. Tables 20 and 21 and Figure 20 gives the same information

TABLE 18

PROPORTION OF JUDGMENTS OF "DIFFERENT" (EQUALS ONE MINUS THE PROPORTION OF JUDGMENTS "SAME") FOR THE PAIR j_k , SUBJECT R_0

	k				
	7	6	5	3.5	3
7	.562	.647	.896	.9621	.9236
6		.548	.607	.776	.808
j 5			.426	.454	.579
3.5				.336	.287
3					.312

TABLE 19

DISTANCE BETWEEN j AND k , DERIVED FROM THE PROPORTIONS OF TABLE 18 BY THE METHOD INDICATED IN APPENDIX 2, SUBJECT R_0

	k				
	7	6	5	3.5	3
7	0	.73	1.92	2.53	2.21
6		0	.76	1.49	1.63
j 5			0	.57	1.03
3.5				0	0
3					0

TABLE 20

PROPORTION OF JUDGMENTS "DIFFERENT" (EQUALS ONE MINUS THE PROPORTION OF JUDGMENTS "SAME") FOR THE PAIR jk , SUBJECT K_a

	k				
	7	6	5	3.5	3
7	.514	.769	.9245	.9175	.9722
6		.615	.639	.611	.663
j 5			.463	.402	.448
3.5				.435	.370
3					.491

TABLE 21

DISTANCE BETWEEN j AND k , DERIVED FROM THE PROPORTIONS OF TABLE 20 BY THE METHOD INDICATED IN APPENDIX 2, SUBJECT K_a

	k				
	7	6	5	3.5	3
7	0	1.34	2.08	2.04	2.58
6		0	.86	.76	.98
j 5			0	0	0
3.5				0	0
3					0

for subject Ka.



Fig. 19 - Scale values from same-different method, analyzed by the procedure given in Appendix 2, subject Ro.



Fig. 20 - Scale values from same-different method, analyzed by the procedure given in Appendix 2, subject Ka.

Comparisons Between the Methods

The same-different method and Coombs' method of similarity

The comparisons of the metric information in the Coombs' unfolding analysis of similarity data with that in the same-different data are presented in Tables 22 and 23 for subjects Ro and Ka, respectively. Since the unfolding technique does not yield an interval scale, a graphical representation of the results is not possible. Also, since the methods generally agree that the order of the five stimuli is 7, 6, 5, 3.5, 3, only metric information aside from that contained in the order of the stimuli is compared here. For example, information that $\overline{73} > \overline{76}$ is not compared. In general, the agreement expressed in Tables 22 and 23 is rather poor. This suggests that the results of the same-different method applied to impoverished stimuli are comparable to the results of the unfolding analysis only up to the level of an ordinal scale of the stimuli.

TABLE 22

METRIC INFORMATION FROM COOMBS' METHOD OF SIMILARITIES
 COMPARED WITH THE INTERVAL SIZES FROM THE SAME-DIFFERENT
 METHOD, SUBJECT Ro

$$\begin{array}{l} \overline{5-6} < \overline{6-7} < \overline{3.5-6} \\ .76 \quad .73 \quad 1.49 \\ \\ \overline{5-6} < \overline{3.5-5} < \overline{5-7} < \overline{3-5} \\ .76 \quad .57 \quad 1.92 \quad 1.03 \\ \\ \overline{3-3.5} < \overline{3.5-5} \\ 0 \quad 1.03 \end{array}$$

TABLE 23

METRIC INFORMATION FROM COOMBS' METHOD OF SIMILARITIES
 COMPARED WITH THE INTERVAL SIZES FROM THE SAME-DIFFERENT
 METHOD, SUBJECT Ka

$$\begin{array}{l} \overline{6-7} < \overline{5-6} \\ 1.34 \quad .86 \\ \\ \overline{5-7} < \overline{3.5-5} \\ 2.08 \quad 0 \\ \\ \overline{3-3.5} < \overline{3.5-5} \\ 0 \quad 0 \end{array}$$

The same-different method and
Torgerson's complete method of triads

Figures 21 and 22 present graphical comparisons of the differences between distances from Torgerson's complete method of triads and the same-different method, for subjects Ro and Ka, respectively. The 40 points plotted in each case represent the 40 differences between distances, determined directly by the complete method of triads (including 10 comparisons with zero distances) versus the 40 differences between distances, determined indirectly from the distances determined by the same-different method. This seems to provide a more direct comparison between the methods than a comparison of scale values would, since the scale values are one or two steps removed from the values directly determined. Note that a direct comparison in this case would mean a comparison of averages of differences (from the same-different method) with averages of differences between averages of differences in distances (from the Torgerson method). Although the present method is a more direct one, for some practical purposes a comparison of scale values might be of more interest. (Note that the steeper slope for subject Ka is not to be interpreted as implying something about a relation between the two methods. The exposure times for the two situations were not in the same ratio for both subjects.) The discrepancies between the two methods are not as great as might be inferred from these plots. In computing scale values, each value would be made up of an average of differences between several of these values, and therefore extreme discrepancies would tend to disappear. With these qualifications, the results seem to indicate a fairly close correspondence between the methods.

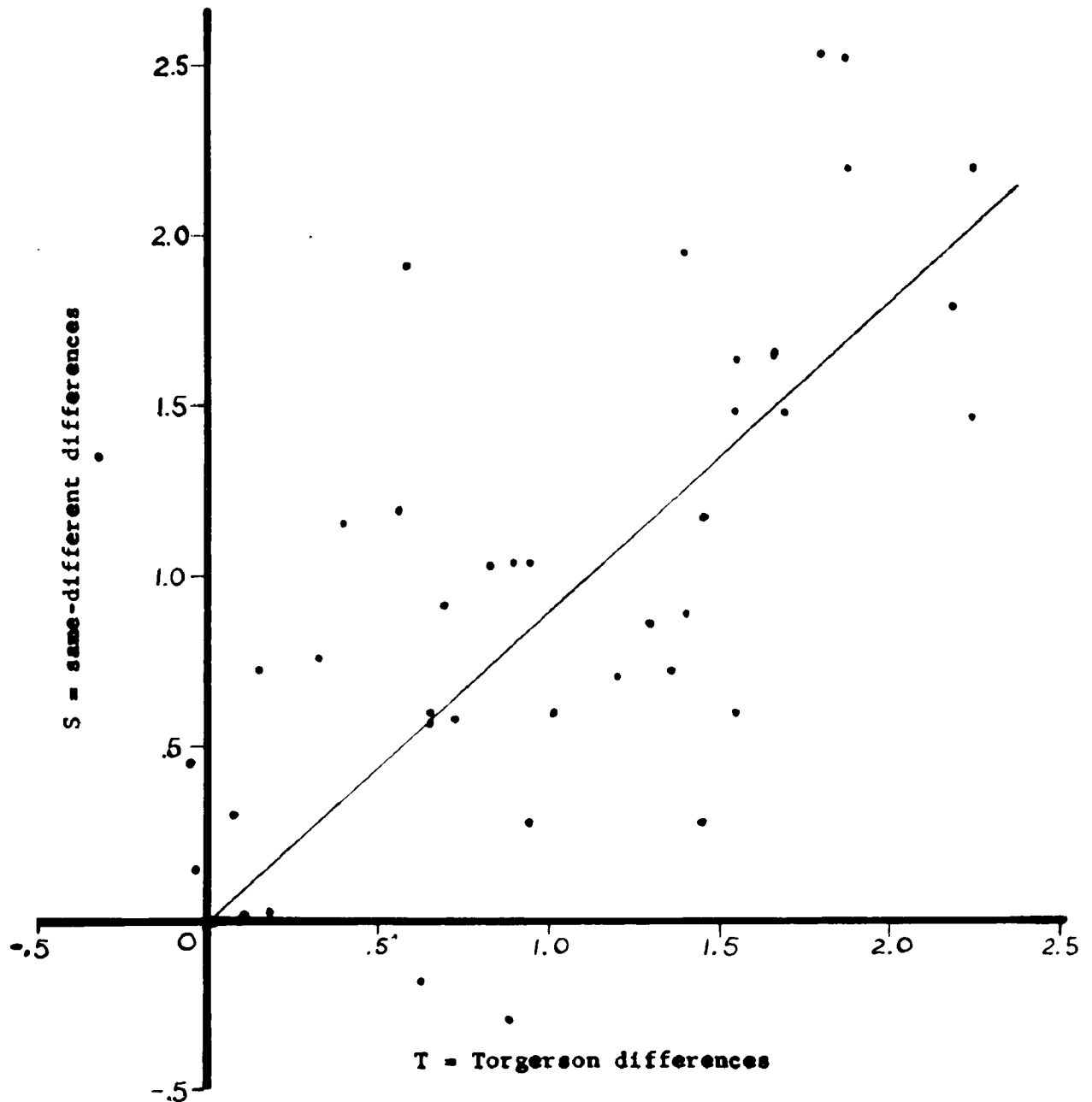


Fig. 21 - The relation between Torgerson differences between distances and same-different differences between distances, subject Ro. The equation for the best-fit-line (method of least squares) is

$$S = .90 T - .001$$

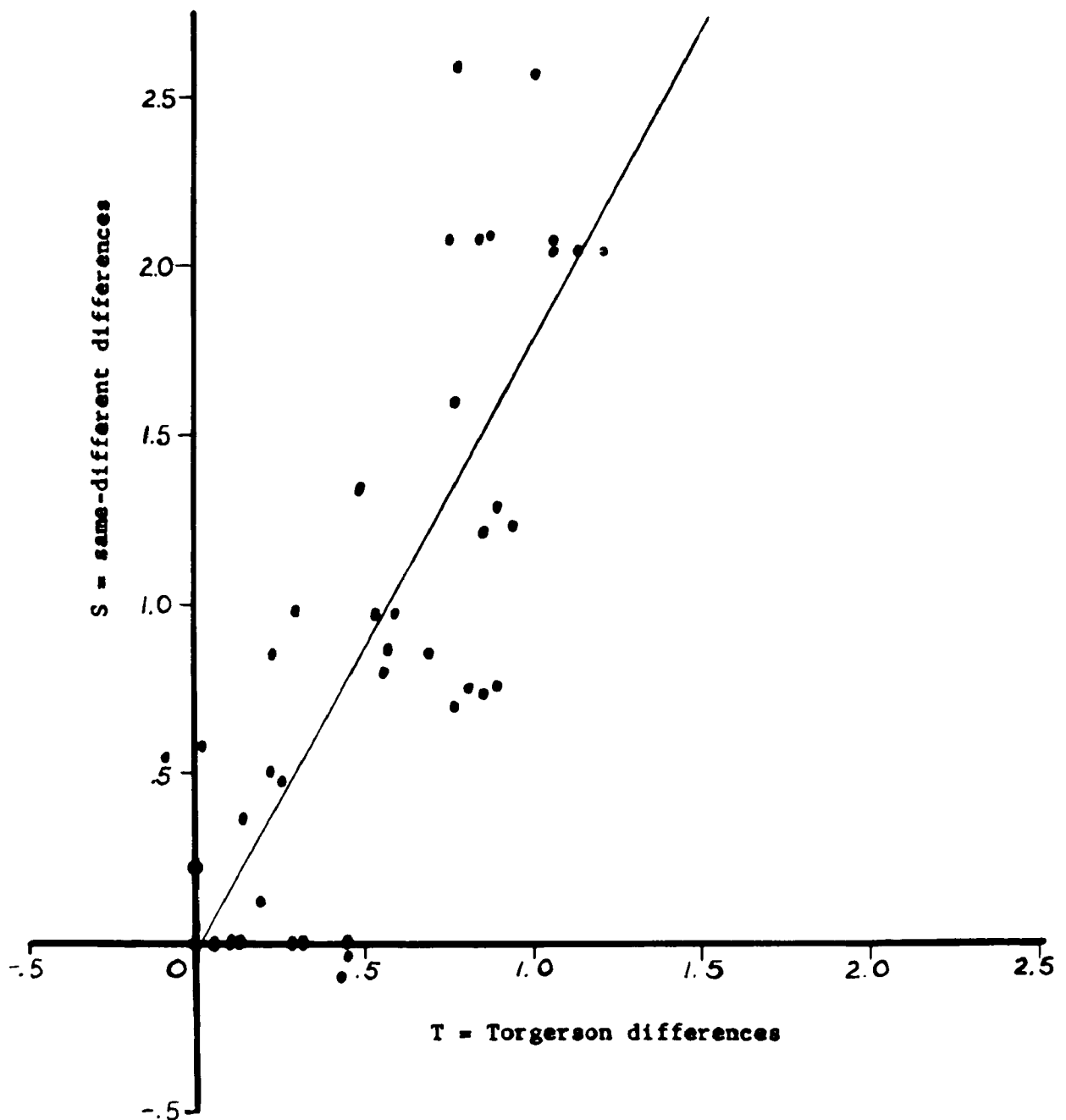


Fig. 22 - The relation between Torgerson differences between distances and same-different differences between distances, subject Ka. The equation for the best-fit-line (method of least squares) is

$$S = 1.83 T - .05$$

The same-different method and
Thurstone's LCJ, Case V

Figures 23 and 24 compare the interpoint distances derived by the same-different method with those derived by Case V of the LCJ applied to judgments of "brighter than," for subjects Ro and Ka, respectively. The relationships between the methods are described here in terms of the interpoint distances directly determined by each method, rather than in terms of the scale values of the stimuli on a continuum, because the latter values are one step further removed from the data. The relationship is well approximated by a linear function in each case. It should be noted that the comparability of these methods, expressed in terms of the scale values recovered, might be reduced by the fact that the LCJ solution is constrained to one dimension, while the same-different method could yield results in more than one dimension.

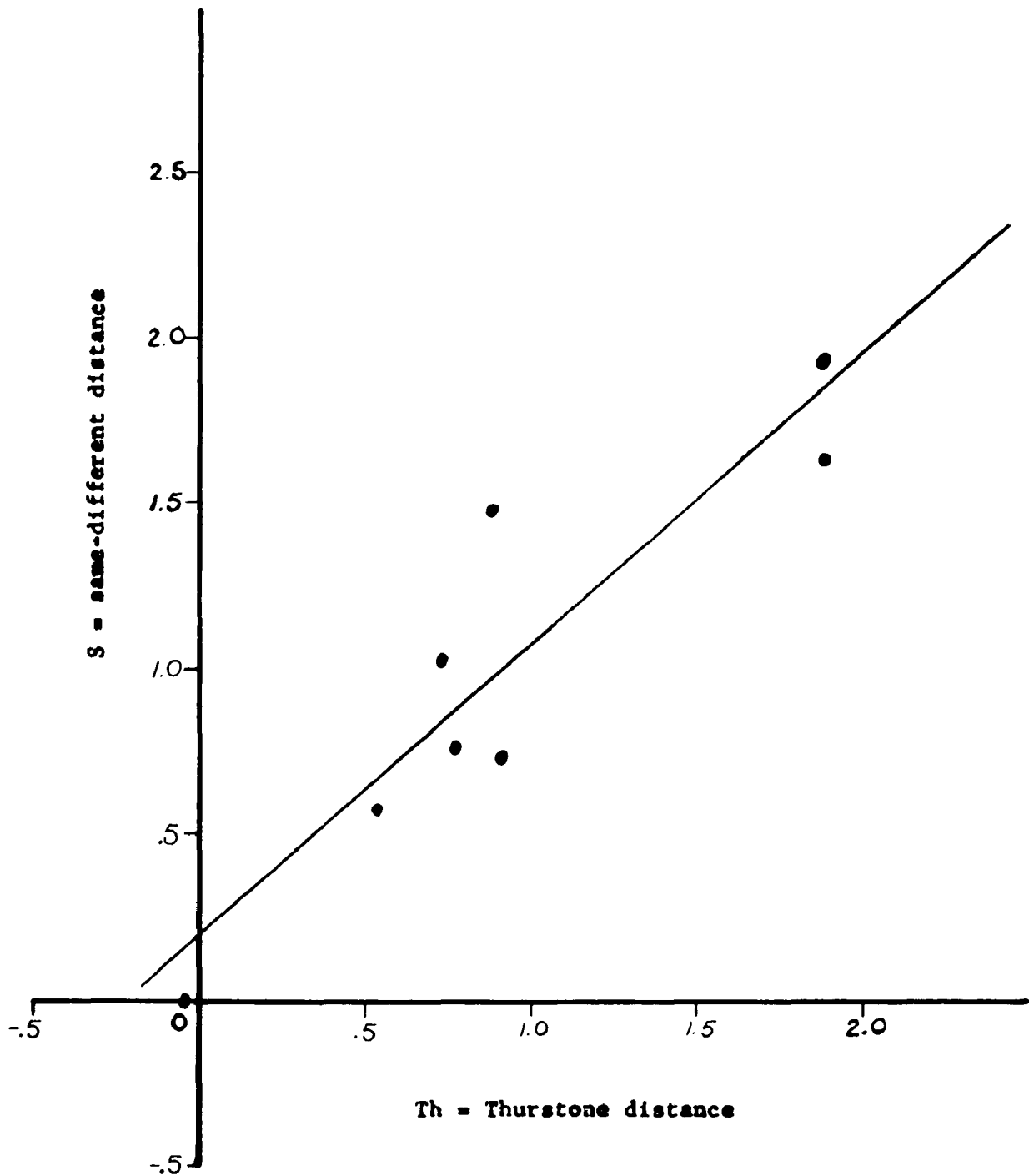


Fig. 23 - The relation between Thurstone interpoint distances and same-different interpoint distances, subject Re. The equation of the best-fit-line (method of least squares) is

$$S = .88 Th + .19$$

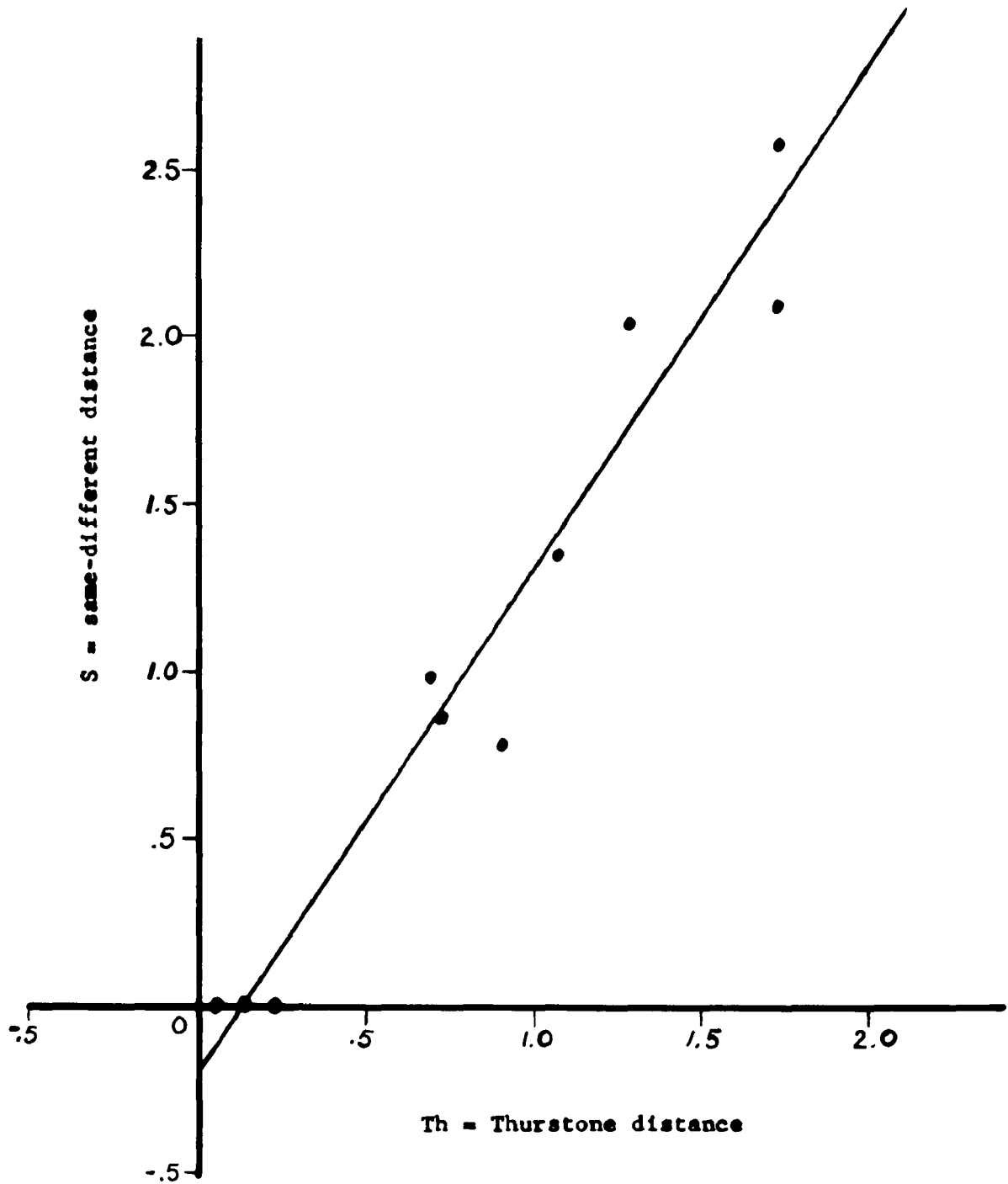


Fig. 24 - The relation between Thurstone interpoint distances and same-different interpoint distances, subject Ka. The equation of the best-fit-line (method of least squares) is

$$S = 1.48 Th - .20$$

CHAPTER VIII

APPLICATION OF THE SAME-DIFFERENT METHOD TO A TWO-DIMENSIONAL EXAMPLE

Several recent studies of the feasibility of multidimensional stimulus scaling techniques have used a set of Munsell red-colored papers on which to test new methods proposed (16, 29, 40, 45). This set of stimuli is presumably approximately located in a two-dimensional plane in psychological color space, and the recovery of something approximating the configuration of points in the Munsell system is regarded as a partial confirmation of the method in question. The general availability and carefully standardized nature of the Munsell papers also make for ready comparability between studies carried out by different investigators. For these reasons, it was decided to use seven of the same Munsell reds in studying the feasibility of the same-different method of collecting data. The particular colors chosen are indicated in Figure 25, which shows the configuration of the points according to the Munsell system. These same seven stimuli have been scaled by other techniques, and the results of this chapter will be compared with these other results.

The equipment and subjects used for this experiment are described in Chapter VI. The instructions to the subject were exactly the same as those described for the same-different method in Chapter VII, except that the subjects were told "1/2 of the pairs will actually

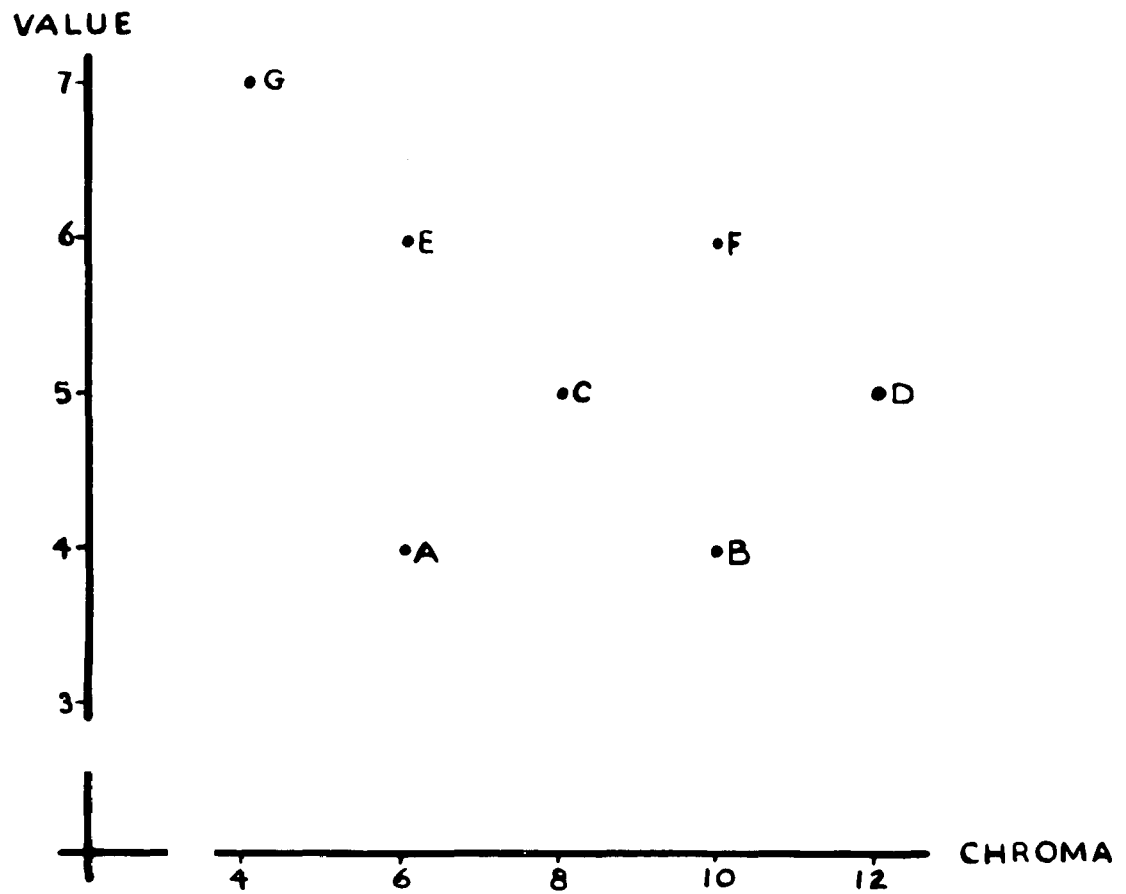


Fig. 25 - Value (brightness) and chroma (saturation) of the Munsell stimuli.

be physically the same and $1/2$ will be different," instead of the $1/3:2/3$ ratio used there. Each pair was replicated at least 100 times. To achieve the desired degree of inconsistency the stimulus pairs were presented for the limited exposure time determined by preliminary experiments to provide satisfactory inconsistency. This time was .0324 seconds for subject Ra and .0510 seconds for subject Vg.

Table 24 gives the proportion of judgments "different" for each pair, for subject Ra. Table 25 gives the rank order of distances determined from Table 24 in the manner described in Appendix 3. Table 26 gives the order of the stimuli on the first dimension, determined by Hays' nonmetric multidimensional unfolding (18) of the rank orders in Table 25. Because the first dimension accounts for nearly all of the variability, there is insufficient information to define a second dimension. The first dimension accounts for the order relations of the distances in 31 of the 35 sets of three stimuli. The four sets remaining unsatisfied might be error, or they might be satisfied by any of several orderings of stimuli on a second dimension. Tables 27, 28, and 29 give the same information for subject Vg. The first dimension accounts for 30 of the 35 sets of three stimuli for subject Vg.

For both subjects, the first dimension is perfectly monotonically related to the brightness dimension of the Munsell system. Since the configuration of these seven stimuli in the other studies, and in the Munsell system, does not suggest greater variability in brightness than in saturation, the interpretation made here is that the impoverishment of the stimuli by brief presentation in the tachistoscope effectively eliminated saturation as a dimension. In other words, at the exposure times used here (.0324 seconds for subject Ra and .0510 seconds for subject Vg), brightness accounted for almost all of

TABLE 24

PROPORTION OF JUDGMENTS "DIFFERENT" (EQUALS ONE MINUS THE PROPORTION OF JUDGMENTS "SAME") FOR THE PAIR j_k , SUBJECT R_a

	A	B	C	^k D	E	F	G
	4/6	4/10	5/8	5/12	6/6	6/10	7/4
A 4/6	.207	.315	.252	.528	.810	.9029	1.000
B 4/10		.216	.202	.245	.842	.9083	1.000
C 5/8			.307	.339	.760	.788	.9528
j D 5/12				.337	.771	.845	.9806
E 6/6					.462	.383	.667
F 6/10						.416	.615
G 7/4							.188

Note: All of the stimuli are Munsell 5R (red); they are therefore referred to here in terms of the value (brightness) and chroma (saturation). The first number is the value, the second is the chroma.

TABLE 25

RANK-ORDER OF DISTANCES BETWEEN j AND k , DERIVED FROM THE PROPORTIONS OF TABLE 24 BY THE METHOD INDICATED IN APPENDIX 3, SUBJECT R_a

LARGEST				
AG, BG (tie)	BE	CE	CD	
DG	DF	EG	AC	
CG	AE	FG	BD	
BF	CF	AD	BC	
AF	DE	AB	EF	
			SMALLEST	

TABLE 26

ORDER OF STIMULI ON THE FIRST DIMENSION, DETERMINED BY HAYS' NONMETRIC MULTIDIMENSIONAL UNFOLDING, SUBJECT R_a

B A D C E F G

TABLE 27

PROPORTION OF JUDGMENTS "DIFFERENT" (EQUALS ONE MINUS THE PROPORTION OF JUDGMENTS "SAME") FOR THE PAIR j k , SUBJECT V_g

		k						
		A	B	C	D	E	F	G
		4/6	4/10	5/8	5/12	6/6	6/10	7/4
A	4/6	.0282	.0286	.119	.302	.9200	.99057	1.000
B	4/10		.0282	.135	.187	.9135	.9816	.99020
C	5/8			.163	.182	.817	.9352	.99000
D	5/12				.267	.9417	.896	.99057
E	6/6					.568	.657	.750
F	6/10						.540	.725
G	7/4							.249

Note: All of the stimuli are Munsell 5R (red); they are therefore referred to here in terms of the value (brightness) and chroma (saturation). The first number is the value, the second is the chroma.

TABLE 28

RANK-ORDER OF DISTANCES BETWEEN j AND k , DERIVED FROM THE PROPORTIONS OF TABLE 27 BY THE METHOD INDICATED IN APPENDIX 3, SUBJECT V_g

LARGEST

AG	BF	DF	AD
BG	CF	CE	BC
CG	DE	EG	BD
DG	AE	FG	AC
AF	BE	EF	AB

CD

SMALLEST

TABLE 29

ORDER OF STIMULI ON THE FIRST DIMENSION, DETERMINED BY HAYS' NONMETRIC MULTIDIMENSIONAL UNFOLDING, SUBJECT V_g

A B C D E F G

the perceived differences between stimuli.

In comparing the results of the same-different method applied to the Munsell reds with other methods of scaling the same stimuli, the most direct comparison is desired. Therefore, the comparison made here is between the rank orders of interpoint distances from the same-different method and the rank orders of interpoint distances from the following three sources: 1. Torgerson's complete method of triads (45); 2. The original spacing of the Munsell colors (33); and 3. The respacing of the Munsell colors undertaken by a committee of the Inter-Society Color Council (34). Table 30 gives the Spearman rank-order correlations of the 21 interpoint distances in Tables 25 and 28, for subjects Ra and Vg respectively, with the interpoint distances determined from these three sources. For comparison, the rank-order correlation of the distances of subject Ra with those of subject Vg is .944, and the rank-order correlation of the Torgerson distances with the original Munsell distances is .843.

TABLE 30
RANK-ORDER CORRELATIONS BETWEEN DISTANCES DERIVED BY SAME-DIFFERENT
METHODS AND DISTANCES FROM THREE OTHER STUDIES

<u>Study</u>	Subject	
	<u>Ra</u>	<u>Vg</u>
Torgerson	.753	.754
Original Munsell	.694	.597
Revised Munsell	.656	.641

All of these results suggest that the same-different method is sensitive to distance relationships in more than one dimension, but that

under impoverished conditions the subjects in the present study responded primarily to brightness differences.

CHAPTER IX

DISCUSSION

The theoretical developments of Chapters IV and V provide two new techniques of stimulus scaling specifically designed to overcome some of the limitations discussed in Chapters I and II. These new techniques are developed as extensions of Thurstone's widely used Law of Comparative Judgment (LCJ). One technique involves tachistoscopic impoverishment of otherwise relatively discriminable stimuli, to provide the inconsistency of judgment necessary for some scaling methods. The other involves judgments of whether pairs of stimuli are "same" or "different" and provides a symmetrical distance for use in multi-dimensional stimulus scaling. The experiments of Chapters VI, VII, and VIII represent the first empirical tests of these methods. In the present chapter the results and further implications of these studies will be discussed.

Relation of Psychological Distance to Exposure Time

In general, the results reported in Chapter VI concerning the relation of psychological distance to exposure time provide confirmation for the "sampling" model of the perceptual process described in Chapter IV. Perhaps the most significant deviation from prediction occurred in the crossing of the linear functions for subject Ro, illustrated in Figure 11. This crossing is contrary to the theory, and indicates that changes in stimulus relationships occur with impoverishment even with stimuli

generally regarded as lying on a unidimensional psychological continuum. It may be that subject Ro was responding to aspects of the stimulus situation other than the brightness of the stimuli, e.g., the texture of the Munsell papers; even so, it is difficult to understand why the effect on these other characteristics should be in opposite directions with impoverishment in the case of the two different pairs of stimuli.

The "sampling" model of the perceptual process has several implications for other aspects of the pair comparison procedure. Conjectures regarding three implications are presented here to illustrate the broad application of the model.

In the usual pair comparison procedure the subject is allowed "unlimited time" to inspect the stimuli and arrive at his decision. According to the sampling theory, he is collecting information regarding the stimuli throughout the entire time of observation (at least up to the limit of his memory capacity). Studies of decision time (10) suggest that, given the opportunity, the subject spends more time looking at more difficult pairs -- that is, pairs closer together. The particular prediction made here, then, is that there will be systematic differences in the scale relationships obtained under conditions where observation time is unlimited, versus conditions where observation time is controlled and equalized. Specifically, the short distances should become relatively shorter when observation periods are equal. If such a prediction were verified, we might ask which scale relationships are to be regarded as more correct. The extension of the LCJ in Chapter IV asserts that distances derived from proportions of judgments "greater" determined with different exposure times must be transformed by $\sqrt{T_o/T_{jk}}$

to recover the scale values of the means of the discriminial processes. Therefore, if different exposure times are used and this correction is not applied, these scale values will not be recovered.

The second conjecture concerns the relationships between psychological distance, decision time, decision confidence, and accuracy of decision. The sampling model provides a rationale for predicting particular relations between these variables. If we regard the subject as behaving as if he were a statistical decision maker, then this theory suggests that he receives sequential information as he continues to observe a stimulus pair. Suppose we assume that he has set a given statistical criterion level, say 10%, and collects information sequentially at a constant rate until he reaches that level of confidence in a decision. When this point is reached, he stops observing and reports his decision. Under these conditions, there would be a perfect inverse monotonic relation between psychological distance (as determined under conditions of equal observation time, or by some independent criterion) and decision time. Decision confidence and decision accuracy would be constant for all pairs. We know that the last two conditions do not obtain; therefore, some additional factor needs to be introduced. One reasonable factor to add would be the cost of obtaining information. All information costs at least the time required to assimilate it, and experimental conditions may be established which add costs controlled by the experimenter (22). The cost of information may be regarded as requiring that the subject progressively lower his criterion level as the period of observation continues, so that he eventually responds, even to very difficult or impossible discrimination situations. Under these conditions, there would be

positive relationships between decision confidence and accuracy, between decision confidence and distance, and between distance and accuracy. There would be negative relationships between time and confidence, between time and distance, and between time and accuracy. If the individual is really a statistical decision maker, then proper specification of the utilities and costs of various decision outcomes, and the cost of information, would allow quantitative prediction of these relations.

The results of the present study may also be used as a guide to quantification of other kinds of stimulus impoverishment. For example, greater observation distance (2) and decreased illumination (45) have both been used as methods for reducing the discriminability of stimuli to scalable ranges. For many purposes it might be desirable to have a quantitative index of the extent of impoverishment. The sampling theory presented here asserts that the distance between relatively indiscriminable stimuli is linearly related to the square root of the number of independent pieces of information obtained from the two distributions. Since it is assumed that information is collected at a constant rate, this predicts that distance is related to the square root of exposure time. The same idea can be used to specify the relative amount of information obtained under various degrees of impoverishment, and also to quantify other aspects of the experimental situation, such as distraction, fatigue, spatial separation of the stimulus pairs compared, etc.

A Scaling Technique Based on Judgments
of "Same" and "Different"

The results reported in Chapter VII indicate general comparability between the same-different method and the methods of Torgerson and Thurstone, when all three methods are applied to stimuli impoverished by brief tachistoscopic presentation. The same-different method resulted in metric information different from that obtained with judgments of supraliminal stimuli analyzed by Coombs' method of similarities. This disagreement is in line with previous evidence of qualitative changes with impoverishment (in Figure 11), and with the disagreements between the method-of-similarities judgments given by the subjects and their dominant-paired comparison judgments in the Torgerson procedure.

It would be possible to compare the same-different method without impoverishment with the Coombs method, but to do so it would be necessary to change the conditions in a different way. The technique is one of augmenting the discriminable stimuli with enough additional stimuli, well distributed along the continuum, to make adjacent stimuli relatively indiscriminable. Figure 26 illustrates such a situation, in which the stimuli A, B, C, D, and E are to be scaled by Coombs' method of similarities, and these five plus the other 15 are to be scaled by the same-different technique.



Fig. 26 - Discriminable stimuli augmented to make adjacent stimuli indiscriminable.

The assumption of unidimensionality would make it possible to determine supraliminal distances, like $\overline{AB}$, from the sum of the adjacent small

intervals that make it up. This procedure has the defect of adding up a large number of terms, each of which will contain a systematic bias unless the distribution of values of the discriminial process is exactly leptokurtic enough to remove the bias (28). If the distribution is normal or platykurtic, small distances are underestimated relative to large ones, and if the distribution is too leptokurtic the opposite occurs.¹

The same-different technique and its associated method of analysis were developed partly because of the unsatisfactory nature of the assumption of a normal distribution of absolute distances in the Torgerson procedure. It was pointed out in Chapter VII that there are several violations of the triangular equality in the results reported here using Torgerson's method, and Torgerson (45) noted the same effect. It was proved in Appendix 1 that the distribution of distances is actually the square root of non-central chi-square. Thus, the distribution that Torgerson assumed to be normal is actually the distribution of differences between the square roots of two non-central chi-square distributions. It is not known whether such a distribution might be sufficiently platykurtic to account for the violations of the triangular equality.

The application of the same-different method in a two-dimensional example yielded results which may also be interpreted as evidence for qualitative changes in stimulus relationships with impoverishment. In this case the dimension of "saturation" was not important in determining the stimulus differences to which the subjects responded. The meaningfulness of the dimension recovered and the close

¹If a distribution which is actually platykurtic is assumed to be normal in a Thurstone-type scaling procedure, the result is to underestimate small distances and overestimate large ones. Such errors would lead to a more extreme violation of the triangular equality than that caused in the situation where the normal assumption is correct (28).

correspondence of the rank-order distances for the two subjects suggest that the same-different method is sensitive to meaningful stimulus relationships. However, both the one-dimensional and the two-dimensional results suggest that significant changes take place in stimulus relationships with impoverishment.

This would seem to be a very important result, because many psychological studies use some form of stimulus impoverishment with what amounts to an assumption that no qualitative change takes place (2, 45). If qualitative changes do take place, then such studies are of limited generality. It would seem that some direct check on the possibility of qualitative change, like described in Chapter IV in connection with Figure 4, would be desirable whenever such impoverishment is to be used. For such checks, a same-different procedure might prove more useful than a judgment that constrains the relationships to one dimension.

Attempts to extend the same-different method to identifiable stimuli give rise to several additional problems. It would not be possible to determine the criterion distance by presenting identical stimuli, since they would be called "same" 100% of the time. The additive constant problem would thus occur in this context. Also, identifiably different stimuli would probably never be called the "same", so the judgmental task might have to be modified to a less restrictive one -- for example, "Tell me if these statements express similar or dissimilar attitudes."

In conclusion, it seems probable that the same-different method, and the method of impoverishing stimuli to make them relatively indiscriminable, will prove to be of more use in investigating the

judgmental process than as practical methods of scaling sets of stimuli.

CHAPTER X

SUMMARY

Alternative psychological models of stimulus scaling differ with respect to certain crucial characteristics which determine the situations to which they are applicable. One of these models is Thurstone's Law of Comparative Judgment, designed for stimuli which are relatively indiscriminable and differ primarily with respect to a single attribute. The present study developed two extensions of the Law of Comparative Judgment intended to make this kind of measurement model applicable to discriminable and multidimensional stimuli. The theoretical developments were then illustrated in studies of the psychological color space.

A review of multidimensional stimulus scaling indicated that most work in this area has centered around the determination of a Euclidean multidimensional metric space by the Young and Householder procedure. This procedure is customarily applied to interpoint distances determined by application of the Law of Comparative Judgment to judgments of distances. However, there are certain difficulties in the usual application of this method: fairly complex judgments are often required of the subject; the assumption of a normal distribution of discriminial processes is necessarily violated when the stimulus elements are distances; an additive constant needs to be estimated to obtain an absolute zero for the distance scale.

In this study, a multidimensional analogue of the Law of Comparative Judgment was developed, designed to overcome these difficulties. The judgment required in this technique is whether two stimuli are "same" or "different". It was proven that the assumption of a normal distribution of discriminative processes of stimuli, here extended to a multivariate normal distribution in r -space, leads to a distribution of distances that is the square root of non-central chi-square. Having this distribution, it became possible to develop a basis for proceeding from the proportions of judgments "different" to the configuration of the points in the space. A routine procedure for this method was described.

In an independent theoretical development, a "sampling" model of the perceptual process involved in pair comparison judgments led to the development of a tachistoscopic technique for making relatively discriminable stimuli become relatively indiscriminable by impoverishing by brief exposure. Thus, the same set of stimuli could be scaled by both the Law of Comparative Judgment and by techniques applicable only to discriminable stimuli. From the theoretical model it was hypothesized that psychological distance and the square root of exposure time should be linearly related, and this hypothesis was tested in experiments using Munsell neutral grey values. The results generally confirmed the prediction.

Using the same Munsell stimuli, the same-different technique was compared with the scaling procedures of Thurstone and Torgerson. General comparability of the results was indicated.

Three specific results pointed to apparent qualitative changes in stimulus relationships when stimuli were impoverished by

tachistoscopic presentation. There were shifts in relative sizes of intervals with change in exposure time; the metric relations of the same-different method applied to impoverished stimuli were different from the metric relations of the Coombs method of similarities applied to the same stimuli visible for unlimited time; and the application of the same-different method to a set of Munsell colored papers varying in brightness and saturation, presented tachistoscopically, led to results accounted for almost entirely in terms of brightness differences. It was suggested that these results have important implications for the interpretation of psychological studies using impoverishment techniques.

In conclusion, it was suggested that the same-different method, and the method of impoverishing stimuli to make them relatively indiscriminable, will probably prove to be of more use in investigating the judgmental process than as practical methods of scaling sets of stimuli.

APPENDIX 1

PROOF OF A LEMMA CONCERNING THE DISTRIBUTION OF DISTANCES IN A MULTIDIMENSIONAL SPACE

Lemma: The distribution function of the square of distances between points drawn at random, pairwise, from two r-variate normal distributions with equal variance in all dimensions, and uncorrelated over dimensions, is non-central chi-square with r degrees of freedom. The non-centrality parameter is the square of the distance between the means of the two r-variate normal distributions.

Proof: Let X be an r-variate normal distribution with a normal projection on each dimension $N(a_d, \sigma^2)$; let Y be an r-variate normal distribution with normal projection on each dimension $N(b_d, \sigma^2)$, where the orthogonal reference dimensions of the space are 1, 2, . . . , d, . . . , r. Now consider a point drawn at random in each distribution:

$X_d = \sigma \alpha_d + a_d$, and $Y_d = \sigma \beta_d + b_d$, where α and β are random deviates drawn from populations that are $N(0,1)$.

The difference between the points on dimension d is

$X_d - Y_d = \sigma(\alpha_d - \beta_d) + (a_d - b_d) = \sigma\sqrt{2}(\delta_d + c_d)$, where δ is a random deviate drawn from a population that is $N(0,1)$ and $c_d = \frac{a_d - b_d}{\sigma\sqrt{2}}$.

The lemma is concerned with the distribution of squares of distances between random samples from the two distributions; this is given by the sum of the squares of the distances on each of the dimensions:

$$D_{XY}^2 = \sum_{d=1}^r (X_d - Y_d)^2 = 2\sigma^2 \sum_{d=1}^r (\delta_d + c_d)^2.$$

Since we are interested in the distances at the level of a ratio scale (the unit being arbitrary), we may determine the unit by setting $\sigma^2 = 1/2$. This unit makes c_d a direct measure of the distance between means on dimension d . Therefore, with the proper choice of the unit distance we have:

$$D_{XY}^2 = \sum_{d=1}^r (\delta_d + c_d)^2.$$

This is the equation of non-central chi-square with r degrees of freedom. The non-centrality parameter, λ , is defined as

$$\lambda = \sum_{d=1}^r c_d^2, \text{ which is the square of the distance between the}$$

means of the two r -variate normal distributions. (See Patnaik, "The non-central χ^2 and F -distributions and their applications."

[35, p. 203])

APPENDIX 2

A ROUTINE PROCEDURE FOR MULTIDIMENSIONAL STIMULUS SCALING BASED UPON REPLICATED, INCONSISTENT JUDGMENTS OF "SAME" OR "DIFFERENT" REGARDING PAIRS OF STIMULI

The procedure described here is a method of analyzing the matrix of the proportion of times stimulus j and stimulus k are called "different". The elements of the matrix are P_{jk} , where $j, k = 1, 2, \dots, n$; including $j=k$. The result of the analysis is expressed in an $n \times r$ matrix of projections of the n stimuli on the r orthogonal dimensions of a Euclidean space.

The procedure is an iterative one in which a particular dimensionality is assumed, interpoint distances are calculated, and then the interpoint distances are used to check the assumption of dimensionality. The numbers of the steps below correspond to the assumed dimensionality.

Step A: Determine if P_{jj} is approximately (within the range of experimental error) constant for all j . If so, then the steps b referred to below may be carried out a single time for all pairs of stimuli. If not, then the assumption of equal variance, or the assumption of equal criterion distances made in the model, or both, are violated. (This is equivalent to the violation of Case V assumptions in the LCJ.) In this latter case the effective criterion distance must be separately determined for each pair, jk , from the average of P_{jj} and

P_{kk} . (This may be regarded as an approximation of the solution of the multidimensional analogue of Case III of the LCJ.)

Step 1a: Assume that $r=1$.

1b: Determine the criterion distance, D_c , which is equal to one-half of the width of the interval centered about zero that contains $(1-P_{jj})$ of the unit normal curve, $N(0, 1)$. This is found from tables of the normal curve by finding the value of the abscissa which contains P_{jj} beyond it in the two symmetrical tails.

1c: Determine the interpoint distances for each pair jk by finding the normal curve $N(D_{jk}, 1)$ which contains $(1-P_{jk})$ within D_c of zero, and P_{jk} outside this interval in the two asymmetrical tails of the normal curve. This value is found by a trial-and-error process of moving an interval of width $2D_c$ along a table of the normal curve until the appropriate values are found.

1d: Check the interpoint distances thus determined to see if they satisfy the requirements of the assumed dimensionality. The modification of the Young and Heuseholder procedure described by Torgerson (45) provides the most general check. Form the matrix B^* , whose elements are the scalar products of the vectors from the centroid of the points to points j and k . If the matrix is positive, semi-definite, and if the rank of the matrix is one, then the points satisfy requirements for one dimension. Factor the matrix to determine the projections of the stimuli on a line, and the analysis is complete. If the matrix B^* does not satisfy these conditions, proceed to step 2.

Step 2a: Assume that $r=2$.

2b: Determine the criterion distance, D_c , which is equal to the square root of the value of the point on the abscissa dividing

chi-square with 2 df into $(1-P_{jj})$ below the point and P_{jj} above the point.

2c: Determine the interpoint distances for each pair jk by finding the non-centrality parameter of the non-central chi-square with 2 df which has $(1-P_{jk})$ of the curve below the point D_c^2 and P_{jk} above the point. The interpoint distance is the square root of the non-centrality parameter.

2d: Form the matrix B^* . If the matrix is positive, semi-definite, and if the rank of the matrix is two, then the points satisfy the requirements for two dimensions. Factor the matrix to determine the projections of the stimuli on the two dimensions, and the analysis is complete. If the matrix B^* does not satisfy these conditions, proceed to step 3.

.

These steps are continued, mutatis mutandis, until a positive, semi-definite matrix is found whose rank is equal to the assumed dimensionality, or until the assumed dimensionality is equal to $n-1$.

APPENDIX 3

A PROCEDURE FOR DETERMINING THE RANK ORDER OF DISTANCE RELATIONS FROM JUDGMENTS OF "SAME" AND "DIFFERENT"

In the application of Hays' nonmetric multidimensional unfolding method to same-different data described by Coombs (9), it was assumed that distances between pairs of stimuli are monotonically related to the proportion of judgments "different" for each pair. This assumption appears to be satisfactory as long as the proportion of judgments "different" for identical stimuli is the same for all stimuli. However, if there are marked deviations from equality of proportion of "different" judgments for identical stimuli, it would appear to be desirable to take this factor into account in transforming the proportion of judgments "different" into a rank order of distance. Such a modification was made in the present study by applying the following transformation:

$$d_{jk} = 1 - \frac{(1-P_{jk})(1-P_{kj})}{(1-P_{jj})(1-P_{kk})}, \text{ where } P_{jk} \text{ is the proportion of judgments of "j different from k".}$$

The rank orders of the d_{jk} were then used in the analysis of the data using Hays' method. This transformation has the property of assuring that the distance from any stimulus to itself is zero. It also relates the proportion of judgments "different" for each stimulus pair to the proportions of judgments "different" for each stimulus with itself. No claim is made that this is the only possible transformation; it is used here to provide

a closer approximation to rank order than is provided by the use of untransformed proportions of judgments.

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