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FOR SEVERAL COLLECTIONS OF SETS.

University of Michigan, Ph.D., 1962
Mathematics

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ON SYSTEMS OF DISTINCT REPRESENTATIVES FOR SEVERAL COLLECTIONS OF SETS

by

Robert Roy Korfhage

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1961

Doctoral Committee:

Assistant Professor Bernard A. Galler, Chairman
Professor Robert C. F. Bartels
Professor P. S. Dwyer
Research Professor Raymond L. Wilder
Doctor Jesse B. Wright

PREFACE

During recent years studies in systems of representatives for collections of sets have been shown to have a bearing on problems in linear programming, game theory, and network analysis. The present study was suggested by Professor Galler as a theoretical background for the extension of such results to broader classes of problems in these areas.

In addition to the members of the doctoral committee, who have read and criticized these findings, the author would like to thank all who have in any way been of assistance. Dr. A. J. Hoffman of RAND Corporation has clarified several points of his "most general" theorem in a letter to the author. Dr. R. E. Bellman and Dr. G. B. Dantzig of RAND Corporation, and Professor R. M. Thrall of the University of Michigan have been of much help in the mathematical programming aspects of the problem. Professor Jack Levine of North Carolina State College has raised several questions which have resulted in improvements in various portions of the paper.

The author is especially indebted to Professor R. Z. Norman of Dartmouth College for giving generously of his time in reading preliminary reports on the study. His many suggestions and references have been invaluable.

Credit is due to the staff of the Computing Center of the University of Michigan for the computations in the body of the paper, and to Mr. Henry Hamilton of the Computing Center of North Carolina State College for handling the computations of Appendix II.

Finally the author would like to acknowledge the less material, though no less real, help which his wife, family, and friends have given through their encouragement and confidence.

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CHAPTER I

INTRODUCTION AND HISTORY

Given a collection of non-empty sets, with elements from some universal set, one question which we might be led to ask is the following. Under what conditions can we select, subject to certain restrictions, elements which can serve as representatives for the sets in the collection? While various restrictions on the elements selected have been studied, relatively little work has been done on the characteristics of the collections of sets. It is our purpose to investigate the basic restriction on the elements - that they be distinct - when the collection of sets is subdivided into several smaller collections.

Defn. 1. Let $\mathcal{A} = \{a_1, \ldots, a_s\}$ be a finite set, and let $\mathcal{S}_1, \ldots, \mathcal{S}_m$ be m collections of not necessarily distinct subsets of $\mathcal{A}$. (Since the sets are not necessarily distinct, we are really concerned with sequences of sets, but shall retain the traditional term "collections.") Further suppose that each $\mathcal{S}_i$ consists of exactly n sets, $S_{i1}, \ldots, S_{in}$, for $i = 1, \ldots, m$. An element $a_k \in \mathcal{A}$ is a representative for the sets $S_{1j_1}, S_{2j_2}, \ldots, S_{mj_m}$ if $a_k \in \bigcap_{i=1}^{m} S_{ij_i}$. The system of collections $\mathcal{S}_1, \ldots, \mathcal{S}_m$ has a (common) system of distinct representatives (SDR) if there exist distinct elements $a_{k_1}, \ldots, a_{k_n}$, and a reordering of the sets within each collection such that a_{k_j} is a representative for the sets $S_{1j}, \ldots, S_{mj}$.

The early results in studies of systems of representatives came from several fields. Miller in 1910 [28] and Chapman in 1913 [1] showed that for any finite group the right and left cosets relative to a given subgroup always have a common set of coset representatives. In 1927

Scorza [34] proved that given two subgroups Q_1 and Q_2 of the same index in a finite group, the right cosets of Q_1 and the left cosets of Q_2 always have a common set of coset representatives.

Frobenius in 1912 [11] was led into the representative problem through the study of partitions of a square matrix of non-negative elements. In 1916 König arrived at Frobenius's result through the medium of graph theory [23, 24]. The statement of the theorem given here is that of Dulmage and Halperin [6].

Defn. 2. A set of k elements in a k by k matrix is a permutation set if there is exactly one element in each row and column.

Theorem 1. (Frobenius-König) A set S of elements in a k by k matrix contains at least one element in each of the $k!$ permutation sets of the matrix if and only if S contains a p by q submatrix of elements, with $p + q = k + 1$.

König arrived at this and other results through application of the following three theorems [23, 24].

Defn. 3. A graph is regular of degree k if each of its vertices is incident to exactly k edges. A graph is bipartite if each of its cycles contains an even number of edges. A subgraph of a given graph is a factor of degree one if its vertices are exactly those of the given graph and each vertex is incident to just one edge of the subgraph.

Theorem 2. (König) The vertices of a graph can be divided into two classes such that the end points of each edge of the graph belong to different classes if and only if the graph is bipartite.

Theorem 3. (König) Each finite regular bipartite graph contains a factor of degree one.

Theorem 4. (König) Each finite regular bipartite graph of degree k factors into k factors of the first degree.

(The proofs of these and other early results will be omitted, as they are special cases of later results.)

The above results were united in a paper by van der Waerden in 1927 [36]. He noted that the Miller-Chapman result is set-theoretic in nature and gave a set-theoretic statement and proof of it. (This also includes Scorza's result, which van der Waerden apparently did not know.)

Theorem 5. (van der Waerden) Any two partitions of a set of pq elements into p classes of q elements each always have a common system of distinct representatives.

Artin [see 36] noted that in this case there are exactly q totally distinct SDRs; and the referee for the paper showed that these results are equivalent to those of König cited above. To establish this correspondence we let the edges of the regular bipartite graph be the points of the set, the two classes of vertices of the graphs be the two partitions, and the vertices themselves be the classes of the partitions. The SDR is then represented by the factor of degree one.

Subsequently Sperner gave an elegant induction proof of van der Waerden's theorem, at the same time generalizing it as follows [35].

Theorem 6. (Sperner) Two partitions of $pq - k$ elements into p classes of at most q elements each, where $0 \leq k \leq q - 1$, always have a common system of distinct representatives.

The above results all pertain to two collections of sets, which are partitions of the universal set, and give sufficient conditions for the existence of a common system of representatives. While no mention is made in the original statements of distinctness, this is assured by the fact that the collections are partitions.

In 1935, P. Hall presented a paper which has provided the basis for most of the work since undertaken [17]. Hall first examined the case $m = 1$ (in the language of Defn. 1) with no restrictions on the sets.

Theorem 7. (P. Hall) Let $m = 1$. Then there exists an SDR for σ_1 if and only if the union of any k sets from σ_1 contains at least k elements, for $k = 1, \ldots, n$.

A proof by Ford and Fulkerson of Theorem 7 will be given below.

Theorem 8. (P. Hall) Let $m = 1$, and suppose also that $\mathcal{A}$ is partitioned into at least n classes, $C_1, \ldots, C_t$, $t \geq n$. Then there exists an SDR for $\sigma_1 = \{S_{11}, \ldots, S_{1n}\}$, no two elements of which belong to the same partition class, if and only if the union of any k of the sets S_{1j} contains elements from at least k classes, for $k = 1, \ldots, n$.

Proof. The 'only if' is obvious. Let us then assume that the given condition holds. Let the partition consist of the classes $C_1, \ldots, C_t$, and let $s_j = \{C_i | C_i \cap S_{1j} \neq \emptyset\}$, $j = 1, \ldots, n$. The given condition on the sets S_{1j} is then equivalent to the statement that the union of

any k of the collections s_j contains at least k of the classes C_i. Hence, applying Theorem 7 to the collections s_j, we may choose an SDR, that is, a collection of n distinct classes, say $C_1, \ldots, C_n$, having the property that $C_j \cap S_{1j} \neq \emptyset$, $j = 1, \ldots, n$. Since these classes are disjoint, the choice of a_j as any element of $C_j \cap S_{1j}$ yields the desired system of representatives.

Using this result, Hall then obtained a necessary and sufficient condition that there exist an SDR for $m = 2$, provided that the collections are partitions, as a special case.

Theorem 9. (P. Hall) Let $m = 2$, and suppose that the two collections are partitions. Then there exists an SDR for $\mathcal{S}_1$ and $\mathcal{S}_2$ if and only if the union of any k sets of $\mathcal{S}_1$ contains elements from at least k sets of $\mathcal{S}_2$, for $k = 1, \ldots, n$.

All of the previous results then are special cases of this theorem, and it is easily shown that under the conditions given for these cases, Hall's condition holds.

Hall's theorems provide a constructive method of obtaining a system of distinct representatives. For example, if $m = 2$ and the collections are partitions, we may apply Theorem 9 successively to the sets S_{11}; S_{11} and S_{12}; $\ldots$; $S_{11}, S_{12}, \ldots, S_{1n}$. The theorem assures us that if an SDR exists at each stage in this process, we have sufficiently many elements available to choose representatives for $S_{11}, \ldots, S_{1k}$ (and the corresponding sets in $\mathcal{S}_2$). The difficulty in this is similar to that which is encountered in constructions involved in the "four color problem." The addition of another set may necessitate reassigning all of the previously chosen representatives. In making this

reassignment one would like to use as much of the previous assignment as possible. M. Hall [16] gives an algorithm for doing this, which is discussed in Chapter IV.

There remained one condition from van der Waerden's theorem to be removed. Hall had successfully removed the restriction on the size of the sets, but he had not been completely successful in removing the requirement of disjointness. This was not needed for the case $m = 1$, but for $m = 2$ it was still required that at least one of the collections be a partition. In 1957 L. R. Ford, Jr. and Fulkerson dropped this condition in a pair of theorems which generalize Hall's work and at the same time present the problem in a new light, that of network theory and linear programming [10].

Defn. 4. A network N is a finite directed linear graph each edge (x, y) of which has associated with it a capacity $c(x, y)$, which is either a non-negative real number or plus infinity. (We assume that if there is no edge (x, y) then $c(x, y) = 0$.) There may exist in a network two distinguished sets of vertices, the sources S and the sinks S', which do not necessarily exhaust the set of vertices, and which have no vertex in common. A flow is an assignment of a non-negative number $f(x, y)$ to each edge (x, y) of the network, such that i) $f(x, y) \leq c(x, y)$ for all edges (x, y), and ii) for each vertex v which is neither a source nor a sink, the total flow into v is equal to the total flow out. The value of a flow f is $\sum_{\sigma \in S} \sum_{x \in N} \left[f(\sigma, x) - f(x, \sigma) \right]$. A cut in N with respect to S and S' is a partition of the vertices into two complementary sets L and L' such that $S \subseteq L$, $S' \subseteq L'$. The value of a cut is $\sum_{x \in L} \sum_{y \in L'} c(x, y)$.

The problem of the existence of an SDR for $m = 1$ or $m = 2$ is equivalent to the existence of a maximal flow for value n in a properly designed network. Ford and Fulkerson proved this for the more general setting of "restricted" representatives. Their proofs invoke the following two theorems from network theory [3, 8, 9, 10, 12, 13].

Theorem 10. Max-flow min-cut. For any network the maximal flow value is equal to the minimal cut value.

Theorem 11. Integrality. For any network, if the capacity function is integer-valued, then there exists a maximal flow which is integer-valued on each edge.

Defn. 5. Let $\mathcal{A}, \mathcal{S}_1, \ldots, \mathcal{S}_m$ be as in Defn. 1. The system of collections $\mathcal{S}_1, \ldots, \mathcal{S}_m$ has a system of restricted representatives (SRR) with restrictions (α_i, β_i), $i = 1, \ldots, s$, if there exist elements $a_{k_1}, \ldots, a_{k_n}$, not necessarily distinct, and a reordering of the sets within each collection, such that a_{k_j} is a representative for the sets $S_{1j}, \ldots, S_{mj}$, subject to the following condition. Each element a_i of $\mathcal{A}$ occurs among the representative set at least α_i and at most β_i times $(0 \leq \alpha_i \leq \beta_i)$.

Proof of Theorem 7. Ford and Fulkerson's proof is based on the network of Fig. 1 [10]. Here we have one source, σ, and one sink, σ'. The vertices $\bar{S}_j$ represent the sets S_{1j} and the vertices $\bar{a}_i$ correspond to the elements a_i of $\mathcal{A}$. The capacities are defined by

$$c(\sigma, \bar{S}_j) = 1, \quad \text{for } j = 1, \ldots, n$$

$$c(\bar{S}_j, \bar{a}_i) = \begin{cases} +\infty & \text{if } a_i \in S_{1j} \\ 0 & \text{otherwise} \end{cases}$$

$$c(\bar{a}_i, \sigma') = 1, \quad \text{for } i = 1, \ldots, s.$$

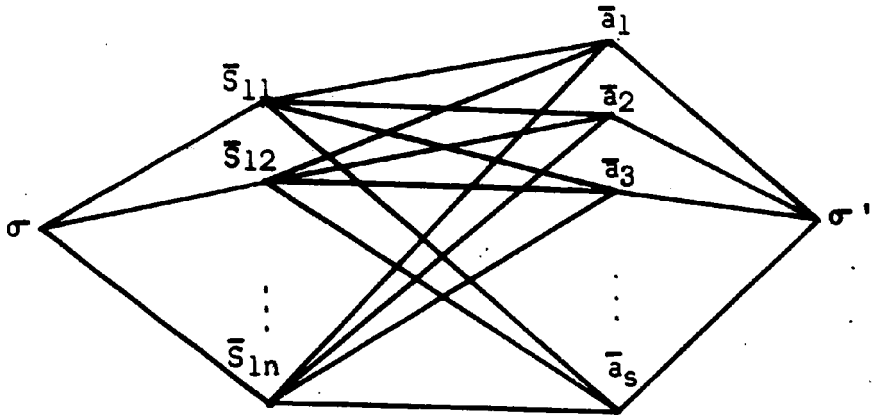


Fig. 1. The network for $m = 1$

If there exists an SDR, we can construct a flow of value n by letting f take on values as follows:

$$f(\sigma, \bar{S}_j) = 1, \text{ for } j = 1, \ldots, n$$

$$f(\bar{S}_j, \bar{a}_i) = \begin{cases} 1 & \text{if } a_i \text{ represents } S_{1j} \\ 0 & \text{otherwise} \end{cases}$$

$$f(\bar{a}_i, \sigma') = \begin{cases} 1 & \text{if } a_i \text{ occurs in the SDR} \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 10 this flow is maximal since there exists a cut of value n (let $L = \{\sigma\}$, $L' = N - \{\sigma\}$). For the converse, if there exists a maximal flow of value n, then there exists an integral flow of value n, by Theorem 11. With such a flow, let a_i represent S_{1j} if and only if $f(\bar{S}_j, \bar{a}_i) = 1$. Then each set S_{1j} is represented since $c(\sigma, \bar{S}_j) = 1$, and no two sets are represented by the same a_i since $c(\bar{a}_i, \sigma') = 1$. Hence we have an SDR.

To obtain Hall's condition from this network, we must examine all possible cuts.

Notation: If X is any set, let $Cd(X)$ denote the cardinality of X. If X is a set of vertices in a network, let $J(X)$ denote the set of

all vertices of N which are joined to some vertex of X by an edge with a non-zero capacity. Given two disjoint subsets X, Y of the vertices of a network, extend the capacity function to these subsets by defining $c(X, Y) = \sum_{x \in X} \sum_{y \in Y} c(x, y)$.

Let L, L' be any cut in the network in Fig. 1, and let $\bar{S} = \{\bar{S}_j\}$, $j = 1, \ldots, n$; $\bar{A} = \{\bar{a}_i\}$, $i = 1, \ldots, s$. Define subsets of the vertices of N by $X = L \cap \bar{S}$, $X' = L' \cap \bar{S}$, $Y = L \cap \bar{A}$, and $Y' = L' \cap \bar{A}$. In order that there exist a flow of value n, the value of any such cut must be at least n:

$$c(L, L') = c(\sigma, X') + c(X, Y') + c(Y, \sigma') \geq n, \quad (1)$$

for all $X \subset \bar{S}$, $Y \subset \bar{A}$. Fig. 2 represents this situation,[1] with the capacity of the cut defined as the sum of the capacities of the edges joining L and L'.

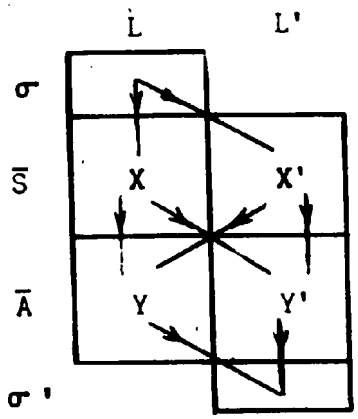


Fig. 2. The partitioning of the vertices of a network

[1]Fig. 2 and similar diagrams elsewhere in the paper are analogous to Venn diagrams. Sets of vertices are represented by areas within simple closed curves, with intersection and union as in the Venn diagrams. The lines between various portions of the figure indicate that in the network there exist one or more edges joining vertices in the indicated subsets.

Since the capacities for the edges from $\bar{S}$ to $\bar{A}$ are infinite, (1) holds automatically unless (X, Y') is empty; and if (X, Y') is empty then $J(X) \cap \bar{A} \subset Y$. Assuming that (X, Y') is empty, (1) is then equivalent to

$$Cd(X') + Cd(Y) \geq n \tag{2}$$

for all $X \subset \bar{S}$ and all $Y \supset J(X) \cap \bar{A}$. Hence

$$Cd(X') + Cd(J(X) \cap \bar{A}) \geq n \tag{3}$$

for all $X \subset \bar{S}$. But since $Cd(X') = n - Cd(X)$, this yields

$$Cd(X) \leq Cd(J(X) \cap \bar{A}) \text{ for all } X \subset \bar{S}. \tag{4}$$

This is Hall's condition, since $J(X) \cap \bar{A}$ is just the set of vertices representing elements of those sets which are represented by points in X.

Using similar techniques with more extensive networks, Ford and Fulkerson obtained the following two results on systems of restricted representatives, which apply without restricting the collections to be partitions. In order to obtain systems of distinct representatives, we would apply these theorems with the restrictions $\alpha_i = 0$, $\beta_i = 1$, $i = 1, \ldots, s$.

Theorem 12. (Ford and Fulkerson). Let $m = 1$. There exists an SRR if and only if for every subset X of the indices $\{1, \ldots, n\}$,

$$Cd(X) \leq \min \left(n - \sum_{i=1}^{s} \alpha_i + \sum_{I(X)} \alpha_i, \sum_{I(X)} \beta_i\right), \tag{5}$$

where $I(X) = \left\{i \middle| a_i \in \bigcup_{j \in X} S_{ij}\right\}$.

Theorem 13. (Ford and Fulkerson) Let $m = 2$. There exists an SRR if and only if for every $X, Y \subset \{1, \ldots, n\}$,

$$Cd(X) + Cd(Y) \leq n - \sum_{i=1}^{s} \alpha_i + \sum_{I(X) \cup I(Y)} \alpha_i + \sum_{I(X) \cap I(Y)} \beta_i, \tag{6}$$

where $I(X) = \{i \mid a_i \in \bigcup_{j \in X} S_{1j}\}$, and $I(Y) = \{i \mid a_i \in \bigcup_{j \in Y} S_{2j}\}$.

Corollary. Let $m = 2$. There exists an SDR if and only if for every $X, Y \subset \{1, \ldots, n\}$,

$$Cd(X) + Cd(Y) \leq n + Cd(I(X) \cap I(Y)), \tag{7}$$

where $I(X)$, $I(Y)$ are as in Theorem 13.

In the years since Hall's paper appeared, a number of modifications have appeared in the literature. These fall mainly into two categories. In the first category, the results are concerned with various restrictions on the elements in the set of representatives. Thus, Hoffman and Kuhn [22] have considered SDRs having intersections with partition classes whose cardinalities are within specified bounds. Gale [13] has discussed representative systems wherein there are bounds on the number of times various elements may occur, as have Ford and Fulkerson in the work cited above. Hoffman and Kuhn [21], and Mann and Ryser [26] have obtained results relating to SDRs having certain specified elements. (These can also be obtained from Ford and Fulkerson's work by letting $\alpha_i = \beta_i = 1$ for these elements.)

The other general category of results concerns the extension of Hall's results to an infinite base set $\mathcal{A}$. Some of these are concerned with finite collections of subsets, others with infinite collections. Papers by Everett and Whaples [7], M. Hall [15], and R. Rado [31; 32, Lemma 2] are representative of contributions in this area.

In addition to the above, there have been a few results along slightly different lines. M. Hall [15] has obtained a lower bound on the number of SDRs for $m = 1$. Some work has been done in applications of Hall's theorems. In addition to the network papers mentioned above, the linear programming papers of Dwyer and Galler [4, 5], and of Hoffman and Kuhn [21, 22] are of interest in this connection.

Many of the results relating to systems of representatives are summarized in the papers by Mann and Ryser [26], and Hoffman [19]. Hoffman's paper deals only with results based on a finite set $\mathcal{A}$. In it Hoffman presents "the 'most general' theorem of the Hall type," with but a brief sketch of its proof. While the comments by Hoffman lead one to believe that this theorem covers all known finitary results, it is stated only for the case $m = 1$. From the statement of the theorem and the discussion of it, it is not clear that the theorem will handle the case $m = 2$, or that it can be modified to handle this case. The argument is made that the case $m = 2$ can be represented by a network, as Ford and Fulkerson have done, and that such networks are covered by the theorem. But it is not known how the two collections of sets will be transformed into the single family stipulated in the theorem [20].

CHAPTER II

REPRESENTATIVES FOR MORE THAN TWO COLLECTIONS OF SETS

While the above results have been phrased in several different terminologies and provide generalizations in several directions, they have in common the fact that they all deal with only one or two collections of subsets of $\mathcal{A}$. In this chapter we consider first the problem of determining the existence of representatives for three or more arbitrary finite collections of sets. We shall be concerned only with distinct representatives over a finite base set $\mathcal{A}$. After considering the problem in general, we shall stipulate that the collections be partitions of $\mathcal{A}$, and shall finally consider in more detail the structure of the individual sets in each collection. We begin with general remarks, definitions, and notation.

If an element of $\mathcal{A}$ is to be a representative it must occur somewhere in each collection. Thus we assume that for each $i = 1, \ldots, m$, the sets S_{ij} cover $\mathcal{A}$. That is, given $a_k \in \mathcal{A}$, there exists a set S_{ij} such that $a_k \in S_{ij}$, for $i = 1, \ldots, m$. We also assume that $\mathcal{A}$ contains at least n elements.

With $\mathcal{X}_i$ denoting any subcollection of $\mathcal{S}_i$, we let $X_i = \{j \mid S_{ij} \in \mathcal{X}_i\}$, and we let $I_i = I(\mathcal{X}_i) = \{k \mid a_k \in \bigcup_{\mathcal{X}_i} S_{ij}\}$. Further we denote the cardinality of any set X by $Cd(X)$, as in Chapter I. We may then assume that $Cd(X_i) \geq 1$ for all i, that is, that the subcollections $\mathcal{X}_i$ are non-empty; for the case of empty subcollections reduces to the case of fewer than m collections. With this notation

and these assumptions we may restate Theorem 7 and the Corollary to Theorem 13 as follows.

Theorem 7'. Let $m = 1$. Then there exists an SDR if and only if for any $\mathcal{X}_1 \subseteq \mathcal{G}_1$, $Cd(X_1) \leq Cd(I_1)$.

Theorem 13'. Let $m = 2$. Then there exists an SDR if and only if for any $\mathcal{X}_1 \subseteq \mathcal{G}_1$ and any $\mathcal{X}_2 \subseteq \mathcal{G}_2$, $Cd(X_1) + Cd(X_2) \leq n + Cd(I_1 \cap I_2)$.

With these results in mind, one is led to the following question. Does there exist some function, $f(m, n)$, such that for any m and any n, and for any $\mathcal{X}_i \subseteq \mathcal{G}_i$, $i = 1, \ldots, m$, an SDR exists if and only if

$$\sum_{i=1}^{m} Cd(X_i) \leq f(m, n) + Cd(\bigcap_{i=1}^{m} I_i)? \tag{8}$$

We shall give examples showing that the answer to this question is negative. In addition, we shall find one function providing a necessary condition for the existence of an SDR and another function providing a sufficient condition, thus providing a partial positive answer to our question. Moreover, we shall find that one may require that the collections be partitions and impose restrictions on the size of the sets without materially changing the results.

We observe that the function f in (8) must satisfy the conditions i) $f(1, n) = 0$ (Theorem 7'); ii) $f(2, n) = n$ (Theorem 13'); and, as will be shown below, iii) $f(m, 1) = m - 1$. With these requirements in mind, we choose $f(m, n) = (m - 1)n - k(m, n)$, where k is a function of m and n still to be determined. In particular, for the

above conditions we want the value of k to be zero. We define Q by $Q(m, n, k) = (m - 1)n - k + Cd(\bigcap_{i=1}^{m} I_i) - \sum_{i=1}^{m} Cd(X_i)$, so that (8) becomes $Q(m, n, k) \geq 0$. We also define Q_0 by $Q_0(m, n) = Q(m, n, 0)$. Given the set $\mathcal{A}$ and the collections $\mathcal{S}_1, \ldots, \mathcal{S}_m$, we shall say that $Q \geq 0$ (or $Q_0 \geq 0$) <u>completely</u> if this is true for all possible choices of non-empty $X_i \subseteq \mathcal{S}_i$, $i = 1, \ldots, m$.

We shall state all theorems with respect to a presupposed finite set $\mathcal{A}$ and presupposed collections $\mathcal{S}_1, \ldots, \mathcal{S}_m$.

<u>Theorem 14</u>. For $n = 1$, there exists an SDR if and only if $Q_0(m, 1) \geq 0$ completely.

<u>Proof</u>. Since $n = 1$, $X_i = \mathcal{S}_i$, $i = 1, \ldots, m$. Thus $Q_0(m, 1) = Cd \bigcap_{i=1}^{m} I_i) - 1$. If there is an SDR then there is at least one element common to all collections, and $Q_0(m, 1) \geq 0$. But if there is no SDR then there is no element in common, and $Q_0(m, 1) = -1$.

<u>Theorem 15</u>. For any m and n, if there exists an SDR then $Q_0 \geq 0$ completely.

<u>Proof</u>. We assume, without loss of generality, that the elements of $\mathcal{A}$ and the sets S_{ij} are so numbered that a_j is a representative for $S_{1j}, S_{2j}, \ldots, S_{mj}$, $j = 1, \ldots, n$. Then $R = \{a_1, \ldots, a_n\}$ is an SDR for $\mathcal{S}_1, \ldots, \mathcal{S}_m$. Let $S'_{ij} = \{a_j\}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$. Then each choice of an $X_i = \{S_{ij_1}, S_{ij_2}, \ldots\}$ induces a choice of an $X'_i = \{S'_{ij_1}, S'_{ij_2}, \ldots\}$, with the property that $\bigcap_{i=1}^{m} I(X'_i) \subseteq \bigcap_{i=1}^{m} I(X_i)$. On the other hand, $\sum_{i=1}^{m} Cd(X'_i) = \sum_{i=1}^{m} Cd(X_i)$.

Thus in order to establish that $Q_0 \geq 0$ completely, it is sufficient to show that

$$(m - 1)n + Cd(\bigcap_{i=1}^{m} I(\mathcal{X}_i')) - \sum_{i=1}^{m} Cd(X_i') \geq 0. \qquad (9)$$

Now in our reduced system each collection $\mathcal{S}_i'$ contains n sets of one element each, hence n elements. In choosing a subcollection $\mathcal{X}_i'$ we omit exactly $n - Cd(X_i')$ elements. Thus in choosing all of the $\mathcal{X}_i'$ we omit at most $\sum_{i=1}^{m} (n - Cd(X_i')) = mn - \sum_{i=1}^{m} Cd(X_i')$ elements from the original n. That is, $Cd(\bigcap_{i=1}^{m} I(\mathcal{X}_i')) \geq n - mn + \sum_{i=1}^{m} Cd(X_i')$, thus establishing (9).

It should be noted that $f(m, n) = (m - 1)n$ is the largest function we need consider, since this provides a necessary condition for the existence of an SDR; and if Q' is a modification of Q by replacing $(m - 1)n - k$ by an arbitrary function g, then $Q' \geq 0$ will provide a sufficient condition only if $Q_0 \geq 0$ is satisfied at the same time. That is, for any particular m and n, $g(m, n)$ need have at most the value $(m - 1)n$.

The following examples show that the necessary condition given by Theorem 15 cannot be improved, i. e., that equality $Q_0 = 0$ is actually attained, and that $Q_0 \geq 0$ completely is not a sufficient condition for the existence of an SDR.

Example 1. In this example we have an SDR, namely $\{2, 3, 4\}$.

$S_{11} = \{1, 2, 3\}$	$S_{21} = \{1, 2, 4\}$	$S_{31} = \{2, 3, 4\}$
$S_{12} = \{2, 3, 4\}$	$S_{22} = \{1, 3, 4\}$	$S_{32} = \{1, 2, 4\}$
$S_{13} = \{3\}$	$S_{23} = \{2\}$	$S_{33} = \{4\}$

Equality ($Q_0 = 0$) is attained by choosing $\chi_1 = \{S_{13}\}$, $\chi_2 = \{S_{21}, S_{23}\}$, and $\chi_3 = \sigma_3$. $\mathcal{A} = \{1, 2, 3, 4\}$.

Example 2. Here there is no SDR.

$S_{11} = \{1, 2\}$ $\quad S_{21} = \{1, 3\}$ $\quad S_{31} = \{1, 4\}$

$S_{12} = \{3, 4\}$ $\quad S_{22} = \{4\}$ $\quad S_{32} = \{2, 5\}$

$S_{13} = \{5\}$ $\quad S_{23} = \{2, 5\}$ $\quad S_{33} = \{3\}$

In this case, $Q_0 = 0$ for $\chi_1 = \{S_{13}\}$, $\chi_2 = \{S_{21}, S_{22}\}$, and $\chi_3 = \sigma_3$. $\mathcal{A} = \{1, 2, 3, 4, 5\}$.

Since the above example shows that the given condition is not sufficient to insure the existence of an SDR, one is led to seek modifications. The most immediate of these is to strengthen the inequality to $Q_0 > 0$ completely. The following example shows that this also is not sufficient.

Example 3.

$S_{11} = \{1, 2\}$ $\quad S_{21} = \{1, 3\}$ $\quad S_{31} = \{1, 4\}$

$S_{12} = \{3, 4\}$ $\quad S_{22} = \{2, 4\}$ $\quad S_{32} = \{2, 3\}$

Here the minimum value of Q_0 is one, when $\chi_1 = \{S_{11}\}$, $\chi_2 = \{S_{21}\}$, and $\chi_3 = \sigma_3$. There is no SDR. $\mathcal{A} = \{1, 2, 3, 4\}$.

The best general sufficient condition which the author has found is obtained by choosing $k = m(n - 1) + 1$, which yields $(m - 1)n - k = m - n + 1$.

Theorem 16. For $n \geq 2$, if $Q(m, n, m(n - 1) + 1) \geq 0$ completely, then there exists an SDR.

Proof. Let the $\mathcal{X}_i$ each consist of one set. Then since $Q \geq 0$, $Cd(\bigcap_{i=1}^{m} I_i) \geq n - 1$. Suppose first that for some such choice of the $\mathcal{X}_i$, say for $\mathcal{X}_i = \{S_{i1}\}$, $i = 1, \ldots, m$, $Cd(\bigcap_{i=1}^{m} I_i) \geq n$. Then the intersections $S_{1j} \cap S_{2j} \cap \ldots \cap S_{mj}$, $j = 2, \ldots, n$, each contain at least $n - 1$ elements. Thus we may choose $n - 1$ distinct elements as representatives for the sets $S_{1j}, S_{2j}, \ldots, S_{mj}$, $j = 2, \ldots, n$. But then at least one of the elements in $\bigcap_{i=1}^{m} S_{i1}$ is not among the chosen ones. Choose one of these to complete the SDR.

Suppose now that for all choices of the $\mathcal{X}_i$, $Cd(\bigcap_{i=1}^{m} I_i) = n - 1$. Thus $S_{1j_1} \cap S_{2j_2} \cap \ldots \cap S_{mj_m} = \{a^{(1)}_{j_1 \ldots j_m}, a^{(2)}_{j_1 \ldots j_m}, \ldots, a^{(n-1)}_{j_1 \ldots j_m}\}$. Now there will exist an SDR unless for each choice of $j_1, \ldots, j_m$

$$S_{1j_1} \cap S_{2j_2} \cap \ldots \cap S_{mj_m} = S_{1,j_1+1} \cap S_{2,j_2+1} \cap \ldots \cap S_{m,j_m+1} = \ldots$$
$$= S_{1,j_1+n-1} \cap S_{2,j_2+n-1} \cap \ldots \cap S_{m,j_m+n-1}, \qquad (10)$$

where the subscript addition is modulo n. If (10) holds for all choices of the $j_1, \ldots, j_m$, let the elements of $\mathcal{A}$ be so numbered that $S_{11} \cap S_{21} \cap \ldots \cap S_{m1} = \{a_1, \ldots, a_{n-1}\}$. Then by (10) $\{a_1, \ldots, a_{n-1}\} \subseteq S_{ij}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$. But by the assumption that $Cd(\bigcap_{i=1}^{m} I_i) = n - 1$, and the assumption that each element occurring in any given set occurs in some set in each collection, it follows that $S_{ij} = \{a_1, \ldots, a_{n-1}\}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$. This contradicts the

fact that each $\mathcal{S}_i$ contains at least n elements. Hence (10) fails for some choice of $j_1, \ldots, j_m$, and an SDR exists.

Example 3 shows that at least for the case $m = 3$, $n = 2$ this result cannot be improved. For Example 3 satisfies $Q(m, n, m(n-1)) \geq 0$ completely, but there is no SDR. Although a number of additional examples are known which satisfy the necessary but not the sufficient condition, and have no SDR, none other than Example 3 is known which satisfies $Q(m, n, m(n-1)) \geq 0$ completely. Thus, although it is believed that such examples exist, there is at present no proof that the sufficient condition cannot be improved to some extent. As we shall see later, any such examples must contain a large number of elements.

If we return to the original restriction of König and Hall that the collections be partitions, we are able to make some improvement on Theorem 16. But here also there is the lack of examples mentioned above. The improvement cannot be complete, i. e., we cannot find a single condition which is both necessary and sufficient, since in Example 3 the collections are partitions. We show first that the solution of the problem in the disjoint case is equivalent to finding the general solution.

Defn. 6. A system of collections $\mathcal{S}'_i = \{S'_{ij}\}_{j=1}^{n}$, $i = 1, \ldots, m$, is a disjoint subsystem of the system $\mathcal{S}_i$, $i = 1, \ldots, m$, if

i) for each $i = 1, \ldots, m$, the sets S'_{ij} are non-empty and pairwise disjoint;

ii) $S'_{ij} \subseteq S_{ij}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$; and

iii) $\bigcup_{j=1}^{n} S'_{ij} = \bigcup_{j=1}^{n} S_{ij} = \mathcal{A}$, $i = 1, \ldots, m$.

Theorem 17. The system of collections $\mathcal{S}_i$, $i = 1, \ldots, m$, has an SDR if and only if at least one of its disjoint subsystems has an SDR.

Proof. Any SDR for a disjoint subsystem is an SDR for the original system of collections. Conversely, if the original system has an SDR, we may form a disjoint subsystem by letting S'_{ij} be the set consisting of the representative for S_{ij} and any elements of S_{ij} which are neither representatives for some set nor included in any set $S'_{ij'}$, $j' < j$, for all i and j. Then the original SDR is also an SDR for this disjoint subsystem.

In view of this result, we shall assume henceforth that all collections mentioned are partitions, unless otherwise specified. An arbitrary system of collections will have at most a finite number of disjoint subsystems, and in fact it is obvious that if no disjoint subsystem exist, then there can be no SDR.

Theorem 18. If the sets in each collection are pairwise disjoint, and $Q(m, n, (m-1)\cdot(n-1) \geq 0$ completely, then there exists an SDR.

Proof. In particular, for a fixed j, choosing χ_i to be the one element set $\{S_{ij}\}$, $i = 1, \ldots, m$, we obtain the inequalities $Cd(\bigcap_{i=1}^{m} I(\{S_{ij}\})) \geq 1$, for $j = 1, \ldots, n$. By the disjointness, these intersections are all distinct. Choose an element from each to form an SDR.

The proof of Theorem 18 utilizes only a small number of the possible χ_i. Thus it would seem possible to improve this result by considering further χ_i. If we have m collections of n sets each, then it is possible to form $(2^n - 1)^m$ combinations of the sets into non-empty subcollections χ_i. Since the proof of Theorem 18 requires only m

of these, we have $(2^n - 1)^m - m$ subcollections which may provide additional information. Actually, many of these are useless, or add nothing new to our knowledge. Clearly, if $\sum_{i=1}^{m} Cd(\chi_i) \geq (m - 1)n - k$, then $Q \geq 0$ holds automatically for these particular χ_i. On the other hand, the fact that inclusion relations exist between some of the χ_i leads one to believe that there may also be informational relationships. This is in fact the case: if $Q \geq 0$ for all χ_i satisfying certain conditions, then $Q \geq 0$ also for certain other χ_i related to the original ones by inclusion.

Theorem 19. If for all χ_{i_o} for which $Cd(\chi_{i_o}) = d$, and for a given class of subcollections χ_i, $i \neq i_o$,

$$0 < \sum_{i \neq i_o} Cd(\chi_i) + d - (m - 1)n + k \leq Cd(I_{i_o} \cap \bigcap_{i \neq i_o} I_i), \tag{11}$$

then for any χ'_{i_o} with $Cd(\chi'_{i_o}) \geq d$, and the same χ_i, $i \neq i_o$,

$$\sum_{i \neq i_o} Cd(\chi_i) + Cd(\chi'_{i_o}) - (m - 1)n + k \leq Cd(I'_{i_o} \cap \bigcap_{i \neq i_o} I_i). \tag{12}$$

Proof. It is sufficient to prove the result for χ'_{i_o} such that $Cd(\chi'_{i_o}) = d + 1$. Let b be the minimum over all χ_{i_o} with $Cd(\chi_{i_o}) = d$ of $Cd(I_{i_o} \cap \bigcap_{i \neq i_o} I_i)$, and suppose that for some χ'_{i_o} with $Cd(\chi'_{i_o}) = d + 1$, say $\chi'_{i_o} = \{S_{i_o j_1}, \ldots, S_{i_o j_{d+1}}\}$, $Cd(I'_{i_o} \cap \bigcap_{i \neq i_o} I_i) = b$. Let $\{a_1, \ldots, a_b\}$ be the set of elements contained in $\chi'_{i_o} \cap \bigcap_{i \neq i_o} \chi_i$. If $b > 1$, suppose that two elements are contained in one of the $S_{i_o j}$, say $a_1, a_2 \in S_{i_o j_1}$. If we now let

$\chi_{i_o} = \{S_{i_o j_2}, \ldots, S_{i_o j_{d+1}}\}$ we have a contradiction to the minimality of b. Hence the elements $a_1, \ldots, a_b$ are distributed throughout b of the sets $S_{i_o j}$. Suppose that $a_k \varepsilon S_{i_o j_k}$, $k = 1, \ldots, b$. Since any d sets of χ'_{i_o} contain at least b elements of the intersection in question, $b \leq d$. Thus at least one of the sets, say $S_{i_o j_{d+1}}$, contains none of the a_i. But then if $\chi_{i_o} = \{S_{i_o j_2}, \ldots, S_{i_o j_{d+1}}\}$, we have again a contradiction to the minimality of b.

Example 4. Suppose $m = 3$. If we know that (11) holds for all χ_1 with $Cd(X_1) = 2$, and for certain χ_2 and χ_3 with $Cd(X_2) = 1$ and $Cd(X_3) = 2$, then we may assert that (12) holds for these same χ_2 and χ_3, and all χ_1 with $Cd(X_1) = 3, 4, \ldots, n$. Moreover, if (11) holds for all χ_2 and χ_3 with the above cardinalities, we may let i_o range successively over 1, 2, and 3, and thus show that (12) holds for all χ_1, χ_2, and χ_3 with $Cd(X_1) \geq 2$, $Cd(X_2) \geq 1$, and $Cd(X_3) \geq 2$.

Corollary. If $Q \geq 0$ for all choices of the χ_i such that $\sum_{i=1}^{m} Cd(X_i) = (m - 1)n - k + 1$, then $Q \geq 0$ completely.

Of course, there may exist no χ_i satisfying the condition in this corollary. In fact, since $\sum_{i=1}^{m} Cd(X_i) \geq m$ for all choices of χ_i, a choice of χ_i satisfying the condition in the corollary will exist if and only if $k \leq (m - 1)(n - 1)$. However, the situation for $k > (m - 1)(n - 1)$ is covered by the following lemma.

Lemma 1. If $k' > k$, then $Q(m, n, k') \geq 0$ for some choice of the χ_i implies that $Q(m, n, k) \geq 0$ for that same choice.

Proof. Clear from the definition of Q, since $(m-1)n-k > (m-1)n-k'$ when $k' > k$.

In particular, if $Q(m, n, k) \geq 0$ completely, for $k > (m-1)(n-1)$, then this is so also for $k = (m-1)(n-1)$, and the collections satisfy the sufficient condition of Theorem 18.

The following two examples show that Theorem 19 cannot be extended.

Example 5. Disjointness is necessary.

$S_{11} = \{1\}$ $S_{21} = \{1, 2\}$ $S_{31} = \{1, 3\}$

$S_{12} = \{1\}$ $S_{22} = \{3, 4\}$ $S_{32} = \{2, 5\}$

$S_{13} = \{2, 3, 4, 5, 6\}$ $S_{23} = \{5, 6\}$ $S_{33} = \{4, 6\}$

Then $m = n = 3$. If $k = 0$, $i_o = 1$, $d = 1$, then $0 < Cd(X_2) + Cd(X_3) + 1 - 6 \leq Cd(I_1 \cap I_2 \cap I_3)$ holds with $Cd(X_2) = Cd(X_3) = 3$, and the hypothesis reduces to $Cd(I(\{S_{1j}\})) \geq 1$, $j = 1, 2, 3$. The conclusion then reads, for $Cd(X_1') = 2$, $Cd(I(\{S_{1p}, S_{1q}\})) \geq 2$, $p \neq q$. But this is false $p = 1$, $q = 2$. $\mathcal{A} = \{1, 2, 3, 4, 5, 6\}$.

Example 6. $\sum_{i \neq i_o} Cd(X_i) + d - (m-1)n + k > 0$ is necessary.

Suppose we let this expression equal zero. Then the hypothesis becomes $Cd(I_{i_o} \cap \bigcap_{i \neq i_o} I_i) \geq 0$, and is trivially satisfied. The desired conclusion reads $Cd(I'_{i_o} \cap \bigcap_{i \neq i_o} I_i) \geq Cd(X'_{i_o}) - 1$. Let $m = n = 3$, $k = 0$, $i_o = d = 1$, $Cd(X_i') = 2$, and consider the following example.

$S_{11} = \{1\}$ $S_{21} = \{1, 2\}$ $S_{31} = \{1, 3\}$

$S_{12} = \{2\}$ $S_{22} = \{3, 4\}$ $S_{32} = \{2, 5\}$

$S_{13} = \{3, 4, 5, 6\}$ $S_{23} = \{5, 6\}$ $S_{33} = \{4, 6\}$

Here we have $Cd(I\{S_{11}, S_{12}\}) \cap I(\{S_{22}, S_{23}\}) \cap I(\{S_{31}, S_{32}, S_{33}\})) = 0$.

But $Cd(X_1') - 1 = 1$. $\mathcal{A} = \{1, 2, 3, 4, 5, 6\}$.

In the Corollary to Theorem 19 we have one of our most powerful tools. Clearly the total number of possible subcollection combinations rapidly become unmanageable, even with large electronic computers. The corollary assures us that we need examine only a relatively few of these combinations in order to establish that $Q \geq 0$ completely, for any particular m, n, and k. And the discussion following Lemma 1 insures that the proper combinations to examine are always present. By Lemma 1 we need only be concerned with values of k no larger than $(m - 1)(n - 1)$ since this provides the sufficient condition, and we know that in this case there always exist subcollections satisfying the conditions of the corollary. The efficacy of the corollary of course depends on exactly how many combinations of subcollections are included in the "relatively few." We shall see that while the number of combinations involved is quite small for small values of m and n, it still becomes quite unwieldy for the values of interest in linear programming applications, etc.

Defn. 7. A composition of a number N is an ordered collection of positive integers whose sum is N. The integers forming the composition are called its parts.

In our problem we are interested in subcollections χ_i such that $\sum_{i=1}^{m} Cd(\chi_i) = (m - 1)n - k + 1$. Thus we are interested in compositions of $(m - 1)n - k + 1$. We have further the restriction that there be exactly m parts to each composition, and that the largest part be no greater than n. The enumerating generating function for such compositions is

$$c_m(5; n) = (t + t^2 + \ldots + t^n)^m. \tag{13}$$

(cf. Riordan [33], p. 124)

The total number of such compositions of a number j is given by the coefficient of t^j in the expansion of (13), namely

$$\sum \frac{m!}{p_1!\, p_2! \cdots p_n!}, \tag{14}$$

where the sum is over all simultaneous solutions of the equations

$$\sum_{\nu=1}^{n} p_\nu = m, \quad \sum_{\nu=1}^{n} \nu p_\nu = j.$$

In our context, p_ν is the number of subcollections χ_i such that $Cd(\chi_i) = \nu$. Hence $\sum_{i=1}^{m} Cd(\chi_i) = \sum_{\nu=1}^{n} \nu p_\nu$. Thus (14) is the number of compositions, i. e., the number of cardinality combinations which we need consider, for $j = (m - 1)n - k + 1$.

Now suppose that $Cd(\chi_i) = c_i$, with $(c_1, \ldots, c_m)$ one of the admissible compositions. Then exactly p_ν of the c_i are equal to ν, $\nu = 1, \ldots, n$. There are $\prod_{i=1}^{m} \binom{n}{c_i}$ ways in which the subcollections χ_i may be chosen with this particular combination of cardinalities. But

$$\prod_{i=1}^{m} \binom{n}{c_i} = \frac{(n!)^m}{\prod_{\nu=1}^{n} [\nu!(n - \nu)!]^{p_\nu}}. \tag{15}$$

Thus there are this many sets to examine for each composition associated with a particular set $p_1, \ldots, p_n$. Since there are $m!/(p_1!\, p_2! \cdots p_n!)$ such compositions, the Corollary to Theorem 19 asserts that we need examine only

$$N = \sum \frac{(n!)^m \, m!}{\prod_{\nu=1}^{n} \left[\nu!(n-\nu)!\right]^{p_\nu} p_\nu!}, \qquad (16)$$

where $\sum_{\nu=1}^{n} p_\nu = m$, $\sum_{\nu=1}^{n} \nu p_\nu = (m-1)n - k + 1$,

subcollections in order to state that $Q(m, n, k) \geq 0$ completely.

The values of N for $m = 2[1]6$, $n = 2[1]6$, $k = 0[1](m-1)(n-1)$ are given in Appendix II, along with the value of $T = (2^n - 1)^m$, the total number of subcollection combinations, for the same m and n.

Upon examining the tables in Appendix II we observe that the ratio N/T is less than 0.5 for all values of m and n given, and that this ratio decreases as m and n increase. Thus as m and n increase, the task of showing that $Q \geq 0$ completely becomes relatively more simple, since we need look at only N set combinations out of a total of T possible ones.

For $m = 2$, only the line $k = 0$ has significance in our work, since this corresponds to Ford and Fulkerson's necessary and sufficient condition. Yet even here we see that their condition is over-stated. For example, if there are six sets in each collection, we need examine only 792 of the 3969 possible set combinations, provided that we examine the proper ones.

In general the sum for N involves several terms. However, in three cases there is only one term in the sum, and only one type of arrangement of the subsets.

Case 1. $n = 2$.

In this case the two equations governing the terms in the sum become

$$\begin{aligned} p_1 + p_2 &= m \\ p_1 + 2p_2 &= 2m - k + 1. \end{aligned} \qquad (17)$$

These have only the solution $p_1 = k - 1$, $p_2 = m - (k - 1)$. That is, the only type of arrangement of the subcollections consists of $k - 1$ subcollections of one set each, and $m - (k - 1)$ subcollections of two sets each. The value of N is $2^{k+1}\binom{m}{k+1}$.

Case 2. $k = (m - 1)(n - 1)$.

Here the two governing equations become

$$\begin{aligned} p_1 + p_2 + \ldots + p_n &= m \\ p_1 + 2p_2 + \ldots + np_n &= m, \end{aligned} \tag{18}$$

which have only the solution $p_1 = m$, $p_2 = \ldots = p_n = 0$. Thus in this case the only type of arrangement of subcollections we need consider consists of one set from each collection, and there are $N = n^m$ such arrangements.

Case 3. $k = (m - 1)(n - 1) - 1$.

For this case the governing equations

$$\begin{aligned} p_1 + p_2 + \ldots + p_n &= m \\ p_1 + 2p_2 + \ldots + np_n &= m + 1 \end{aligned} \tag{19}$$

have the solution $p_1 = m - 1$, $p_2 = 1$, $p_3 = \ldots = p_n = 0$. Thus we have to examine arrangements of $m - 1$ subcollections of one set each, and one subcollection of two sets. There are $N = \frac{1}{2} m(n - 1)n^m$ such arrangements.

We turn now to consideration of the parameter k. Heretofore this parameter has been quite arbitrary, and we have seen that different values of it lead us to the necessary condition (Theorem 15) and the two sufficient conditions (Theorems 16 and 18). However, there has been nothing said about the relationship of this parameter to the collections and sets involved in the problem. As we shall now show, if we assume

k to be a constant function of m and n, then k is related to s, the number of elements in $\mathcal{A}$, and to the manner in which they are divided among the sets S_{ij}. We shall see that choosing a value of k sets a lower bound on the size of these sets, and also sets limits on the intersections between sets in the various collections. We shall continue to assume that the collections are partitions, although many of these results also hold in the more general case.

Lemma 2. If $Q \geq 0$ completely, then for all i and all j, $Cd(I\{S_{ij}\})) \geq k + 1$, and hence $s \geq (k + 1)n$.

Proof. Given i_o, j_o, let $\chi_{i_o} = \{S_{i_o j_o}\}$, and let $\chi_i = \sigma_i$ for $i \neq i_o$. The inequality $Q \geq 0$ reduces to $Cd(\bigcap_{i=1}^{m} I_i) =$

$$Cd(I(\{S_{i_o j_o}\})) \geq \sum_{i=1}^{m} Cd(\chi_i) - (m - 1)n + k = (m - 1)n + 1 - (m - 1)n + k =$$

$k + 1$. Also $s = \sum_{j=1}^{n} Cd(I(\{S_{ij}\})) \geq (k + 1)n$.

Lemma 3. If $Q \geq 0$ completely, and if $k \geq n - 1$, then every two sets $S_{i_1 j_1}$ and $S_{i_2 j_2}$, $i_1 \neq i_2$, intersect in at least $k + 2 - n \geq 1$ elements.

Proof. Let $\chi_{i_1} = \{S_{i_1 j_1}\}$, $\chi_{i_2} = \{S_{i_2 j_2}\}$, and $\chi_i = \sigma_i$ for $i \neq i_1, i_2$. The inequality $Q \geq 0$ reduces to

$$Cd(I(\{S_{i_1 j_1}\}) \cap I(\{S_{i_2 j_2}\})) \geq \sum_{i=1}^{m} Cd(\chi_i) - (m - 1)n + k =$$

$$(m - 2)n + 2 - (m - 1)n + k = k + 2 - n.$$

Corollary. If $k \geq n - 1$, then $s \geq n^2(k + 2 - n)$.

Proof. Since the sets in each collection form a partition of $\mathcal{A}$,

$$Cd(I(\{S_{i_1j_1}\})) = Cd\Big(\bigcup_{j_2=1}^{n} (I(\{S_{i_1j_1}\}) \cap I(\{S_{i_2j_2}\}))\Big) =$$

$$\sum_{j_2=1}^{n} Cd(I(\{S_{i_1j_1}\}) \cap I(\{S_{i_2j_2}\})) \geq n(k + 2 - n).$$ The result then follows from the fact that there are n sets in each collection.

If we now return to Theorem 18, we find that the hypothesis can be satisfied only if $s \geq n^2(m - 1)(n - 1) + 2 - n) = n^2(mn - m - 2n + 3)$. Thus even for the case $m = n = 3$, examples satisfying the sufficient condition given by Theorem 18 must contain at least 27 elements. Clearly the number of ways in which three collections of three pairwise disjoint sets each may be formed from these 27 elements precludes a search for such examples. Allowing non-disjoint sets further complicates matters.

Although we cannot guarantee an intersection between $S_{i_1j_1}$ and $S_{i_2j_2}$ if $k < n - 1$, we can guarantee that $S_{i_1j_1}$ will intersect the union of all of the sets of $\mathcal{S}_{i_2}$ except $S_{i_2j_2}$. The following three lemmas establish this fact and provide more exact information on the distribution of the elements of a given set among sets of another collection.

Lemma 4. If $Q \geq 0$ completely, then any set S_{ij} intersects the union of any $n - 1$ sets of any other collection in at least k points.

Proof. Let $Cd(X_{i_1}) = 1$, $Cd(X_{i_2}) = n - 1$, $Cd(X_i) = n$ for $i \neq i_1, i_2$. The inequality $Q \geq 0$ reduces to $Cd\Big(\bigcap_{i=1}^{m} I_i\Big) = Cd(I_{i_1} \cap I_{i_2}) \geq$

$$\sum_{i=1}^{m} Cd(X_i) - (m - 1)n + k = 1 + (n - 1) + (m - 2)n - (m - 1)n + k = k.$$

Lemma 5. If $Q \geq 0$ completely, and for given i_1 and j_1, $Cd(I(\{S_{i_1j_1}\})) = k + 1 + t_{i_1j_1}$, $t_{i_1j_1} \geq 0$, then for any set $S_{i_2j_2}$, $i_2 \neq i_1$, $Cd(I(\{S_{i_1j_1}\}) \cap I(\{S_{i_2j_2}\})) \leq t_{i_1j_1} + 1$.

Proof. By Lemma 4, at least k of the elements of $S_{i_1j_1}$ are included in $\sigma_{i_2} - \{S_{i_2j_2}\}$.

Lemma 6. If $n \geq 2$ and $Q \geq 0$ completely, and if for given i_1 and j_1, $Cd(I(\{S_{i_1j_1}\})) = k + 1 + t_{i_1j_1}$, $t_{i_1j_1} \geq 0$, then given any collection σ_{i_2}, $i_2 \neq i_1$, the elements of $S_{i_1j_1}$ are distributed over at least $[(k + t_{i_1j_1})/(t_{i_1j_1} + 1)] + 1 = [(k - 1)/(t_{i_1j_1} + 1)] + 2$ sets of σ_{i_2} and at most $\min(n, k + 1 + t_{i_1j_1})$ sets of σ_{i_2}.

Proof. The fact that the elements are distributed over at most $\min(n, k + 1 + t_{i_1j_1})$ sets of σ_{i_2} is clear. The fact that they are distributed over at least $[(k + t_{i_1j_1})/(t_{i_1j_1} + 1)] + 1$ sets of σ_{i_2} follows from Lemma 5. For any set $S_{i_2j_2} \in \sigma_{i_2}$, $Cd(I(\{S_{i_1j_1}\}) \cap I(\{S_{i_2j_2}\})) \leq t_{i_1j_1} + 1$.

If $k \geq n$, then Lemmas 3 and 5 yield further restrictions on the size of the individual sets allowable if $Q \geq 0$ completely. For suppose that $k = n + p$, $p \geq 0$. Then by Lemmas 3 and 5, $1 + t_{i_1j_1} \geq Cd(I(\{S_{i_1j_1}\}) \cap I(\{S_{i_2j_2}\})) \geq n + p + 2 - n = p + 2$. Hence $t_{i_1j_1} \geq p + 1$. In Lemma 6 this implies that $[(k - 1)/(t_{i_1j_1} + 1)] + 2 = [(n + p - 1)/(t_{i_1j_1} + 1)] + 2 \leq [(n + p - 1)/(p + 2)] + 2 = [(n - 3)/(p + 2)] + 3 \leq [(n - 3)/2] + 3$. But this is at most n only if $n \geq 2$. Hence the restriction on n in Lemma 6 is necessary.

The above lemmas are useful in constructing examples to satisfy $Q \geq 0$ completely. For example, if $m = n = 3$, and we wish to construct an example satisfying the condition given in Theorem 18, we then have

$k = 4$. By these lemmas we know that any such example must have the following properties: i) each set contains at least 9 elements; ii) there must be at least 27 elements in all; iii) each set from any one collection must intersect each set from any other collection in at least 3 and at most $1 + t$ elements, where t is the minimum number of elements over 5 in either set involved; iv) each set from any one collection must intersect each pair of sets from any other collection in at least 4 elements. While these restrictions limit greatly the number of collections among which one might find an example, there are still far too many of these to make a search feasible. In addition, Theorem 19 says that we may have to examine up to 27 subcollection combinations for each example before we can decide whether or not $Q \geq 0$ completely.

If we further restrict s to be a multiple of n, in particular $s = (k + 1)n$, we are able to extend the above lemmas slightly, and to obtain a bound on the number of collections with which we may work.

Lemma 7. If $Q(m, n, k) \geq 0$ completely and $s = (k + 1)n$, then $k \leq n - 1$ and $s \leq n^2$.

Proof. By Lemma 2, $Cd(I(\{S_{ij}\})) = k + 1$ for all i and j; and by Lemma 5, $Cd(I(\{S_{i_1 j_1}\}) \cap I(\{S_{i_2 j_2}\})) \leq 1$ for all $i_1 \neq i_2$, j_1, j_2. But since for $k \geq n$ we have seen that $t_{ij} \geq p + 1 \geq 1$ for all i and j, it follows that $k \leq n - 1$.

This lemma enables us to strengthen the Corollary to Lemma 3 as follows.

Lemma 8. If $Q \geq 0$ completely, then if $k = n - 1$, $s \geq n^2$, and if $k \geq n$, $s > n^2(k + 2 - n)$.

In addition, if we restrict s to be a multiple of n we are then able to extend Lemma 5 to cover intersections of several sets from each of two collections.

Lemma 9. If $Q \geq 0$ completely and $s \leq (k+1)n$, then any set of one collection intersects any q sets of any other collection in at least $k + q + 1 - n$ points and in at most $\min(q, k+1)$ points, for $q = n - k, \ldots, n$.

Proof. A proof for the sets S_{11} and $S_{21}, \ldots, S_{2q}$ will suffice. By Lemma 4, $Cd(I(\{S_{11}\}) \cap \bigcup_{j=1}^{n-1} I(\{S_{2j}\})) \geq k$. But by Lemma 5, $Cd(I(\{S_{11}\}) \cap I(\{S_{2j}\})) \leq 1$, for all j. In particular, S_{11} intersects $\bigcup_{j=q+1}^{n-1} S_{2j}$ in at most $n - q - 1$ points. Hence $Cd(I(\{S_{11}\}) \cap \bigcup_{j=1}^{q} I(\{S_{2j}\})) \geq k - (n - q - 1) = k + q + 1 - n$. The upper bound is evident.

Lemma 10. If $Q \geq 0$ completely and $s = (k+1)n$, then any p sets of one collection intersect any q sets of any other collection in at least $p(k + q + 1 - n)$ points, and in at most $p \cdot \min(q, k+1)$ points, for $p = 1, \ldots, n$; $q = n - k, \ldots, n$.

Proof. Clear from Lemma 9 since the p sets are pairwise disjoint.

Probably the most significant result in this section is a restriction on the size of m, that is, on the number of collections. We are able to guarantee that if $s = (k+1)n$ then $Q \geq 0$ cannot hold completely for more than a certain maximum number of collections.

Theorem 20. If $q \geq 0$ completely and $s = (k + 1)n$, then $m \leq [((k + 1)n - 1)/k]$. Moreover, an SDR exists only if $m \leq n$.

Proof. Choose an element, say a_1, from S_{11}. This element must occur in some set in each $\mathcal{S}_i$, $i = 2, \ldots, m$. We may assume that $a_1 \in S_{i1}$, $i = 2, \ldots, m$. Now each S_{i1} contains $k + 1$ elements, by Lemma 2, and for $i \neq 1$, $Cd(I(\{S_{11}\}) \cap I(\{S_{i1}\})) \leq 1$, by Lemma 5. From the remaining $(k + 1)n - 1$ elements (other than a_1) we must have disjoint groups of k elements each filling out the sets S_{i1}, $i = 1, \ldots, m$. The disjointness is required by Lemma 5 which asserts that $Cd(I(\{S_{i_1 1}\}) \cap I(\{S_{i_2 1}\})) \leq 1$ for all $i_1 \neq i_2$, and the fact that a_1 is known to be in this intersection. But there cannot be more than $[((k + 1)n - 1)/k] = n + [(n - 1)/k]$ such disjoint groups. Hence $m \leq n + [(n - 1)/k]$.

Since we know that $k \leq n - 1$ (Lemma 7), it follows that the upper bound on m is at least $n + 1$. In the sets $S_{11}, \ldots, S_{m1}$ there occur exactly $1 + km$ elements from the total number, $(k + 1)n$. Suppose that there exists an SDR. Since the choice of a_1 and of $S_{11}, \ldots, S_{m1}$ is dependent only on how the elements and sets are numbered, we may assume also that a_1 is one of the representatives. If a_2 is any other representative, then it cannot be among the $1 + km$ elements in the sets S_{i1}, $i = 1, \ldots, m$. For if it were, then some set S_{i1} would be represented by both a_1 and a_2, and some other set in $\mathcal{S}_i$ would have no representative. Since there are $n - 1$ representatives other than a_1, it follows that $(k + 1)n - (1 + km) \geq n - 1$. This easily reduces to $m \leq n$.

Corollary. If s is a multiple of n, $s = (k + 1)n$, and $m > n$, then either $Q(m, n, k) < 0$ for some choice of the χ_i, $i = 1, \ldots, m$, or there does not exist an SDR.

It is not known whether or not the condition $m \leq n$ in the above theorem is sufficient to guarantee the existence of an SDR.

The importance of the restriction $s = (k + 1)n$ lies primarily in the fact that Lemma 2 then tells us the exact number of elements in each set S_{ij}, namely $k + 1$. Thus there are relatively few ways in which the collections $\mathcal{S}_i$ can be composed, and many of these are eliminated by the restrictions on the intersections.

When $s > (k + 1)n$ we have quite a different problem, for here we have only a lower bound on the size of any individual set. When $s = (k + 1)n + 1$ the situation is not extremely bad, for we can say that in each collection one set contains $k + 2$ elements and the rest contain $k + 1$ elements each. But for larger values of s the elements in excess of $(k + 1)n$ may in each collection be distributed over one or more of the sets, so that there is not even a guarantee that the collections "have the same shape."

We might try to obtain other facts about the situation when s is not a multiple of n by suitably modifying the sets so that the number of elements is a multiple of n. We could embed $\mathcal{A}$ in a set satisfying this condition, scattering the new elements among the sets S_{ij} in some suitable manner. But this is a futile procedure, for even if there were an SDR for this new system, we could not guarantee that all of the representatives are from $\mathcal{A}$. On the other hand, we can make the following reduction, which holds with no disjointness restrictions on the S_{ij}.

Lemma 11. If there exists an SDR for a system $\mathcal{S}_1, \ldots, \mathcal{S}_m$ with elements in a set $\mathcal{A}$, then there exists a subset $\mathcal{A}'$ whose cardinality is a multiple of n, and a system $\mathcal{S}_1', \ldots, \mathcal{S}_m'$ with sets defined by $S_{ij}' = S_{ij} \cap \mathcal{A}'$, $i = 1, \ldots, m$; $j = 1, \ldots, n$, such that the given SDR is also an SDR for $\mathcal{S}_1', \ldots, \mathcal{S}_m'$.

Proof. Let, for example, S_{ij}' consist solely of the representative for S_{ij}.

The difficulty in using this reduction is that we cannot now say that each set contains exactly $k + 1$ elements. Some may contain more, and in fact the composition may vary from collection to collection. Thus $\mathcal{S}_1$ may contain sets of cardinalities 6, 6, 4, and 4, while the sets in $\mathcal{S}_2$ have cardinalities 7, 5, 4, 4. Under these circumstances it will generally be impossible to eliminate the excess of elements over $(k + 1)n$, leave each set with exactly $k + 1$ elements, and still have $Q(m, n, k) \geq 0$ completely.

The above discussion is oriented towards the question: how does one find a system $\mathcal{S}_1, \ldots, \mathcal{S}_m$ which satisfies $Q(m, n, k) \geq 0$ completely, for given m, n, and k? We may also ask, given a system $\mathcal{S}_1, \ldots, \mathcal{S}_m$, for what values of k is $Q(m, n, k) \geq 0$ completely? A partial answer to this is given by the contrapositive to Lemma 2.

Lemma 12. If in a given system $\mathcal{S}_1, \ldots, \mathcal{S}_m$ the minimum number of elements in any set S_{ij}, $i = 1, \ldots, m$; $j = 1, \ldots, n$, is p, then $Q(m, n, k) \geq 0$ completely can only hold for $k < p$.

There is, of course, no guarantee that we will have $Q \geq 0$ completely for any of these values.

To conclude this chapter we present in Figs. 3 - 5 the known examples for $m = 3, 4$. These, with one exception, were computed on the University of Michigan IBM 704 computer. The computation takes the form of $y = \max(\sum_{i=1}^{m} Cd(X_i) - Cd(\bigcap_{i=1}^{m} I_i))$. This is the minimum value which $(m - 1)n - k$ may assume if we are to have $Q \geq 0$ completely. In the graphs the points where there are SDRs are symbolized by "x," those where there are no SDRs by "o," and those where there are known examples both with and without SDRs by "xo." The points on or below the top line, AB, satisfy the necessary condition of Theorem 15; and those on or below the bottom line, AC, satisfy the sufficient condition of Theorem 16 or 18, depending on whether or not the collections are partitions of $\mathcal{A}$. The intermediate lines on Fig. 3, labelled with values of k, are those given by Lemma 2 and Lemma 12. If $k_0 \leq n - 1$ and $s < (k_0 + 1)n$, or if $k_0 > n - 1$ and $s \leq n^2(k_0 + 2 - n)$, then the corresponding point on the graph lies above the line $k = k_0$.

The accompanying tables, 1 - 3, give the number of known examples, with and without SDRs, for each pair (n, y). The distribution of these examples, including the preponderance of examples with no SDR, is indicative of the manner in which the examples were chosen rather than the natural distribution.

Because of the length of the computation, no examples were computed for $m > 4$.

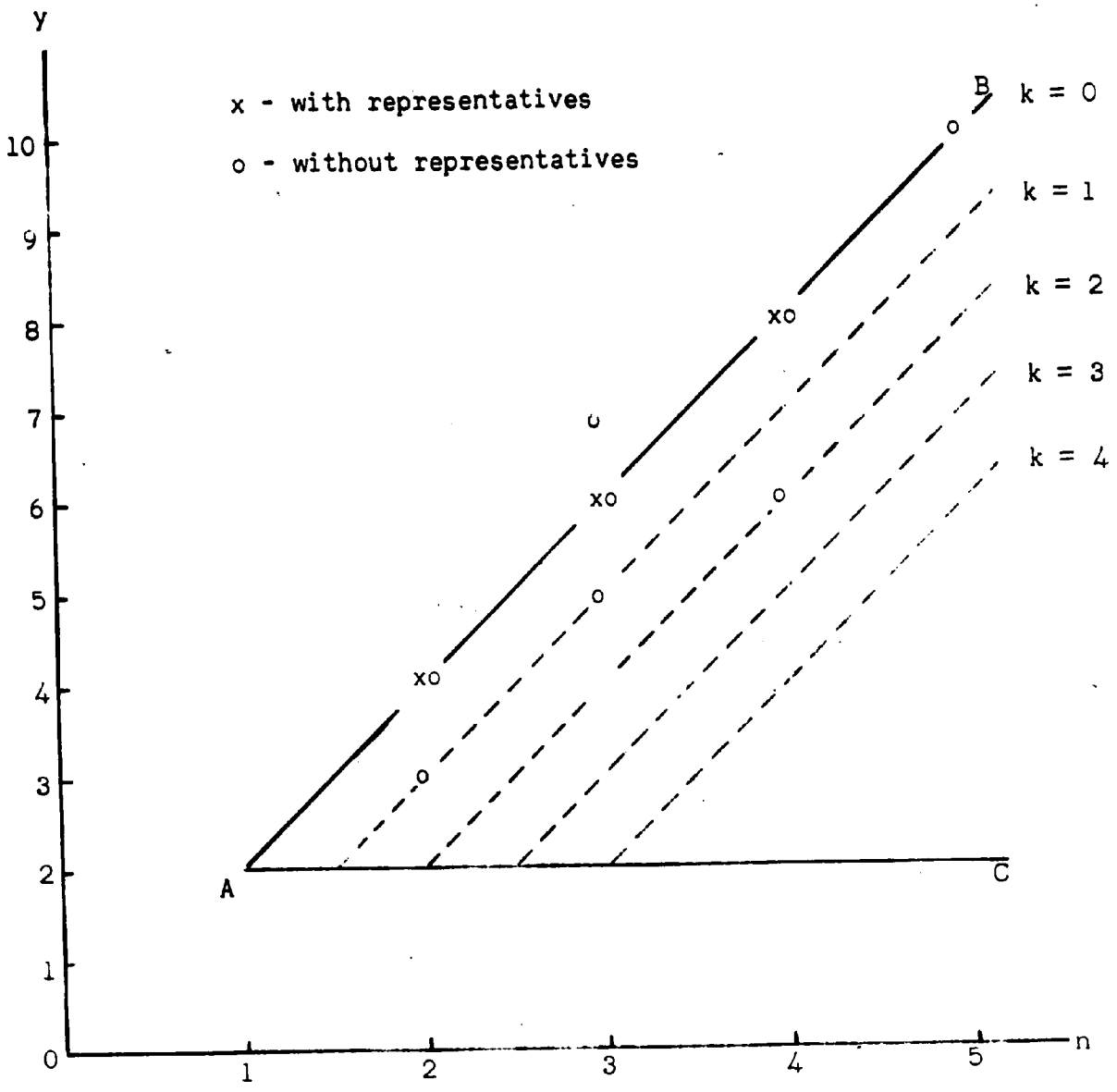


Fig. 3. Distribution of known systems of three collections of sets, when the collections are partitions

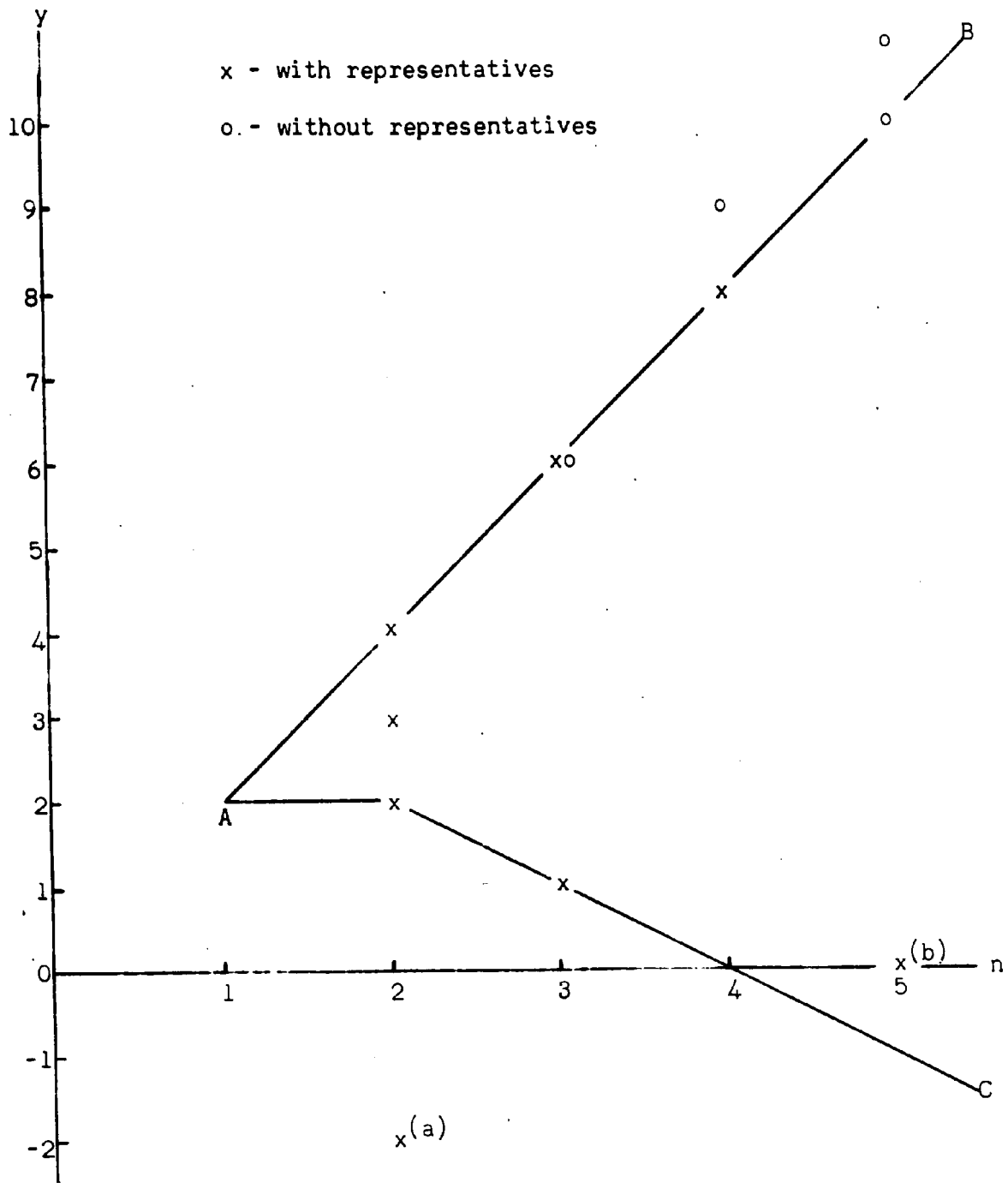


Fig. 4. Distribution of known systems of three collections of sets, when the collections are not partitions

(a): This point was hand-computed.
(b): This point may have $y < 0$; the computer program was limited by zero.

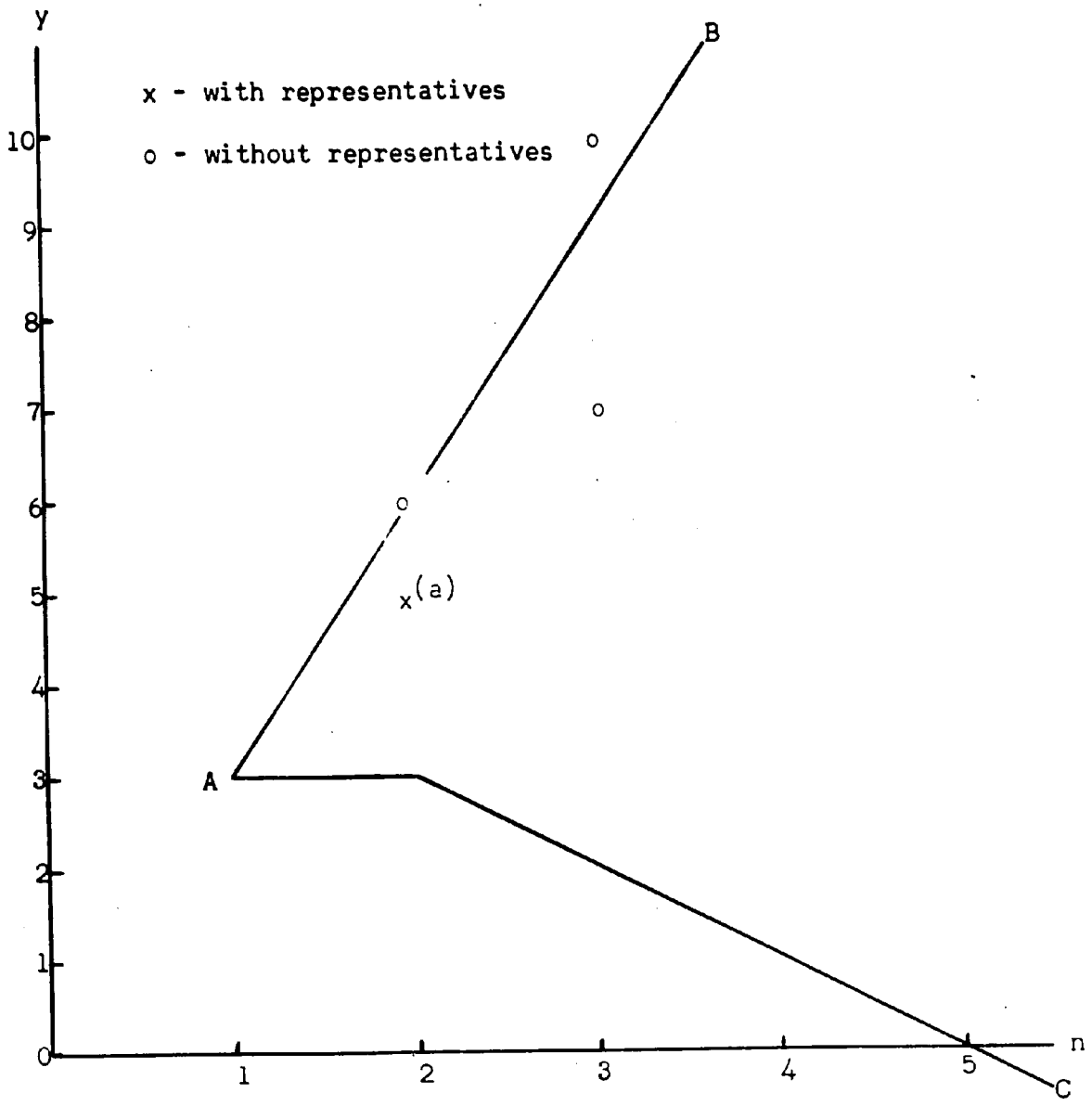


Fig. 5. Distribution of known systems of four collections of sets

(a): The collections for this point were not partitions; in all other cases they were.

Table 1. The number of known systems of three collections of sets, when the collections are partitions

y \ n	2	3	4	5
3				
4	1	1		
5				
6		3		
7				
8			1	
9				
10				

A) with representatives

y \ n	2	3	4	5
3	1			
4	1			
5		10		
6		16	2	
7		1		
8			6	
9				
10				1

B) without representatives

Table 2. The number of known systems of three collections of sets, when the collections are not partitions

y \ n	2	3	4	5
-2	1			
-1				
0				1[a]
1		1		
2	1			
3	1			
4	1			
5				
6		1		
7				
8			2	
9				
10				
11				

A) with representatives

y \ n	2	3	4	5
-2				
-1				
0				
1				
2				
3				
4				
5				
6		2		
7				
8				
9			1	
10				1
11				1

B) without representatives

[a]The y value of this point may be less than zero.

Table 3. The number of known systems of four collections of sets

y \ n	2	3	4	5
5	1			
6				
7				
8				
9				
10				

A) with representatives

y \ n	2	3	4	5
5				
6	1			
7		1		
8				
9				
10		1		

B) without representatives

CHAPTER III

ALTERNATIVE APPROACHES TO THE PROBLEM

In Chapter II we considered the problem of systems of distinct representatives for more than two collections of sets from a set-theoretic point of view. We began with inequality (7) of Chapter I,

$$Cd(X) + Cd(Y) \leq n + Cd(I(X) \cap I(Y)) \qquad (7)$$

given originally by Ford and Fulkerson, and attempted to generalize this, observing the relationship between this and the sets under consideration. In this chapter we shall examine a more direct extension of the work of Ford and Fulkerson. They constructed a class of networks for $m = 1$ and another class for $m = 2$. These classes have the property that a specific network in one of the classes has a maximum flow of value n if and only if the corresponding system of collections of sets has an SDR. We shall examine several types of networks which correspond to three collections of sets, and which can be extended to m collections, for $m > 3$. In each case we will find that additional non-linear restrictions must be placed on the network so that it will function in the desired manner, that is, so that a maximal flow of a specified value will exist if and only if there is an SDR. We shall conclude with some remarks on the application of more general mathematical programming techniques to the problem of finding necessary and sufficient conditions for the existence of an SDR.

Let us recall Fig. 1. This is the network which Ford and Fulkerson used to prove Theorem 7. The corresponding network for $m = 2$ (Fig. 6) contains two sets of vertices representing $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, and between these sets two sets of vertices representing the elements

of $\mathcal{A}$. These latter vertices are paired so as to insure that no one element a_i is chosen to represent two sets in the same collection.

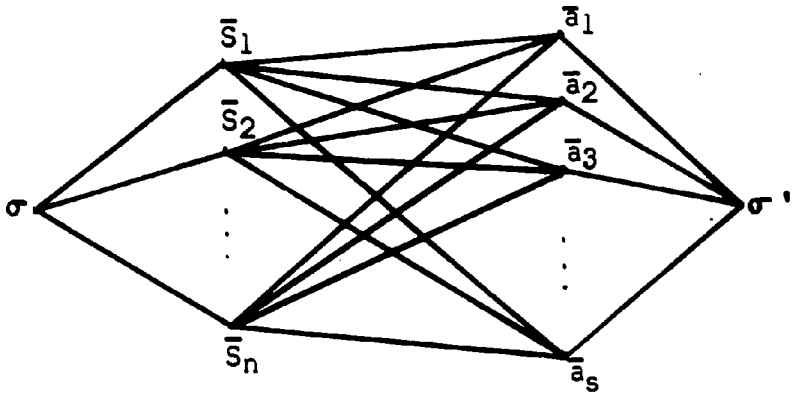


Fig. 1. The network for m = 1

The capacities on the edges are

$$c(\sigma, \bar{S}_{1j}) = 1, \quad \text{for } j = 1, \ldots, n$$

$$c(\bar{S}_{1j}, \bar{a}_i) = \begin{cases} +\infty & \text{if } a_i \in S_{1j} \\ 0 & \text{otherwise} \end{cases}$$

$$c(\bar{a}_i, \bar{a}'_k) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$$c(\bar{a}'_k, \bar{S}_{2j}) = \begin{cases} +\infty & \text{if } a_k \in S_{2j} \\ 0 & \text{otherwise} \end{cases}$$

$$c(\bar{S}_{2j}, \sigma') = 1, \quad \text{for } j = 1, \ldots, n.$$

This is a special case of the network for SRRs given by Ford and Fulkerson [10].

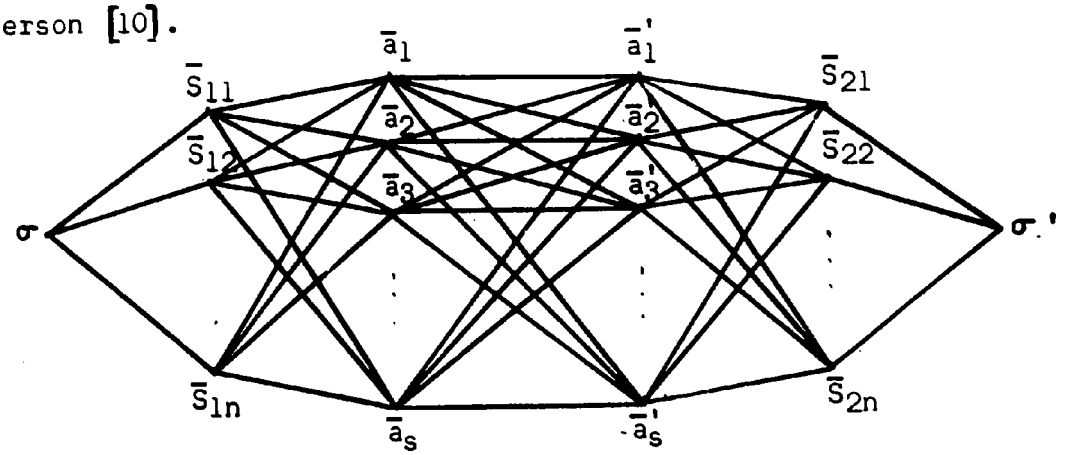


Fig. 6. The network for m = 2

In order to extend the use of networks to more than two collections of sets, it is useful to introduce the following symbolism. By letting a block labelled $\mathcal{S}_i$ stand for the set of vertices associated with the sets S_{ij} and letting blocks labelled $\mathcal{A}$ and $\mathcal{A}'$ symbolize the vertices associated with the elements a_k, we may abbreviate Figs. 1 and 6 as Figs. 1' and 6'. The numbers then indicate the capacities of the individual edges, except that the zero-capacity edges between $\mathcal{S}_i$ and $\mathcal{A}$ are not indicated. For our purposes these may be considered as not being in the network. We turn now to Figs. 7, 8, 9, and 10 which are possible representations of the problem for three collections of sets. Fig. 7 is a direct extension of Fig. 1, in which an SDR is chosen for each set, and these are then compared for identity. Figs. 8, 9, and 10 are similar extensions of Fig. 6, obtained by placing the third collection in various positions.

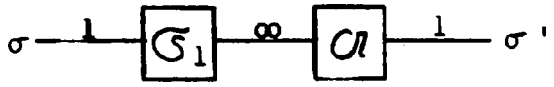


Fig. 1'. The network for m = 1

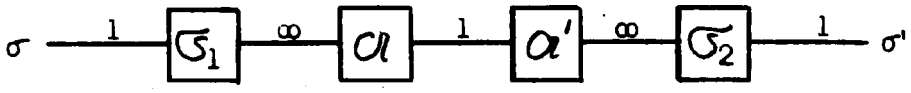


Fig. 6'. The network for m = 2

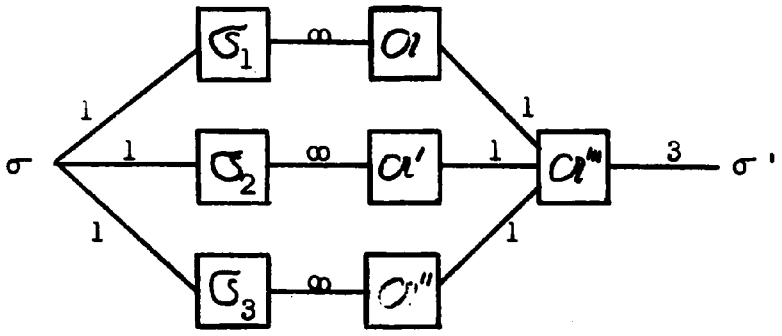


Fig. 7. A network for m = 3

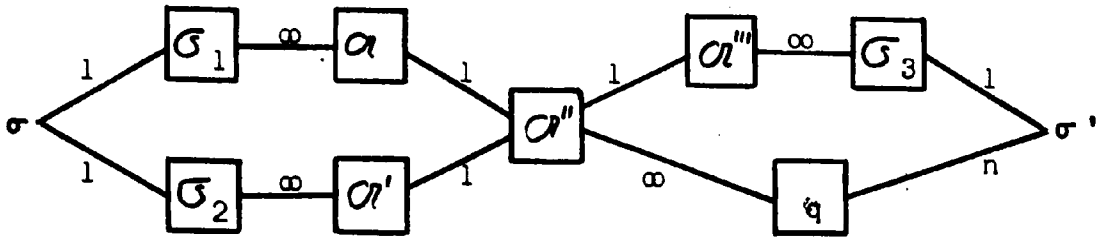


Fig. 8. A network for m = 3

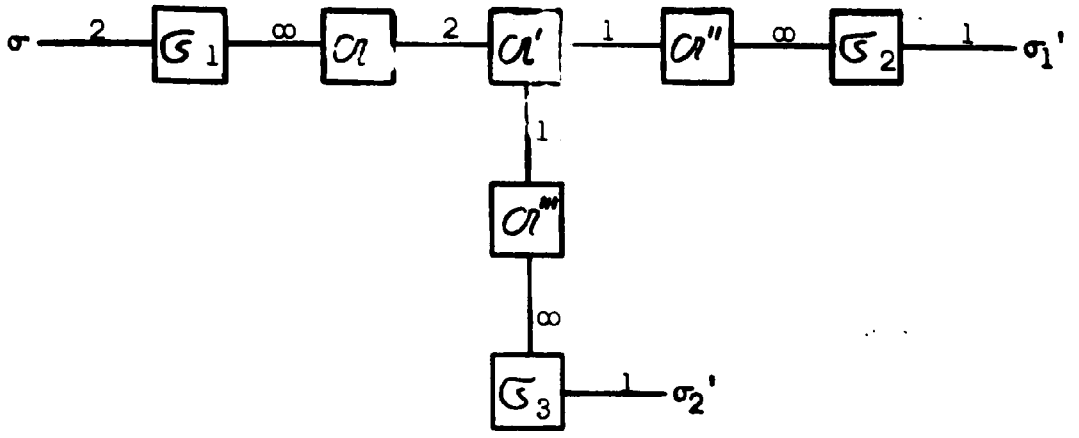


Fig. 9. A network for m = 3

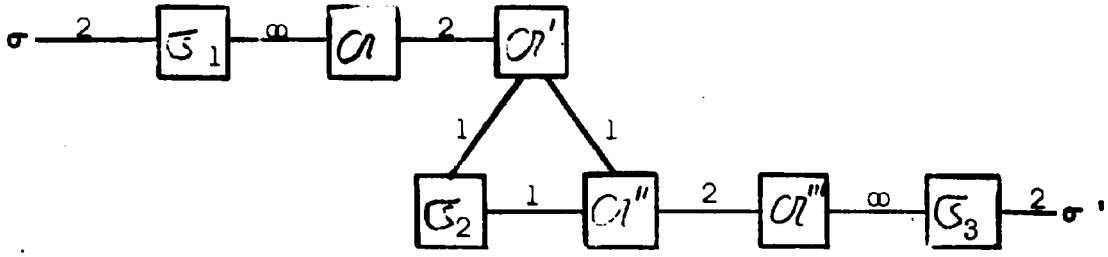


Fig. 10. A network for m = 3

In the network represented by Fig. 7 the maximum flow value is at most 3n (in general mn) since this is the maximum flow out of σ. Similarly, in Figs. 8 and 9 the maximum flow value is at most 2n (in general (m - 1)n); and in Fig. 10 the maximum possible value is 2n, independent of the number of collections. It is desired that this maximum flow occur if and only if there is a common SDR for the corresponding collections of sets.

For the network of Fig. 7 there is obviously no guarantee that we will have a common SDR. A different system of representatives could be chosen for each of the collections of sets, and a flow of value $3n$ can still be achieved. With the added restriction that the flow from each vertex of $\mathcal{A}'''$ be either zero or at least three, this network will function properly, for then a vertex $\bar{a}_k''' \in \mathcal{A}'''$ can be chosen in a flow if and only if there is flow from each collection to it, thus making it the representative for each collection. A flow of one or two implies it does not represent all collections. This insures that if a vertex corresponds to a representative, the element is then a representative for all three collections.

In succeeding figures we have shown networks which place more restrictions on the number of vertices involved in a maximal flow. Thus in Fig. 8 we have introduced the vertex "q" to serve as a counter. This is an attempt to insure that exactly n vertices $\bar{a}_k$ are involved in the flow. For a flow of value $2n$ in the network there must be a flow of value n through the vertex q since there can be at most a flow of value n from $\mathcal{S}_3$ to σ'. In Fig. 9 the same effect is achieved by separating $\mathcal{S}_2$ and $\mathcal{S}_3$ and allowing two sinks, σ_1' and σ_2'. In Fig. 10 we have introduced the vertices of $\mathcal{A}''$, which duplicate the vertices of $\mathcal{A}$ and $\mathcal{A}'$, except for the associated edges. Here the edges $(\bar{a}_j', \bar{a}_k'')$, $\bar{a}_j' \in \mathcal{A}'$, $\bar{a}_k'' \in \mathcal{A}''$, have a capacity of one if $j = k$, and zero otherwise, so that if $\bar{a}_k'$ is involved in a flow, then $\bar{a}_k''$ is also.

In terms of the process of choosing representatives, each of these last three networks corresponds to a more sequential choice. Thus the network in Fig. 7 corresponds to a simultaneous choice of representatives

for all collections - with no guarantee that the same choice is made for each. In Fig. 8, we choose SDRs for $\mathcal{S}_1$ and $\mathcal{S}_2$, and then check these with $\mathcal{S}_3$. But again there is nothing to prevent a different choice being made for $\mathcal{S}_2$ than for $\mathcal{S}_1$: this can be achieved by adding the restriction that the flow from a vertex of $\mathcal{A}''$ must be either zero or at least two (in general $(m - 1)n$).

Fig. 9 represents a network corresponding to the choice of an SDR for $\mathcal{S}_1$ which is then simultaneously checked in $\mathcal{S}_2$ and $\mathcal{S}_3$; and in Fig. 10 the SDR for $\mathcal{S}_1$ is checked first in $\mathcal{S}_2$, and then in $\mathcal{S}_3$. But in each of these cases the twofold choice of SDRs in $\mathcal{S}_1$ implied by the capacity being two leads us to apply the same type of restriction as above to the vertices of $\mathcal{A}'$, and also $\mathcal{A}''$ in Fig. 10. For Fig. 9 the flow through each of these vertices is either zero or at least $(m - 1)n$ (in general); and for Fig. 10 the flow must be either zero or at least two, independent of m.

Thus in each of these representations there arises the same problem: we cannot assure that the n points chosen on one branch of the network are the same as those for the other branch without restrictions such as those mentioned. That is, without such conditions there is nothing to prevent, for example, a flow of 1 through vertices $\bar{a}_1$ and $\bar{a}_4$ and of 2 through $\bar{a}_2$ and $\bar{a}_3$, with $\bar{a}_1$, $\bar{a}_2$, and $\bar{a}_3$ contributing to one branch, and $\bar{a}_2$, $\bar{a}_3$, and $\bar{a}_4$ contributing to the other. Such a flow would satisfy the condition that exactly n vertices (representatives) are used for each branch, for $n = 3$, but would not satisfy this for the network as a whole. Example 7 illustrates this point.

Example 7. For this example we use the collections given in Example 2 and the network form of Fig. 8.

$S_{11} = \{1, 2\}$ $S_{21} = \{1, 3\}$ $S_{31} = \{1, 4\}$

$S_{12} = \{3, 4\}$ $S_{22} = \{4\}$ $S_{32} = \{2, 5\}$

$S_{13} = \{5\}$ $S_{23} = \{2, 5\}$ $S_{33} = \{3\}$

For these collections there is no SDR; yet for the network there is a flow of value 2n, which is obtained by assigning the flows indicated on Fig. 11. $\mathcal{A} = \{1, 2, 3, 4, 5\}$.

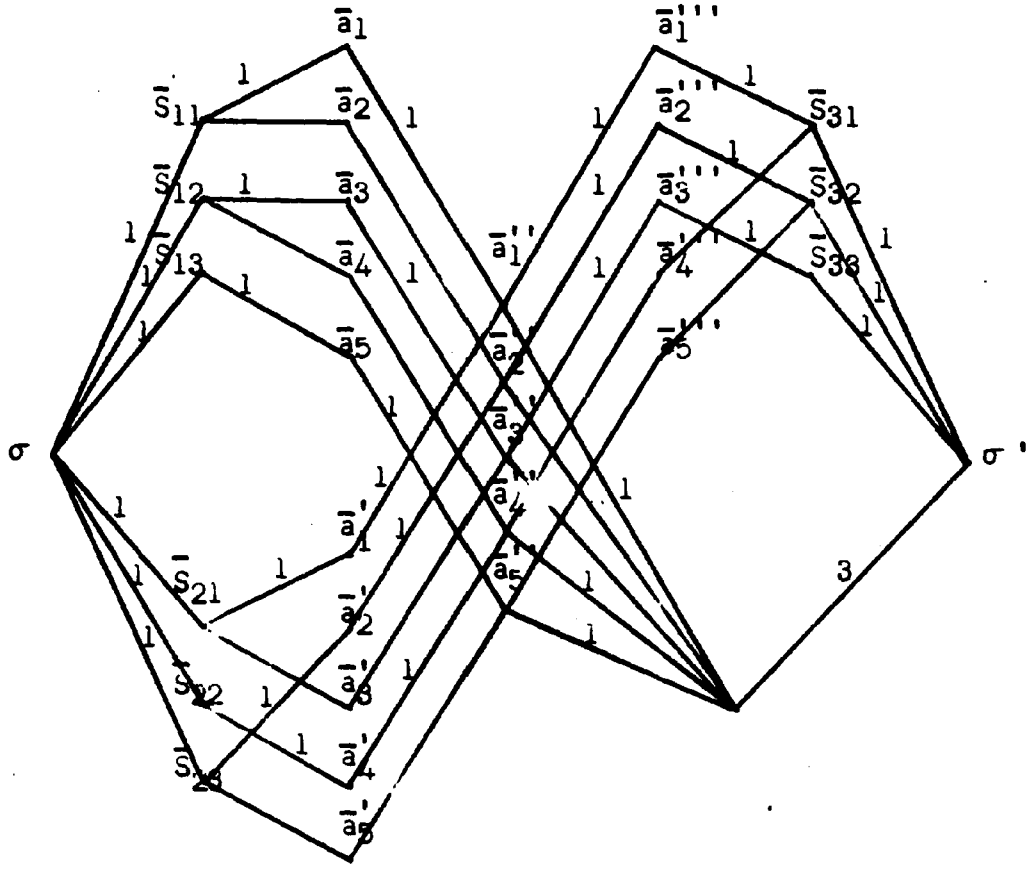


Fig. 11. The network for Example 7

It is evident that the same type of situation would prevail if we used the network form of Fig. 9 or Fig. 10.

The situation in Example 7 is typical. We have SDRs for the three collections individually, but these are not correlated. Some of the elements in the SDRs for $\mathcal{S}_1$ and $\mathcal{S}_2$ are used to form the SDR for $\mathcal{S}_3$,

and the remaining elements are used for the counter q. That is, the vertices which are counted are not always those which are used to form the SDR for $\mathcal{S}_3$. It is in order to prevent such occurrences that the "threshold" concept mentioned above is introduced. Through this concept we require that the flow through any one of the vertices be either zero or at least a certain minimum depending on m and the type of network. That is, we require that if a vertex $\bar{a}_k''$ is used in the SDR for $\mathcal{S}_1$, then it is used in the common SDR for all collections, and also in any counting paths which occur in the network. The threshold condition accomplishes this since the flow into a vertex $\bar{a}_k''$ can be at most one from either the set $\mathcal{A}$ or the set $\mathcal{A}'$ (by the capacity restrictions). Hence there will be a non-zero flow through this vertex (by the threshold condition) only if the vertices $\bar{a}_k$, $\bar{a}_k'$ have been chosen as representatives for sets in their respective collections. As far as the author has been able to determine, every network for three or more collections of sets involves this concept. The determination of the conditions for a specified maximal flow in such networks is an unsolved problem.

The network approach to the problem of finding systems of representatives leads naturally to the use of mathematical programming techniques. In fact, Ford and Fulkerson's proofs are based on the systems of linear inequalities and equations relating to the networks used. Unfortunately the threshold statement leads to quadratic constraints.

We may refer to Fig. 11 as a guide in stating the mathematical programming problem. While the particular constants and equations apply only to networks represented by Fig. 8, the general discussion applies equally well to all of the networks which we have discussed. It is clear that any flow from σ to σ' must involve vertices of the set $\bar{a}_1''$, ..., $\bar{a}_s''$,

and that an upper bound on the flow value is $(m - 1)n$, since this is the maximum flow from σ. Since the flow into any vertex is equal to the flow out, except for σ and σ', our problem reduces to that of maximizing the flow into the vertices $\bar{a}_1''$, ..., $\bar{a}_s''$. If we represent the flow from $\bar{a}_k$ to $\bar{a}_k''$ by x_k, $k = 1, \ldots, s$, and the flow from $\bar{a}_k'$ to $\bar{a}_k''$ by x_{s+k}, $k = 1, \ldots, s$, etc., the problem is then the following.

Maximize the linear expression $x_1 + \ldots + x_{(m-1)s}$ subject to the conditions

$$0 \leq x_k \leq 1, \quad k = 1, \ldots, (m - 1)s$$

$$\left(\sum_{j=0}^{m-2} x_{k+js} \right) \left(\sum_{j=0}^{m-2} x_{k+js} - m + 1 \right) \geq 0 \qquad (20)$$

the flow equalization equations

the capacity restrictions on all other flows.

The first set of restrictions in (20) are the capacity restrictions on the x_k. The second set of conditions are those corresponding to the threshold requirement. The flow into a vertex $\bar{a}_k''$ is $\sum_{j=0}^{m-2} x_{k+js}$ and the threshold statement requires that this be either zero or at least $m - 1$ for networks of this type. The flow equalization equations are those equations stating that the flow into any vertex of the network, other than the source and sink, is equal to the flow out. These equations relate the flows x_k to all other flows in the network. These latter are of course limited by the capacity restrictions on their respective edges.

For any particular instance of the above problem, a method given by G. Dantzig [2] will reduce the quadratic conditions to linear ones at the expense of an added variable for each such condition. A technique due to Gomory [14] will then yield a sequence of solutions converging to an integral solution. We then have an SDR if and only if the maximum value of $x_1 + \ldots + x_{(m-1)s}$ is $(m - 1)n$. However, nothing is known at present

about the general solution to this type of problem, so that we are unable to state the conditions under which this maximum value will be attained.

CHAPTER IV

ALGORITHMS FOR FINDING SYSTEMS OF REPRESENTATIVES

The criteria developed for determining the existence of an SDR, even in the cases where a single necessary and sufficient condition exists, are not of much help in finding an SDR. The intersections developed generally have no relation to the SDR, if it exists. Moreover, as we showed in Chapter II, one must examine a large number of combinations of sets even to show the existence of an SDR. In this chapter we undertake the task of searching for an actual SDR. This is a task of the same order of magnitude as establishing the existence, and is, in a sense, a more positive task. In the existence of proofs we need to examine the full complement of set combinations given by (16)

$$N = \sum \frac{(n!)^m\, m!}{\prod_{\nu=1}^{n} \left[\nu!(n-\nu)!\right]^{p_\nu}\, p_\nu!}, \tag{16}$$

where the sum is over all simultaneous solutions of the equations

$$\sum_{\nu=1}^{n} p_\nu = m, \quad \sum_{\nu=1}^{n} \nu p_\nu = (m-1)n - k + 1, \quad \text{whenever an SDR exists}$$

(and even then we aren't positive one exists); in searching for an SDR we need examine all possible candidates only when no SDR exists, and hopefully not many if an SDR exists.

While we cannot necessarily develop a search procedure which is efficient in all possible cases, nevertheless it is possible to develop some heuristics to guide the search. We begin with a discussion of the situation when the collections are partitions, which is relatively simple. Following this we take up the general case, starting with a

discussion of two algorithms for finding an SDR when there is only one collection.

If the sets in each collection are pairwise disjoint, we may use the following technique, which involves a reduction and rearrangement before searching for the SDRs. If any set consists of exactly one element, then in order that there be an SDR, this element must be a representative. Suppose, for example, that $S_{11} = \{a_1\}$. The element a_1 must be in some set of $\mathcal{S}_2$, say S_{2j_1}, and hence must represent this set. Thus we may eliminate all other members of S_{2j_1} as possible representatives and drop them from all sets in which they occur. In particular, S_{2j_1} is replaced by $S'_{2j_1} = \{a_1\}$. After a finite number of repetitions of this procedure one of three possibilities occurs. If one of the sets is reduced to the empty set, there is no SDR. If each set in every collection is reduced to a single element, then these elements form the only SDR. Finally, if there still exist sets which contain more than one element, we must search for possible SDRs.

This search is facilitated by the fact that the possible sets of SDRs form a nested sequence, in the following sense. If M_k is the set of SDRs for $\mathcal{S}_1, \ldots, \mathcal{S}_k$, $k = 1, \ldots, m$, then $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_m$. If the collections are partitions, then the number of SDRs for any one collection by itself is exactly the product of the cardinalities of the sets in that collection. We arrange the collections $\mathcal{S}_i$ in increasing order relative to this product of cardinalities. Then $\mathcal{S}_1$ has the least number of SDRs of any of the collections individually, and by the nested sequence property, the maximum number of possible SDRs for the collections together. We have then merely to list the

SDRs of $\mathcal{S}_1$ and examine each to see if it is an SDR for $\mathcal{S}_2, \mathcal{S}_3, \ldots, \mathcal{S}_m$. Those SDRs not eliminated by this procedure, if any, are the only SDRs for the collections together.

Because of the disjointness, the search is relatively simple. If, for example, $S_{21} = \{a_{k_1}, \ldots, a_{k_t}\}$, then we may eliminate from the of SDRs of $\mathcal{S}_1$ any which contain one or more of the pairs (a_{k_i}, a_{k_j}), $i = 1, \ldots, t-1$; $j > i$. If $\mathcal{S}_2$ contains sets of cardinalities $c_1, \ldots, c_n$, the search for these pairs among the SDRs of $\mathcal{S}_1$ involves $\sum_{j=1}^{n} c_j(c_j - 1)/2$ pairs, with a similar number involved for each $\mathcal{S}_i$.

If the collections are not partitions, the above procedure is complicated considerably. Even for just one collection, the finding of an SDR becomes more difficult. If the collection is a partition, then the choice of one element from each set automatically yields an SDR. If the collection is not a partition, then such an arbitrary choice may lead to a duplication of one or more elements. An algorithm given by M. Hall for finding an SDR for one collection is of use here [16].

Algorithm 1. (M. Hall) To find an SDR for the sets $S_1, \ldots, S_n$, beginning with S_1, choose distinct elements a_i arbitrarily to represent S_i. If such a choice can be made for all n sets, we have the required SDR. If not, we arrive at a set $S_{j_0} = b_1, \ldots, b_t$, all of whose elements are representatives of some previous sets. We form an auxiliary list of elements, beginning with $b_1, \ldots, b_t$. If there are any unlisted elements in the set represented by b_1, we add these to the list as $b_{t+1}, \ldots, b_v$. Continuing thus, at the i-th step, if b_i represents the set S_j, we add to the list all the elements of S_j

which are not already listed. One of two events ultimately occurs: (1) we arrive at a b_i which is not a representative for any set, or (2) the list terminates with all of the b_i representatives. In the latter case it can be shown that P. Hall's condition (Theorem 7) is violated and hence there is no SDR. In the former case, b_i is an element of a set represented by some $b_{i'}$, $i' < i$. Let b_i represent this set, thus freeing $b_{i'}$ to represent some other set in which it occurs. Continuing thus, we ultimately free one of the elements $b_1, \ldots, b_t$. This may then be used to represent S_{j_0}, and the search is continued with S_{j_0+1}.

Hall's algorithm makes use of a rather obvious procedure - when we have a partial SDR and are unable to complete it, rather than begin anew, we attempt to modify what we already have so that we may continue. Thus this algorithm is a "hunting" algorithm, and as such, relatively inefficient. The efficiency of course depends largely on how the sets are numbered. Thus the sets should first be ordered in such a way as to decrease the amount of hunting necessary. Although not much is known about an algorithm to accomplish this, it would probably be based on a restrictive principle - dealing first with those sets in which there is the least freedom of choice. Thus fundamental to such an algorithm would be the requirement that the sets be listed in order of increasing cardinality. The arrangement of sets of like cardinality would probably be based on their intersections, but this problem has not been thoroughly studied.

A restrictive principle is behind the following algorithm, which does not require any rearrangement of the sets.

Algorithm 2. To find an SDR for the sets $S_1, \ldots, S_n$, with elements $a_1, \ldots, a_s$. We assume that the sets are ordered by their subscripts, and that the elements in any one set are ordered by their subscripts. For each element a_k, let n_k be the number of sets containing a_k. Let N be the minimum value of the $n_k > 0$, and let $K = \{a_k | n_k = N\}$. For all sets S_j such that $K \cap S_j \neq \emptyset$, let $h_j = \sum_{a_k \in S_j} n_k$; and let H be the minimum value of the h_j. If S_{j_0} is the first set such that $h_{j_0} = H$, represent S_{j_0} by the first element in S_{j_0} which is also in K, say a_{k_0}. Delete the set S_{j_0} and all occurrences of a_{k_0} in the remaining sets, compute new values for the n_k, and iterate on the remaining sets. After a finite number of repetitions of this process all of the n_k will be equal to zero. If we have by that time chosen n representatives, we have the required SDR; if not, no such SDR exists.

The verification of this algorithm depends on the following lemma.

Lemma 13. If $\mathcal{S}$ has an SDR, then it has an SDR containing a specified element.

Proof. Let $\bar{a}$ be the specified element, and R the given SDR. If $\bar{a}$ is an element of R we are done. If not, let $\bar{S}$ be a set which contains $\bar{a}$. $\bar{S}$ is then represented by some element a'. Form a new SDR which is identical with R except that a' is replaced by $\bar{a}$.

(Note that we cannot always specify more than one element, nor require that a certain element not occur in the SDR.)

Theorem 21. Given a finite collection of arbitrary finite sets, Algorithm 2 will yield an SDR if and only if one exists.

Proof. Suppose that there exists an SDR, and let a_{k_0} be chosen as in the algorithm. By Lemma 13, there exists an SDR containing a_{k_0}. Let

R be such an SDR. It is obvious that the reduced subcollection obtained by deleting a_{k_o} and the set it represents has an SDR, namely $R - \{a_{k_o}\}$. But by Lemma 13 this subcollection then has an SDR containing a specified element, and hence we may specify a new element by the algorithmic process and iterate the procedure. This process will terminate only when there is no $n_k > 0$. But since at each stage we have a subcollection with an SDR, and hence an element occurring in at least one set, the process will end only when we have deleted all sets of the original collection. Theorem 7 assures us that there are sufficiently many occurrences of the various elements that only one set will be deleted at each stage. Hence we will make n choices of elements, thus forming an SDR.

If there is no SDR, then by Theorem 7 there are some k sets containing $p < k$ elements. When these elements are chosen, as they must be since we must reduce all of the n_k to zero, all of the k sets will be deleted. Hence we will have reduced all of the n_k to zero before choosing n representatives.

Example 8. In this example an SDR is computed using Algorithm 2. Since there are five sets in the collection, five stages are required for this particular example. In each stage S_i refers to the i-th set, even though this set is modified in the course of the calculation. The values of n_k, h_j, N, K, and H are all computed by the algorithm. If the value "n. c." is given for h_j, it means that h_j was not computed because $K \cap S_j = \emptyset$. For each k, n_k refers to the element "k." It should be noted that for this example Algorithm 1 gives the SDR $\{1, 2, 4, 3, 6\}$. $\mathcal{A} = \{1, 2, 3, 4, 5, 6\}$.

Stage 1

$S_1 = \{1, 2, 3\}$ $n_1 = n_3 = n_5 = n_6 = 2$; $n_2 = n_4 = 3$.

$S_2 = \{2, 4\}$ Therefore $N = 2$ and $K = \{1, 3, 5, 6\}$.

$S_3 = \{1, 4, 5, 6\}$ $h_1 = h_4 = 7$; h_2 = n. c.; $h_3 = 9$; $h_5 = 5$.

$S_4 = \{2, 3, 5\}$ Therefore $H = 5$.

$S_5 = \{4, 6\}$ Select 6 to represent S_5.

Stage 2

$S_1 = \{1, 2, 3\}$ $n_1 = n_3 = n_4 = n_5 = 2$; $n_2 = 3$.

$S_2 = \{2, 4\}$ Therefore $N = 2$ and $K = \{1, 3, 4, 5\}$.

$S_3 = \{1, 4, 5\}$ $h_1 = h_4 = 7$; $h_2 = 5$; $h_3 = 6$.

$S_4 = \{2, 3, 5\}$ Therefore $H = 5$.

Select 4 to represent S_2.

Stage 3

$S_1 = \{1, 2, 3\}$ $n_1 = n_2 = n_3 = n_4 = 2$.

$S_3 = \{1, 5\}$ Therefore $N = 2$ and $K = \{1, 2, 3, 5\}$.

$S_4 = \{2, 3, 5\}$ $h_1 = h_3 = 6$; $h_2 = 4$.

Therefore $H = 4$.

Select 1 to represent S_3.

Stage 4

$S_1 = \{2, 3\}$ $n_2 = n_3 = 2$; $n_5 = 1$.

$S_4 = \{2, 3, 5\}$ Therefore $N = 1$ and $K = \{5\}$.

h_1 = n. c.; $h_4 = 5$.

Therefore $H = 5$.

Select 5 to represent S_4.

Stage 5

$S_1 = \{2, 3\}$ $\quad n_2 = n_3 = 1.$

Therefore $N = 1$ and $K = \{2, 3\}$.

$h_1 = 2.$

Therefore $H = 2$.

Select $\underline{2}$ to represent S_1.

The selected SDR is thus $\{2, 4, 1, 5, 6\}$.

The primary advantage of this algorithm over the first one is that it leads directly to an SDR. No hunting is necessary. Its disadvantage is that it will not yield all possible SDRs. By reordering the sets and rearranging the elements within each set it is possible to obtain all SDRs through Algorithm 1. A similar procedure will yield a number of SDRs through Algorithm 2, but generally not all since the representatives are chosen by the minimality of n_k. Thus if $S_1 = \{1, 2\}$, $S_2 = \{2, 3, 4\}$, and $S_3 = \{3, 4\}$, the second algorithm will yield the SDRs $\{1, 2, 3\}$ and $\{1, 2, 4\}$, but never the SDRs $\{1, 3, 4\}$, $\{1, 4, 3\}$, $\{2, 4, 3\}$, and $\{2, 3, 4\}$. For many (if not most) applications one SDR is sufficient. However, if we are trying to find a common SDR for several collections of sets, we may need to try several from the first collection before meeting with success.

Since the lemma on which Algorithm 2 is based makes no reference to the minimality condition, it should be possible to formulate related algorithms to obtain SDRs containing an element or elements specified in some other way, if such exist. Thus one should be able to obtain algorithms relating to the problems of Hoffman and Kuhn [21] and Mann and Ryser [26] mentioned in Chapter I. **See Appendix IV in this regard.**

Even after we have found an SDR for one collection there is no guarantee that it is simultaneously an SDR for the other collections with which we are concerned, and we must verify that this is indeed the case. However, the reduction procedure used in the partition case does not go through in the same simple manner, and more possibilities are involved. If, for example, $S_{11} = \{a_1\}$, then as above a_1 must be a representative. But if it occurs in two or more sets of $\mathcal{S}_2$, we cannot say which set it will represent, and hence cannot eliminate other elements as we could in the partition case. Even if a_1 occurs in only one set of $\mathcal{S}_2$, we cannot necessarily eliminate the other elements of this set, as some of them may occur in, and represent, other sets of $\mathcal{S}_2$.

The nested sequence property may be used as in the partition case, but we cannot make such an efficient arrangement of the collections. Mann and Ryser [26] give a lower bound on the number of SDRs for one collection, and there is an obvious upper bound, namely the product of the cardinalities of the sets in the collection. Either of these could be used to order the collections. Or one might use a measure of disjointness, such as $\sum_{j=1}^{n-1} \sum_{k>j} Cd(I(\{S_{ij}\}) \cap I(\{S_{ik}\}))$, for each i, and order the collections on this basis. Finally, from the few examples checked, $D = \sum_{j=1}^{n} \sum_{a_k \in S_j} n_k = \sum_{k=1}^{s} n_k^2$ appears to correlate quite well with the number of SDRs. But none of these arrangements is known to guarantee that $\mathcal{S}_1$ will have the minimal number of SDRs of all $\mathcal{S}_i$. In fact, the correlation of D is known to be less than perfect, since

for example any partition of $\mathcal{A}$ will have the same D value, regardless of the manner in which the elements are divided.

Finally, again because of the possibility that an element may occur in more than one set in each collection, one must alter the procedure for checking each SDR through successive collections. It is sometimes possible to state that the elements of a set containing more than one element must all be representatives. For if any k sets of one collection contain exactly k elements, then in any SDR these sets must be represented by these k elements. This fact enables us, in such cases, to ignore these sets and concentrate on finding representatives for the remaining n - k sets in the collection. Once these have been found we may then investigate the manner in which the original k sets are represented. If in examining the other collections we find one in which the given k elements cannot occur together in an SDR, we then know immediately that there is no SDR for the collections together.

Similarly, if one set is disjoint from the remainder of a collection, we then know that one and only one of its elements can occur in each SDR. But in general no such tricks are available, and the best we can do is essentially to check each potential SDR against all possible permutations of the sets in each collection until it is found to be an SDR for all collections, or is eliminated. This may involve a maximum of $(n!)^{m-1}$ searches for each SDR.

In summary, although a straight-forward procedure for finding the SDRs is available, it tends to be "brute-force," and its very length makes it impractical. If one desires to check one possible SDR, then the methods given above are useable, at least for small values of m

and n. But the general search for SDRs is not feasible without the development of a more powerful search technique.

CHAPTER V

CONCLUSIONS

Of all the approaches to the problem for proving the existence or non-existence of an SDR for three or more collections of sets, that of set theory has led to the most useful results. But even these results are incomplete, and there is undoubtedly room for refinement. We have a necessary condition, and a different sufficient condition. And we have examples, both with and without representatives, which satisfy the former but not the latter. If we restrict the collections to be partitions we are able to improve the sufficient condition, but not enough to match the necessary condition. If in addition we place restrictions on the cardinalities and the intersections of the various sets, and on the size of $\mathcal{A}$, we are able to make further small improvements in our results. But still there remain the examples with and without representatives, satisfying the necessary but not the sufficient condition.

One might suggest that the form of Q be modified by changing the expression $(m - 1)n - k$, that is, by assuming k to be other than a constant. As the following two examples show, this does not seem to be the fundamental difficulty. In these examples we have pairwise disjoint sets with no special relations between them (e. g., no one set belongs to two collections). The two examples differ only in that two elements are exchanged in sets of the third collection, and they both satisfy $Q(m, n, k) \geq 0$ completely only for $k = 0$. Yet one example has an SDR and the other does not. **For both, $\mathcal{A} = \{1, 2, 3, 4, 5, 6\}$.**

Example 9. This example has an SDR, namely $\{1, 4, 5\}$.

$S_{11} = \{1, 2\}$ $\quad$ $S_{21} = \{1, 3\}$ $\quad$ $S_{31} = \{1, 6\}$

$S_{12} = \{3, 4\}$ $S_{22} = \{2, 5\}$ $S_{32} = \{2, 4\}$

$S_{13} = \{5, 6\}$ $S_{23} = \{4, 6\}$ $S_{33} = \{3, 5\}$

Example 10. This example has no SDR, although each pair of collections has an SDR.

$S_{11} = \{1, 2\}$ $S_{21} = \{1, 3\}$ $S_{31} = \{1, 6\}$

$S_{12} = \{3, 4\}$ $S_{22} = \{2, 5\}$ $S_{32} = \{2, 3\}$

$S_{13} = \{5, 6\}$ $S_{23} = \{4, 6\}$ $S_{33} = \{4, 5\}$

Thus we are led to conclude that no function in place of $(m - 1)n - k$, which is wholly dependent on m, n, and the size and intersections of the individual sets, will serve our purpose. For such a function could not distinguish between the above two examples.

Another change one could make in Q is to use other combinations of sets than those specified in the expression. For example, one could at some stage use set unions rather than intersections. This was tried with no success. It seems unlikely that such a move would yield any results, for several reasons. In the first place, switching to unions at some stage seems to imply a new level of complexity in the problem. Yet we are always dealing with collections of sets of elements, whether we have one, two, or arbitrarily many collections. Secondly, because of the nested sequence property mentioned in Chapter IV, as we add more collections the necessary and sufficient condition, if such exists, must become more stringent. Replacing the intersections used in Q by unions, at any stage, seems to be a weakening, rather than a strengthening of the condition. Finally, any SDR for m collections is a fortiori an SDR for any $m - 1$ of these collections. Hence, by setting $\mathcal{X}_{i_o} = \mathcal{T}_{i_o}$ for any specific i_o, any necessary and sufficient

condition for m collections must reduce to that for m - 1 collections. In particular, in this situation any necessary and sufficient condition for three collections must reduce to one for two collections. While this does not have to be the known one, it certainly must be one logically equivalent to it.

There are, as mentioned above, places for refinement. Prime among these are the demonstrations that the various sufficient conditions are the best possible under the given restrictions, or the improvement of these conditions. But as was shown in Chapter II, the examples necessary to this work, and the proofs of properties which these examples indicate, are cumbersome and difficult to find. It is, of course, possible to place further restrictions on the sets and obtain better sufficient conditions in these cases. But at this point the theory loses much of its generality. Another area for improvement lies in the redundancy which we have shown to exist in Ford and Fulkerson's criterion and its generalization. A comparison of the N and T values given in Appendix II shows that the ratio N/T is less than one half for the values given and appears to become small as m and n increase. Not only does the small value of N decrease the necessary work to establish whether or not our various conditions hold, but also the large number of unused set combinations may provide a means of obtaining more exact conditions for the existence of an SDR.

While the set theory approach has been most fruitful thus far, it appears that for further developments an attack through the methods of mathematical programming may prove most useful. In fact, since we have been able to formulate the problem completely and concisely in such language, it appears that once the programming theory for linear

functions with quadratic constraints is fully elaborated the solution to the present problem will be immediate. At present it appears possible to handle certain special cases by such techniques as dynamic programming or a reduction to linear constraints. But for the general solution there is currently only the "brute force" attack - tantamount to the search for SDRs.

As long as we are dealing with collections which are partitions, we have a number of useful algorithms for actually finding SDRs. These involve first a reduction and rearrangement of the collections involved, followed by a sieving of possible SDRs from the first collection. In this manner we can determine all SDRs for the collections together. When the collections are not partitions, the situation is not so fine. We have two algorithms for finding SDRs for one collection. One of these will yield all SDRs, but only with some rearrangement and hunting. The other will yield an SDR directly, with no hunting or rearrangement, but will not generally give us all SDRs for the collection. We can make use of the same reduction and reordering algorithms as in the partition case, and similar ones; but since we have no very good measure of the restrictiveness of the various collections, we cannot use these algorithms to full advantage. In this phase of the work the major outstanding problem is the determination of such a measure. Until this is settled the search algorithms and methods we have will remain only partially feasible for locating one SDR, and virtually useless for determining all SDRs because of the amount of computation for problems of large size.

APPENDIX I

NOTATION

Expressions of the type $[x]$ denote the greatest integer less than or equal to x wherever they occur in the equations and inequalities of this paper.

The following special notation has been used.

Notation			Introduced on page
$\mathcal{A}$	-	the base set	1
a_k	-	an element of $\mathcal{A}$	1
$\mathcal{S}_i$	-	the i-th collection	1
S_{ij}	-	the j-th set of $\mathcal{S}_i$	1
m	-	the number of collections	1
n	-	the number of sets in each collection	1
s	-	the number of elements in	1
f	-	a flow in a network	6
c	-	a capacity in a network	6
L, L'	-	the parts of a cut in a network	6
σ	-	a source in a network	6
σ'	-	a sink in a network	6
Cd(X)	-	the cardinality of X	8
J(X)	-	the vertices of a network adjacent to vertices of X	8
$\mathcal{X}_i$	-	a subset of $\mathcal{S}_i$	13

Notation			Introduced on page
X_i	-	the set of indices of sets in $\mathcal{X}_i$	13
I_i, $I(\mathcal{X}_i)$	-	the set of indices of points in sets of $\mathcal{X}_i$	13
k	-	a function of m and n	14
Q	-	the expression involved in a Ford and Fulkerson type of inequality	15
Q_o	-	Q when $k = 0$	15
N	-	the minimum number of subcollections needed to assert $Q \geq 0$ completely	25
T	-	the total number of subcollections	26
K	-	the set of minimal elements	57

APPENDIX II

TABLES OF N AND T

$$N = \sum \frac{(n!)^m \, m!}{\prod_{\nu=1}^{n} [\nu!(n-\nu)!]^{p_\nu} \, p_\nu!}$$

and $$T = (2^n - 1)^m$$

Appendix Table 1. N and T for m = 2

k \ n	2	3	4	5	6
0	4	15	56	210	792
1	4	18	68	250	922
2		9	48	200	780
3			16	100	465
4				25	180
5					36
T	9	49	225	961	3969

Appendix Table 2. N and T for m = 3

k \ n	2	3	4	5	6
0	6	36	220	1365	8568
1	12	81	492	3000	18561
2	8	108	768	4975	31788
3		81	840	6300	43560
4		27	624	6075	47960
5			288	4375	42273
6			64	2250	29448
7				750	15795
8				125	6210
9					1620
10					216
T	27	343	3375	29791	250047

Appendix Table 3. N and T for m = 4

k \ n	2	3	4	5	6
0	8	66	560	4845	42504
1	24	216	1816	15500	134592
2	32	459	4320	38700	346032
3	16	648	7744	77100	734859
4		594	10560	124150	1304240
5		324	10896	162500	1949016
6		81	8320	172750	2461872
7			4480	148000	2629906
8			1536	100500	2368920
9			256	52500	1786620
10				20000	1114344
11				5000	563409
12				625	223560
13					65880
14					12960
15					1296
T	81	2401	50625	923521	15752961

Appendix Table 4. N and T for m = 5

k \ n	2	3	4	5	6
0	10	105	1140	12650	142506
1	40	450	4840	53125	593770
2	80	1305	15424	177000	2035680
3	80	2673	38160	479750	5851545
4	32	3915	74720	1075875	14297030
5		4050	116880	2018750	29991885
6		2835	146240	3191250	54414780
7		1215	145376	4263750	85820255
8		243	112960	4813750	118029510
9			66560	4575000	141746850
10			28160	3631250	148588160
11			7680	2375000	135646995
12			1024	1253125	107364810
13				515625	73158075
14				156250	42464700
15				31250	20675655
16				3125	8253630
17					2608200
18					615600
19					97200
20					7776
T	243	16807	759375	28629151	992436543

Appendix Table 5. N and T for m = 6

k \ n	2	3	4	5	6
0	12	153	2024	27405	376992
1	60	810	10620	142500	1947786
2	160	2970	42384	593625	8347500
3	240	7938	133456	2034000	30257730
4	192	15849	339264	5839125	94118920
5	64	23814	706416	14231250	254022426
6		26730	1214720	29726250	599950260
7		21870	1730496	53565000	1248115065
8		12393	2039424	83611875	2298575160
9		4374	1975616	113287500	3761183790
10		729	1553664	133237500	5482090020
11			971520	135737500	7127761410
12			466944	119259375	8270370360
13			162816	89718750	8558194870
14			36864	57171875	7884128700
15			4096	30375000	6446365065
16				13143750	4656793680
17				4468750	2953118790
18				1125000	1629501300
19				187500	772909425
20				15625	309847356
21					102461436
22					26943840
23					5307120
24					699840
25					46656
T	729	117649	11390625	887503681	62523502209

APPENDIX III

MATROIDS AND REPRESENTATIVES

In 1942, R. Rado published a paper [30] concerned with systems of distinct representatives for sets having an independence relation defined on them. In particular, he showed that P. Hall's condition for one collection of sets, stated relative to the independence relation, is a necessary and sufficient condition for the existence of an SDR whose elements are independent under the given relation. In 1949, he extended this result to subsets of an infinite base set $\mathcal{A}$ through the use of the concept of rank.

Defn. 8. Given an arbitrary set $\mathcal{A}$, let I be a function which assigns to each finite ordered sequence $x_1, \ldots, x_t$ of elements from $\mathcal{A}$ a number $I(x_1, \ldots, x_t)$ which is either zero or one. If $\emptyset$ is the null set, we assume that $I(\emptyset) = 1$. I is an independence relation, with $I(x_1, \ldots, x_t) = 1$ if and only if $x_1, \ldots, x_t$ are independent, if I has the following properties:

i) I is decreasing, i. e., $I(x_1, \ldots, x_t) \geq I(x_1, \ldots, x_t, x_{t+1})$ for every t and every sequence $x_1, \ldots, x_t$.

ii) I is commutative, i. e., $I(x_1, \ldots, x_t) = I(x_{j_1}, \ldots, x_{j_t})$, for every permutation $j_1, \ldots, j_t$ of the numbers $1, \ldots, t$.

iii) I is non-reflexive, i. e., $I(x, x) = 0$ for all $x \in \mathcal{A}$.

iv) I is distributive, i. e.,

$$I(x_1, \ldots, x_t)\, I(y_1, \ldots, y_{t+1}) \leq \sum_{p=1}^{t+1} I(x_1, \ldots, x_t, y_p)$$

for every $t \geq 0$ and every $x_1, \ldots, x_t, y_1, \ldots, y_{t+1}$.

Defn. 9. A collection of sets $\mathcal{A} = \{S_1, \ldots, S_n\}$ has property H relative to an independence relation I if for any numbers t, $\nu_1, \ldots, \nu_t$ satisfying $1 \leq t \leq n$, $1 \leq \nu_1 < \nu_2 < \ldots < \nu_t \leq n$, the union $S_1 \cup \ldots \cup S_t$ contains elements $x_1, \ldots, x_t$ for which $I(x_1, \ldots, x_t) = 1$. The collection has property R relative to I if we can choose from each set S_i an element a_i such that $I(a_1, \ldots, a_n) = 1$.

Rado proved the following theorem.

Theorem 22. (R. Rado) For any independence relation I, the properties H and R are equivalent.

Three special cases of this theorem may be noted.

Case 1. $I(x_1, \ldots, x_t) = 1$ if and only if $x_1, \ldots, x_t$ are distinct. I is clearly an independence relation, and thus the theorem holds. This is, in fact, P. Hall's original result.

Case 2. Suppose that $\mathcal{A}$ is a finite dimensional vector space, and let $I(x_1, \ldots, x_t) = 1$ if and only if the vectors $x_1, \ldots, x_t$ are linearly independent. Theorem 22 then asserts that if no k of the sets $S_1, \ldots, S_n$ are contained in any subspace of dimension $k - 1$, for $k = 1, \ldots, n$, it is possible to select one vector a_i from each set S_i in such a way that $a_1, \ldots, a_n$ are linearly independent.

Case 3. Suppose that $\mathcal{A}$ is a finite set of polynomials in the variables $v_1, \ldots, v_p$, with coefficients in some field F. Given polynomials $a_1, \ldots, a_k$, let $I(a_1, \ldots, a_k) = 1$ if no polynomial $f(w_1, \ldots, w_k)$ exists with coefficients in F not all zero, such that $f(a_1, \ldots, a_k) = 0$ identically in $v_1, \ldots, v_p$. Then if any k of the sets Si

contain between them at least k independent polynomials, for $k = 1, \ldots, n$, it is possible to select polynomials $a_i \in S_i$ such that $a_1, \ldots, a_n$ are independent.

The idea of a "matroid," introduced by Whitney in 1935 [37], has a role in the study of SDRs. Since any linear graph has a matroid associated with it, the results of König and others can be stated in terms of this associated matroid. Moreover, it was through the concept of a "rank function," used in defining a matroid, that Rado was led to his generalization of $\mathcal{C}\mathcal{L}$ to an infinite set.

Defn. 10. A matroid is an ordered pair (M, r), where M is a set and r is a rank function satisfying the following properties:

R1. $r(\emptyset) = 0$.

R2. For any subset $A \subset M$ and any element e not in A, $r(A) \leq r(A \cup \{e\}) \leq r(A) + 1$.

R3. For any subset $A \subset M$ and any elements e_1 and e_2 not in A, if $r(A \cup \{e_1\}) = r(A \cup \{e_2\}) = r(A)$, then $r(A \cup \{e_1, e_2\}) = r(A)$.

The nullity $n(A)$ of a set $A \subset M$ is defined by $n(A) = Cd(A) - r(A)$. A set $A \subset M$ is independent if $n(A) = 0$, and is dependent otherwise. A base for the matroid is a maximal independent submatroid; a circuit is a minimal dependent submatroid.

It is easy to show that the rank and nullity are non-negative monotone functions. That is, if $A \subset B \subset M$, then $0 \leq r(A) \leq r(B)$ and $0 \leq n(A) \leq n(B)$. Whitney also showed that the concepts of independence, base, and circuit are equivalent to that of rank.

Theorem 23. (Whitney) Given a set M, it is possible to define a rank function r such that (M, r) is a matroid if and only if any one of the following three sets of conditions holds:

(independence)

I1. Any subset of an independent set is independent.

I2. If $A = \{e_1, \ldots, e_p\}$ and $A' = \{e'_1, \ldots, e'_{p+1}\}$ are independent sets, then for some i such that e'_i is not in A, $A \cup \{e'_i\}$ is independent.

(base)

B1. No proper subset of a base is a base.

B2. If B and B' are bases and e is an element of B, then for some element e' in B', $B - \{e\} \cup \{e'\}$ is a base.

(circuit)

C1. No proper subset of a circuit is a circuit.

C2. If P_1 and P_2 are circuits, $e_1 \varepsilon P_1 \cup P_2$, $e_2 \varepsilon P_1 - P_2$, then there is a circuit $P_3 \subset P_1 \cup P_2$ containing e_2 but not e_1.

It is clear that any matrix has associated with it a matroid whose elements are the column vectors of the matrix with the rank function taken as the column rank of the matrix. With this matroid the usual definitions of base and (linear) independence apply as above. In order to associate a matroid with a graph, we may take the edges of the graph as the elements of M and define a circuit as a simple closed curve in the graph. An independent set is then a set of edges of the graph containing no circuit, and a base will be the usual edge basis for the

graph - a tree containing all points of the graph if the graph is connected, or a forest otherwise.

Let us recall P. Hall's second result: given a partition $C_1, \ldots, C_t$ of $\mathcal{A}$, and sets $S_1, \ldots, S_n$, $n \leq t$, it is possible to find an SDR for $S_1, \ldots, S_n$ having at most one element in each equivalence class if and only if any k of the sets S_i contain elements from at least k of the classes C_j, $k = 1, \ldots, n$. We may state this in terms of a matroid by using the equivalence classes to define the rank. For any set A, let $r(A)$ be the number of equivalence classes which A intersects. It is clear that the properties R1, R2, and R3 are satisfied. Also $n(A) = 0$ if and only if the number of classes which A intersects is equal to the cardinality of A. Thus Hall's result is that we can find an SDR, the elements of which form an independent set, if and only if the rank of the union of any k sets S_i is at least k, $k = 1, \ldots, n$.

While the above rank function was defined from the partition, it is not possible to define all rank functions in this manner. However, Rado showed [32] that even if we allow $\mathcal{A}$ to be an infinite set, Hall's result holds for an arbitrary rank function.

Theorem 24. Let $\mathcal{A}$ be an arbitrary (possibly infinite) set, and let $\{S_\nu\}$ be a system of finite subsets of $\mathcal{A}$, with indices in some index set M'. Let r be a rank function defined on subsets of $\mathcal{A}$. Then in order that there exist an independent set of distinct elements $a_\nu \in S_\nu$, it is necessary and sufficient that for any finite subset $K \subset M'$, $r(\bigcup_{\nu \in K} S_\nu) \geq Cd(K)$.

APPENDIX IV

EXTENSIONS OF ALGORITHM 2

There are two directions in which extensions of Algorithm 2 may be attempted. First, one may, as mentioned in Chapter IV, attempt to formulate algorithms yielding SDRs with specific elements, or even systems of restricted representatives, in the sense of Ford and Fulkerson. Second, one may try to extend the algorithm to more than one collection of sets. Since this latter generalization is more pertinent to our problem, we shall consider it first.

Let us call an element a_k a <u>minimal element</u> if it is a member of the set K of Algorithm 2. If we have several collections of sets, each collection will have its own set of minimal elements. These sets are not necessarily identical, and may even be disjoint. Thus the best statement we could hope for is that if a common SDR exists, then there exists one, one of whose elements is a minimal element for at least one of the collections. The following example shows that this in unattainable, even with as few as two collections.

<u>Example 11</u>. In this example $\mathcal{A} = \{1, 2, 3, 4, 5\}$ and $K = \{1\}$ for both collections of sets. The only common SDR is $\{2, 3, 4, 5\}$, and this does not intersect K.

$S_{11} = \{1, 2, 3, 4, 5\}$ $\quad$ $S_{21} = \{1, 2, 3, 4, 5\}$

$S_{12} = \{2, 3\}$ $\quad$ $S_{22} = \{4, 5\}$

$S_{13} = \{4\}$ $\quad$ $S_{23} = \{2\}$

$S_{14} = \{5\}$ $\quad$ $S_{24} = \{3\}$

Turning to the other generalization, we face an immediate problem. Supposing that an SDR exists with a specified element and that the given element is in more than one set, which set does this element represent? Thus in the first collection of the above example, we can always find an SDR one of whose elements is 4, but we can never find an SDR in which 4 represents S_{11}. In general, if we wish an element $\bar{a}$ to be in an SDR, we arbitrarily let this represent one of the sets in which it occurs. Eliminating this set, and $\bar{a}$ from the rest of the sets, we may then apply Algorithm 2 to the remaining modified collection. If this yields an SDR, we are done; if not, we restore the original collection and begin again, letting $\bar{a}$ represent another set in which it occurs. If after letting $\bar{a}$ represent all possible sets we have found no SDR, then it is clear that none exists.

It is evident that the above procedure can be extended to handle more than one specified element, and even to handle systems of restricted representatives by allowing more than one occurrence of an element to be specified. (We may specify that an element not occur simply by eliminating it from all sets before beginning the search for SDRs.) However, it is also apparent that this process involves a hunting situation similar to that which occurs in Algorithm 1. Thus an efficient use of this method awaits the development of a good method of ordering the sets so as to reduce the amount of hunting which is necessary.

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