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ON THE HYDRODYNAMIC STABILITY OF SOME SPATIALLY PERIODIC FLOWS

by
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CHAPTER I

FORMULATION OF THE PROBLEM

1.1 Introduction

The bulk of this thesis will be concerned with the stability of the incompressible, boundary-free, parallel flow whose velocity profile is the cosine. It is a special case of some two dimensional exact solutions of the equations of motion discussed in Chapter 5 and might be thought of as an infinite number of evenly spaced free boundary layers. The flow is an exact solution of the Navier-Stokes equations provided the appropriate external body forces are maintained. From the discussion of Section 5.1 it will be seen that if the body forces are turned off the flow continues to be sinusoidal but is exponentially damped because of viscous dissipation. In Chapter 5 a two dimensional cellular flow will be considered but because of the problem's complexity not treated as thoroughly as the parallel flow.

The method of small perturbations is used to linearize the equations of motion about the basic flow. The boundary condition imposed is that the perturbation shall be bounded in space. This is somewhat weaker than the boundary conditions usually imposed in the case of unbounded flows, viz., that the perturbation vanish at infinity. The time dependence of the perturbation, assumed exponential, is chosen so that stability depends on the imaginary part of a parameter, denoted by c, which is considered to be the eigenvalue. If c_i is positive the disturbance grows in time and the flow is considered unstable. Other parameters of interest are the Reynolds' number, R, and, for the parallel flow, the wave number of the perturbation, α .

In Chapter 2 the behavior of the parallel flow is investigated in the limits of small α and large $|c|$ by imposing the boundary conditions at two finite points. The same problems are treated in Chapter 3 by formulating the eigenvalue problem as the vanishing of an infinite determinant. A critical Reynolds' number, $R_c = \sqrt{2}$, is found below which the flow is stable with respect to all perturbations. The method of Chapter 3 is found to be convenient for the calculation of the neutral curve, defined by $c_i(\alpha, R) = 0$, near the critical Reynolds' number. It is found that the curve intersects the R axis at R_c. In Chapter 4 the problem is studied for large values of the Reynolds' number and the neutral curve is found to approach $\alpha = 1$ asymptotically as $R \longrightarrow \infty$. A general description of the eigenvalue spectrum may be obtained from the material of Chapters 2, 3, and 4. It consists of an infinity of segments or bands, separated by small gaps, lying along the imaginary axis of the c-plane. It is found that there is a single band which can lie in the unstable part of the c-plane, given appropriate values of α and R.

In Chapter 5 the stability of a two dimensional square cellular flow is considered. The eigenvalue spectrum has not been thoroughly investigated. However, certain modes have been found in the neighborhood of the origin of the c-plane which are unstable for values of the Reynolds' number greater than $\sqrt{2}$.

The linear theory of stability dates back to Kelvin. Because of the long history of the theory, it is perhaps surprising that it has only recently been recognized that the stability of flows in which no solid boundary is present is quite different from that of bounded flows. The salient feature of the boundary-free flows is their small critical Reynolds' numbers (below which all disturbances are damped). The

literature on the stability of these flows is not large and is primarily concerned with flows of the free boundary layer type, e.g., jets, wakes, and the mixing of parallel streams. We may contrast critical Reynolds' numbers of zero for the free boundary layer between parallel streams [19]* and of about 4 for a two dimensional jet [20] with values of 420 for the boundary layer along a flat plate [12] and 5314 for plane Poiseuille flow [12].

Helmholtz and Rayleigh were probably the first to investigate the stability of unbounded flows for perfect fluids. Helmholtz [10] showed that a vortex sheet of zero thickness is unstable and noted its tendance to "roll up" into asymmetric spirals. Rayleigh [16] treated the vortex sheet of finite thickness by using a piecewise linear velocity profile. Subsequently, the problem of the stability of viscous fluids attracted a great deal of attention, but until recently in all of the problems considered there has been at least one solid boundary present.

The first attempts to include viscosity in the treatment of boundary free flows were by Foote and Lin [6], Lessen [11], and Chiarulli [2]. Foote and Lin considered flows of the free boundary layer type from a general point of view. Lessen solved the boundary layer equation numerically for the mixing of parallel streams and investigated its stability at high Reynolds' numbers but did not obtain a critical Reynolds' number. Chiarulli dealt with the problem using analytical methods.

More recently the two dimensional laminar jet has been treated by Pai [15], Curle [3], Howard [8], and Tatsumi and Kakutani [20]. In addition to the work of Lessen and Chiarulli, free boundary

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*Numbers in square brackets refer to the bibliography at the end of the thesis.

layers between uniform streams have been considered by Esch [5], who used a piecewise linear velocity profile, and Tatsumi and Gotoh [19], who did not introduce a specific profile but were able to solve the problem generally for this type of flow. The stability properties of the free boundary layer between uniform streams appear to resemble those to be discussed in Chapters 2, 3, and 4. Further discussion of this point will be found in Section 4.6.

1.2 The Fundamental Equation

In Chapters 2, 3, and 4 we shall be dealing with the steady flow whose velocity components are

$$U_{*1} = A_* \cos k_* y_*$$

$$U_{*2} = U_{*3} = 0$$

where A_* and k_* are constants. Asterisks as subscripts are used to denote dimensional quantities. In order to put our equations into dimensionless form we introduce the characteristic velocity

$$V_* = A_*$$

and the characteristic length

$$L_* = 1/k_*$$

The dimensionless variables are defined as follows:

$$U_i = U_{*i}/V_* \qquad x = x_*/L_*$$

$$p = p_*/\rho_* V_*^2 \qquad y = y_*/L_*$$

$$F_i = F_{*i} L_*^2 /V_*^2 \qquad z = z_*/L_*$$

$$t = t_* V_*/L_*$$

Here the U_i are the velocity components, p the pressure, ρ_* the density, F_i the components of the external force per unit mass, and x, y, z, t the spacial and time coordinates. In dimensionless form the velocity profile is simply cos y. The equations of motion are the

Navier-Stokes equations and the incompressibility condition. They may be put into dimensionless form by formally setting $\rho_* = 1$ and $\nu = R^{-1}$,

$$\frac{\partial \vec{U}}{\partial t} + (\vec{U} \cdot \nabla)\,\vec{U} = -\nabla p + \frac{1}{R}\,\nabla^2 \vec{U} + \vec{F} \tag{1.2.1}$$

$$\nabla \cdot \vec{U} = 0$$

where ν is the kinematic viscosity, and R the Reynolds' number defined by

$$R = \frac{V_* L_*}{\nu}$$

Squire [18] has shown that for parallel flows the least stable perturbation is in the plane of the basic flow. Therefore, only the two dimensional problem will be considered and the equations of motion can be written in terms of the stream function ψ defined by

$$U_1 = -\frac{\partial \psi}{\partial y} \qquad U_2 = \frac{\partial \psi}{\partial x}$$

Taking the curl of (1.2.1),

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{U} \cdot \nabla)\vec{\omega} - (\vec{\omega} \cdot \nabla)\,\vec{U} = \frac{1}{R}\,\nabla^2 \vec{\omega} + \nabla \times \vec{F} \tag{1.2.2}$$

where $\vec{\omega}$ is the vorticity

$$\vec{\omega} \equiv \text{curl } \vec{U}$$

In two dimensions $\vec{\omega}$ has only one component,

$$\omega = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = -\nabla^2 \psi$$

and the vorticity equation (1.2.2) becomes

$$\left\{ \frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{1}{R}\nabla^2 \right\} \nabla^2 \psi = \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \tag{1.2.3}$$

To study the behavior of small disturbances we substitute

$$\psi = \bar{\psi} + \psi'$$

into (1,2,3) and neglect terms quadratic in ψ'. Here $\bar{\psi}$ is the stream function of the basic flow and ψ' is the stream function of a small perturbation. For parallel flows $\partial\bar{\psi}/\partial x = 0$ and the coordinates x and t are cyclic. The solution is, therefore, written in the form

$$\psi' = \varphi(y)\, e^{i\alpha(x - ct)}$$

Substituting into the linearized equation we obtain the equation which governs the stability of parallel flows and which is usually known as the Orr-Sommerfeld equation,

$$(D^2 - \alpha^2)^2\varphi = i\alpha R \left\{ (u - c)(D^2 - \alpha^2) - u'' \right\}\varphi \qquad (1.2.4)$$

where $D \equiv \frac{d}{dy}$ and u(y) is the velocity profile of the basic flow. α may be thought of as the (dimensionless) wave number of the perturbation and c will be considered to be the eigenvalue. That is, we shall regard α and R as being at our disposal so that c must be adjusted to satisfy the boundary conditions. It will be noticed that c is defined so that stability is determined by its imaginary part. If $c_i > 0$ the perturbation grows in time and the basic flow is considered unstable. If $c_i < 0$ the perturbation is damped and the flow is considered stable. Neutral stability occurs when $c_i = 0$ and the perturbation is oscillatory. The curve

$$c_i(\alpha, R) = 0$$

is called the curve of neutral stability, or simply the neutral curve, and separates the stable from the unstable regime. The calculation of this curve is the primary objective in stability problems.

1.3 Formulation of the Eigenvalue Equation

The boundary conditions to be imposed are that the components of the perturbation, $-\frac{\partial\psi'}{\partial y}$ and $\partial\psi'/\partial x$, be bounded as $x, y \to \pm\infty$.

That is, $\varphi'(y)$ and $\alpha\varphi(y)$ should be bounded as $y \to \pm\infty$, where $\varphi(y)$ is the solution of the fundamental equation (1.2.4) with $\mu(y) = \cos y$. The theory of differential equations with periodic coefficients is generally known as Floquet theory and is developed in references on differential equations, for example, in Ince [9] p. 381.

The following theorem is proved in [9]. Let $\varphi_1(y)$, $\varphi_2(y)$, . . . $\varphi_n(y)$ be a fundamental set of solutions of an n^{th} order linear differential equation whose coefficients are periodic with period 2π and whose only singularities are poles in the finite y-plane. Then $\varphi_1(y+2\pi)$, $\varphi_2(y+2\pi)$ . . . $\varphi_n(y+2\pi)$ are also solutions and there exists a set of relations

$$\varphi_r(y+2\pi) = \sum_{p=1}^{n} a_{rp}\,\varphi_p(y) \qquad r = 1, 2, \ldots n$$

The nature of the solutions of the equation is determined by the roots of the polynomial

$$\begin{vmatrix} a_{11}-s & a_{12} & \ldots a_{1n} \\ a_{21} & a_{22}-s & \ldots a_{2n} \\ \ldots & & \\ a_{n1} & a_{n2} & \ldots a_{nn}-s \end{vmatrix} = 0 \qquad (1.3.1)$$

If (1.3.1) has n distinct roots $s_1, s_2, \ldots s_n$ then there exists a fundamental set of solutions

$$\varphi_r = e^{i\nu_r y}\, S_r(y) \qquad r = 1, 2, \ldots n$$

where $s_r \equiv e^{2\pi i \nu_r}$ and the functions $S_r(y)$ are periodic with period 2π. If s_1 is a ν-fold repeated root then there is a sub-set of solutions corresponding to it:

$$\varphi_1 = e^{i\nu_1 y}\, S_1(y)$$

$$\varphi_2 = e^{i\nu_1 y}\left\{ P_1(y)\, S_1(y) + S_2(y) \right\}$$

. . .

$$\varphi_\nu = e^{i\tau y}\left\{P_{\nu-1}(y)\,\zeta_1(y) + P_{\nu-2}(y)\,\zeta_2(y) + \cdots \zeta_\nu(y)\right\}$$

where $\zeta_1(y) \cdots \zeta_\nu(y)$ are periodic with period 2π and

$$P_\nu(y) = \frac{y\,(y-2\pi)\cdots\{y-(2\nu-2)\pi\}}{(2\pi)^\nu\,\nu!}$$

Thus, we see that a bounded solution of (1.2.4) must be of the form

$$\varphi(y) = e^{i\tau y}\,\zeta(y) \tag{1.3.2}$$

where τ is real and $\zeta(y)$ has period 2π. This result is the basis of our formulation of the eigenvalue problem. (It could be regarded as a special case of the theorem proved in Appendix V). The eigenvalue equation will be written in a number of ways, but always so that a general solution of (1.2.4) will be of the form (1.3.2). The general approach will be to assume c of some order of magnitude, solve the eigenvalue equation for c, and then verify that the assumption is satisfied, if necessary by suitably restricting the range of τ.

The parameter τ may be thought of as "labeling" the eigenvalues. That is, for fixed α and R we expect the real axis of the τ-plane to map into the eigenvalue spectrum in the c-plane (although the mapping is not necessarily one to one). Thus, the spectrum is a continuum, as would be expected in an eigenvalue problem involving an infinite domain. (In the case of bounded flows the spectra are discrete.)

We shall have occasion to use equation (1.3.1) as the eigenvalue equation together with the stipulation that τ must be real ($s \equiv e^{2\pi i\tau}$), the a_{ij} being given by*

$$\varphi_i^{(K-1)}(y + 2\pi) = \sum_{j=1}^{4} a_{ij}\,\varphi_j^{(K-1)}(y) \qquad i, K = 1,2,3,4$$

*Superscripts in parentheses as well as primes will be used to indicate differentiation.

The boundary conditions can also be imposed by

$$\varphi^{(j)}(y_o + 2\pi) = s\,\varphi^{(j)}(y_o) \qquad j = 0,1,2,3 \qquad (1.3.3)$$

where y_o is some definite value of y, $s \equiv e^{2\pi i \tau}$, and τ must be real. The above is simply the requirement that $\zeta(y)$ be periodic, i.e.,

$$\zeta^{(j)}(y_o + 2\pi) = \zeta^{(j)}(y_o) \qquad j = 0,1,2,3$$

This can be seen by differentiating $\zeta(y)$ three times. From (1.3.2)

$$\zeta(y) = \varphi(y)\ e^{-i\tau y}$$

$$= \zeta(y + 2\pi) = \varphi(y + 2\pi)\ e^{-i\tau(y + 2\pi)}$$

Differentiating,

$$\zeta'(y) = -i\tau\,\zeta(y) + \varphi'(y)\ e^{-i\tau y}$$

$$= \zeta'(y + 2\pi) = -i\tau\,\zeta(y + 2\pi) + \varphi'(y + 2\pi)\ e^{-i\tau(y + 2\pi)}$$

$$= -i\tau\,\zeta(y) + \varphi'(y + 2\pi)\ e^{-i\tau(y + 2\pi)}$$

etc.

From these equations we obtain

$$\varphi(y + 2\pi) = e^{2\pi i\tau}\varphi(y)$$

$$\varphi'(y + 2\pi) = e^{2\pi i\tau}\varphi'(y)$$

etc.

i.e., we obtain the relations (1.3.3).

Since the coefficients of (1.2.4) are analytic in y, α, R, and c solutions are entire functions of these quantities and the eigenvalue equation obtained from (1.3.1) or(1.3.3) is a relation

$$F(\alpha, R, c, s) = 0$$

where F is an entire function of α, R, c, and s. For fixed α, R, s

there is a discrete set of solutions for c because the zeros of an entire function must be discrete unless the function is identically zero. But for fixed α and R if τ is allowed to vary over all permitted values, i.e., all real values, a continuous spectrum of eigenvalues is obtained.

The above formulation of the eigenvalue problem will be used in Chapters 2 and 4. The function $F(\alpha, R, c, s)$ is a fourth order polynomial in s and, as we have remarked, its roots determine the nature of the solutions of (1.2.4). Of course, for the calculation of the eigenvalue spectrum and the neutral curve we are only interested in roots corresponding to bounded solutions.

In Chapter 3 $\zeta(y)$ will be expanded in a Fourier series and the problem formulated as the vanishing of a certain infinite determinant. The method is more convenient for the calculation of the spectrum and the neutral curve when αR is small, but produces less information about the solutions of (1.2.4). To this extent the two methods are complementary.

CHAPTER II

THE METHOD OF FINITE BOUNDARY CONDITIONS

In this chapter we shall treat the problem of the parallel flow using the method of finite boundary conditions. We shall be primarily interested in the case in which α and $|c|$ are small, since this is the case in which unstable modes are found. We shall begin, however, by examining the eigenvalue spectrum for the cases in which $|c|$ is large and of order unity.

2.1 Eigenvalue Spectrum When $|c|$ Is Large

With $\mathcal{U}(y) = \cos y$, the Orr Sommerfeld equation (1.2.4) can be written:

$$(D^2 - \alpha^2)^2 \varphi + i\alpha R c (D^2 - \alpha^2) \varphi = i\alpha R \cos y (D^2 - \alpha^2 + 1) \varphi \qquad (2.1.1)$$

In what follows it is not necessary to make any assumption about α^2 or αR provided $|c|$ is sufficiently large. We shall treat (2.1.1) as an inhomogeneous equation. In "zero$^{\text{th}}$ order" the inhomogeneous part (right side) of (2.1.1) is neglected and solutions are $\exp \pm \alpha y$, $\exp \pm my$ where $m = \sqrt{-i\alpha Rc + \alpha^2}$. To satisfy the boundary conditions the first two solutions must be rejected and c must be pure imaginary and negative. Thus, for $|c|$ sufficiently large the spectrum lies along the negative imaginary axis in the c-plane.

By solving (2.1.1) as an inhomogeneous equation using the method of variation of parameters integral equations may be obtained for φ and its first three derivatives.

$$\varphi^{(i)}(y) = A\alpha^i e^{\alpha y} + B(-\alpha)^i e^{-\alpha y} + C m^i e^{my} + D(-m)^i e^{-my}$$

$$+ \int_0^y K(y,y') L \varphi(y') dy' \quad i = 0,1,2,3 \qquad (2.1.2)$$

where $L \equiv d^2/dy'^2 - \alpha^2 + 1$

$$K(y,y') = \alpha^i e^{\alpha y} K_1(y') + (-\alpha)^i e^{-\alpha y} K_2(y')$$

$$+ m^i e^{my} K_3(y') + (-m)^i e^{-my} K_4(y')$$

$$K_1(y') = -\frac{i\alpha R}{m^2 - \alpha^2} \frac{\cos y'}{2\alpha} e^{-\alpha y'}$$

$$K_2(y') = \frac{i\alpha R}{m^2 - \alpha^2} \frac{\cos y'}{2\alpha} e^{\alpha y'}$$

$$K_3(y') = \frac{i\alpha R}{m^2 - \alpha^2} \frac{\cos y'}{2m} e^{-my'}$$

$$K_4(y') = -\frac{i\alpha R}{m^2 - \alpha^2} \frac{\cos y'}{2m} e^{my'}$$

The lower limit of integration is arbitrary but for convenience has been taken to be zero. A,B,C, and D are arbitrary constants.

With appropriate choices for the constants A,B,C, and D, four solutions of (2.1.2) can be obtained as uniformly convergent series in m^{-1} by iteration. The proper choice for the constants has been discussed by Schensted [17]. Since the boundary conditions will be applied at the points $y = 0, 2\pi$ the constants will be chosen so that the series converge uniformly on the interval $0 \leq y \leq 2\pi$. Let the branch of m be chosen so that Re $m \geq 0$ A solution, φ_1 say, can be constructed by setting

$A = 1 \quad B = D = 0$

$$C = C_1 = - \int_0^{2\pi} K_3(y') \, L \, \varphi_1(y') \, dy'$$

Then (2.1.2) becomes

$$\varphi_1^{(i)}(y) = \alpha^i e^{\alpha y} + \int_0^{y} \left\{ \alpha^i e^{\alpha y} K_1(y') + (-\alpha)^i e^{-\alpha y} K_2(y') \right.$$

$$\left. + (-m)^i e^{-my} K_4(y') \right\} L \varphi_1(y') \, dy'$$

$$- \int_y^{2\pi} m^i e^{my} K_3(y') \, L \, \varphi_1(y') \, dy' \qquad (2.1.3)$$

A solution of (2.1.3) for $\varphi_1^{(o)}(y) \equiv \varphi_1(y)$ can be obtained by iteration.

$$\varphi_1 = \sum_{n=0}^{\infty} \varphi_1^n \quad \text{where } \varphi_1^o = e^{\alpha y} \qquad (2.1.4)$$

$$\varphi_1^n = \int_0^{y} \left\{ e^{\alpha y} K_1(y') + e^{-\alpha y} K_2(y') + e^{-my} K_4(y') \right\} L \varphi_1^{n-1}(y') dy'$$

$$- \int_y^{2\pi} e^{my} K_3(y') \, L \, \varphi_1^{n-1}(y') \, dy' \qquad n \geq 1$$

The reason for the particular choice made for C is now clear. If C had simply been set equal to zero, the remainder after a finite number of terms in the series would be exponentially large. The uniform convergence of (2.1.4) for $0 \leq y \leq 2\pi$ is proved in Appendix II. It is also shown there that convergence is rapid if $|m|$ is large, in fact that $\varphi_1^n = \mathcal{O}\left(\frac{1}{|m|^{2n}}\right)$. Since the series (2.1.4) converges uniformly it can be integrated term by term. Therefore, it can be verified that (2.1.3) is satisfied by direct substitution.

Three other solutions can be constructed as follows. To construct φ_2 set

$B = 1$, $A = D = 0$

$$C = C_2 = - \int_0^{2\pi} K_3(y') \, L \, \varphi_2(y') \, dy'$$

To construct φ_3 set

$$C = 1, \quad A = B = D = 0$$

To construct φ_4 set

$$A = A_4 = - \int_0^{2\pi} K_1(y') \, L \, \varphi_4(y') \, dy'$$

$$B = B_4 = - \int_0^{2\pi} K_2(y') \, L \, \varphi_4(y') \, dy'$$

$$C = C_4 = - \int_0^{2\pi} K_3(y') \, L \, \varphi_4(y') \, dy'$$

$$D = 1$$

These solutions can also be obtained as series by iteration. To summarize, we have

$$\varphi_\gamma = \sum_{n=0}^{\infty} \varphi_\gamma^{\,n} \qquad \gamma = 1,2,3,4 \tag{2.1.5}$$

where $\varphi_1^{\,0} = e^{\alpha y} \quad \varphi_2^{\,0} = e^{-\alpha y} \quad \varphi_3^{\,0} = e^{my} \quad \varphi_4^{\,0} = e^{-my}$

$$\varphi_j^{\,n} = \int_0^{y} \left\{ e^{\alpha y} K_1(y') + e^{-\alpha y} K_2(y') + e^{-my} K_4(y') \right\} L \, \varphi_j^{\,n-1}(y') dy'$$

$$- \int_y^{2\pi} e^{my} K_3(y') \, L \, \varphi_j^{\,n-1}(y') \, dy' \qquad n \geq 1 \qquad j = 1,2$$

$$\varphi_3^{\,n} = \int_0^{y} \left\{ e^{\alpha y} K_1(y') + e^{-\alpha y} K_2(y') \right.$$

$$\left. + e^{my} K_3(y') + e^{-my} K_4(y') \right\} L \, \varphi_3^{\,n-1}(y') \, dy' \qquad n \geq 1$$

$$\varphi_4^{\,n} = -\int_y^{2\pi} \left\{ e^{\alpha y} K_1(y') + e^{-\alpha y} K_2(y') + e^{my} K_3(y') \right\} L\,\varphi_4^{\,n-1}(y')\,dy'$$
$$+ \int_0^{y} e^{-my} K_4(y')\, L\,\varphi_4^{\,n-1}(y')\,dy' \qquad n \geq 1$$

The uniform convergence of the series (2.1.5) on the interval $0 \leq y \leq 2\pi$ is proved in Appendix II.

The boundary conditions will be imposed in the form (1.3.1) by the equation

$$\begin{vmatrix} a_{11}-s & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22}-s & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33}-s & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}-s \end{vmatrix} = 0 \qquad (2.1.6)$$

together with the condition that γ must be real where $s \equiv e^{2\pi i \gamma}$.

The a_{ij} are given by

$$\varphi_i^{(K-1)}(2\pi) = \sum_{j=1}^{4} a_{ij}\,\varphi_j^{(K-1)}(0) \qquad i, K = 1,2,3,4 \qquad (2.1.7)$$

For example, the expression for a_{11} is

$$a_{11} = \frac{1}{W} \begin{vmatrix} U+V+Z & 1+C_2 & 1 & 1+A_4+B_4+C_4 \\ \alpha U-\alpha V-mZ & -\alpha+mC_2 & m & -m+\alpha A_4-\alpha B_4+mC_4 \\ \alpha^2 U+\alpha^2 V+m^2 Z & \alpha^2+m^2 C_2 & m^2 & m^2+\alpha^2 A_4+\alpha^2 B_4+m^2 C_4 \\ \alpha^3 U-\alpha^3 V-m^3 Z & -\alpha^3+m^3 C_2 & m^3 & -m^3+\alpha^3 A_4-\alpha^3 B_4+m^3 C_4 \end{vmatrix}$$

where W is the Wronskian of φ_1, φ_2, φ_3, φ_4 and

$$U = e^{2\gamma\alpha}\left\{ 1 + \int_0^{2\pi} K_1(y')\, L\,\varphi_1(y')\,dy' \right\}$$

$$V = e^{-2\pi\alpha} \int_0^{2\pi} K_2(y')\, L\,\varphi_1(y')\,dy'$$

$$Z = e^{-2\pi m} \int_0^{2\pi} K_4(y')\, L\,\varphi_1(y')\,dy'$$

After the obvious simplifications of the determinant we find

$$a_{11} = e^{2\pi\alpha}\left\{ 1 + \int_0^{2\pi} K_1(y')\, L\, \varphi_1(y')\, dy' \right\} - A_4\, e^{-2\pi m} \int_0^{2\pi} K_4(y')\, L\, \varphi_1(y')\, dy'$$

Similarly, we find

$$a_{22} = e^{-2\pi\alpha}\left\{ 1 + \int_0^{2\pi} K_2(y')\, L\, \varphi_2(y')\, dy' \right\} - B_4\, e^{-2\pi m} \int_0^{2\pi} K_4(y')\, L\, \varphi_2(y')\, dy'$$

$$a_{33} = e^{2\pi m}\left\{ 1 + \int_0^{2\pi} K_3(y')\, L\, \varphi_3(y')\, dy' \right\} - C_1\, e^{2\pi\alpha} \int_0^{2\pi} K_1(y')\, L\, \varphi_3(y')\, dy' - C_2\, e^{-2\pi\alpha} \int_0^{2\pi} K_2(y')\, L\, \varphi_3(y')\, dy' - (C_4 - C_1 A_4 - C_2 B_4)\, e^{-2\pi m} \int_0^{2\pi} K_4(y')\, L\, \varphi_3(y')\, dy'$$

$$a_{44} = e^{-2\pi m}\left\{ 1 + \int_0^{2\pi} K_4(y')\, L\, \varphi_4(y')\, dy' \right\}$$

Expressions for the off diagonal elements of (2.1.6) are, of course, found in the same way. Using the solutions (2.1.5) it is found that the offdiagonal elements are all of order $|m|^{-2}$. Now terms of order $|m|^{-4}$ will be neglected in the expansion of (2.1.6). The off diagonal elements may, therefore, be neglected since any term in the expansion which contains an off diagonal element must contain at least two. Then (2.1.6) reduces to

$$(a_{11} - s)(a_{22} - s)(a_{33} - s)(a_{44} - s) = 0 \qquad (2.1.8)$$

Using the solutions (2.1.5) we find

$$a_{11} = e^{2\pi\alpha} + \mathcal{O}\left(\frac{1}{|m|^4}\right)$$

$$a_{22} = e^{-2\pi\alpha} + \mathcal{O}\left(\frac{1}{|m|^4}\right)$$

$$a_{33} = e^{2\pi m}\left\{1 + 2\pi K + \mathcal{O}\left(\frac{1}{|m|^4}\right)\right\}$$

$$a_{44} = e^{-2\pi m}\left\{1 - 2\pi K + \mathcal{O}\left(\frac{1}{|m|^4}\right)\right\}$$

where $K = -\dfrac{\sqrt{-i\alpha R c}}{16 c^2}$

From the first two factors of (2.1.8) we have

$$i\tau \approx \pm\alpha$$

Since τ is not real, these two roots correspond to unbounded solutions which, according to the discussion of section 1.3, are of the form $e^{\pm\alpha y}F(y)$ where $F(y + 2\pi) = F(y)$.

From the second two factors of (2.1.8) we have

$$1 - s\left\{2\cosh 2\pi m + 4\pi K \sinh 2\pi m\right\} + s^2 + \mathcal{O}\left(\frac{1}{|m|^4}\right) = 0$$

In order to satisfy the boundary conditions we see that c must be pure imaginary and negative. We have, then,

$$\sin^2\pi\tau = \sin^2\pi\sqrt{\alpha R|c_i| - \alpha^2} + \frac{\pi}{16}\frac{\sqrt{\alpha R}}{|c_i|^{3/2}}\sin 2\pi\sqrt{\alpha R|c_i| - \alpha^2}$$

Now for values of c near $c = -i n^2/\alpha R$, n an integer, $n^2 >> \alpha^2$, the right side of this equation is negative. Thus, since τ must be real, there are gaps in the spectrum near these values, i.e., the spectrum consists of an infinity of bands separated by small intervals, the length of the intervals being approximately

$$\Delta c_i = \frac{1}{4}\frac{\alpha R}{n^2}$$

As $c \longrightarrow -i\infty$ the gaps shrink to zero and the spectrum goes over into the "zero$^{\text{th}}$ order" approximation.

2.2 Nature of the Solutions for Small α

We turn to the case in which α is small. We begin by assuming $|c|$ and R of order unity and obtain solutions as power series in α * such that

$$\varphi_j^{(i-1)}(0) = \delta_{ij} \qquad i, j = 1,2,3,4 \qquad (2.2.1)$$

When $\alpha = 0$ the Orr-Sommerfeld equation becomes simply $D^4 \varphi = 0$

Therefore, let

$$\varphi_j(y) = \frac{1}{(j-1)!} y^{(j-1)} + \mathcal{O}(\alpha) \qquad j = 1, 2, 3, 4$$

A sample calculation will illustrate the method. Since $\varphi_1(y) = 1 + \mathcal{O}(\alpha)$ we have from (1.2.4):

$$\varphi_1^{(4)}(y) = i\alpha R \cos y + \mathcal{O}(\alpha^2)$$

$$\varphi_1'''(y) = i\alpha R \sin y + \mathcal{O}(\alpha^2)$$

$$\varphi_1''(y) = -i\alpha R (\cos y - 1) + \mathcal{O}(\alpha^2)$$

$$\varphi_1'(y) = -i\alpha R (\sin y - y) + \mathcal{O}(\alpha^2)$$

$$\varphi_1(y) = 1 + i\alpha R (\cos y - 1 + \tfrac{1}{2} y^2) + \mathcal{O}(\alpha^2)$$

The process is repeated as many times as necessary. The boundary conditions will again be imposed in the form (1.3.1). From (2.2.1) it is seen that the a_{ij}, defined by (2.1.7), are given by

$$a_{ij} = \varphi_j^{(i-1)}(2\pi) \equiv \varphi_j^{(i-1)}$$

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*Since the coefficients of (1.2.4) are analytic in α and y solutions are entire functions of these quantities and the power series converges uniformly in the finite α - and y-planes.

The eigenvalue equation, therefore, may be written

$$\begin{vmatrix} \varphi_1 - s & \varphi_2 & \varphi_3 & \varphi_4 \\ \varphi_1' & \varphi_2' - s & \varphi_3' & \varphi_4' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' - s & \varphi_4'' \\ \varphi_1''' & \varphi_2''' & \varphi_3''' & \varphi_4''' - s \end{vmatrix} = 0 \qquad (2.2.2)$$

or, expanding (2.2.2),

$$x^4 + A\,x^3 + B\,x^2 + C\,x + D = 0 \qquad (2.2.3)$$

where $x \equiv 1-s$ and A,B,C,D are obtained as power series in α.

$$A = \sum_{n=0}^{\infty} A_n \alpha^n \qquad C = \sum_{n=0}^{\infty} C_n \alpha^n$$

$$B = \sum_{n=0}^{\infty} B_n \alpha^n \qquad D = \sum_{n=0}^{\infty} D_n \alpha^n$$

The calculation of these series is quite complicated but straightforward and will not be reproduced here. The solutions φ_j carried out to the required order in α are given in Appendix I.

Equation (2.2.3) is a quartic in s whose constant term is $1 + A + B + C + D$. From (2.2.2) this term is the Wronskian of $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$, $\varphi_4(y)$ evaluated at $2\tilde{\pi}$. Abel's identity for the Wronskian of an n^{th} order linear differential equation is [9] p. 119.

$$W(y) = W_o \exp - \int_{y_o}^{y} \frac{p_1(y)}{p_o(y)}\, dy$$

where p_o is the coefficient of the n^{th} derivative in the differential equation, p_1 is the coefficient of the $(n-1)^{th}$ derivative, and W_o is the value of W at y_o. Since in the present case $p_1 \equiv 0$ the Wronskian is a constant. From (2.2.1) it is equal to unity. Therefore,

$$A_i + B_i + C_i + D_i = 0 \qquad i = 1,2,\ldots$$

This provides a check on the calculations.

We find for the coefficients:

$$A_1 = -B_1 = -i\alpha R c (2\pi)^2 \qquad (2.2.4)$$

$$C_1 = D_1 = 0$$

$$A_2 = -B_2 = \frac{7}{2}(\alpha R)^2 (2\pi)^2 + 2\alpha^2 (2\pi)^2 - \frac{1}{12}(\alpha R)^2 c^2 (2\pi)^4$$

$$C_2 = D_2 = 0$$

$$D_3 = -i\alpha^3 R c (2\pi)^4$$

$$D_4 = \left\{ \alpha^4 - \frac{1}{2}\alpha^2 (\alpha R)^2 \right\} (2\pi)^4$$

The roots of the quartic (2.2.3) are near $x = \pm\alpha$ and $x = \pm\sqrt{\alpha}$. Near $x = \pm\alpha$ we have

$$x^2 = (2\pi\alpha)^2 + \mathcal{O}(\alpha^3)$$

$$i\gamma \approx \pm\alpha$$

Since γ is not real these roots correspond to unbounded solutions, solutions of the form $e^{\pm\alpha y} F(y)$ where $F(y + 2\pi) = F(y)$. Near $x = \pm\sqrt{\alpha}$, $i\gamma \approx \sqrt{-i\alpha R c}$. γ is real, and hence these roots correspond to bounded solutions, when c is pure imaginary and negative. Thus, we find, as in the last section, two unbounded solutions proportional to $e^{\pm\alpha y}$ and two bounded solutions proportional to $\exp \pm\sqrt{-i\alpha R c}\, y$.

2.3 The Unstable Modes

We have seen that for $|c|$ large and of order unity there are eigenvalues lying along the negative imaginary axis in the c-plane. The question of primary interest is whether the spectrum can penetrate into the half plane $c_i > 0$, i.e., whether unstable modes can exist.

Finally, therefore, the case in which both α and $|c|$ are small will be studied. It will be assumed that $|c| = \mathcal{O}(\alpha)$. Now $C_3 = \mathcal{O}(\alpha^4)$. The roots of the quartic (2.2.3) are all near $x = \pm \alpha$.

Using (2.2.4),

$$x^4 + B_1 x^2 + D_3 + D_4 + \mathcal{O}(\alpha^5) = 0$$

$$x^2 = \frac{(2\pi)^2}{2} \left\{ - i\alpha R c + \frac{7}{2} (\alpha R)^2 + 2\alpha^2 \pm \sqrt{\left[i\alpha R c - \frac{7}{2} (\alpha R)^2 \right]^2 + 16 \alpha^2 (\alpha R)^2} \right\} \qquad (2.2.5)$$

If x is of order α , τ must be of order α since $x = 1-s = -2\pi i \tau + \ldots$ Thus, $x^2 = - (2\pi)^2 \tau^2 + \mathcal{O}(\alpha^3)$. There are two bounded solutions, i.e., two real values of τ, if the right side of (2.2.5) is real and negative. This condition will be satisfied if c is pure imaginary and if the negative sign is taken in (2.2.5). Two unbounded solutions then correspond to the positive sign in (2.2.5). Solving for c,

$$i\alpha R c = \frac{4\tau^2 (\alpha R)^2}{\tau^2 + \alpha^2} + \tau^2 + \alpha^2 - \frac{1}{2} (\alpha R)^2$$

The least stable eigenvalue is given by $\tau = 0$. A small portion, of the spectrum may lie along the positive imaginary axis of the c-plane if $R > \sqrt{2}$. The critical Reynolds' number is, therefore, $R_c = \sqrt{2}$ and the neutral curve in the α-R plane intersects the R - axis at this point. The calculation could, in principle, be carried further to obtain a portion of the neutral curve. This has not been done because the calculation at this point has become algebraically very complicated. A better method for obtaining the neutral curve using the method of infinite determinants is described in the next chapter.

CHAPTER III

THE METHOD OF INFINITE DETERMINANTS

The eigenvalue problem will next be formulated as the vanishing of an infinite determinant. The expansion of the determinant will be found to converge quickly when α is small and the critical Reynolds' number and a portion of the neutral curve will be obtained with considerably less difficulty than by the method of Chapter 2. The cases $|c|$ large and of order unity will also be treated and the previous results corroborated (and in the latter case slightly extended).

3.1 Formulation of the Eigenvalue Equation

We require the solution of the Orr-Sommerfeld equation (1.2.4), φ, to be of the form $e^{i\gamma y} S(y)$ where γ is real and $S(y)$ has period 2π. Expanding $S(y)$ in a Fourier series, we substitute

$$\varphi = \sum_{-\infty}^{\infty} a_n e^{i(n+\gamma)y}$$

into (1.2.4) and obtain a three term recursion relation for the Fourier coefficients.

$$A_n' a_{n-1} + C_n' a_n + B_n' a_{n+1} = 0 \qquad (3.1.1)$$

where

$$A_n' = \tfrac{1}{2} i\alpha R \left[(n + \gamma - 1)^2 + \alpha^2 - 1 \right]$$

$$B_n' = \tfrac{1}{2} i\alpha R \left[(n + \gamma + 1)^2 + \alpha^2 - 1 \right]$$

$$C_n' = \left[(n+\gamma)^2 + \alpha^2 \right] \left[(n+\gamma)^2 + \alpha^2 - i\alpha R c \right]$$

The relations (3.1.1) are a set of homogeneous equations for the a_n $(-\infty \leq n \leq \infty)$. The condition for non-trivial solutions is, of course, that the determinant of the coefficients vanish. In this case the determinant is infinite so the recursion relation (3.1.1) is first

divided by C_n' to secure the proper convergence properties. Then the determinant $\Delta(\tau)$ of its coefficients is formed.

$$\Delta(\tau) = \lim_{m\to\infty} \begin{vmatrix} 1 & B_{-m} & 0 & \cdots & & & & 0 \\ \cdots & & & & & & & \\ 0 \cdots & A_{-1} & 1 & B_{-1} & 0 & 0 & \cdots & 0 \\ 0 \cdots & 0 & A_0 & 1 & B_0 & 0 & \cdots & 0 \\ 0 \cdots & 0 & 0 & A_1 & 1 & B_1 & \cdots & 0 \\ \cdots & & & & & & & \\ 0 \cdots & & & & & 0 & A_m & 1 \end{vmatrix}$$

where

$$A_n = A_n'/C_n', \quad B_n = B_n'/C_n'$$

An infinite determinant converges if the product of its diagonal elements converges absolutely and the sum of its non-diagonal elements converges absolutely [23] p. 36. These conditions are satisfied by $\Delta(\tau)$. Then with the condition that τ must be real, we may take as the eigenvalue equation the vanishing of $\Delta(\tau)$.

We proceed now to evaluate the determinant. To begin it is convenient to list the following properties of $\Delta(\tau)$:

1. $\Delta(i\infty) = 1$

Off diagonal elements are all zero.

2. $\Delta(\tau + 1) = \Delta(\tau)$

τ occurs only as $n + \tau$. Thus, replacing τ by $\tau + 1$ is equivalent to replacing n by n + 1, which does not change the determinant. Hence, $\Delta(\tau)$ has period one.

3. $\Delta(\tau) = \Delta(-\tau)$

Since $A_n(-\tau) = B_{-n}(\tau)$ and $B_n(-\tau) = A_{-n}(\tau)$ replacing τ by $-\tau$

is equivalent to reversing first the rows and then the columns of the determinant, which does not alter its value. Hence, $\Delta(\tau)$ is an even function.

4. The only singularities of $\Delta(\tau)$ are simple poles at

$$\tau = -n \pm i\alpha, \quad \tau = -n \pm \sqrt{i\alpha R\, c - \alpha^2}$$

where n takes all integral values from $-\infty$ to $+\infty$.

Let K_1 be the residue at $\tau = \sqrt{i\alpha R\, c - \alpha^2}$.

Let K_2 be the residue at $\tau = i\alpha$.

Then by the above properties of $\Delta(\tau)$:

The residues at $-n + \sqrt{i\alpha R\, c - \alpha^2}$ are K_1

The residues at $-n - \sqrt{i\alpha R\, c - \alpha^2}$ are $-K_1$

The residues at $-n + i\alpha$ are K_2

The residues at $-n - i\alpha$ are $-K_2$

Next we form a new function $D(\tau)$ by subtracting from $\Delta(\tau)$ its poles.

$$D(\tau) \equiv \Delta(\tau) - K_1 \sum_{-\infty}^{\infty} \left\{ \frac{1}{\tau + n - p} - \frac{1}{\tau + n + p} \right\} - K_2 \sum_{-\infty}^{\infty} \left\{ \frac{1}{\tau + n - i\alpha} - \frac{1}{\tau + n + i\alpha} \right\}$$

where $p \equiv \sqrt{i\alpha R\, c - \alpha^2}$. The series can be summed giving

$$D(\tau) = \Delta(\tau) - \pi K_1 \left\{ \cot \pi(\tau - p) - \cot \pi(\tau + p) \right\} - \pi K_2 \left\{ \cot \pi(\tau - i\alpha) - \cot \pi(\tau + i\alpha) \right\}$$

Since $D(\tau)$ has no singularities it is a constant. By letting $\tau \longrightarrow i\infty$ we see that $D(\tau) = 1$. Hence,

$$\Delta(\tau) = 1 + \pi K_1 \left\{ \cot \pi (\tau - p) - \cot \pi (\tau + p) \right\}$$
$$+ \pi K_2 \left\{ \cot \pi (\tau - i\alpha) - \cot \pi (\tau + i\alpha) \right\} \quad (3.1.2)$$
$$= 1 + \pi K_1 \frac{2 \sin \pi p \cos \pi p}{\sin^2 \pi \tau - \sin^2 \pi p} + \pi K_2 \frac{2 \sin \pi i\alpha \cos \pi i\alpha}{\sin^2 \pi \tau - \sin^2 \pi i\alpha}$$

The residues K_1 and K_2 will be calculated directly, e.g.,

$$K_1 = \lim_{\tau \to p} (\tau - p) \Delta(\tau).$$

The central row of the determinant is multiplied by $\tau - p$. Then τ is set equal to p making the center element zero, which means that in the three cases we are about to consider the determinant is given to first order in the appropriate small quantity by its central nine elements.

3.2 Eigenvalue Spectrum When |c| is Large

We are now prepared to examine the eigenvalue spectrum in various limits. We shall begin by taking |c| large. For convenience, α and R may be considered to be of order unity, although this is not a necessary assumption provided |c| is sufficiently large. The residues K_1 and K_2 are obtained by multiplying the central row of $\Delta(\tau)$ by $(\tau - p)$ or $(\tau - i\alpha)$ and taking the appropriate limit. The diagonal elements of the determinants so formed are unity except for the center element, which is zero, and their off diagonal elements are all small. Terms of the first order in the expansion of such a determinant contain factors of two off diagonal elements, terms of the second order contain factors of four off diagonal elements, etc. It will be noticed that the only terms of first order are those formed by the minor whose elements are the central nine elements of the determinant.

$$K_1 = \lim_{\gamma \to p} \begin{vmatrix} 1 & B_{-1} & 0 \\ (\gamma - p) A_0 & 0 & (\gamma - p) B_0 \\ 0 & B_1 & 1 \end{vmatrix} + \mathcal{O}\left(\frac{1}{|c|^{7/2}}\right)$$

$$= \frac{1}{16} \frac{1}{c} \frac{i\alpha R}{\sqrt{i\alpha R c}} + \mathcal{O}\left(\frac{1}{|c|^2}\right)$$

$$K_2 = \lim_{\gamma \to i\alpha} \begin{vmatrix} 1 & B_{-1} & 0 \\ (\gamma - i\alpha) A_0 & 0 & (\gamma - i\alpha) B_0 \\ 0 & A_1 & 1 \end{vmatrix} + \mathcal{O}\left(\frac{1}{|c|^4}\right)$$

$$= \mathcal{O}\left(\frac{1}{|c|^2}\right)$$

Neglecting the term involving K_2,

$$\Delta(\gamma) = 1 + \pi K_1 \frac{2 \sin \pi \sqrt{i\alpha R c - \alpha^2} \cos \pi \sqrt{i\alpha R c - \alpha^2}}{\sin^2 \pi \gamma - \sin^2 \pi \sqrt{i\alpha R c - \alpha^2}} + \mathcal{O}\left(\frac{1}{|c|^2}\right)$$

from which we obtain

$$\sin^2 \pi \gamma = \sin^2 \pi \sqrt{\alpha R |c_i| - \alpha^2} + \frac{\pi}{16} \frac{\sqrt{\alpha R}}{|c_i|^{3/2}} \sin 2\pi \sqrt{\alpha R |c_i| - \alpha^2}$$

This agrees with the result of Section 2.1. A discussion of it will be found there.

3.3 Eigenvalue Spectrum When |c| Is of Order Unity, α Small

If α is small and |c| and R are of order unity the residues may be found using the method of the previous section. They are

$$K_1 = -\frac{7}{4}\frac{i\alpha R\sqrt{i\alpha R\, c}}{c} + \mathcal{O}(\alpha^2)$$

$$K_2 = -\frac{2\alpha^2 R}{c} + \mathcal{O}(\alpha^3)$$

Substituting these expressions into (3.1.2) , and neglecting the term involving K_2,

$$i\alpha R\, c = \gamma^2 + \alpha^2 + \frac{7}{2}(\alpha R)^2$$

The result agrees with the conclusion of Section 2.2 that c must be pure imaginary and negative. The calculation has been carried to a higher order here, but perhaps the only further point of interest is that there are no gaps in this part of the spectrum.

3.4 The Unstable Modes

If α is small, $|c| = \mathcal{O}(\alpha)$, and R is of order unity the residues are found, by the technique of Section 3.2, to be

$$K_1 = \frac{-1}{4}\frac{(\alpha R)^2}{\sqrt{i\alpha Rc-\alpha^2}} - \frac{2}{c} i\alpha R\sqrt{i\alpha Rc-\alpha^2} + \mathcal{O}(\alpha^2)$$

$$K_2 = \frac{-2}{c}\alpha^2 R + \mathcal{O}(\alpha^2)$$

Substituting these expressions into (3.1.2) and solving for c we have, finally,

$$i\alpha R\, c = \frac{4\gamma^2(\alpha R)^2}{\gamma^2+\alpha^2} + \gamma^2 + \alpha^2 - \tfrac{1}{2}(\alpha R)^2$$

This is identical with the result of Section 2.3 and no further discussion will be given here.

3.5 The Neutral Curve

It can be seen from the result of the last section that the critical Reynolds' number is $\sqrt{2}$ and the end point of the spectrum corresponds to $\gamma = 0$. Accordingly, we shall take $\Delta(o) = 0$ as the equation for the least stable eigenvalue and calculate that portion of the neutral curve in the α-R plane on which α is small.

Before forming the determinant $\Delta(\gamma)$ we divided the recursion relation (3.1.1) by C_n'. In particular we divided by $C_o' = (\gamma^2 + \alpha^2)(\gamma^2 + \alpha^2 - i\alpha R\, c)$, thus, excluding $\gamma = \alpha = 0$. We now multiply $\Delta(o) = 0$ by C_o' (i.e., multiply the central row by C_o') and keeping terms to order α^6 obtain

$$\begin{vmatrix} 1 & B_{-1} & 0 \\ A_o' & C_o' & B_o' \\ 0 & A_1 & 1 \end{vmatrix} + C_o' \left\{ \sum_{n=-1}^{-\infty} A_n B_{n-1} + \sum_{n=2}^{\infty} A_n B_{n-1} \right\}$$

$$- \left\{ B_o' A_1 + B_{-1} A_o' \right\} \left\{ \sum_{n=-2}^{-\infty} A_n B_{n-1} + \sum_{n=3}^{\infty} A_n B_{n-1} \right\}$$

$$+ \mathcal{O}(\alpha^7) = 0$$

We may drop the terms n = - 1 in the first sum and n = 2 in the second because B_{-2} and A_2 are $\mathcal{O}(\alpha^3)$. Now,

$$\sum_{n=-2}^{-\infty} A_n B_{n-1} + \sum_{n=3}^{\infty} A_n B_{n-1} = 2 \sum_{n=2}^{\infty} A_{n+1} B_n$$

$$= -\tfrac{1}{2} (\alpha R)^2 \sum_{n=2}^{\infty} \frac{(n-1)(n+2)}{n^3 (n+1)^3} = -0.0145 \; (\alpha R)^2$$

Thus,

$$\begin{vmatrix} 1 & B_{-1} & 0 \\ A_o' & C_o' & B_o' \\ 0 & A_1 & 1 \end{vmatrix} \left\{ 1 - 0.0145 \; (\alpha R)^2 \right\} + \mathcal{O}(\alpha^7) = 0$$

The second term in the brackets is at least three orders of magnitude smaller than unity. Since the 3 x 3 determinant is of order α^4,

$$C_o' - B_o' A_1 - B_{-1} A_o' + \mathcal{O}(\alpha^7) = 0$$

Finally setting $c_i = 0$ we obtain the neutral curve:

$$R^2 = \frac{2(1+\alpha^2)^2}{1-\alpha^2} + \mathcal{O}(\alpha^3)$$

or

$$R^2 = 2(1+3\alpha^2) + \mathcal{O}(\alpha^3)$$

The curve intersects the R axis of the α - R plane at $R_c = \sqrt{2}$ with infinite slope. In contrast with many flows, e.g., the two dimensional jet [20], plane Poiseuille flow [12], the boundary layer along a flat plate [12], the curve has no lower branch. Further discussion of the neutral curve will be postponed until the end of the next chapter when the portion of the curve on which R is large has been obtained.

CHAPTER IV

THE LIMIT OF LARGE REYNOLDS' NUMBER

We shall complete our treatment of the parallel flow by considering the part of the α-R plane for which αR is very large. We shall investigate the eigenvalue spectrum in the neighborhood of the origin of the c-plane and obtain a portion of the neutral curve for these modes. The case in which $|c|$ is of order unity and α^2 is small will also be considered briefly.

4.1 Asymptotic Solutions of the Orr-Sommerfeld Equation

Solutions of the Orr-Sommerfeld equation (1.2.4) can be obtained as asymptotic expansions by formally substituting the series

$$\phi = \sum_{n=0}^{\infty} \left(\frac{1}{\alpha R}\right)^n \phi_n \tag{4.1.1}$$

into the equation. Comparing powers of $(\alpha R)^{-1}$ we have

$$(u - c)(D^2 - \alpha^2)\phi_0 - u''\phi_0 = 0 \tag{4.1.2}$$

$$(u - c)(D^2 - \alpha^2)\phi_n - u''\phi_n = -i(D^2 - \alpha^2)^2\phi_{n-1} \qquad n \geq 1 \tag{4.1.3}$$

The first term of the expansion satisfies the inviscid equation ($\alpha R = \infty$). Since this is a second order equation the method gives only two solutions, usually called the "inviscid solutions." Two other solutions of (1.2.4) may be obtained by formally substituting into the equation the expansion

$$\phi = \exp(\alpha R)^{\frac{1}{2}} Q(y) \sum_{n=0}^{\infty} \frac{\sigma_n(y)}{(\alpha R)^{n/2}} \tag{4.1.4}$$

from which we find

$$Q(y) = \pm \int_{y_o}^{y} \sqrt{i(u-c)}\, dy \qquad \sigma_o = (u-c)^{-5/4}$$

where y_o is an arbitrary point in the complex y-plane. These are usually called the "viscous solutions".

At this point the four solutions (4.1.1,4) are purely formal. The expressions (4.1.5) have branch points where u(y) - c = 0. (These points are usually called "critical points".) Solutions of (4.1.2,3) will also, in general, have branch points at the critical points since the equations have singularities there, The Orr-Sommerfeld equation, however, has no singularities in the finite plane. Its solutions must, therefore, be single valued. It is clear that around the critical points the multiple valued expressions (4.1.1,4) cannot be uniform limits of the solutions of the full Orr-Sommerfeld equation. At these points there is a "Stokes phenomenon": the asymptotic expression for a given solution (of the Orr-Sommerfeld equation) has different forms in different sectors of the complex plane.

Regions of validity of the asymptotic solutions are determined by the lines

$$Re \int_{y_c}^{y} \sqrt{i(u-c)}\, dy = 0$$

where y_c is a critical point. These lines (hereafter, called Stokes lines) branch off from the critical points with three branches at equal angles. Let us denote the Stokes lines by C_k (k = 1,2,3) and the sector opposite C_k by S_k. Then S_k is bounded by two curves C_j ($j \neq k$) and we shall consider these curves to be part of S_k. We denote by S the annular region

$$y_c < \left| \int_{y_c}^{y} \sqrt{i(u-c)}\, dy \right| \leq K$$

where K is chosen small enough so that S does not contain any zeros of u(y) - c. Wasow [22] has shown that in any region S - S_k there exist solutions of the Orr-Sommerfeld equation represented asymptotically by solutions of the inviscid equation (4.1.2). He has further shown that in any region S - C_k there exist solutions represented asymptotically by the viscous solutions (4.1.4), where the real part of Q(y) has negative real part in S_k.

We define the integral path in (4.1.5) by requiring that everywhere on the path the viscous solutions (4.1.4) represent the same solutions of the Orr-Sommerfeld equation. Foote and Lin [6] have shown that on such a path the real part of Q(y) is monotonically increasing*. The viscous solutions, then, are exponentially unbounded and we shall not use them. Hence, the effect of viscosity will enter only through the higher order terms in $(\alpha R)^{-1}$ of the inviscid solutions.

4.2 The Inviscid Limit

A natural starting point for the study of problems involving large Reynolds' numbers is the inviscid limit. Since in the present case $u(y) = \cos y$, the inviscid equation (4.1.1) may be written

$$\phi_0'' + (1 - \alpha^2)\phi_0 = \frac{-c}{\cos y - c}\phi_0$$

---

*Foote and Lin discussed the case in which there are two critical points, but the extension of their work to the present case is obvious.

Using the method of variation of parameters this can be transformed into an integral equation.

$$\phi_0(y) = f(y) + c \int_{-\pi}^{y} K(y, y')\, \phi_0(y')\, dy' \qquad (4.2.1)$$

where

$$f(y) = A\, e^{i\sqrt{1-\alpha^2}\, y} + B\, \frac{e^{-i\sqrt{1-\alpha^2}\, y}}{-i\sqrt{1-\alpha^2}}$$

$$K(y, y') = \frac{-1}{\sqrt{1-\alpha^2}}\, \frac{\sin \sqrt{1-\alpha^2}\,(y-y')}{\cos y' - c}$$

and A and B are arbitrary constants. The lower limit of integration is arbitrary but will be taken for convenience to be $-\pi$. Since the kernel is singular we must next define the integral path. We do so by applying the criterion of Wasow [22], as extended by Foote and Lin [6] to the case of two critical points, that everywhere on this path a given solution of the Orr-Sommerfeld equation must be represented by the same asymptotic expression. Specifically, Foote and Lin have shown that the neighborhood of a path Γ which crosses only one Stokes line of each critical point is a region of validity of the asymptotic solutions.

We shall be interested in evaluating the asymptotic expressions at $y = \pm\pi$ since the boundary conditions will be applied at these points. Since u(y) = cos y the critical points are given by

$$y_c = \pm \cos^{-1} c = \pm \left\{ \frac{\pi}{2} - c + \ldots \right\}$$

Let $y_1 = \frac{\pi}{2} - c$

$$y_2 = -\frac{\pi}{2} + c$$

Near y_1 the Stokes lines are given by

$$R_e \int_{y_{\gamma 1}}^{y_\gamma} \sqrt{-i\,(y - y_1)}\; dy = 0$$

Let $Z_1 = y - y_1$

then

$$R_e \left[\frac{2}{3} \; e^{-\frac{\pi}{4} i} \; Z_1^{3/2} \right] = 0$$

and

$$\arg Z_1 = \frac{2n\pi}{3} + \frac{\pi}{2} \qquad n = \text{integer}$$

Near y_2 we find

$$\arg Z_2 = \frac{2n\pi}{3} + \frac{\pi}{6} \qquad n = \text{integer}$$

where $Z_2 = y - y_2$. In Fig. 1 the Stokes lines and the path are shown when $c_1 < 0$.

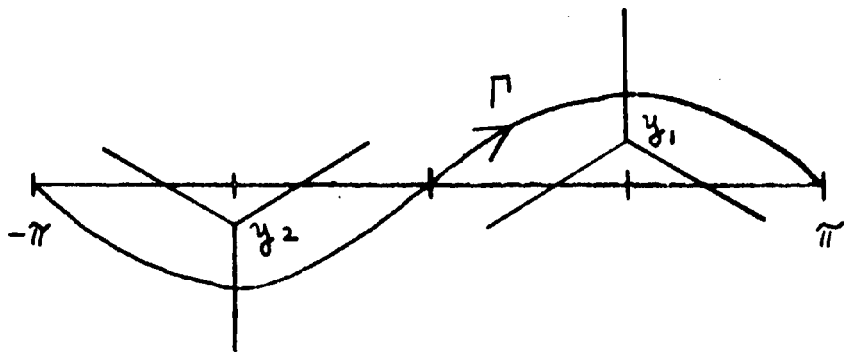


Fig. 1
The Integral Path Γ in the
Complex y-plane

4.3 Eigenvalue Spectrum for Small |c|

Solutions of the integral equation (4.2.1) may be obtained by iteration as series,

$$\varphi_o = \sum_{n=0}^{\infty} \varphi_o^{\,n}$$

where

$$\varphi_o^{\,0} = f(y)$$

and

$$\varphi_o^{\,n} = c^n \int_{-\pi}^{y} dt_1\, K(y,t_1) \int_{-\pi}^{t_1} dt_2\, K(t_1,t_2) \cdots \int_{-\pi}^{t_{n-1}} dt_n\, K(t_{n-1},t_n)\, f(t_n)$$

$$(n \geq 1)$$

The uniform convergence of this series in the neighborhood of Γ is proved for |c| sufficiently small in Appendix III. Two solutions of (4.1.2), φ_{o1} and φ_{o2}, are obtained by setting A = 1, B = 0 and A = 0, B = 1.

The boundary conditions are imposed by the equation

$$\begin{vmatrix} \varphi_{o1}(\pi) - s\,\varphi_{o1}(-\pi) & \varphi_{o2}(\pi) - s\,\varphi_{o2}(-\pi) \\ \varphi_{o1}'(\pi) - s\,\varphi_{o1}'(-\pi) & \varphi_{o2}'(\pi) - s\,\varphi_{o2}'(-\pi) \end{vmatrix} = 0 \qquad (4.3.1)$$

together with the condition that τ must be real where $s \equiv e^{2\pi i \tau}$

Since the Wronskian of φ_{o1}, φ_{o2} is 2, we have

$$2 - s\left\{ \varphi_{o1}(-\pi)\,\varphi_{o2}'(\pi) + \varphi_{o2}'(-\pi)\,\varphi_{o1}(\pi) - \varphi_{o1}'(-\pi)\,\varphi_{o2}(\pi) - \varphi_{o2}(-\pi)\,\varphi_{o1}'(\pi) \right\} + 2s^2 = 0$$

or, keeping terms to order |c| and dividing by s,

$$4\cos 2\pi\sqrt{1-\alpha^2} - \frac{2c}{\sqrt{1-\alpha^2}} \sin 2\pi\sqrt{1-\alpha^2} \int_{\Pi} \frac{dy}{\cos y - c} - 2\left(\frac{1}{s} + s\right) = 0$$

The integral may be evaluated by deforming the path as shown in Fig. 2.

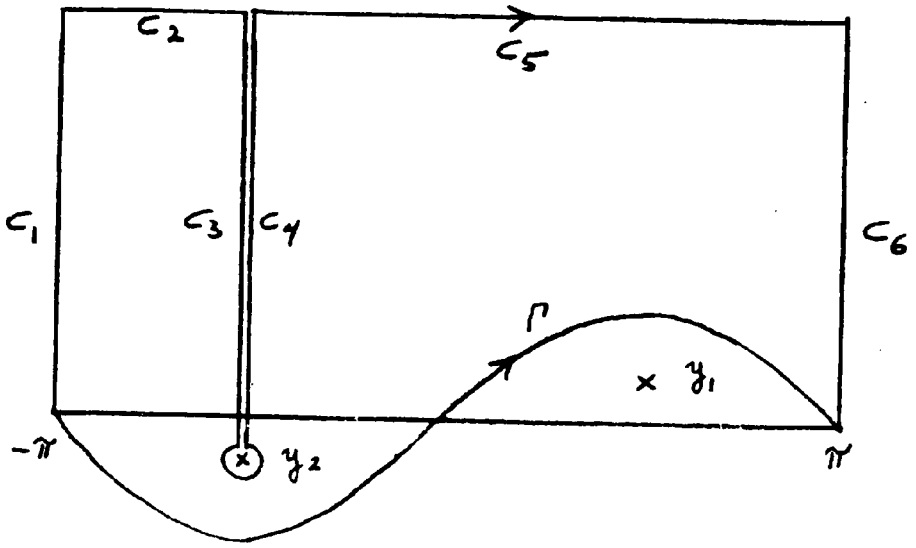


Fig. 2
Deformation of the Integral Path Γ

The contributions to the integral from c_1 and c_6 cancel because the integrand has period 2π. The path $c_2 + c_5$ is from $-\pi + i\infty$ to $+\pi + i\infty$ where the integrand vanishes. The contributions from c_3 and c_4 obviously cancel and so the path reduces to a path about the simple pole at y_2.

$$\oint \frac{dy}{\cos y - c} = 2\pi i \left. \frac{y - y_2}{\cos y - c} \right|_{y=y_2} = \frac{2\pi i}{-\sin y_2} = \frac{2\pi i}{\sqrt{1-c^2}}$$

Solving the eigenvalue equation for c, we have,

$$c = i \frac{\sqrt{1-\alpha^2}}{\pi} \frac{\cos 2\pi\gamma - \cos 2\pi\sqrt{1-\alpha^2}}{\sin 2\pi\sqrt{1-\alpha^2}}$$

Of course, since we have assumed $|c|$ to be small this result is valid only within a small circle about the origin in the c-plane. Within this circle if α is sufficiently small we find

$$c \approx i\,\frac{\cos 2\pi\gamma - 1 + \pi^2\alpha^4/2}{-\pi^2\alpha^2}$$

In order to satisfy the assumption of small $|c|$ we must have $\gamma \ll \alpha$. The spectrum lies almost entirely in the unstable region, penetrating a distance $\alpha^2/2$ into the stable region.

If α is approximately one we find (again γ must be small)

$$c \approx i\,\frac{\cos 2\pi\gamma - 1 + 2\pi^2(1-\alpha^2)}{2\pi^2}$$

The spectrum lies almost entirely in the stable region, penetrating for $\alpha < 1$ a distance $1 - \alpha^2$ into the unstable region. For $\alpha = 1$ the spectrum just touches the real axis of the c-plane (neutral stability) and for $\alpha > 1$ it recedes into the stable region. Hence, the neutral curve in the α-R plane should approach $\alpha = 1$ asymptotically as $R \to \infty$.

We have noted in Chapters 2 and 3 the banded character of the eigenvalue spectrum. It would appear from the above results that as α varies from $\alpha > 1$ to $\alpha = 0$ the last band of the spectrum moves from the stable region into the unstable region until when $\alpha = 0$ it lies entirely in the unstable region with its lowest point just touching the real axis of the c-plane. This suspicion will be substantiated by the next section. The situation described is illustrated in Fig. 3.

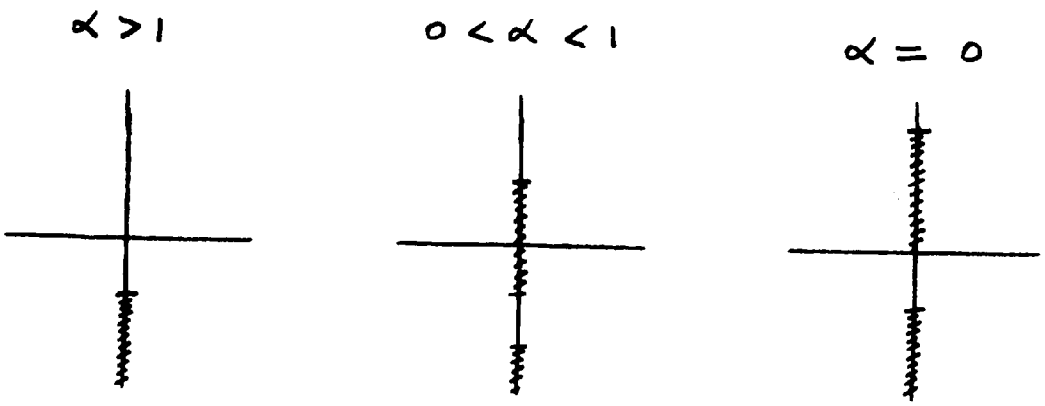


Fig. 3
Illustration of a Part of the Eigenvalue
Spectrum in the c-plane

4.4 Eigenvalue Spectrum for Small α

Heisenberg [7,13] has obtained solutions of the inviscid equation as a convergent series in α^2.

$$\varphi_0 = (u-c) \sum_{n=0}^{\infty} g_n \alpha^{2n}$$

where

$$g_{n+1} = \int^{y} (u-c)^{-2} dy' \int^{y'} (u-c)^2 g_n(y'') dy''$$

and

$$g_0 = 1 \quad \text{or} \quad \int^{y} (u-c)^{-2} dy$$

we shall call φ_{01} the solution corresponding to $g_0 = 1$ and φ_{02} the other solution. The lower limit of integration is arbitrary and will be taken to be $-\pi$. The integral paths are defined as in Section 4.2. We shall consider the case in which $\alpha^2 << 1$ and $|c|$ is or order unity. From the eigenvalue equation (4.3.1) we have

$$1 + s \left\{ (1 + c) \varphi_{02}'(\pi) + \frac{\varphi_{01}(\pi)}{1 + c} \right\} + s^2 = 0$$

Dividing the equation by s,

$$2\cos 2\pi\gamma = 2 + \alpha^2 \left\{ \int_{-\pi}^{\pi} \frac{dy}{(\cos y - c)^2} \int_{-\pi}^{y} dy' \, (\cos y' - c)^2 \right.$$

$$\left. + \int_{-\pi}^{\pi} dy \, (\cos y - c)^2 \int_{-\pi}^{y} \frac{dy'}{(\cos y' + c)^2} \right\} + O(\alpha^4)$$

Integrating by parts,

$$2(\cos 2\pi\gamma - 1) = \alpha^2 \left[\int_{-\pi}^{\pi} dy \, (\cos y - c)^2 \right] \left[\int_{-\pi}^{\pi} \frac{dy}{(\cos y - c)^2} \right] + O(\alpha^4)$$

Using a result of Section 4.4,

$$\int_{-\pi}^{\pi} \frac{dy}{(\cos y - c)^2} = \frac{d}{dc} \int_{-\pi}^{\pi} \frac{dy}{\cos y - c} = \frac{d}{dc} \frac{2\pi i}{\sqrt{1-c^2}} = \frac{2\pi i c}{(1-c^2)^{3/2}}$$

We have, finally,

$$2 (\cos 2\pi\gamma - 1) = 2\pi^2 \alpha^2 \frac{ic(1 + 2c^2)}{(1 - c^2)^{3/2}} + O(\alpha^4).$$

If c is small this agrees with the result of the previous section. Since γ must be real the right hand side of this equation must satisfy

$$-4 \leq \text{RHS} \leq 0$$

This is possible if c is pure imaginary and

$$0 \leq c_i \leq \frac{1}{\sqrt{2}}$$

or

$$c_i \leq -\frac{1}{\sqrt{2}}$$

Thus, there is a band which lies entirely in the unstable region as we conjectured in the last section. The situation is somewhat analogous to the Poiseuille flow between parallel planes in which case

there is a single discrete eigenvalue which can be unstable.

4.5 The Neutral Curve

In order to calculate the part of the neutral curve for which αR is large we must obtain higher order terms of the inviscid solutions. These are given by (4.1.3). Assuming $c = \mathcal{O}\left(\frac{1}{\alpha R}\right)$,

$$\phi_1 = \phi_{01} + \frac{1}{2}\frac{i}{\alpha R}\phi_{01}^{\circ}\int \frac{\phi_{02}^{\circ}\phi_{01}^{\circ}}{\omega_2 y - c}\,dy$$

$$- \frac{1}{2}\frac{i}{\alpha R}\phi_{02}^{\circ}\int \frac{[\phi_{01}^{\circ}]^2}{\omega_2 y - c}\,dy + \mathcal{O}\left[\left(\frac{1}{\alpha R}\right)^2\right] \qquad (4.5.1)$$

$$\phi_2 = \phi_{02} + \frac{1}{2}\frac{i}{\alpha R}\phi_{01}^{\circ}\int \frac{[\phi_{02}^{\circ}]^2}{\omega_2 y - c}\,dy$$

$$- \frac{1}{2}\frac{i}{\alpha R}\phi_{02}^{\circ}\int \frac{\phi_{01}^{\circ}\phi_{02}^{\circ}}{\omega_2 y - c}\,dy + \mathcal{O}\left[\left(\frac{1}{\alpha R}\right)^2\right]$$

The integral paths are defined in Section 4.2. The boundary conditions are imposed by the equation*.

$$\begin{vmatrix} \phi_1(\pi) - s\phi_1(-\pi) & \phi_2(\pi) - s\phi_2(-\pi) \\ \phi_1'(\pi) - s\phi_1'(-\pi) & \phi_2'(\pi) - s\phi_2'(-\pi) \end{vmatrix} = 0 \qquad (4.5.2)$$

where $s \equiv e^{2\pi i\gamma}$ together with the condition that γ must be real. Now,

$$\begin{vmatrix} \phi_1(\pi) & \phi_2(\pi) \\ \phi_1'(\pi) & \phi_2'(\pi) \end{vmatrix} = 2 + \mathcal{O}\left[\left(\frac{1}{\alpha R}\right)^2\right]$$

*Since the boundary conditions are imposed at two finite points, the viscous solutions (4.1.4) could be formally included. The eigenvalue equation would then be a 4 x 4 determinant. The inclusion of these solutions, however, would be found to have no effect. This will be shown in Appendix IV.

and so the eigenvalue equation can be written

$$2 - s\left\{\begin{vmatrix} \phi_1(-\pi) & \phi_2(\pi) \\ \phi_1'(-\pi) & \phi_2'(\pi) \end{vmatrix} + \begin{vmatrix} \phi_1(\pi) & \phi_2(-\pi) \\ \phi_1'(\pi) & \phi_2'(-\pi) \end{vmatrix}\right\} + 2s^2 + \theta\left[\left(\frac{1}{\alpha R}\right)^2\right] = 0$$

Keeping terms to order $(\alpha R)^{-1}$, the result of the calculation is

$$ic = -\frac{\sqrt{1-\alpha^2}}{\pi}\,\frac{\cos 2\pi\gamma - \cos 2\pi\sqrt{1-\alpha^2}}{\sin 2\pi\sqrt{1-\alpha^2}} + \frac{1}{\alpha R} \qquad (4.5.3)$$

Again the range of γ must be restricted in order to satisfy the assumption about c. If αR is set equal to ∞ this gives the result of section 4.3. The least stable eigenvalue corresponds to $\gamma = 0$. Letting $\gamma = 0$, $c_i = 0$ we obtain the neutral curve.

$$\alpha R = \frac{\pi}{\sqrt{1-\alpha^2}}\,\frac{\sin 2\pi\sqrt{1-\alpha^2}}{1 - \cos 2\pi\sqrt{1-\alpha^2}}$$

Of course, the physically meaningful part of this curve is the part in the first quadrant of the α-R plane and there only the part for which αR is large is valid. The curve approaches $\alpha = 1$ asymptotically as $R \longrightarrow \infty$ as we anticipated in Section 4.4. The curve is sketched in Fig. 4 together with the portion obtained in Chapter 3 for small α.

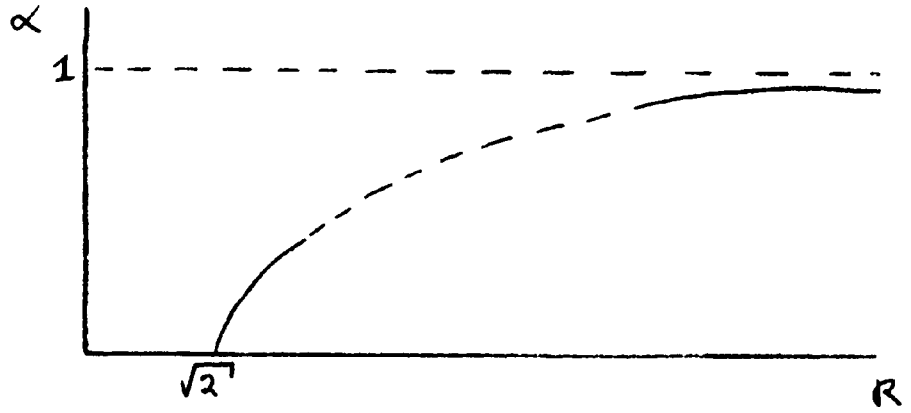


Fig. 4
The Neutral Curve in the α-R plane

The unstable region is the region between the neutral curve and the R-axis. A dotted line indicates a reasonable conjecture concerning the part of the curve which has not been calculated.

The general shape of the neutral curve resembles those for free boundary layers between uniform streams. Various expressions have been used for the velocity profile of this type of flow. A method of normalizing them has been suggested by Tatsumi and Gotoh [19] so that the asymptotic form of the neutral curve in the vicinity of the critical Reynolds' number is universal. Specifically, they find that the critical Reynolds' number is zero and that the neutral curve is $\alpha = R/4\sqrt{3}$ for small R. It is found that at large Reynolds' numbers α approaches a constant value, denoted by α_s, on the neutral curve. The value of α_s depends on the velocity profile used. Lessen [11] obtained a value of 0.90 (with the normalization used in [19]) by solving the boundary layer equation numerically. This can be compared with unity for the present problem.

CHAPTER V

STABILITY OF A CELLULAR FLOW

In this chapter the stability of a square cellular flow with respect to small disturbances is investigated. As in Chapter 3 the eigenvalue problem is formulated as the vanishing of an infinite determinant. The treatment is incomplete, but certain modes are found which can be unstable and with respect to which the critical Reynolds' number is $\sqrt{2}$.

5.1 Cellular Solutions of the Navier-Stokes Equations

Exact, two dimensional, cellular solutions of the Navier-Stokes equations for incompressible fluids were first discovered by Taylor [21]. The essence of his method is to choose the spacial dependence so that the non-linear "inertial" terms of the equations of motion cancel leaving a linear equation to be solved. These solutions are for the force free case and are essentially a system of eddies decaying exponentially due to viscous dissipation.

Meecham [14] has generalized Taylor's treatment to include a cellular force field whose time dependence is quite general and has also obtained cellular solutions of the hydromagnetic equations for a highly conducting, incompressible fluid. Meecham's treatment will be sketched briefly.

The basic equations of incompressible magnetohydrodynamics for a medium of high electrical conductivity are, in cgs units,

$$\frac{\partial u_i}{\partial t} + \partial_\alpha (u_i u_\alpha - h_i h_\alpha) = - \partial_i \omega + \nu \nabla^2 u_i + F_i$$

$$\frac{\partial h_i}{\partial t} + \partial_\alpha (h_i u_\alpha - u_i h_\alpha) = \lambda \nabla^2 h_i$$

$$\partial_\alpha u_\alpha = 0$$

$$\partial_\alpha h_\alpha = 0$$

where (5.1.1)

$$h_i = \left(\frac{\mu}{4\pi\rho}\right)^{\frac{1}{2}} H_i$$

$$\omega = \frac{p}{\rho} + \frac{1}{2} h_\alpha h_\alpha$$

$$\lambda = \frac{1}{4\pi\rho\sigma}$$

$$\partial_\alpha \equiv \partial / \partial x_\alpha$$

The summation convention for repeated indices has been used. In these equations ν is the kinematic viscosity, ρ the density, p the pressure, U_i the velocity, H_i the magnetic field intensity, μ the magnetic permeability, F_i the external force per unit mass, and σ the electrical conductivity of the medium.

The Navier-Stokes equations are obtained by setting $h_i = 0$. After the pressure term has been eliminated the equations of motion in vector notation are

$$\nabla^2 \left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \vec{u} = \nabla \times [\nabla \times (\vec{u} \cdot \nabla) \vec{u}] + \nabla^2 \vec{F} - \nabla (\nabla \cdot \vec{F}) \quad (5.1.2)$$

$$\nabla \cdot \vec{u} = 0 \quad (5.1.3)$$

These equations are solved for the force field

$$F_1 = f_1(t) \cos K y + \dot{a}(t)$$
$$F_2 = f_2(t) \cos K x + \dot{b}(t) \quad (5.1.4)$$
$$F_3 = 0$$

where K is constant with the conditions

$$f_1 = f_2 = a = b = 0 \qquad t \leq 0$$

$$\vec{U} = 0 \qquad t \leq 0$$

The fluid is at rest for negative times. Initial value problems are solved by using an impulsive force (Dirac delta function). The functions f_1, f_2, a, b should be twice differentiable but are otherwise arbitrary. The solution of (5.1.2) for the force field (5.1.4) is of the form

$$U_1 = a(t) + a_c(t) \cos K y + a_s(t) \sin K y$$

$$U_2 = b(t) + b_c(t) \cos K x + b_s(t) \sin K x$$

$$U_3 = 0$$

If this velocity field is substituted into the equations of motion we find the (5.1.3) is automatically satisfied and the non-linear terms in (5.1.2) cancel leaving a linear system which can be solved to give

$$U_1 = a(t) + \int_0^t ds\, f_1(t-s)\, e^{-K^2 \nu s} \cos\left[\int_{t-s}^{t} K\, b(t'')\, dt'' - K y \right]$$

$$U_2 = b(t) + \int_0^t ds\, f_2(t-s)\, e^{-K^2 \nu s} \cos\left[\int_{t-s}^{t} K\, a(t'')\, dt'' - K x \right]$$

Similar solutions of the hydromagnetic equations can be obtained. Assuming a square cellular force field,

$$F_1 = f(t) \cos K y$$

$$F_2 = f(t) \cos K x$$

with $f(t) = 0 \qquad t < 0$

the solution can be written

$$U_1 = a_c(t) \cos K y$$

$$U_2 = a_c(t) \cos K x$$

$$h_1 = h_o \exp(-\lambda K^2 t) \cos K y$$

$$h_2 = h_o \exp(-\lambda K^2 t) \cos K x$$

where h_o is determined by the initial value of the magnetic field and

$$a_c(t) = e^{-\nu K^2 t} \int_o^t e^{\nu K^2 t'} f(t')\, d t'$$

The solution can be verified by direct substitution into equations (5.1.1). We notice that the velocity field is unaffected by the magnetic field which simply decays exponentially from its initial value because of Joulian heating.

Meecham has discussed two special cases of the solutions of the Navier-Stokes equations. A magnetic field may be included in these examples if the mean flow vanishes. In the first case it is assumed that the force is purely impulsive,

$$f_1 = f_2 = U \delta(t - t_o)$$

which corresponds to the initial value problem in which the maximum value of the initial periodic components of the velocity is U. It is also assumed that the mean flow is constant,

$$\dot{a} = \dot{b} = 0$$

The solutions obtained from equations (5.1.2) are $(t > t_o)$

$$U_1 = a + U \cos\left[K b(t - t_o) - K y\right] \exp\left[-K^2 \nu (t - t_o)\right]$$

$$U_2 = b + U \cos\left[K a(t - t_o) - K x\right] \exp\left[-K^2 \nu (t - t_o)\right] \qquad (5.1.5)$$

This is similar to the special case discussed by Taylor [21] . The initial cellular structure is blown downstream by the mean flow with

components (a,b) while decaying exponentially.

In the second case it is assumed that the force is turned on at t = 0 and maintained.

$$f_1 = f_2 = F \qquad t > 0$$
$$= 0 \qquad t \leq 0 \tag{5.1.6}$$

It is also assumed that the mean flow is constant.

$$\dot{a} = \dot{b} = 0$$

The solutions obtained from equations (5.1.2) are, for $t > 0$,

$$U_1 = a + \frac{F}{K^2(b^2+K^2\nu^2)} \left\{ \left[K b \sin K y + K^2 \nu \cos K y \right] \right.$$
$$\left. - \left[K b \sin K(y - bt) + K^2 \nu \cos K (y - bt) \right] e^{-K^2 \nu t} \right\} \tag{5.1.7}$$
$$U_2 = b + \frac{F}{K^2(a^2+K^2\nu^2)} \left\{ \left[K a \sin K x + K^2 \nu \cos K x \right] \right.$$
$$\left. - \left[K a \sin K(x - at) + K^2 \nu \cos K(x - at) \right] e^{-K^2 \nu t} \right\}$$

In this case the field is composed of two parts, a transient which decays as it is blown downstream and a steady state solution which remains fixed in space.

5.2 Formulation of the Problem

We shall assume that the force field (5.1.6) has been maintained long enough for the transients to have died out. It is further assumed that the mean flow vanishes. From (5.1.7) the velocity components are

$$U_1 = \frac{F}{K^2 \nu} \cos K y$$

$$U_2 = \frac{F}{K^2 \nu} \cos K x$$

From(5.1.5) we may note that if at some later time t_o the force field is turned off the cellular flow will simply be exponentially damped.

In order to put our equations into dimensionless form we introduce the characteristic length

$$L_* = 1/K$$

and the characteristic velocity

$$U_* = F/K^2 \nu$$

Then the equations in dimensionless form are obtained by formally setting $\rho = 1$ and $\nu = R^{-1}$ where R is the Reynolds' number defined by

$$R = U_* L_* / \nu$$

In dimensionless form the velocity components are simply

$$U_1 = \cos y, \qquad U_2 = \cos x \tag{5.2.1}$$

In order to study the effect of small disturbances on the basic flow we substitute

$$U = \overline{U} + U' , \qquad p = \overline{p} + p'$$

into the Navier-Stokes equations where $\overline{U}$ is the basic flow $(U_1, U_2, 0)$ and U' is a perturbation (u, v, w) assumed sufficiently small for quadratic terms to be neglected. p' is a perturbation of the basic pressure p. In dimensionless form the linearized Navier-Stokes equations together with the incompressibility condition are

$$\frac{\partial u}{\partial t} + U_1 \frac{\partial u}{\partial x} + U_2 \frac{\partial u}{\partial y} + v \frac{\partial U_1}{\partial y} = - \frac{\partial p'}{\partial x} + \frac{1}{R} \nabla^2 u$$

$$\frac{\partial v}{\partial t} + U_1 \frac{\partial v}{\partial x} + U_2 \frac{\partial v}{\partial y} + u \frac{\partial U_2}{\partial x} = - \frac{\partial p'}{\partial y} + \frac{1}{R} \nabla^2 v \qquad (5.2.2)$$

$$\frac{\partial w}{\partial t} + U_1 \frac{\partial w}{\partial x} + U_2 \frac{\partial w}{\partial y} = - \frac{\partial p'}{\partial z} + \frac{1}{R} \nabla^2 w$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Squire's theorem, which states that the least stable perturbation is in the plane of the basic flow, has not been proved for this problem. Nevertheless, since it does not seem possible to simplify equations (5.2.1) any further, we shall immediately simplify the problem by considering only perturbations in the plane of the basic flow. Since the flow is two dimensional and incompressible the equations of motion can be written in terms of the stream function ψ, defined by

$$U_1 = - \frac{\partial \psi}{\partial y} \qquad U_2 = \frac{\partial \psi}{\partial x}$$

as the vorticity equation (1.2.3), which in dimensionless form is

$$\left\{ \frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} - \frac{1}{R} \nabla^2 \right\} \nabla^2 \psi = \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \qquad (5.2.3)$$

To determine the effect of small disturbances on the basic flow (5.2.3) is linearized by the substitution

$$\psi = \bar{\psi} + \psi'$$

where $\bar{\psi}$ is the stream function of the basic flow and ψ' is the stream function of a small perturbation. Since in the present case $\bar{\psi}$ satisfies

$$\nabla^2 \bar{\psi} + \bar{\psi} = 0$$

the linearized vorticity equation in dimensionless form may be written

$$\left(\frac{\partial}{\partial t} - \frac{1}{R}\nabla^2\right)\nabla^2\psi' + \left(\frac{\partial\bar{\psi}}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\bar{\psi}}{\partial y}\frac{\partial}{\partial x}\right)(\nabla^2\psi' + \psi') = 0 \tag{5.2.4}$$

5.3 The Eigenvalue Equation

We shall take for the stream function of the basic flow

$$\bar{\psi} = \cos x + \cos y$$

which is obtained from (5.2.1) with a translation of the coordinate system. For the boundary conditions we require that solutions of (5.2.2) be bounded as $x, y \longrightarrow \pm\infty$. The time dependence will be assumed to be $\exp(-i c t)$. As for the parallel flow, if $c_i > 0$ the perturbation grows in time and is called unstable. If $c_i < 0$ the perturbation is damped and is called stable. Neutral stability occurs when $c_i = 0$ and the perturbation is oscillatory. It is shown in Appendix V that bounded solutions of (5.2.2) must have the spatial dependence $\chi(x, y) \exp i(\alpha x + \beta y)$ where α and β are real and $\chi(x, y)$ is periodic in x and y with period 2π.

$$\chi(x + 2\pi, y) = \chi(x, y + 2\pi) = \chi(x, y)$$

Thus, α and β play the same role as the parameter called τ in the case of the parallel flow, i.e., α and β "label" the eigenvalues. The domain $-\infty \leq \alpha, \beta \leq \infty$ maps into the eigenvalue spectrum in the c-plane.

Expanding χ in a double Fourier series,

$$\psi' = e^{-ict}\, e^{i(\alpha x + \beta y)} \sum_{m,n=-\infty}^{\infty} a_{mn}\, e^{i(mx + ny)}$$

and substituting into (5.2.2) we obtain a recursion relation for the Fourier coefficients,

$$A_{mn}\ a_{m-1,n} + B_{mn}\ a_{m,n-1} + C_{mn}\ a_{mn} + D_{mn}\ a_{m,n+1} + E_{mn}\ a_{m+1,n} = 0 \qquad (5.3.1)$$

where

$$A_{mn} = \tfrac{1}{2} R (n + \beta) \left[1 - (m + \alpha - 1)^2 - (n + \beta)^2 \right]$$

$$B_{mn} = -\tfrac{1}{2} R (m + \alpha) \left[1 - (m + \alpha)^2 - (n + \beta - 1)^2 \right]$$

$$C_{mn} = \left[(m + \alpha)^2 + (n + \beta)^2 \right] \left[(m + \alpha)^2 + (n + \beta)^2 - i R c \right]$$

$$D_{mn} = \tfrac{1}{2} R (m + \alpha) \left[1 - (m + \alpha)^2 - (n + \beta + 1)^2 \right]$$

$$E_{mn} = -\tfrac{1}{2} R (n + \beta) \left[1 - (m + \alpha + 1)^2 - (n + \beta)^2 \right]$$

Equations (5.3.1) form a doubly infinite set of homogeneous equations for the a_{mn}. The condition for non-trivial solutions is, of course, that the determinant of their coefficients vanish. The equations may be ordered by holding m fixed and letting n vary from $-\infty$ to $+\infty$, then increasing m by one, etc. The determinent will then be the determinent $\Delta(\alpha, \beta, c)$ of an infinite partitioned matrix each element of which is itself an infinite matrix.

$$\Delta(\alpha, \beta, c) = \left| \Gamma_{pq} \right| \qquad -\infty \leq p, q \leq \infty$$

where

$$\Gamma_{pq} = L_p \delta_{p,q+1} + M_p \delta_{pq} + N_p \delta_{p,q-1}$$

$$(L_p)_{rs} = A_{ps} \delta_{rs}$$

$$(M_p)_{rs} = B_{p,s+1} \delta_{r,s+1} + C_{ps} \delta_{rs} + D_{p,s-1} \delta_{r,s-1} \qquad (5.3.3)$$

$$(N_p)_{rs} = E_{ps} \delta_{rs} \qquad -\infty \leq r, s \leq \infty$$

It will be seen that (T_{pq}) is "quasi tri-diagonal", i.e., it is a tri-diagonal matrix whose elements are matrices. L_p and N_p are diagonal and M_p is a tri-diagonal matrix quite similar to the matrix obtained in Chapter 3 for the parallel flow.

The determinant $\Delta(\alpha, \beta, c)$ does not converge as it stands, but can be made to do so by dividing it by its diagonal elements. In order to show this, it is necessary to use a weaker condition than was used in Chapter 3. After the division the determinant will have ones along its main diagonal. Off diagonal elements corresponding to large values of m or n will be of order $\frac{1}{2}R/p$ or smaller where p is a (large) integer. Because of the particular symmetry of the determinant, in the expansion such terms are always multiplied by terms of order $\frac{1}{2}R/p$ or smaller. Thus, the expansion converges as $\sum 1/p^2$.

The boundary conditions will be imposed by setting $\Delta(\alpha, \beta, c)$ equal to zero together with the condition that α and β must be real. From (5.3.2) we note that $\Delta(0, 0, c) = 0$. Assuming α and β small, c is to be obtained as an expansion

$$iRc = C_1\alpha + C_2\beta + C_3\alpha^2 + C_4\beta^2 + C_5\alpha\beta + \ldots$$

It will first be proved that $C_1 = C_2 = C_5 = 0$, $C_3 = C_4$. From (5.3.2) we notice that the following relations hold:

$$A_{-n,\,m}(-\beta, \alpha) = D_{mn}(\alpha, \beta)$$

$$B_{-n,\,m}(-\beta, \alpha) = A_{mn}(\alpha, \beta)$$

$$C_{-n,\,m}(-\beta, \alpha) = C_{mn}(\alpha, \beta)$$

$$D_{-n,m}(-\beta,\alpha) = E_{mn}(\alpha,\beta)$$

$$E_{-n,m}(-\beta,\alpha) = B_{mn}(\alpha,\beta)$$

Now if the argument of the coefficients of (5.3.1) is $(-\beta,\alpha)$ and the equations are arranged by holding n fixed and letting m vary from $+\infty$ to $-\infty$, then increasing n by one, etc. the determinent of the coefficients will be

$$\Delta(-\beta,\alpha,c) = |T'_{pq}| \qquad -\infty \le p,q \le \infty$$

where $$T'_{pq} = L'_p \delta_{p,q+1} + M'_p \delta_{pq} + N'_p \delta_{p,q-1}$$

$$(L'_p)_{rs} = B_{-s,p}(-\beta,\alpha)\,\delta_{rs}$$

$$(M'_p)_{rs} = E_{-s-1,p}(-\beta,\alpha)\,\delta_{r,s+1} + C_{-s,p}(-\beta,\alpha)\,\delta_{rs} + A_{-s+1,p}(-\beta,\alpha)\,\delta_{r,s-1}$$

$$(N'_p)_{rs} = D_{-s,p}(-\beta,\alpha)\,\delta_{rs}$$

$$-\infty \le r,s \le \infty$$

From the relations (5.3.2) this determinent is equal to $\Delta(\alpha,\beta,c)$ i.e.,

$$\Delta(-\beta,\alpha,c) = \Delta(\alpha,\beta,c)$$

and

$$c(-\beta,\alpha,) = c(\alpha,\beta)$$

It follows immediately that $C_1 = C_2 = C_5 = 0$, $C_3 = C_4$.

5.4 Unstable Modes

Next $\Delta(\alpha, 0, c) = 0$ will be solved for c assuming $|c|$ and α small. Referring to (5.3.3), a Laplacian expansion will be made using the rows corresponding to p = 0: from these 3n non-zero columns (i.e., the columns of L_o, M_o, N_o) all possible n x n determinants must be formed multiplied by their (signed) minors where n is the number of rows corresponding to p = 0. We begin with the determinant of the center block $|M_o|$ and its minor m_o. Since $\beta = 0$, $|M_o|$ is identical with the determinant obtained in Chapter 3 for the parallel flow and we know that it is given to order α^4 by its central nine elements Δ_o. Other terms of the expansion are formed by replacing columns of $|M_o|$ by columns from the two off-diagonal blocks L_o and N_o, the determinant so formed being multiplied by its minor, of course.

Referring to (5.3.3) and remembering that $\beta = 0$, it will be seen that terms of order $|M_o| m_o$ are formed only when diagonal elements of $|M_o|$ not contained in Δ_o are replaced, other terms being of higher order in α. Terms of order α^4, then, all contain a factor of Δ_o and we, therefore, approximate the eigenvalue equation by

$$\Delta_o = \begin{vmatrix} C_{o-1} & D_{o-1} & 0 \\ B_{o\,o} & C_{o\,o} & D_{o\,o} \\ 0 & B_{o\,1} & C_{o\,1} \end{vmatrix} = 0$$

Solving for c,

$$1\ R\ c = \alpha^2 (1 - \tfrac{1}{2} R^2) + \vartheta(\alpha^3)$$

and therefore, using the result of the previous section,

$$i R c = (\alpha^2 + \beta^2) (1 - \tfrac{1}{2} R^2) + \mathcal{O}(\alpha^3)$$

Instability occurs if the Reynolds' number is greater than $\sqrt{2}$. It should be emphasized that it has not been proved that these are the only unstable modes. We can only state that with respect to these modes, viz., modes corresponding to $\alpha, \beta \ll 1$, the critical Reynolds' number is $\sqrt{2}$.

CHAPTER VI

CONCLUSIONS AND DISCUSSIONS

We shall now summarize our results briefly, and discuss possibilities for future research. The thesis has been primarily concerned with the stability of the unbounded flow whose velocity profile is the cosine. The problem has been formulated so that stability depends on the sign of the imaginary part of the eigenvalue, denoted by c. A mode having a positive imaginary part corresponds to a small perturbation which grows in time and is unstable. An interesting feature of the problem is the nature of the eigenvalue spectrum, which consists of an infinity of bands, separated by small gaps, lying along the imaginary axis of the c-plane. We have seen that there is a single band which can penetrate into the half plane $c_i > 0$ with suitable values for the wave number of the perturbation, α, and the Reynolds' number R. The neutral curve, $c_i(\alpha, R) = 0$, which separates the stable from the unstable regime, has been discussed in Chapter 4. A similarity with the neutral curve for the free boundary layer between parallel streams has been noted. It has not been proved that the eigenfunctions form a complete set. Since our conclusion that there is a critical Reynolds' number ($R_c = \sqrt{2}$) below which the flow is stable depends on completeness, it would clearly be important to prove such a theorem.

We have seen that the formulation of the eigenvalue problem as the vanishing of an infinite determinant is more convenient for obtaining the portion of the neutral curve near the critical Reynolds' number than the more conventional methods used in Chapter 2. This method could be applied to any spatially periodic flow. Its success

depends on approximating the determinant by a few terms of its expansion. In general, the number of non-zero super- and sub-diagonals will be equal to the number of Fourier components of the velocity profile. Assuming that the off diagonal and center elements are small, as is the case, for example, when α is small, the number of terms needed to give the determinant to first order is equal to the number of non-zero elements of the center row (or column). Thus, the method is particularly useful for flows whose velocity profiles have a small number of Fourier components and for which the onset of instability occurs at small values of α.

In Chapter 5 the stability of a two dimensional, cellular flow has been considered. The material of this chapter should be considered as preliminary to a thorough treatment of the problem. The eigenvalue spectrum has been obtained only in the neighborhood of the origin of the c-plane. Modes have been found there which are unstable if $R > \sqrt{2}$. However, it has not been shown that these are the only modes in this neighborhood. A more complete study of the eigenvalue spectrum would clearly be desirable.

Moreover, the perturbations have been restricted to the plane of the basic flow, although it has not been shown that these are the least stable perturbations. It would be interesting to study the more difficult problem of the effect of three dimensional perturbations (cf. equations (5.2.2)). The wave number of the component of the perturbation perpendicular to the plane of the basic flow, γ say, would then play a role analogous to that of α for the parallel flow and it would be possible to define a neutral curve in the γ-R plane.

Finally, it would be interesting to include the effects of compressibility and heat conduction with a view to applications to flows

of the atmosphere and oceans. Eckart [4], for example, has shown that internal waves in the ocean are possible which are two dimensional and cellular in character.

APPENDIX I

The solutions of (1.2.4) are given in powers of α .

$$\varphi_j = \sum_{n=0}^{\infty} \varphi_j^{\,n} \qquad j = 1,2,3,4$$

$$\varphi_1^{\,0} = 1 \qquad \varphi_1^{\,1} = i\alpha R \left\{ \cos y - 1 + \tfrac{1}{2} y^2 \right\}$$

$$\varphi_1^{\,2} = -\tfrac{1}{2} (\alpha R)^2 \left\{ -20 (\cos y - 1) - 3y^2 - 8 y \sin y + y^2 \cos y \right\}$$

$$- (\alpha R)^2 c \left\{ \cos y - 1 + \tfrac{1}{2} y^2 - \tfrac{1}{4!} y^4 \right\}$$

$$\varphi_1^{\,3} = - i \alpha^3 R \left\{ 3 (\cos y - 1) + 3/2\, y^2 - 1/12\, y^4 \right\} + 1/4!\, i \alpha^3 R c\, y^4$$

$$- 1/2\, i(\alpha R)^3 \left\{ - 7/4!\, y^4 + 119/8\, y^2 + 11/8 \sin^2 y + 74(\cos y - 1) + 24 y \sin y \right.$$

$$\left. - 3 y^2 \cos y - 1/4\, y \sin y \cos y \right\} + i (\alpha R)^3 c \left\{ - 1/8\, y^4 + 56 (\cos y - 1) \right.$$

$$\left. + 15/2\, y^2 + 26 y \sin y - 11/2\, y^2 \cos y - 2/3\, y^3 \sin y + 1/4!\, y^4 \cos y \right\}$$

$$+ i (\alpha R)^3 c^2 \left\{ - (\cos y - 1) - 1/2\, y^2 + 1/4!\, y^4 - 1/6!\, y^6 \right\}$$

$$
\begin{aligned}
\varphi_1^{4'''} = &- \alpha^4 y + \alpha^2 (\alpha R)^2 \left\{ 13/2\, y + 1/2 \sin y \cos y - 17 \sin y + 10\, y \cos y \right. \\
&\left. + 3\, y^2 \sin y - 1/3\, y^3 \cos y - 2/4!\, y^4 \sin y \right\} \\
&+ 1/2\, (\alpha R)^4 \left\{ - 7/6\, y^3 \cos y + 119/4\, y \cos y - 7/4!\, y^4 \sin y + 119/8\, y^2 \sin y \right. \\
&- 143/2 \sin y - 25/24 \sin^3 y + 18\, y + 24 \sin y \cos y + 6\, y \sin^2 y \\
&\left. - 1/4\, y \cos^3 y - 1/12 \sin^3 y \right\} + (\alpha R)^4 c^3 \left\{ \sin y - y + 1/6\, y^3 - 1/5!\, y^5 \right\} \\
&+ \alpha^2 (\alpha R)^2 c \left\{ - 1/6\, y^3 \cos y - 1/4!\, y^4 \sin y - 2 \sin y + 2y + 1/2\, y^3 \right\} \\
&+ (\alpha R)^4 c \left\{ 1/2\, y^3 \cos y - 24\, y \cos y + 1/8\, y^4 \sin y - 9\, y^2 \sin y \right. \\
&+ 81 \sin y - 92/3\, y - 631/24 \sin y \cos y - 113/12\, y \sin^2 y \\
&+ 13/12\, y^3 + 11/12\, y^2 \sin 2y - 9/16 \sin 2y - 9/16 \sin^2 y + 11/12\, y \cos 2y \\
&\left. + 1/6\, y^3 \sin^2 y + 1/6\, y^2 \sin y \cos y \right\} \\
&+ (\alpha R)^4 c^2 \left\{ 1/120\, y^5 \cos y - 2/3\, y^3 \cos y + 16\, y \cos y + 1/6!\, y^6 \sin y \right. \\
&\left. - 1/12\, y^4 \sin y + 4\, y^2 \sin y - 31 \sin y + 15\, y \right\} + 1/6\, \alpha^2 (\alpha R)^2 c^2 y^3
\end{aligned}
$$

$$\varphi_2^0 = y \qquad \varphi_2^1 = i\alpha R \left\{ -4 \sin y + 3y - 1/3!\, y^3 + y \cos y \right\}$$

$$\varphi_2^2 = -(\alpha R)^2 \left\{ -7/12\, y^3 + 129/8\, y - 1/8 \sin y \cos y - 28 \sin y + 12 y \cos y \right.$$

$$\left. + 2 y^2 \sin y - 1/3!\, y^3 \cos y \right\}$$

$$+ (\alpha R)^2 c \left\{ 6 \sin y - 5y + 1/2\, y^3 - 1/5!\, y^5 - y \cos y \right\}$$

$$\varphi_2^{3'} = -i(\alpha R)^3 \left\{ -175/16\, y^2 + 7/48\, y^4 - 1747/24 (\cos y - 1) - 269/8\, y \sin y \right.$$

$$+ 21/4\, y^2 \cos y + 7/12\, y^3 \sin y + 1/72 (\cos^3 y - 1) + 67/16 \sin^2 y$$

$$\left. - 5/4\, y \sin y \cos y - 1/8\, y^2 \sin^2 y \right\} + 1/4!\, i \alpha^3 R c y^4$$

$$+ i(\alpha R)^3 c \left\{ -12 y^2 \cos y + 92 (\cos y - 1) - 3/2\, y^3 \sin y + 46 y \sin y \right.$$

$$\left. + 1/8\, y^4 \cos y + 173/16\, y^2 + 1/5!\, y^5 \sin y - 5/16 \sin^2 y - 7/48\, y^4 \right\}$$

$$- i(\alpha R)^3 c^2 \left\{ -7 (\cos y - 1) - 5/2\, y^2 + 1/8\, y^4 - 1/6!\, y^6 - y \sin y \right\}$$

$$+ i \alpha^3 R \left\{ 13(\cos y - 1) + 7/2\, y^2 - 1/12\, y^4 + 3 y \sin y \right\}$$

$$\varphi_3^{\,0} = 1/2\, y^2$$

$$\varphi_3^{\,1} = i\alpha R \left\{ -9(\cos y - 1) - y^2 - 4y \sin y + 1/2\, y^2 \cos y \right\} - 1/4!\; i\alpha R c\, y^4$$

$$\varphi_3^{\,2} = -(\alpha R)^2 \left\{ -7/48\, y^4 + 95/16\, y^2 + 11/16 \sin^2 y + 27(\cos y - 1) + 8y \sin y \right.$$

$$\left. - y^2 \cos y - 1/8\, y \sin y \cos y \right\} + 1/12\, \alpha^2 y^4$$

$$+ (\alpha R)^2 c \left\{ 45(\cos y - 1) + 22y \sin y - 5y^2 \cos y + 11/2\, y^2 \right.$$

$$\left. - 2/3\, y^3 \sin y + 1/4!\; y^4 \cos y - 1/12\, y^4 \right\} - 1/6!\; (\alpha R)^2 c^2 y^6$$

$$\phi_3^{3''} = -i(\alpha R)^3 c^2 \left\{ 1/8\, y^4 \cos y - 6\, y^2 \cos y + 47(\cos y - 1) + 2/5!\, y^5 \sin y \right.$$

$$\left. - y^3 \sin y + 24\, y \sin y - 1/6!\, y^6 \cos y + 11/2\, y^2 - 1/12\, y^4 \right\} + 1/6!\, i(\alpha R)^3 c^3 y^6$$

$$- i(\alpha R)^3 \left\{ -151/16\, y^2 \cos y + 2735/48(\cos y - 1) - 7/6\, y^3 \sin y + 245/8\, y \sin y \right.$$

$$+ 7/48\, y^4 \cos y - 29/144(\cos^3 y - 1) + 7/2\, y^2 + 9/2 \sin^2 y - y \sin y \cos y$$

$$\left. + 1/24\, y \sin^3 y \right\}$$

$$+ i \alpha^3 R \left\{ 7/2\, y^2 \cos y - 25(\cos y - 1) + 2/3\, y^3 \sin y - 14 \sin y \right.$$

$$\left. - 1/12\, y^4 \cos y - 2y^2 \right\}$$

$$+ i(\alpha R)^3 c \left\{ 349/24\, y^2 + 641/48 \sin^2 y + 97(\cos y - 1) - 53/12\, y \sin y \cos y \right.$$

$$- 7/24\, y^4 - 11/12\, y \sin 2y + 11/24\, y^2 \cos 2y - 67/96(\cos 2y - 1) + 33\, y \sin y$$

$$- 17/2\, y^2 \cos y + 1/12\, y^3 \sin y \cos y - 5/24\, y^2 \sin^2 y - 2/3\, y^3 \sin y$$

$$\left. + 1/12\, y^4 \cos y \right\} - 1/8\, i \alpha^3 R c\, y^4$$

$$f_4^0 = 1/6\, y^3$$

$$f_4^1 = i \propto R \left\{ 16 \sin y - 7y - 9y \cos y - 2y^2 \sin y + 1/6\, y^3 \cos y \right\} - 1/5!\; i \propto R\, c\, y^5$$

$$f_4^2 = -(\propto R)^2 \left\{ -7/240\, y^5 + 23/12\, y^3 - 369/16\, y + 33/16 \sin y \cos y + 28 \sin y \right.$$

$$\left. - 7y \cos y + 11/16\, y \sin^2 y - 1/16\, y^2 \sin y \cos y \right\}$$

$$+ (\propto R)^2 c \left\{ 45\, y \cos y + 11 y^2 \sin y - 86 \sin y + 41\, y - 5/3\, y^3 \cos y \right.$$

$$\left. - 1/6\, y^4 \sin y + 1/5!\; y^5 \cos y - 7/6\, y^3 \right\} + 1/60 \propto^2 y^5 - 1/7!\; (\propto R)^2 c^2 y^7$$

$$\varphi_4^{3'''} = -i(\alpha R)^3 \left\{ -7/48\, y^4 \cos y + 23/4\, y^2 \cos y - 1377/48 (\cos y - 1) - 7/240\, y^5 \sin y \right.$$

$$+ 23/12\, y^3 \sin y - 225/8\, y \sin y + 198/144 (\cos^3 y - 1) + 7 \sin^2 y$$

$$\left. - 27/48\, y \sin^3 y - 3/48\, y^2 \cos^3 y \right\}$$

$$+ i(\alpha R)^3 c \left\{ 561/48\, y^2 + 93/4\, y \sin y \cos y - 211/4 \sin^2 y + 41/8\, y^2 \sin^2 y \right.$$

$$+ 62 (\cos y - 1) + 48\, y \sin y - 13/48\, y^4 - 2/3\, y^3 \sin y \cos y$$

$$\left. - 1/24\, y^4 \sin^2 y - 7/2\, y^2 \cos y - 7/6\, y^3 \sin y \right\}$$

$$- i(\alpha R)^3 c^2 \left\{ 6\, y^2 \cos y - 41 (\cos y - 1) + y^3 \sin y - 23\, y \sin y - 1/8\, y^4 \cos y \right.$$

$$\left. - 7/2\, y^2 + 1/7!\, y^7 \sin y - 1/5!\, y^5 \sin y + 1/6!\, y^6 \cos y \right\}$$

$$+ i \alpha^3 R \left\{ 15 (\cos y - 1) + 11\, y \sin y - 7/2\, y^2 \cos y - 1/2\, y^3 \sin y \right.$$

$$\left. + 1/12\, y^4 \cos y + 1/60\, y^5 \sin y \right\}$$

$$- 1/8\, i \alpha^3 R\, c\, y^4 + 1/6!\, i (\alpha R)^3 c^3 y^6$$

APPENDIX II

In Chapter 2 four solutions of the integral equation (2.1.2) were obtained as the series (2.1.5). We shall now prove that these series converge uniformly on the interval $0 \leq y \leq 2\pi$ provided $|m|$ is sufficiently large. Consider first the solution ϕ_1. The equation for ϕ_1, (2.1.3), may be written

$$\phi_1(y) = e^{\alpha y} - \frac{i\alpha R}{m^2-\alpha^2} \int_0^y \left\{ \frac{1}{\alpha} \cosh \alpha(y-y') + \frac{1}{2m} e^{-m(y-y')} \right\} \cos y' \, L\phi_1(y') \, dy' - \frac{i\alpha R}{m^2-\alpha^2} \int_y^{2\pi} \frac{1}{2m} e^{m(y-y')} \cos y' \, L\phi_1(y') \, dy' \tag{II.1}$$

$$L \equiv \frac{d^2}{dy'^2} - \alpha^2 + 1$$

The series obtained for ϕ_1 is

$$\phi_1 = \sum_{n=0}^{\infty} \phi_1^n \quad \text{where} \quad \phi_1^0 = e^{\alpha y} \tag{II.2}$$

$$\phi_1^n = -\frac{i\alpha R}{m^2-\alpha^2} \left\{ \int_0^y \frac{1}{\alpha} \cosh \alpha(y-y') + \frac{1}{2m} e^{-m(y-y')} \right\} \cos y' \, L\phi_1^{n-1}(y') \, dy' + \int_y^{2\pi} \frac{1}{2m} e^{m(y-y')} \cos y' \, L\phi_1^{n-1}(y') \, dy'$$

We recall that the branch of m has been chosen so that Re $m \geq 0$.

It will first be proved that

$$\varphi_1(y) = \mathcal{O}(e^{\alpha y})$$

Let $$\varphi_1(y) = e^{\alpha y} F(y) \qquad \text{(II.3)}$$

Let μ be the largest of the maxima of $|F(y)|, |F'(y)|, |F''(y)|$ on $0 \leq y \leq 2\pi$.

Let $|e^{-\alpha(y-y')}| \leq M \quad 0 \leq y \leq 2\pi, 0 \leq y' \leq 2\pi$.

Substituting (II.3) into (II.1)

$$F(y) = 1 - \frac{i\alpha R}{m^2-\alpha^2}\int_0^y \left\{\frac{1}{2\alpha}\left[1 - e^{-2\alpha(y-y')}\right] + \frac{1}{2m} e^{-(m+\alpha)(y-y')}\right\} \cos y' \left[F''(y') + 2\alpha F'(y') + F(y')\right] dy'$$

$$- \frac{i\alpha R}{m^2-\alpha^2}\int_0^{2\pi} \frac{1}{2m} e^{(m-\alpha)(y-y')} \cos y' \left[F''(y') + 2\alpha F'(y') + F(y')\right] dy'$$

$$\mu \leq 1 - \left|\frac{i\alpha R}{m^2-\alpha^2}\right| \int_0^{2\pi} \left\{\frac{1}{\alpha}(1+m^2) + \frac{M}{|m|}\right\} \mu(1+\alpha)\, dy'$$

$$- \left|\frac{i\alpha R}{m^2-\alpha^2}\right| \int_0^{2\pi} \frac{M}{|m|} \mu(1+\alpha)$$

$$\leq \frac{1}{1 - \left|\frac{i\alpha R}{m^2-\alpha^2}\right| \left\{\frac{1+m^2}{\alpha} + \frac{2M}{|m|}\right\} (2\pi)(1+\alpha)}$$

Thus, $\varphi_1(y) = \mathcal{O}(e^{\alpha y})$ provided $|m|$ is sufficiently large. Substituting this result into (II.1)

$$\varphi_1(y) = e^{\alpha y} + \mathcal{O}\left(\frac{1}{|m|^2}\right)$$

The process may be repeated. Thus,

$$\varphi_1^{\,n} = \mathcal{O}\left(\frac{1}{|m|^{2n}}\right)$$

and the series converges absolutely and uniformly on the interval $0 \leq y \leq 2\pi$ for sufficiently large $|m|$. Moreover, we see that each iteration produces higher order terms in m^{-1} than the last, i.e., the series converges rapidly if $|m|$ is large.

The convergence of the series for φ_2 can be proved in exactly the same way. The proof for φ_3 and φ_4 is essentially the same but will be given because it differs in detail from the above proof. After two integrations by parts, the equation for φ_3 may be written

$$\begin{aligned}\varphi_3(y) = e^{my} &+ \frac{i\alpha R}{m^2-\alpha^2}\left\{\varphi_3'(0)\left[\frac{\sinh \alpha y}{\alpha} - \frac{\sinh my}{m}\right]\right. \\ &\left. + \varphi_3(0)\left[\cosh \alpha y - \cosh my\right]\right\} \qquad \text{(II.4)} \\ &+ \int_0^y \left\{\frac{2i\alpha R}{m^2-\alpha^2}\left[\cosh m(y-y') - \cosh \alpha(y-y')\right]\sin y'\right. \\ &\left. + \frac{i\alpha R}{m}\sinh m(y-y')\cos y'\right\}\varphi_3(y')\,dy'\end{aligned}$$

We shall first prove that

$$\varphi_3(y) = \mathcal{O}(e^{my})$$

Let

$$\varphi_3(y) = e^{my}\,F(y)$$

Let μ be the largest of the maxima of $|F(y)|$, $|F^{-1}(y)|$ on $0 \leq y \leq 2\pi$. Let $|\cosh \alpha(y - y')| \leq M \quad 0 \leq y \leq 2\pi$, $0 \leq y' \leq 2\pi$.

Substituting (II.5) into (II.4)

$$\mu \leq 1 + \left|\frac{i\alpha R}{m^2-\alpha^2}\right| \left\{ \mu(|m|+1)\left[\frac{M}{\alpha} + \frac{1}{|m|}\right] + \mu(M+1) \right\}$$

$$+ \left|\frac{2i\alpha R}{m^2-\alpha^2}\right| \int_0^{2\pi} \mu(1+M)\,dy' + \left|\frac{i\alpha R}{m}\right| \int_0^{2\pi} \mu\,dy'$$

$$\leq \frac{1}{1 - \left|\frac{i\alpha R}{m^2-\alpha^2}\right| \left\{ (|m|+1)\left(\frac{M}{\alpha} + \frac{1}{|m|}\right) + (1+M)(4\pi+1) + \left|\frac{i\alpha R}{m}\right| (2\pi) \right.}$$

Thus, $\varphi_3(y) = \mathcal{O}(e^{my})$ if $|m|$ is sufficiently large. Substituting this result into (II.4)

$$\varphi_3(y) = e^{my}\left\{ 1 + \mathcal{O}\left(\frac{1}{|m|}\right) \right\}$$

The process may be repeated. Thus,

$$\varphi_3^{\,n}(y) = \mathcal{O}\left(\frac{e^{my}}{|m|^n}\right)$$

and the series converges absolutely and uniformly on $0 \leq y \leq 2\pi$ for sufficiently large $|m|$.

The proof of the convergence of the series for $\varphi_4(y)$ is similar to that for $\varphi_3(y)$.

APPENDIX III

In Section 4.4 a solution of the integral equation

$$\varphi_o(y) = f(y) + c \int_{-\pi}^{y} K(y, y')\, \varphi_o(y')\, dy'$$

$$f(y) = A\, e^{i\sqrt{1-\alpha^2}\; y} + B\, \frac{e^{-i\sqrt{1-\alpha^2}\; y}}{-i\sqrt{1-\alpha^2}}$$

$$K(y,y') = \frac{-1}{\sqrt{1-\alpha^2}}\, \frac{\sin \sqrt{1-\alpha^2}\,(y-y')}{\cos y' - c}$$

where A and B are arbitrary constants was obtained as a series

$$\varphi_o = \sum_{n=0}^{\infty} \varphi_o^{\,n} \text{ where } \varphi_o^{\,o} = f(y) \text{ and}$$

$$\varphi_o^{\,n} = c^n \int_{-\pi}^{y} K(y,t_1)\, dt_1 \int_{-\pi}^{t_1} K(t_1,t_2)\, dt_2 \cdots \int_{-\pi}^{t_{n-1}} K(t_{n-1}, t_n)\, f(t_n)\, dt_n$$

The integral path Γ is defined in Section 4.3. We wish to prove the uniform convergence of the series in the neighborhood of Γ. The method will be the same as that used in Appendix I.

Let $|f(y)| \leq M$ on Γ

$|K(y,y')| \leq L$ y, y' on Γ

$|\varphi_o(y)| \leq \mu$ on Γ

We begin by proving $\varphi_o(y) = \mathcal{O}(1)$. From the integral equation we have,

$$\mu \leq M + |c| \int_{\Gamma} L\, \mu\, dy$$

$$\leq \frac{M}{1 - |c|\, l\, L}$$

where l is the "length" of Γ . Since M may be taken to be of order unity, $\varphi_o = \mathcal{O}(1)$ for $|c|$ sufficiently small. Substituting this result into the integral equation we have

$$\varphi_o = f(y) + \mathcal{O}(|c|)$$

The process may be repeated indefinitely so that

$$|\varphi_o^n| \leq |c|^n M$$

and therefore, the series converges uniformly and absolutely in a neighborhood of Γ for $|c|$ sufficiently small. Moreover, we see that the series converges rapidly.

APPENDIX IV

In the formulation of the eigenvalue equation (4.5.2) in Section 4.5 the viscous solutions (4.1.4)

$$\phi_{\frac{3}{4}} = (u - c)^{-5/4} \exp \pm \sqrt{\alpha R} \int_{y_0}^{y} \sqrt{i(u - c)}\, dy \quad (IV.1)$$

of the Orr-Sommerfeld equation (1.2.4) for large αR were rejected because they are exponentially unbounded. However, since the boundary conditions are applied at finite points in the y-plane, these solutions could be included formally. It will now be shown that the effect of formally including them is negligible. Using all four solutions, (4.5.1) and(IV.1), the eigenvalue equation is a fourth order polynomial in s

$$\begin{vmatrix} \phi_1(\pi) - s\,\phi_1(-\pi) & \cdots & \phi_4(\pi) - s\,\phi_4(-\pi) \\ \cdots & & \\ \phi_1'''(\pi) - s\,\phi_1'''(-\pi) & \cdots & \phi_4'''(\pi) - s\,\phi_4'''(-\pi) \end{vmatrix} = 0 \quad (IV.2)$$

Since in the present case u(y) has period 2π, from the third and fourth columns of (IV.2) we can factor out

$$1 - 2s \cosh \int_{\Gamma} \sqrt{i\alpha R(u - c)}\, dy + s^2 \quad (IV.3)$$

The integral path Γ is defined in Section 4.2. We have noted that on this path the real part of Q(y) is monotonically increasing where

$$Q(y) = \int_{y_c}^{y} \sqrt{i\,(u - c)}\, dy$$

$$u\,(y_c) - c = 0$$

Therefore,

$$\text{Re} \int_{\Gamma} \sqrt{i\,(u - c)}\, dy = \text{Re}\left[\, Q(\pi) - Q(-\pi) \,\right] \neq 0$$

Since αR is large, the roots of (IV.3) correspond to unbounded solutions of (1.2.4) ($|s| \neq 1$) by the theorem quoted in Section 1.3.

After the above factorization the third column of (IV.2) is subtracted from the fourth, the fourth added to the third, and both divided by $-i\alpha R(1 + c)$. Assuming c of order $(\alpha R)^{-1}$ (as we did in Section 4.5) and neglecting terms of order $(\alpha R)^{-2}$,

$$\begin{vmatrix} \cdots & i/\alpha R & 0 \\ \cdots & 0 & i/\alpha R \\ \cdots & 1 + 5/4\ i/\alpha R & 0 \\ \cdots & 0 & 1 + 13/4\ i/\alpha R \end{vmatrix} = 0$$

Expanding and dividing by s,

$$ic = -\frac{\sqrt{1-\alpha^2}}{\pi}\,\frac{\cos 2\pi\tau - \cos 2\pi\sqrt{1-\alpha^2}}{\sin 2\pi\sqrt{1-\alpha^2}} + \frac{1}{\alpha R}$$

$$-\frac{i}{\alpha R}\,\frac{\sqrt{1-\alpha^2}}{\pi}\,\frac{\cos 2\pi\tau - \cos 2\pi\sqrt{1-\alpha^2}}{\sin 2\pi\sqrt{1-\alpha^2}}\left[\frac{9}{2}\right.$$

$$\left. + 2(1-\alpha^2)\right] + \mathcal{O}\left[\left(\frac{1}{\alpha R}\right)^2\right]$$

The expression (4.5.3) for c_i is not changed. (The range of τ must be restricted so that the assumption $c = \mathcal{O}\left(\frac{1}{\alpha R}\right)$ is satisfied.) It is seen that c_r is of order $c_i/\alpha R$, i.e., of order $(\alpha R)^{-2}$. Thus, the effect of the viscous solutions is of the order neglected.

APPENDIX V

Consider a linear partial differential equation in two variables x and y whose coefficients are periodic functions of x and y with the period a. It will be proved that a solution $\psi(x,y)$ of the equation which is bounded as $x,y \longrightarrow \pm \infty$ must be of the form

$$\psi(x,y) = e^{i(\alpha x + \beta y)} \chi(x,y)$$

where α and β are real and $\chi(x,y)$ is periodic in x and y with the period a

$$\chi(x + a,y) = \chi(x, y + a) = \chi(x,y)$$

The proof is an adaptation of the method of Bloch [1] . Bloch imposed cyclic boundary conditions. Here we only require boundedness.

Suppose that for a given eigenvalue there are L linearly independent eigenfunctions ψ_α ($\alpha = 1 \ldots L$). If ψ_K is a solution of the equation then obviously so are $\psi_K(x + a,y)$ and $\psi_K(x,y + a)$ and there exist relations

$$\psi_K(x + a,y) = \sum_{\alpha=1}^{L} a_{K\alpha} \psi_\alpha(x,y)$$

$$\psi_K(x, y + a) = \sum_{\alpha=1}^{L} b_{K\alpha} \psi_\alpha(x,y)$$

In group theoretical language, let T_1 and T_2 be translation operators such that

$$T_1 : x \longrightarrow x + a$$

$$T_2 : y \longrightarrow y + a$$

then $$\mathcal{T}_1 \psi = A \psi \qquad (V.1)$$

$$\mathcal{T}_2 \psi = B \psi$$

where ψ is the (column) vector ψ_K ($k = 1 \ldots L$) and the ψ_K's are the basis functions of the group. $A = (a_{k\alpha})$ and $B = (b_{k\alpha})$ are the representations of $\mathcal{T}_1$ and $\mathcal{T}_2$.

$\mathcal{T}_1$ and $\mathcal{T}_2$ obviously commute and, therefore, A and B commute and can be simultaneously diagonalized. Let the similarity transformation which diagonalizes them be

$$\mathcal{T}_A \mathcal{T}^{-1} = A'$$

$$\mathcal{T}_B \mathcal{T}^{-1} = B'$$

where A' and B' are diagonal matrices.

Next we define L new linearly independent functions by forming linear combinations of the ψ_K's:

$$\psi' \equiv \mathcal{T} \psi$$

then from (V.1)

$$\mathcal{T}_1 \psi' = \psi'(x+a, y) = \mathcal{T} A \psi$$

$$= \mathcal{T} A \mathcal{T}^{-1} \mathcal{T} \psi = A' \psi'(x, y)$$

similarly,

$$\psi'(x, y+a) = B' \psi'(x, y)$$

Since A' and B' are diagonal,

$$\psi'_K(x+a, y) = a'_{KK} \psi'_K(x, y)$$

$$\psi'_K(x, y+a) = b'_{KK} \psi'_K(x, y) \qquad K = 1 \ldots L$$

i.e., the new functions all have the property

$$\psi'(x + a,y) = s \quad \psi'(x,y)$$

$$\psi'(x,y + a) = p \quad \psi'(x,y)$$

where s and p are constants. Define

$$s \equiv e^{\mu a}$$

$$p \equiv e^{\sigma a}$$

then

$$\psi'(x + a,y) = e^{\mu a} \quad \psi'(x,y)$$

Multiplying both sides by $e^{-\mu(x+a)}$

$$e^{-\mu(x + a)} \quad \psi'(x + a,y) = e^{-\mu x} \quad \psi'(x,y)$$

Therefore, the function $e^{-\mu x} \quad \psi'(x,y)$ has the period a in x, i.e.,

$$\psi' = e^{\mu x} \quad \varphi(x,y)$$

where φ is periodic in x. Similarly,

$$\psi' = e^{\sigma y} \quad \theta(x,y)$$

where θ is periodic in y. In order for the solutions to be bounded, μ and σ must be pure imaginary. Thus, all bounded solutions are of the form

$$\psi' = e^{i(\alpha x + \beta y)} \chi(x,y)$$

where $\chi(x + a,y) = \chi(x,y + a) = \chi(x,y)$ and α and β are real.

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