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A CHARACTERIZATION OF THE VISIBILITY PROCESS AND ITS EFFECT ON SEARCH POLICIES

by
Michael Leo Moore

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Industrial Engineering)
in The University of Michigan
1972

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Chapter 1

INTRODUCTION AND LITERATURE REVIEW

The topic of this dissertation arose out of research performed by the Systems Research Laboratory (SRL) under contract number N00014-67-A-0181-0012 with the Office of Naval Research. The overall research program is concerned with the development of more generalized mathematical structures of military processes, although as will soon become evident, the results are applicable to nonmilitary processes as well. Emphasis has been directed to descriptive modeling of combat processes and has assumed perfect intelligence-gathering capabilities of the units involved. It was thought that many of the existing search and reconnaissance theories would be useful for predicting the amount of intelligence-gathering capability processed by a tactical unit. However, a thorough literature review indicated that most search and reconnaissance modeling efforts have been devoted to the development of strategies for the optimal allocation of search effort and little to the development of realistic descriptive models of intelligence-gathering process. The literature review and preliminary modeling efforts (Moore, 1970) indicated the need to consider more realistically both environmental effects and search objectives in developing

descriptive structures of the search process and analysis of optimal search strategies. This dissertation addresses one dimension of the environmental effects--the "visibility process." The main thrust of the work is the development of descriptive search models which include the visibility process, and the analysis of these models to gain physical insight into the intelligence-gathering process, and to determine strategies for optimal allocation of search effort.

In this chapter, some fundamental terms are defined and a taxonomy of possible classifications of search problems is introduced in order to establish a common vocabulary. A classification of search problems is then introduced to familiarize the reader with the general structure of search problems. Next the results of an extensive literature review are summarized to provide the reader with background in the area as well as a ready source of those results which are referred to in the body of the thesis. Finally, the specific area of research is outlined and the broad applicability of the visibility models discussed.

1.1 Characterization of the Elements of Search Problems

1.1.1 Definitions

This section contains some basic definitions and notations used throughout the paper. Any additional notation and exceptions to those specified herein will be specifically noted in the text.

Detection - The act of gathering information pertaining to the object being sought, the sifting out of what is important information and the relaying of that information in some efficient form to the decision maker.

Discrete Detection Model - Let $(1 - q_i)$ be the instantaneous probability of detection of the ith scan of an area. Given n such scans, the probability of detection is

$$P(D) = 1 - \prod_{i=1}^{n} q_i \ .$$

The q_i are referred to as "overlook" probabilities.

Continuous Detection Model - The probability of detecting the target in the interval $(t, t + dt)$, given no detection up to t, is given by $\gamma(t)dt$. Given continuous observance over an interval $(0,\tau)$, the probability of detection is

$$P(\tau) = 1 - e^{-\int_0^{\tau} \gamma(t)dt} \ .$$

Search Strategy - The decision made on the basis of information obtained from the detection process. A "search strategy" will be that set of rules which associates a decision with every conceivable result of the detection process, e.g., the next region to be searched and how much effort to expend there.

Target - The object of the search: a military target, a mineral deposit, or any other object about which information is desired.

Search Space - The region containing the target or targets. It can be *discrete* consisting of boxes or subregions, or it may be *continuous*. It need not be considered in a strictly geometrical sense, e.g., the search space may consist of the possible frequencies of an unknown signal.

Cumulative Detector - A detection device having either of the following characteristics:

(a) No loss of information - Given that x units of effort have been continuously applied, all the information thus gained is retained, when at some later time, additional effort is placed in the same region.

(b) Partial loss of information - Given that x units of effort have been continuously applied, only certain portions of the information thus gained are retained when at some later time, additional effort is placed in the region. The portion retained could be a function of:

(1) the length of time since the first trial

(2) for moving targets, the motion structure.

Non-Cumulative Detector - A detection device having the property that all the information gained from a previous search is lost, when at some later time additional effort is placed in the same region.

Visibility - That condition under which the sensor (detector) signals can reach the target and be received.

Search Objectives - The two common search objectives are:

(a) Given a constraint on the available search effort, maximize the probability of detecting the target.

(b) Given unlimited search effort, minimize the expected effort required until detection occurs.

The first is an effectiveness measure, while the second is a cost measure.

1.1.2 Classification of Detectors and Targets

Models of search and reconnaissance processes treat detectors and targets with varied combinations of properties or assumptions regarding their behavior. This section presents a classification of analytic assumptions that can be used to describe the behavior of detectors and targets.

Detectors

1. Single Detector with a Single Scan[1]

   A) Discrete detection

   B) Continuous detection

---

[1]Aerial photographic reconnaissance provides an example of this situation.

1) Non-cumulative probability of detection[1] (Complete loss of information)

2) Cumulative probability of detection

a) partial loss of information

b) no loss of information

2. Single Detector with Multiple Scan Capability

A) Discrete detection

B) Continuous detection

1) Non-cumulative probability of detection[1]

2) Cumulative probability of detection

a) partial loss of information

b) no loss of information

3. Multiple Detectors with Single Scan Capability

A) Discrete detection

1) Detectors act statistically independently

2) Detectors act dependently

B) Continuous detection

1) Independent action

a) non-cumulative probability of detection[1]

b) cumulative probability of detection

(1) partial loss of information

(2) no loss of information

2) Dependent action

---

[1] These conditions can also hold for a discrete detector.

a) non-cumulative probability of detection

b) cumulative probability of detection

(1) partial loss of information

(2) no loss of information

4. Multiple Detectors with Multiple Scan Capability

A) Discrete detection

1) Independent action

2) Dependent action

B) Continuous detection

1) Independent action

a) non-cumulative probability of detection[1]

b) cumulative probability of detection

(1) partial loss of information

(2) no loss of information

2) Dependent action

a) non-cumulative probability of detection

b) cumulative probability of detection

(1) partial loss of information

(2) no loss of information

Targets

1. Non-Conscious Evasion

A) Single target

1) Stationary

a) continuously visible

b) not continuously visible

[1]These conditions can also hold for a discrete detector.

2) Moving

a) continuously visible

b) not continuously visible

B) Multiple Targets

1) Stationary

a) number

b) location

(1) independent actions

(a) continuously visible

(b) not continuously visible

(2) dependent actions

(a) continuously visible

(b) not continuously visible

2) Moving

a) number

b) initial distribution; description of the motion

(1) independent actions

(a) continuously visible

(b) not continuously visible

(2) dependent actions

(a) continuously visible

(b) not continuously visible

2. Conscious Evasion

(Same as 1 above)

1.1.3 Classification of Search Problems

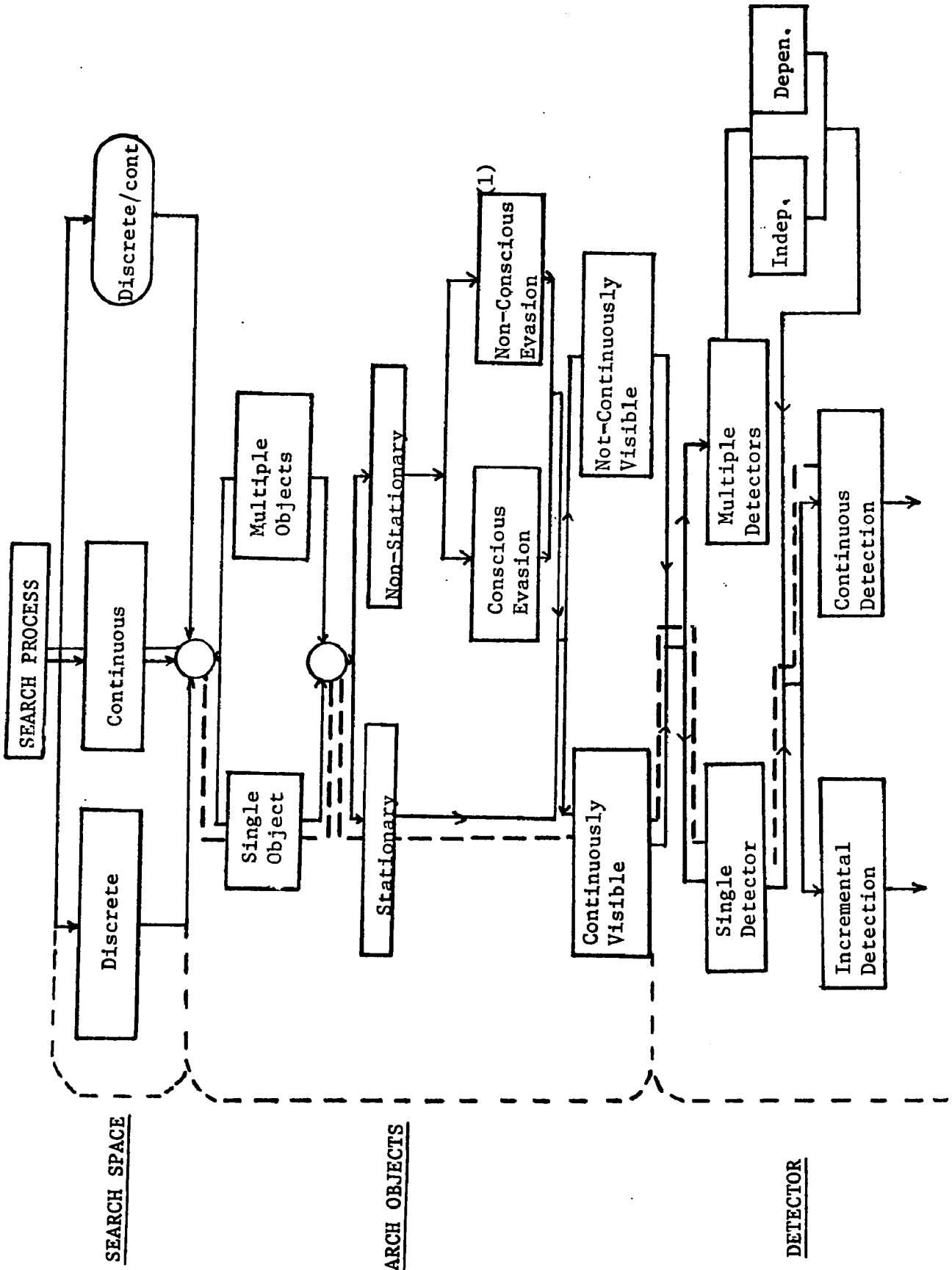
Many variations of search problems appear in the open literature. The following decision diagram[1] presents, in flow chart format, the various attributes of search problems that may be considered. It is introduced at this point in the paper to familiarize the reader with the general structure of search problems and to provide a vehicle for the location of the proposed research topics within the structure. Given any of the problems discussed in the literature, one can characterize it by a path through the decision diagram. For example, the classic search problem of Koopman (1946) which involves a single searcher with a continuous detection device looking for a stationary target, continuously visible, located on the real line is characterized by the dotted path.

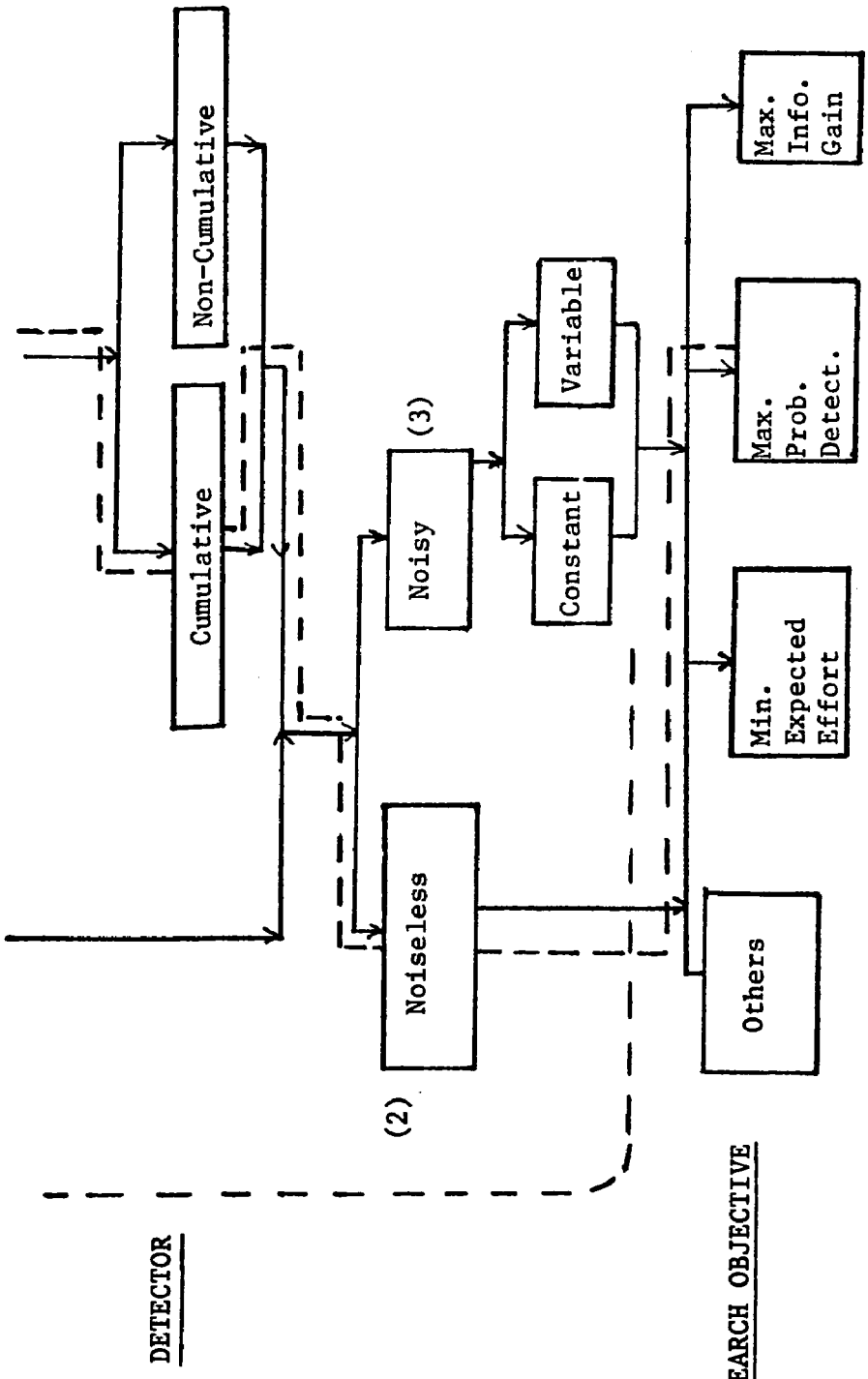
1.2 Literature Review

The author has compiled a reasonably complete review of the open literature in search theory, (Moore, 1970), in an effort to provide a base for the research performed in this dissertation and to indicate fruitful areas of research for other investigators. This section summarizes the results of the review. The following classification scheme will be

---

[1]The chart given is a modification of one originally given by H. Beiman, "An Investigation of Sequential Search Algorithms," Oper. Res. Inc., Silver Spring, Maryland, AD 657050, January, 1967.


SEARCH PROCESS
Continuous
Discrete/cont
Discrete
Single Object
Multiple Objects
Stationary
Non-Stationary
Conscious Evasion
Non-Conscious Evasion
(1)
Continuously Visible
Not-Continuously Visible
Single Detector
Multiple Detectors
Indep.
Depen.
Continuous Detection
Incremental Detection
SEARCH SPACE
SEARCH OBJECTS
DETECTOR



(1) Includes the case of perfect information on the part of the searcher, i.e., searcher is aware of all the strategies available to the target.

(2) Perfect detector, (i.e., no false alarms).

(3) Includes false alarms with the possibility of a variable false alarm rate.

utilized to outline the historical development of what we shall term "classical search theory:"

(A) Non-Conscious Evasion

These papers deal with the problem of determining the optimal allocation of effort to find a target when the probability distribution of the target location is known to the searcher

Two sub-headings are considered under this category.

(1) Stationary targets, i.e., the target is assumed stationary although some authors consider targets that suddenly appear and remain visible.

(2) Moving targets, i.e., the target is moving without conscious evasion and the searcher knows, or is willing to assume, the motion or distribution of motion of the target.

(B) Conscious Evasion

These papers, which usually include game theoretic concepts, consider the search problem with a conscious evader. Included in this category are the search/evasion problems in which the searcher and evader can alter their motions differentially by choices of continuously varying parameters.

(C) Search and the Visibility Process

This section includes papers containing important results, utilized in our work, in the development of search and detection theory and methodology or the application of search concepts to the operations of reconnaissance and surveillance.

1.2.1 Non-Conscious Evasion

Koopman (1946) laid the foundation for the entire field of optimal search and detection. In a series of fundamental papers he described certain basic detection processes and the procedure for the optimum allocation of effort in the search for a stationary target. We shall define this problem and its solution in some detail in order to better understand the later developments in the field, and to make some comparisons with our results.

Let a stationary target be located in a known region A with known complete probability density function p(x,y) continuous over A. It is assumed that the searcher has certain constraints on the amount of effort, Φ, that can be directed towards the search. Let $\phi(x,y)$ be a search density function defined on the region A, with the properties that

$$\iint_A \phi(x,y)dxdy = \Phi \ ,$$

$$\phi(x,y) \geq 0 \text{ on } A \ .$$

If one assumes that the searcher is employing a continuous detector, then the probability of detecting the target with effort ϕ, $P(\phi)$, is given by

$$P(\phi) = \iint_A p(x,y)\left(1 - e^{-\phi(x,y)}\right)dxdy \ ,$$

where $1 - e^{-\phi(x,y)}$ represents the conditional probability of detecting a target at (x,y) with effort $\phi(x,y)$, conditioned on the target being present, and is derived as a consequence of Koopman's Law of Random Search (Koopman, 1946).

The fundamental problem is to determine from the class of functions satisfying the constraints that search density function which maximizes the objective function. Koopman obtained the result:[1]

$$\phi(x,y) = \begin{cases} \log p(x,y) - \frac{1}{\hat{A}} \iint_{\hat{A}} \log p(x,y)\,dx\,dy + \frac{\Phi}{\hat{A}}, & (x,y) \in \hat{A} \\ 0, & (x,y) \in A - \hat{A} \end{cases}.$$

where

$$\hat{A} = \left\{ (x,y) : p(x,y) \geq b \text{ \& } \log b - \frac{1}{\hat{A}} \iint_{\hat{A}} \log p(x,y)\,dx\,dy + \frac{\Phi}{\hat{A}} = 0 \right\}.$$

By considering A to be composed of subregions $A_1, A_2, \ldots, A_N$, one can obtain the solution to the N-region discrete search space problem. Some generalizations suggested by Koopman

---

[1]The solution gives one the optimal allocation of the effort. One is not concerned here with the sequence of allocations.

include the case of visibility varying from position to position, i.e., let v(x) denote the probability that a target in (x,x + dx) is visible, weighting the probability of detection by a function dependent upon where the target is detected, e.g., the detection of saboteurs, and weighting the search density function by a cost function dependent upon the region being searched.

Charnes and Cooper (1958) developed an algorithm for the solution of a discrete search space/continuous detector version of Koopman's problem, i.e., the search region is divided into n subregions with an a priori probability vector on the target position. The algorithm is obtained from the application of the Kuhn-Tucker conditions for optimality to the resulting convex programming problem.

Blackman (1959) considered the following discrete search space variation on the Koopman problem. The target is not present at the beginning of the search, but the searcher has a prior distribution of arrival times, and the objective is not the maximization of the probability of detection, but minimization of the expected time to detect the target after arrival. If the time of appearance of the target is uniformly distributed over a long interval of length T, the author determines the order in which the various possible locations should be scanned to minimize the expected time

between the appearance of the target and its detection[1]. Blackman and Proschan (1959) studied similar problems of objects arriving in accordance with a Poisson process. Having arrived, the objects appear (and remain until detected) in region i with probability p_i. A single scan of the ith region costs c_i, takes time t_i, and, if the object is present at the start of the scan, will detect it with probability $(1-q_i)$. The resultant gain $g_i(t)$ is a non-increasing function of t, the time between the arrival and the beginning of the detecting look. Considering only cyclic search schedules, i.e., those which repeat after D units of time, where D is arbitrary, the authors derive the optimum search schedule.

Gilbert (1959) considers a two-region search problem, with continuous detection and including nonzero switching times, with the objective of minimizing the expected time until detection. His results are summarized in two theorems: The first gives necessary conditions on the optimum solution (sequence and allocations); the second, upper and lower bounds on the value of the objective function. In the special case in which the detection probabilities satisfy the Law of Random Search and are identical, the first theorem will in fact yield the optimal solution.

---

[1]Here the objective is to determine the optimal *sequence*, i.e., a sequential rather than a parallel search problem.

MacQueen and Miller (1960) deal with the problem of whether or not a search activity should be started, and, if started, whether or not it should be continued. Their model gives rise to a general functional equation for which existence and uniqueness conditions are given.

Pollock (1960) introduces a discrete detection model for the two-region search problem and determines the optimal sequential strategies for this model. As the search progresses, the a posteriori probabilities of target position are obtained using Bayes' Theorem. Switching rules are derived for the search sequence which minimizes the expected length of search. The author makes some comparisons of the optimal values of the expected length of the search for discrete and continuous detection models, deriving the conditions for similarity. He also makes the important observation that, *for the models under consideration*, the criteria of (a) maximizing the probability of detection by the end of a fixed time and (b) minimizing the expected length of time until detection; both lead to the same results in terms of the allocation of effort.

The detection processes in Koopman's formulations were restricted to those which satisfied the Law of Random Search; de Guenin (1961) generalized these processes. He made the following assumptions regarding $P(\phi)$, the probability of detecting the target with an effort $\phi(x)$ when the target is at x:

(1) $P(0) = 0$

(2) $\frac{dP(\phi)}{d\phi} = P'(\phi) \geq 0$

(3) $P'(\phi)$ is a decreasing, continuous function of ϕ

(4) $P'(0) > 0$, $P'(\infty) = 0$.

From the above properties, $P'(\phi)$ has an inverse function $\phi = f(P')$.

The basic problem becomes

$$\max \int_{-\infty}^{\infty} p(x)\, P[\phi(x)]dx \ ,$$

$$\text{S.T.} \int_{-\infty}^{\infty} \phi(x)dx = \Phi$$

$$\phi(x) \geq 0$$

where $p(x)$ is the probability density function for the target location. de Guenin's major result is: "whenever the distribution of effort is optimum, the marginal effort required to increase the detection probability at any point is proportional to the probability density, $p(x)$, of the location of the object."

Zahl (1963) considered the following general class of problems:

$$\max \int f[x,y(x)]dx$$

$$\text{S.T.} \int g[x,y(x)]dx = \text{constant}$$

$$a(x) \leq y(x) \leq b(x) \ .$$

He gives necessary and sufficient conditions for a maximizing function y(x) under fairly weak conditions. One may readily interpret the above problem in the search theory context, i.e., with the notation used for de Guenin, take

$$\phi(x) = y(x) \ ,$$

$$f[x,y(x)] = g(x)P[\phi(x)] \ ,$$

$$g[x,y(x)] = \phi(x) \ .$$

The author requires only that the conditional detection function $P(\phi)$ be nondecreasing in $\phi(x)$.

Dobbie (1963) develops sufficient conditions for Koopman's additive property to hold. This property states that the distribution which maximizes the detection probability with a given amount of effort, ϕ, has the interesting property that it is the sum of the optimal distribution of effort ϕ_1 and the conditionally optimal distribution of effort ϕ_2, $(\phi_1 + \phi_2 = \phi)$ given that the target has not been found with the previous distribution of effort ϕ_1. The sufficient condition requires that the conditional detection function be an increasing concave function of the search density

function. The author also points out that one can express the expected effort required to detect the target as

$$\bar{E} = \int_0^\infty Q(\phi)d\phi = \int_0^\infty (1 - P(\phi))d\phi$$

where $P(\phi)$ is the probability of detecting the target with effort distributed according to a particular effort density function. From this equation, one can see that the expected effort, e.g., the expected time to detect the target, is minimized by always distributing the effort to maximize the probability of detection with the effort expended thus far. This suggests, under the conditions of the concave objective function, that the strategies for both problems are identical.

Pollock (1964) develops search strategies to minimize the expected cost of search. The search process is represented in terms of a stochastic dynamic program. The optimal search strategies as well as the associated minimum costs are given. It is shown that the optimal solution is the Wald sequential probability ratio test.

Matula (1964) derives conditions for the existence of an "ultimately periodic" search program in the following context. An object is located in one of a finite number of possible locations with a priori probability p_i . Associated with each location i is a cost for searching that location, c_i, and an overlook probability q_i. The problem is to determine a sequence of locations to be searched such that the

expected cost of finding the object is minimized. A program is called ultimately periodic if after a transient period of length T, the sequence of locations to be searched is repeated with the length of the period denoted by θ. The author obtains the following theorem.

Periodic Search Theorem

A necessary and sufficient condition for the existence of an ultimately periodic optimal program is that the ratios $[\log q_i / \log q_j]$ all be rational.

It is interesting to note that the limiting frequency of search of a location for any optimal program depends only on the overlook probabilities, not on the initial probability distribution on target location or even the relative costs.

Kadane (1968) studies the problem of choosing a strategy to maximize the probability of finding a stationary object in a discrete search space when a budget ceiling is imposed. He also assumes that the overlook probability, a discrete detector is assumed, may depend upon the region and the *number of previous looks*. The major result in this paper is the following extension of the Neyman-Pearson Lemma,

Theorem

Let $\{P_i\}$ and $\{C_i\}$ be arbitrary nonnegative sequences such that $\Sigma P_i < \infty$. Let X be the class of sequences X_i, such that $0 \leq X_i \leq 1$, $\forall_i$, then the maximum of

$$\sum X_i P_i$$

$$\text{S.T.} \quad \sum X_i C_i \leq C$$

$$X_i \in X$$

is attained. It occurs when and only when

$$X_i = \begin{cases} 0, & P_i < rC_i \\ 1, & P_i > rC_i \end{cases}$$

for some r, $0 < r < \infty$, and $\Sigma X_i C_i = C$.

The author describes an integer programming algorithm (branch and bound variety) adapted to the above problem. The ratio P_i/C_i is the appropriate decision variable in the sense that searches with large P_i/C_i should be included and those with small P_i/C_i excluded. P_i/C_i plays the role of a cost-effectiveness criterion. Zahl (1963) gave the continuous analog of these results. Black (1965) presents a graphical argument for the optimal sequential search procedure for the minimum cost problem. He shows that the policy with the minimum expected cost is generated by the rule: "Always look in the region for which the posterior probability (given the failure of earlier looks) of finding the object divided by the cost is maximum." Chew (1967) gives the following optimal strategy: to maximize the probability of finding the object in a fixed number, N, of searches, choose those

N searches for which the posterior probability (given the failure of earlier glimpses) of finding the object is largest. It is assumed that the overlook probability does not depend upon the number of previous searches.

Koopman (1946) was the first to examine situations in which a search is to be conducted for a moving target. Among these are the barrier patrol search procedures for a target moving through a straight channel with the vector velocities at all points parallel and equal. Another situation studied is that in which an initial distribution of target location is given at the time of fix, but with the target moving in a random direction with an estimated constant speed. The objective is the construction of a search after a large amount of time has elapsed from the time the target is fixed to the initiation of the search. It is shown that in order to obtain the maximum probability of detection per unit time, the ideal track for a "cookie cutter" detector is an equiangular or logarithmic spiral which is approximated by the "retiring square search procedure."

Klein (1968) considered the following moving-target problem. An object moves about within a finite number of regions, one per time unit, according to the transition matrix

$$H = \{h_{ij}\}$$

where i denotes the searcher's current location and j the target's next, (obviously the target discovers the searcher's location at the end of each period). A single searcher, using a continuous detection system whose effectiveness is a function of the amount of effort used and the region searched, checks one region at a time until the object is found, his budget exhausted, or he decides it is "uneconomical" to continue. The problem is to find an optimal sequential search policy, i.e., one which tells the searcher, at each point in time, whether to search, where to search, and how much effort to use, for the following problems:

(a) Minimize the expected cost subject to achieving a specified level of the probability of detection.

(b) Minimize the expected time until detection subject to achieving a specified level of the probability of detection, and an upper bound on the budget.

(c) Maximize the probability of detection subject to upper bounds on the expected duration of the search and the expected cost.

The author doesn't solve these problems, but suggests that certain of these formulations can be transformed into linear programming problems.

Pollock (1970) considers a target moving in a Markovian fashion between two regions. Dynamic programs for the

standard problems of minimizing the expected time until detection and maximizing the probability of detection under a constraint on search effort are solved. For certain special forms of the transition matrix, decision rules are derived for the minimum expected time problem as well as upper and lower bounds for the minimum expected time.

Wagner (1969) develops the following theorem which is applicable to either continuous or discrete search space problems.

Theorem

Let e and c be real-valued functions (of two variables) defined on

$$\{(x,j) \mid a < x < b,\ \ell(x) \leq j \leq u(x),\ j \text{ an integer}\} .$$

Let ϕ be the set of all integer-valued functions f on (a,b) such that $\ell(x) \leq f(x) \leq u(x)$ for $a < x < b$ for which

$$-\infty < E(f) = \int_a^b e(x,f(x))dx < \infty$$

$$-\infty < C(f) = \int_a^b c(x,f(x))dx < \infty .$$

Suppose that $g \in \phi$ has the following property: there exists a $\lambda > 0$ such that for all $x \in (a,b)$ and integers j

$$e(x,j) - e(x,j-1) \leq \lambda[c(x,j) - c(x,j-1)] \text{ whenever}$$

$$g(x) < j \leq u(x)$$

$e(x,j) - e\ x,j-1) \geq \lambda[c(x,j) - c(x,j-1)]$ whenever

$$\ell(x) \leq j - 1 < g(x)$$

then

$$E(g) = \max \{E(f) \mid f \in \phi \text{ and } C(f) \leq C(g)$$

$$C(g) = \min \{C(f) \mid f \in \phi \text{ and } E(f) \geq E(g)$$

Onaga (1971) studied the problem of the minimization of the expected time until detection for general detection functions. He also includes penalty or switch times which occur whenever the searcher changes regions. The major results of the paper are summarized in two theorems. The first is a necessary condition for optimality and is applicable to either a continuous or discrete detection function, $P(\phi)$. It consists of two characterizations: the first regulates the optimal lengths of visit times and the second determines the optimal search order. The second result is a necessary and sufficient condition stated in constructive form which is applicable to unimodal probability density functions $\rho(\phi)$. Of special importance to the results of our work is the following theorem.

Theorem 7

If the density function, $\rho(\phi)$, is unimodal, one can use the minimal concave majorant of $\rho(\phi)$ for obtaining the optimal policies.

1.2.2 Conscious Evasion

Norris (1962) appears to have been among the first to consider the two-sided search problem. The search is conducted against a conscious evader who is able to observe the searcher's actions and capitalize on any errors he makes. The evasion device of moving between looks is treated. The game is zero-sum and incorporates a fairly general reward structure which can include discounting. The reward coefficients associated with this structure, as well as the detection probabilities, are known to both players. Three levels of moving costs for the evader are considered for a two-region search problem. In the case of an infinite moving cost, the author derives a condition which is a special case of the Periodic Search Theorem of Matula (1964). The searcher's "good strategy" in the case of a finite moving cost is generated by a finite Markov process. Finally, when no moving cost is incurred by the evader, the searcher cannot gain any information concerning the evader's position from his past sequence of unsuccessful looks. Therefore, each look should be made according to the same probability distribution. In the N-region formulation of the finite moving cost game, a "good search" strategy cannot be generated by a finite Markov process.

Koopman (1968) extended his original work to the two-sided search situation. It is shown that a uniform distribution

of search effort on the part of the searcher and a uniform position density on the part of the target form the optimal strategies for the resulting zero-sum game.

Neuts (1963) develops, among other things, stationary minimax strategies for a multistage search game. The objective is the minimization of the expected discounted return to the target. A stationary strategy for the hunter is an n-tuple

$$Y = (Y_1, \ldots, Y_N) \; ; \; Y_i \geq 0, \; \sum_{i=1}^{N} Y_i = 1 \; ,$$

which denotes a probability distribution, chosen once, and by which the region to be examined at each stage is selected. It is shown that the optimal stationary policy for the searcher is independent of all parameters except the discrete detection probabilities, a result which also holds for the objective of minimizing the expected duration of the game. Such stationary minimax strategies correspond to the following cases:

(a) a memoryless searcher,

(b) the target is allowed to move after each region is searched.

Charnes and Schroder (1967) develop models and methods to find optimal tactics in an idealization of antisubmarine warfare, viewed as a game of pursuit between the hunter force

and a submarine. The objective function of minimizing the expected duration of the search can be expressed as a stochastic game. The solution of this game is accomplished by solving a sequence of linear programming problems. In the event the hunter knows the behavior of the submarine, the game becomes a one-person game and may be treated as a discounted Markovian decision process of the type studied by Howard (1960).

1.2.3 Search and the Visibility Process

As noted earlier, Koopman (1946) briefly examined the situation in which the probability of target presence is modified by the inclusion of the probability that the target is visible given that it is present.

Stollmack (1968) determined by both field and laboratory experimentation that the exponential detection function with constant rates is a valid model of visual detection. The study centered on the visual detection of tanks by experienced personnel in the terrain surrounding Fort Knox, Kentucky. The detection rates obtained by Stollmack were shown to be statistically dependent upon range and background (i.e., the number of confusing forms, ruggedness of the terrain, etc.). The empirical relationships indicated that at mid to high ranges the detection rate is as sensitive to changes in the background as it is to changes in range. It was shown that differing detection rates and

visibility conditions were the rule rather than the exception over the local terrain.

Bonder (1970) and Disney (1970) formulated descriptive models of the situation in which the target visibility changes over time. Bonder considered the situation in which the target and the searcher (detector) may not be continuously visible during the period of time in which the searcher is examining the subregion containing the target. The searcher has a detection capability only when the target is visible.

The author considered the following situations within each search area:

(a) The target may be visible to the searcher for the entire search interval with some known probability p (this is the Koopman suggestion),

(b) the target may be visible at the start of the search period, the length of the visible period being a random variable with known probability density function, and not reappear,

(c) a single period of visibility may be exhibited starting at some random time during the search interval and lasting a random amount of time.

In each of these cases it is assumed that the target is stationary in the sense that it remains in a given region, although its motion within a region may give rise to the visibility process. These models are the first to interface the visibility and detection processes. In each of these cases, the probability density functions for the time until detection, the time spent searching the area until a fixed

number of detections occur, and the time spent searching the total area, including switching times, are derived.

Disney described the visibility process in which the target alternates between visible and invisible states as an alternating renewal process. He did not consider the interface between the visibility and the detection processes. The transition matrix for this process is

$$\begin{array}{c} \\ \text{Vis.} \\ \\ \text{Masked} \end{array} \begin{array}{c} \begin{array}{cc} \underline{\text{Vis.}} & \underline{\text{Masked}} \end{array} \\ \begin{pmatrix} 0 & f_1(t) \\ f_2(t) & 0 \end{pmatrix} \end{array}$$

where $f_1(t)$ is the probability density function for the time in the visible state and $f_2(t)$ the probability density function for the time in the masked state.

Employing some renewal theory arguments, the author obtained among other things:

(a) $\pi_1(t)$, the probability that the target is visible in $(t, t + dt)$,

(b) for a fixed time interval of length τ_d, the distribution of:
   (1) the number of times the target is visible,
   (2) the total time of visibility.

1.2.4 Summary and Conclusions from the Literature Review

The review of the open literature on search and reconnaissance theory indicated that the bulk of the research activity has dealt with purely mathematical extensions of the

work of Koopman. In nearly every case, no attempt was made to relate the mathematical models to real world situations. It seems apparent that more effort should be directed toward the study of search models which afford more realistic descriptions of search scenarios.

A recent research report (Vector (1970)) has noted the importance of modeling the effects of terrain on combat processes. Since the reconnaissance activity is a prelude to such activities, one should also be concerned with the effects of terrain on the search activity, i.e., how does the inclusion of the visibility process effect the optimal search strategies and returns?

Realistically, the optimization criteria should depend upon the objective of the operation. Research should be devoted to the structuring of the total activity, which includes search, detection, tracking, and ensuing action, before selecting the optimization criteria.[1]

Research is required to understand the behavior of operationally useful devices, e.g., the effect of multiple scans, independence between successive looks, coupling of various types of detectors, etc. In general, a study of the structure and capabilities of operational detectors is required.

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[1]The relationship between search and ensuing action is being studied under this contract and is being published as a separate report (Kronz, 1971).

The output of many of the "classical" search studies has been a fixed amount of time to spend searching a region. The likelihood that searchers will not (or cannot) follow optimal search procedures suggests research be devoted to the problem of converting theoretical results into practical rules of application.

The open literature on search and reconnaissance is virtually vacuous in the important areas of multiple detectors and/or multiple targets with varying degrees of dependency within each group. Clearly all of the topics outlined in the previous paragraphs are of interest when viewed in the context of multiple detectors and/or targets.

1.3 Research Area

This dissertation addresses the problem of characterizing the interaction between the detection and visibility processes. The work of Bonder (1970) and Disney (1970) is extended and the resultant models analyzed to determine strategies for the optimal allocation of search effort. Comparisons with existing search theory strategies are made to determine conditions under which different strategies, which consider the visiblity process effects, are required. The models are further analyzed to gain insight into the intelligence-gathering process.

Specifically, the problem considered is that of searching for a stationary target located in one of N regions (discrete search space) with prior probability vector $P = (p_1, \ldots, p_N)$. Analogous formulations for the continuous search space are discussed in an appendix. Although the solution procedures and (we believe) the results carry through for the N region case, explicit policies and returns are given only for the two-region situation to facilitate both the analysis and interpretation of results. We principally consider the case of a continuous detector. Discrete detector formulations and associated optimal policies also are contained in an appendix. Within the region in which the target is located, it is assumed that during a given period of time in which a search for the target is underway, the target may exhibit one of the following types of behavior as indicated in Bonder (1970) and Disney (1970).

(a) The target may be visible to the searcher for the entire interval with some fixed probability v. This case is also referred to as the "binary" intervisibility process in the following discussions.

(b) The target may be visible at the start of the search period, the length of the visible period being a random variable with known probability density function.

(c) A single period of visibility may be exhibited, starting at some random time during the search interval and lasting a random amount of time.

(d) During the searching interval, the target may exhibit alternating periods of visibility and invisibility, the durations of each being random variables.

Physically, these visibility structures can be considered in the following contexts:

(1) Search for a target assigned to one region in which the local terrain, foliage, weather, etc., contributes to masking effect.

(2) Search for a submarine in which environmental conditions between the surface searching vessel and the target generate the visibility periods, e.g., thermal barriers to sonar detection devices.

(3) In the case of a single interval of visibility, the random length of this period could be considered the length of time until the target discovers the presence of the searcher in his region.

(4) The multiple periods of visibility could also reflect the target's strategy with regard to exposing himself in order to obtain information, supplies, etc.

(5) The multiple periods of visibility might correspond to the periods a raiding party of guerillas, operating from a neutral or safe zone, spends in a region in which they are susceptible to detection.

(6) Searching for schools of fish which periodically submerge to depths which preclude their detection.

(7) The situation in which a data bank is simultaneously accessed by many users. When one user has access to certain information, the others are temporarily denied access.

(8) The problem of detecting a disease, the symptoms of which exhibit intermittent remission, may be of interest.

(9) The time until a crime, e.g., a robbery, may be considered a random variable, as well as the length of time required to carry out the act.

(10) The general formulation of the search problem can be interpreted readily in a research and development context, i.e., one interprets the probability of detection as the probability of "discovery." However, the single interval of visibility with a random start time lends itself to some interesting interpretations in the research and development context. In many research situations, one may have some random period of time, effort, etc., in which the probability of discovery is essentially zero. Of course, there may be some alternatives for which this "settling in" period is unnecessary. Given that one attains the end of the "settling in" period, one may then estimate, based perhaps on current market conditions, the probability density function for the time required to complete the job in order that the end product will be competive, timely, etc.

Although (d) essentially includes structures (a)-(c) of the visibility process, the descriptive models, associated allocation strategies, and analyses of the models are presented in order of increasing complexity for pedagogical reasons at the expense of increasing the quantity of

information, since the research was performed in this fashion.[1] Because of this writing approach, the principal new results in search theory are not encountered until the latter part of Chapter 3 and thereafter.

Although the current research was motivated by the desire to learn more about the effects of the visibility processes on reconnaissance in the military context, a number of non-military interpretations of the visibility processes may be made as well.

[1] Morris (1967), Bonder (1971), and Wagner (1971) have indicated that a principal capability needed, but usually not developed, in new analysts is the process of model development. Writing the paper in this fashion will indicate "how" it is done, in contrast to presentation of just the end product which typically appears in journal articles.

Chapter 2

THE BINARY VISIBILITY MODEL

The Binary Visibility model is described and analyzed in this chapter. It assumes that, upon the entry of the searcher into a region containing the target, the target is visible to the searcher for the duration of the search with some fixed probability which may be regionally dependent. The problem of maximizing the probability of detection under a constraint on the available search time is studied in detail. Optimal solutions to this problem are obtained for the situation in which the searcher utilizes a continuous detection device. Comparisons are then made between these optimal policies and those for which the Binary process is ignored (the "Koopman" models). A sensitivity analysis of both the optimal policy and its return is carried out. The objective of minimizing the expected time until detection is also examined.

2.1 Description

We consider only the discrete search space version of this problem throughout this disseration. The discrete search space version of the problem of maximizing the probability of detecting the target is

$$\max \sum_{i=1}^{N} p_i v_i P_i(t_i)$$

$$\text{S.T.} \sum_{i=1}^{N} t_i \leq T \quad , \tag{1}$$

and

$$t_i \geq 0 \quad ,$$

where

p_i = probability that the target is in the ith region,

v_i = probability that, in the ith region, the target and the searcher are intervisible for the entire period,

$P_i(t)$ = the conditional probability of detecting a target in the ith region at or before time t.

As noted in Chapter 1, Koopman (1946) gave a mathematical justification (The Law of Random Search) for a detection function of the form

$$P_i(t) = 1 - e^{-k_i t} \tag{2}$$

where k is the conditional detection rate.

More recently, Stollmack (1968), studying visual detection, established the validity of the above expression via experimental data. Therefore, (2) is used as the standard form for a continuous detection device. Observe that (2) is a concave function of search time t. Hence, upon substitution of (2) into (1), one obtains a concave maximization problem.

2.2 *Allocation of Effort to Maximize the Probability of Detection*

2.2.1 Model Solution

In this section, we shall develop the optimal policy for allocating search time for the Binary model given by (1) and (2) of 2.1. The reader is referred to Appendix A for the proof of the modified Charner-Cooper Algorithm which can be used to determine the optimal allocation $(t_1, \ldots, t_N)$ for the N-region problem. We shall discuss only the attributes of the 2-region situation for the purposes of economy of notation and the inherent symmetry of the situation.

In Appendix A, we develop the rule for determining the order in which regions begin to receive search effort (First Allocation Rule [FAR]). For this model the FAR may be stated:

Choose the region j for which

$$p_j k_j v_j = \max_{1 \le i \le N} \{ p_i k_i v_i \} .$$

Equation 8 of Appendix A suggests that all effort be placed in region 1 (assuming $p_1 k_1 v_1 \ge p_2 k_2 v_2$) when $T < T^{**}$, where T is the total available search time and

$$T^{**} = \frac{1}{k_1} \ln \left(\frac{p_1 k_1 v_1}{p_2 k_2 v_2} \right) . \qquad (3)$$

For $T \ge T^{**}$, and using an analog of (9) in Appendix A, one obtains the following for the optimal allocations to the two regions,

$$t_1 = \frac{1}{k_1 + k_2}\left\{\ln\left(\frac{p_1 k_1 v_1}{p_2 k_2 v_2}\right) + k_2 T\right\},$$

$$t_2 = \frac{1}{k_1 + k_2}\left\{\ln\left(\frac{p_2 k_2 v_2}{p_1 k_1 v_1}\right) + k_1 T\right\}. \qquad (4)$$

The optimal value for the probability of detection is then given by[1]

$$P(T) = p_1 v_1\left[1 - \left[\frac{p_1 v_1 k_1}{p_2 v_2 k_2}\right]^{-\frac{k_1}{k_1+k_2}} e^{-\left(\frac{k_1 k_2}{k_1+k_2}\right)T}\right] + p_2 v_2\left[1 - \left[\frac{p_2 v_2 k_2}{p_1 v_1 k_1}\right]^{-\frac{k_2}{k_1+k_2}} e^{-\left(\frac{k_1 k_2}{k_1+k_2}\right)T}\right]. \qquad (5)$$

2.2.2 Comparison with the Koopman Model

In this section, we examine the situation in which a searcher, being aware of the earlier results of Koopman (which assume continuously visible targets), applies them to situations in which the target is not continuously visible because he is (a) aware of the visibility process but unwilling or unable to obtain estimates of the visibility parameters, or (b) unaware of the visibility process. We utilize the classical Koopman model since most efforts in search

[1] Formulations and results for discrete detections are given in Appendix B. Continuous search space versions of the model are discussed in Appendix C.

theory have been essentially embellishments of it and such comparisons will provide landmarks regarding the robustness of the classical models as guides for search decision makers. The following paragraphs, which form a conceptual summarization of the searcher's position in this respect as well as the options available to him, are introduced only for illustrative purposes.

First we assume that the searcher can obtain estimates of the visibility parameters at some additional cost (e.e., weather studies, terrain analysis, etc.). In this situation, the searcher may take the following actions:

(a) Allocate the search effort according to the Koopman scheme (without obtaining any estimates on the visibility parameters).

(b) Spend additional funds to obtain the required estimates on the visibility parameters).[1]

(c) Expend additional search effort under the Koopman policy in an attempt to make up for the defects resulting from using the "Koopman" policy.[2]

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[1] One might assume that various levels of spending yield varying degrees of accuracy in the parameter estimates. In this situation, it is of interest to determine the sensitivity of the optimal allocations and returns to changes in the visibility parameters.

[2] This will be shown to be a meaningful alternative.

The outcomes of such actions are, respectively:

(a) A difference in the probability of detecting the target under a constraint on the total search time resulting from the use of the nonoptimal policy, denoted as ΔP.

(b) By obtaining the estimates of the visibility parameters, he can apply the optimal policy. A measure of the effectiveness of this option is given by $\Delta P/C_E$, where C_E is the cost of obtaining the estimates of the visibility parameters.

(c) The additional search effort needed to achieve the optimal return has an associated cost C_T. The gain from such an action is computed from the function which expresses the difference in the optimal return under the two policies, E(T), as a function of the total available search effort, T. Note the gain from this action will be computed from the changes in return under the additional effort allocated via the Koopman policy (we label this gain $\Delta P(T)$); however, the costs, C_T, may be less than C_E. A measure of the effectiveness of this action is given by $\Delta P(T)/C_T$.

In the situation in which the searcher is unable to obtain estimates of the visibility parameters or is unaware of the visibility process, the searcher may take the following actions:

(a) (Unaware) Allocate the search effort according to the Koopman scheme.

(b) (Unable) Expend additional search effort.

The consequences of such actions are described by outcomes (a) and (c), respectively, of the first case.

The solution for the Koopman problem is given by (4), with $v_i = 1$, for every i. One also observes that in these equations if $v_1 = v_2 \neq 1$, then the optimal allocation for the Binary model is identical to that of the Standard Koopman Allocation (SKA); hence, no error results from the use of the SKA in the Binary model. (The result is also true in the n-region situation, see Appendix A.)

Since we have closed-form expressions for both allocation procedures, they can be substituted into the objective function, to form the difference function

$$E(T) = P_B(T) - P_{SKA}(T). \tag{6}$$

$$E(T) = e^{-\frac{k_1k_2}{k_1+k_2}T}\left\{ p_1v_1\left(\frac{p_1k_1}{p_2k_2}\right)^{-\frac{k_1}{k_1+k_2}}\left[1 - \left(\frac{v_1}{v_2}\right)^{-\frac{k_1}{k_1+k_2}}\right] + p_2v_2\left(\frac{p_2k_2}{p_1k_1}\right)^{-\frac{k_2}{k_1+k_2}}\left[1 - \left(\frac{v_2}{v_1}\right)^{-\frac{k_2}{k_1+k_2}}\right]\right\}.$$

We note that $\lim_{T\to\infty} [P(T) - P_{SKA}(T)] \to 0$. Equation 6 only holds where, under both policies, both regions are receiving positive allocations of search effort. In the event this situation doesn't hold, one has four cases to consider.

The switch points (the values of total available search effort, T, at which the second region begins to receive some effort) are given by:

(a) for SKA

$$T^* = \frac{1}{k_1} \ln \frac{p_1 k_1}{p_2 k_2}$$

(b) for the Binary model [7]

$$T^{**} = \frac{1}{k_1} \ln \frac{p_1 k_1}{p_2 k_2} + \frac{1}{k_1} \ln \frac{v_1}{v_2} \quad .$$

If it is assumed that Region 1 is selected first by both policies, the following situations can arise as a result of the difference in switch points:

(a) $T < T^* \leq T^{**}$ the error is zero, since under both policies all the effort is placed in Region 1.

(b) $T^* < T < T^{**}$ under SKA the effort is divided according to equation 4, while under the Binary allocation all effort is still placed in Region 1, resulting a nonzero error.

(c) One could have $v_1 < v_2$, in which case $T^{**} \leq T < T^*$ and the reverse of (a) and (b) would occur, again the error term is nonzero.

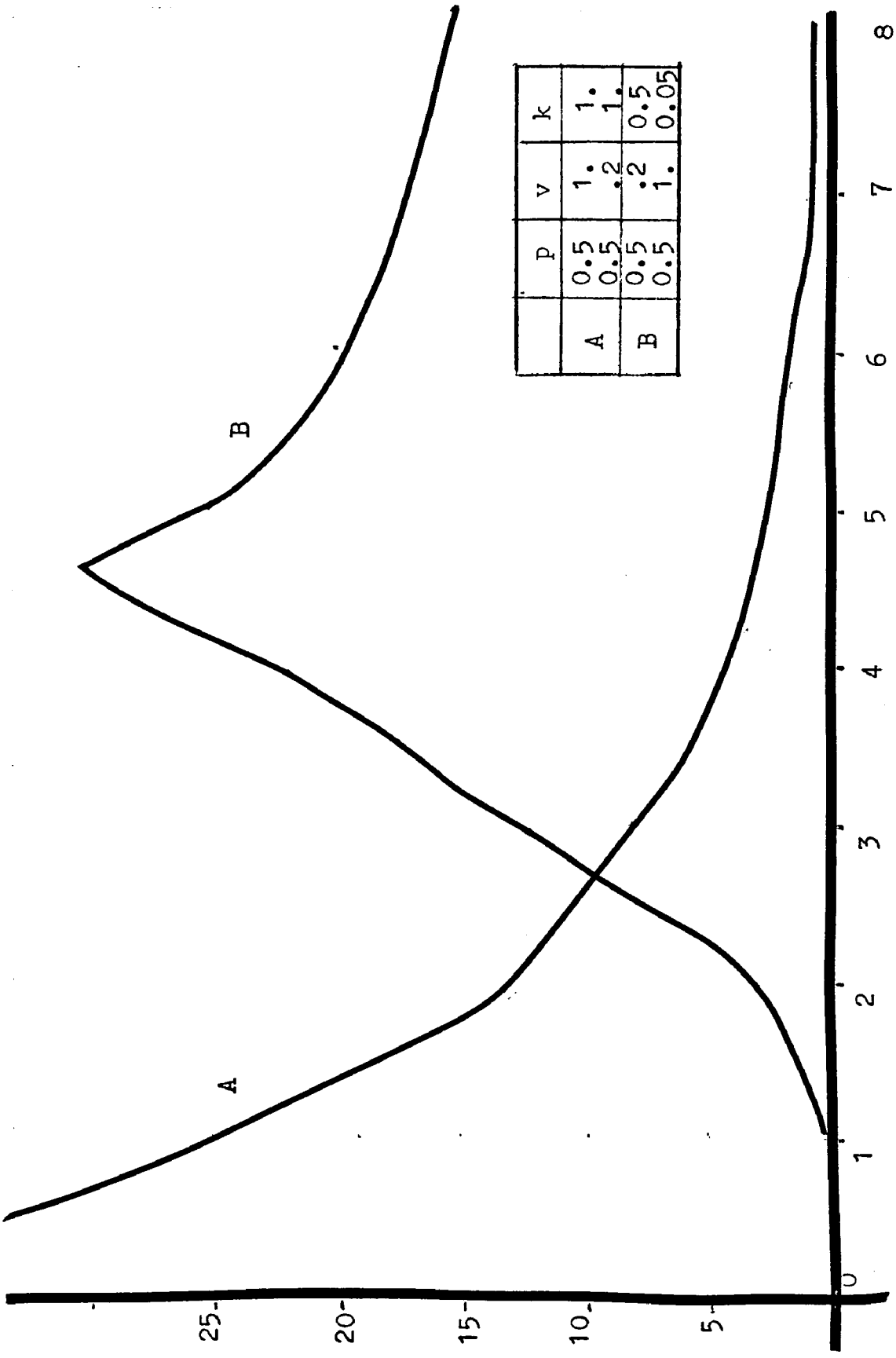
(d) $T^* < T^{**} \leq T$, one obtains equation 6 as the error expression.

Suppose $p_1k_1 > p_2k_2$, but $p_1k_1v_1 < p_2k_2v_2$, i.e., the infinitesimal detection rate is largest in the first region; however, the probability that the target is visible is much higher in the second region. In this situation the SKA selects Region 1 and allocates all effort to that region until $T > T^*$, while the optimal policy for the Binary model selects Region 2 and allocates all effort (T) to that region until $T > T^{**}$. In Figures 1 and 2 we illustrate the above points, by plotting the percent relative error, $E(T)/P_B(T)$, in the probability of detection using the Koopman policy versus the total available search time. Figure 1 contains examples where both policies begin searching the same region. One may note that the peak percent relative error appears to occur at the switch point for the Koopman model, T^*. The peak error functions and their times of occurrence will be discussed later in this section.

Figure 2 contains examples in which the Koopman policy results in the wrong region being searched. These are situations in which the total available search is highly constrained. This suggests that, in such situations, the decision maker must be aware of the visibility probabilities for the search space in order to conduct an effective search.

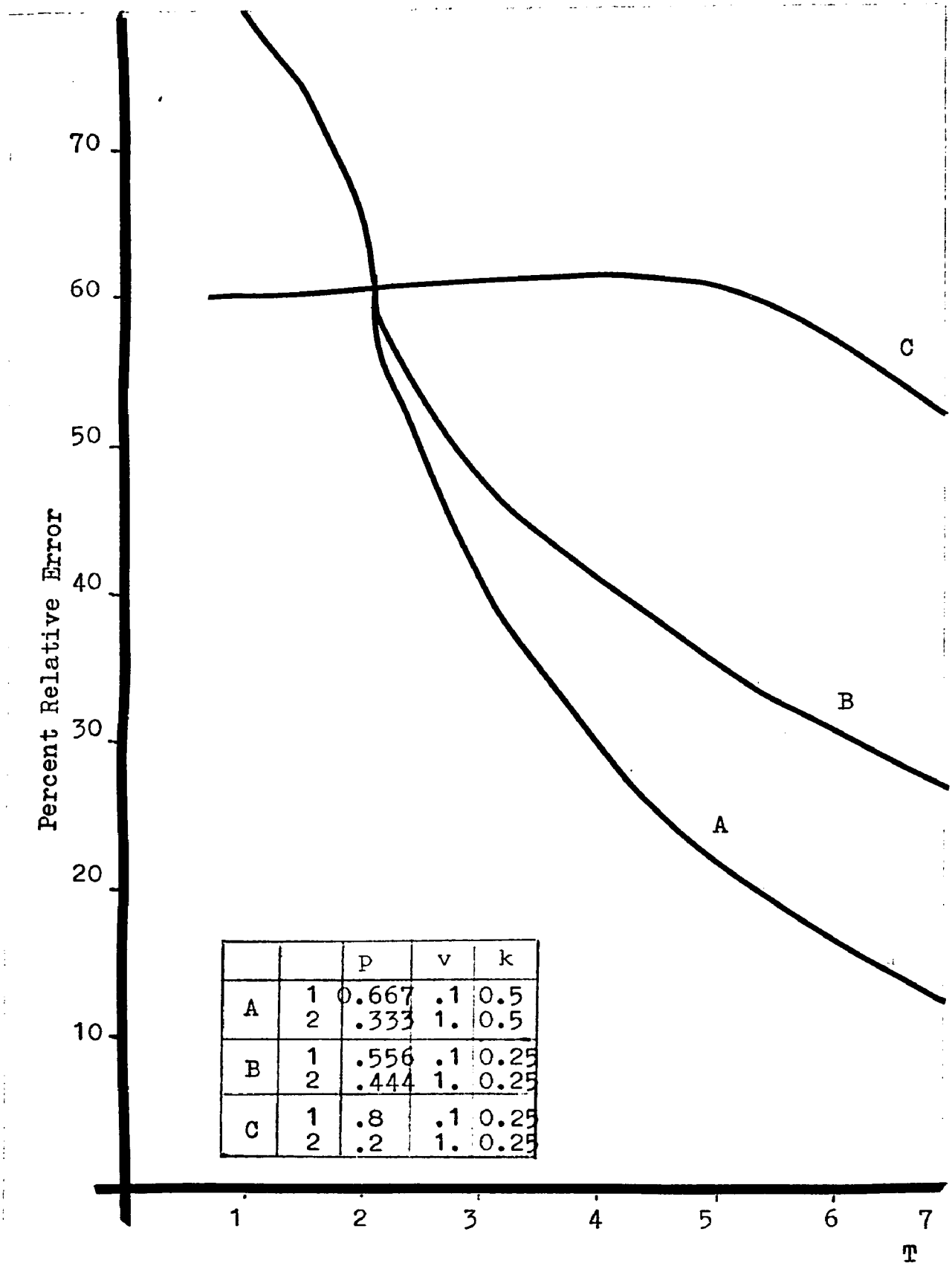
An analysis of the error function, E(T), was undertaken to determine:

(a) At what levels of total available time (T) the peak differences occur.



	p	v	k
A	0.5 0.5	1. .2	1. 1.
B	0.5 0.5	.2 1.	0.5 0.05

Figure 1 - Percent Relative Error Versus Search Time



		p	v	k
A	1	0.667	.1	0.5
	2	.333	1.	0.5
B	1	.556	.1	0.25
	2	.444	1.	0.25
C	1	.8	.1	0.25
	2	.2	1.	0.25

Figure 2 - Sensitivity of the Error Function

(b) What determines their shape, i.e., height, rate of decay, etc.

For the two-region Binary Search Model, there are essentially four sets of parametric relationships, since one need only interchange the labels to obtain the others. These relationships are outlined below.

For the SKA we assume that

$$\frac{p_1}{p_2} \geq \frac{\frac{1}{k_1}}{\frac{1}{k_2}} ,$$

Case I: $p_1 k_1 v_1 \geq p_2 k_2 v_2$, $(v_1/v_2) \geq 1$, and $T^{**} \geq T^*$

Case II: $p_1 k_1 v_1 \geq p_2 k_2 v_2$, $(v_1/v_2) < 1$, and, $T^* > T^{**}$

CaseIII: $p_1 k_1 v_1 \leq p_2 k_2 v_2$, $(v_1/v_2) < 1$, and $T^* \leq T^{**}$

Case IV: $p_1 k_1 v_1 \leq p_2 k_2 v_2$, $(v_1/v_2) < 1$, and $T^* > T^{**}$.

Expressions for the peak error functions and the level of search time at which the peaks occur are developed and analyzed below.

Case I

In the first situation, the peak difference occurs at

$$T_p = T^* - \frac{(k_1 + k_2)}{k_1^2} \ln \left[\frac{k_2}{k_1 + k_2} + \left(\frac{v_2}{v_1}\right) \frac{k_1}{k_1 + k_2} \right],$$

substituting this expression into the error function gives

$$E(T_p) = p_1 v_1 \left(\frac{p_1 k_1}{p_2 k_2}\right)^{-1} \left[1 - \left(\frac{k_2}{k_1 + k_2} + \left(\frac{v_2}{v_1}\right) \frac{k_1}{k_1 + k_2}\right)\right]\left(\frac{k_2}{k_1 + k_2}\right.$$

$$+ \left(\frac{v_2}{v_1}\right) \frac{k_1}{k_1 + k_2}\Bigg)^{k_2/k_1} - p_2 v_2 \left[1 - \left(\frac{k_2}{k_1 + k_2}\right.\right.$$

$$+ \left.\left.\left(\frac{v_2}{v_1}\right) \frac{k_1}{k_1 + k_2}\right)^{k_2/k_1}\right] .$$

Case II and IV yield the following results

$$T_p = T^* = \frac{1}{k_1} \ln \left(\frac{p_1 k_1}{p_2 k_2}\right) ,$$

and substitution into the error function gives

$$E(T_p) = p_2 v_1 \left(\frac{k_2}{k_1}\right)\left[1 - \left(\frac{v_2}{v_1}\right)^{\frac{k_1}{k_1+k_2}}\right] + p_2 v_2 \left[1 - \left(\frac{v_2}{v_1}\right)^{\frac{k_2}{k_1+k_2}}\right]$$

The peak error is identical for Cases II and IV; however, the shapes of the error functions are different since in Case IV the Koopman policy starts in the wrong region, while in Case II both policies start in the same region.

Case III

In the third case, some subcases must be examined: for

$$\left(\frac{k_2}{k_1} + 1\right) \ln\left(\frac{p_1 k_1}{p_2 k_2}\right) < -\left(1 + \frac{k_1}{k_2}\right) \ln\left[\frac{k_1}{k_1 + k_2} + \left(\frac{v_1}{v_2}\right) \frac{k_2}{k_1 + k_2}\right]$$

$$T_p = \frac{1}{k_2} \ln\left(\frac{p_2 k_2}{p_1 k_1}\right) - \frac{k_1 + k_2}{k_2^2} \ln\left[\frac{k_1}{k_1 + k_2} + \left(\frac{v_1}{v_2}\right) \frac{k_2}{k_1 + k_2}\right]$$

and

$$E(T_p) = v_2 p_1 \frac{k_1}{k_2}\left[\frac{k_1}{k_1 + k_2} + \left(\frac{v_1}{v_2}\right) \frac{k_2}{k_1 + k_2}\right]^{k_1/k_2}$$

$$\cdot \left\{1 - \left[\frac{k_1}{k_1 + k_2} + \left(\frac{v_1}{v_2}\right) \frac{k_2}{k_1 + k_2}\right]\right\}$$

$$- p_1 v_1 \left\{1 - \left[\frac{k_1}{k_1 + k_2} + \left(\frac{v_1}{v_2}\right) \frac{k_2}{k_1 + k_2}\right]^{k_1/k_2}\right\},$$

In the event this condition doesn't hold,

$$T_p = T^* = \frac{1}{k_1} \ln\left(\frac{p_1 k_1}{p_2 k_2}\right),$$

and

$$E(T_p) = p_2 v_2 \left[1 - \left(\frac{p_1 k_1}{p_2 k_2}\right)^{-k_2/k_1}\right] - p_1 v_1 \left[1 - \left(\frac{p_1 k_1}{p_2 k_2}\right)^{-1}\right].$$

It can also be shown that the peak error, at least in the two-region case, will always occur at or before the larger of (T*,T**) as follows. One need only observe that for T > max (T*,T**) the error function given by equation 6, is a decreasing function of the total available effort T. This implies that in general the maximum error appears in situations in which the search effort is highly constrained.

Having closed-form expressions for the peak error functions, one can conceptually take partial derivatives to determine the sensitivity of these functions to changes in (p_i, v_i, k_i). However, this is algebraically messy. A straightforward numerical search for extrema over the entire search parameter space was undertaken and produced the results summarized in Table 1. From Table 1, it is apparent that the critical factor is the ratio of v_1/v_2, e.g., for areas having homogeneous visibility (equal visibility parameter values) the peak differences are small, suggesting the adequacy of the Koopman policy. Note that, in general, one obtains greater errors whenever the ratios of the prior probabilities and the visibility probabilities are opposite in magnitude, suggesting, in the case $k_1 \simeq k_2$, that the peak error results from an erroneous selection of the initial region searched.

A secondary analysis produced differences in the probability of detection in the range (.12, .17) for the following

Table 1

Maximum Differences in the Probability of Detection

$p_1/p_2 = .1$

k_2/k_1 \ v_1/v_2	.1	1.	10.	100.
.1	.027	0.	.045	.063

$p_1/p_2 = 1.$

k_2/k_1 \ v_1/v_2	.1	1.	10.	100.
.1	.059	0.	.025	.035
1.	.101	0.	.10	.1225

$p_1/p_2 = 10.$

k_2/k_1 \ v_1/v_2	.1	1.	10.	100
.1	.0106	0.	.0048	.006
1.	.0425	0.	.018	.0225
10.	.048	0.	.027	.031

$p_1/p_2 = 100.$

k_2/k_1 \ v_1/v_2	.1	1.	10.	100.
.1	.0118	0.	.0005	.0007
1.	.0468	0.	.002	.0024
10.	.0644	0.	.003	.0034
100.	.065	0.	.003	.0036

values of the search parameters

$$1 \leq p_1/p_2 \leq 3,\ .01 \leq v_1/v_2 \leq .1,\ .5 \leq k_1/k_2 \leq 1.5\ .$$

Finally, we note that the above analysis deals only with the peak differences and yields no information about the rate of decay of the error function with increasing available effort.

Table 2 presents some additional measures of the adequacy of using the Koopman policy when in fact the target is not continuously visible. The measures presented in Table 2 consist of:

(a) The expected time to detect using the Koopman strategy and model to allocate an unlimited amount of effort, labeled KET (Appendix A). It is conjectured that a searcher unaware of the visibility process may use such a measure to characterize a given search situation.

(b) Since in the Binary Model the ultimate probability of detection is less than unity, one can only make comparisons on the basis of the *conditional* expected times to detect, conditioned on detection occurring. Table 1 contains the ratio of these conditional expected times until detection. The numerator is computed using the Koopman strategy in the Binary model situation while the denominator is computed from the optimal policy for the Binary Model. (See 2.3.)

(c) Given that one is using the Koopman strategy in the context of a Binary search situation, it is of interest to determine how much search time must be

Table 2

Binary Model Sensitivity

P_1/P_2	k_2/k_1	v_1/v_2 0.05	v_1/v_2 0.1
1	0.5	1.42 / 1.53 / .04 / 4.	1.42 / 1.38 / .03 / 4.
	1.0	2. / 1.62 / .049 / 5.	2. / 1.44 / .037 / 2.
	1.5	1.64 / 1.59 / .03 / 5.	1.64 / 1.41 / .042 / 4.
2	0.5	1.23 / 1.61 / .025 / 5.	1.23 / 1.39 / .035 / 4.
	1.0	1.9 / 1.86 / .04 / 5.	1.9 / 1.56 / .05 / 4.
	1.5	1.65 / 1.95 / .036 / 5.	1.65 / 1.60 / .05 / 4.
3	0.5	1.1 / 1.6 / .028 / 5.	1.1 / 1.36 / .037 / 4.
	1.0	1.78 / 1.9 / .047 / 6.	1.78 / 1.55 / .032 / 6.
	1.5	1.6 / 2.05 / .045 / 5.	1.6 / 1.63 / .031 / 5.

1. KET-Koopman Expected Time. 

2. E(T)/P(T): Value of E(T)/P(T) and time at which E(T)/P(T) < .05.

3. Ratio of the conditional expected times until detection under the respective policies.

available in order that the ratio of the difference function to the probability of detection under the optimal policy be less than some specified value. Assuming a level of 0.05, the table gives the total search time required to reach that level using the Koopman scheme.

Thus, it would appear, from Table 2, that for situations in which the searcher (decision maker) has large quantities of search time, (T > 2 KET), he need not be concerned with having good estimates of the visibility parameters. Qualitatively, equation 6 enables one to answer such questions over a range of possbile values for the visibility parameters. For example, assume one wishes to determine that value of T, the total available time for which the percent relative error is below some preassigned level, say P_0. By equating the error function, equation 6, to this value and solving for T, one obtains the required level of time. Table 2 contains some numerical analyses of this nature. For example, in the situation in which $p_1/p_2 = 3$ and $k_1/k_2 = 1$, if the decision maker estimates the ratio of v_1/v_2 at 0.1 and assigns 6 units of search time, such an assignment will yield a relative error of less than 0.05 even if v_1/v_2 turned out to be as low as 0.05. Of course, if the ratio turned out to in the interval [.1,1], the actual percent relative error will be even less. Note also from Table 2 that the 6 units of search time are approximately three times the Koopman expected time until detection. The searcher has the choice

of obtaining estimates of v_1 and v_2 at some cost (e.g., weather information), or expending extra effort by disregarding the visibility process and following the Koopman scheme. If the decision maker has means of obtaining estimates of v_1 and v_2, the above analysis provides a sensitivity study of the cost to him of using the Koopman scheme, i.e., failing to obtain them.

2.2.3 Model Sensitivity

In the following paragraphs, we present the results of a sensitivity analysis with respect to the model visibility parameters for both the optimal policy and its associated return. In order that the following results hold, it is necessary to assume that the available search time is larger than T**.

Consider first, the sensitivity of the optimal policy. The relevant partial derivatives are listed below. For the optimal allocation to region 1, one has

$$\frac{\partial t_1}{\partial v_1} = \frac{1}{k_1 + k_2}\left(\frac{1}{v_1}\right) \text{ and } \frac{\partial t_1}{\partial v_2} = \frac{-1}{k_1 + k_2}\left(\frac{1}{v_2}\right).$$

For region 2, we have

$$\frac{\partial t_2}{\partial v_2} = \frac{1}{k_1 + k_2}\left(\frac{1}{v_2}\right) \text{ and } \frac{\partial t_2}{\partial v_1} = \frac{-1}{k_1 + k_2}\left(\frac{1}{v_1}\right).$$

First, some general observations. Note that the larger v_i $(0 < v_i \leq 1)$, the less sensitive the optimal allocation to changes in v_i. Also, the optimal policy is fairly robust

with respect to changes in the visibility parameters whenever the detection rates k_1, k_2 are large. Finally, we note that the above partials are independent of the level of available search effort. This suggests that the optimal policy will be much more sensitive to changes in the visibility parameters whenever the total search effort is limited. We note in advance that such independence will not be the case for the models of Chapters 3 and 4. These results imply that:

(a) the decision maker need not be concerned about small errors in his estimates of the v_i in the situation in which the detection rates are high, and

(b) as the estimates of v_i tend toward unity, small errors in them will have little effect on the optimal policy unless the detection rates are very small.

Next we examine the sensitivity of the optimal return function to changes in the visibility parameters. In order that results remain valid, the available search time must be larger than T**. The partial derivatives are listed below with comments whenever appropriate:

$$\frac{\partial P(T)}{\partial v_1} = p_1 \left[1 - e^{-\frac{k_1 k_2}{k_1+k_2} T} \left(\frac{p_1 k_1 v_1}{p_2 k_2 v_2} \right)^{-\frac{k_1}{k_1+k_2}} \right] \geq 0 ,$$

$$\frac{\partial P(T)}{\partial v_2} = p_2 \left[1 - e^{-\frac{k_1 k_2}{k_1+k_2} T} \left(\frac{p_2 k_2 v_2}{p_1 k_1 v_1} \right)^{-\frac{k_2}{k_1+k_2}} \right] \geq 0 .$$

Here we note that the sensitivity of the optimal return increases with increasing search time, T. However, the maximum sensitivity is determined by the prior probabilities on target presence. The greater the prior probability of target presence in a region, the more sensitive the optimal return to changes (errors) in the estimates of the visibility parameters. The above interpretation doesn't imply that the optimal policy is robust for small values of total available search time, since, on a relative basis, relative to the optimal return, the sensitivity is higher for small values of search time.

We can summarize our analysis of the Binary process by noting that the important effects of this type of process occur under nonhomogeneous visibility conditions and for search situations with limited total available search time.

2.3 Minimization of the Expected Time Until Detection

2.3.1 Model Solution

Recall the arguments given in Appendix A for the value of the objective function in problem of minimizing the expected time to detect. These will hold here also with the following important difference. The optimal value of the probability of detection for the Binary model, under a constraint on the total search time, is in the limit as $T \to \infty$.

$$P(\infty) = p_1 v_1 + p_2 v_2 < 1, \quad \text{for } 0 < v_1, v_2 < 1 .$$

Then with probability $(1 - P(\infty))$ the search is unsuccessful, and the expected time to detect increases without limit. In order to see this more clearly, consider the following expression for the expected time to detect:

$$E = \lim_{T\to\infty} \left\{ \int_0^T t\,dP(t) + T(1 - P(T)) \right\} .$$

For the Binary model

$$\lim_{T\to\infty} (1 - P(T)) \to \varepsilon > 0 ,$$

hence, the unbounded expected time to detect.

One can, however, consider the conditional expected time to detect, given that detection ultimately occurs, i.e.,

$$E = \int_0^\infty \frac{t\,dP(t)}{P(\infty)} .$$

Proceeding as in Appendix A under the assumption that the searcher starts in region 1, we obtain for the optimal value of this objective function.

$$E = \frac{1}{P(\infty)} \left[p_1 v_1 \left[\frac{1}{k_1} \left(1 - e^{-k_1 T^{**}} \right) + \left[\frac{p_1 k_1 v_1}{p_2 k_2 v_2} \right]^{-\frac{k_1}{k_1+k_2}} \left\{ 1 + \frac{k_1}{k_2} \right\} e^{-\frac{k_1 k_2}{k_1+k_2} T^{**}} \right] + p_2 v_2 T^{**} \right] ,$$

where T** is given by (3).

For the case in which the infinitesimal detection rates are identical, i.e., $p_1 v_1 k_1 = p_2 v_2 k_2$, one has that

$$E = \frac{k_1 + k_2}{k_1 k_2} ,$$

which is identical to the unconditional expected time to detect for the standard model. Of course, E is the *conditional* expected time to detect; with probability $(1 - \bar{P}(\infty))$ the target is not detected.

2.3.2 Comparison with the Koopman Model

In this section we shall again consider the case of the partially informed searcher; however, the objective of the search will be the minimization of the conditional expected time until detection. Recall that our mode of comparison is that of a searcher who is aware of the Koopman (1946) results but is not aware of the target behavior and/or visibility conditions. In such a situation, he would make use of Dobbie's (1962) results for the expected time problem, i.e., the policy that maximizes the probability of detection is also the one which minimizes the expected time until detection, if the conditional detection functions are concave. Proceeding as in Appendix A, using the Koopman allocation policy in the Binary model, one obtains for the optimal value of the

conditional expected time until detection for a 2-region search situation,

$$E\cdot(p_1v_1 + p_2v_2) = p_1v_1\left\{\frac{1}{k_1}\left[1 - e^{-k_1T^*}\right] - Te^{-k_1T^*}\right\} + \left[p_1v_1\left(\frac{p_1k_1}{p_2k_2}\right)^{-p_1/p_1+p_2} + p_2v_2\cdot\left(\frac{p_2k_2}{p_1k_1}\right)^{-k_2/k_1+k_2}\right]\cdot\left[T^*e^{-\frac{k_1k_2}{k_1+k_2}T^*} + \frac{k_1 + k_2}{k_1k_2}e^{-\frac{k_1k_2T^*}{k_1+k_2}}\right],$$

where $T^* = \frac{1}{k_1}\ln\left(\frac{p_1k_1}{p_2k_2}\right)$.

If region 2 is searched first, then T^* is defined accordingly and one makes the obvious changes in the first term of the above expression.

In order to complete our comparison with the Koopman model, we form the ratio of the measure just obtained to that resulting from using the optimal policy for the Binary model in the computation of the expected time until detection. Table 2 (repeated here for convenience) contains the results of several such comparison. It is interesting to note that:

(a) Although in certain cases the Koopman policy yields a small maximum difference in the probability of

Table 2

Binary Model Sensitivity

p_1/p_2	k_2/k_1	v_1/v_2 = 0.05	v_1/v_2 = 0.1
1	0.5	1.42 / 1.53 / .04 / 4.	1.42 / 1.38 / .03 / 4.
	1.0	2. / 1.62 / .049 / 5.	2. / 1.44 / .037 / 2.
	1.5	1.64 / 1.59 / .03 / 5.	1.64 / 1.41 / .042 / 4.
2	0.5	1.23 / 1.61 / .025 / 5.	1.23 / 1.39 / .035 / 4.
	1.0	1.9 / 1.86 / .04 / 5.	1.9 / 1.56 / .05 / 4.
	1.5	1.65 / 1.95 / .036 / 5.	1.65 / 1.60 / .05 / 4.
3	0.5	1.1 / 1.6 / .028 / 5.	1.1 / 1.36 / .037 / 4.
	1.0	1.78 / 1.9 / .047 / 6.	1.78 / 1.55 / .032 / 6.
	1.5	1.6 / 2.05 / .045 / 5.	1.6 / 1.63 / .031 / 5.

1. KET-Koopman Expected Time. 

2. E(T)/P(T): Value of E(T)/P(T) and time at which E(T)/P(T) < .05.

3. Ratio of the conditional expected times until detection under the respective policies.

detection, the same policy yields an increase in excess of 50 percent in the conditional expected time until detection. For the Binary model, the long-term allocation rates are identical to those resulting from the Koopman policy. The larger ratios of the expected time in Table 2 occur, for the most part, as a result of the Koopman policy selecting the wrong initial region.

(b) The allocation of search time on the order of $\sim$3 KET, reduces the error function to less than 0.05; however, cost-wise, in terms of the minimum cost of searching, the Koopman policy is extremely inefficient for the ranges of Table 2.

(c) For the parameter ranges of Table 2, and a constant value of p_1/p_2, the ratio of the conditional expected times until detection tends to increase with increasing k_2/k_1. The effect of increasing k_2/k_1 is greater than that of increasing the ratio of prior probabilities.

Chapter 3

A RANDOM INTERVAL OF VISIBILITY: INITIALLY VISIBLE

The "single-interval-of-visibility" model is described and analyzed in this chapter. It is assumed that the target is visible upon the searcher's entry into the appropriate region; however, the length of time it remains visible is a random variable, the distribution of which is possibly regionally dependent. This model might describe a situation in which a target, operating in the area about to be searched, possesses detection gear. The length of the "visible" period is analogous to the length of time required for the target to become aware of the searcher's presence in his region.

The problem of maximizing the probability of detection under a constraint on the available search time is examined in detail. A comparison of both optimal policies and associated returns for the standard Koopman and random interval models is made. First-order sensitivity analyses on the optimal policy and return for this model are then described. The minimization of the conditional expected time to detect, given unlimited search effort, is also considered.

3.1 Description

As Bonder (1970) points out, the probability of detection in a given region under this model, in an infinitesimal interval $d\tau$, is given by

$$P(\tau)d\tau = Pr\{\tau \leq \tau_d \leq \tau + d\tau | \text{target presence } \& \; t_v > \tau\} Pr\{t_v > \tau\} ,$$

$$= \rho(\tau)d\tau \int_{\tau}^{\infty} h(t)dt ,$$

where

τ_d = time until detection,

$\rho(\tau)$ = probability density function for τ_d,

t_v = time that the target remains visible, and

$h(\tau)d\tau = Pr\{\tau \leq t_v \leq \tau + d\tau\}$.

Then the probability of detection in the ith region given t, the time spent searching, is

$$P_i(t) = \int_0^t p_i \rho(\tau)\bar{H}(\tau)d\tau , \qquad (8)$$

where

$$\bar{H}(\tau) = \int_{\tau}^{\infty} h(t)dt \ .$$

The problem of maximizing the probability of detection over a discrete search space is then given by

$$\max \sum_{i=1}^{N} p_i \int_{0}^{t_i} \rho(\tau)\bar{H}(\tau)d\tau$$

$$\text{S.T.} \sum_{i=1}^{N} t_i \leq T \ , \qquad (9)$$

$$t_i \geq 0 \ .$$

3.2 Allocation of Search Time to Maximize the Probability of Detection

3.2.1 Model Solution

Recall the comments made in Chapter 2 concerning the theoretical and empirical relevance of the form of the conditional detection function we have assumed, i.e.,

$$\rho_i(\tau) = k_i e^{-k_i \tau} \ , \quad \tau \geq 0.$$

In view of the interpretation of the visible period as the length of time required for the target to detect the searcher, it is reasonable to assume that the target has similar capabilities[1], i.e.,

[1]Additionally, personal correspondence between Professor Bonder and the staff of the Defense Operations Analysis Establishment indicate that visibility periods in terrain examined by the British do follow an exponential law.

$$h_i(\tau) = \lambda_i e^{-\lambda_i \tau} , \quad \tau \geq 0 .$$

Substitution into equation 8 yields

$$P_i(T) = \frac{k_i}{\lambda_i + k_i} (1 - e^{-(\lambda_i + k_i)T}) ,$$

and the maximization problem of (9) becomes

$$\max \sum_{i=1}^{N} p_i \frac{k_i}{\lambda_i + k_i} (1 - e^{-(\lambda_i + k_i)t_i})$$

$$\text{S.T.} \sum_{i=1}^{N} t_i \leq T , \qquad (10)$$

$$t_i \leq 0 ,$$

which is a concave maximization problem. The modified Charnes-Cooper algorithm given in Appendix A is again applicable. We again restrict our discussions to two regions for the reasons already noted. The solution to the N-region problem is obtained by following the procedure given in Appendix A.

The condition on available search time which places all the search time in Region 1 is $T < T^{**}$,

$$\text{where} \quad T^{**} = \frac{1}{k_1 + \lambda_1} \ln \frac{p_1 k_1}{p_2 k_2} . \qquad (11)$$

We have assumed that Region 1 is selected by the First Allocation Rule (Appendix A), i.e., $p_1 k_1 \geq p_2 k_2$. The First

Allocation Rule for this model is identical to that of the Koopman model.

Choose that region j for which $p_j k_j = \max_{1 \le i \le N} p_i k_i$.

Using an analog of equation 9, Appendix A, for the case $T \geq T^{**}$, one obtains, for the optimal allocations,

$$t_1 = \frac{1}{\lambda_1 + k_1 + \lambda_2 + k_2} \left\{ \ln\left(\frac{p_1 k_1}{p_2 k_2}\right) + (\lambda_2 + k_2)T \right\},$$

and (12)

$$t_2 = \frac{1}{\lambda_1 + k_1 + \lambda_2 + k_2} \left\{ \ln\left(\frac{p_2 k_2}{p_1 k_1}\right) + (\lambda_1 + k_1)T \right\}.$$

Note that in the limit as λ_1 & $\lambda_2 \to 0$ (the mean length of the visible period increases without limit) one obtains the Koopman model. The probability of detection under the optimal policy is then given by[1]

$$p(T) = \frac{p_1 k_1}{k_1 + \lambda_1}\left[1 - \left[\frac{p_1 k_1}{p_2 k_2}\right] e^{-\frac{(\lambda_1+k_1)}{\lambda_1+k_1+\lambda_2+k_2} - \frac{(\lambda_1+k_1)(\lambda_2+k_2)}{\lambda_1+k_1+\lambda_2+k_2}T}\right]$$

$$+ \frac{p_2 k_2}{k_2 + \lambda_2}\left[1 - \left[\frac{p_2 k_2}{p_1 k_1}\right] e^{-\frac{(\lambda_1+k_2)}{\lambda_1+k_1+\lambda_2+k_2} - \frac{(\lambda_1+k_1)(\lambda_2+k_2)}{\lambda_1+k_1+\lambda_2+k_2}T}\right]. \quad (13)$$

[1] Formulations and results for discrete detectors are given in Appendix B. Continuous search space (continuous detector) versions of the model are discussed in Appendix C.

3.2.2 Comparison with the Koopman Model

In this section we examine the situation in which a partially informed searcher being aware of the earlier results of Koopman (which assume continuously visible targets), applies them to situations in which the target behavior is actually characterized by the random interval of visibility process.[1] The consequences (in terms of the probability of detecting the target under a constraint on the total available search time) of this erroneous application of the Koopman results will be compared to those obtained from the optimal allocations for these models.

First, we note that in equations 12 as $\lambda_1, \lambda_2 \to 0$, the optimal allocation policy becomes equivalent to the standard Koopman allocation (SKA). Consider the following comparisons between the optimal allocations of (12) and the SKA: for $p_1k_1 = p_2k_2$, and $k_1 = k_2 = k$, $\lambda_1 = \lambda_2 = \lambda$, the optimal allocations are identical and no error results from using the SKA. When $p_1k_1 = p_2k_2$, and $k_1 = k_2 = \lambda_1 = \lambda_2 = \gamma$, the same conclusion holds. These results also hold for n-regions

[1] The reader is referred to Section 2.2.3 for a detailed discussion of the implications of such an analysis to the decision maker (searcher).

(See Appendix A).[1] As shown earlier, the switch points are given by (assuming $p_1k_1 \leq p_2k_2$):

(a) for SKA

$$T^* = \frac{1}{k_1} \ln\left(\frac{p_1k_1}{p_2k_2}\right),$$

(b) for the single interval model

$$T^{**} = \frac{1}{k_1 + \lambda_1} \ln\left(\frac{p_1k_1}{p_2k_2}\right).$$

We observe that $T^{**} \leq T^*$ for $\lambda_1 > 0$. Intuitively, this is reasonable since the target will not remain visible forever in the first region, hence one must switch earlier.

The following situations can arise as a result of the difference in switch points (both policies will always select identical starting regions, i.e., the FAR for this model is identical to the Koopman FAR).

(a) $T < T^{**} \leq T^*$, no error, both policies allocate all effort to Region 1.

---

[1] Under the interpretation of searching for a target with detection capabilities, this result implies that, for identical detectors for both the target and searcher, not necessarily equal, the Koopman policy is optimal. Hence if the decision maker (searcher) is willing to make this assumption, he may proceed as though he were searching for a continuously visible target. This also implies that for an identical visibility process, the Koopman policy is optimal only if $p_1 = p_2$ and $k_2 = k_1$. Recall that in the Binary process, the Koopman policy was optimal whenever $v_1 = v_2$.

(b) $T^{**} < T < T^{*}$, the SKA concentrates all effort in Region 1, while the single interval optimal solution follows (12).

(c) $T^{**} < T^{*} < T$, both policies split the available effort. The error term in this situation also tends to zero as the available search effort increases without limit, as in Chapter 2.

Figure 3 illustrates some of the above situations. The ordinate again represents the percent relative error, while the abscissa gives the total available search effort. The results of a parametric analysis on the effects of the relative values of the prior probabilities on the error function are given. Note that as the ratio p_1/p_2 tends toward unity the error function decreases.[1] Also note that in each case the peak error occurs at $T = T^{*}$, the Koopman switch point. The peaks will be analyzed later in this section. It is also interesting to observe the reduction in the error as $\lambda_1/\lambda_2 \rightarrow 1$, as was noted earlier. Thus, a decision maker (searcher), operating in this scenario with an amount of search effort T^{*}, will be required to:

(a) obtain estimates of λ_1 and λ_2 in order to conduct an effective search, or

[1]This suggests that the effect of nonhomogeneous visibility may be lessened in the situation in which the searcher has no prior knowledge on target location. For example, a high likelihood on target position could be offset by a low visibility likelihood.

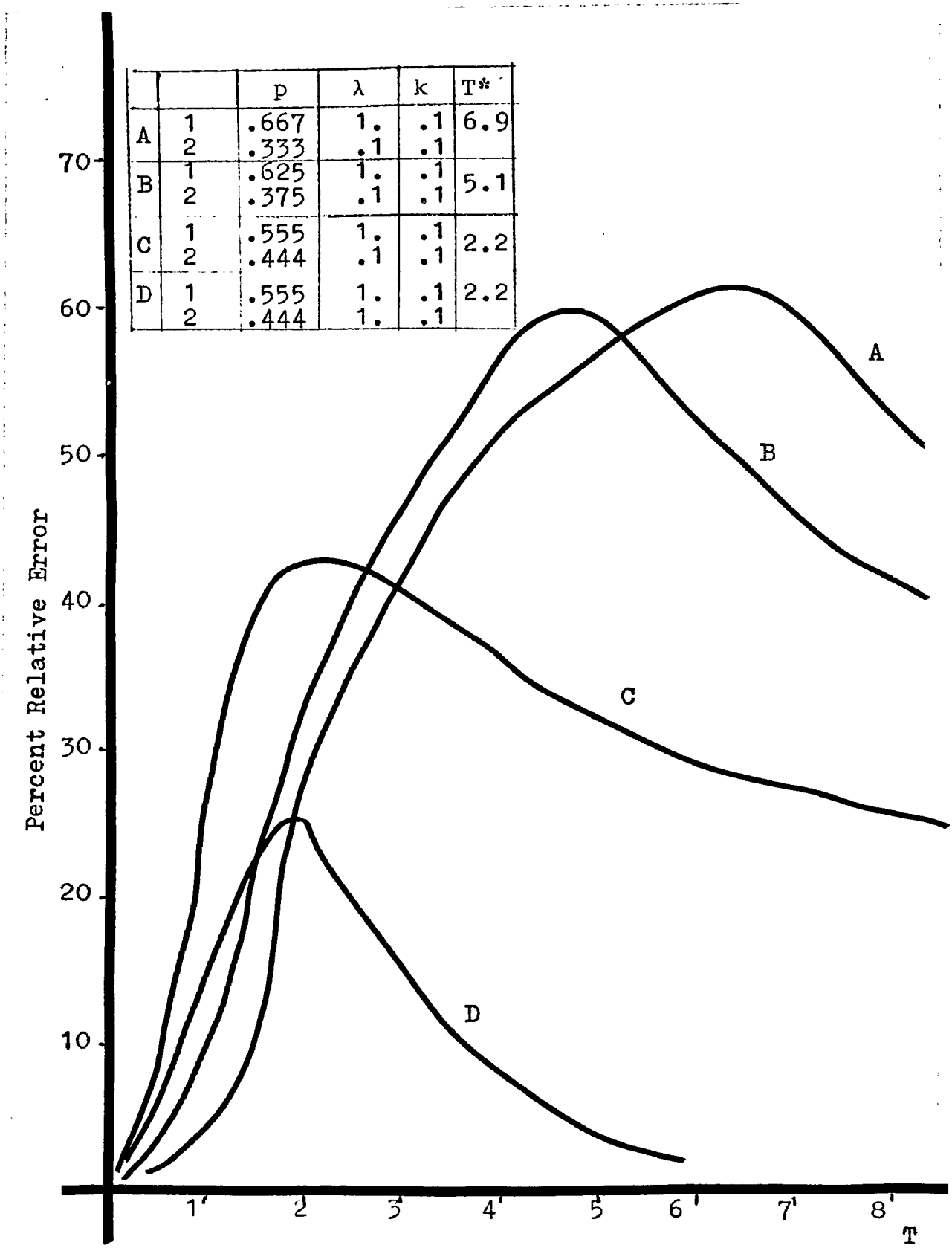


Figure 3 - Error Function Sensitivity

(b) increase the available effort far beyond T*.

An analysis of the error function was undertaken to determine:

(a) At what levels of total available effort the peak differences occur?

(b) What influences their shape, i.e., height, rate of decay, etc.?

For the random-interval, 2-region, initially visible model, one has only two cases to consider, since both policies always select the same initial region and the optimal policy for this model starts splitting the effort before the Koopman policy. These cases are

$$\text{Case (1)} \quad \frac{\partial E(T)}{\partial T} \leq 0 \text{ for } T > T^*, \text{ and}$$

$$\text{Case (2)} \quad \frac{\partial E(T)}{\partial T} \geq 0 \text{ for } T > T^*.$$

where E(T) denotes the difference in the probability of detection under the two policies. In the first case, we observe that the peak always occurs at T*, since the error will increase until the Koopman policy begins to allocate in Region 2 (recall T** < T*). While in the second case, one must utilize numerical techniques to solve for the time at which the peak difference occurs.[1] The following are

[1]The unimodality of the error function is established by noting that it can be expressed as the difference of two exponential functions.

sufficient conditions for each of the above cases:

Case (1) $$\frac{\lambda_1}{k_1} = \frac{\lambda_2}{k_2}$$

Case (2) $$p_1 k_1 = p_2 k_2 \rightarrow T^* = T^{**} = 0.$$

This implies that the relationship between the detection and visibility processes is homogeneous across regions. Physically, one might view this as representative of the situation in which whenever the searcher's detection gear is affected by regional conditions, the target's gear is also affected in the same degree. The error function evaluated at the peak for Case (1) is

$$E_p = \frac{p_1 k_1}{\hat{k}_1}\left[\left(\frac{p_1 k_1}{p_2 k_2}\right)^{-\hat{k}_1/k_1} - \left(\frac{p_1 k_1}{p_2 k_2}\right)^{-\frac{\hat{k}_1}{\hat{k}_1+\hat{k}_2}(1+(\hat{k}_2/k_1))}\right]$$

$$+ \frac{p_2 k_2}{\hat{k}_2}\left[1 - \left(\frac{p_1 k_1}{p_2 k_2}\right)^{\frac{\hat{k}_2}{\hat{k}_1+\hat{k}_2}(1-(\hat{k}_1/k_1))}\right],$$

where $\hat{k}_i = \lambda_i + k_i$.

Given two search scenarios with the following properties, where the first subscript denotes the region, the second the search situation, $p_{11} = p_{12} = p_{21} = p_{22}$.

$$k_{11} = k_{21},\ k_{12} = k_{22},\ \text{and}\ \frac{\lambda_{11}}{k_{11}} = \frac{\lambda_{12}}{k_{12}},\ \frac{\lambda_{21}}{k_{21}} = \frac{\lambda_{22}}{k_{22}}$$

then the peak error functions for both cases are identical even though the times of occurence are not necessarily identical. Thus as long as the relationship between the detection and visibility process is homogeneous across regions, the decision maker can expect the same maximum error in using the Koopman policy, without having knowledge of the visibility parameters.

Since in Case (2) an explicit expression for the error function cannot be given, one must resort to numerical techniques. The following numerical procedure was utilized:

(1) Starting at $T = T^*$ compute $E'(T + \Delta T)$ until that value of T for which $E'(T) \leq 0$.

(2) Use an approximate Fibonacci search on the interval obtained in step (1) to obtain T.

(3) Substitute that value of T into the expression for E(T).

(4) For large values of T (i.e., $T >> T^{**}$), E(T), in this case, is approximated by

$$E(T) = \frac{k_1 p_1}{\hat{k}_1}\left(e^{-\frac{\hat{k}_1 k_2}{k_1+k_2}T} - e^{-\frac{\hat{k}_1 \hat{k}_2}{k_1+k_2}T}\right) + \frac{k_2 p_2}{\hat{k}_2}\left(e^{-\frac{\hat{k}_2 k_1}{k_1+k_2}T} - e^{-\frac{\hat{k}_1 \hat{k}_2}{\hat{k}_1+\hat{k}_2}T}\right) \tag{14}$$

Table 3 gives the results for a series of computer runs which were made in an effort to study the sensitivity of the above functions to changes in the search parameters. It contains the measures introduced in Section 2.3.

First we note that in a great many cases the peak difference in the probability of detection is extremely small, suggesting the adequacy of the Koopman policy. The notable exceptions to this observation occur when the visibility parameters are nonhomogeneous. Note also that the assumption $p_1 = p_2$ is not critical relative to the optimality of the Koopman policy in those situations in which $\lambda_1 = \lambda_2$ and $k_1 = k_2$.

The results of Table 3 deal only with the peak errors. One must utilize equation 14 in order to determine their rate of decay with increasing search effort. For example, if $\lambda_1 = 10$, $\lambda_2 = .1$, and $k_1 = k_2 = .1$ (see Table 3), we determine that value of total search effort for which E(T), given by equation 14, is less than .01. Equating E(T) to .01 and solving for T, one obtains $T \simeq 28$, and $T^* \simeq 7$.

3.2.3 Model Sensitivity

In the following paragraphs, we present the results of a sensitivity analysis on the visibility parameters for both the optimal policy and its associated return. In order

Table 3

Random Interval Sensitivity Results

$p_1/p_2=2$		$T^*=6.9$	$T^*=3.$	$T^*=1.6$	$T^*=0.7$
λ_1	λ_2	$k_1=.1$ $k_2=.1$	$k_1=1.$ $k_2=.1$	$k_1=.1$ $k_2=1.$	$k_1=1.$ $k_2=1.$
.1	.1	.05 1.03 .014	.002 1. .001	.002 1.02 .0006	.001 1. .0004
	1.	.016 1.12 .004	.001 1.2 .0006	.13 1.1 .02	.013 1.1 * .01 3.58
	10.	.002 1.23 .0005	.025 1.4 * .015 5.72	.72 1.3 .08	.095 1.3 * .05 2.25
1.	.1	.62 2.2 .10	.08 1.1 .028	.001 1.7 .0004	.05 1.1 * .025 1.89
	1.	.32 2. .03	.04 1.1 .015	.08 1.11 .014	.05 1. .014
	10.	.05 1.2 .003	.007 1. .002	.60 2.1 .05	.016 1.1 .004
10.	.1	.95 3.2 .12	.53 1.6 .07	.12 2.5 * .04 4.3	.60 2.2 .13 .83
	1.	.82 7.3 .03	.32 2.7 .03	.02 1.3 .003	.62 2.2 .10
	10.	.33 13. .003	.05 2.2 .003	.18 2.5 .007	.32 2. .03
		1 3 2		*Case 2	

1. Relative error at the peak difference.
2. Peak difference & time of occurence, if not equal to T^*.
3. Ratio of conditional expected times (Koopman/optimal).

that the following partial derivatives hold, it is necessary to assume that the available search time is larger than T**. The relevant partials are listed below. Consider first the sensitivity of the optimal policy. For the optimal allocation to Region 1, one has

$$\frac{\partial t_1}{\partial \lambda_1} = - \frac{t_1}{(\lambda_1 + k_1 + \lambda_2 + k_2)} \leq 0$$

and

$$\frac{\partial t_1}{\partial \lambda_2} = \frac{t_2}{(\lambda_1 + k_1 + \lambda_2 + k_2)} \geq 0 .$$

Symmetrically, for Region 2, we have

$$\frac{\partial t_2}{\partial \lambda_2} = - \frac{t_2}{(\lambda_1 + k_1 + \lambda_2 + k_2)} \leq 0$$

and

$$\frac{\partial t_2}{\partial \lambda_1} = \frac{t_1}{(\lambda_1 + k_1 + \lambda_2 + k_2)} \geq 0 .$$

First we observe that the higher the detection and/or visibility rate, the less sensitive the optimal policy to changes in the visibility parameters, i.e., the policy is fairly robust for good detectors and visibility situations in which

the rate is high. It is also apparent, as was *not* the case in the Binary model, that the sensitivity of the optimal policy increases linearly with T, the total available search effort (since t_1 is a linear function of T). Increasing the visibility rate decreases the allocation for that region. Intuitively, this is reasonable since an increase in λ implies a decrease in the mean length of the visible period.

Next we examine the sensitivity of the optimal return function (the probability of detection) to changes in the visibility parameters. In order that the results remain valid, the available search time must be larger than T**. The partial derivatives are listed below. For Region 1, we have

$$\frac{\partial P(\tau)}{\partial \lambda_1} = \frac{p_1 k_1}{k_1 + \lambda_1} \left\{ e^{-\alpha\tau} \left[t_1 + \frac{1}{k_1 + \lambda_1} \right] \left(\frac{p_1 k_1}{p_2 k_2} \right)^{-\alpha} - \frac{1}{\lambda_1 + k_1} \right\}$$

and

$$\lim_{\tau \to \infty} \frac{\partial P(\tau)}{\partial \lambda_1} = - \frac{p_1 k_1}{k_1 + \lambda_1} \left(\frac{1}{k_1 + \lambda_1} \right),$$

where

$$\alpha = \frac{\lambda_1 + k_1}{\lambda_1 + k_1 + \lambda_2 + k_2}$$

and

$$\tau = (\lambda_2 + k_2)T.$$

From the symmetry of the two-region search situation, one can obtain $\frac{\partial P(\tau)}{\partial \lambda_2}$ from the above expression by interchanging the regional labels. At $T = T^{**}$, the above expression becomes

$$\frac{\partial P(T^{**})}{\partial \lambda_1} = \frac{p_1 k_1}{k_1 + \lambda_1} \left[\left(\frac{p_2 k_2}{p_1 k_1} \right)^{-\alpha} \left(T^{**} + \frac{1}{\lambda_1 + k_1} \right) e^{-\alpha T^{**}} - \frac{1}{\lambda_1 + k_1} \right],$$

which is nonpositive if $T^{**} \geq 0$. Thus we observe that the decision maker's analysis of the effects of errors in the visibility parameters rests heavily on the amount of search effort available to him. As $T \to \infty$, the magnitude of $\frac{\partial P(\tau)}{\partial \lambda}$ is the product of two factors:

$\frac{p_1 k_1}{k_1 + \lambda_1}$ = the ultimate probability of detection in the ith region,

$\frac{1}{k_1 + \lambda_1}$ = the conditional mean time until detection in the ith region for the single-interval process, conditioned on detection occurring.

Although the optimal policy becomes increasingly sensitive to changes in the visibility parameters with increasing search time, the sensitivity of the return function is limited by the above expressions. Hence, beyond a certain level of total available search time, no matter how great the change in the λ_i, the change in the optimal return is constant.

Furthermore, for large detection and/or visibility rates that constant approaches zero; hence, under these conditions the return is fairly robust. Finally, we use some numerical examples to illustrate the above results. Assume search scenarios characterized by:

#1	#2
$\mathbb{P}$ = (2/3,1/3)	$\mathbb{P}$ = (2/3,1/3)
$\mathbb{K}$ = (1,0.1)	$\mathbb{K}$ = (10,1)
λ = (1,1)	λ = (10,10)
T = 10.	T = 2.0

Then one obtains the following sensitivity results.

#1	#2
T** = 1.5	T** = 0.15
t_1 = 4.5	t_1 = 0.72
t_2 = 5.5	t_2 = 1.28
KET = 5.7	KET = 0.45
$\frac{\partial t_1}{\partial \lambda_1}$ = -1.45	$\frac{\partial t_1}{\partial \lambda_1}$ = -0.024
$\frac{\partial t_1}{\partial \lambda_2}$ = 1.76	$\frac{\partial t_1}{\partial \lambda_2}$ = 0.0415
$\frac{\partial P(T)}{\partial \lambda_1}$ = -0.167	$\frac{\partial P(T)}{\partial \lambda_1}$ = 0.0167

Summarizing our analysis of the random interval process, we note:

(a) the important differences induced by this visibility process occur under nonhomogeneous visibility conditions (as did those for the Binary model);

(b) while the peak error in using the Koopman policy occurs in limited search time situations, the long-term allocation rates may be sufficiently different from the Koopman rates to induce very slow rates of decay (recall that for the Binary model the long-term allocation rates are identical to those of the Koopman model).

At this point is is of interest to recall that from the results of Stollmack (1968) on visual detection processes, differing detection rates were the rule rather than the exception. Furthermore, the data also indicate heterogeneous visibility conditions over the local terrain.

3.3 Minimization of the Expected Time Until Detection

3.3.1 Model Solution

Recall the discussion of the problem of minimizing the expected time until detection in Chapter 2. The same situation holds for this model, namely, that there is a nonzero probability of failing to detect the target given unlimited search time. The optimal value of the probability of detection for the single interval model, under a constraint on the total search time, is given by (13).

In the limit, equation 6 becomes

$$\lim_{T\to\infty} P(T) = \frac{p_1 k_1}{k_1 + \lambda_1} + \frac{p_2 k_2}{k_2 + \lambda_2} < 1,$$

for $\lambda_1, \lambda_2 > 0$.

As in Chapter 2, we consider the conditional expected time to detect given that detection occurs, i.e.,

$$E = \int_0^\infty t \, \frac{dP(t)}{P(\infty)} \quad .$$

Using the procedure of Appendix A, the optimal value of the objective function is

$$E = \frac{1}{P(\infty)} \left\{ \frac{p_1 k_1}{k_1 + \lambda_1} \left[\frac{1}{\hat{k}_1}(1 - e^{-\hat{k}_1 T^{**}}) + \left(\frac{p_1 k_1}{p_2 k_2}\right)^{-\frac{\hat{k}_1}{\hat{k}_1+\hat{k}_2}} \left\{ \frac{\hat{k}_1 + \hat{k}_2}{\hat{k}_1 \hat{k}_2} \right\} \left\{ \frac{1 + \hat{k}_1}{\hat{k}_2} \right\} e^{-\frac{\hat{k}_1\hat{k}_2}{\hat{k}_1+\hat{k}_2} T^{**}} \right] + \left[\frac{p_2 k_2}{k_2 + \lambda_2} \right] T^{**} \right\} ,$$

where $\hat{k}_i = k_i + \lambda_i$.

For the case in which $p_1 k_1 = p_2 k_2$, one has that $T^{**} = 0$ and that

$$E = \frac{\hat{k}_1 + \hat{k}_2}{\hat{k}_1 \hat{k}_2} = \frac{k_1 + \lambda_1 + k_2 + \lambda_2}{(\lambda_1 + k_1)(\lambda_2 + k_2)} ,$$

which in the limit, as λ_1 & $\lambda_2 \to 0$, tends toward the corresponding result for the Koopman model. This is the conditional

expected time to detect, with probability $(1 - \bar{P}(\infty))$ the target is not detected.

3.3.2 Comparison with the Koopman Model

As in Chapter 2, we consider the case of the partially informed searcher who wishes to minimize the conditional expected time until detection, given unlimited effort and conditioned on detection occurring. The partially informed searcher allocates search effort according to the Koopman policy in an effort to minimize the expected time until detection. The return from such a policy for the single interval model in a 2-region situation is

$$E\bullet\left(\frac{p_1 k_1}{k_1 + \lambda_1} + \frac{p_2 k_2}{k_2 + \lambda_2}\right) = \frac{p_1 k_1}{k_1 + \lambda_1}\left\{\frac{\left(1 - e^{-\hat{k}_1 T^*}\right)}{\hat{k}_1} - T^* e^{-k_1 T^*}\right\}$$

$$+ p_1\left(\frac{p_1 k_1}{p_2 k_2}\right)^{-\frac{\hat{k}_1}{k_1+k_2}}\left[\frac{k_1}{\hat{k}_1}\right] e^{-\frac{\hat{k}_1 k_2 T^*}{k_1 + k_2}}$$

$$\left\{T^* + \frac{k_1 + k_2}{\hat{k}_2 k_1}\right\} + p_2\left(\frac{p_2 k_2}{p_1 k_1}\right)^{-\frac{\hat{k}_1}{k_1+k_2}}$$

$$\left[\frac{k_2}{\hat{k}_2}\right] e^{-\frac{k_2}{\hat{k}_1+k_2} T^*}\left\{T^* + \frac{k_1 + k_2}{\hat{k}_2 k_1}\right\},$$

where $T^* = \frac{1}{k_1} \ln\left(\frac{p_1 k_1}{p_2 k_2}\right)$.

Tables 4-6 contain the results of several comparisons with the Koopman search model, suggesting the following observations:

(a) From Table 3, Section 3.2.3, note that peak relative errors of less than .05 do not, in general, imply that the Koopman policy is "good" for the expected time problem.

(b) For homogeneous visibility rates; the Koopman policy works well for small visibility rates, but poorly for the larger visibility rates (large relative to the detection rates).

(c) For homogeneous detection rates, the Koopman policy works well whenever the visibility rates are either homogeneous or heterogeneous and less than or equal to the detection rates.

(d) At $p_1/p_2 = 10$, note that Koopman policy works well for $\lambda_1 = 0.1$ over all levels of detection rates. Note also the trend, since it begins improving at $p_1/p_2 = 2$. On the other hand, for $\lambda_1 = 10$, in general, the opposite effect occurs implying increasing errors with a increasing prior probability ratio.

Some implications of these results for the decision maker are listed below.

(a) Given a great deal of certainty on target location and a detection rate in the primary region at least as large as the target's visibility rate, the searcher need not be concerned with obtaining estimates of the visibility rates.

Table 4

Random Interval Sensitivity Results: Ratio of Conditional Expected Times Until Detection (Koopman/Optimal)

p1/p2 = 1

λ_1	λ_2	.1 .1	1. .1	.1 1.	1. 1.	k_1 k_2
.1	.1	1.	1.+	1.+	1.	
	1.	1.5	1.4	1.1	1.1	
	10.	1.9	1.7	1.5	1.7	
1.	.1	1.5	1.1	1.4	1.1	
	1.	1.	1.1	1.1	1.	
	10.	1.6	1.1	2.4	1.5	
10.	.1	1.9	1.5	1.7	1.7	
	1.	1.6	2.4	1.1	1.5	
	10.	1.	2.5	2.5	1.	

Table 5

Random Interval Sensitivity Results:
Ratio of Conditional Expected Times
Until Detection (Koopman/Optimal)

p1/p2 = 2

λ_1	λ_2	.1 .1	1. .1	.1 1.	1. 1.	k_1 k_2
.1	.1	1.0^+	1.0^+	1.0^+	1.0^+	
	1.	1.1	1.2	1.1	1.1	
	10.	1.2	1.4	1.3	1.3	
1.	.1	2.2	1.1	1.7	1.1	
	1.	2.0	1.1	1.1	1.0^+	
	10.	1.2	1.0^+	2.1	1.1	
10.	.1	3.2	1.6	2.5	2.2	
	1.	7.3	2.7	1.3	2.2	
	10.	13.	2.2	2.5	2.0	

Table 6

Random Interval Sensitivity Results:
Ratio of the Conditional Expected
Times Until Detection (Koopman/Optimal)

p1/p2 = 10

λ_1	λ_2	.1 .1	1. .1	.1 1.	1. 1.	k_1 k_2
.1	.1	1.1$^+$	1.$^+$	1.$^+$	1.$^+$	
	1.	1.$^+$	1.$^+$	1.	1.$^+$	
	10.	1.$^+$	1.1	1.1	1.1$^+$	
1.	.1	3.3	1.1	3.$^+$	1.2	
	1.	2.4	1.$^+$	1.8	1.1	
	10.	1.2	1.$^+$	1.	1.$^+$	
10.	.1	6.$^+$	1.8	9.	3.	
	1.	17.3	2.7	7.7	3.3	
	10.	16.	1.4	2.8	2.4	

(b) On the other hand, if the target's detection rate is greater than that of the searcher, it is important the searcher obtain estimates of it.

One can also make the following observations concerning the implications of these results to the target.

(a) Under complete uncertainty on the searcher's part, the target should choose his visibility rates, assuming they are in some sense under his control, as follows:
   (1) for heterogeneous detection rates, choose homogeneous visibility rates, as large as possible.
   (2) for homogeneous detection rates, choose heterogeneous visibility rates.

(b) If the searcher has increasing prior knowledge on the target's position, the target should use his highest visibility rate in the prime region.

(c) If the maximum available visibility rate in the prime region is less than or equal to the searcher's detection rate in that region, the tactic of remaining in the visible state a random length of time is not effective, at least in terms of ratio of the conditional expected times until detection. Certainly, from a unconditional standpoint, the tactic always has merit inasmuch as it results in a nonzero probability of failing to detect the target.

Chapter 4

A RANDOM INTERVAL OF VISIBILITY: RANDOM INITIATION AND LIMIT

In this chapter we shall consider the situation in which the target is masked when the searcher enters the appropriate region; however, the target becomes visible after some random length of time. The distributions of the starting time and the length of the visible period may be regionally dependent. One may view this as a model of a situation in which the searcher is not sure that the target has arrived in the search zone at the time the search is initiated. The length of the visible period could then correspond to the length of time required for the target to become aware of the searcher's presence.[1]

4.1 Description

Assume that the searcher spends a total of T time units searching a region. Given that the target becomes visible at some time $\mu\varepsilon(0,T)$, the probability of detecting the target is (using the notation of the previous chapter)

[1] The model also has an analogy in a police patrol situation in which the time until a crime is committed is taken to be a random variable as well as the length of time required to carry out the act.

$$\int_{\mu}^{T} \rho(\tau)\bar{H}(\tau)d\tau = \int_{0}^{T-\mu} \rho(s - \mu)\bar{H}(s - \mu)ds$$

Since the above description hinges on the target becoming visible at μ (which is itself a random variable with probability density function $f(\mu)$), the conditional probability of detection, conditioned on the target presence, is given by

$$P(T) = \int_{0}^{T} f(\mu) \int_{0}^{T-\mu} \rho(s - \mu)\bar{H}(s - \mu)ds d\mu \qquad (15)$$

For an N-region search situation, the problem of maximizing the probability of detecting the target, given T time units, is

$$\max \sum_{i=1}^{N} p_i \int_{0}^{t_i} f(\mu) \int_{0}^{t_i-\mu} \rho(s - \mu)\bar{H}(s - \mu)ds \; d\mu \qquad (16)$$

$$\text{S.T.} \sum_{i=1}^{N} t_i \leq T,$$

$$t_i \geq 0 .$$

The following special cases are of interest: taking $f(\mu)$ as an impulse function at $\mu = 0$ yields the model of Chapter 3; setting $\bar{H}(\tau) = 1$ for $\tau \geq 0$, yields the situation in which the target arrives late, but never departs (the model of Blackman (1959)). In this chapter $h(t)$ has an infinite domain, situations in which $h(\tau)$ or both $f(\tau)$ and $h(\tau)$ have finite domains are considered in Appendix D.

The solutions for this model depend upon the distributions of the time until detection, visibility period, and the start of the visibility period. We have already remarked on the appropriateness of

(a) exponential detectors

(b) an exponential distribution on the length of the visible period.

We shall consider two examples of start-time distributions.

Consider first the uniform distribution of starting times in the ith region, given by

$$f_i(\mu) = \begin{cases} \frac{1}{S_i}, & 0 \leq \mu \leq S_i \\ 0, & \mu > S_i \end{cases} .$$

Referring to equation 15, and dividing the region of integration into the appropriate parts one obtains[1]

[1]Formulations for discrete detectors are given in Appendix B. Continuous search space versions of the model are discussed in Appendix C.

$$P_i(t) = \begin{cases} \frac{1}{S_i}\left(\frac{k_i p_i}{\lambda_i + k_i}\right)\left[t - \frac{\left(1 - e^{-(\lambda_i+k_i)t}\right)}{(\lambda_i + k_i)}\right], \\ \qquad t \leq S_i , \\ \\ \frac{1}{S_i}\left(\frac{k_i p_i}{\lambda_i + k_i}\right)\left[S_i - \frac{\left(e^{-(\lambda_i+k_i)(t-S_i)} - e^{-(\lambda+k)t}\right)}{\lambda_i + k_i}\right] \\ \qquad \text{for } t > S_i \end{cases} \quad (17)$$

Note that the limit of (17) as $t \to \infty$ is

$$\lim_{t\to\infty} P_i(t) = \frac{k_i p_i}{\lambda_i + k_i} ,$$

which is equal to the limiting result for the model of Chapter 3. Finally we note that since

$$\frac{\partial^2 P(t)}{\partial t^2} = \begin{cases} \frac{k_i p_i}{S_i} e^{-(\lambda_i + k_1)t} \geq 0 , & t \leq S_i , \\ - \frac{k_i p_i}{S_i} \left[e^{-(\lambda_i + k_i)t} (e^{(\lambda_i + k_i)S_i} - 1) \right] , & t > S_i , \end{cases}$$

the conditional detection function is convex until $t = S_i$ and concave thereafter. We refer to this type of function as *pseudo-concave*.[1] Consider next the situation in which the start of the visibility period is exponentially distributed, i.e., $f(\mu) = \beta e^{-\beta \mu}$ for $\mu \geq 0$. Employing equation 15 for the ith region one obtains

$$P(t) = \begin{cases} \frac{k_i p_i}{\lambda_i + k_i} \left[1 - \frac{(\beta_i e^{-(\lambda_i + k)t} - (\lambda_i + k_i) e^{-\beta_i t})}{(\beta_i - (\lambda_i + k_i))} \right] , \\ \qquad \text{assuming } \beta_i \neq \lambda_i + k_i , \\ \frac{k_i p_i}{\lambda_i + k_i} \left[1 - e^{\beta_i t} (1 + \beta_i t) \right] , \\ \qquad \text{if } \beta_i = \lambda_i + k_i . \end{cases} \tag{18}$$

[1] These have been explored by Zahl (1963) and Onaga (1971); however, no attempt was made to provide examples of search situations which would yield such functions.

One can show that (assuming $\beta_i \neq \lambda_i + k_i$)

$$\frac{\partial^2 P(t)}{\partial t^2} \gtrless 0 \text{ for } t \lessgtr \bar{T} = \frac{1}{\beta_i - (\lambda_i + k_i)} \ln\left(\frac{\beta_i}{\lambda_i + k_i}\right),$$

and is negative for $t > \bar{T}$.

while for $\beta_i = \lambda_i + k_i$, the point $\bar{T}$ is given by

$$\bar{T} = \frac{1}{\beta_i} = \text{mean length of time until the target becomes visible.}$$

4.2 *Allocation of Search Effort to Maximize the Probability of Detection*

4.2.1 Model Solution

Consider first the situation in which the start time is uniformly distributed on [0,S] (see equation 17). An exact analytic (explicit) solution for this model cannot be attained.[1] The structure of the discrete search space version of the problem of maximizing the probability of detection under a constraint on the available search time is ideally suited for a

[1]This is primarily due to the fact that for $T < S$, $P(T)$ does not possess an inverse function.

dynamic programming approach. Hence, any subsequent numerical results for this model (regardless of the assumption on start times) will have been determined via dynamic programming, unless otherwise noted.[1]

Several important features can be determined from a marginal analysis of the model. The First Allocation Rule for this case may be stated as: Choose the region j for which

$$\frac{p_j k_j}{S_j} = \max_{1 \le i \le N} \frac{p_i k_i}{S_i} .$$

While we are unable to specify an analytic solution in the more general situation, it is possible to determine the solution provided T is large enough so that $t_1 > S_1$ and $t_2 > S_2$ in the 2-region case. The conditional detection probability can be written as

$$P(t_i) = \frac{k_i}{k_i + \lambda_i} \left\{ 1 - e^{-(\lambda_i + k_i)t_i} \left[\frac{e^{(\lambda_i + k_i)S} - 1}{(\lambda_i + k_i)S} \right] \right\}.$$

Setting

$$d_i = \frac{e^{(\lambda_i + k_i)S_i} - 1}{(\lambda_i + k_i)S_i} ,$$

[1] The program descriptions and listings are contained in Appendix E.

which is a constant, we note that the above function differs from that of the previous chapter by the constant d_i (independent of t_i). Accordingly, the techniques discussed in Appendix A are applicable here and one obtains for the 2-region case

$$t_1 = \frac{1}{\lambda_1 + k_1 + \lambda_2 + k_2} \left\{ \ln \frac{p_1 k_1 d_1}{p_2 k_2 A_2} + (\lambda_2 + k_2)T \right\},$$
$$t_2 = \frac{1}{\lambda_1 + k_1 + \lambda_2 + k_2} \left\{ \ln \frac{p_2 k_2 d_2}{p_1 k_1 A_1} + (\lambda_1 + k_1)T \right\}. \quad (19)$$

Extension to the N-region case is also easily obtained via the results of Appendix A.

Consider next the case of the exponential distributions leading to the conditional detection functions given by (18). The First Allocation Rule (FAR) in this case is given as: Choose the region j for which

$$p_j k_j \beta_j = \max_{1 \le i \le N} p_i k_i \beta_i$$

We also note that as $\beta_j \to +\infty$ for every j, this model tends toward that of Chapter 3. An interesting property of the

[1] Note the similarity to the optimal allocations for the model of Chapter 3.

pseudo-concave detection function, is that (in the case of identical regions) one doesn't start the allocation procedure by dividing the search effort equally among the identical regions as in the SKA or any other model with a concave conditional detection function (see Figure 8).

Again, while we are unable to determine an analytic solution in the most general situation, it is possible to approximate it provided one can make the following assumptions. First we note that for large quantities of search effort, (18), under the assumption $(\lambda + k) > \beta$, becomes

$$P(T) \simeq \frac{k\,p}{k + \lambda}\left[1 - \frac{(\lambda + k)e^{-\beta T}}{(\lambda + k) - \beta}\right] . \tag{20}$$

Also if one has $(\lambda + k) >> \beta$, then the requirement of large quantities of search effort may be dropped and (20) becomes

$$P(T) \simeq \frac{kp}{k + \lambda}\left[1 - e^{-\beta T}\right] . \tag{21}$$

Next we observe that in the situation in which $\beta > \lambda + k$ or $\beta >> (\lambda + k)$ the analogous equations (from (18)) become

$$P(T) \simeq \frac{pk}{k + \lambda}\left[1 - \frac{e^{-(\lambda+k)T}}{\beta - (\lambda + k)}\right], \tag{22}$$

and

$$P(T) \simeq \frac{pk}{k + \lambda}\left[1 - e^{-(\lambda+k)T}\right]. \tag{23}$$

In an N-region search problem, there are 2^N possible cases to consider. Here we shall exhibit one possible result for the 2-region situation.

Certainly, under both of the above assumptions, the resulting optimization problem has the same general form as those studied in Appendix A. In a 2-region search problem where $(\lambda_1 + k_1) > \beta_1$ and $\beta_2 > (\lambda_2 + k_2)$, one obtains for the approximate allocation scheme

$$t_1 = \frac{1}{(\lambda_2 + k_2 + \beta_1)} \left[\ln [z_1/z_2] + (\lambda_2 + k_2)T \right], \qquad (24)$$

and $t_2 = T - t_1$. Assuming $z_1 > z_2$, the approximate switch points are given by

$$T^* = \frac{1}{\beta_1} \ln [z_1/z_2] ,$$

where

$$z_1 = \frac{k_1 p_1 \beta_1}{(k_1 + \lambda_1) - \beta_1} \quad \text{and} \quad z_2 = \frac{k_2 p_2 \beta_2}{\beta_2 - (\lambda_2 + k_2)} .$$

The results for the other three cases in the 2-region problem follow from the appropriate substitutions into (24). Likewise, the extension to the N-region case is obtained via the results of Appendix A.

Assuming that

$$\lim_{T \to \infty} \frac{t_1}{t_2} \to \phi, \text{ a constant},$$

for the 2-region case in which $\beta_i = \lambda_i + k_i$, one obtains as approximate allocation policies

$$t_1 = \frac{1}{\beta_1 + \beta_2}\left[\ln \frac{p_1 k_1}{p_2 k_2} + 2 \ln \frac{\beta_1}{\beta_2} + \beta_2 T\right]$$

and

$$t_2 = \frac{1}{\beta_1 + \beta_2}\left[\ln \frac{p_2 k_2}{p_1 k_1} + 2 \ln \frac{\beta_2}{\beta_1} + \beta_1 T\right] .$$

Finally, we note that the above results can be specialized to the situation in which the target arrives at some random time after the start of the search and doesn't leave or become masked. One need only choose $\lambda_i = 0$ in each of the above examples, and, in general, assume that

$$H(\tau) = 1, \text{ for } \tau \geq 0.$$

4.2.2 Comparison with the Koopman Model

In this section we examine the situation in which a partially informed searcher applies the Koopman policies to situations in which the target behavior is actually characterized by the random interval of visibility process. Recall that the First Allocation Rule for the Koopman search problem is

Choose the region j for which

$$p_j k_j = \max_{1 \le i \le N} p_i k_i \ ,$$

while for the uniform-start-time model, one has as the FAR,

Choose the region j for which

$$\frac{p_j k_j}{S_j} = \max_{1 \le i \le N} \frac{p_i k_i}{S_i} \ .$$

Thus one could have the following situations:

(a) $p_1 k_1 \geq p_2 k_2$ and $S_1 > S_2$, which under SKA implies that Region 1 is selected first, but under the FAR for this model, assuming $\frac{p_1 k_1}{S_1} < \frac{p_2 k_2}{S_2}$, Region 2 is selected first.

(b) $p_1 k_1 > p_2 k_2$, $\frac{p_1 k_1}{S_1} > \frac{p_2 k_2}{S_2}$ and $T^* < S_1$, thus

(1) for $T \in [0, T^*]$ no error

(2) $T \in [T^*, S_1]$, SKA divides effort while the single interval policy allocates to Region 1 only.

(3) $T > S_1$, both policies divide the effort.

Figure 4 illustrates the respective policies in the situation in which, for T sufficiently small, a searcher using the Koopman policies will place all the search effort in the

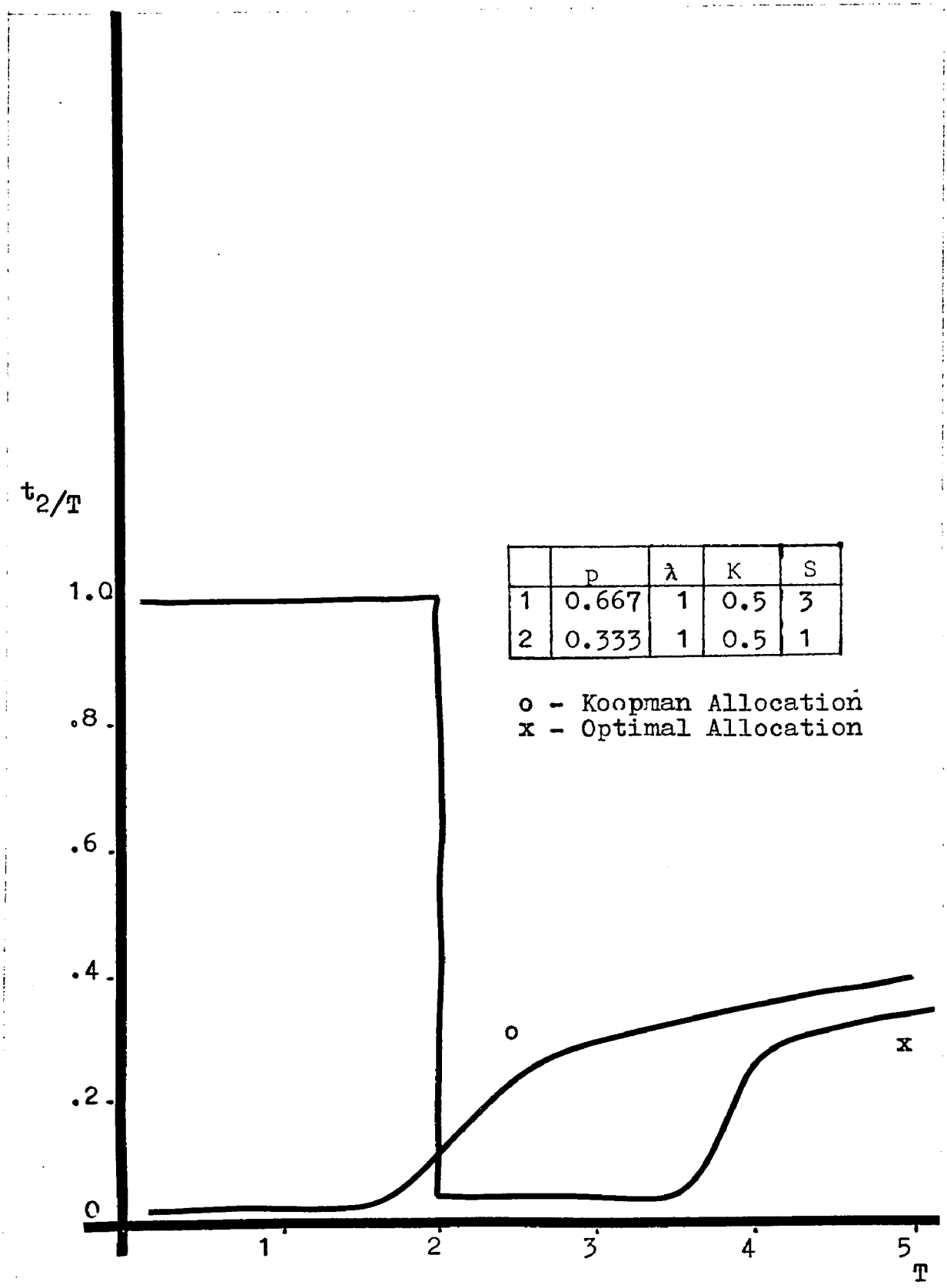
wrong region. It also illustrates a phenomenon heretofore unencountered in the search literature, vis., an optimal policy which is decreasing with increasing total available effort.[1] In the example shown, the optimal policy is such that for small values of search time all the effort is placed in Region 1, and finally for larger levels of search time the optimal policy begins to divide the time. We note that the Koopman policy is nondecreasing with increasing effort, as shown in Figure 4. Figure 5 is a plot of the percent relative error in the probability of detection when the Koopman policy is used for the allocations described in Figure 4. Note that for the levels of visibility parameters of Figure 4, once sufficient search effort is available, the parameters satisfy the conditions noted in Section 3.2.2 for agreement with the Koopman allocation--hence, the rapid decay of the error function.

One can make use of the analytic expression obtained in Section 4.2.1 to exhibit an approximate error function. For 2-regions, one obtains as an approximate error expression,[2]

---

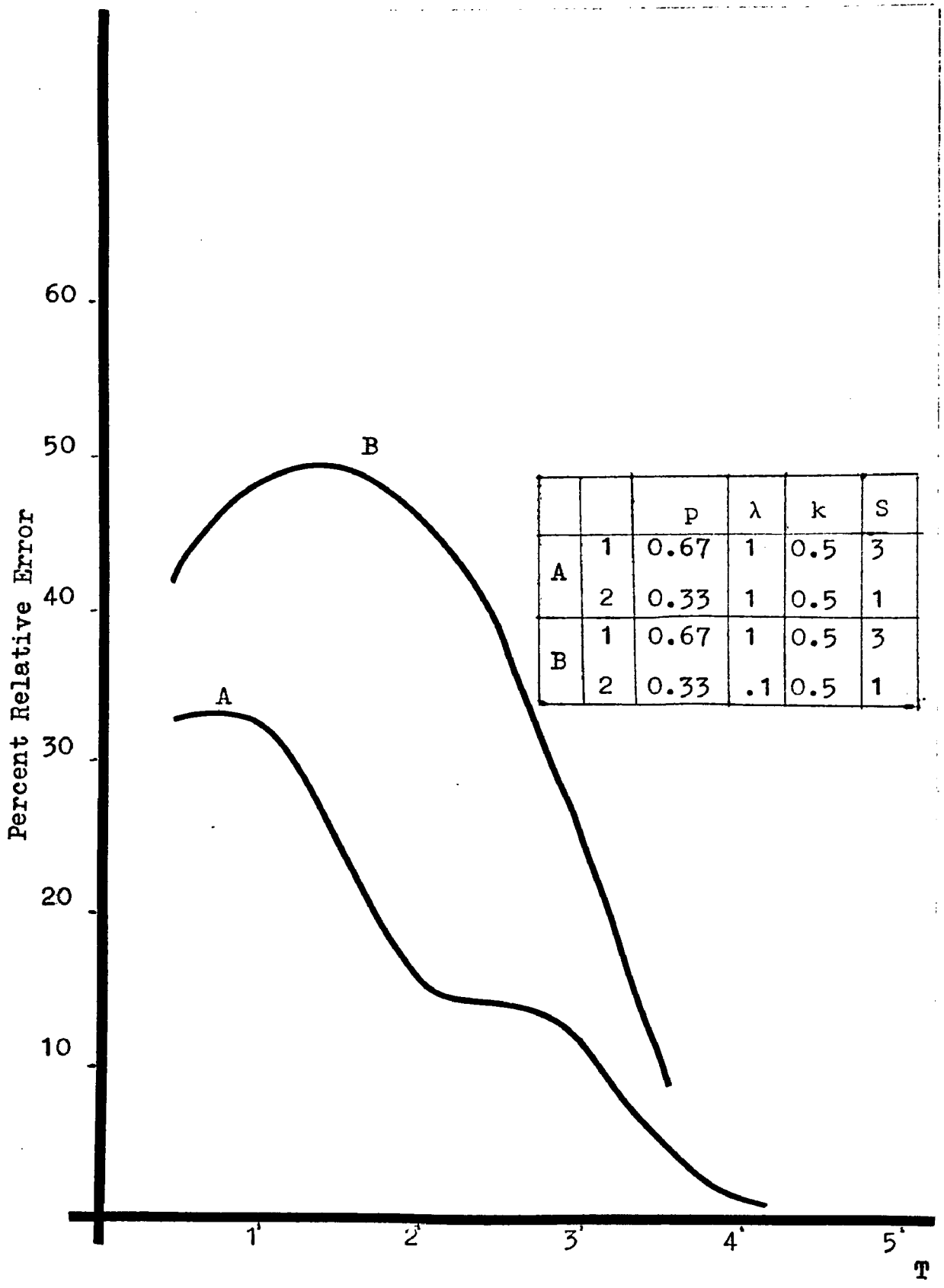
[1] An exception to this statement is the work of Kronz (1971) conducted simultaneously with this effort at the Systems Research Laboratory, The University of Michigan.

[2] Note the similarity between (25) and the error function for the model of Chapter 3, equation 14.



	p	λ	K	S
1	0.667	1	0.5	3
2	0.333	1	0.5	1

Figure 4 - A Comparison of Allocation Policies



		p	λ	k	S
A	1	0.67	1	0.5	3
	2	0.33	1	0.5	1
B	1	0.67	1	0.5	3
	2	0.33	.1	0.5	1

Figure 5 - Percent Relative Error versus Search Time

$$E(T) = \frac{p_1 k_1 d_1}{k_1 + \lambda_1}\left[e^{-\frac{(\lambda_1+k_1)k_2 T}{k_1+k_2}} - e^{-\frac{(\lambda_1+k_1)(\lambda_2+k_2)T}{\lambda_1+k_1+\lambda_2+k_2}}\right] + \frac{p_2 k_2 d_2}{k_2 + \lambda_2}\left[e^{-\frac{(\lambda_2+k_2)k_1 T}{k_1+k_2}} - e^{-\frac{(\lambda_1+k_1)(\lambda_2+k_2)T}{\lambda_1+k_1+\lambda_2+k_2}}\right] \quad (25)$$

We use the term "approximate" because in developing the above expression it was assumed that the total available effort, T, was allocated as follows by the respective policies:

Koopman	Single Interval
$t_1 \simeq \frac{k_2}{k_1 + k_2} T$	$t_1 \simeq \frac{(\lambda_2 + k_2)T}{k_2 + \lambda_2 + k_1 + \lambda_1}$
$t_2 \simeq \frac{k_1}{k_1 + k_2} T$	$t_2 \simeq \frac{(\lambda_1 + k_1)T}{k_2 + \lambda_2 + k_1 + \lambda_1}$

(26)

These allocations were obtained from equations 19 by neglecting the constant terms.

Considering the exponential-start-time model, recall that the First Allocation Rule is

Choose the region j for which

$$p_j k_j \beta_j = \max_{k_i \leq N} p_i k_i \beta_i \, .$$

The combination of differing FAR and differing switch points can lead to the following situations when one searches according to the Koopman policy.

(a) $p_1k_1 > p_2k_2$, $\beta_2 > \beta_1$, and $\beta_1p_1k_1 < \beta_2p_2k_2$, thus the SKA chooses the wrong region under an extremely constrained search time situation.

(b) Or, one could have $\beta_1p_1k_1 > \beta_2p_2k_2$ and $p_1k_1 > p_2k_2$ and

$$\bar{T} >> T^*.$$

In this case, one has

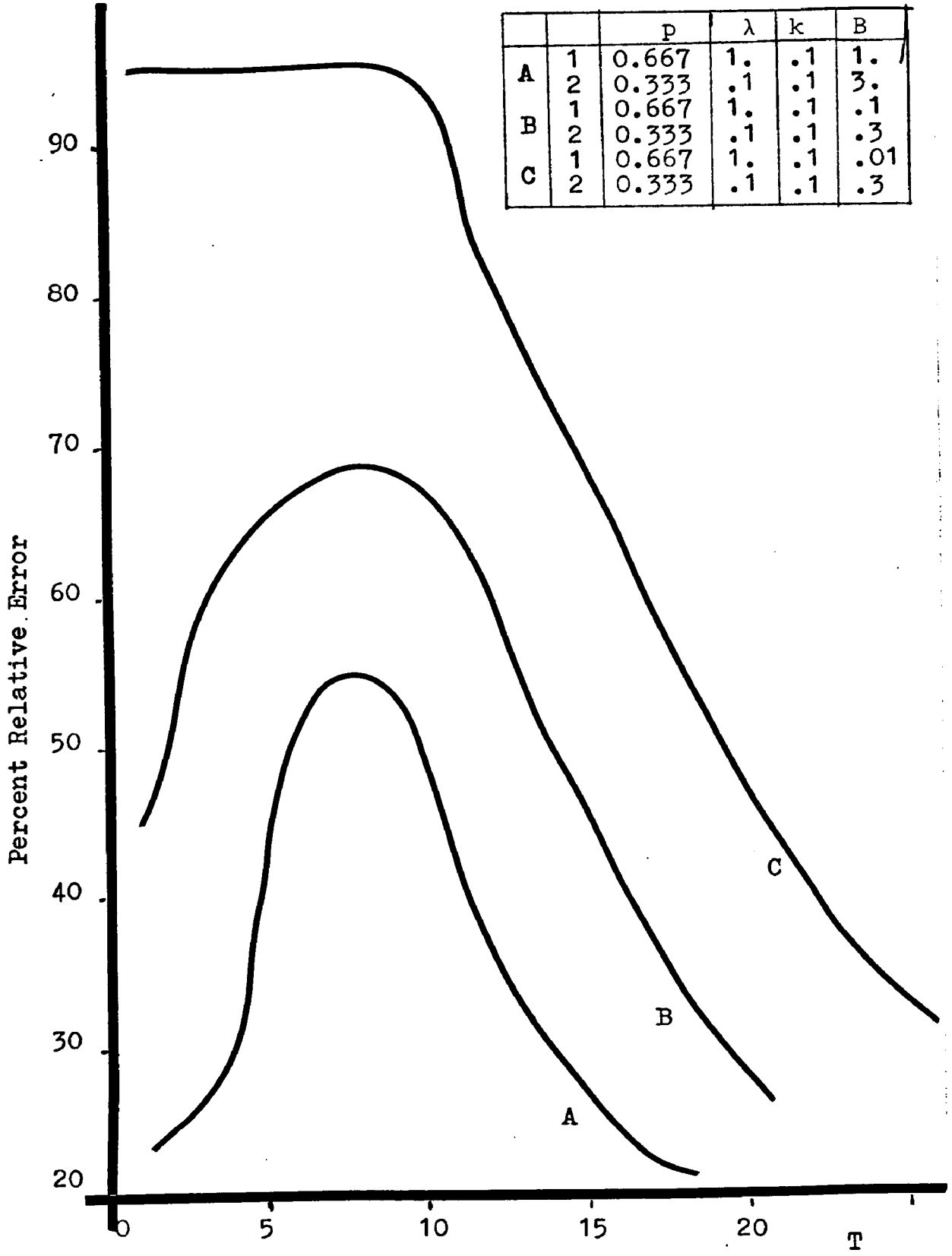
(1) $t \in [0,T^*]$, no error

(2) $t \in [T^*,\bar{T})$, a division of effort under SKA, all effort in region (1) under the optimal policy,

(3) $t \in [\bar{T},\infty)$, a division of effort under both policies.

Figure 6 is a plot of percent relative error in the probability of detection under the SKA policy, and Figure 7 gives the probability of detection under the optimal policy.

Figure 6 presents the results of an analysis on the effects of variations in the rates of the process controlling the length of time until the target becomes visible. Observe that as the respective rates decrease, the percent relative error curve increases everywhere; and that in each case, a searcher using the Koopman policy chooses the wrong initial region. Also note that at $T \simeq T^*$, the Koopman policy begins to allocate some effort to Region 1 and the relative error function begins to decrease. Figures 6 and 7



		p	λ	k	B
A	1	0.667	1.	.1	1.
	2	0.333	.1	.1	3.
B	1	0.667	1.	.1	.1
	2	0.333	.1	.1	.3
C	1	0.667	1.	.1	.01
	2	0.333	.1	.1	.3

Figure 6 - Error Function Sensitivity

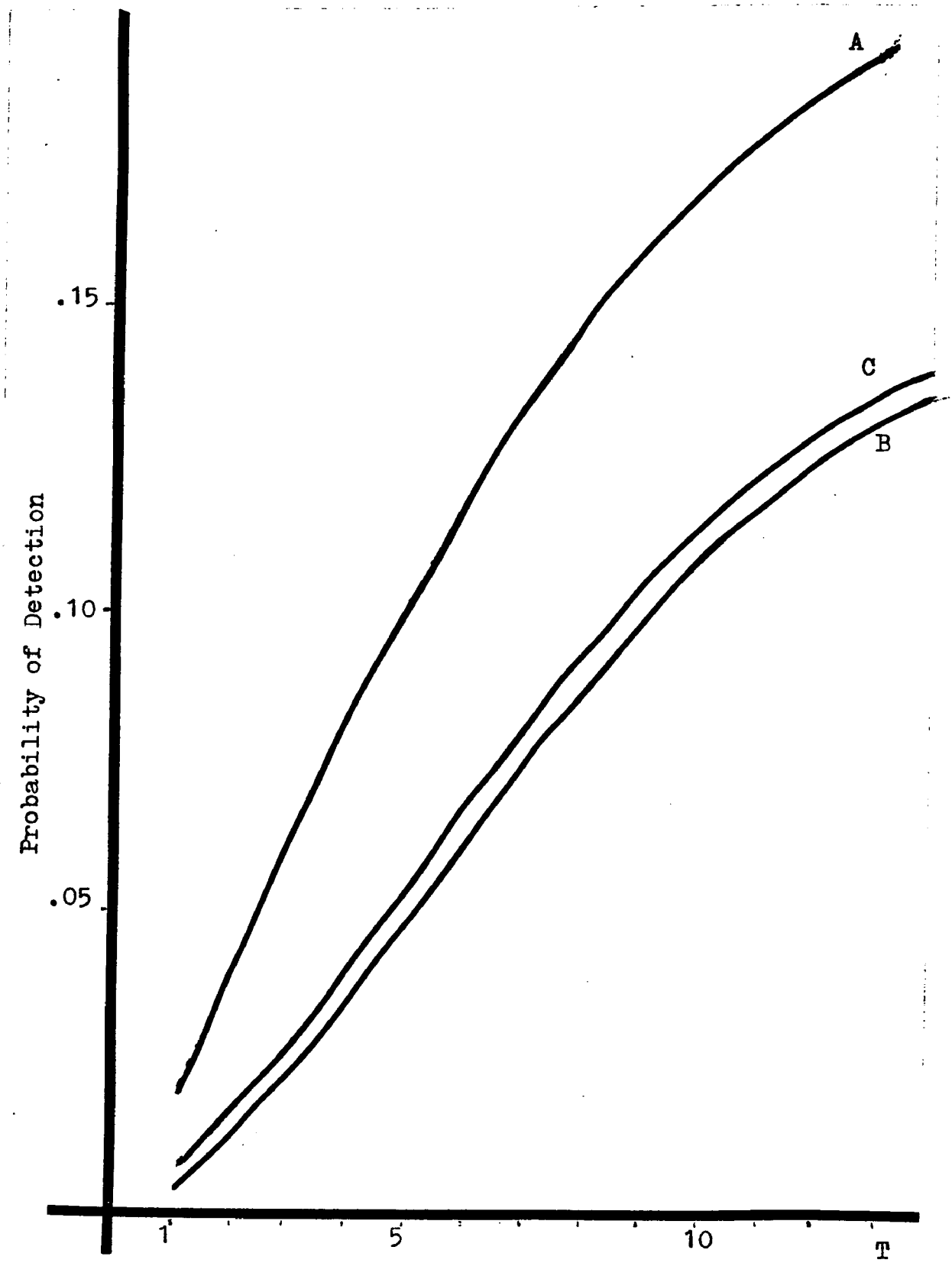


Figure 7 - Optimal Return Versus Search Time

enable one to determine the actual magnitudes of these errors, in every case the limiting value of the probability of detection is 0.2265 (obtained from equation (16) when $T \to \infty$).

Figure 8 contains some results which are characteristic of the pseudo-concave detection functions for the random interval model. The figure depicts the optimal allocation policy for a 4-region problem in which the regional search parameters are identical. The Koopman policy in this situation divides the search time evenly among the regions, but in the random interval model such is not the case. For the 2-region case in which

$$(\lambda_1 + k_1) > \beta_1, \beta_2 > (\lambda_2 + k_2) ,$$

and T is large, one obtains the following approximate error function[1]

$$E(T) = \frac{p_1 k_1}{k_1 + \lambda_1} \left\{ e^{-\frac{\beta_1 k_2 T}{k_1 + k_2}} - e^{-\frac{\beta_1 (\lambda_2 + k_2) T}{\beta_1 + \lambda_2 + k_2}} \right\} + \frac{p_2 k_2}{k_2 + \lambda_2} \left\{ e^{-\frac{(\lambda_2 + k_2) k_1 T}{k_1 + k_2}} - e^{-\frac{\beta_1 (\lambda_2 + k_2) T}{\beta_2 + \lambda_2 + k_2}} \right\} . \quad (27)$$

---

[1] One can interpret $\lambda_i + k_i$ as the combined detection rate under the visibility process, hence $1/(\lambda_i + k_i)$ is the conditional mean time until detection, conditioned on detection occuring in that region. Thus the approximate allocation policies are always determined by using the smaller of the two rates, the combined detection rate, and the rate for the start of the visible period.

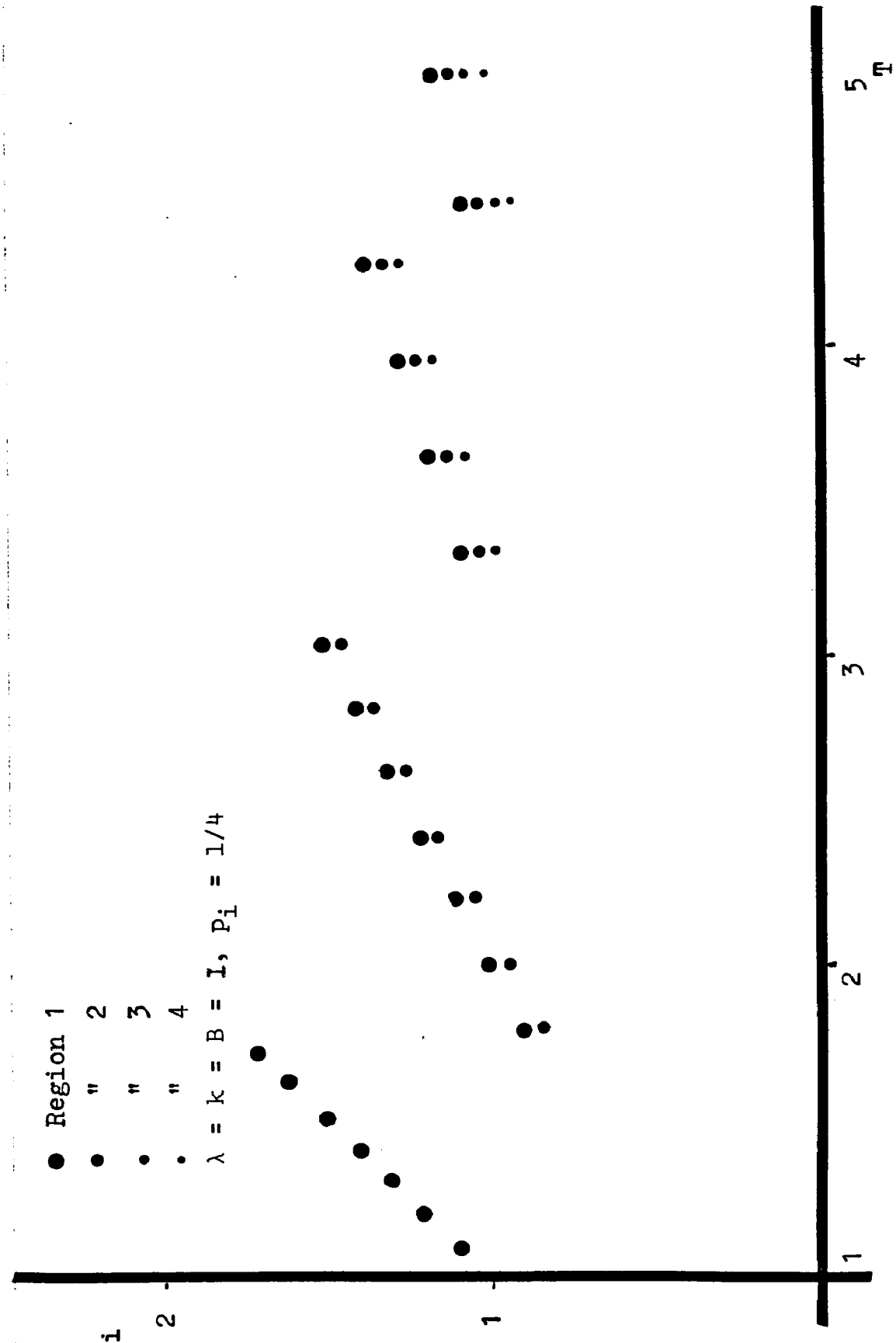


Figure 8 - The Allocation Policy for Four Identical Regions

Again, in deriving the above expression it was assumed that the total search time T was allocated as follows by the respective policies (see equations 24):

Koopman	Single Interval
$t_1 \simeq \frac{k_2 T}{k_1 + k_2}$	$t_1 = \frac{(\lambda_2 + k_2)T}{\beta_1 + (\lambda_2 + k_2)}$
$t_2 \simeq \frac{k_1 T}{k_1 + k_2}$	$t_2 = \frac{\beta_1 T}{\beta_1 + (\lambda_2 + k_2)}$

(28)

Equation 27 can be used to study the sensitivity of the error function to changes in the search parameters as well as to determine the amount of search time required to reduce the error function to some specified value.

Figure 9 is an example of the situation in which the proper combination of parameters cam yield an error function which continues to increase over a large range of total available search effort. In Figure 9, one has $\lambda_1 + k_1 > \beta_1$ and $\lambda_2 + k_2 > \beta_2$, hence, as we have already observed, the approximate long-term allocations become

$$t_1 = \frac{\beta_2}{\beta_1 + \beta_2} T \text{ and } t_2 = \frac{\beta_1}{\beta_1 + \beta_2} T \quad ,$$

which for this situation are the reverse of the long-term Koopman allocation policy. In other words the dominant process in this example is the length of time until the target becomes visible.

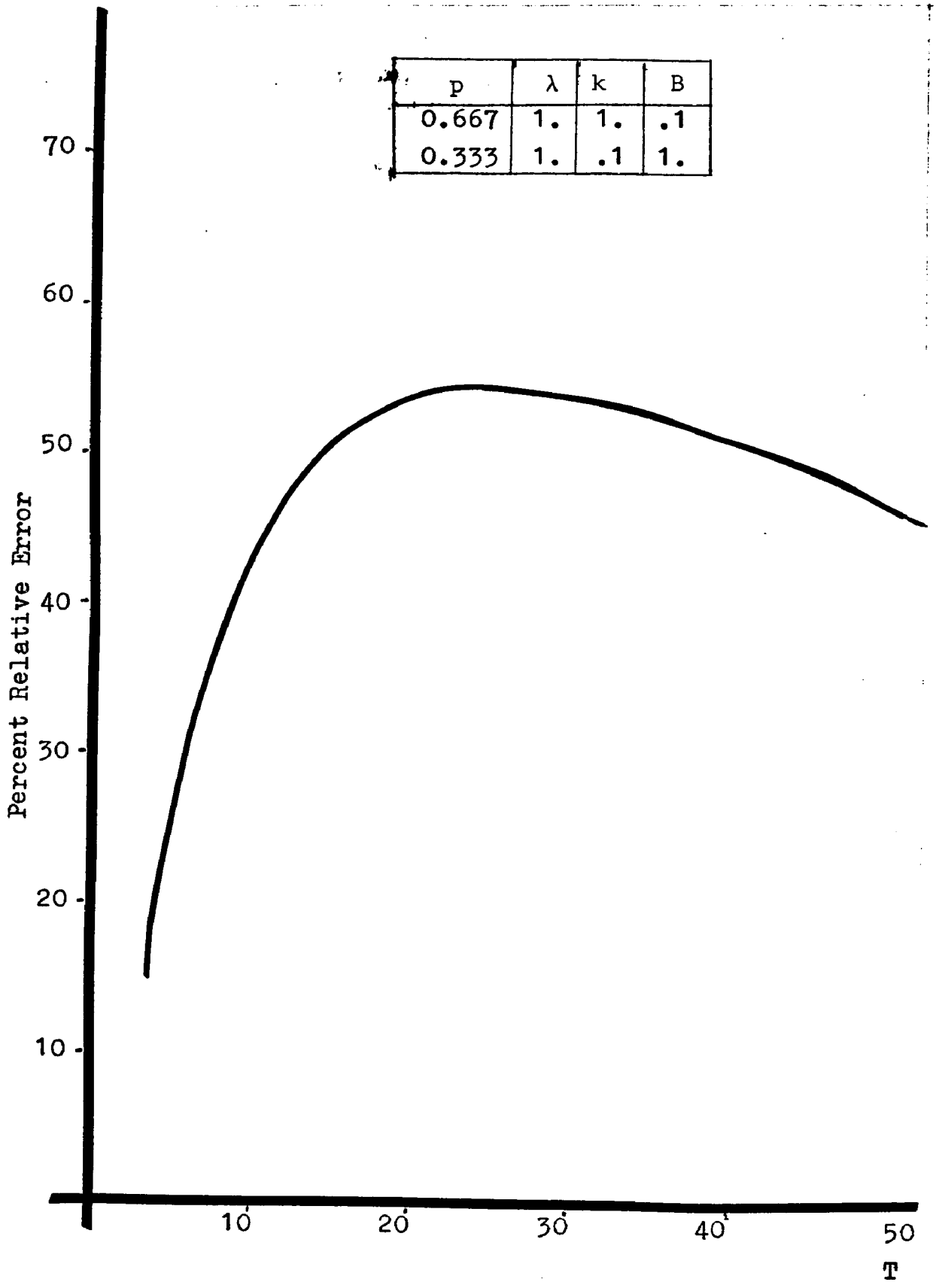


Figure 9 - Percent Relative Error Versus Search Time

4.2.3 Model Sensitivity

In this section we present the results of an analysis of the sensitivity of both the optimal policy and its associated return to changes (or errors) in the visibility parameters. The approximate allocation policies (equation 26) and the resulting returns obtained in Section 4.2.2 are used. Thus we are implicitly assuming that sufficient search time is available for these approximations to hold.

As was noted, the optimal policy for the uniform start time model, under the assumption that $T >> 1$, is identical to that for the model of Chapter 3. Hence we need only consider the exponential start time model.

For the exponential-start-time model, the sensitivity of the optimal allocations is determined from the approximate solution. For the example of this section (i.e., $(\lambda_1 + k_1) > \beta_1$ and $\beta_2 > (\lambda_2 + k_2)$), one has the following partial derivatives:

$$\frac{\partial t_1}{\partial \beta_1} = - \frac{t_1}{\lambda_2 + k_2 + \beta_1} , \qquad \frac{\partial t_1}{\partial \lambda_2} = \frac{t_2}{(\lambda_2 + k_2 + \beta_1)}$$

$$\frac{\partial t_2}{\partial \lambda_2} = - \frac{t_2}{\lambda_2 + k_2 + \beta_1} , \qquad \frac{\partial t_2}{\partial \beta_1} = \frac{t_1}{(\lambda_2 + k_2 + \beta_1)} .$$

Since the t_i are linear functions of the total available search effort, the sensitivity of the optimal allocations to the visibility parameters increases with increasing available search effort. In view of the assumptions required for the approximate solution, i.e., $\lambda_1 + k_1 > \beta_1$ and $\beta_2 > \lambda_2 + k_2$, for a fixed

level of search time, the optimal policy will be highly sensitive to changes in the relevant visibility parameters λ_2 and β_1.

The ramifications of the sensitivity of the optimal policy must be evaluated in light of the robustness of the return function, since we are only considering situations in which the total search time is large enough to permit one to utilize the approximate solutions. Accordingly, the partial derivatives of the return function with respect to the visibility parameters are given by

$$\frac{\partial P(T)}{\partial \beta_1} = \frac{(\lambda_2 + k_2)^2 T}{(\beta_1 + \lambda_1 + k_2)^2} e^{-\frac{\beta_1(\lambda_2+k_2)}{\beta_1+\lambda_2 k_2} T} \left\{ \frac{p_1 k_1}{\lambda_1 + k_1} + \frac{p_2 k_2}{\lambda_2 + k_2} \right\}$$

$$\geq 0$$

$$\frac{\partial P(T)}{\partial \lambda_2} = -\frac{p_2 k_2}{(\lambda_2 + k_2)^2} \left(1 - e^{-\frac{\beta_1(\lambda_1+k_1)T}{\lambda_2+k_2+\beta_1}} \right) + \frac{\beta_1 t_2}{\beta_1 + \lambda_2 + k_2}$$

$$\cdot \left\{ \frac{p_1 k_1}{\lambda_1 + k_1} + \frac{p_2 k_2}{\lambda_2 + k_2} \right\} e^{-\frac{\beta_1(\lambda_2+k_2)T}{\beta_1+\lambda_2+k_2}} .$$

As $T \to \infty$, one has

$$\frac{\partial P(T)}{\partial \beta_1} \simeq 0 \text{ and } \frac{\partial P(T)}{\partial \lambda_2} = -\frac{p_2 k_2}{(\lambda_2 + k_2)^2} .$$

Intuitively, the signs of these partials are reasonable since (a)

increasing β_1 decreases the mean time until the target becomes visible in the first region allowing (in a fixed total time T) more time for searching while the target is visible, thus increasing the probability of detection; (b) increasing λ_2 has just the opposite effect in region 2, thus reducing the overall probability of detection. Thus, although the optimal policy becomes increasingly sensitive to changes in the visibility parameters with increasing search time, the sensitivity of the return function is limited by the above expressions. Hence, beyond a certain level of search effort, no matter how great the changes in the optimal policy due to changes in the λ_i, the change in the optimal return is essentially constant. Thus whenever λ and k are large it is robust.

Suppose the searcher had some means of controlling the visiblity parameters with an associated cost, but not eliminating them. The above expressions yield the loss or gains in the return function per unit change in these parameters. On the other hand, the decision maker could weigh this action against that of using additional search time and then determine the most cost-effective option. To illustrate the relative magnitudes of these partial derivatives, we consider the search situation determined by $\mathbb{P}$ = (2/3, 1/3), $\mathbb{K}$ = (1, .1), λ = (1, .1), β = (.1, 1), one has then

$$\left.\frac{\partial t_1}{\partial \lambda_2}\right|_{T=20} = 22.2 \ , \quad \left.\frac{\partial t_2}{\partial \beta_1}\right|_{T=20} = 44.4 \ ,$$

$$\left.\frac{\partial P(T)}{\partial \lambda_2}\right|_{T=20} = -0.122 \text{ and } \left.\frac{\partial P(T)}{\partial \beta_1}\right|_{T=20} = 1.18 \ .$$

Thus in situations in which the detection and visibility rates are low, both the optimal policy and return are sensitive to changes in the visiblity parameters.

Appendix F contains the description and results of a numerical study of the error function over the entire range of total available search effort in contrast to the above study of the approximate results. These results are briefly summarized as follows. In contrast to the models of Chapters 2 and 3,

(a) the use of the Koopman policy will not be adequate in situations in which the visibility conditions are homogeneous across the regions of interest,

(b) the error function is not reduced by increasing the available search time to some realistic level,

(c) the maximum errors do not occur in situations in which the total available search time is highly constrained (i.e., less than or equal to the Koopman switch time).

(d) the availability of extremely good detectors (high rates) will not imply that the Koopman policy will yield small errors.

These results imply that the decision maker, in general, must obtain accurate estimates of the visibility parameters in order to conduct an effective search, since his option of

increasing the available search time may no longer be cost-effective.

4.3 Minimization of the Expected Time Until Detection

4.3.1 Model Solution

In its most general form this model has the property that the limiting value of the probability of detection under the optimal policy is less than unity. Thus the expected time to detect the target increases without limit. However, one can, as in Section 3.3.1, consider the conditional expected time until detection, conditioned on detection occurring. We consider first the special case in which the target may arrive (or appear) at some random time after the start of the search and remain (stay visible). From the general expressions of the detection probabilities of Section 4.1 we observed that for $T \to \infty$, the limiting value of the probability of detection is unity. For the purpose of the following discussion, we restrict attention to this case.

Recall the discussion of the early results of Dobbie (1963) in Chapter 1. Namely, that one can always express the expected time until detection as

$$E = \int_0^\infty (1 - P(\phi(T))dT,$$

where $P(T)$ = probability of detection under the allocation policy $\phi(T)$. If $\phi(T)$ is nondecreasing in T, and maximizes

$P(\phi(T))$ for each value of T, it also is the optimal policy for the expected time problem. From the examples of Section 4.2.2, we recall that the optimal policies for the maximization of the probability of detection did not necessarily have the property that the amount allocated a given region was non-decreasing as a function of the available search effort (see Figure 4). The reason for this was the fact that the detection functions for this model are pseudo-concave. Thus from the above we may conclude that the optimal allocation policies for the two search objectives are different, at least for small values of search effort.

How then does one generate the optimal policy for the expected time problem in this situation? Intuitively, it is clear that the two policies will have to agree almost everywhere. Onaga (1971) studied the most general form of the expected time problem. As a result of his work, one may obtain the optimal policy by solving the problem of maximizing the probability of detection using a modified form of the detection function. In particular, the form used is the "minimal concave majorant[1]" of the detection function which is constructed as follows. Graphically the majorant function for a pseudo-concave function (dashed curve) is shown as the solid curve in the figure below.

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[1]This term was apparently defined by Stone (1971) since it is not widely used in the mathematical literature.

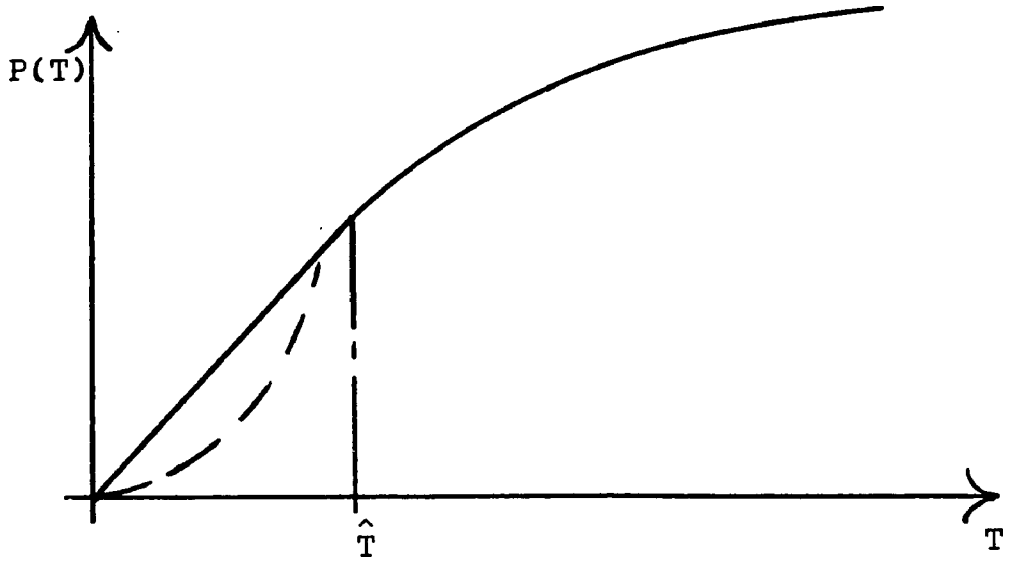

Figure 10. Minimal Concave Majorant

The point $\hat{T}$ is of some interest since the composite curve formed by the straight line from the origin to $(\hat{T}, P(\hat{T}))$ and the curve $P(T)$ for $T > \hat{T}$ is the minimal concave majorant for the entire function. $\hat{T}$ is determined by observing that the straight line of Figure 10 has the form

$$P(T) = P'(\hat{T})T \ , \qquad (29)$$

in order that it be tangent to the curve $P(T)$ at $T = \hat{T}$. The solution of (29)then yields $\hat{T}$. In the event one has to resort to numerical techniques, a lower bound on $\hat{T}$ exists, namely, the upper bound on the region of convexity for the detection function , $\overline{T}$.

The modified optimization problem which yields the optimal policy for the original expected time problem is

$$\text{Max } P(T) = \sum_{i=1}^{N} \begin{cases} P'(\hat{T}_i)t_i, & t_i \leq \hat{T}_i \\ P(t_i), & t_i > \hat{T}_i \end{cases} \qquad (30)$$

$$\text{S.T.} \sum_{i=1}^{N} t_i \leq T$$

$$t_i \geq 0 \ ,$$

for all $T > 0$.

The solution is obtained by applying the Kuhn-Tucker (K-T) conditions to the above concave programming problem. First the K-T conditions require that for the optimal solution, when $T < \text{Min } (\hat{T}_i)$ (where Ψ is the Kuhn-Tucker multiplier)

$$P_i(\hat{T}_i) = P'_i(t_i) = \Psi$$

for all $t_i > 0$, and when

$$\min (\hat{T}_i) < T < \sum_{i=1}^{N} \hat{T}_i \ ,$$

$$P'_i(t_i) = \Psi = P'_j(\hat{T}_j)$$

for some j for which $t_j < \hat{T}_i$. The last statement implies that once one starts allocating according to the FAR, say in the ith region, one continues to allocate to that region until

$$P'_i(t_i) = P'_j(\hat{T}_j),$$

when $t_i \geq \hat{T}_i$. Then one begins to place *all* the additional time into the next region, j. This process is repeated

until that value of total available time, T, is attained such that the optimal policy has the property that

$$t_i > \hat{T}_i$$

for every i. Note this procedure is utilized to determine the allocation policy, the return for that policy is, of course, obtained by substituting the policy into the *original* problem.

To briefly summarize, we have discussed the procedure for solving the expected-time problem. Algorithmically,

1. Solve equation 29 for the $\hat{T}_i$, i = 1,2,...,N.
2. Solve the maximization problem (equation 30) for all T > 0.
3. Using the results of Step (2), form

$$E = \int_0^\infty (1 - P(T))dT \ .$$

In the following paragraphs, we discuss the approach taken in each of the above steps. The solution of equation 29 requires that one solve a set of equations for the models of this chapter. First, the situation in which the distribution on the start of the visible period is uniform on (0,S) yields the following equation in $\hat{T}$, for the ith region,

$$\hat{T}_i = \left\{ \frac{S_i}{(e^{(\lambda_i+k_i)S_{i-1}} - 1)e^{-(\lambda_i+k_i)\hat{T}_i}} - 1 \right\} \frac{1}{\lambda_i + k_i} \cdot$$

For the exponentially-distributed-start-time models, one obtains

(a) $(\lambda_i + k_i) \neq \beta_i$

$$\hat{T} = \frac{1}{\beta_i(\lambda_i+k_i)} [\beta_i-(\lambda_i+k_i)]e^{(\lambda_i+k_i)\hat{T}} + (\lambda_i+k_i) \cdot$$

$$(\beta_i\hat{T} + 1)e^{-(\beta_i-(\lambda_i+k_i)\hat{T}} - \beta_i$$

(b) $(\lambda_i + k_i) = \beta_i$

$$\hat{T} = \frac{1}{\beta_i}(e^{-\hat{T}} - \beta_i^2T^2 - 1)$$

Since the above expressions were derived from the pseudo-concave detection functions of this chapter we are assured of positive roots, other than $\hat{T} = 0$.

Having obtained the $\hat{T}_i$ for each region, we solve the resulting maximization problem using dynamic programming. The very nature of a dynamic programming solution allows us to obtain the optimal policy for all values of T up to some maximum, say T_{max}.[1]

Finally, the output from Step (2), the value of the probability of detection for each level of total effort using the

[1]Appendix E contains the program listings for these computations .

above policy, becomes the input to a numerical integration routine in order to compute the resulting expected time until detection.

There are two types of errors in this procedure: the first is the error in using numerical quadrature; and the second, possibly more important error, is that obtained from the restriction of the region of integration to the range $[0,T_{max}]$. Since expressions for the first type of error are readily obtained (Brand, 1960), we shall make some observations concerning the second error source. In Section 4.2, it was observed that the asymptotic form of the expression for the probability of detection was exponential in nature (See equation 22), the rate being a function of the search parameters. Given these facts, it seems reasonable to assume that one may readily determine an ξ such that

$$e^{-\xi T} \geq 1 - P(T) \text{ for } T > T_{max} .$$

Then the following relationship holds

$$\int_0^{\infty} e^{-\xi T}\, dT - \int_0^{T_{max}} e^{-\xi T}\, dT \geq \int_0^{\infty} (1 - P(T))\, dT - \int_0^{T_{max}} (1 - P(T))dT,$$

or

$$\frac{e^{-\xi T_m}}{\xi} \geq \int_0^{\infty} (1 - P(T))dT - \int_0^{T_{max}} (1- P(T))dT$$

Thus the above expression forms an upper bound on the second source of error in the numerical procedure. Of course, if we specify an upper bound for this source of error, say ε, then one may solve for the corresponding value of T_{max}, e.g.,

$$T_{max} > \frac{-1}{\xi} \ln (\varepsilon\xi) .$$

Finally, if one removes the assumptions made at the beginning of this section, i.e., that once the target appears it remains visible, then the expected time to detect the target increases without limit under the optimal policy. However, if we restrict our attention to the conditional expected time to detect, conditioned on detection occurring, then one can use the above procedure to determine the optimal allocation policy and the resulting value for the conditional expected time until detection.

4.3.2 Comparison with the Koopman Model

The procedure just outlined can also be utilized to determine how the Koopman allocation policies perform when minimizing the conditional expected time until detection. That is, one generates the values of the objective function for the problem of maximizing the probability of detection using the Koopman allocation policy for increasing levels of total available search time, then carries out the numerical integration required to compute the conditional expected time

until detection, conditioned upon detection ultimately occurring. Table 7 contains the results of a study of the sensitivity of the Koopman policy to changes in the detection rates for a 2-region search situation in which the

(a) searcher has the maximum uncertainty in the target's location ($p_1 = p_2$), and

(b) visibility parameters are identical.

Observe that the error (the ratio of the conditional expected times until detection) is minimized in the situation in which the detection rates are identical. Furthermore, when the detection rates are large relative to the visibility rates, the error is acceptable. Finally, the significant errors occur in the situation in which the detection rates are unequal[1].

4.4 General Single Interval of Visibility Model

4.4.1 Description

In this section we consider the situation in which the target may be either visible or masked with respect to the searcher whenever the latter enters the appropriate region. The probability vector describing these target states is

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[1]The reader is referred to Figure 21, 5.3.2 for a comparison of the late-arrival version of the single interval model to the analogous multiple interval models. In addition, Section F.4 contains the maximum errors and their times of occurrence for the parameters of Table 7, as well as other combinations of parameters.

Table 7

The Sensitivity of the Conditional Expected Time Until Detection to the Detection Rates for the Random Interval Model

k_1 \ k_2	.1	1.
.1	1.3	1.9
1.	1.9	1.+

Ratio of Conditional Expected Times Until Detection

k_2 \ k_1	.1	1.
.1	28.9	30.1
1.	30.1	20.3

The conditional expected time until detection under the optimal policy, where $p_1 = p_2 = 1/2$, $\lambda_1 = \lambda_2 = 1.$, and $\beta_1 = \beta_2 = 0.1$.

$\overline{\Pi} = (\Pi, 1-\Pi)$, where Π is the visibility probability and $1-\Pi$ the masking probability. The vector $\overline{\Pi}$ is assumed known for each of the regions of interest. One might, for example, consider $\overline{\Pi}_j = (\Pi_j, 1 - \Pi_j)$ to be a vector representing environmental conditions affecting the searcher's detection gear. Physically, the single interval of visibility could represent the length of time required for the target to become aware of the searcher's presence. As an example, consider the scenario of a search for a submarine. The masked state is generated by the fact that the submarine may not have arrived on station at the time at which the search begins. The visible period could be generated by his length of time on station. The vector $\overline{\Pi}$ represents the searcher's prior probability on whether or not the target is on station yet; the vector P, the searcher's prior probability on target location.

In general, the target is visible with probability Π upon the entry of the searcher into the appropriate region. Given that the target is visible, then one has from Chapter 3 that the conditional probability of detection, given that t time units are spent searching, is

$$P(t) = \int_0^t \rho(\tau) H(\tau) d\tau .$$

On the other hand, given that the target is masked upon the searcher's entry, one has that

$$P(t) = \int_0^t f(u)M(t-u)du$$

where

$$M(t-u) = \int_0^{t-u} (s-u)\bar{H}(s-u)ds .$$

Then for the ith region, we have that the probability of detection, given t time units spent searching, is

$$P_i(t) = P_i \Pi_i M_i(t) + (1 - \Pi_i) \int_0^t f_i(u)M_i(t-u)du \quad ,$$

$$= P_i \Pi_i \int_0^t \rho_i(\tau)H_i(\tau)d\tau + (1 - \Pi_i) \int_0^t f_i(u)M_i(t-u)du \quad (31).$$

4.4.2 Model Solution

Let $\Pi = (\Pi_1, \ldots, \Pi_N)$ be a vector containing the probabilities that the target is visible at the start of the search in each region. We shall consider the example of Section 4.2 in which the lengths of visible periods and the times until the start of the same, given that the target is not visible with probability $1 - \Pi_i$, are exponentially distributed. For $\beta_i \neq (\lambda_i + k_i)$ the probability of detection under these assumptions is

$$P_i(t) = \Pi_i\left(\frac{p_i k_i}{k_i + \lambda_i}\right)\left\{1 - e^{-(\lambda_i+k_i)t}\right\} \tag{32}$$

$$+ (1 - \Pi_i)\left\{\frac{k_i p_i}{\lambda_i + k_i}\left[1 - \frac{[\beta_i e^{-(\lambda_i+k_i)t} - (\lambda_i + k_i)e^{-\beta_i t}]}{\beta_i - (\lambda_i + k_i)}\right]\right\},$$

and for $\beta = \lambda_i + k_i$

$$P_i(t) = \Pi_i\frac{p_i k_i}{(k_i + \lambda_i)}\left\{1 - e^{-(\lambda_i+k_i)t}\right\} \tag{33}$$

$$+ (1 - \Pi_i)\left\{\frac{k_i p_i}{(\lambda_i + k_i)}\left[1 - e^{-\beta_i t}(1 + \beta_i t)\right]\right\}.$$

These detection functions are either strictly concave or pseudo-concave, i.e., convex until some point, $\bar{T}$, and concave thereafter. For equation 32 the convex region is

$$t_i \leq \frac{1}{\beta_i - (\lambda_i + k_i)} \ln\left\{\frac{(1 - \Pi_i)\beta_i^2}{(\lambda_i + k_i)[\beta_i - \Pi_i(\lambda_i + k_i)]}\right\}, \tag{34}$$

In the event the right-hand side is less than or equal to zero, the detection function is strictly concave. For equation 33, the condition for pseudo-concavity becomes

$$t_i \leq \frac{1 - 2\Pi_i}{\beta_i(1 - \Pi_i)}. \tag{35}$$

The First Allocation Rule (FAR) for this model is:

Choose that region j for which

$$p_j k_j \Pi_j = \max_{1 \le i \le N} p_i k_i \Pi_i \quad .$$

Note that this is the Koopman FAR weighted by the probability that the target is visible. For small values of T, the rule becomes:

Choose j such that

$$p_j k_j \Pi_j + p_j k_j T(\beta_j(1 - \Pi_j) - \Pi_j(\lambda_j + k_j))$$

$$= \max_{1 \le i \le N} \left\{ p_i k_i \Pi_i + p_i k_i T(\beta_i(1 - \Pi_i) - \Pi_i(\lambda_i + k_i)) \right\} \quad .$$

If Π_i is viewed as the limiting probability of an alternating renewal process (the states being the visible and masked conditions of the target), then $\Pi_i = \left(\frac{\beta_i}{\beta_i + \lambda_i}\right)$.[1] In this case, it can be shown that the detection functions are concave.

Where the detection functions are concave, the switch time T* (from region 1 to 2) is determined from the solution of

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[1] See Section 5.2.1 for the proof of this assertion. The situation in which the visibility process is modeled as an alternating renewal process is treated in Chapter 5.

$$P_1(T^*) = p_2 k_2 \Pi_2 \ .$$

The extension of these ideas to N-regions basically involves the solution of the Kuhn-Tucker conditions for the N-region problem.

Again, we can specify an approximation for large values of T, the total available search time. Since this model is a weighted combination of the two previous ones, the conditions for an approximate solution are similar to those already given by equations 20-23. Consider the 2-region example of the previous section wherein $(\lambda_1 + k_1) > \beta_1$ and $\beta_2 > (\lambda_2 + k_2)$. The conditional detection functions are then approximated by

$$P_1(t_1) \simeq \frac{\Pi_1 k_1 p_1}{\lambda_1 + k_1} + (1 - \Pi_1)\left(1 - \frac{(\lambda_1 + k_1)e^{-\beta_1 t_1}}{[(\lambda_1 + k_1) - \beta_1]}\right)\frac{p_1 k_1}{k_1 + \lambda_1}$$

$$P_2(t_2) \simeq \frac{\Pi_2 k_2 p_2}{\lambda_2 + k_2}(1 - e^{-(\lambda_2+k_2)t_2}) + \frac{(1-\Pi_2)p_2 k_2}{k_2 + \lambda_2}\left(1 - \frac{\beta_2 e^{-(\lambda_2+k_2)t_2}}{\beta_2 - (\lambda_2 + k_2)}\right)$$

One obtains then for the approximate allocation scheme

$$t_1 = \frac{1}{\beta_1 + \lambda_2 + k_2}\left\{\ln[U_1/\xi_2] + (\lambda_2 + k_2)T\right\}, \qquad (36)$$

$$t_2 = T - t_1 \ ,$$

where

$$\xi_2 = P_2\Pi_2 k_2 + \frac{(1 - \Pi_2)P_2 k_2 \beta_2}{[\beta_2 - (\lambda_2 + k_2)]} \quad , \text{ and}$$

$$U_1 = \frac{(1 - \Pi_1)k_1 P_1 \beta_1}{[(\lambda_1 + k_1) - \beta_1]} \cdot$$

4.4.3 Comparison with the Koopman Model

In this section, we examine the situation in which a partially informed searcher, being aware of the results of Koopman (which assume continuously visible targets), applies them to situations in which the target behavior is actually characterized by the random interval visibility process.

The different FAR, can lead to the erroneous selection of the initial region to receive the search effort, and the differing switch points can influence the shape of the error function. Figures 11 and 12 illustrate some of these remarks. Figure 11 contains the results of a parametric study of the effects of the prior probabilities on whether or not the target is visible (present) at the start of the search. The results indicate that the greatest errors in using the Koopman policy occur whenever it is highly probable that the target will be masked (or not present) at the start of the search. The limiting case of the single interval process is shown and represents the worst situation in terms of the use of the Koopman policy. Note that as the $\Pi_i \rightarrow 1$, the maximum percent

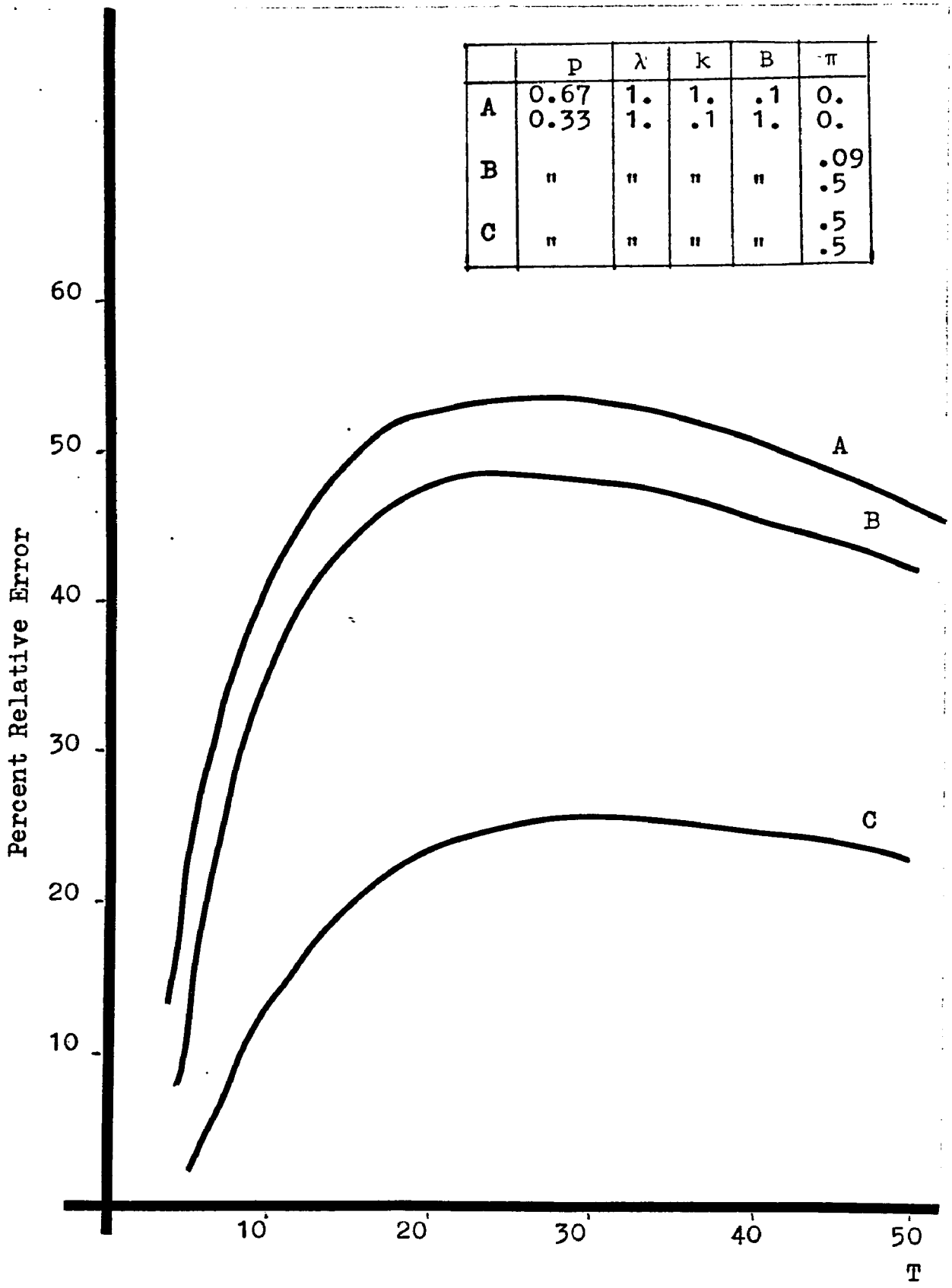


Figure 11 - Sensitivity to the Visibility Parameters

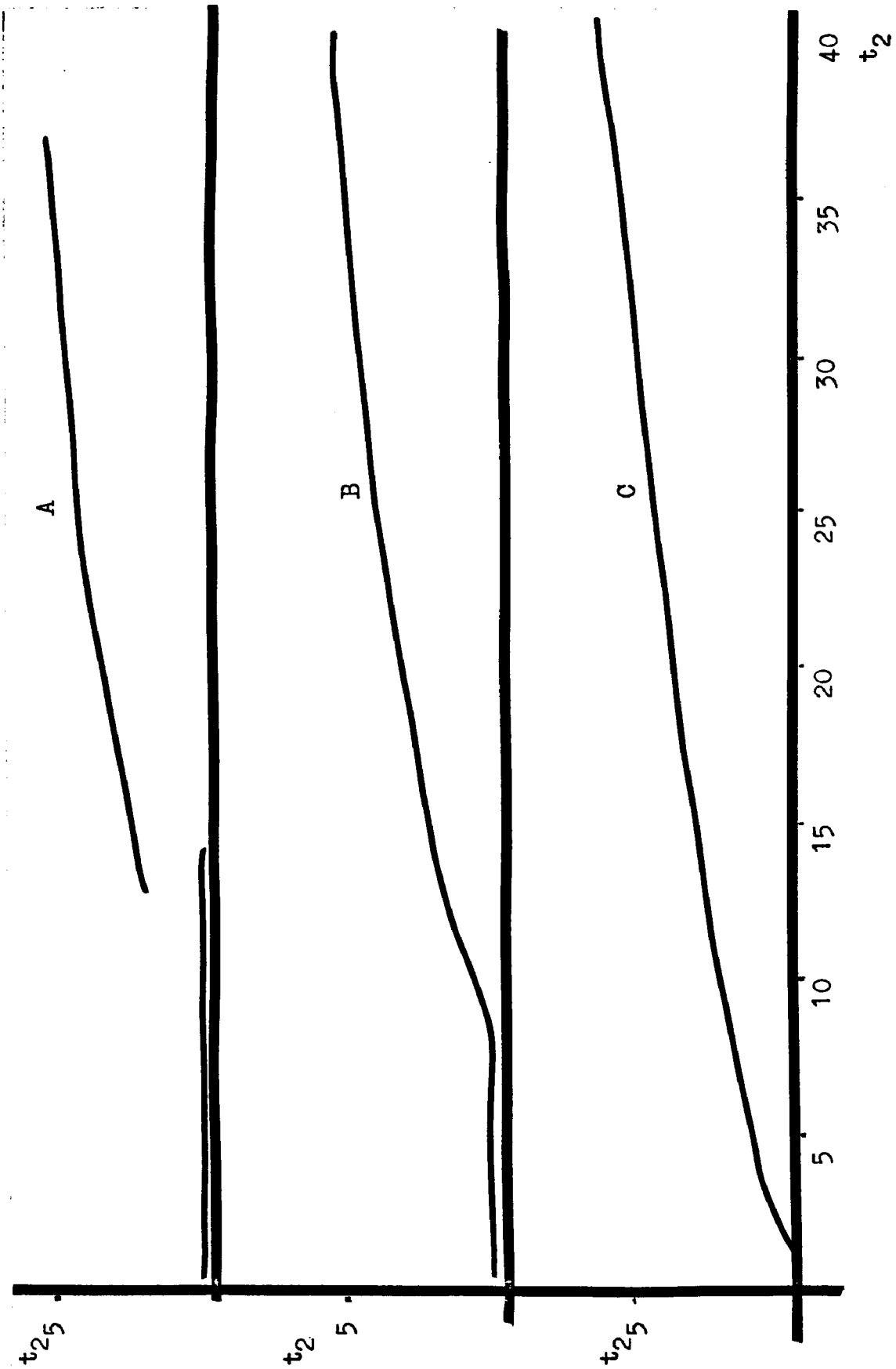


Figure 12- A Comparison of Allocation Policies

relative errors decrease, but the rate of decay of the error function is also decreasing. The decision maker who cannot estimate the visibility parameters will be concerned in such situations, since the lower rates imply that the option of increasing the available search effort to reduce the errors in using the Koopman policy may not be cost-effective.

Figure 12 displays the effects of variations in Π's on the optimal policies used in the determination of Figure 11. The higher the likelihood that the target is masked at the start of the search, the greater the length of time spent in Region 1, for any fixed level of total available effort. Note that the Koopman policy (which is determined from k's), under large quantities of available search time, allocates the majority of the search time to the second region.

The approximate allocation policies given by equation 36 give rise to an approximate difference in the probability of detection under the two policies, optimal and the Koopman, as a function of time. For the conditions given on page 144, one obtains

$$E(T_{\cdot}) = \frac{(1 - \Pi_1)k_1 p_1}{\lambda_1 + k_1} \left\{ \frac{\lambda_1 + k_1}{\lambda_1 + K_1 - \beta_1} \right\} \left\{ e^{-\frac{\beta_1 k_2 T}{k_1 + k_2}} - e^{-\frac{\beta_1(\lambda_2 + k_2)T}{(\beta_1 + \lambda_2 + k_2)}} \right\}$$

$$+ \left\{ e^{-\frac{(\lambda_2 + k_2)k_1 T}{k_1 + k_2}} - e^{-\frac{\beta_1(\lambda_2 + k_2)T}{(\lambda_2 + k_2 + \beta_1)}} \right\} \left\{ \frac{\Pi_2 k_2 p_2}{\lambda_2 + k_2} \right.$$

$$\left. + \left[\frac{(1 - \Pi_2)k_2 p_2}{(\lambda_2 + k_2)} \right] \left[\frac{\beta_2}{\beta_2 - (\lambda_2 + k_2)} \right] \right\}.$$

We observe that the sensitivity results of this model are weighted combinations of the sensitivity results of Sections 3.2.3 and 4.2.3.

Appendix F contains the description and results of a numerical study of the error function. These are summarized below. It was noted that the Koopman policy *can* be used effectively in

(a) the homogeneous detector, homogeneous visibility parameter scenario, and

(b) situations in which the mean time to detect is much less than the mean length of the visible period.

The values of Π, the prior probability vector on target visibility at the start of the search, are of importance here. Since for $\Pi_i \to 1$, the general model reduces to the model of Chapter 3, which has, under the Koopman policy, errors when

T is small. If, on the other hand, $\Pi_i \to 0$, one has the random-interval, random-start-time model which may have significant errors when T is large under the Koopman policy.

Chapter 5

MULTIPLE PERIODS OF VISIBILITY

In this chapter we consider the situation in which the target may exhibit alternate periods of visibility and invisibility during the time which the searcher spends in the appropriate region. As noted in Chapter 1, this type of process can be used to characterize the behavior of a submarine on patrol within a region; a patrol moving through rugged terrain, foliage, etc.; a school of tuna or other fish. Initially, a simple Markovian visibility model is introduced for pedagogical purposes. Next the target's behavior is characterized as an alternating renewal process. The interaction between the target and the searcher is characterized as a Markov-renewal process. Having characterized the detection processes, we then study the problem of maximizing the probability of detection under a constraint on the available search time for certain special cases. Finally, the minimization of the expected time until detection, given unlimited search effort, is investigated.

5.1 Description

5.1.1 A Simple Markovian Model of the Multiple Interval Process

Assume that we are searching for a stationary target located in one of N regions with prior positional probability

vector P. Within the region containing the target, the visibility process is going on with transitions, according to the matrix

$$\begin{array}{c} \\ \text{Visible} \\ \text{Masked} \end{array} \begin{array}{c} \begin{array}{cc} \text{Visible} & \text{Masked} \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} ,$$

between the visible and masked states occurring at fixed, known time intervals. The presence of the searcher within the region in which the target is operating induces an absorbing state which is the detection state. The transition matrix for this new process is

$$\begin{array}{c} \\ \text{Visible} \\ \text{Masked} \\ \text{Detected} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Visible} & \text{Masked} & \text{Detected} \end{array} \\ \begin{bmatrix} 0 & 1 - p_{D_i} & p_{D_i} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} ,$$

where p_{D_i} is the probability of detection within the ith subregion.

Assume that a total of M glimpses are to be optimally allocated. We allocate m_j glimpses to region j, then seek the probability of absorption (detection). Let $f_{ij}^{(n)}$ denote the first passage probability from state i to state j, and X_n denote the state at the nth transition, i.e.,

$$f_{ij}^{(n)} = \Pr\left\{X_n = j,\ X_m \neq j,\ m = 1, \ldots, n-1 \mid X_0 = i\right\} .$$

We can express $P_{ij}^{(n)}$, where

$$P_{ij}^{(n)} = \Pr\left\{X_n = j \mid X_0 = i\right\} ,$$

as

$$P_{ij}^{(n)} = \sum_{K=0}^{n} f_{ij}^{(n-K)}\, P_{jj}^{(K)} ,$$

where $f_{ij}^{(0)} = 0$. Let j denote the absorbing state. Then

$$\Pr\left\{\text{absorption} \mid n \text{ looks}, X_0 = i\right\} = \sum_{K=1}^{n} f_{ij}^{(K)} .$$

Consider the above expression and note that since j is an absorbing state, we have

$$P_{jj}^{(K)} = 1 ,$$

for $K = 0, 1, 2, \ldots$, hence

$$P_{ij}^{(n)} = \sum_{K=0}^{n} f_{ij}^{(n-K)} = \sum_{K=1}^{n} f_{ij}^{(K)} ,$$

since $f_{ij}^{(0)} = 0$.

From the above results we can formulate the following variation on Koopman's original problem.

$$\max \sum_{j=1}^{N} \Pr\left\{\text{absorption (detection) in the } j^{th} \text{ region} \,\middle|\, m_j \text{ looks}\right\}$$

$$\text{S.T.} \sum_{j=1}^{N} m_j \leq M$$

$$m_j \geq 0 \ .$$

where

$$\Pr\left\{\text{absorption (detection) in the } j^{th} \text{ region} \,\middle|\, m_j,\ X_0 = i\right\} = \sum_{K=1}^{m_j} f_{ij}^{(K)}$$

$$= P_{ij}^{(m_j)} \ .$$

The steady state probabilities of being in state i (the visible, and masked states of the visibility process) are (see Parzen (1962), p. 256),

$$(\Pi_1, \Pi_2) = (1/2, 1/2) \ ,$$

therefore

$$\Pr\left\{\text{absorption (detection) in the } j^{th} \text{ region} \,\middle|\, m_j\right\} = \sum_{j=1}^{2} P_{ij}^{(m_j)} \Pi_i p_j \ ,$$

where p_j is the probability that the target is in the jth subregion.

For our simple process (for the ith region)

$$P^1 = \begin{bmatrix} 0 & 1-P_D & P_D \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad P^2 = \begin{bmatrix} 1-P_D & 0 & P_D \\ 0 & 1-P_D & P_D \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & (1-P_D)^2 & P_D(1-P_D)+P_D \\ (1-P_D) & 0 & P_D \\ 0 & 0 & 1 \end{bmatrix},$$

$$P^4 = \begin{bmatrix} (1-P_D)^2 & 0 & P_D(1-P_D)+P_D \\ 0 & (1-P_D)^2 & P_D(1-P_D)+P_D \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 0 & (1-P_D)^3 & P_D(1-P_D)^2+P_D(1-P_D)+P_D \\ (1-P_D)^2 & 0 & P_D(1-P_D)+P_D \\ 0 & 0 & 1 \end{bmatrix}$$

For ℓ even and $\ell > 1$, the entries in P^{ℓ} are

$$P_{23}^{(\ell)} = P_{13}^{(\ell)} = P_{13}^{(\ell-1)}$$

and for ℓ odd and $\ell > 1$, we have

$$P_{23}^{(\ell)} = P_{23}^{(\ell-1)}$$

$$P_{13}^{(\ell)} = P_D(1 - P_D)^{(\ell-[\frac{\ell}{2}+1])} + P_{13}^{(\ell-2)},$$

where [x] denotes the greatest integer in the term in brackets. The optimization problem becomes

$$\max \sum_{i=1}^{N} \sum_{K=1}^{2} P_{K3}^{(m_i)} \Pi_K \; P_i$$

$$\text{S.T.} \sum_{i=1}^{N} m_i \leq M$$

$$m_i \geq 0 \;, \quad i = 1, 2, \ldots, N \;,$$

where $P_{K3}^{(M_i)}$ is defined above. The above problem is readily solved using dynamic programming or the results of Wagner (1969) given in Appendix A.[1]

---

[1] A generalization of the transition matrix is

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} .$$

Then, given the presence of the searcher, the transition matrix for the search problem becomes

$$\begin{bmatrix} \alpha_i(1-P_D) & \beta_i(1-P_D) & P_D \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

5.1.2 Characterization of the Multiple Interval Search Scenario as a Semi-Markov Process

Disney (1970) characterized a visibility process in which the target alternates between visible and masked states as an alternating renewal process. The transition matrix for this process is

$$\begin{pmatrix} 0 & f_1(t) \\ f_2(t) & 0 \end{pmatrix}$$

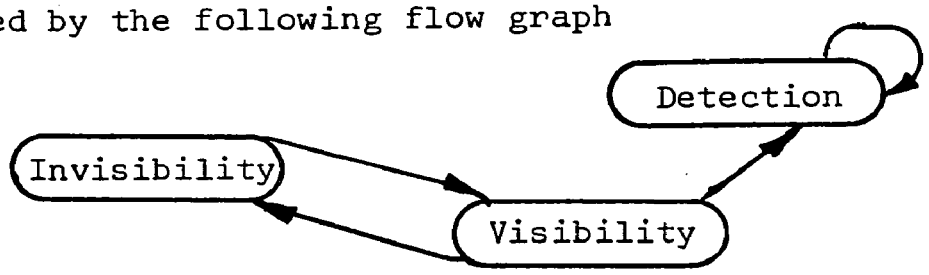
where $f_1(t)$ is the probability density function for the time in the visible state and $f_2(t)$ the probability density function for the time in the masked state.

Using some renewal theory results, the author was able to obtain, among other things,

(a) $\Pi_1(t)$ the probability that the target is visible in $(t,t+dt)$.

(b) For a fixed time interval of length d, the distributions of
   (1) the number of times the target is visible,
   (2) the total time of visibility.

When the searcher enters the region in which the target is operating, we obtain the three-state process characterized by the following flow graph



The associated semi-Markov transition matrix is

$$Q = \begin{array}{c} \\ V \\ I \\ D \end{array} \begin{array}{c} \begin{array}{ccc} V & I & D \end{array} \\ \begin{pmatrix} 0 & R(t) & L(t) \\ F(t) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array},$$

where

$R(t)$ = distribution function for the time spent in the visible state

$F(t)$ = distribution function for the time in the masked state

$L(t)$ = distribution function for the time until detection

$l(t) = \frac{dL(t)}{dt}$

$r(t) = \frac{dR(t)}{dt}$.

Next we derive the expressions for $l(t)$ and $r(t)$. Let $\rho(\tau)$ denote the conditional detection function, i.e., if τ_d is the time until detection, then

$$\rho(\tau)d\tau = \Pr\left\{\tau \leq \tau_d \leq \tau + d\tau \,\middle|\, \begin{array}{l}\text{target present and visible} \\ \text{for } \tau \text{ or more time units}\end{array}\right\} .$$

Let $h(t)$ be the probability density function for t_v, the length of the visible period.

Assuming that the process just entered the visible state, in order for the target to be detected in t time units, t_d must be in the interval (t,t+dt) and t_v must have been at least equal to t, i.e.,[1]

$$\ell(t) = \rho(t)\bar{H}(t), \text{ where } \bar{H}(t) = \int_t^\infty h(t)dt \ .$$

If the target is to re-enter the masked state in t time units, then t_d must have been greater than t and t_v must equal t, i.e.,

$$v(t) = \bar{P}(t)h(t), \ \bar{P}(t) = \int_t^\infty \rho(\tau)dt \ .$$

The density function for the length of the masked period is f(t).

Suppose the search must be carried out under a constraint on the total available search time, T. The problem of determining the optimal policy to maximize the probability of detection is

$$\max \sum_{i=1}^{N} \Pr\left\{\begin{matrix}\text{Time until absorption in} \\ \text{region (i)} \leq t_i\end{matrix}\right\}$$

$$\text{S.T.} \sum_{i=1}^{N} t_i \leq T$$

$$t_i \geq 0 \ .$$

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[1] Looking at the process just after a transition, for the situation in which one considers the searcher entering at an arbitrary time, the process becomes a delayed Markov renewal process with the relevant p.d.f. being the forward recurrance times of the visibility and invisibility processes.

We now introduce the distribution function for the first passage times as well as the marginal distribution function of the semi-Markov process. Let $G_{ij}(t)$ denote the distribution function for the first passage time, i.e.,

$$G_{ij}(t) = \Pr[N_j(t) > 0 | X_0 = i] \equiv \Pr[t_{j_1} \leq t | X_0 = i] ,$$

where t_{j_1} denotes the time of the first passage to state j and $N_j(t)$ denotes the number of entries to state j in time t. Let $P_{ij}(t)$ be defined as

$$P_{ij}(t) = \Pr\{X_t = j | X_0 = i\} .$$

Appendix G contains a number of theorems due to Pyke (1961) which assist us in relating P and G. From Theorem 1 of Appendix G, we have[1]

$$P_{ij}(t) = P_{jj}(t) * G_{ij}(t) + \delta_{ij}(1 - H_i(t)) .$$

Let j denote the detection state, an absorbing state, i.e., $P_{jj}(t) = 1$, for $t \geq 0$. Since the search process will never start in state j, $\delta_{ij} = 0$, and from Theorem 1 of Appendix G, we have

$$P_{ij}(t) = 1 * G_{ij}(t) = G_{ij}(t) .$$

The procedure for solving the optimization problem follows directly from the above results:

(1) By Theorem 2 we may determine $\Pi_{ij}(S)$ for the visible and masked states,

[1]See Appendix G for notation.

(2) inversion of $\Pi_{ij}(S)$ will yield $dG_{ij}(t)$ by virtue of the previous comments,

(3) having obtained $dG_{ij}(t)$ we are ready to study the properties of optimization problem.

The optimization problem can be restated in terms of the G functions as

$$\max \sum_{i=1}^{N} G_i(t_i)$$

$$\text{S.T.} \sum_{i=1}^{N} t_i \leq T$$

where $G_j(t) = p_j \cdot \sum_{\substack{\ell=1 \\ j\neq\ell}}^{2} G_{\ell j}(t)\Pi_\ell$, $\Pi_\ell = \Pr\{X_0=\ell\}$ and p_j is the probability that the target is in the jth region.

Next we consider the following special case of the multiple interval process. Let the conditional detection density function be of the form

$$\rho(t) = k.e^{-kt} ,$$

and the density functions for the lengths of the visible and masked times be, respectively[1]

[1]Bonder's (1970) discussion with the British on combat data indicated that this form for these processes appears to have an experimental basis.

$$h(t) = \lambda e^{-\lambda t}, \quad f(t) = \beta e^{-\beta t} .$$

From the above results the density function for the transition time from the visible to the detected state is given by

$$\ell(t) = \rho(t)\bar{H}(t) = ke^{-(\lambda+k)t} .$$

Likewise, the density function for the transition time from the visible to the masked state is determined as

$$r(t) = p(t)h(t) = \lambda e^{-(\lambda+k)t} .$$

The following list gives the Laplace transforms for each of the above density functions:

Density Function	Laplace Transform
$g(t)$	$g(S) = \frac{k}{k + S}$
$h(t)$	$h(S) = \frac{\lambda}{\lambda + S}$
$f(t)$	$f(S) = \frac{\beta}{\beta + S}$
$\ell(t)$	$\ell(S) = \frac{k}{\lambda + k + S}$
$r(t)$	$r(S) = \frac{\lambda}{\lambda + k + S}$

The Laplace transform of the semi-Markov matrix Q(t) is

$$q(S) = \begin{pmatrix} 0 & \frac{\lambda}{\lambda + k + S} & \frac{k}{\lambda + k + S} \\ \frac{\beta}{\beta + S} & 0 & 0 \\ 0 & 0 & e^{-S} \end{pmatrix},$$

and

$$I - q(S) = \begin{pmatrix} 1 & \frac{-\lambda}{\lambda + k + S} & \frac{-k}{\lambda + k + S} \\ \frac{-\beta}{\beta + S} & 1 & 0 \\ 0 & 0 & 1 - e^{-S} \end{pmatrix}$$

Defining the matrix $\Lambda(t)$ as follows

$$\Lambda(t) = (\delta_{ij}\Lambda_i), \quad \Lambda_i(t) = \sum_{j=1}^{3} Q_{ij}(t) .$$

Then

$$d\Lambda_i(t) = \sum_{j=1}^{3} dQ_{ij}(t)$$

and from the convolution property of the Laplace transform

$$\Lambda(S) = \delta_{ij}\mathcal{L}(d\Lambda_i(t)) = \prod_{j=1}^{3}\mathcal{L}(dQ_{ij}(t)),$$

or

$$\Lambda(S) = \begin{bmatrix} \frac{\lambda k}{(\lambda + k + S)^2} & 0 & 0 \\ 0 & \frac{\beta}{\beta + S} & 0 \\ 0 & 0 & 1 - e^{-S} \end{bmatrix}$$

$$I - \Lambda(s) = \begin{bmatrix} 1 & \frac{-\lambda k}{(\lambda + k + S)^2} & 0 & 0 \\ 0 & & 1 - \frac{\beta}{\beta + S} & 0 \\ 0 & & 0 & 1-e^{-S} \end{bmatrix}$$

Pyke's Theorem of Appendix G states that

$$\Pi(S) = (I - q)^{-1}(I - \Lambda) ,$$

where

$$\Pi(S) = \mathcal{L}(P(t)) .$$

First we compute the required inverse,

$$(I - q)^{-1} = \frac{1}{D} \begin{bmatrix} 1 - e^{-S} & \frac{\lambda(1 - e^{-S})}{\lambda + k + S} & \frac{k}{\lambda + k + S} \\ \frac{\beta(1 - e^{-S})}{\beta + S} & 1 - e^{-S} & \frac{\beta k}{(\beta + S)(\lambda + k + S)} \\ 0 & 0 & 1 - \frac{\beta k}{(\beta+S)(\lambda+k+S)} \end{bmatrix}$$

where

$$D = |(I - q)^{-1}| = (1 - e^{-S})\left(1 - \frac{\lambda\beta}{(\beta + S)(\lambda + k + S)}\right)$$

We are interested in determining Π, where

$$\Pi = (I - q)^{-1}(I - \Lambda(S)) ,$$

in particular Π_{13} and Π_{23}. Carrying out the indicated matrix multiplication,

$$\Pi_{13}(S) = (1 - e^{-S}) \left(\frac{k}{\lambda + k + S} \right) \frac{1}{D}$$

$$= \frac{k(\beta + S)}{[(\beta + S)(\lambda + k + S) - \lambda\beta]}$$

$$\Pi_{23}(S) = (1 - e^{-S}) \frac{\beta k}{(\beta + S)(\lambda + k + S)} \frac{1}{D}$$

$$= \frac{\beta k}{[(\beta + S)(\lambda + k + S) - \lambda\beta]} .$$

Next the mean times to absorption are obtained from (Theorem 3, Appendix G)

$$\mu = \lim_{S \to 0+} \frac{1}{S} (1 - q),$$

where

$$q(S) = \int_0^\infty e^{-St} \, dG_{ij}(t) = \int_0^\infty e^{-St} \, dP_{ij}(t) = \Pi(S),$$

for j = 3 the absorbing state.[1]

[1] The μ_{i3} is determined from $\mu_{13} = \lim_{S \to 0+} \frac{1}{S} \left(1 - \frac{k(\beta + S)}{[(\beta + S)(\lambda + k + S) - \lambda\beta]} \right)$, or

$$\mu_{13} = \frac{\beta + \lambda}{\beta k} , \text{ and}$$

$$\mu_{23} = \lim_{S \to 0+} \frac{1}{S} \left(1 - \frac{\beta k}{[(\beta + S)(\lambda + k + S) - \lambda\beta]} \right), \text{ or}$$

$$\mu_{23} = \frac{\lambda + \beta + k}{\beta k}$$

The inversion of $\Pi_{13}(S)$ and $\Pi_{23}(S)$ is accomplished by partial fraction expansion. Since the denominator is common to both terms, we can factor it into a product of linear terms. Let

$$D(S) - (\beta + S)(\lambda + k + S) - \lambda\beta = S^2 + (\lambda + k + \beta)S + \beta k ,$$

the discriminant is then

$$b^2 - 4AC = (k + \lambda + \beta)^2 - 4\beta k = \lambda^2 + 2\lambda\beta + (k - \beta)^2 + 2\lambda k > 0 ,$$

since $k, \lambda, \beta > 0$. Both roots are negative since $\left(b^2 - 4AC\right) < b^2$, $4AC > 0$, implies $\left[-b \pm \sqrt{b^2 - 4AC}\right] < 0$. Denote the roots by γ_1 and γ_2, where γ_2 is the larger. This is possible since $b^2 - 4AC \neq 0$ for $k, \lambda, \beta > 0$. Expand $\Pi_{13}(S)$ by partial fractions to obtain

$$\Pi_{13}(S) = \frac{k(\beta + S)}{(S + \gamma_1)(S + \gamma_2)} = \frac{\frac{k(\beta - \gamma_1)}{(\gamma_2 - \gamma_1)}}{S + \gamma_1} - \frac{\frac{k(\beta - \gamma_2)}{(\gamma_2 - \gamma_1)}}{S + \gamma_2} .$$

Inverting we obtain

$$dP_{13}(t) = \frac{k(\beta - \gamma_1)e^{-\gamma_1 t}}{(\gamma_2 - \gamma_1)} - \frac{k(\beta - \gamma_2)e^{-\gamma_2 t}}{(\gamma_2 - \gamma_1)}$$

$$= \frac{k}{\gamma_2 - \gamma_1}\left[(\beta - \gamma_1)e^{-\gamma_1 t} - (\beta - \gamma_2)e^{-\gamma_2 t}\right] .$$

Expanding $\Pi_{23}(S)$ in the same fashion, yields

$$\Pi_{23}(S) = \frac{\beta k}{(S + \gamma_1)(S + \gamma_2)} = \frac{\frac{\beta k}{\gamma_2 - \gamma_1}}{S + \gamma_1} - \frac{\frac{\beta k}{\gamma_2 - \gamma_1}}{S + \gamma_2}$$

and inverting

$$dP_{23}(t) = \frac{\beta\, k}{\gamma_2 - \gamma_1} \left(e^{-\gamma_1 t} - e^{-\gamma_2 t}\right).$$

Integrating with respect to time, the desired conditional detection probabilities are

$$P_{23}(t) = \frac{\beta k}{\gamma_2 - \gamma_1} \left[\frac{1}{\gamma_1} (1 - e^{-\gamma_1 t}) - \frac{1}{\gamma_2} (1 - e^{-\gamma_2 t})\right] ,$$

and

$$P_{13}(t) = \frac{k}{\gamma_2 - \gamma_1} \left[\frac{(\beta - \gamma_1)}{\gamma_1} \left(1 - e^{-\gamma_1 t}\right) - \frac{(\beta - \gamma_2)}{\gamma_2} \left(1 - e^{-\gamma_2 t}\right)\right].$$

The limiting detection probabilities are,

$$\lim_{t\to\infty} P_{13}(t) = \frac{k\beta}{\gamma_1\gamma_2} = \lim_{t\to\infty} P_{23}(t) .$$

Also note the following facts:

(1) $\lim_{t\to 0^+} P_{\perp 3}(t) = 0 = \lim_{t\to 0^+} P_{23}(t)$, and

(2) $\lim_{t\to\infty} P_{\perp 3}(t) = \lim_{t\to\infty} P_{23}(t) = \frac{k\beta}{\gamma_1\gamma_2} = 1,$

Since

$$\gamma_2 = \frac{b + \sqrt{b^2 - 4AC}}{2A}, \qquad \gamma_1 = \frac{b - \sqrt{b^2 - 4AC}}{2A},$$

$$\gamma_1\gamma_2 = C/A .$$

But $D(s) = S^2 + (\lambda + k + \beta)S + k\beta$, hence

$$\gamma_1\gamma_2 = k\beta.$$

We still require the probabilities of being in the visible and masked states at the start of the search. For the derivation of these probabilities, assume that the visibility process, as characterized by Disney (1970), has been going on for some time before the searcher enters the region containing the target, i.e., that this process has attained steady state. Let $\Pi_1(t)$ denote the probability of being visible at time t and $\Pi_2(t)$ the probability of being masked at time t. Assuming the process starts in visible state, then from Disney's results

$$\Pi_1(t) = F_1^c(t) + \int_0^t m_y(\mu)F_1^c(t - \mu)d\mu$$

$$\Pi_2(t) = \int_0^t m_z(\mu)F_2^c(t - \mu)d\mu ,$$

where:

$F_1^C(t)$ = complementary cumulative distribution for the length of the visible period

$F_2^C(t)$ = complementary cumulative distribution for the length of the masked period

y = denotes the times between the renewals of the visibility events

z = denotes the times between the renewals of the masking events

$m_y(t)$ = the renewal density function for the y process

$$m_y(t) = f_y(t) + \int_0^t m_y(t - \lambda) f_y(\lambda) d\lambda \ .$$

$m_z(t)$ = the renewal density function for the z process

Under the assumptions made about the density functions in this example

$$m_y(t) = \frac{\lambda\beta}{\lambda + \beta} \left[1 - e^{-(\lambda+\beta)t}\right] ,$$

$$m_z(t) = \frac{\lambda}{\lambda + \beta} \left[\beta + \lambda e^{-(\lambda+\beta)t}\right] ,$$

and

$$\Pi_1(t) = \frac{1}{\lambda + \beta} \left[\beta + \lambda e^{-(\lambda+\beta)t}\right]$$

$$\lim_{t\to\infty} \Pi_1(t) = \frac{\beta}{\lambda + \beta} = \Pi_1(\infty)$$

$$\Pi_2(t) = \frac{\lambda}{\lambda + \beta}\left[1 - e^{-(\lambda+\beta)t}\right]$$

$$\lim_{t\to\infty} \Pi_2(t) = \frac{\lambda}{\lambda + \beta} = \Pi_2(\infty) \ .$$

Since the visibility process is a continuous parameter Markov process for this special case, these results follow from the theory of such processes[1].

Now given that a specific region is being searched, the probability of detection as a function of the time spent searching is

$$P(t) = P_{13}(t)\Pi_1(\infty) + P_{23}(t)\Pi_2(\infty) \ .$$

By summing across regions, we obtain the objective function for the problem of maximizing the probability of detection.

5.2 *Allocation of Effort to Maximize the Probability of Detection*

5.2.1 Model Solution

In this section the target motion, within a region, from the masked to the visible state and back again

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[1]From the solution of

$$\frac{dP(t)}{dt} = P(t)\begin{pmatrix} -\lambda & \lambda \\ \beta & -\beta \end{pmatrix} .$$

is described as an alternating renewal process. The interaction of the searcher and the target is then modeled as a semi-Markov process. As an example of such a situation, consider an alternating renewal process (whose associated probability density functions are both exponential). The problem of maximizing the probability of detection under a constraint on the total available errort is given by[1]

$$\max \sum_{i=1}^{N} p_i[\Pi_i P_{v_i}(t_i) + (1 - \Pi_i)P_{m_i}(t_i)]$$

$$\text{S.T.} \quad \sum_{i=1}^{N} t_i \leq T \quad ,$$

$$t_i \geq 0 \quad , \quad \forall i \quad ,$$

where Π_i denotes the probability that the target is visible and

$$P_{v_i}(t) = \frac{k_i}{\gamma_{i_1} - \gamma_{i_2}} \left[\frac{(\beta_i - \gamma_{i1})}{\gamma_{i1}}[1 - e^{-\gamma_{i1}t}] \right. \tag{37}$$

$$\left. - \frac{(\beta_i - \gamma_{i2})}{\gamma_{i2}}[1 - e^{-\gamma_{i2}t}] \right] \quad ,$$

[1]Unless otherwise noted, the parameters are identical to those introduced in the previous sections.

and

$$P_{m_i}(t) = \frac{\beta_i k_i}{\gamma_{i2} - \gamma_{i1}} \left\{ \frac{1}{\gamma_{i1}} (1 - e^{-\gamma_{i1} t}) - \frac{1}{\gamma_{i2}} (1 - e^{-\gamma_{i2} t}) \right\}, \tag{38}$$

where

$$\gamma_{i2} = \frac{b + \sqrt{b^2 - 4c}}{2}, \qquad \gamma_{i1} = \frac{b - \sqrt{b^2 - 4c}}{2} \tag{39}$$

and

$$b = \lambda_i + k_i + \beta_i$$

$$c = \beta_i k_i .$$

All of the visibility models with the exception of the one just presented have the following property.

$$\lim_{T \to \infty} P_i(T) \to \phi < 1 .$$

We have seen that, for the multiple interval model,

$$\lim_{T \to \infty} P_i(T) \to 1 .$$

The detection function for any region is convex over the interval $(o,\bar{t})$ where

$$\bar{t} \leq \frac{1}{(\gamma_2 - \gamma_1)} \ln[\gamma_2(\beta - \Pi_2)/\gamma_1(\beta - \Pi\gamma_1)] , \qquad (40)$$

and concave thereafter.

A necessary consequence of having a detection function which is convex on $(0,\bar{t})$ is that

$$\frac{\beta}{\beta + \lambda + k} \geq \Pi . \qquad (41)$$

If Π is the limiting probability that the alternating renewal process is in the visible state, then

$$\Pi = \frac{\beta}{\beta + \lambda} ,$$

and the above condition doesn't hold, i.e., the detection function is concave over the entire range of search effort.

The first allocation rule for this model is:

Choose the region j such that

$$p_j k_j \Pi_j = \max_{1 \leq i \leq N} p_i k_i \Pi_i , \qquad (42)$$

which is identical to that of the general single interval model.

Let Π be the limiting probability that the alternating renewal process is in the visible state, then the switch time T_s is obtained from the solution of

$$P_1'(T_s) = p_2 k_2 \Pi_2 .$$

The solution of the above equation can be greatly simplified by making the following observations:

(a) $\gamma_2 > \gamma_1$

(b) $$P_1'(T) = \frac{p_1 k_1}{(\gamma_2 - \gamma_1)} \left\{ e^{-\gamma_1 T}(\beta_1 - \gamma_1 \Pi_1) - e^{-\gamma_2 T}(\beta_1 - \gamma_2 \Pi_1) \right\}$$

(c) $(\beta_1 - \gamma_1 \Pi_1) > 0$

$(\beta_1 - \gamma_2 \Pi_1) < 0$

Now if $\gamma_2 \Pi_1 - \beta_1 >> \beta_1 - \gamma_1 \Pi_1$ or, equivalently,

$$\frac{k_1}{\lambda_1 + \beta_1} >> 1 \quad \text{then}$$

one would expect the γ_2 term to dominate and the switch times will be "short." On the other hand, if $\gamma_2 \Pi_1 - \beta_1 < \beta_1 - \gamma_1 \Pi_1$ or $\frac{k_1}{\lambda_1 + \beta_1} \leq 1$, then one expects the γ_1 term to dominate and "longer" switch times will occur.

By making use of the observation that $\gamma_2 > \gamma_1$, approximate solutions can be developed which are applicable when "large" amounts of total search effort[1] are available. Under the above assumptions the conditional detection function becomes

$$P_v(t) \simeq \frac{k}{\gamma_2 - \gamma_1} \left\{ \frac{\beta - \gamma_1}{\gamma_1} [1 - e^{-\gamma_1 t}] - \frac{\beta - \gamma_2}{\gamma_2} \right\}$$

$$P_m(t) \simeq \frac{\beta k}{\gamma_2 - \gamma_1} \left\{ \frac{1}{\gamma_1} (1 - e^{\gamma_1 t}) - \frac{1}{\gamma_2} \right\} .$$

These approximations to the conditional detection functions are of the same general form as those investigated in Appendix A. For the 2-region case, the optimal allocations become

$$t_1 \simeq \frac{1}{\gamma_{11} + \gamma_{21}} \left\{ \ln[\phi_1/\phi_2] + \gamma_{21} T \right\},$$

$$t_2 \simeq \frac{1}{\gamma_{11} + \gamma_{21}} \left\{ \ln[\phi_2/\phi_1] + \gamma_{11} T \right\}, \qquad (43)$$

where for γ_{il}, the i denotes the region, and

$$\phi_1 = \frac{p_1 k_1 [\beta_1 - \Pi_1 \gamma_{11}]}{\gamma_{12} - \gamma_{11}}, \qquad \phi_2 = \frac{p_2 k_2 [\beta_2 - \Pi_2 \gamma_{21}]}{\gamma_{22} - \gamma_{21}} .$$

and assuming $\phi_1 > \phi_2$ the switch point is given by $T^* \simeq \frac{1}{\gamma_{11}} \ln\left[\frac{\phi_1}{\phi_2}\right]$

To facilitate the comparison of these results with those of the previous chapters, recall that the γ's are given by

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[1] These results will also be applicable in the situation in which $\gamma_2 >> \gamma_1$.

$$\gamma_{i1} = \frac{\lambda_i + \beta_i + k_i - \sqrt{(\lambda_i + \beta_i + k_i)^2 - 4\beta_i k_i}}{2} \qquad (44)$$

$$\gamma_{i2} = \frac{\lambda_i + \beta_i + k_i + \sqrt{(\lambda_i + \beta_i + k_i)^2 - 4\beta_i k_i}}{2} \quad .$$

The techniques developed in Appendix A are used to obtain a solution to the N-region problem under the above assumptions. Consider first the expression under the radical in equation 44.

$$(\lambda + k + \beta)^2 - 4k\beta = (k - \beta)^2 + \lambda(\lambda + 2k + 2\beta) \quad .$$

Since all the parameters are positive, we can minimize this expression by choosing $k = \beta$ and $\lambda << 1$. In which case, γ_1 and γ_2 are given by $\gamma_1 \simeq k + \lambda/2 - \sqrt{k\lambda}$ and $\gamma_2 \simeq k + \lambda/2 + \sqrt{k\lambda}$. Physically this corresponds to the situation in which the mean length of the visible period is much larger than the mean length of the masked period which equals the mean time to detect within a region. Since the γ_i's are approximately equal to the k's, the Koopman allocation policy will provide a good approximation for large quantities of search time. If one has $\lambda = \beta$ and $k << 1$, then $\gamma_1 \simeq k/2$ and $\gamma_2 \simeq 2\beta$. From the previous discussion on switch times, this situation could lead to some very long switch times.

The condition $\lambda = k = \beta$ leads to the following values for γ_1 and γ_2

$$\gamma_1 = \frac{k}{2}(3 - \sqrt{5}) \text{ and } \gamma_2 = \frac{k}{2}(3 + \sqrt{5}).$$

This situation will also yield long term allocation policies which will agree with the Koopman scheme. Finally, by requiring $k = \lambda$ and $\beta << 1$, γ_1 and γ_2 become

$$\gamma_1 \simeq \beta/2 \text{ and } \gamma_2 \simeq 2k.$$

Since in this case $k \neq \beta$, one might again expect the Koopman policy to be inadequate for long-term allocations.

It was noted earlier in this section that the choice of Π_i as the limiting visibility probability of the alternating renewal process lead to a concave conditional detection function. The reaminder of this section will deal with the extreme limits on Π_i, i.e., $\Pi = 1.0$ or 0, since these cases are, as will be shown, direct generalizations of the single interval models. In Section 4.4 only a single interval of visibility was considered, here the target may exhibit multiple intervals; however, it is either initially masked or visible. In the former case, the conditional detection function is

$$P_i(t) = \frac{k}{\gamma_2 - \gamma_1}\left\{\frac{\beta - \gamma_1}{\gamma_1}\left[1 - e^{-\gamma_1 t}\right] - \frac{(\beta - \gamma_2)}{\gamma_2}\left[1 - e^{-\gamma_2 t}\right]\right\},$$

note that

$$P_v(\infty) = 1.$$

It is also noted that this function is concave for all values of

T since $\frac{\beta}{\beta + \lambda + k} < 1$.

Since $\Pi_i = 1$ for every i, the FAR is identical to the Koopman FAR, i.e.,

$$p_i k_i = \max_j p_j k_j \ .$$

The approximate solutions for large quantities of effort for a 2-region problem are

$$t_1 = \frac{1}{\gamma_{11} + \gamma_{21}} \left\{ \ln \frac{p_1 k_1}{p_2 k_2} + \ln \left[\frac{(\beta_1 - \gamma_{11})}{(\beta_2 - \gamma_{21})} \left(\frac{\gamma_{22} - \gamma_{21}}{\gamma_{12} - \gamma_{11}} \right) \right] + \gamma_{21} T \right\}$$

and $t_2 = T - t_1$.

This model can be interpreted as a direct generalization of the single-interval-start-at-time-zero model, i.e., a multiple interval, start at time zero model.

One may interpret the case ($\Pi = 0$) as a direct generalization of the single-interval, random-start-time model. The conditional detection function for this case is given by

$$P_m(T) = \frac{\beta k}{\gamma_2 - \gamma_1} \left\{ \frac{1}{\gamma_1} \left(1 - e^{-\gamma_1 t} \right) - \frac{1}{\gamma_2} \left(1 - e^{-\gamma_2 t} \right) \right\} \ .$$

This function is pseudo-concave, the region of convexity is the interval $(0, \overline{T})$ where $\overline{T}$ is given by

$$\overline{T} = \frac{1}{\gamma_2 - \gamma_1} \ln \left[\gamma_2 / \gamma_1 \right] \ .$$

The FAR for this model is given by

$$p_i k_i \beta_i = \max_{1 \le j \le N} p_j k_j \beta_j$$

which is the FAR for the single-interval random-start-time model. The approximate allocations for large quantities of total effort for a 2-region problem are

$$t_1 = \frac{1}{\gamma_{11} + \gamma_{21}} \left\{ \ln \frac{p_1 k_1 \beta_1}{p_2 k_2 \beta_2} + \ln \left[\frac{\gamma_{22} - \gamma_{21}}{\gamma_{21} - \gamma_{11}} \right] + \gamma_{21} T \right\}$$

and $t_2 = T - t_1$.

5.2.2 Comparison with the Koopman Model

In this section we examine the situation in which a partially informed searcher being aware of the results of Koopman (which assume continuously visible targets) applies them to situations in which the target behavior is actually characterized by the multiple interval inter-visibility process[1].

[1] The reader is referred to Section 2.2.3 for a detailed discussion of the implications to the decision-maker (searcher) of such an analysis.

It is clear that, with the exception of the situation in which $\Pi = 1$, the FAR for this model could lead to the selection of a different initial region than that selected via the Koopman policy. It was also noted that obvious differences in switch points are possible.

Figure 13 is a composite plot of percent relative error versus available search time and the probability of detection under the optimal policy, versus available search time. We note that the peak percent relative error occurs when, under the optimal policy there is a large amount of available time i.e., the probability of detection is 0.86. Even at T = 200, the relative error is still 23 percent of the optimal return. Figure 14 compares the allocation policies, Koopman and optimal, utilized in obtaining the results shown in Figure 13. In addition, the effects of variations in the prior probabilities of target visibility on the allocation policies are shown. Finally, by way of comparison, an example in which the target arrives late but remains visible is included in Figure 14. (This is a special case of the random-single-interval model of Chapter 4). Observe that the sustained error function of Figure 13 results from the fact that the Koopman policy allocates the majority of any given level of search effort to Region 2, while the optimal solution does exactly the opposite (the approximately optimal solutions are given by equation 43). Figure 15 exhibits the probability of detection over a range of levels of available search time for each of the policies

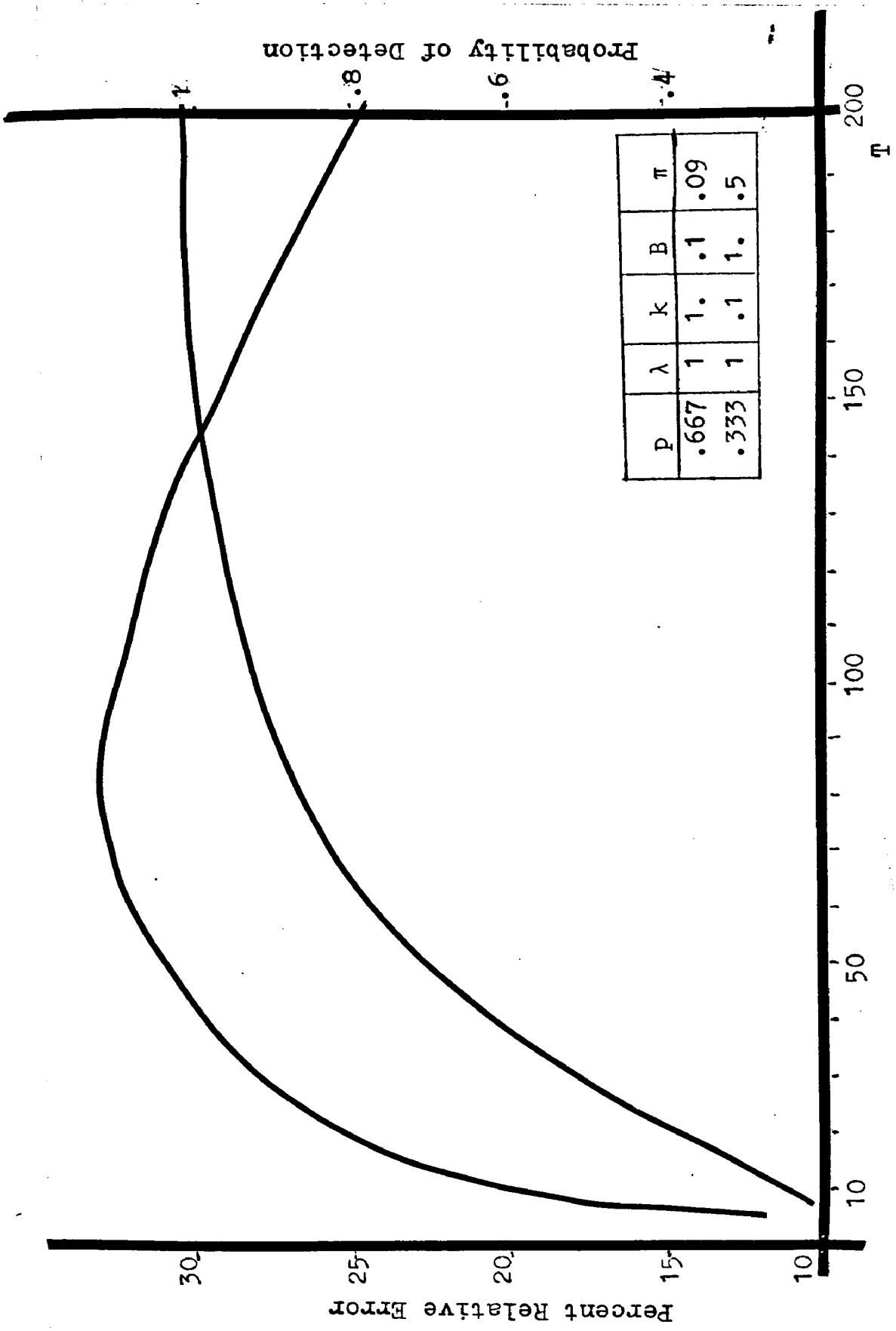


Figure 13 - Percent Relative Error and Optimal Return Versus Search Time

		p	λ	k	B	π
A	1	0.667	1.	1.	.1	1.
	2	0.333	1.	.1	1.	1.
B	"	"	"	"	"	.09
						.5
C	"	"	"	"	"	0.
						0.

K - Koopman Allocation Scheme

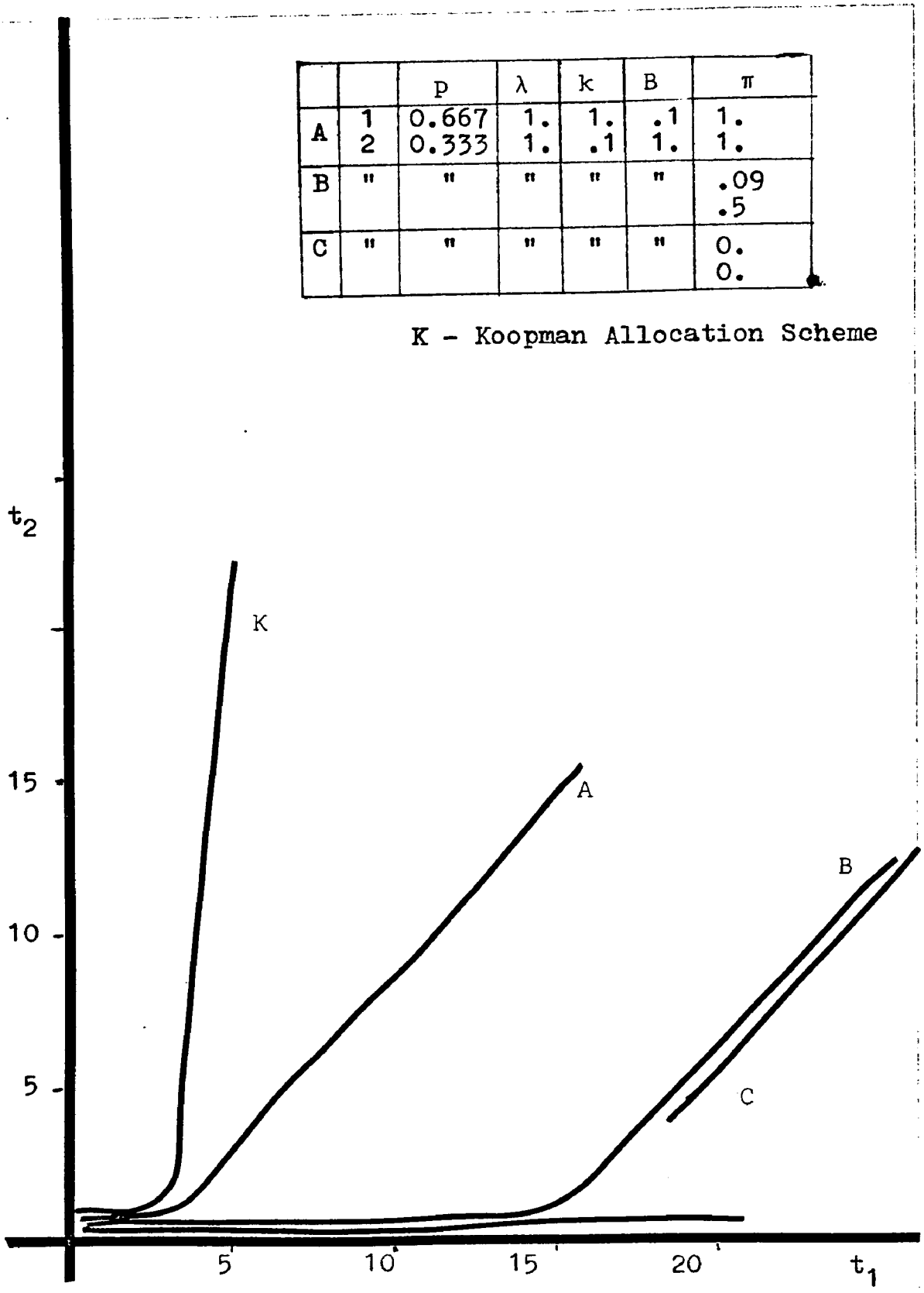
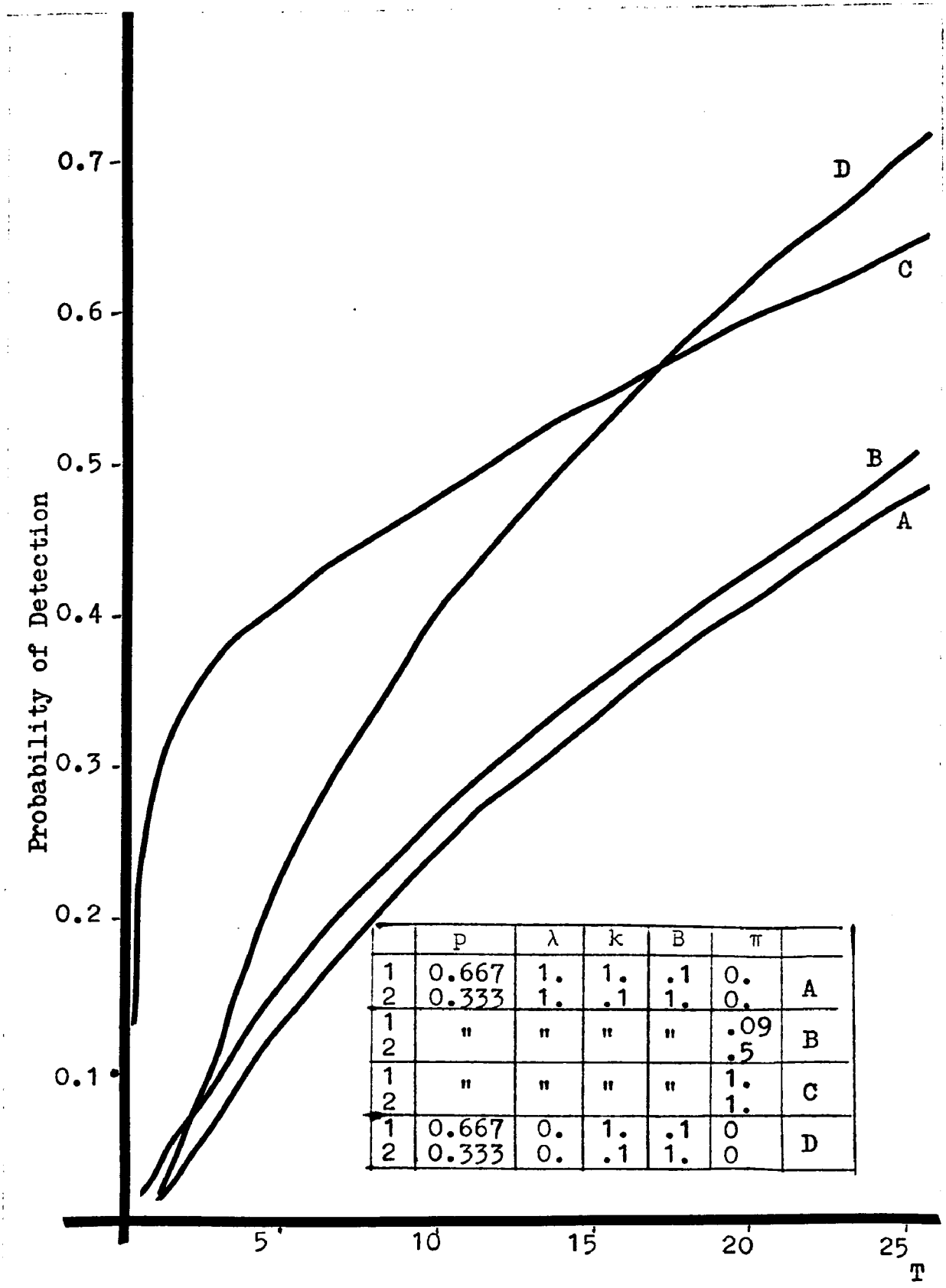


Figure 14 - Sensitivity of the Optimal Policy to Prior Estimates of Target Visibility



	p	λ	k	B	π	
1 2	0.667 0.333	1. 1.	1. .1	.1 1.	0. 0.	A
1 2	"	"	"	"	.09 .5	B
1 2	"	"	"	"	1. 1.	C
1 2	0.667 0.333	0. 0.	1. .1	.1 1.	0 0	D

Figure 15 - Sensitivity of the Optimal Return To Prior Estimates of Target Visibility

described in Figure 14. The following observations are of interest:

(a) The assumption that the visibility probabilities are obtained from the limiting alternating renewal process results yields little difference in the optimal return when compared with the assumption that the target is initially masked. Thus the return, for these parameter levels, is insensitive to these assumptions.

(b) Comparing the return under the assumption that the target is visible at the start of the search to the return in the late-arrival situation, one notes the rapid initial rise in the detection probability due to the initial visibility interval. However, since in the late arrival situation the target is always visible, the return in this case soon overtakes that for the multiple interval model. This indicates the advantage of intervisibility tactics. Also note that in comparing the late arrival situation to the multiple interval case in which the target is initially masked, the returns agree when the total available search effort is highly constrained. Thus in such situations the assumptions on target behavior are not critical.

Figure 16 compares the return (the probability of detection versus total available search time) from several allocation policies, the optimal policy, the approximate solution (equation 43), and the Koopman policy. In the example shown the approximate solution agrees very closely with the optimal solution over the entire range of available search time, the maximum error occurs in the vicinity of the switch point, T≈15. Figure 17 gives the allocation policies associated with the

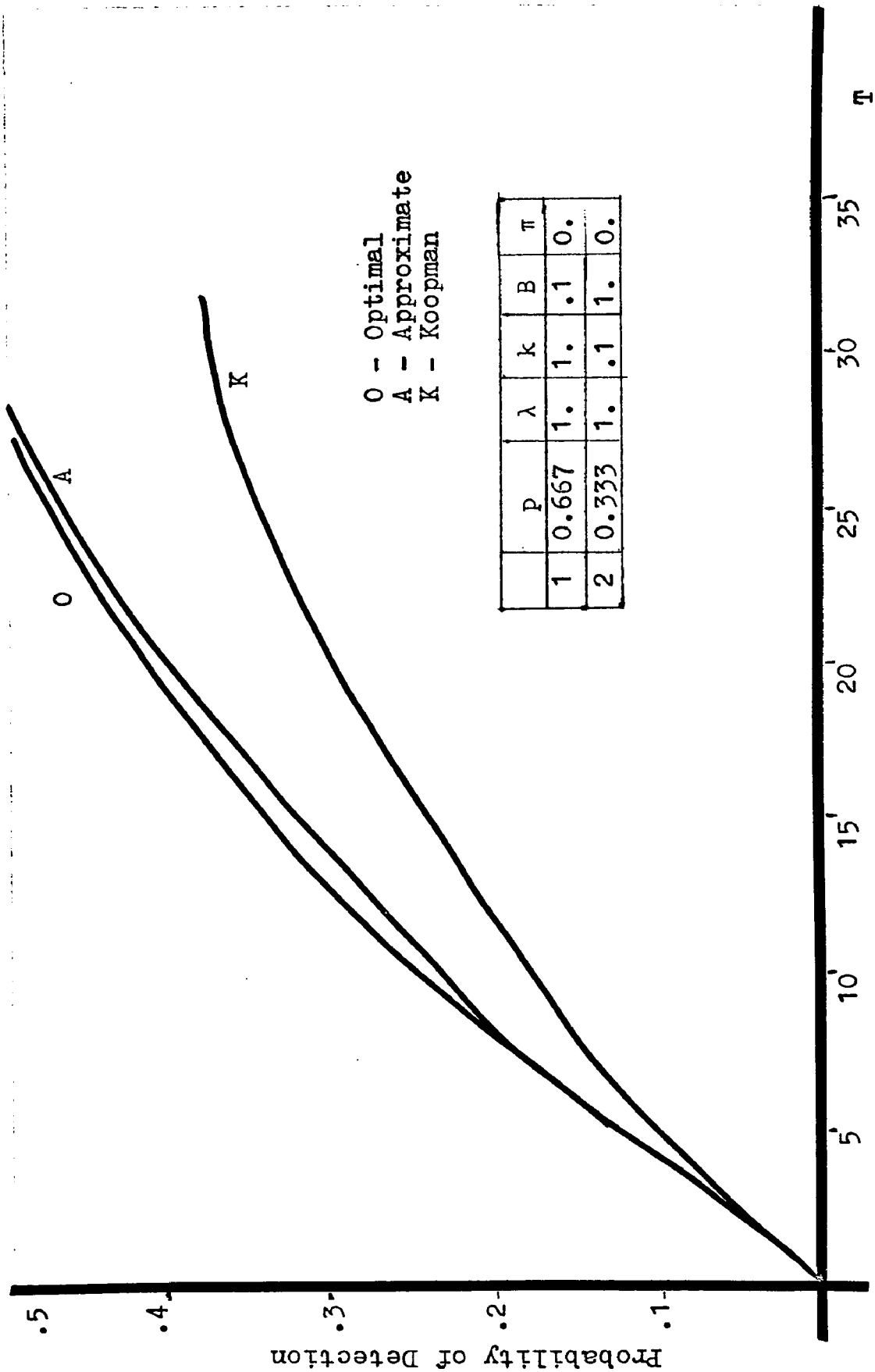
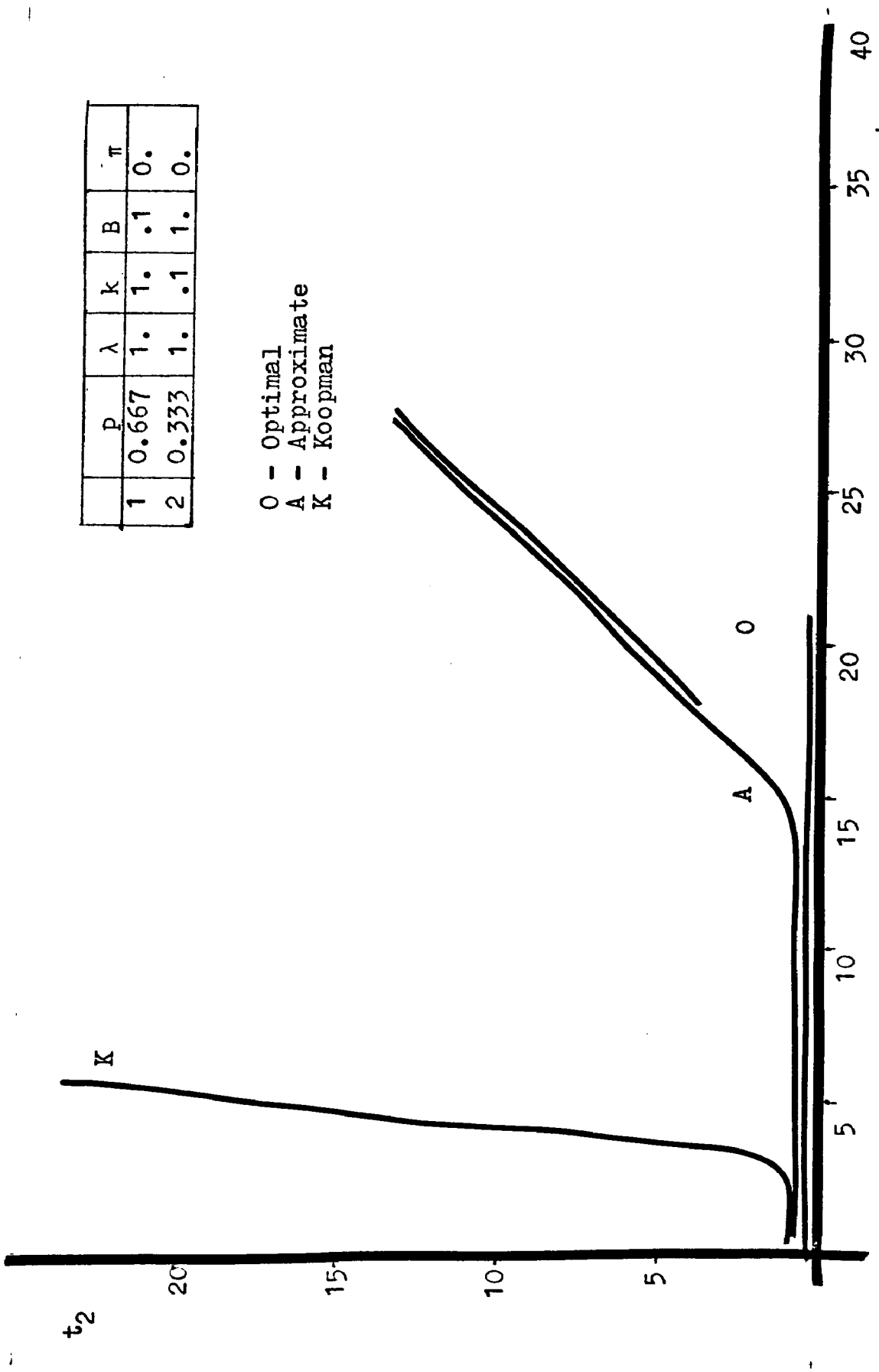


Figure 16 - A Comparison of the Results of Several Allocation Policies



	p	λ	k	B	π
1	0.667	1.	1.	.1	0.
2	0.333	1.	.1	1.	0.

O - Optimal
A - Approximate
K - Koopman

Figure 17 - A Comparison of Allocation Policies

return functions of Figure 16. We note that the optimal policy is not strictly increasing with increasing available search time, while the approximate policy is. Figure 16 shows that the maximum error in using the approximate policy occurs for $15 \leq T \leq 20$, the two policies differ over the interval because the optimal policy is not strictly increasing.

Figure 18 displays the probability of detection under the optimal and Koopman policies. Note that in this case the peak difference occurs at a level of search effort which is far greater than the Koopman switch point ($T^* = 3$). The reader will recall that for the models of Chapters 2 and 3, the use of the Koopman policy lead to small errors for larger quantities of search time. Thus the earlier option of the decision-maker, to expend additional effort under the Koopman policy in order to reduce the errors in the probability of detection, is no longer cost-effective, i.e., an inordinate amount of additional effort would be required to achieve the desired reduction of the error function. Figure 19, a plot of the probability of detection under the optimal and Koopman policies, illustrates the points just made for the late-arrival, no-departure model.

From the earlier discussions of Section 5.2.2, one can construct approximate expressions for the long term error in using the Koopman policy in searching for an intervisible target. The two-region search situation leads to the following error expression

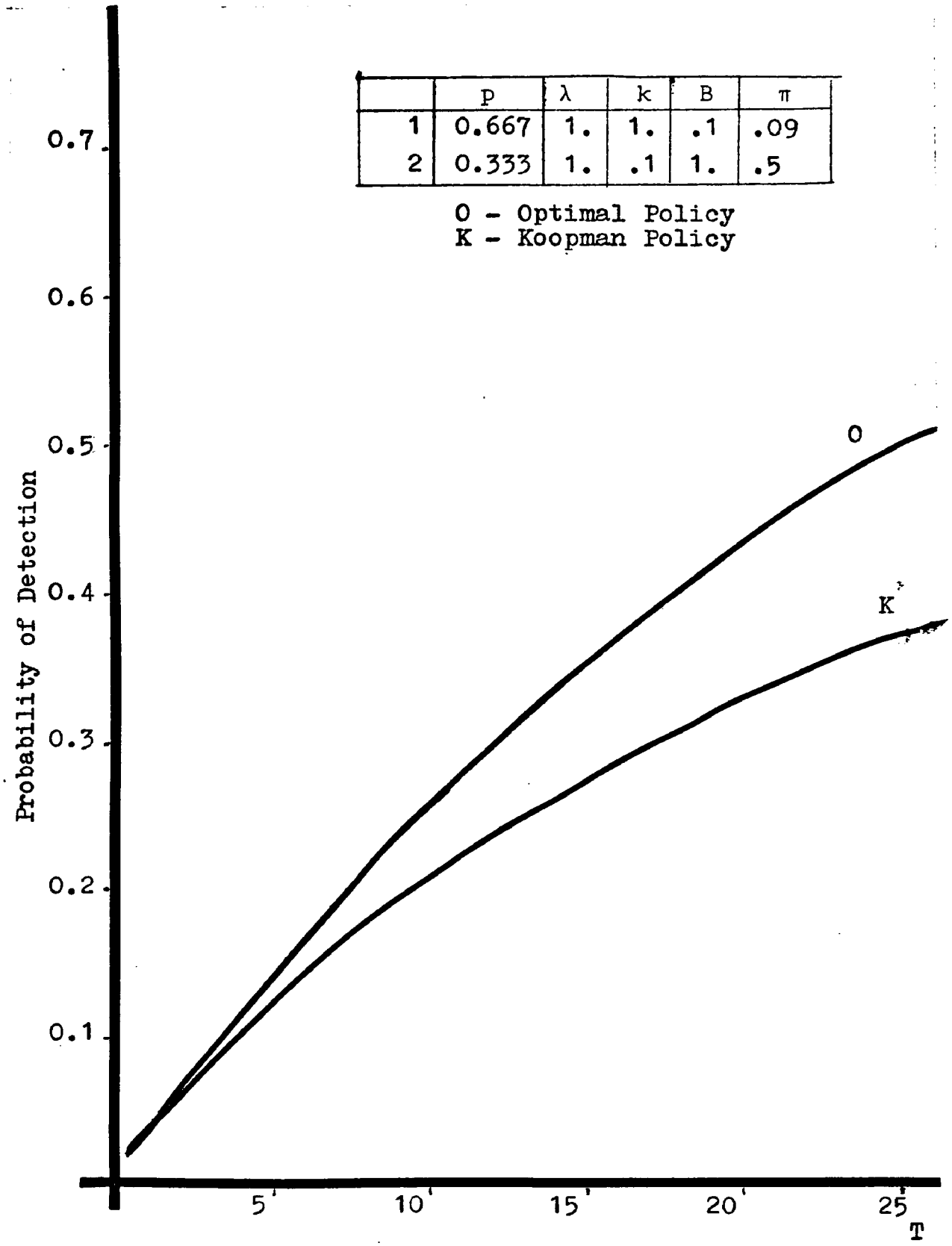
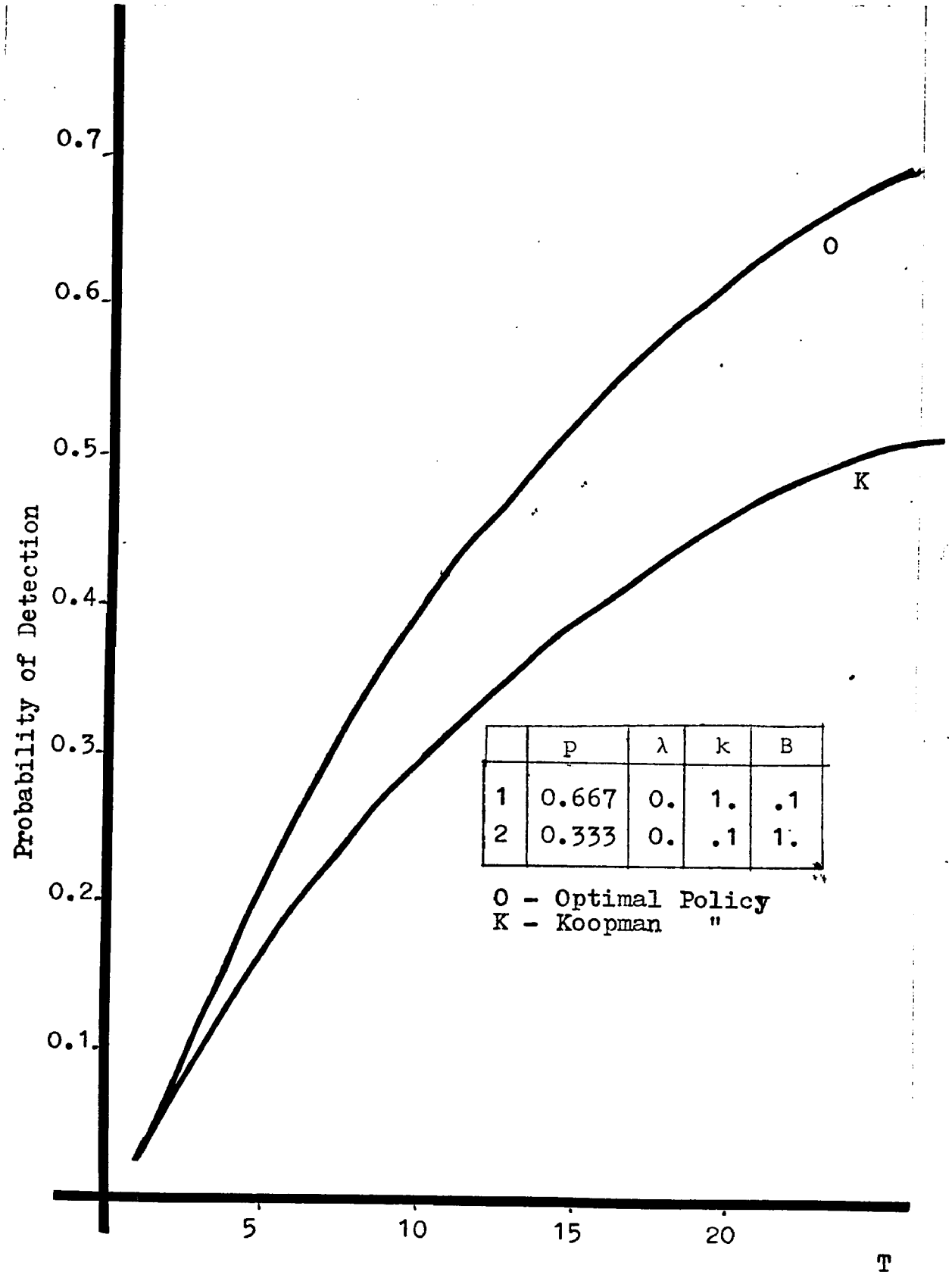


Figure 18 - A Comparison of Returns



	p	λ	k	B
1	0.667	0.	1.	.1
2	0.333	0.	.1	1.

Figure 19 - Probability of Detection versus Search Time

$$E(T) \simeq \left(e^{-\frac{\gamma_{11} k_2 T}{k_1+k_2}} - e^{-\frac{\gamma_{11}\gamma_{21} T}{\gamma_{11}+\gamma_{21}}} \right) \left\{ \left(\frac{\beta_1-\gamma_{11}}{\gamma_{11}} \right) \left(\frac{p_1 k_1 \Pi_1}{\gamma_{12}-\gamma_{11}} \right) \right.$$

$$\left. + \frac{(1-\Pi_1) p_1 k_1 \beta_1}{(\gamma_{12}-\gamma_{11})\gamma_{11}} \right\}$$

$$+ \left(e^{-\frac{\gamma_{21} k_1 T}{k_1+k_2}} - e^{-\frac{\gamma_{11}\gamma_{21} T}{\gamma_{11}+\gamma_{21}}} \right) \left\{ \left(\frac{\beta_2-\gamma_{21}}{\gamma_{21}} \right) \left(\frac{p_2 k_2 \Pi_2}{\gamma_{22}-\gamma_{21}} \right) \right.$$

$$\left. + \frac{(1-\Pi_2) p_2 \beta_2 k_2}{(\gamma_{22}-\gamma_{21})\gamma_{21}} \right\} .$$

Again the knowledge of the above function provides answers to the following questions over a wide range of values of the parameters λ_i, k_i, and β_i.

(a) What is the sensitivity of the error function to changes in the parameters?

(b) How much time, T, must be available to insure that the error function is less than some specified value, E_0? (i.e., solve $E(T) = E_0$, for T).

5.2.3 Model Sensitivity

In the following paragraphs, we present the results of a sensitivity analysis with respect to the visibility parameters for both the optimal policy and its associated return. We shall be utilizing the approximate allocation policies and the resulting returns obtained in Section 5.2.2. Thus, it is implicitly assumed that sufficient search time is

available for the approximations to hold. The sensitivity of the allocation policy to changes in the visibility parameters is computed from

$$t_1 = \frac{\gamma_{21}^T}{\gamma_{11} + \gamma_{21}} \quad \text{and} \quad t_2 = \frac{\gamma_{11}^T}{\gamma_{11} + \gamma_{21}} \quad \text{as follows:}$$

$$\frac{\partial t_1}{\partial \lambda_1} = \frac{\partial t_1}{\partial \gamma_{11}} \frac{\partial \gamma_{11}}{\partial \lambda_1} ,$$

where

$$\frac{\partial t_1}{\partial \gamma_{11}} = - \frac{t_1}{\gamma_{11} + \gamma_{22}}$$

and

$$\frac{\partial \gamma_{11}}{\partial \lambda_1} = \frac{1}{2} \left(1 - \frac{(\lambda_1 + k_1 + \beta_1)}{(\lambda_1 + k_1 + \beta_1)^2 - 4\beta k_{11}} \right) < 0 .$$

In the same fashion, one obtains

$$\frac{\partial t_1}{\partial \beta_1} = \frac{\partial t_1}{\partial \gamma_{11}} \frac{\partial \gamma_{11}}{\partial \beta_1} ,$$

where

$$\frac{\partial \gamma_{11}}{\partial \beta_1} = \frac{1}{2} \left(1 - \frac{[(\lambda_1 + k_1 + \beta_1) - 4k_1]}{(\lambda_1 + k_1 + \beta_1)^2 - 4\beta k_{11}} \right) > 0 .$$

One obtains for the sensitivity of t_1 to the visibility parameters of region 2,

$$\frac{\partial t_1}{\partial \lambda_2} = \left(-\frac{t_2}{\gamma_{11} + \gamma_{21}}\right) \frac{\partial \gamma_{21}}{\partial \lambda_2} \quad \text{and} \quad \frac{\partial t_1}{\partial \beta_2} = \left(\frac{t_2}{\gamma_{11} + \gamma_{21}}\right) \frac{\partial \gamma_{21}}{\partial \beta_2} .$$

The relative sensitivity of the λ's and β's for a given region is readily determined by the comparison of $\frac{\partial \gamma_{i1}}{\partial \lambda_i}$ and $\frac{\partial \gamma_{i1}}{\partial \beta_i}$, since the other terms are constant.

Considering the example of Figure 17, we have

$$\frac{\partial \gamma_{11}}{\partial \lambda_1} = .025 \text{ and } \frac{\partial \gamma_{11}}{\partial \beta_1} \simeq 1 ,$$

indicating the dominance of the rate for the masked period at any appropriate level of total available search time. From the symmetry of the two-region search situation, the sensitivity calculations for t_2 are apparent.

The explicit computation of the partial derivatives of the optimal return with respect to the visibility parameters is prohibitive, e.g., one must compute

$$\frac{\partial P(T)}{\partial \lambda_1} = \frac{\partial P(T)}{\partial \gamma_{11}} \cdot \frac{\partial \gamma_{11}}{\partial \lambda_1} + \frac{\partial P(T)}{\partial \gamma_{12}} \cdot \frac{\partial \gamma_{12}}{\partial \lambda_1} , \text{ etc.},$$

However, the dynamic programming solution technique is utilized to obtain some numerical approximations via difference functions. Again, considering the example of Figure 17, we obtain the

following results for T = 30;

$$\frac{\partial P(T)}{\partial \lambda_1} = -0.14, \quad \frac{\partial P(T)}{\partial \beta_1} = 0.69, \quad \frac{\partial P(T)}{\partial \lambda_2} = -0.05, \text{ and } \frac{\partial P(T)}{\partial \beta_2} = 0.$$

Again we note the dominance of the rate for the masked period.

A numerical study of the error function is contained in Appendix F and is summarized below. The Koopman policy is adequate in the following situations:

(a) a search scenario characterized by identical detection rates and identical visibility parameters.

(b) whenever the mean time-to-detect is much less than the mean length of the visible periods.

On the other hand, in the search situations characterized by heterogeneous visibility conditions, the resultant approximate rates (γ_i's) may differ significantly from the detection rates used in the Koopman policy, resulting in error functions which continue to increase far beyond the Koopman expected time to detect. Recall that (a) and (b) also held in the General Single Interval Model of Section 4.2, when one used the limiting probabilities of an alternating renewal process as the probabilities of target visibility at the start of the search. In contrast, the Koopman policy was not adequate for the single interval model with random starting times in Cases (a) and (b).

5.3 Minimization of the Expected Time Until Detection

5.3.1 Model Solution

As was noted in Section 5.2, the forms of the detection functions for the examples therein ranged from pseudo-concave to concave depending upon the choice of the initial state probabilities for the masked and visible states. Regardless of the form of the detection function one has, for this model, the result that the limiting value of the probability of detection is unity. If the detection function is concave, Dobbie's (1963) results imply that the policy which maximizes the probability of detection for all levels of effort is also the policy which minimizes the expected time until detection, under unlimited available search time. On the other hand, if the initial state probabilities are such that the detection function is pseudo-concave, the results described in [4.3.1] are employed to solve the expected time problem.

Computationally then, one first utilizes the criteria of Section 5.2 to determine the form of the detection function. If it is pseudo-concave, Steps 1-3 of the procedure given in [4.3.1] are followed; if concave, the output from the dynamic programming solution to the constrained time problem is taken as the input to the numerical integration routine in order to determine the minimum expected time.

5.3.2 Comparison with the Standard Model

Again we determine the effectiveness of the Koopman allocation in this situation by generating the value of the probability of detection under the Koopman policy and carrying out the required numerical integrations. The approximate solutions discussed in Section 5.2 may also be utilized, as was the Koopman policy, in generating the expected time until detection in order to compare this result with the actual minimum. Figure 20 illustrates these points (See also Figures 15-19) and compares the multiple interval models with a late-arrival, no-departure version of the random interval models of Chapter 4. Note that, for the parameters of Figure 20, the assumptions that the prior probabilities on target visibility are the limiting results of an alternating renewal process, and that the target is masked at the start of the search, lead to essentially the same value for the expected time until detection. Hence the appropriateness of the Koopman policy is unaffected, in this case, by the difference in the prior probabilities on the target's visibility status at the start of the search. Figure 21 shows the sensitivity of the Koopman policy to variations in the detection rates. Since the visibility parameters λ and β are not identical over the regions of interest, one is not assured that under identical detection rates (of the order of the visibility rates) the Koopman policy will yield good results for the

expected time problem. Appendix F contains the maximum errors and their times of occurrence for the parameters of Figure 21, as well as other combinations of visibility parameters.

Ident. Symbol	Regions (2) R	Prior on-tgt. Loc. p	Visibility Rates λ	Detection Rates k	Masking Rates β	Prior on Visibility Π	Expt. Time (optimal)	Expt. Time (Koopman)	Model Desc.
	1 2	2/3 1/3	1 1	1. .1	.1 1.	0. 0.	* 39.4	88.2	Multiple Interval
×	1 2	"	"	"	"	0.09091 0.5	38.2	84.5	"
0	1 2	"	"	"	"	1. 1.	28.9	49.2	"
4	1 2	"	0. 0.	"	"	- -	* 20.4	42.3	Late Arrival No Departure

*Using the allocation policy generated by the minimal concave majorant of the detection functions.

All the values for the expected search time were obtained via numerical integration over a finite interval, the lengths of the intervals as well as the terminal values of the probability of detection under the various policies are given below

Ident. Symbol	T_{max}	$P_D(T_{max})$ Optimal	$P_D(T_{max})$ Koopman
	200	.99	.75
×	200	.99	.76
0	200	.99	.87
4	100	.99	.77

Figure 20 Sensitivity of the Expected Time Until Detection To the Prior Estimates On Target Visibility

k_2 \ k_1	.1	1.
.1	1.2	2.2
1.	1.+	1.3

Ratio of the Expected Times Until Detection

k_2 \ k_1	.1	1.
.1	108	38.2
.1	77.2	18.8

The expected time until detection under the optimal policy for

p	λ	β
0.667	1.	.1
0.333	1.	1.

Figure 21 The Sensitivity of the Expected Time Until Detection to the Detection Rates

Chapter 6

SUMMARY, CONCLUSIONS AND AREAS FOR FUTURE RESEARCH

In this chapter the important results for each of the search models presented in Chapters 2-5 are summarized and several areas for future research introduced. The results are presented in the order in which the models were described. The areas for future research are then presented.

6.1 Summary and Conclusions

6.1.1 The Binary Visibility Model

In the study of the Binary Visibility model of Chapter 2, the concept of the First Allocation Rule (FAR) was introduced and the FAR for this model was shown to differ from that of the classical Koopman results, i.e., the use of the Koopman FAR could lead to incorrect selection of the initial region to be searched. It was noted that the switch points in the 2-region search situation may differ from those of the classical search problem in either a positive or negative direction. In a situation in which the visibility probabilities are identical across the N-regions (not necessarily equal to one), the optimal solutions are identical to those obtained from classical search theory in which the target is assumed to be continuously visible. In comparing the optimal policies, Binary versus Koopman,

it was observed that the long-term allocation rates are identical. Thus the errors, when one uses the classical results in this situation, are maximized in situations in which the visibility conditions are heterogeneous and the available search effort is highly constrained. Hence, a searcher may compensate for a lack of knowledge of the visibility parameters at the cost of increasing the amount of effort allocated to the search.

Sensitivity analyses indicated that the optimal *policy* is robust with respect to changes in the visibility parameters whenever the detection rates are large. Sensitivity of the optimal policy to changes in the visibility parameters is independent of the level of available search effort.

Explicit expressions are provided for the optimal policies in the discrete detection and visibility process situation and it is shown that when one equates the infinitesimal detection and visibility probabilities to their discrete counterparts, the optimal allocations are approximately equal.

In considering the problem of minimizing the expected time until detection, one is forced to restrict attention to the conditional version of this problem, since the ultimate probability of detection is less than unity. Although there are many situations in which the use of the Koopman policy is approximately optimal in maximizing the

probability of detection, it may be inefficient when used in the expected time problem in a search situation in which the target is not continuously visible.

6.1.2 Random Interval of Visibility: Initially Visible

In this case the First Allocation Rule is identical to the classical results; however, the switch point for this model is always less than that of the classical search situation, i.e., one always begins to allocate some effort to the second region at a lower level of total available search time. Viewing this model as representing a search for a target which has its own detection capabilities, it is shown that wherever the searcher and target have identical regional capabilities (not necessarily equal), and the state of maximum prior uncertainty on the target position exists, the Koopman or classical policy is optimal. Hence, if the searcher is willing to make this assumption, or has data to verify these conditions, he may proceed as though he were searching for a continuously visible target. Furthermore, the assumption of maximum uncertainty in the prior location was later shown in specific numerical examples not to be critical. These results are valid for the N-region search situation.

In comparing optimal policies and returns, classical versus the Random Interval, it was noted that the peak relative error may occur at a level of search time approximately equal to the expected time to detect the target (denoted KET) under the optimal policy in situations in which the target is continuously visible. In such situations, one may expect relative errors of 10 and 5 percent at levels of search effort equal to two and three times the KET, respectively. Thus the searchers option of increasing the search time to offset a lack of knowledge of the visibility parameters may (depending on the acceptable error) no longer be cost-effective. These situations can occur whenever the visibility conditions are regionally heterogeneous.

The use of the classical policies in the problem of minimizing the conditional expected time until detection can lead to large errors even though the classical policy is approximately optimal in the problem of maximizing the probability of detection. It was observed in this context that, if the target's detection rates are greater than those of the searcher, it is important that the searcher obtain estimates of these in order to conduct an effective search.

Sensitivity analyses indicated that, in contrast to the results for the Binary model, the sensitivity of the

optimal *policy* to variations in the visibility parameters increases with increasing available search effort; however, the regional change as a percentage of the total allocation to that region remains constant. The sensitivity of the optimal allocation in the first region to variations in the visibility parameters of the second region is proportional to the current level of search effort in the second region. It was noted that the limiting sensitivity, as $T \to \infty$, of the optimal *return* to variations in the visibility parameters of a given region is proportional to the ultimate probability of detection in the region, where the constant of proportionality is the conditional mean time until detection in that region (conditioned on detection occurring).

The situation in which the detection and visibility processes are discrete was modeled and the optimal policies explicitly obtained. The optimal policies for the discrete and continuous detection models are shown to be approximately equal whenever the infinitesimal detection and visibility probabilities are approximately equal to their discrete analogs.

6.1.3 Random Interval of Visibility: Random Initiation and Limit

Application of the classical FAR in this situation could lead (in many cases) to the erroneous selection of the initial region to be searched. The model gave rise to pseudo-concave conditional detection functions, thus presenting the first "practical" examples of this more general class of detection functions. The optimal allocation policies resulting from such models have the property that they are not strictly non-decreasing with increasing total available search effort. Such policies have not previously appeared in the open literature on search and reconnaissance. Furthermore, in the case of identical regions it is shown that one doesn't start the allocation procedure by dividing the effort equally among the identical regions. The nature of these optimal policies implies that in the random-interval, random-start time model, one must have a minimum level of available time (which can be numerically determined from the process parameters) before beginning to search a new region. In the event that level of total available search time is insufficient in this respect, the optimal policy is to continue placing all the effort in one region rather than a small quantity in a new region. Recall that in this model that target is not visible (or present) at the time the searcher arrives in the

appropriate region, but is to become visible (or arrive) after some random length of time. Hence the return from placing small additional amounts of effort into a region in which one has been searching is greater than that obtained from allocating the same amount to a second, previously unsearched region, provided the total available search time is less than the sum of the minimum levels of search time for each of the two regions.

The pseudo-concave character of the conditional detection functions precludes the derivation of explicit expressions for the optimal policies. However, approximate policies are developed and the requisite conditions for their applicability discussed. It is shown that these policies are determined from the lesser of the two rates associated with the regional visibility processes--the first being the rate associated with the length of the masked period (β); the second, the conditional detection rate ($\lambda + k$) conditioned on detection occurring in the visibility interval. One can easily determine the conditions under which the classical optimal allocation policies will coincide with the approximate policies, or will greatly differ from them. Such observations are directly applicable to N-region search situations. Thus, in contrast to the models of Chapters 2 and 3, it is shown that

(a) the use of the Koopman policy will not be adequate in situations in which the visibility conditions are homogeneous across the regions of interest,

(b) the error function is not reduced by increasing the available search time to some realistic level,

(c) the maximum errors do not occur in situations in which the total available search time is highly constrained, and

(d) the availability of extremely good detectors (high rates) will not imply that the Koopman (or classical) policy yields small errors.

These results suggest that the searcher, in general, must obtain accurate estimates of the visibility parameters in order to conduct an effective search.

The model was shown to specialize to a late-arrival, no-departure model (by choosing $\lambda = 0$) and all of the analyses are directly applicable. It, in turn, was shown to be a special case of what was termed the General Single Interval model. This model has detection functions ranging from concave to pseudo-concave depending upon the levels of the a priori probabilities of initial target visibility, Π. The approximate policies just discussed were shown to be applicable here also.

Numerical studies comparing the classical and optimal search policies indicated that the classical policy, while not necessarily optimal, could be used effectively for the

General Random Interval model in situations in which (a) the mean time required to detect the target is much less than the mean length of the visible period or (b) the detection and visibility rates are identical across regions.

Sensitivity analyses indicated that the sensitivity of the optimal policy to changes in the visibility process parameters increases with increasing total available search effort; however, the change as a percent of the total allocation remains constant. The sensitivity of the optimal allocation for the first region to changes in the visibility parameters of the second region is directly proportional to the current level of search effort within that region. One can, of course, directly determine the effects of these changes in the visibility parameters upon the optimal return. For example, increasing in the rate of the masking period causes an increase in the optimal return. Also, as one might expect, this effect diminishes to zero as the total available search time increases. On the other hand, the sensitivity of the optimal return to changes in the visibility rate increases with increasing available search time to an asymptotic value which is identical to that obtained for the previous model.

The situation in which the detection and visibility processes are discrete is modeled. The solution techniques utilized in the previous chapters were again applicable; however, the nature of the resulting detection functions

precluded the derivation of explicit expressions for the optimal allocation policies. Accordingly, numerical techniques were used. All of the extensions and special cases associated with the continuous detection models were shown to be valid in this situation.

6.1.4 Multiple Intervals of Visibility

In this model the First Allocation Rule (FAR) can differ from the classical FAR. Thus, the use of the latter in this situation could lead to erroneous selection of regions to be searched. The conditional detection functions range from concave to pseudo-concave depending upon the levels of Π, the prior probability on target visibility at the start of the search. For example, the assumption that the Π's are the limiting probabilities of an alternating renewal process leads to a concave conditional detection function.

The nature of the conditional detection functions for this model, conditioned on the target presence, preclude the derivation of explicit expressions for the optimal allocation policies. The pseudo-concave conditional detection functions give rise to optimal allocation policies which require minimum levels of search time before a new region is searched (as did the model of Chapter 4, which is, of course, a special case of this model). Approximate

solutions were derived from an analysis of the regional composite detection rates, which are functions of the visibility and detection parameters.

From these studies it is easy to determine the conditions under which the approximate allocation policy (which is accurate for large quantities of search effort) will be close to the classical allocation policies, as well as those for which the opposite conclusion holds. Thus, the classical or Koopman policy was shown (both numerically and analytically) to be adequate whenever (a) the search scenario is characterized by identical regional detection rates and identical regional visibility parameters, or (b) whenever the mean time until detection is much less than the mean length of the visibility periods.[1]

By choosing Π to be either zero or one, direct generalizations of the models of Chapters 2 and 3 are obtained. For example, for Π equal to unity, one obtains a multiple-interval-start-at-time-zero model, while the opposite situation yields a multiple-interval, random-start-time model.

Sensitivity analyses indicate that the sensitivity of the optimal allocation in a given region to changes in

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[1]These results hold under the assumption that the Π's are the limiting probabilities of an associated alternating renewal process.

the visibility parameters is proportional to the amount of effort allocated to the region containing the visibility parameter, the constants of proportionality are given by the rates of change of the composite process rate with respect to the visibility parameters. These constants can be compared readily to determine the relative influence of the various visibility parameters.

It was also shown that the probability of detection tends toward unity with increasing search effort. The other models did not have this property, with the exception of the late-arrival, no departure model of Chapter 4. In light of these results, one may study the expected time problem without introducing the conditioning devise required for the other models. Numerical results indicate that, while the classical policy is adequate in many cases, for the problem of maximizing the probability of detection, it may not be efficient when minimizing the expected time until detection.

Finally, some general observations on our models are as follows. It was observed that (with a single exception), under the restrictions of

(a) a uniform distribution on target location,

(b) uniform regional detection capabilities,

(c) uniform visibility parameters, and

(d) levels of search time equal to twice the KET;

the Koopman or classical policies are approximately optimal in maximizing the probability of detection under a constraint on the total available search time. The exception occurs with the model of Chapter 4. Thus, the classical policies are applicable to a number of situations in which their applicability was not apparent. On the other hand, the inadequacy of the classical policies was demonstrated when one or more of the above restrictions is violated.

For each of the situations analyzed, it was assumed that the visibility and detection processes had very specific forms. The insights developed and summarized above were based on the use of these forms. However, the model structures are not restricted to these special forms, and accordingly, the optimal policies and returns can be readily obtained, via numerical techniques (e.g., dynamic programming), for any detection and visibility processes.

6.2 Areas of Future Research

In this section, several areas for future research are discussed. We briefly introduce each area and suggest some avenues of approach.

(1) Enrichments of the multiple interval model

We shall consider three directions of extension for this model:

(1) the target has a detection capability, i.e., if the target detects the searcher first, he exits;

(2) the number of transitions to the masked or visible state is controlled by an external random process, and

(3) the introduction of various levels of visibility.

We shall explore only the first direction.

One could consider two cases:

(1) the target can detect only while it is in the "visible" state,

(2) the target can detect the searcher in either state.

This situation gives a complete spectrum of visibility models, since, loosely speaking, it is equivalent to allowing a finite number of visibility intervals. The model of Chapter 5 allows for an infinite number and the models of Chapters 3 and 4 treat the single interval case. Note that the ultimate probability of detection in these situations is less than unity.

Let the visibility process of the target (within a region) be modeled as an alternating renewal process, the steady state probabilities of which give the probabilities

of the target being visible. First, we assume that the target's detection gear is working only while in the visible state (e.g., passive air search for a submarine). If the searcher enters the region and finds the target in the visible state, then the length of time the target spends in the visible state is a random variable given by the smaller of either:

(a) the forward recurrence time of the visibility period, or

(b) length of time required for the target to detect the searcher.

It might be assumed that the searcher has another chance at the target in case (a), while in case (b) the target has escaped.

Let the transition matrix for the model be given by

$$Q(T) = \begin{array}{c} \begin{matrix} (V) & (I) & (D) & (E) \end{matrix} \\ \begin{pmatrix} 0 & R(t) & D(t) & H(t) \\ F_m(t) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} ,$$

where the states are

V, visible;

I, masked;

D, target detected; and

E, target escaped.

Expressions for the distribution functions may be derived as in Chapter 5. Having obtained the transition matrix, one need only apply the procedures of Chapter 5 to obtain the conditional detection functions.

(2) Avalanche search model

In this model the searcher knows that a climber is lost in an avalanche or a storm somewhere on a mountain. The length of time that he may be expected to survive is a random variable dependent upon his precise location. The objective is to assign a limited amount of search effort in such a way as to maximize the probability that he will be found alive.

There are several other analogies of this model, the most obvious being the stricken submarine, e.g., the *Thresher* incident. Another interesting analogue is in the area of medical screening for a terminal disease. One is interested in the sequence, and quantity of treatment which maximizes the probability of survival.

A description of this problem may be obtained by starting with the general formulation of the model for a single interval of visibility starting at time zero, i.e., for the jth region,

$$P(T_1) = p_j \int_0^{T_1} \rho_j(\tau)\bar{H}_j(\tau)d\tau \ , \qquad (45)$$

where $\rho(\tau)$ is density function for the time until detection, $\bar{H}(\tau)$ is complementary cumulative distribution function on the length of the visible period, and p_j is the probability that the target is in the jth region. Equation 45 is appropriate if the jth region is searched first. Next, assume it has been determined that region K is next in sequence. It shall also be assumed that in going from region j to K, one incurs a switching time t_{jK}. The probability of detection is this region, assuming one allocates T_2 units of time, is given by

$$P_D(T_2) = p_K \int_0^{T_2} \rho_K(\tau)\bar{H}_K(\tau+T_1+t_{jK})d\tau \ , \qquad (46)$$

where p_K reflects the fact that the previous search of region j was unsuccessful. Conceptually, this process could be continued for successive numbers of regions. However, even if the sequence is given, one is still faced with the problem of determining the optimal quantities of the total available effort to assign to each region in the given sequence. This problem is more difficult than the previous formulations because of the non-separability aspects. If one were able to explicitly solve for the optimal allocations of a given sequence, then the substitution of

these results into the return function for the sequence would yield an expression which contained only the known search parameters. Having accomplished this, one could then test for dominance among all such return functions, hopefully, arriving at a criteria (in terms of the known parameters of the problem), which would enable the a priori determination of the optimal sequence.

This ambitious program was attempted for a 2-region search situation, with exponential distributions on the detection and visibility phenomena, with little success. The reason for the lack of success was the failure to obtain explicit expressions for the optimal allocations under a given sequence. However, this brief exercise did produce some interesting results and conjectures. In certain cases one places all the effort in the first region regardless of the total amount available, while in others, there is an upper bound on the amount of effort placed in the initial region. It was also noted that the First Allocation Rule will determine the proper order for very small quantities of total available search effort. Finally, the research suggests the following conjecture: "For the N-region problem with $p_i k_i = p_j k_j$, $j \neq i$, the optimal sequence is obtained from the ranking as to increasing expected lengths of the respective visibility periods."

(3) Random search times

One might consider either of the following random search time situations: First, the case in which the total available effort is a random variable with known distribution; second, the situation in which the searcher spends a random amount of time in each region. The former situation could arise in context of an aerial search in which unknown or changing weather conditions randomly influence the total search time. An approach to this situation might be to formulate it as a stochastic programming problem of either the 2-stage or chance-constraint variety depending upon the particular situation. One approach to the second problem might be to solve it, assuming the search times to be fixed. Then use these optimal values as nominal values with some known, possibly regionally dependent, variation to compute the expected probability of detection. Or as an approximation to this, one could compute the partial derivatives (evaluated at the nominal levels) of the return function with respect to the optimal allocations. These scenarios become especially interesting in view of the allocation policies for the models of Chapters 4 and 5, since models gave rise to situations in which the optimal policy has the property that the amount of effort put in a given region is not monotone non-decreasing as a function of the total available search effort (e.g., see Figure 4, Chapter 4).

(4) Target motion

The process of target transitions from one region to another might be realistically modeled as a semi-Markov process with the distribution of transition times depending upon the current state and the next state. Such a model seems to be appropriate in the sense that a commander can choose both the next region and the transit speed to that region at random thus generating the semi-Markov transition matrix. In actual transit situations this appears to be a more reasonable model than, say, Brownian motion. In addition to the motion consideration, target intervisibility could occur due to changes in velocity. In this same context, tracking states (i.e., entrances to and exits from) are also of interest. With the introduction of such states, some new search objective functions might be of interest, e.g., the maximization of the probability that the time spent in the tracked state is greater than some specified level.

Appendix A

METHODOLOGY

A.1 Purpose

The intent of this appendix is to develop in detail the solution methodology for the two major search objectives:

(a) maximizing the probability of detection under a constraint on the available effort, and

(b) minimizing the expected time until detection.

First, we shall present a modified version of the Charnes-Cooper Algorithm for the discrete search space version of Koopman's problem. This algorithm is used in solving the approximations to many of the models presented in Chapters 2-5. The discrete detection version of the Koopman problem is then discussed and related to the continuous detection analog. The technique presented will be useful in analyzing the discrete versions of many of the visibility models.[1] Finally, we look at minimizing the expected time until detection for both the continuous and discrete cases.

A.2 The Allocation of Search Effort to Maximize the Probability of Detection

A.2.1 Continuous Detectors

In this section we modify the Charnes-Cooper (1958) Algorithm for solving the discrete search space problem.

[1] See Appendix B for discrete detector formulations and results.

The target of Koopman's problem is located in one of N regions with prior probability vector $P = (p_1, p_2, \ldots, p_N)$ and the searcher has detection rates for each of these regions given by the vector $K = (k_1, k_2, \ldots, k_n)$. The basic optimization problem becomes

$$\max \sum_{i=1}^{N} p_i \left(1 - e^{-k_i t_i}\right) ,$$

$$\text{S.T.} \sum_{i=1}^{N} t_i = T ,$$

$$t_i \geq 0 .$$

The Charnes and Cooper algorithm for the above problem has the restriction that the $k_i = k$ for each region, which can be easily removed. The application of the Kuhn-Tucker conditions to the above problem will yield the optimal solution. Then the conditions are necessary and sufficient in this problem since we have a concave objective function. Reformulating the problem as a minimization one:

$$\max \sum_{i=1}^{N} p_i \left(1 - e^{-k_i t_i}\right) \equiv \min \sum_{i=1}^{N} p_i e^{-k_i t_i}$$

Scaling the allocations by T;

$$\frac{t_i}{T} = \tau_i \text{ , and } \sum_{i=1}^{N} \tau_i = 1,$$

one obtains as the problem

$$\min \sum_{i=1}^{N} p_i e^{-Tk_i \tau_i}$$

$$\text{S.T.} \sum_{i=1}^{N} \tau_i = 1$$

$$\tau_i \geq 0 \cdot$$

Form the Lagrangian

$$W(\tau, w, \psi) = \sum_{j=1}^{N} p_j e^{-Tk_j \tau_j} - \sum_{j=1}^{N} w_j \tau_j + \psi \sum_{j=1}^{N} (\tau_j - 1) .$$

Application of the Kuhn-Tucker conditions to the above equation yields

$$Tk_j p_j e^{-Tk_j \tau_j} = \psi - w_j \quad \text{if } w_j \geq 0, \quad \tau_j \geq 0 ,$$

$$Tk_j p_j e^{-Tk_j \tau_j} = \psi, \text{ i.e., for } \tau_j > 0, \text{ then } w_j = 0 .$$

From these equations one obtains that

$$Tk_j p_j = \psi - w_j \text{ when } \tau = 0 , \quad w_j \geq 0 .$$

$$Tk_j p_j e^{-Tk_j \tau_j} = \psi \text{ when } \tau_j > 0 .$$

Taking logarithms, we have

$$\ln T + \ln k_j + \ln p_j \leq \ln \psi \tag{1}$$

$$\ln T + \ln k_j - T k_j \tau_j + \ln p_j = \ln \psi, \quad \tau_j > 0 . \tag{2}$$

Dividing (2) by k_j, one obtains

$$\frac{\ln T}{k_j} + \frac{\ln k_j}{k_j} - T\tau_j + \frac{\ln p_j}{k_j} = \frac{\ln \psi}{k_j}, \quad \tau_j > 0 . \tag{3}$$

Let the set J be the set of indices of those regions which receive a positive allocation of search effort, i.e., note that

$$J = \left\{ i \middle| \tau_i > 0 \right\} ,$$

and

$$\sum_{j=1}^{N} \tau_i = \sum_{j \in J} \tau_i = 1 .$$

Summing equation 3 over all $j \in J$,

$$\sum_{j \in J} \frac{\ln T}{k_j} + \sum_{j \in J} \frac{\ln k_j}{k_j} - T \sum_{j \in J} \tau_j + \sum_{j \in J} \frac{\ln p_j}{k_j} = \sum_{j \in J} \frac{\ln \psi}{k_j}$$

or

$$\ln \psi = \ln T + \frac{1}{\sum_{j \in J} \frac{1}{k_j}} \left(\sum_{j \in J} \frac{\ln k_j}{k_j} + \frac{\ln p_j}{k_j} - T \right) \tag{4}$$

Denote $\ln x$ by $\hat{x}$ and note that $Tk_j\tau_j > 0$ for $j \in J$. Then, using (1) and (4)

$$\hat{T} + \hat{k}_S + \hat{p}_S \leq \hat{T} + \frac{1}{\sum_{j\in J} \frac{1}{k_j}} \left(\sum_{j\in J} \frac{\hat{k}_j}{k_j} + \frac{\hat{p}_j}{k_j} - T \right) \tag{5}$$

$$< \hat{T} + \hat{k}_r + \hat{p}_r$$

where $S \notin J$ and $r \in J$

Substitution of (4) into (2) yields

$$\tau_j = \frac{1}{Tk_j} \left[\hat{p}_j + \hat{k}_j - \frac{1}{\sum_{i\in J} \frac{1}{k_j}} \left(\sum_{j\in J} \frac{\hat{k}_j}{k_j} + \frac{\hat{p}_j}{k_j} - T \right) \right]. \tag{6}$$

Now, if the members of the set J were known, the solutions are given by (6). Toward this end we modify a lemma by Charnes and Cooper.

From (5) we have

$$\min_{r\in J} \hat{p}_r + \hat{k}_r > \frac{1}{\sum_{j\in J} \frac{1}{k_j}} \left(\sum_{j\in J} \frac{\hat{p}_j}{k_j} + \frac{\hat{k}_j}{k_j} - T \right) \tag{7}$$

$$> \max_{S\in J} \hat{p}_S + \hat{k}_S$$

Replace $\hat{p}_j + \hat{k}_j$ by β_j. The resultant lemma is

Lemma: If $\beta_{m+1} > \frac{1}{\sum_{i=1}^{m} \frac{1}{k_j}} \left(\sum_{i=1}^{m} \frac{\beta_i}{k_i} - T \right)$,

then

$$\beta_{m+1} > \frac{1}{\sum_{i=1}^{m+1} \frac{1}{k_i}} \left(\sum_{i=1}^{m+1} \frac{\beta_i}{k_i} - T \right).$$

Proof: By hypothesis

$$\sum_{i=1}^{m} \frac{1}{k_i} \beta_{m+1} > \sum_{i=1}^{m} \frac{\beta_i}{k_i} - T$$

$$\sum_{i=1}^{m} \frac{1}{k_i} \beta_{m+1} + \frac{\beta_{m+1}}{k_{m+1}} > \sum_{i=1}^{m} \frac{\beta_i}{k_i} + \frac{\beta_{m+1}}{k_{m+1}} - T$$

$$\beta_{m+1} > \frac{1}{\sum_{i=1}^{m+1} \frac{1}{k_{m+1}}} \left(\sum_{i=1}^{m+1} \frac{\beta_i}{k_i} - T \right)$$

Q.E.D.

To apply these results, to our minimization problem

(a) Arrange the values of $\beta_j = \hat{p}_j + \hat{k}_j$ in decreasing order, and relabel regions according to this order.

(b) If $\hat{p}_1 + \hat{k}_1 > k_1 \left(\frac{\hat{p}_1 + \hat{k}_1}{k_1} - T \right)$, region 1 is selected.

(c) If $\hat{p}_2 + \hat{k}_2 \leq k_1 \left(\frac{\hat{p}_1 + \hat{k}_1}{k_1} - T \right)$, stop.

Otherwise region 2 is selected and one continues via the lemma until the set J is determined.

The results for the two-region model follow directly. We shall assume that $p_1k_1 \geq p_2k_2$. First, we determine under what conditions all the search effort is placed in region 1. From the algorithm for the selection of indices for the set J, one stops at region 1 if

$$\ln(p_2k_2) \leq k_1\left(\frac{\ln(p_1k_1)}{k_1} - T\right)$$

leading to

$$T^* \leq \frac{1}{k_1} \ln\left(\frac{p_1k_1}{p_2k_2}\right) . \tag{8}$$

Then for total search time $T \leq T^*$, one spends the entire time searching region 1. For $T \geq T^*$, one obtains, using (6), the optimal allocations to the two regions as

$$t_1 = \frac{1}{k_1 + k_2}\left[\ln\left(\frac{p_1k_1}{p_2k_2}\right) + k_2T\right] ,$$

and

$$t_2 = \frac{1}{k_1 + k_2}\left[\ln\left(\frac{p_2k_2}{p_1k_1}\right) + k_1T\right] . \tag{9}$$

We observe that for large T, one has

$$t_1 \simeq \frac{k_2T}{k_1 + k_2} , \quad t_2 \simeq \frac{k_1T}{k_1 + k_2}$$

The optimal value for the probability of detection is then given by

$$P(T) = p_1 \left[1 - \left(\frac{p_1 k_1}{p_2 k_2}\right)^{-\frac{k_1}{k_1+k_2}} e^{-\frac{k_1 k_2}{k_1+k_2} T}\right] + p_2 \left[1 - \left(\frac{p_2 k_2}{p_1 k_1}\right)^{-\frac{k_2}{k_1+k_2}} e^{-\frac{k_1 k_2}{k_1+k_2} T}\right].$$

To handle the N-region case, one merely follows the procedure outlined in the proof of the algorithm

A.2.2 Discrete Detectors

Recall the following definition given in Chapter 1, Incremental Detection: Let $1 - q$ be the instantaneous probability of detection on any scan of a region. Given m such scans, the conditional probability of detection is

$$P(m) = 1 - q^m .$$

Let $P = (p_1, p_2, \ldots, p_N)$ be the prior probabilities on target location. Then the N-region search problem may be stated as

$$\max \sum_{i=1}^{N} p_i(1 - q_i^{m_i})$$

$$\text{S.T.} \quad \sum_{i=1}^{N} m_i \leq M \qquad (10)$$

$$m_i \geq 0 \text{ , integers.}$$

This formulation may be solved using Dynamic Programming, (or the results of Chew (1967)), however, some results (Wagner, 1969) enable one to take a more direct approach. These results are summarized in the following theorem.

Wagner's Theorem

Let e and c be real-valued functions (of two variables) defined on

$\{(x,j) | a < x < b, \ell(x) \leq j \leq u(x), j \text{ an integer}\}$.

Let Φ be the set of all integer-valued functions f on (a,b) such that $\ell(x) \leq f(x) \leq u(x)$ for $a < x < b$ for which

$$-\infty < E(f) = \int_a^b e(x,f(x))dx < \infty$$

$$-\infty < C(f) = \int_a^b c(x,f(x))dx < \infty .$$

Suppose that $g \in \phi$ has the following property: there exists a $\lambda > 0$ such that for all $x \in (a,b)$ and integers j

$$e(x,j) - e(x,j-1) \leq \lambda[c(x,j) - c(x,j-1)]$$

whenever $g(x) < j \leq u(x)$, and

$$e(x,j) - e(x,j-1) \geq \lambda[c(x,j) - c(x,j-1)]$$

whenever $\ell(x) \leq j - 1 < g(x)$, then

$$E(g) = \max \{E(f) | f \in \phi \text{ and } C(f) \leq C(g)\}$$
$$C(g) = \min \{C(f) | f \in \phi \text{ and } E(f) \geq E(g)\} .$$

Application of the theorem to (10) gives the following necessary conditions on the solution:

(a) for the ith region

$$m_i = \left[\frac{\log \left(\frac{\lambda}{p_i(1-q_i)} \right)}{\log q_i} + 1 \right] , \qquad (11)$$

where the bracket implies the largest integer therein,

(b) The objective is to choose a $\lambda > 0$, such that (a) is satisfied for all i for which $m_i > 0$, and

$$\sum_{i=1}^{N} m_i = M .$$

One does this iteratively by assuming a λ, checking (b), etc.

In the two-region situation, one has for the optimal allocations

$$m_1 = \frac{1}{\log q_1 + \log q_2}\left[(M-2)\log q_2 + \log\left(\frac{p_2(1-q_2)}{p_1(1-q_1)}\right)\right] + 1 ,$$

$$m_2 = \frac{1}{\log q_1 + \log q_2}\left[(M-2)\log q_1 + \log\left(\frac{p_1(1-q_1)}{p_2(1-q_2)}\right)\right] + 1 . \tag{12}$$

Next, we shall use these results to answer some questions about the limiting values of the m_i. In the limit as $M \to \infty$, from (11) one has that

$$m_i \simeq \frac{\log \lambda}{\log q_i} . \tag{13}$$

For the two-region situation, this results in the following allocations

$$m_1 = \frac{(\log q_2)\, M}{\log q_1 + \log q_2}$$

and (14)

$$m_2 = \frac{(\log q_1)\, M}{\log q_1 + \log q_2}$$

If one assumes

(a) $k_i \Delta t = 1 - q_i$

(b) $q_i \to 1^-$,

then the optimal allocations for the discrete and continuous detectors are approximately equal. This agrees with Pollock's (1960) observations on the discrete detection problem. In general, one might make good use of the continuous analog in order to obtain an initial solution for the discrete necessary conditions, even in the N-region case.

A.3 Minimization of the Expected Time Until Detection

A.3.1 Continuous Detectors

Here we consider the other classic objective function, the minimization of the expected time until detection with no constraint on the available search time. Dobbie (1963) has shown that a concave conditional detection function is sufficient to guarantee that the policy which maximizes the probability of detection for each value of T, the available search time, is also the policy which minimizes the expected time until detection. Certainly, the conditional detection function obtained from the "law of random search" satisfies the above condition. We shall use this result to obtain an expression for the minimum expected time until detection.

The optimal value of the probability of detection for a two-region search situation, given T, is

$$P(T) = \sum_{i=1}^{2} p_i(1 - e^{-k_i t_i(T)}) ,$$

and

$$\frac{dP(T)}{dT} = \sum_{i=1}^{2} p_i k_i e^{-k_i t_i(T)} \frac{dt_i(T)}{dT}$$

where, from the previous section, the $t_i(T)$ are given by (assuming region 1 is selected first)

$$t_1(T) = \begin{cases} T, & T < T^* \\ \frac{1}{k_1 + k_2}\left[\ln\left(\frac{p_1 k_1}{p_2 k_2}\right) + k_2 T\right], & T \geq T^* \end{cases}$$

$$t_2(T) = \begin{cases} 0, & T < T^* \\ \frac{1}{k_1 + k_2}\left[\ln\left(\frac{p_2 k_2}{p_1 k_1}\right) + k_1 T\right], & T \geq T^* \end{cases}$$

The minimal expected search time will then be given by

$$E \equiv \int_0^\infty (1 - P(T))dT .$$

Carrying out the above integration, one obtains for the minimal expected time to detect

$$E = p_1 \left\{ \frac{1}{k_i} \left(1 - e^{-k_i T^*} \right) + \left(\frac{p_1 k_1}{p_2 k_2} \right)^{-\frac{k_1}{k_1+k_2}} \right. \tag{14}$$
$$\left. \cdot \left(1 + \frac{k_1}{k_2} \right) \left(\frac{k_1+k_2}{k_1 k_2} \right) e^{-\frac{k_1 k_2}{k_1+k_2} T^*} \right\} + p_2 T^* ,$$

where T^* is given by (8).

For the situation in which the infinitesimal detection probabilities are identical, i.e., $p_1 k_1 = p_2 k_2$, the above expression simplifies to

$$E = \frac{k_1 + k_2}{k_1 k_2} .$$

A.3.2 Discrete Detectors

Here we need only recall the important results of Black (1965). He shows that the policy with the minimum expected cost is generated by the rule" "Always look in the region for which the posterior probability (given the failure of earlier looks) of finding the object divided by the cost (time) is maximum."

A.4 First Allocation Rule

This section considers the following question: For the N-region, discrete search space situation, how does one determine the order in which the regions receive search effort as the amount of total available search time increases? As the amount of available search time is

increased one can begin to place the additional effort in regions which have yet to receive any effort. In the previous section, it was noted that this rule, which we shall designate as the *First Allocation Rule* was determined by choosing as the initial region that region i for which

$$p_i k_i = \max_{1 \leq j \leq N} p_j k_j \quad .$$

The First Allocation Rule for the general case

$$\max \sum_{i=1}^{N} p_i P_i(t_i)$$

$$\text{S.T.} \sum_{i=1}^{N} t_i \leq T \ ,$$

$$t_i \geq 0 \ ,$$

can be established as follows. First, we observe that if $T = \varepsilon << 1$, then it is important to allocate T to that region for which the rate of return is maximized,

$$P_i'(T) = \max_{1 \leq j \leq N} P_j'(T) \quad .$$

Thus we can determine the initial region. Suppose next that one has T sufficiently large to warrant the search of an additional region. Once again the goal is the maximization of the return from that additional effort, hence one chooses

$$P_i'(\varepsilon) = \max_{k \notin J} P_k'(\varepsilon) ,$$

where $J = \{i | t_i > 0\}$. If the conditional detection function has the property that $P'(0) > 0$, then one can let $\varepsilon \to 0$ in the above rules; if $P'(0) = 0$, then one merely chooses $\varepsilon \ll 1$ and applies the rules as stated. We remark that this technique does not enable one to determine the value of T at which the next region begins to receive effort, but merely the ordering of the regions.

Appendix B

DISCRETE DETECTION AND VISIBILITY PROCESSES

B.1 Introduction

This appendix contains the formulations and results for the discrete analogues of the models of Chapters 2-4 respectively. Explicit expressions for the optimal allocation policies are developed for each of the models.

B.2 The Binary Model

If the searcher is utilizing a discrete (glimpse) detection device, two versions of the Binary model may be considered. First assume that upon the entry of the searcher the target holds his current visibility status throughout the search. In this situation, the problem of maximizing the probability of detection is given by

$$\max \sum_{i=1}^{N} p_i v_i (1 - q_i^{m_i}) \quad , \tag{1}$$

$$\text{S.T.} \sum_{i=1}^{N} m_i \leq M \quad ,$$

where M and m_i are positive integers and $1 - q_i$ is the glimpse detection probability in the ith region. The second situation occurs under the assumption that the target's visibility status changes with each glimpse. The problem

of maximizing the probability of detection is stated as

$$\max \sum_{i=1}^{N} p_i \left\{1 - (q_i v_i)^{m_i}\right\}$$

$$\text{S.T.} \sum_{i=1}^{N} m_i \leq M, \; m_i \geq 0. \tag{2}$$

These discrete detection models can be solved in a variety of ways. In order to directly compare the discrete and continuous detector models, we shall present the solution given in Appendix A. First, under the assumptions leading to (1), the optimal allocations (for the two-region problem) are

$$m_1 = \frac{1}{\ln q_1 + \ln q_2} \left\{\ln q_2(M-2) + \ln \left(\frac{p_2 v_2 (1-q_2)}{p_1 v_1 (1-q_1)}\right)\right\} + 1,$$

and

$$m_2 = \frac{1}{\ln q_1 + \ln q_2} \left\{\ln q_1(M-2) + \ln \left(\frac{p_1 v_1 (1-q_1)}{p_2 v_2 (1-q_2)}\right)\right\} + 1.$$

As $M \to \infty$, one has

$$m_1 = \frac{(\ln q_2)M}{\ln q_1 + \ln q_2}$$

and

$$m_2 = \frac{(\ln q_1)M}{\ln q_1 + \ln q_2} \quad . \tag{3}$$

Under the assumptions leading to (2), the analogous policies are

$$m_1 = \frac{1}{\ln v_2q_2 + \ln v_1q_1}\left\{(M-2)\ln(v_2q_2) + \ln\frac{p_2(1-v_2q_2)}{p_1(1-v_2q_2)}\right\}+1$$

and

$$m_2 = M - m_1.$$

As $M \to \infty$, we have[1]

$$m_1 = M\frac{\ln v_2q_2}{\ln v_2q_2 + \ln v_1q_1},$$

and (4)

$$m_2 = M\frac{\ln v_1q_1}{\ln v_1q_1 + \ln v_2q_2}.$$

If one assumes that $k_i\Delta t \simeq 1 - q_i$, where Δt is the "glimpse" time, then as $q_i \to 1$, one has from (3) that the optimal allocations for the continuous and discrete cases are approximately equal.

Finally, note that the results of Chew (1967) given in Chapter 1 are directly applicable here. Namely, "to maximize the probability of finding the object in a fixed number of searches, choose those n-searches for which the posterior probability (given the failure of earlier looks) of finding the object is largest." We calcualte these probabilities using the operator θ, i.e., if $\mathbb{P} = (p_1,\ldots,p_N)$ then

[1] In each of the above expressions for m_i, one must choose the largest integer in the right hand side of the appropriate expression.

$(\theta_i P)_j$, is the posterior probability that the target is in the jth region given the failure of the past search in region i. For the model given by (1),

$$(\theta_i P)_j = \left\{ \begin{array}{ll} \dfrac{q_i v_i p_i}{1 - (1 - q_i) p_i v_i}, & j = i \\ \\ \dfrac{p_j}{1 - (1 - q_i) p_i v_i}, & j \neq i \end{array} \right\} . \qquad (5)$$

The model given by (2) yields the following expressions

$$(\theta_i P)_j = \left\{ \begin{array}{ll} \dfrac{(1 - (1 - q_i) v_i) p_i}{1 - (1 - q_i) v_i p_i}, & j = i \\ \\ \dfrac{p_i}{1 - (1 - q_i) v_i p_i}, & j \neq i \end{array} \right\} . \qquad (6)$$

Clearly, as $v_i \to 1$ for all i, both relate to the model first studied by Pollock (1960).

B.3 A Random Interval Process: Target Initially Visible

Let $0 < h_j < 1$ denote the probability that the target is visible on a glimpse in the jth region. It is assumed that the target is visible at the start of the search. Given that the target is visible at the start of the search, and that the target is present in the jth region, the conditional probability of detecting it, at or before the mth look, is given by

$$P_j(m) = h_j\bar{q}_j + \bar{q}_j h_j^2 q_j + \cdots + \bar{q}_j h_j^m q_j^{m-1}$$

where $\bar{q}_j = (1 - q_j)$ = the glimpse probability of detection in the jth region, and

$$P_j(m) = \frac{h_j\bar{q}_j}{1 - h_j q_j}\left(1 - (h_j q_j)^m\right) .$$

Of course as $h \to 1$, this yields Pollock's (1960) model. The problem of maximizing the probability of detection, given a total of M glimpses, is

$$\text{MAX} \sum_{i=1}^{N} P_i\left(\frac{h_i\bar{q}_i}{1 - q_i h_i}\right)\left(1 - (h_i q_i)^{m_i}\right)$$

$$\sum_{i=1}^{N} m_i \leq M ,$$

where m_i are non-negative integers.

Again the results of Appendix A are applicable. We replace

$$p_i \text{ by } \frac{p_i h_i \bar{q}_i}{1 - q_i h_i} , \text{ and } q_i \text{ by } h_i q_i$$

in equation 12, Section A.2 to obatin (in the two-region case) the optimal allocations

$$m_1 = \left[\frac{1}{\ln (h_1 q_1) + \ln (h_2 q_2)} \left\{ [M - 2] \ln (h_2 q_2) + \left(\frac{p_1 h_1 (1 - q_1)}{p_2 h_2 (1 - q_2)} \right) \right\} + 1 \right] , \tag{8}$$

and

$$m_2 = [M - m_1] .$$

As $M \to \infty$, (8) yields

$$m_1 \simeq \frac{M \ln (h_2 q_2)}{\ln (h_1 q_1) + \ln (h_2 q_2)} , \tag{9}$$

and

$$m_2 \simeq \frac{N \ln (h_1 q_1)}{\ln (h_1 q_1) + \ln (h_2 q_2)} .$$

Setting up the correspondence $k\Delta t = 1 - q$ and $\lambda \Delta t = 1 - h$, where Δt is the "glimpse" time, one can obtain the corresponding allocation policy for continuous detectors under the assumption that k and $\lambda << 1$. The continuous policy may be useful in obtaining initial estimates for the discrete policy via the above correspondence.

B.4 Random Interval Process: Random Initiation and Limit

Making use of the discrete model just obtained, we observe, given that the target becomes present at the jth glimpse, that the probability of detection in m - j + 1 identical glimpses is given by

$$P(m|j) = h\bar{q} + h^2\bar{q}q + \cdots + \bar{q}h^{m-j+1}q^{m-j}$$

$$= \frac{h\bar{q}(1 - (hq)^{m-j+1}}{1 - hq} ,$$

where

$\bar{q} = 1 - q$ = glimpse probability of detection,

h = probability target remains visible.

Let A denote the probability that the target appears on any glimpse, then the probability that the target becomes visible on the ith glimpse is $P(i) = A(1 - A)^{i-1}$. The probability of detection in m glimpses is

$$P(m) = \sum_{i=1}^{m} P(m|i)P(i) = \sum_{i=1}^{m} \frac{h\bar{q}}{(1 - hq)} (1 - (hq)^{m-j+1})A(1 - A)^{i-1} \tag{10}$$

If $A = 1$, $P(m) = \frac{h\bar{q}}{1 - hq}(1 - (hq)^m)$, which is the previous model. Carrying out the summation in (10)

$$P(m) = \frac{sA\bar{q}}{(1 - sq)}\left[\frac{1 - (1 - A)^m}{A} - sq\,\frac{(sq)^m - (1 - A)^m)}{(sq - (1 - A))}\right].$$

The discrete, N-region optimization problem is

$$\max \sum_{i=1}^{N} P_i \frac{h_i A_i \bar{q}_i}{1 - h_i q_i}\left[\frac{1 - (1 - A_i)^{m_i}}{A_i} - \frac{h_i q_i (h_i q_i)^{m_i} - (1 - A_i)^{m_i}}{(h_i q_i - (1 - A_i))}\right] \tag{11}$$

$$\text{S.T.} \sum_{i=1}^{N} m_i \leq M$$

$$m_i \geq 0, \text{ integers}.$$

We make the following observations concerning P(m):

(a) $\lim_{A \to 1^-} P(m) = \frac{h\bar{q}}{1 - hq}\left\{(1 - hq)^m\right\}$, the model of B.2

(b) $\lim_{m \to \infty} P(m) = \frac{h\bar{q}}{1 - hq}$, the limiting result for the model of B.2.

(c) $\lim_{\substack{h \to 1^- \\ A \to 1^-}} P(m) = \left\{1 - q^m\right\}$, standard discrete detector model of Pollock (1960)

(d) $\lim_{h \to 1^-} P(m) = \left\{1 - \frac{Aq^{m+1}}{(q - (1 - A))} + (1 - A)^m\left\{\frac{Aq}{q - (1 - A)} - 1\right\}\right\}$

a late arrival, no departure model.

The results of Wagner, given in Appendix A are again applicable to equation 11; however, just as in the continuous analog of this model, the solution techniques are considerably more difficult. The following conditions lead to limiting allocation policies similar to those already obtained:

(a) $M >> 1$, and $A_i \to 1^- \ \forall i$, then the limiting allocations for two regions are

$$m_1 \simeq \left[\frac{M \ln[h_2 q_2]}{\ln[h_2 q_2] + \ln [h_1 q_1]} \right] \tag{12}$$

and

$$m_2 \simeq \left[\frac{M \ln[h_1 q_1]}{\ln[h_2 q_2] + \ln[h_1 q_1]} \right].$$

(b) $M >> 1$, and $A_i << 1$, $\forall i$, we obtain for the two-region case[1]

$$m_1 \simeq \left[\frac{M \ln[1 - A_2]}{\ln[1 - A_2] + \ln[1 - A_1]} \right] \tag{13}$$

and

$$m_2 \simeq \left[\frac{M \ln[1 - A_1]}{\ln[1 - A_1] + \ln[1 - A_1]} \right]$$

which simplifies to

$$m_1 \simeq \frac{A_2 M}{A_1 + A_2} \text{ and } m_2 \simeq \frac{A_1 M}{A_1 + A_2} .$$

To set up the correspondence to the continuous model, let

[1] [ ] denotes the largest integer therein.

$$\lambda\Delta t \sim (1 - s)$$

$$k\Delta t \sim (1 - q)$$

$$\beta\Delta t \sim (A) ,$$

where Δt is the "glimpse" time, then as $h \to 1^-$, $q \to 1^-$, and $A \to 0^+$, the continuous detection policy approximates the discrete policy.

Appendix C

CONTINUOUS SEARCH SPACE FORMULATIONS

C.1 Introduction

In this appendix we present the continuous search space formulations for the models of Chapters 2-4. We restrict our attention to comtinuous detection models throughout our discussions. The models are derived in the next sections in the order in which they are introduced in the text. Solution techniques are investigated in the final section.

C.2 Model Development

In order to present the continuous search space version of the Binary visibility model, we introduct the following notation. Let

- $p(x)$ be the probability density function on target location, defined on a region A;
- $v(x)dx$ denote the probability that a target in $(x, X + dx)$ is visible; and
- $\phi(x)$ denote the search intensity function, defined as the amount of search effort (time) allocated to an interval $(x, x + dx)$.

The conditional probability of detecting (conditioned on the target being visible and present) a target at x with effort of intensity $\phi(x)$ is denoted by $P(x, \phi(x))$. Using the above definitions, the problem of maximizing the probability of detection over a continuous search space is

$$\max \int_A v(x)p(x)P(x,\phi(x))dx \tag{1}$$

$$\text{S.T.} \int_A \phi(x)dx = T \quad ,$$

and

$$\phi(x) \geq 0.$$

The continuous search space version of the random interval (initially visible) model is

$$\max \int_A p(x)M(x,\phi(x)) \tag{2}$$

$$\text{S.T.} \int_A \phi(x)dx \leq T \quad ,$$

$$\phi(x) \geq 0 \quad ,$$

where

$$M(x,\phi(x)) = \int_0^{\phi(x)} \rho(x,\tau)\bar{H}(x,\tau)d\tau = \text{conditional probability}$$

of detecting a target in the interval $(x,x + dx)$, conditioned on the target being present,

$\bar{H}(x,\tau)$ = probability that the length of the visibility period at x is greater than or equal to τ.

$\rho(x,\tau)d\tau$ = probability that the time until detection at x is in the interval $(\tau, \tau + d\tau)$, given that the target is continuously visible at x up to $\tau + d\tau$.

The random interval (random initiation and limit) model in the continuous search space situation is

$$\max \int_A p(x) \int_0^{\phi(x)} f(x,\mu)M(x,\phi(x) - \mu)\, d\mu dx \qquad (3)$$

$$\text{S.T.} \int_A \phi(x)dx \leq T$$

$$\phi(x) \geq 0 \ ,$$

where

$f(x,\mu)d\mu$ = probability that a target at x becomes visible in the interval $(\mu,\mu+d\mu)$.

$M(x,\phi(x)-\mu) = \int_0^{\phi(x)-u} \rho(x,\tau-\mu)\bar{H}(x,\tau-\mu)d\tau$, in the notation of the previous paragraph.

C.3 Solution Techniques

As noted in the literature review Zahl (1963) gave necessary and sufficient conditions for a general version of the search problem, i.e.,

$$\max F(\phi) = \int_c^d f(x,\phi(x))dx$$

$$\text{S.T. } G(\phi) = \int_c^d g(x,\phi(x))dx = T \ ,$$

and

$$a(x) \leq \phi(x) \leq b(x) \ .$$

His major result is the following theorem:

Theorem

A necessary and sufficient condition that a function $\bar{y}(\cdot)$ maximize $F(\phi(\cdot))$, under the above restrictions, is that there exist a constant λ such that for almost every x, $\bar{y}(x)$ maximizes

$$f(x,\phi) - \lambda g[x,\phi] \text{ over } a(x) \leq \phi(x) \leq b(x) \ .$$

The correspondence to our work is

$$f(x,\phi(x)) = p(x)M(x,\phi(x)) ,$$

where $p(x)$ is the probability density function on target location and $M(x,\phi(x))$ is the conditional detection function which results from one of our models. The function $g(x,\phi(x))$ becomes, in our case, just $\phi(x)$. Hence we are concerned with finding a constant λ and a function $\bar{y}(x)$ which maximizes

$$p(x)M(x,\phi(x)) - \lambda\phi(x) .$$

Thus we shall be interested in finding the solution to the functional equation

$$p(x)M_{\phi(x)}(x,\phi(x)) - \lambda = 0 . \qquad (4)$$

In general, one cannot expect to be able to solve (4) for $\phi(x)$ as a function of λ. However, in the situation in which $M(x,\phi(x))$ satisfies the conditions of deGuenin (1961), then the existence of an inverse function for $M_{\phi(x)}(x,\phi(x))$, is guaranteed,[1] i.e.,

$$\phi(x) = M^{-1}_{\phi(x)} \left(\frac{\lambda}{p(x)}\right) .$$

Now we proceed as follows

(a) Choose $\lambda_1 > 0$, which determines a subset $A \subset [c,d]$ such that

$$\phi(x) > 0 \text{ for } x \in A .$$

[1] See Chapter 1 for a complete description of these conditions

(b) then form

$$\int_A \phi(x)dx = T_1$$

(c) If

$T_1 = T$, $\phi(x)$ is optimal

$T_1 > T$, $\lambda_2 > \lambda_1$ and go to step a

$T_1 < T$, choose $\lambda_2 < \lambda_1$ and go to step a

We investigate the applicability of de Guenin's conditions to the models just developed. First, for the Binary model, one has the following optimization problem.

$$\max \int_R p(x)v(x)\left[\int_0^{\phi(x)} \rho(x,\tau)d\tau\right]dx$$

$$\text{S.T.} \int_R \phi(x)dx = T \ ,$$

$$\phi(x) \geq 0 \ .$$

Since $v(x) \geq 0$ for all x, if $P(x, \phi(x)) = \int_0^{\phi(x)} \rho(x,\tau)d\tau$ satisfies de Guenin's conditions, then certainly $v(x)P(x,\phi(x))$ will also. For the model of Chapter 3, the problem statement is given by

$$\max \int_R p(x)\left[\int_0^{\phi(x)} \rho(x,\tau)\bar{H}(x,\tau)d\tau\right]dx \ .$$

$$\text{S.T.} \int_R \phi(x)dx = T$$

$$\phi(x) \geq 0 \ .$$

Now conditions (1), (2), and (4) (See Chapter 1) certainly hold; if one assumes that $P(x,\phi(x))$ satisfies the conditions of de Guenin, condition (3) follows. For the model of Chapter 4, we have

$$\max P(x,\phi(x)) = \int_R p(x)\left[\int_0^{\phi} f(x,\mu) \cdot \left[\int_0^{\phi-\mu} \rho(x,\tau)\bar{H}(x,\tau)d\tau\right]d\mu\right]dx$$

$$\text{S.T.} \int_R \phi(x)dx = T$$

$$\phi(x) \geq 0 \ .$$

Here an analysis of the second partial derivative of $P(x,\phi(x))$ with respect to $\phi(x)$ yields the result that $dP(x,\phi(x))/d\phi(x)$ is *not* a decreasing function of ϕ. Hence the required inverse function does not exist. For the models of Chapter 5, one cannot make a general statement; however, several of the examples therein violate Condition (3) under certain conditions.

In the event the conditions of de Guenin fail to hold, one proceeds as follows:

(a) Choose a λ and solve [4] for $\phi^*(x,)$ and $\phi^{**}(x,\lambda)$ which are the smallest and largest solutions of [4] for a given x and λ, respectively; thus determining A* and A**.

(b) Next, compute

$$\int_{A^*} \phi^*(x,\lambda)dx = T^* \text{ , and } \int_{A^{**}} \phi^{**}(x,\lambda)dx = T^{**} \text{ ,}$$

if the current λ, say λ_0, is the greatest lower bound on the set λ for which

$$T^* \leq T \leq T^{**},$$

then let

$$\phi_t(x,\lambda_0) = \left\{ \begin{array}{ll} \phi^*(x,\lambda_0), & x \leq t \\ \phi^{**}(x,\lambda_0), & x > t \end{array} \right\}$$

by the continuity of the integral there exists a t such that

$$\int_c^d \phi_t(x,\lambda_0)dx = T \text{ .}$$

Otherwise change λ, and return to step a[1],

---

[1] Note that in a discrete search space situation, one may have determined a $\phi_t(x,\lambda_0)$ which satisfies [4] but for which $\sum_x \phi_t(x,\lambda_0) \neq T$. Hence in this case the condition of Zahl is sufficient but not necessary

Appendix D

RANDOM INTERVAL MODELS WITH FINITE VISIBLE PERIODS

D.1 Introduction

This appendix deals with some random interval models with the characteristic that the range of the random variable which denotes the length of the visible period is finite.[1] Some examples are given, their particular properties examined, and finally, a comparison with the Koopman results is made.

D.2 Random Interval of Visibility: Initially Visible

First, it is assumed that the probability density function on the length of the visibility period is given by

$$h(t) = \begin{cases} \dfrac{\lambda e^{-\lambda t}}{C} & , \quad 0 \leq t \leq T_u \\ 0 & , \quad t > T_u \end{cases},$$

where $C = 1 - e^{-\lambda T_u}$.

Applying the results of Chapter 3, we obtain for the conditional detection function

[1] If one interprets the length of the visible period as the length of time the target spends in the regions to be searched, such distributions more accurately reflect the search situation.

$$\rho(T) = \frac{1}{C}\left[\frac{k}{\lambda + k}\left(1 - e^{-(\lambda+k)T}\right) - e^{-\lambda T_u}\left(1 - e^{-kT}\right)\right], \quad T \leq T_u,$$

$$\rho(T) = \rho(T_u), \text{ for } T > T_u. \qquad (1)$$

Note that if $T_u \to \infty$, one obtains the model of Chapter 2; if, in addition, $\lambda \to 0$, one has the Koopman model.

The First Allocation Rule is readily shown to be

$$p_i k_i = \max_{1 \leq j \leq N} p_j k_j .$$

The conditional detection function is concave over the interval $(0, T_u)$; however, the resulting Kuhn-Tucker conditions don't yield explicit solutions for T. A study of the switching criteria gives the following expression for the switch point (in the 2-region case, assuming the search started in Region 1)

$$T = \frac{1}{k_1} \ln\left(\frac{p_1 k_1}{p_2 k_2 C_1}\right) - \frac{1}{k_1} \ln\left\{\frac{1}{e^{-\lambda_1 T} - e^{-\lambda_1 T_{u_1}}}\right\}, \quad T < T_u \qquad (2)$$

An upper limit on the switch time obtained from the above expression is given by

$$\min\left\{T_{u_1}, \left| -\frac{1}{\lambda_1}\left\{\frac{p_2 k_2 C_2}{p_1 k_1} + e^{-\lambda_1 T_{u_1}}\right\}\right|\right\}.$$

Next, we assume that the probability density function on the length of the visibility period is uniform, i.e.,

$$h(t) = \begin{cases} \frac{1}{S} & 0 < t \leq S \\ 0 & t > S \end{cases} .$$

Applying the results of Chapter 3, one obtains for the conditional detection function

$$P(T) = 1 - e^{-kT} + \frac{Te^{-kT}}{S} - \frac{1}{kS}\left[1 - e^{-kT}\right], T \leq S \qquad (3)$$

$$P(T) = P(S) \text{ for } T > S .$$

Note that as $S \to \infty$, i.e., the target is always visible, one obtains the Koopman model. The First Allocation Rule is identical to the one already given. Again, the conditional detection function is concave on (0,S) and the resulting Kuhn-Tucker conditions fail to yield explicit solutions for T. A study of the switching criteria gives the following expression for the switch point (in the two-region case assuming the search started in Region 1)

$$T = \frac{1}{k_1} \ln\left[\frac{p_1 k_1}{p_2 k_2}\right] - \frac{1}{k_1} \ln\left(1 - \frac{T}{S_1}\right), \quad 0 \leq T < S_1 \qquad (4)$$

Since the term

$$-\frac{1}{k_1} \ln\left[1 - \frac{T}{S_1}\right] > 0 ,$$

the solution of the above expression, when it exists, is always larger than the "Koopman Switch Time." Also under conditions $p_1 k_1 = p_2 k_2$, the Koopman policy splits the effort, but from (4)

$$T = -\frac{1}{k_1} \ln\left(1 - \frac{T}{S_1}\right) \quad .$$

Assuming the search started in Region 1, this implies that once a region is chosen it receives all the effort up to T_s, the solution to (4).

As noted earlier the Kuhn-Tucker conditions don't yield explicit solutions for the optimal allocations; however, one can gain some insight into certain properties of this model by making the following assumptions. In a two-region situation, it is assumed that in the first region the target is always visible (i.e., Koopman model), while in the second region the length of the visible period is uniformly distributed on the interval $(0,S_2)$. Let T denote the total available search time. Then the expression for the amount of effort to be allocated to Region 1 is given by

$$T_1 = \left(\frac{1}{k_1 + k_2}\right)\left\{\ln\left(\frac{p_1 k_1}{p_2 k_2}\right) + k_2 T - \ln\left(1 - \frac{(T - T_1)}{S_2}\right)\right\}. \quad (5)$$

Note that

(a) $(T - T_1) < S_2$, i.e., one never saturates the second region, and

(b) since $-\ln\left(1 - \frac{(T - T_1)}{S_2}\right) \geq 0$ the first region always receives more effort in this mixed situation (more relative to the KSA).

D.3 A Random Interval of Visibility: Random Initiation & Limit

In this section the following assumptions are made:

(1) The random variable corresponding to the start of the visible period is exponentially distributed,

(2) The length of the visibility period is uniformly distributed with parameter S, and

(3) The detection function is exponential.

From the second example of Section D.2 (assuming the visibility period starts at μ and is of length S) the conditional detection function is

$$M(T - \mu) = (1 - e^{-(T-\mu)}) + \left(\frac{T - \mu}{S}\right) e^{-k(T-\mu)} - \frac{1}{kS}(1 - e^{-k(T-\mu)}),$$

for $T - \mu \leq S$, and $M(T - \mu) = M(S)$ for $(T - \mu) > S$.

In Chapter 4, the general expression for the probability of detection for this type of visibility model was given by

$$P(t) = \int_0^t f(\mu)M(t - \mu)d\mu ,$$

the change of variable $\tau = t - \mu$, yields

$$P(t) = \int_0^t f(t - \tau)M(\tau)d\tau ,$$

$$M(\tau) = M(S), \text{ for } \tau > S, \text{ and for } \tau \leq S$$

$$M(\tau) = (1 - e^{-k\tau}) + \frac{\tau}{S} e^{-k\tau} - \frac{1}{kS} (1 - e^{-k\tau}) .$$

Because of the form of $M(\tau)$, in evaluating $P(t)$ one specifies whether $t > S$, or not. Furthermore, special consideration must be given the situation in which $k = \beta$. The density function on the start time is, of course,

$$f(t) = \beta e^{-\beta t}, \quad t \geq 0 .$$

Under the above restrictions, one obtains the following expressions for $P(t)$.

First for $t \leq S$,

(a) $k \neq \beta$

$$P(t) = \left\{1 - \frac{1}{kS}\right\}\left[\left[1 - e^{-\beta t}\right] - \frac{\beta e^{-\beta t}}{k - \beta}\left\{1 - e^{-(k-\beta)t}\right\}\right] \quad (6)$$

$$+ \frac{\beta e^{-\beta t}}{S}\left\{-\frac{te^{-(k-\beta)t}}{(k - \beta)} + \frac{1}{(k - \beta)^2}(1 - e^{-(k-\beta)t})\right\},$$

(b) $k = \beta$,

$$P(t) = (1 - \frac{1}{kS})\left((1 - e^{-\beta t}) - t\beta e^{-\beta t}\right) + \frac{\beta t^2}{2S} e^{-\beta t} \quad (7)$$

Note that by assuming $S \to \infty$, the late arrival cases discussed in Chapter 4 are obtained. Also if $\beta \to \infty$, part (a) tends to the model in Section D.2. For $T > S$, the respective expressions are

(a) $k \neq \beta$

$$P(t) = (1 - \frac{1}{kS})\left\{e^{-\beta(t-S)} - e^{-\beta t}\right\} - (1 - \frac{1}{kS})\frac{\beta e^{-\beta t}}{k - \beta}(1 - e^{-(k-\beta)S})$$

$$+ \frac{1}{S} e^{-\beta t}\left\{\frac{-Se^{-(k-\beta)S}}{(k - \beta)} + \frac{1}{(k - \beta)^2}(1 - e^{-(k-\beta)S})\right\} \quad (8)$$

$$+ (1 - e^{-\beta(t-S)})(1 - \frac{1}{kS}(1 - e^{-kS})).$$

(b) $k = \beta$

$$p(t) = \left(1 - \frac{1}{kS}\right)\left\{e^{-\beta(t-S)} - e^{-\beta t}\right\} - \left(1 - \frac{1}{kS}\right) e^{-\beta t_S} + \frac{S}{2}\ \beta e^{-\beta t} + \left(1 - e^{-\beta(t-S)}\right)\left(1 - \frac{1}{kS}\left(1 - e^{-kS}\right)\right) . \quad (9)$$

As $t \to \infty$, a limiting value of P (t) is attained which is equivalent to that obtained in Section D.2, namely

$$P(\infty) = 1 - \frac{1}{kS}\left(1 - e^{-kS}\right) .$$

The First Allocation Rules for these models are identical to those given in Chapter 4, i.e., choose as the initial region

$$p_i k_i \beta_i = \max_j p_j k_j \beta_j .$$

These functions have the pseudo-concave properties discussed in Chapter 4. Explicit expressions for the optimal allocations cannot be obtained from the Kuhn-Tucker conditions. However, some insight can be gained from the following special case. First, assuming that sufficient total search time is available to insure that either equation 8 or 9 is valid, we observe that the derivatives with respect to time of these expressions are of the form

$$P'(T) = \beta e^{-\beta t} \{\cdot\} , \tag{11}$$

where

$\{\cdot\}$ is a constant with respect to time. The significance of this is that the methods given in Appendix A are applicable and explicit representations of the optimal allocations for the N-region search problem are available. Also, from the approximation discussions of Chapters 3 and 4, the Koopman allocation scheme will be effective for the case $k = \beta$, while for $k \neq \beta$ such may not be true.

The final special case results from the following assumptions. First, consider a two-region case in which $k_1 = \beta_1$, while in the second region, the target is continuously visible, i.e., the standard Koopman situation. Given this scenario with the restriction that $T \leq S_1$, the optimal allocation to the first region is determined from

$$t_1 = \frac{1}{k_1 + k_2} \left\{ \ln \left(\frac{p_1 k_1}{p_2 k_2} \right) + k_2 T \right\} + \frac{1}{k_1 + k_2} \ln \left(\beta_1 \left(T - \frac{t_1^2}{2S_1} \right) \right) , \quad 0 \leq t_1 \leq S_1 . \tag{12}$$

The above expression contains the standard Koopman allocation plus an extra term. Analysis of this term yields the following information.

(a) the expression $f(t) = \beta(t - \frac{t^2}{2S})$ has its maximum at $t = S$, that maximum is $\beta S/2$.

(b) Assuming that T_1 as determined from equation 12 is not on the boundary i.e., $T_1 \neq 0$ or $T_1 \neq S_1$, then

(1) if $f(S) = \beta S/2 < 1$, the optimal allocation is less than the Koopman scheme

(2) if $f(T_1) > 1$, then the reverse is true.

Appendix E

COMPUTER PROGRAMS

E.1 Description and Listing of the Dynamic Programming Computer Program

In this section we shall briefly describe the programs utilized in obtaining the numerical results of the main text and Appendix F. This will be followed by a listing of the FORTRAN level G source statements for the program. First we simply note that the dynamic program for the solution of the problem of maximizing the probability of detection under a constraint on the total available search time uses the standard dynamic programming approach to the constrained maximization problems for each of the models discussed in the text. (See Bellman and Dreyfus (1962)). A glossary of variables unique to our models is given below:

GLOSSARY: Dynamic Programming Source Program

1. Binary Model (MTYPE = 1)

   P(I) = prior probability on target location $1 \leq I \leq 50$.

   PV(I) = prior probability on target visibility in the Ith region, $1 \leq I \leq 50$.

R(I) = the detection rate for the Ith region

2. Random Interval Model: Initial Visibility (MTYPE = 2)

PV(I) = the rate for the length of the visible period in the Ith region

R(I) = the detection rate for the Ith region

3. Random Interval Model: Random Initiation and Limit (Uniform Distribution on Start Times) (MTYPE = 3)

PV(I) = (See MTYPE = 2)

R(I) = " "

S(I) = the length of the uniform interval on the start of the visible period in region I.

4. Random Interval Model: Random Initiation and Limit (Exponential Start Times) (MTYPE = 4)

PV(I) = (See MTYPE = 2)

R(I) = " "

S(I) = the rate for distribution of starting times for the Ith region.

5. General Random Interval Model (MTYPE = 5)

PV(I) = (See MTYPE = 4)

R(I) = " "

S(I) = " "

PHI(I) = prior probability on target visibility at the start of the search in region I.

6. Multiple Interval Model (MTYPE = 6)

PV(I) = (See MTYPE = 5)

R(I) = " "

S(I) = " "

PHI(I) = " "

7. Miscellaneous

NAREA = the number of areas to be searched

DELTA = incremental value of search time (effort) used in the dynamic programming solution of the search problem

TMAX = the total available search time (effort)

PSAVE(I,J) = an intermediate storage array used in the dynamic programming computations

$1 \leq I \leq 500$

$1 \leq J \leq 2$

PSAVE(I,1) = the optimal return

PSAVE(I,2) = the optimal allocation

(These results are printed for each region in this order as they are obtained.)

This program contains a subroutine called DPROB which computes the probability of detection (for any of our models) for a given region with a specified level of search time. The pertinent variables (which were not previously defined) are listed below.

KK = that region for which the probability of detection is to be evaluated ($1 < KK < 50$)

TAL = the level of search time for which the probability of detection is to be evaluated.

RESULT = the detection probability for region KK when TAL units of search time are allocated to that region.

```
C          MAIN PROGRAM
C           MODEL 1/PV(I)=PROB. TGT. VIS.
C          MODEL 2 / PV(I)=RATE EXP P.D.F. FOR LENGTH VIS. PERIOD
C        MODEL 3 / PV(I)=  RATE "   "        #          "      "
C    MODEL 4 / S(I)=RATE FOR EXP START TIMES
C    MODEL 5 / PHI(I)=PROB. TGT. VIS. COMBINED SINGLE INT. MODEL
C      MODEL 6  MULTIPLE INTERVAL
C       R(I)=RHO=DETECTION RATE
C      PV(I)=LAMBDA=VIS.  P.D.F.
C      S(I)=MU=MASK  P.D.F.
       DIMENSION P(50),PV(50),R(50),PMAX(500,2),PSAVE(500,2)
      1,S(50),PHI(50)
     1 READ(2,1000) MTYPE,NAREA,DELTA,TMAX
       WRITE(3,1002)
       WRITE(3,1000) MTYPE,NAREA,DELTA,TMAX
       GO TO (2,2,2,2,2,2,60),MTYPE
     2 DO 5 I=1,NAREA
     5 READ(2,1001) P(I),PV(I),R(I),S(I),PHI(I)
       DO 6 I=1,NAREA
     6 WRITE(3,1001) P(I),PV(I),R(I),S(I),PHI(I)
C          INITIALIZE REGIONAL DETECTION PROBABILITY SUBROUTINE
       CALL DPROB1(MTYPE,NAREA,P,PV,R,S,PHI)
       NCHK=(TMAX/DELTA)+.5
       IF (NCHK.GT.500) GO TO 100
C          ZERO STARTING VALUES OF OBJ. FCN. AND CORRS. ALLOC.
       NZERO=NCHK+1
       DO 7 J=1,2
       DO 7 I=1,NZERO
     7 PMAX(I,J)=0.
C          SET REGIONAL COUNTER
       K=1
C          INITIALIZE TOTAL AVAILABLE SEARCH TIME
    10 TNOW=0.
C          SET CURRENT MAXIMUM
    15 BETA=-1.E-05
C          INITIALIZE CURRENT VALUE OF SEARCH TIME
       TK=0.
    20 INDEXC=((TNOW-TK)/DELTA)+0.5+1
C          OBTAIN DETECTION PROB FOR CURRENT ALLOC.
       CALL DPROB(K,TK,GT)
       IF(GT.LT.0.) GT=0.
       ALPHA=GT+PMAX(INDEXC,1)
C          TEST FOR NEW MAXIMUM
       IF(ALPHA.LT.BETA) GO TO 30
       BETA=ALPHA
       GAMMA=TK
    30 TK=TK+DELTA
       IF (TK.GT.TNOW) GO TO 40
       GO TO 20
```

```
C          SAVE NEW MAX. & CORRES. ALLOCATION
   40 INDEXS=(TNOW/DELTA)+0.5+1
      PSAVE(INDEXS,1)=BETA
      PSAVE(INDEXS,2)=GAMMA
      TNOW=TNOW+DELTA
      IF (TNOW.GT.TMAX) GO TO 50
      GO TO 15
C          PRINT RESULTS FOR THIS REGION
   50 WRITE(3,1002) K
      NPRINT=NCHK+1
      DO 51 I1=1,NPRINT

   51 WRITE(3,1003) PSAVE(I1,1),PSAVE(I1,2)
C          TRANSFER P9EVIOUS RESULTS
      DO 55 J=1,2
      DO 55 I=1,NPRINT
   55 PMAX(I,J)=PSAVE(I,J)
      K=K+1
C          TEST FOR MORE REGIONS
      IF (K.GT.NAREA) GO TO 1
      GO TO 10
C          ERROR RETURNS
  100 WRITE(3,1004)
   60 CALL EXIT
 1000 FORMAT(2I2,6X,3F10.5)
 1001 FORMAT(5F10.5)
 1002 FORMAT(1H1,I3)
 1003 FORMAT(10X,F10.5,5X,F10.5)
 1004 FORMAT(1H1,5X,' ERROR IN MODEL SPECIFICATIONS ' )
      END
      SUBROUTINE DPROB1(MTYPE,NAREA,P,PV,R,S,PHI)
      DIMENSION P(50),PV(50),RATE(50),PVIS(50),R(50),PL(50)
     1,S(50),SR(50),PHI(50),SPHI(50)
      REAL NUM1
C          SUBRT. CALC. DET. PROB./ AN ALLOC. OF TIME(TAL)
      DO 5 I=1,NAREA
      PL(I)=P(I)
      PVIS(I)=PV(I)
      SR(I)=S(I)
      SPHI(I)=PHI(I)
    5 RATE(I)=R(I)
      RETURN
      ENTRY DPROB (KK,TAL,RESULT)
      IF(TAL.GT.0.) GO TO 7
      RESULT=0.
      RETURN
    7 GO TO (10,20,20,40,20,60),MTYPE
C          MODEL 1 BINARY INTERVISIBILITY
   10 RESULT=PL(KK)*PVIS(KK)*(1.-EXP(-RATE(KK)*TAL))
      RETURN
```

```
C          MODEL 2 SINGLE INTERVAL,START AT TIME ZERO
   20 RESULT=PL(KK)*(RATE(KK)/(RATE(KK)+PVIS(KK)))
      IF (MTYPE.EQ.3) GO TO 30
      RESULT=RESULT*(1.-EXP(-(RATE(KK)+PVIS(KK))*TAL))
      IF(MTYPE.EQ.5) GO TO 50
      RETURN
C          MODEL 3 SINGLE INTERVAL UNIFORM START TIME
C        SR(KK) DENOTES LENGTH OF UNIFORM INT. ON START TIME
   30 IF (TAL.NE.0.) GO TO 31
      RESULT=0.
      RETURN
   31 IF(TAL.GT.SR(KK)) GO TO 310
      TEMP=1.-EXP(-(RATE(KK)+PVIS(KK))*TAL)
      TEMP=TEMP/(RATE(KK)+PVIS(KK))
      RESULT=RESULT*(TAL-TEMP)/SR(KK)
      RETURN
  310 TEMP=EXP(-(RATE(KK)+PVIS(KK))*TAL)
      TEMP=EXP(-(RATE(KK)+PVIS(KK))*(TAL-SR(KK)))-TEMP
      TEMP=SR(KK)-TEMP/(RATE(KK)+PVIS(KK))
      RESULT=RESULT*TEMP/SR(KK)
      RETURN
C     MODEL 4 SINGLE INT. EXP START TIME

C     PV(I)=RATE FOR VIS. DIST
C     RATE(I)=RATE FOR COND. DETECT. FUNCT.
C     SR(I)=RATE FOR START VIS. PERIOD
   40 DEN=SR(KK)-RATE(KK)-PVIS(KK)
      NUM1=RATE(KK)+PVIS(KK)
      IF(ABS(DEN).LT.1.E-4) GO TO 410
      RESULT=SR(KK)*EXP(-NUM1*TAL)-NUM1*EXP(-SR(KK)*TAL)
      RESULT=PL(KK)*RATE(KK)*(1.-RESULT/DEN)/NUM1
      IF(MTYPE.EQ.5) GO TO 52
      RETURN
  410 TEMP=EXP(-SR(KK)*TAL)*(1.+SR(KK)*TAL)
      RESULT=PL(KK)*RATE(KK)*(1.-TEMP)/NUM1
      IF(MTYPE.EQ.5) GO TO 52
      RETURN
C     MODEL 5 COMBO. ZERO &NON-ZERO START TIMES
   50 SAVE1=RESULT
      GO TO 40
   52 RESULT=SPHI(KK)*SAVE1+(1.-SPHI(KK))*RESULT
      RETURN
C     MODEL 6 MULTIPLE INTERVAL
C      SOLVE FOR ROOTS
   60 B=SR(KK)+PVIS(KK)+RATE(KK)
      C=SR(KK)*RATE(KK)
      A=1.
      DISC=B*B-4.*A*C
      IF(DISC.LT.0.) GO TO 100
      ROOT2=(B+SQRT(DISC))/(2.*A)
```

```
      ROOT1=(B-SQRT(DISC))/(2.*A)
      DROOT=ROOT2-ROOT1
C     COMPUTE ABSORPTION PROB
C---------P23(T)
      TEMP1=(1.-EXP(-TAL*ROOT1))/ROOT1
      TEMP2=(1.-EXP(-TAL*ROOT2))/ROOT2
      TEMP=SR(KK)*RATE(KK)*(TEMP1-TEMP2)/DROOT
C-------P13(T)
      SAVE=RATE(KK)*(SR(KK)-ROOT1)*TEMP1/DROOT
      SAVE=SAVE-(SR(KK)-ROOT2)*TEMP2*RATE(KK)/DROOT
C        COMPUTE STATE PROB.
      RESULT=TEMP*(1.-SPHI(KK))
      RESULT=RESULT+SAVE*SPHI(KK)
      RESULT=PL(KK)*RESULT
      RETURN
  100 RESULT=-20.
      RETURN
      END
```

E.2 Description and Listing of the Computer Program for the Comparison of the Optimal Search Policy with the Associated Koopman Policy

Here the program used in the determination of Section F.4 of Appendix F is briefly described. This is followed by a listing of the FORTRAN level G source statements for this program. The approach taken may be summarized as follows:

1. The dynamic programming routine described in Section E.1 is used to generate the optimal search policy, region by region.
2. When the results for the last region have been attained, one then computes the allocations under the Koopman policy.[1]
3. The difference in the return from the two policies is then computed as well as the percent relative error for each level of search time up to the maximum specified level.

Since this program makes use of the one just described, we shall list in the following glossary only those variables which were not previously defined.

---

[1] Here we are restricted to two regions; however, this restriction can easily be removed by extending the Koopman calculations to N-regions via the techniques described in Appendix A.

GLOSSARY:

LPRNT = Flags used to control the printing of
IBACK the difference curve, i.e.,

LPRNT \ IBACK	= 0	≠ 0
= 0	peak only	peak up to T_{max}
≠ 0	up to the peak	complete curve

ITEST = Flag to indicate that the computation is to be continued until the percent relative error is less than PERROR

PERROR = Limiting value of the percent relative error

Output Description

(a) Field 1 = optimal return
Field 2 = return under the Koopman policy
Field 3 = difference (1-2)
Field 4 = allocation to region 1 under the Koopman policy
Field 5 = allocation to region 2 under the Koopman policy
Field 6 = total available search time

(b) If IBACK = 0 or ITEST = 0,

Field 1 = difference in the optimal return (optimal-Koopman)

Field 2 = total available search time

Field 3 = percent relative error at the peak difference

Field 4 = optimal return (probability of detection)

Field 5 = Koopman expected time until detection

Field 6 = Koopman switch time.

```
C          MAIN PROGRAM
C           MODEL 1/PV(I)=PROB. TGT. VIS.
C          MODEL 2 / PV(I)=RATE EXP P.D.F. FOR LENGTH VIS. PERIOD
C        MODEL 3 / PV(I)=  RATE "   "        #          "      "
C    MODEL 4 / S(I)=RATE FOR EXP START TIMES
C    MODEL 5 / PHI(I)=PROB. TGT. VIS. COMBINED SINGLE INT. MODEL
C      MODEL 6  MULTIPLE INTERVAL
C       R(I)=RHO=DETECTION RATE
C      PV(I)=LAMBDA=VIS.  P.D.F.
C      S(I)=MU=MASK  P.D.F.
       DIMENSION P(50),PV(50),R(50),PMAX(500,2),PSAVE(500,2)
      1,S(50),PHI(50),XL(8)
     1 READ(2,1000) MTYPE,NAREA,LPRNT,IBACK,ITEST,DELTA,TMAX,PERROR
       WRITE(3,1000) MTYPE,NAREA,LPRNT,IBACK,ITEST,DELTA,TMAX,PERROR
       GO TO (2,2,2,2,2,2,60),MTYPE
     2 DO 5 I=1,NAREA
     5 READ(2,1001) P(I),PV(I),R(I),S(I),PHI(I)
       DO 6 I=1,NAREA
     6 WRITE(3,1001) P(I),PV(I),R(I),S(I),PHI(I)
C          INITIALIZE REGIONAL DETECTION PROBABILITY SUBROUTINE
       CALL DPROB1(MTYPE,NAREA,P,PV,R,S,PHI)
       SAVE1=-1.E+05
       SAVE2=0.
       SAVE3=0.
       SAVE4=0.
       IOVER=0
       IEXIT=0
   498 Z1=ALOG(P(1)*R(1)/(P(2)*R(2)))
       Z2=Z1/R(1)
       IF(Z2.LT.0.) GO TO 5100
       IFIRST=1
       GO TO 66
  5100 Z2=-Z1/R(2)
       IFIRST=2
```

```
   66 CALL TKOOP(P(1),P(2),R(1),R(2),SAVE5)
      NCHK=(TMAX/DELTA)+.5
      IF (NCHK.GT.500) GO TO 100
C          ZERO STARTING VALUES OF OBJ. FCN. AND CORRS. ALLOC.
      NZERO=NCHK+1
      DO 7 J=1,2
      DO 7 I=1,NZERO
    7 PMAX(I,J)=0.
C          SET REGIONAL COUNTER
      K=1
C          INITIALIZE TOTAL AVAILABLE SEARCH TIME
   10 TNOW=0.
C          SET CURRENT MAXIMUM
   15 BETA=-1.E-05
C          INITIALIZE CURRENT VALUE OF SEARCH TIME
      TK=0.
      IF(K.EQ.1) TK=TNOW
   20 INDEXC=((TNOW-TK)/DELTA)+0.5+1
C          OBTAIN DETECTION PROB FOR CURRENT ALLOC.
      CALL DPROB(K,TK,GT)
      IF (GT.LT.0.) GO TO 110
      ALPHA=GT+PMAX(INDEXC,1)
C          TEST FOR NEW MAXIMUM
      IF(ALPHA.LT.BETA) GO TO 30
      BETA=ALPHA

      GAMMA=TK
   30 TK=TK+DELTA
      IF (TK.GT.TNOW) GO TO 40
      GO TO 20
C          SAVE NEW MAX. & CORRES. ALLOCATION
   40 INDEXS=(TNOW/DELTA)+0.5+1
      PSAVE(INDEXS,1)=BETA
      PSAVE(INDEXS,2)=GAMMA
      IF(K.EQ.NAREA) GO TO 499
   45 TNOW=TNOW+DELTA
      IF (TNOW.GT.TMAX) GO TO 50
      GO TO 15
C          PRINT RESULTS FOR THIS REGION
   50 NPRINT=NCHK+1
      IF(LPRNT.EQ.0) GO TO 52
      WRITE(3,1002) K
      DO 51 I1=1,NPRINT
   51 WRITE(3,1003) PSAVE(I1,1),PSAVE(I1,2)
C          TRANSFER PREVIOUS RESULTS
```

```
   52 CONTINUE
      DO 55 J=1,2
      DO 55 I=1,NPRINT
   55 PMAX(I,J)=PSAVE(I,J)
   57 K=K+1
C          TEST FOR MORE REGIONS
      IF(K.GT.NAREA) GO TO 505
      GO TO 10
C          ERROR RETURNS
  100 WRITE(3,1004)
      GO TO 60
  110 WRITE(3,1005)
   60 CALL EXIT
C------ COMPUTE RESULTS FROM KOOPMAN SCHEME
  499 CONTINUE
      IF(TNOW.GT.Z2) GO TO 5012
      GO TO (5008,5009),IFIRST
 5008 T1=TNOW
      T2=0.
      GO TO 5014
 5009 T2=TNOW
      T1=0.
      GO TO 5014
 5012 T1=(Z1+R(2)*TNOW)/(R(1)+R(2))
      T2=(-Z1+R(1)*TNOW)/(R(1)+R(2))
 5014 IF(MTYPE.LT.3) GO TO 100
      K=1
      CALL DPROB(K,T1,G1)
      K=2
      CALL DPROB(K,T2,G2)
      PROB=G1+G2
      TEMP=PSAVE(INDEXS,1)-PROB
      IF((ITEST.EQ.0).OR.(IOVER.EQ.0)) GO TO 5007
      IF(PSAVE(INDEXS,1).EQ.0.) GO TO 5007
      TEST=TEMP/PSAVE(INDEXS,1)
      IF(TEST.GT.PERROR) GO TO 5007
      IEXIT=1
      GO TO 5013
 5007 IF((IOVER.EQ.1).AND.(IBACK.NE.0)) GO TO 5013
      IF(LPRNT.EQ.0) GO TO 5015
 5013 WRITE(3,1001) PSAVE(INDEXS,1),PROB,TEMP,T1,T2,TNOW
      IF(IEXIT.EQ.1) GO TO 1
 5015 IF(TEMP.LT.0.) GO TO 500
      IF(TEMP.LT.SAVE1) GO TO 510
      SAVE1=TEMP
      SAVE2=TNOW
      IF(PSAVE(INDEXS,1).EQ.0.) GO TO 500
      SAVE3=TEMP/PSAVE(INDEXS,1)
      SAVE4=PSAVE(INDEXS,1)
```

```
  500 GO TO 45
  505 IF((IBACK.NE.0).OR.(ITEST.NE.0)) GO TO 507
      WRITE(3,1006)
      WRITE(3,1001) SAVE1,SAVE2,SAVE3,SAVE4,SAVE5,Z2
      GO TO 1
  507 WRITE(3,1007) PERROR
      WRITE(3,1001) PSAVE(INDEXS,1),PROB,TEMP,T1,T2,TNOW
      GO TO 1
  510 CONTINUE
      IF(IOVER.EQ.1) GO TO 511
      WRITE(3,1001) SAVE1,SAVE2,SAVE3,SAVE4,SAVE5,Z2
      IOVER=1
  511 IF((IBACK.NE.0).OR.(ITEST.NE.0)) GO TO 45
      GO TO 1
 1000 FORMAT(5I2,3F10.5)
 1001 FORMAT(8F10.5)
 1002 FORMAT(1H1,I3)
 1003 FORMAT(10X,F10.5,5X,F10.5)
 1004 FORMAT(1H1,5X,' ERROR IN MODEL SPECIFICATIONS ' )
 1005 FORMAT(1H1,5X,' ERROR IN DET. PROB. SUBROUTINE' )
 1006 FORMAT(' DIFFERENCE FCN STILL INCREASING')
 1007 FORMAT('PERCENT RELATIVE ERROR LARGER THAN  ',F10.5)
      END
      SUBROUTINE DPROB1(MTYPE,NAREA,P,PV,R,S,PHI)
      DIMENSION P(50),PV(50),RATE(50),PVIS(50),R(50),PL(50)
     1,S(50),SR(50),PHI(50),SPHI(50)
      REAL NUM1
C          SUBRT. CALC. DET. PROB./ AN ALLOC. OF TIME(TAL)
      DO 5 I=1,NAREA
      PL(I)=P(I)
      PVIS(I)=PV(I)
      SR(I)=S(I)
      SPHI(I)=PHI(I)
    5 RATE(I)=R(I)
      RETURN
      ENTRY DPROB (KK,TAL,RESULT)
      GO TO (10,20,20,40,20,60),MTYPE
C           MODEL 1 BINARY INTERVISIBILITY
   10 RESULT=PL(KK)*PVIS(KK)*(1.-EXP(-RATE(KK)*TAL))
      RETURN
C           MODEL 2 SINGLE INTERVAL,START AT TIME ZERO
   20 RESULT=PL(KK)*(RATE(KK)/(RATE(KK)+PVIS(KK)))
      IF (MTYPE.EQ.3) GO TO 30
      RESULT=RESULT*(1.-EXP(-(RATE(KK)+PVIS(KK))*TAL))
      IF(MTYPE.EQ.5) GO TO 50
      RETURN
C           MODEL 3 SINGLE INTERVAL UNIFORM START TIME
C         SR(KK) DENOTES LENGTH OF UNIFORM INT. ON START TIME
   30 IF (TAL.NE.0.) GO TO 31
      RESULT=0.
      RETURN
   31 IF(TAL.GT.SR(KK)) GO TO 310
```

```
      TEMP=1.-EXP(-(RATE(KK)+PVIS(KK))*TAL)
      TEMP=TEMP/(RATE(KK)+PVIS(KK))
      RESULT=RESULT*(TAL-TEMP)/SR(KK)
      RETURN
  310 TEMP=EXP(-(RATE(KK)+PVIS(KK))*TAL)
      TEMP=EXP(-(RATE(KK)+PVIS(KK))*(TAL-SR(KK)))-TEMP
      TEMP=SR(KK)-TEMP/(RATE(KK)+PVIS(KK))
      RESULT=RESULT*TEMP/SR(KK)
      RETURN
C     MODEL 4 SINGLE INT. EXP START TIME
C     PV(I)=RATE FOR VIS. DIST
C     RATE(I)=RATE FOR COND. DETECT. FUNCT.
C     SR(I)=RATE FOR START VIS. PERIOD
   40 DEN=SR(KK)-RATE(KK)-PVIS(KK)
      NUM1=RATE(KK)+PVIS(KK)
      IF(ABS(DEN).LT.1.E-4) GO TO 410
      RESULT=SR(KK)*EXP(-NUM1*TAL)-NUM1*EXP(-SR(KK)*TAL)
      RESULT=PL(KK)*RATE(KK)*(1.-RESULT/DEN)/NUM1
      IF(MTYPE.EQ.5) GO TO 52
      RETURN
  410 TEMP=EXP(-SR(KK)*TAL)+SR(KK)*TAL*EXP(-NUM1*TAL)
      RESULT=PL(KK)*RATE(KK)*(1.-TEMP)/NUM1
      IF(MTYPE.EQ.5) GO TO 52
      RETURN
C     MODEL 5 COMBO. ZERO &NON-ZERO START TIMES
   50 SAVE1=RESULT
      GO TO 40
   52 RESULT=SPHI(KK)*SAVE1+(1.-SPHI(KK))*RESULT
      RETURN
C     MODEL 6 MULTIPLE INTERVAL
C      SOLVE FOR ROOTS
   60 B=SR(KK)+PVIS(KK)+RATE(KK)
      C=SR(KK)*RATE(KK)
      A=1.
      DISC=B*B-4.*A*C
      IF(DISC.LT.0.) GO TO 100
      ROOT2=(B+SQRT(DISC))/(2.*A)
      ROOT1=(B-SQRT(DISC))/(2.*A)
      DROOT=ROOT2-ROOT1
C     COMPUTE ABSORPTION PROB
C---------P23(T)
      TEMP1=(1.-EXP(-TAL*ROOT1))/ROOT1
      TEMP2=(1.-EXP(-TAL*ROOT2))/ROOT2
      TEMP=SR(KK)*RATE(KK)*(TEMP1-TEMP2)/DROOT
C-------P13(T)
      SAVE=RATE(KK)*(SR(KK)-ROOT1)*TEMP1/DROOT
      SAVE=SAVE-(SR(KK)-ROOT2)*TEMP2*RATE(KK)/DROOT
```

```
C          COMPUTE STATE PROB.
        RESULT=TEMP*(1.-SPHI(KK))
        RESULT=RESULT+SAVE*SPHI(KK)
        RESULT=PL(KK)*RESULT
        RETURN
  100   RESULT=-20.
        RETURN
        END
C-----  COMPUTE EXPT. TIME TO DETECT KOOPMAN MODEL
        SUBROUTINE TKOOP(X1,X2,Y1,Y2,EC)
        P1=X1
        P2=X2
        R1=Y1
        R2=Y2
        R=P1*R1/(P2*R2)
        Q=(R1+R2)/(R1*R2)
        IF(R.EQ.1.) GO TO 15
        IF(R.GT.1.) GO TO 10
C------ INTERCHANGE PARAMETERS
        SAVE1=P1
        SAVE2=R1
        P1=P2
        R1=R2
        P2=SAVE1
        R2=SAVE2
   10   R=P1*R1/(P2*R2)
        T=ALOG(R)/R1
        Z=(R1+R2)/R2
        EC=Z*Q*EXP(-T/Q)
        EC=EC*R**(-(1.-1./Z))+(1.-EXP(-R1*T))/R1
        EC=EC*P1+P2*T
        GO TO 20
   15   EC=Q
   20   RETURN
        END
```

Appendix F

NUMERICAL COMPARISON WITH THE KOOPMAN MODEL

F.1 Introduction

In this appendix we investigate the situation in which a partially informed searcher being aware of the earlier results of Koopman (which assume continuously visible targets) applies them to situations in which the target behavior is actually characterized by one of the intervisibility processes. The consequences (in terms of the probability of detecting the target under a constraint on the total available search effort) of this application of the Koopman results will be compared to those obtained from the optimal allocation procedures for these models. In this appendix these comparisons are made from the results of a numerical analysis, rather than the approximation analyses of Chapters 4 and 5.

This method of comparison has a great deal of practical significance (in addition to being a useful mathematical device for comparing the two models). For example, one may view each of the models under study as being composed of two types of parameters: those which the searcher may be expected to readily obtain or generate, e.g., the prior probabilities on target location, and the regional detection

capabilities; and those which essentially characterize the visibility processes, e.g., the distributions on the lengths of the masked and visible periods. The former group may be labeled the "searcher" parameters; the latter, the "target behavior/environmental" parameters. It is not unreasonable to assume that the searcher may

(a) have imperfect knowledge of "target behavior/environmental" parameters, and

(b) not be able to obtain estimates of these parameters or be unaware of the fact that the target is not continuously visible.[1]

F.2 Description of the Technique

In making a comparison such as this, the decision maker might generate the following list of questions relative to the appropriateness of the Koopman policy (denoted SKA) in situations in which the target behavior is characterized by one of the visibility processes.

(1) Under what conditions, if any, does the use of the SKA policy lead to small errors? By small errors, the decision maker may mean a relative error < 5%, at the peak difference in the probability of detection.

(2) Which, if any, of the following combinations of "searcher" and "target" parameters yield small errors?

---

[1] In Section 2.2.3, we discuss, conceptually, the implications of this analysis in terms of the various options open to the searcher as well as the associated cost-effectiveness measures.

Searcher { Identical regional detection capability / Non-identical regional detection capability }

Target { Identical regional behavior/environment / Non-identical regional behavior/environment }

(3) In those situations in which the mean time to detect is much less than the mean length of the visible period, can the searcher safely assume the SKA to be adequate?

(4) In those situations in which the SKA is shown to be inadequate, are there any simple relationships between the detection and target behavior parameters which indicate this inadequacy?

These questions apply to two situations: a *short-term* search situation in which the total available search time is less than the Koopman expected time for detection; and, a *long-term* situation in which the decision maker has an amount of time much greater than the Koopman expected time.

In order to answer any of these questions, one has to decide on certain measures of the effectiveness of using the SKA. As a measure of the worst case situation in using the SKA, consider

(a) the percent relative error at the peak difference in the probability of detection (% rel.).

However, the decision-maker also needs to know at what level of search effort this peak occurs in order to judge its

significance relative to the total amount of search time he has available. Thus we take as our second measure,

(b) that level of total available search time at which the peak difference occurs (Peak Time).

The analysis of the models of Chapters 2 and 3 indicated that the error resulting from the SKA could be reduced by increasing the total available search time. Thus the decision maker (searcher) is interested in determining

(c) the level of total available search time at which the percent relative error is less than some specified value.

Assuming that the searcher's strategy is to allocate an amount of time *at least* equal to the expected time to detect the target under the Koopman strategy, the following results might occur: the percent relative error is moderate but the peak time is much greater than the Koopman expected time; or the percent relative error may be high and occur at, or before, the Koopman expected time. Both of these cases would appear to be significant to the decision maker. The following measure is used to describe these situations:

$$\text{(d)} \qquad R = \frac{\text{Peak Time}}{\text{KET}} \cdot (\%\ \text{rel.}) \ .$$

In a certain sense, the Koopman expected time may be used to characterize the classical search situation as to the

degree of difficulty. Thus it was chosen to normalize the time at which the peak differences occur.

The limited analytical results for the models of Chapters 4 and 5 provide "explicit" answers to the above questions in very specific cases. In particular, the approximations (valid for large quantities of search effort) yield analytical expressions which can be studied to provide the required answers. Situations in which the available effort is limited to the extent that under the optimal policy all the effort is placed in a single region will also lend themselves to such analysis. However, in order to study the error function over the spectrum of search time, numerical techniques are required. Since this type of problem is readily solved using dynamic programming, this procedure is used in the following numerical analysis.

Having obtained the optimal allocation policy and return for a given level of search effort, the return from the SKA at this level of effort and the difference in the two returns is computed. If interest centers on the peak differences, as soon as the difference function begins to decrease the computation is ended. If one is interested in determining the amount of effort required to reduce the

percent relative error below some specified value, the computation continues until that point.[1] In the peak difference situation, the nature of a dynamic programming solution requires that we only compute the optimal solution for the last region until the peak is passed. A flow chart for these computations is given in Figure F.1.

F.3 Summary of the Numerical Results

The charts and tables given in this section contain the results of a series of computer runs for each of the models represented in Figure F.1. Because the dimensionality required to describe the search and target-environment parameter space is extremely large, only the two-region case is considered. Figure F.2 is a schematic representation of the parameter ranges used.

A summary of the results for each model relative to the questions formulated in Section F.2 is given. These results are identified by the ratio of prior probabilities on target location and the values of the detection rates for each region. Finally, a series of tables containing the measures introduced in Section F.2 are given.

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[1] The unimodality of the error function is determined via an analysis of the approximate error functions given for each of the models.

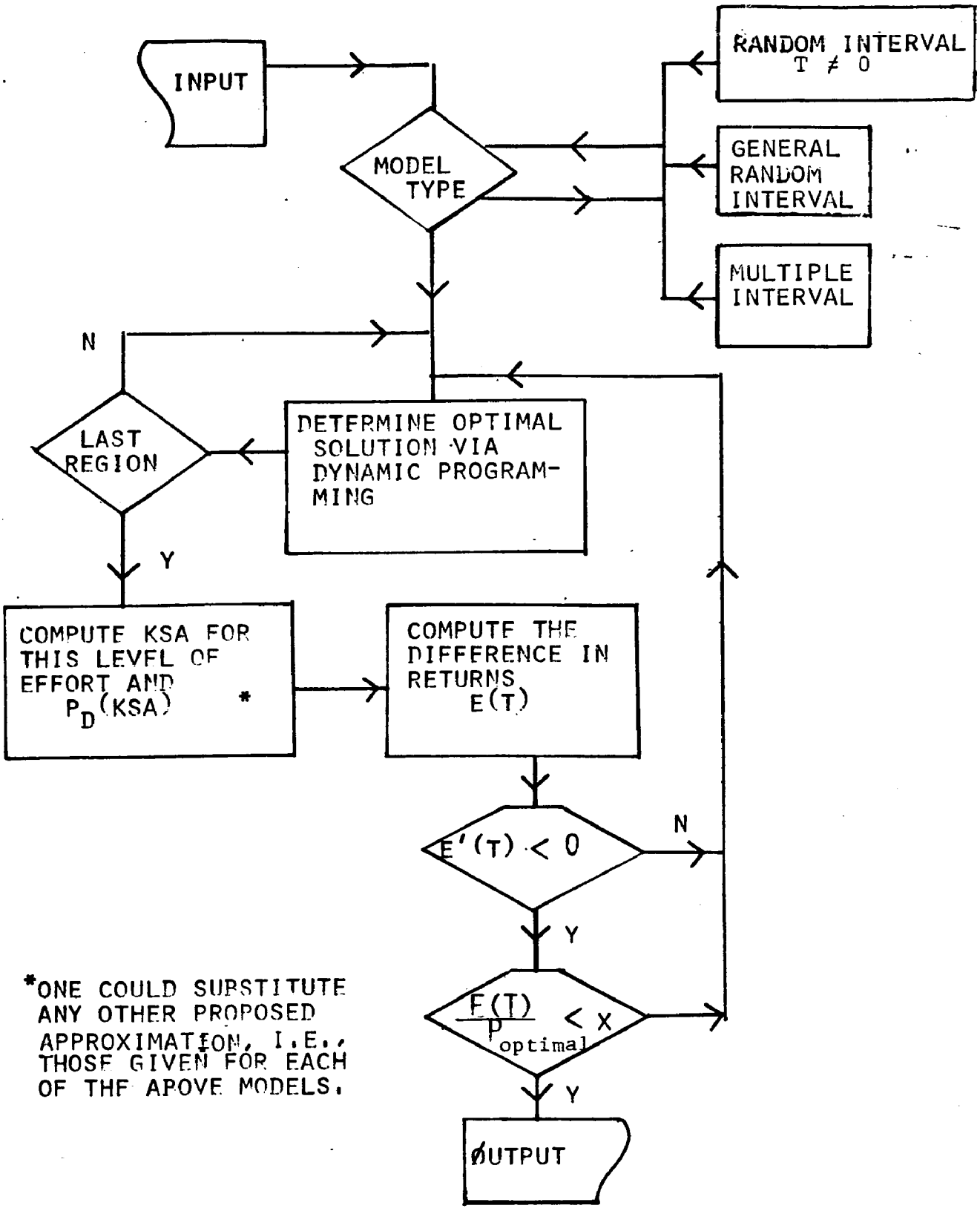


Figure F.1 Sensitivity Computations

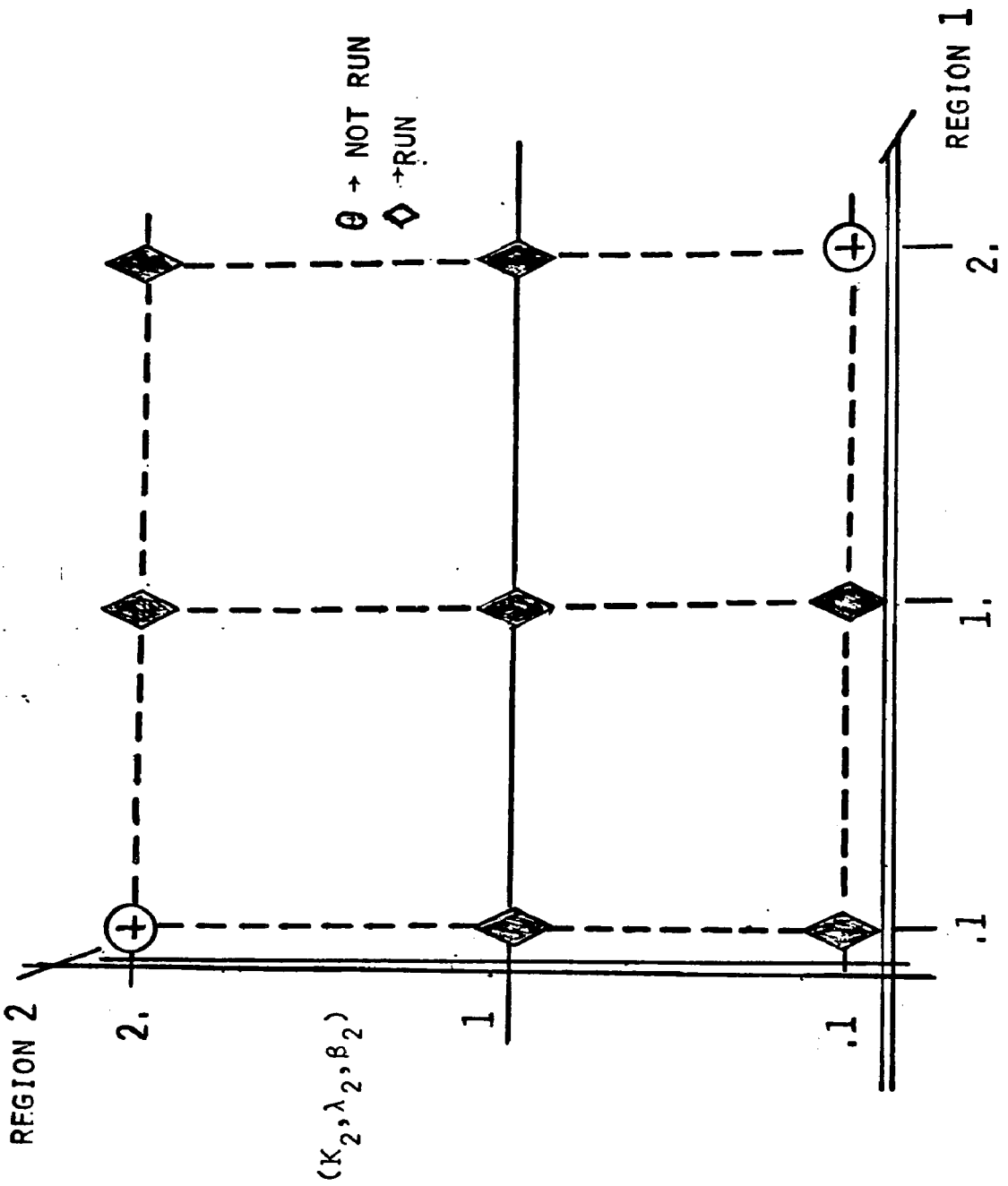


Figure F.2 Visibility Parameter Space

(I) Random Interval/Random Initiation and Limit

Exponential distributions are assumed on the start time as well as on the length of the visible period and the detection devices. The following results were obtained. The labeling corresponds to the respective question number of Section F.2.

(1) The availability of search time on the order of 2-KET (Koopman expected time) will *not* reduce the relative error below 5% in using the SKA.

(2) In those cases in which the mean time to detect is much less than the mean length of the visibility period, large errors are obtained from the use of SKA.

(3) For the short-term search situation:

$$p_1/p_2 = 1$$

k_1 \ k_2	0.1	1.0
0.1	Min % R.E. = 21[1]	See tables
1.0	See table[2] F.5	Min % R.E. = 19

[1]Over the visibility parameter space under investigation, none of the combinations studied led to small short-term errors in using the Koopman policy.

[2]Note, in many of these cases, the extreme sensitivity of the error function measures to changes in the visibility parameters.

$p_1/p_2 = 2$

k_1 \ k_2	0.1	1.0
0.1	See table F.6	Min % R.E. = 11
1.0	See table F.6	Min % R.E. = 16

(4) From the tables, we observe that the greatest errors in using the SKA occur with non-identical detectors and that level of total available search effort at which they occur is, in many cases, much greater than the Koopman expected time (KET). The approximation analysis of Section 4.2.2 gives insight into the causes of such sustained errors. Consider the following example:

p	λ	k	β
2/3	.1	1.	.1
1/3	1	.1	.1

From the analysis of Section 4.2.2, the long-term approximate policies are

$$t_1 = \frac{\beta_2 t}{\beta_1 + \beta_2} \quad \text{and} \quad t_2 = \frac{\beta_1 t}{\beta_1 + \beta_2},$$

while the similar results for the Koopman policy are

$$t_1 = \frac{k_2}{k_1 + k_2} t \quad \text{and} \quad t_2 = \frac{k_1 t}{k_1 + k_2}.$$

Thus, under the Koopman policy region 1 doesn't receive enough of the search effort. Also note that if $\beta_1 = 1$, then the long-term policies are identical under both methods.[1] To summarize, an investigation of the relative magnitudes of the conditional detection rate ($\lambda_i + k_i$) and the rate of masking interval, β_i, determines the long-term allocation policy. When this policy greatly differs from the Koopman policy, the latter will be inadequate over a large range of total available search effort.

Tables F.1 and F.2 summarize the main effects due to the various search parameters; measure (d) is used in the tables. The following observations are of interest.

(a) For $k_1 = k_2 = .1$, note the increasing sensitivity of measure (d) to λ_1 and λ_2 with increasing prior positional probability ratios p_1/p_2.

(b) For $k_1 = .1$, and $k_2 = 1.$, note the sensitivity to β_2 and the reversal of its effect at $p_1/p_2 = 10$.

(c) For $k_1 = 1.$, $k_2 = .1$, note the *extreme* sensitivity of measure (d) to β_1. If one uses the approximate solutions of Section 4.2.2, then it is apparent that at the lower level of β_1 the Koopman and optimal policies are very different.

(d) For $k_1 = 1. = k_2$, we note the increasing sensitivity to β_1.

---

[1] Which explains the senstivity to β_1 in the tables.

Table F.1

Random Interval Sensitivity

$p_1/p_2=1$	$k_1 = .1$ $k_2 = .1$	$k_1 = .1$ $k_2 = 1.$	$k_1 = 1.$ $k_2 = .1$	$k_1 = 1.$ $k_2 = 1.$
$\lambda_1 = .1$	92.9	149.6	570.3	315.5
$\lambda_1 = 1.$	71.3	945.8	525.2	291.
$\lambda_2 = .1$	92.9	569.9	149.7	315.5
$\lambda_2 = 1.$	71.3	525.5	945.8	291.
$\beta_1 = .1$	115.1	582.9	1035.5	355.5
$\beta_1 = 1.$	52.7	513.	60.	251.
$\beta_2 = .1$	112.5	1035.5	589.2	355.5
$\beta_2 = 1.$	54.6	60.2	506.3	251.
$p_1/p_2=2$				
$\lambda_1 = .1$	63.8	103.8	1137.9	540.6
$\lambda_1 = 1.$	168.3	485.2	662.5	193.7
$\lambda_2 = .1$	167.8	306.6	588.6	445.
$\lambda_2 = 1.$	64.3	255.4	1211.9	289.3
$\beta_1 = .1$	131.5	497.2	1747.1	522.4
$\beta_1 = 1.$	100.6	64.8	53.4	211.9
$\beta_2 = .1$	138.4	491.7	812.7	342.3
$\beta_2 = 1.$	93.7	70.3	987.8	392.

Table F.2

Random Interval Sensitivity

p_1/p_2=3	k_1 = .1 k_2 = .1	k_1 = .1 k_2 = 1.	k_1 = 1. k_2 = .1	k_1 = 1. k_2 = 1.
λ_1 = .1	49.2	91.6	1418.6	492.1
λ_1 = 1.	195.6	431.6	1197.7	339.5
λ_2 = .1	188.2	291.2	1143.4	390.4
λ_2 = 1.	56.5	232.0	1472.9	44.2
β_1 = .1	124.4	457.4	2566.0	639.4
β_1 = 1.	120.3	65.8	50.5	191.4
β_2 = .1	135.2	447.8	1090.8	559.9
β_2 = 1.	109.5	75.4	1525.7	270.9
p_1/p_2=10				
λ_1 = .1	54.0	56.9	2620.0	1113.
λ_1 = 1.	218.9	80.2	2455.	721.
λ_2 = .1	205.9	91.8	2454.	784.
λ_2 = 1.	57.0	45.3	2621.	1050.
β_1 = .1	107.8	88.8	5035.	1710.
β_1 = 1.	148.6	48.8	40.	124.
β_2 = .1	124.2	35.1	2368.	980.
β_2 = 1.	148.7	102.5	2707.	854.

Table F.2 suggests the following comments:

(a) Note the insensitivity of the situation in which $k_1 = k_2 = .1$ to changes in the prior probabilities on target location, as contrasted with the approximate linear effect of such changes when $k_1 = k_2 = 1$. Essentially, this difference results from differences in the peak time relative to the Koopman expected times in the two cases. The peaks occur early in the former case.

(b) Note the decreasing rate of change of measure (d) with increasing p_1/p_2 for the conditions $k_1 = 1.$, $k_2 = .1$.

In contrast to the models of Chapters 2 and 3, we observe that

(a) the use of the Koopman policy will not be adequate in cases in which one has homogeneous visibility conditions,

(b) the error function is not reduced by increasing the available search time (unless extremely large increases are made),

(c) the maximum errors do not occur in situations in which the total available search time is constrained.

(d) the availability of extremely "good" detection devices will not imply that the Koopman scheme will yield small errors.

The above results imply that the decision maker, in general, must obtain accurate estimates of the visibility parameters in order to conduct an effective search, since

the option of increasing the available search time may no longer be cost-effective. The sensitivity analysis of 4.2.3 will give some insight into the sensitivity of the optimal policy and return to these estimates as well as the numerical results just reviewed.[1]

(II) General Single Interval Model

Here the two single-interval models of Chapters 3 and 4 are combined using the limiting values for the state probabilities of an associated alternating renewal process (see Section 4.4.2, $\Pi_i = \frac{\beta_i}{\lambda_i + \beta_i}$).[2] These results were obtained,

(1) The availability of search time on the order of 2-KET (Koopman expected time) will not, in general, reduce the relative error below 5%, in using the SKA. The only exception is the situation in which $p_1/p_2 = 1$ and $k_1 = k_2 = 1$.

(2) In those cases in which the mean time to detect is much less than the mean length of the visibility period (e.g., $k_1 = k_2 = 1$, and $\lambda_1 = \lambda_2 = .1$), the SKA can be used. Under these conditions the probability that the target is visible at the start of the search is at least 0.5 for the levels of β used.

[1] All of the above observations hold for the situation in which the start of the visibility period in the ith region is uniformly distributed on $(0,S_i)$. Section F.4 contains the results of a sensitivity analysis of this situation (Tables F.8 and F.9).

[2] One could, of course, use any prior estimate on Π_i, e.g., one which is independent of λ_i and β_i.

(3) For the short-term search situation, one has the following results:

$$p_1/p_2 = 1$$

k_1 \ k_2	0.1	1.0
0.1	Homogeneous visibility conditions yield small errors	See table F.10
1.0	See table F.10	Homogeneous visibility parameters yield small errors[1]

$$p_1/p_2 = 2$$

k_1 \ k_2	0.1	1.0
0.1	Homogeneous visibility conditions ($\beta_1 = \beta_2 = .1$) → small errors	See table F.11
1.0	See table F.11	Homogeneous visibility conditions with ($\beta_1 = \beta_2 = 1.$) → small errors[2]

[1] Recall that for the model of Chapter 3, such conditions implied the optimality of the Koopman policy.

[2] Here one notes that the long-term allocation policies are identical and that the probability of the target being visible at the start of the search is almost 0.5.

Table F.3 summarizes the main effects in terms of measure (d) due to the various search parameters. It suggests the following comments.

(a) For $k_1 = k_2 = .1$, note that the sensitivity to changes in λ_1 and λ_2 increases with increasing p_1/p_2. Changes in the λ's affect the prior probabilities on target visibility at the start of the search. Thus, the general model tends either toward the model of Chapter 3 or Chapter 4 depending upon whether Π can effect the FAR for the general random interval model.

(b) For $k_1 = 1.$ and $k_2 = 0.1$, note the extreme sensitivity to β_1. This causes the general model to tend toward that of Chapter 3 by increasing Π. The reader will recall from our earlier discussions that the model of Chapter 3 has primarily short-term errors. From approximation policies of Section 4.4.2, observe that for $\beta_1 = 1$, the approximate long-term allocation tends to agree with the Koopman allocation.

To summarize, we have observed that

(a) the Koopman policy can be used effectively in the homogeneous detection, homogeneous visibility condition situation, and

(b) it will also prove adequate in the situations in which the mean time to detect is much less than the mean length of the visible period.

The values of Π, the prior probability vector on target visibility at the start of the search, are of importance here. Since for $\Pi_i \to 1$, the general model reduces to the random

Table F.3

General Random Interval Sensitivity

$p_1/p_2=1$	$k_1 = .1$ $k_2 = .1$	$k_1 = .1$ $k_2 = 1.$	$k_1 = 1.$ $k_2 = .1$	$k_1 = 1.$ $k_2 = 1.$
$\lambda_1 = .1$	55	205	443	37
$\lambda_1 = 1.$	51	665	427	59
$\lambda_2 = .1$	55	443	205	37
$\lambda_2 = 1.$	51	427	665	59
$\beta_1 = .1$	60	433	856	60
$\beta_1 = 1.$	46	437	14	36
$\beta_2 = .1$	60	856	433	60
$\beta_2 = 1.$	46	14	437	36
$p_1/p_2=2$				
$\lambda_1 = .1$	35	17	618	101
$\lambda_1 = 1.$	130	396	782	152
$\lambda_2 = .1$	105	210	543	99
$\lambda_2 = 1.$	60	203	857	154
$\beta_1 = .1$	88	295	1383	185
$\beta_1 = 1.$	77	118	17	68
$\beta_2 = .1$	73	392	792	126
$\beta_2 = 1.$	92	21	608	127

interval, initially visible model, and this model has primarily short-term errors under the Koopman policy. If, on the other hand, $\Pi_i \to 0$, one has the random-interval, random-start-time model which has significant long-term errors under the Koopman policy.

(III) Multiple Interval Model

Here it is assumed that the visibility process is characterized by an alternating renewal process with the residency times being exponentially distributed. Again, the probabilities of the target being either visible or masked are taken to be the limiting state probabilities of the associated alternating renewal process. The following results were obtained.

(1) The availability of search time on the order of 2-KET (Koopman expected time) will *not*, in general, reduce the relative error below 5% in using the SKA.

(2) The SKA can be effectively used in those cases in which the mean time to detect is much less than the mean length of the visible periods.

(3) For the short-term situation, one has the following results:

$p_1/p_2 = 1$

k_1 \ k_2	0.1	1.0
0.1	Homogeneous visibility conditions → small errors	See table F.12
1.0	See table F.12	Homogeneous visibility conditions → small errors

$p_1/p_2 = 2$

k_1 \ k_2	0.1	1.0
0.1	Homogeneous visibility conditions → small errors	See table F.13
1.0	See table F.13	Homogeneous visibility conditions with ($\lambda_1 = \lambda_2 = .1$) → small errors

Table F.4 summarizes the main effects in terms of measure (d) due to the various search parameters. It suggests the following comments.

(a) For $k_1 = k_2 = 0.1$, note the increasing sensitivity to λ_1 and β_1 with increasing p_1/p_2 (an increase in p_1 increases the contribution to the optimal return function of the first region).

(b) For $k_1 = 1.$ and $k_2 = 0.1$, note the increasing sensitivity to changes in β_1 (increasing β_1 tends to

Table F.4.

A Multiple Interval Sensitivity

$p_1/p_2=1$		$k_1 = .1$ $k_2 = .1$	$k_1 = .1$ $k_2 = 1.$	$k_1 = 1.$ $k_2 = .1$	$k_1 = 1.$ $k_2 = 1.$
λ_1 =	.1	51	76	153	36
	1.	93	333	256	91
λ_2 =	.1	51	153	76	35
	1.	93	256	333	92
β_1 =	.1	91	334	348	75
	1.	53	75	61	52
β_2 =	.1	91	348	334	75
	1.	53	61	75	52
$p_1/p_2=2$					
λ_1 =	.1	43	30	204	33
	1.	109	125	467	190
λ_2 =	.1	60	28	204	69
	1.	92	127	467	154
β_1 =	.1	108	129	585	156
	1.	44	26	86	67
β_2 =	.1	91	111	489	162
	1.	61	44	182	61

yield long-term approximation policies which are close to the Koopman policy).

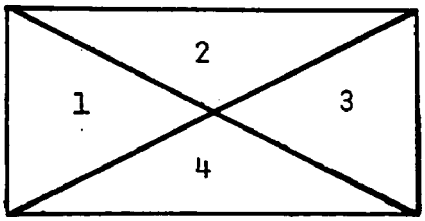
(c) For $k_1 = k_2 = 1$, note the general increase in the sensitivity of all the visibility parameters λ_1, β_1, λ_2, and β_2.

The SKA yields small to moderate errors for homogeneous detection and visibility conditions. Homogeneous visibility conditions will yield, via equation 44 of Section 5.2.1, equal rates (γ_i) for use in the long-term approximate allocations. Thus whenever the detection rates are identical, only short-term errors will occur. On the other hand, heterogeneous visibility conditions yield heterogeneous rates (γ_i) which may differ significantly from the detection rates used by the Koopman policy, resulting in error functions which increase beyond the Koopman expected time. Thus, in certain cases, the decision maker has the full range of options given in Section 2.2.3. For the situation in which the visibility conditions are homogeneous the most cost-effective approach may well be that of increasing the available search effort as opposed to obtaining estimates of the visibility parameters. The data also indicate the sensitivity of the error function to changes in the visibility parameters in the heterogeneous case. Thus, an erroneous assumption on the homogeneity of the visibility conditions could lead to extreme errors if the searcher is

using the Koopman scheme. The sensitivity analysis of Section 5.2.3 and the data of Table F.4 provide insight into the sensitivity of the optimal policy and return to variations in the estimates of the visibility parameters.

F.4 Tables

This section contains the data summarized in the preceding section. For each model the data are arranged in order of increasing ratios of p_1/p_2. The data are arranged in the following order within each cell of the table (for a given ratio of p_1/p_2):



,

where

1 - maximum difference in the probability of detection,

2 - time at which the maximum difference occurs,

3 - percent relative error at the peak difference,

4 - measure (d), defined in Section F.2.

Table F.5

Random Interval Sensitivity Analysis

(Exponential Start Times)

$p_1/p_2=1$	k_1	.1				1.			
	k_2	.1		1.		.1		1.	
λ_1 λ_2 β_2	β_1	.1	1.	.1	1.	.1	1.	.1	1.
.1 .1	.1	10 .023 23 12	5 .06 43 11	13 .13 40 68	4 .02 16 8	13 .126 40 68	5 .09 20 13	3 .02 23 35	2 .136 48 48
.1 .1	1.	5 .06 43 11	3 .02 21 3	5 .089 20 13	4 .054 13 7	4 .02 16 8	4 .05 13 7	2 .14 48 48	1 .04 31 16
.1 1.	.1	18 .06 35 32	7 .07 38 13	7 .039 33 30	4 .032 29 15	27 .24 58 205	5 .09 21 14	6 .04 21 63	2 .14 50 50
.1 1.	1.	2 .013 46 5	6 .04 24 7	4 .026 11 6	3 .013 6 2	33 .27 57 246	5 .07 15 10	2 .07 37 37	2 .05 19 19
1. .1	.1	18 .06 35 32	2 .01 46 5	27 .24 58 204	33 .269 57 246	7 .04 33 30	4 .03 11 6	6 .04 21 63	2 .07 37 37
1. .1	1.	7 .07 38 13	6 .04 24 7	5 .094 21 14	5 .067 15 10	4 .03 29 15	3 .01 6 2	2 .14 50 50	2 .05 19 19
1. 1.	.1	3 .002 23 4	2 .014 48 5	41 .12 45 241	34 .14 50 222	41 .12 45 241	4 .03 12 6	2 .01 21 21	2 .07 39 39
1. 1.	1.	2 .014 48 5	1 .004 31 2	4 .03 12 6	3 .014 6 2	34 .14 50 222	3 .014 6 2	2 .07 39 39	1 .02 23 23

Table F.6

Random Interval Sensitivity Analysis

(Exponential Start Times)

$p_1/p_2=2$	k_1 / k_2	.1		1.		.1		1.	
λ_1 λ_2 β_2	β_1	.1	1.	.1	1.	.1	1.	.1	1.
.1 .1	.1	16 .035 17 14	11 .02 6 3	7 .05 38 28	3 .05 4 14	20 .23 44 156	5 .07 12 11	7 .06 23 80	3 .14 28 41
	1.	5 .04 45 11	8 .01 2 1	4 .09 33 15	3 .05 21 7	44 .30 41 319	5 .05 8 7	2 .11 56 59	2 .08 21 22
.1 1.	.1	21 .05 21 23	12 .02 7 5	4 .02 31 14	3 .06 48 16	29 .33 57 292	6 .08 12 12	10 .09 25 126	3 .14 29 43
	1.	2 .01 47 5	9 .01 3 1	3 .03 21 8	2 .02 11 3	33 .34 57 332	5 .06 10 9	2 .06 32 44	2 .09 22 26
1. .1	.1	17 .05 40 36	23 .07 37 45	40 .16 50 213	1 .01 24 4	13 .08 34 80	5 .02 6 5	4 .02 21 38	2 .06 23 27
	1.	8 .09 74 31	10 .08 51 26	5 .10 35 17	4 .07 24 10	5 .01 13 11	4 .01 3 2	2 .12 57 66	2 .03 17 13
1. 1.	.1	10 .002 5 3	19 .01 10 10	48 .08 38 194	2 .02 43 9	28 .16 51 252	5 .02 6 5	6 .03 19 60	2 .06 24 28
	1.	3 .013 56 8	7 .02 26 10	4 .04 23 9	2 .02 12 3	34 .18 51 306	4 .01 3 2	2 .06 59 50	2 .04 16 14

Table F.7

Random Interval Sensitivity Analysis
(Exponential Start Times)

$p_1/p_2=10$	k_1 / k_2	.1		.1		1.		.1		1.		.1		1.	
λ_1	λ_2 β_2 β_1	.1		1.		.1		1.		.1		1.		.1		1.	
.1	.1	31 .014 8	3	42 .01 7	2	13 .02 8	8	5 .02 4	8	29 .37 580	48	7 .026 9	3	14 .15 240	24	4 .06 23	8
	1.	26 .004 2	1	25 .02 7	4	3 .04 16	58	4 .01 3	7	36 .37 690	46	6 .02 5	2	22 .12 251	16	4 .05 17	6
1.	.1	35 .02 11	4	- -	-	13 .02 9	8	5 .02 4	8	31 .40 659	51	7 .03 9	3	15 .16 279	26	4 .06 23	8
	1.	41 .015 10	3	24 .003 1	1	2 .02 11	60	4 .01 3	7	31 .40 659	51	7 .02 9	3	18 .14 257	20	4 .05 23	7
.1	.1	36 .01 28	11	29 .03 58	28	5 .002 3	6	2 .004 2	8	27 .18 484	43	6 .005 3	1	12 .06 171	20	4 .02 12	4
	1.	24 .02 41	24	23 .04 56	34	4 .05 25	68	10 .06 33	36	41 .19 683	40	5 .002 1	1	4 .01 63	7	3 .01 7	3
1.	.1	26 - 1	1	24 .006 12	7	5 .002 3	7	2 .004 2	8	31 .21 620	48	7 .005 4	1	14 .08 220	22	4 .02 12	4
	1.	23 .004 8	5	23 .008 15	9	2 .025 14	7	2 .003 1	6	33 .21 660	48	6 .002 1	1	20 .07 229	16	3 .01 7	3

Table F.8

Random Interval Sensitivity Analysis
(Uniform Start Times)

$p_1/p_2=1$	k_1 / k_2	.1		1.		.1		1.	
λ_1 λ_2 S_2	S_1	1	11	1	11	1	11	1	11
.1	1	2 .02 27 3	5 .05 36 9	3 .02 5 2	4 .04 9 5	3 .02 5 2	3 .03 26 10	1 .07 41 20	2 .17 46 46
	11	5 .05 36 9	12 .05 32 19	3 .03 26 10	12 .24 54 84	4 .04 9 5	12 .24 54 84	2 .17 46 46	11 .04 9 50
1.	1	11 .04 16 8	2 .02 45 5	4 .01 3 2	3 .004 2 1	3 .02 5 2	16 .34 69 140	1 .08 45 23	1 .09 64 32
	11	6 .06 36 12	14 .10 49 35	3 .05 47 18	11 .09 38 54	4 .04 10 5	13 .32 70 118	2 .17 47 47	11 .11 27 148
.1	1	11 .04 16 8	6 .06 36 12	3 .02 5 2	4 .04 10 5	4 .01 3 2	3 .05 47 18	1 .08 45 23	2 .17 47 47
	11	2 .02 45 5	14 .10 49 35	16 .34 68 140	13 .32 70 118	3 .004 2 1	11 .09 38 54	1 .09 64 32	11 .11 27 148
1.	1	1 .01 41 2	2 .02 46 5	3 .002 1 1	3 .004 2 1	3 .002 1 1	1 .005 28 4	1 .05 35 18	1 .09 65 33
	11	2 .02 46 5	11 .004 9 5	1 .005 28 4	12 .15 62 97	3 .004 2 1	12 .16 62 97	1 .09 65 33	11 .01 5 28

Table F.9

Random Interval Sensitivity Analysis
(Uniform Start Times)

$p_1/p_2=2$	k_1 / k_2	.1		.1		1.		.1		1.		1.	
λ_1	λ_2	S_2	S_1	1	11	1	11	1	11	1	11	1	11
.1	.1	1	1	.01 9/4/2	.05 4/55/11	.03 2/10/2	.06 3/20/6	.01 4/2/1	.37 22/50/193	.07 2/15/15	.15 2/62/62
			11	.02 11/5/3	.06 13/25/17	.06 3/44/14	.10 11/36/42	.02 5/4/4	.37 12/62/130	.12 2/26/26	.17 11/30/175
		1.	1	.004 8/2/1	.01 2/56/6	.01 2/4/1	.01 3/7/2	.01 4/2/1	.42 15/67/161	.08 2/17/17	.08 1/80/40
			11	.02 12/6/4	.07 14/26/19	.07 3/50/16	.02 5/34/17	.03 5/4/4	.41 13/69/160	.12 2/26/26	.21 11/38/220
	1.	1	1	.09 8/60/24	.09 8/69/29	.03 3/10/3	.06 4/20/9	.01 4/4/3	.19 24/38/160	.04 1/20/10	.16 2/65/65
			11	.10 19/50/50	.06 14/48/35	.02 2/42/9	.20 12/68/87	.001 4/.3/-	.17 11/53/100	.05 2/15/15	.04 11/12/70
		1.	1	.03 7/31/11	.02 2/73/7	.01 2/4/1	.01 3/7/2	.001 4/.3/-	.22 15/61/160	.04 1/20/10	.08 2/52/52
			11	.02 16/20/17	.01 11/14/8	.03 2/56/12	.09 12/53/67	.001 4/.3/-	.21 12/64/135	.05 2/15/15	.085 11/27/156

Table F. 10

General Random Interval
Sensitivity Analysis

$p_1/p_2=1$	k_1 / k_2	.1				1.				.1				1.			
λ_1 λ_2 β_2	β_1	.1		1.		.1		1.		.1		1.		.1		1.	
.1	.1	- / -	-	.01 / 4	12	.09 / 91	14	.11 / 105	16	.09 / 91	14	.004 / -	1	- / -	-	.01 / 2	3
	1.	.01 / 4	12	- / -	-	.004 / -	1	.001 / -	-	.11 / 105	16	.001 / -	-	.01 / 2	3	- / -	-
1.	.1	.04 / 14	25	.05 / 9	30	.002 / 1	3	- / -	-	.11 / 122	24	.024 / 3	5	.04 / 14	27	.09 / 16	31
	1.	.04 / 17	17	.03 / 7	13	.025 / 8	32	.005 / -	2	.14 / 117	28	.02 / 5	4	.002 / 1	-	.01 / 2	4
.1	.1	.04 / 14	25	.04 / 17	17	.11 / 122	24	.024 / 3	5	.002 / 1	3	- / -	-	.04 / 14	27	.002 / 1	-
	1.	.05 / 9	30	.03 / 7	13	.14 / 117	28	.02 / 5	4	.025 / 8	32	.005 / -	2	.09 / 16	31	.01 / 2	4
1.	.1	- / -	-	.01 / 2	24	.11 / 214	40	.13 / 198	40	.11 / 244	40	.01 / 2	4	- / -	-	.04 / 13	26
	1.	.01 / 2	24	- / -	-	.01 / 2	4	.01 / 4	4	.13 / 198	46	.01 / 4	4	.04 / 13	26	- / -	-

Table F.11

General Random Interval
Sensitivity Analysis

$p_1/p_2=2$	k_1 / k_2	.1		.1		1.		.1		1.		.1		1.	
λ_1 λ_2	β_1 / β_2	.1	1.	.1	1.	.1	1.	.1	1.	.1	1.	.1	1.	.1	1.
.1	.1	- / -	.07 / 3	34 / 2	- / -	- / -	.003 / -	2 / 2	.13 / 150	50 / 17	.002 / -	6 / -	.002 / 1	6 / -	.013 / 2	1 / 4
.1	1.	.016 / 3	.01 / 2	7 / 4	.01 / -	2 / 2	- / -	- / -	.14 / 157	47 / 19	.001 / -	10 / -	.007 / 2	2 / 2	- / -	- / -
1.	.1	.03 / 10	.014 / 3	17 / 10	.02 / 6	12 / 5	.05 / 10	2 / 30	.15 / 147	32 / 26	.02 / 3	5 / 3	.04 / 12	2 / 12	.06 / 12	2 / 12
1.	1.	.03 / 11	.01 / 3	27 / 8	- / -	20 / 2	.01 / 1	2 / 4	.17 / 157	33 / 27	.013 / 4	10 / 2	.05 / 70	21 / 7	.01 / 2	1 / 3
.1	.1	.05 / 23	.05 / 24	7 / 63	.08 / 107	46 / 22	.10 / 92	32 / 27	.12 / 230	50 / 26	- / -	- / -	.05 / 22	1 / 43	.03 / 40	23 / 4
1.	1.	.09 / 28	.09 / 22	7 / 75	.03 / 5	4 / 10	.03 / 6	7 / 7	.003 / 2	3 / 3	.001 / 4	5 / 4	.11 / 29	1 / 58	.01 / 3	2 / 3
.1	1.	- / -	.01 / 10	17 / 10	.07 / 174	48 / 34	.01 / 3	2 / 15	.14 / 260	34 / 44	.007 / 2	5 / 2	.014 / 28	8 / 7	.03 / 9	2 / 9
1.	1.	.02 / 13	.02 / 10	7 / 28	.01 / 3	3 / 9	.01 / 6	10 / 6	.16 / 280	34 / 47	.007 / 4	10 / 2	.04 / 21	1 / 42	- / -	- / -

Table F.12

Multiple Interval
Sensitivity Analysis

$p_1/p_2=1$	k_1	.1				1.			
	k_2	.1		1.		.1		1.	
λ_1 λ_2 β_2	β_1	.1	1.	.1	1.	.1	1.	.1	1.
.1 .1	.1	- / - / -	6 / .03 / 4	12 / 50 / 46	7 / .06 / 46	- / - / -	- / - / -	3 / .01 / 1	1 / .01 / 2
	1.	6 / .03 / 4	12 / - / -	- / 3 / 1	.01 / - / -	- / - / -	- / - / -	1 / .01 / 2	4 / - / -
.1 1.	.1	21 / .06 / 22	22 / 12 / 18	29 / - / -	- / 3 / 8	22 / .03 / 22	18 / .10 / 54	23 / .04 / 5	8 / .04 / 14
	1.	27 / .09 / 16	32 / - / -	5 / 6 / 12	12 / 22 / 4	12 / .04 / 12	50 / .06 / 46	7 / .01 / 1	3 / - / -
	1.	- / .009 / 2	3 / - / -	21 / - / -	22 / 18 / 23	54 / 50 / .06	7 / - / -	46 / .05 / 9	12 / 1 / .04
1. .1	.1	26 / - / -	- / .06 / 22	22 / - / -	- / .10 / 29	18 / 12 / 5	.02 / 13 / 3	3 / 5 / .04	8 / 3 / .01
	1.	5 / 1 / .09	32 / .009 / 2	3 / 3 / .03	23 / 22 / .04	4 / 1 / .09	32 / .009 / 2	3 / - / -	19 / 33 / .08
1. 1.	.1	25 / .18 / 45	194 / - / -	- / 33 / .18	45 / 8 / .12	25 / - / -	25 / 3 / .07	20 / 30 / -	- / - / -
	1.	19 / .08 / 25	25 / - / -	8 / .12 / 25	25 / 5 / .05	12 / - / -	8 / 5 / .05	12 / 3 / .07	20 / 30 / -

Table F.13

Multiple Interval
Sensitivity Analysis

$p_1/p_2=2$	k_1		.1				1.			
λ_1 λ_2 β_2 β_1	k_2		.1	1.	.1	1.	.1	1.	.1	1.
.1	.1	.01	20 2 2	.02	13 5 3	.001	3 1 0	.004	2 3 0	.07	20 12 41	.01	4 2 1	-	- - -	.02	1 4 3
.1	1.	.014	7 7 3	-	- - -	.01	3 4 1	-	- - -	.05	20 8 26	.003	4 4 -	.02	1 8 3	-	- - -
1.	.1	.07	20 18 18	.07	20 13 13	.018	2 37 6	.06	2 65 11	.16	20 27 95	.03	6 5 5	.04	2 11 11	.07	2 13 13
1.	1.	.003	12 1 1	.02	12 4 3	.03	8 7 7	.02	20 2 5	.06	20 10 35	.01	4 2 1	-	- - -	.02	1 4 3
.1	.1	.05	7 56 20	.02	20 4 11	.05	10 19 20	-	- - -	.12	20 29 102	.08	7 13 16	.05	1 54 22	.03	5 4 11
1.	1.	.12	7 75 27	.006	7 3 1	.05	4 14 6	.01	3 4 1	-	- - -	.05	16 6 18	.12	1 70 28	.01	1 6 2
.1	1.	.01	20 13 13	.08	20 18 18	.07	20 32 68	.02	2 38 6	.24	20 56 198	.13	9 20 31	.03	14 10 70	.09	4 18 32
1.	1.	.06	10 48 24	.01	14 3 2	.09	7 30 21	.02	4 9 3	.11	20 15 88	.08	7 13 15	.06	1 56 22	.01	3 2 3

Appendix G

SOME RESULTS FROM THE THEORY OF SEMI-MARKOV PROCESSES

In this appendix, we list some theorems on Semi-Markov processes due to Pyke (1961). First some definitions, let $G_{ij}(t)$ denote the distribution function for the first passage time, i.e.,

$$G_{ij}(t) = \Pr[N_j(t) > 0 | X_0 = i] \equiv \Pr\{t_{j_1} \leq t | X_0 = i\} ,$$

where t_{j_1} denotes the time of the first passage to state j and $N_j(t)$ denotes the number of entries to state j in time t. Let $P_{ij}(t)$ be defined as

$$P_{ij}(t) = \Pr\{X_t = j | X_0 = i\} .$$

The Laplace transforms of these are, respectively,

$$g_{ij}(s) = \int_0^\infty e^{-st} \, dG_{ij}(t) ,$$

and

$$\Pi_{ij}(s) = \int_0^\infty e^{-st} \, dP_{ij}(t) .$$

The Laplace transform of Q, the associated semi-Markov matrix, is denoted by

$$q = \{q_{ij}(s)\} = \int_0^\infty e^{-st} \, dQ_{ij}(t) .$$

Pyke (1961) proves the following theorems relating P and G.

Lemma 1.0

For all $t \geq 0$, $s > 0$

$$P_{ij} = \delta_{ij} - \sum_{K=1}^{m} [\delta_{ij} - P_{Kj}(t)]*Q_{iK}(t)$$

$$\Pi_{ij}(s) = \delta_{ij} - \sum_{K=1}^{m} [\delta_{ij} - \Pi_{Kj}(s)]q_{iK}(s) \ .$$

Lemma 2.0

For $t \geq 0$, $s > 0$,

$$G_{ij}(t) = \sum_{K=1}^{m} G_{Kj}(t)*Q_{iK}(t) + [1 - G_{jj}(t)]*Q_{ij}(t)$$

$$g_{ij}(s) = \sum_{K=1}^{m} g_{Kj}(s)q_{iK}(s) + [1 - g_{jj}(s)]q_{ij}(s) \ .$$

Theorem 1

For $t \geq 0$, $s > 0$,

$$P_{ij}(t) = P_{jj}(t)*G_{ij}(t) + \delta_{ij}[1 - \Lambda_i(t)]$$

and

$$\Pi_{ij}(s) = \Pi_{jj}(s) g_{ij}(s), \quad i \neq j$$

$$\Pi_{jj}(s) = \frac{1 - h_j(s)}{1 - g_{jj}(s)}$$

where

$$\Lambda_i(t) = \sum_{j=1}^{M} Q_{ij}(t), \quad h_{ij}(s) = \int_0^{\infty} e^{-st} d\Lambda_j(t) \ .$$

Next we define the convolution on matrix-valued functions as

$$(K*L)_{ij} = \sum_{K=1}^{m} K_{iK} * L_{Kj} \ ,$$

and

$$K^{\circ} = I, \ K^{(N)} = K^{(N-1)} * K, \ (I - K)^{(-1)} = \sum_{n=0}^{\infty} K^{(N)}.$$

Theorem 2

Given $P = \{P_{ij}\}$, $Q = \{Q_{ij}\}$, $H = \{\delta_{ij}\Lambda_i\}$ and their L - S transforms Π, q, h, then the following relationship holds

$$P = (I - Q)^{-1} * (I - \Lambda), \quad \Pi = (I - q)^{-1}(I - h) \ .$$

For any square matrix (or matrix valued function) $A = (a_{ij})$, define the diagonal and off-diagonal parts of A by

$$dA = (\delta_{ij} A_{ij})$$

$$_{o}A = A - dA \ .$$

Theorem 3

As defined on $s \in (0,\infty)$,

$$g = q\Pi(d\Pi)^{-1}$$

As a result of this theorem, one can obtain the mean recurrence times $u = (u_{ij})$ from

$$u = \lim_{s \to 0} \frac{1}{s} [\mathbb{1} - g] \ ,$$

where $\mathbb{1}$ is an nxn matrix each entry being unity.

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