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STUDIES IN NONLINEAR MODELING V: NONLINEAR MODELING FUNCTIONS OF A SPECIAL TYPE

by

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FOREWORD

Linear modeling is the technique which is usually employed in scaling physical phenomena to dimensions feasible for laboratory experiments, and has found many applications in cases where existing theoretical and experimental methods are inadequate to describe the full scale behavior. It has been, and continues to be, an important tool in situations where a suitable physical model is easily constructed and the parameters of interest can be measured with laboratory equipment. Nevertheless, linear modeling is only a particular form of a more general type, and there are many instances where a valid scaling can only be obtained by modeling in a nonlinear manner. Theoretically at least, this is always true if the parameters of the object under investigation are functionally related to a linear dimension in other than a linear way. An example of this is provided by the surface conductivity, which is a nonlinear function of the wavelength, and if the physical behavior is not adequately described by treating the conductivity as infinite, a linear modeling process is inapplicable. Such a situation is of frequent occurrence in investigations of dielectric and plasma materials. Even in cases where linear modeling would not imply a fundamental error of this type it may be practically undesirable inasmuch as it leads to a reduction in the magnitude of the quantity to be measured to a level which is below the sensitivity of the available equipment. This can occur in attempts to measure the scattering from a low cross section target within the laboratory and if a nonlinear modeling process could be found which would replace the target by one which was more measurable it could be of considerable practical importance.

In 1959 an investigation of the application of nonlinear modeling to selected problems was initiated at the Radiation Laboratory under the sponsorship of the Air Force Cambridge Research Laboratories. The ultimate goal was the use of nonlinear modeling to transform by analytical methods a physical problem for which neither experimental nor theoretical techniques are appropriate into another problem which is more amenable to experiment or theory or both. The pursuit of this goal has involved many mathematical obstacles, and a variety of new analyses, but though a considerable amount of useful work has been completed, much remains to be done before a practical technique can be specified.

This report is the fifth of a series detailing the research carried out on nonlinear modeling at the Radiation Laboratory, and is the final one devoted to this topic as such under the present contract. The four earlier reports are:

1 "Studies in Radar Cross Sections XXXVII - Nonlinear Modeling of Maxwell's Equations," J. E. Belyea, R. D. Low and K. M. Siegel, Radiation Laboratory Report 2871-4-T, December 1959. Contract AF 19(604)-4993.

2 "Studies in Radar Cross Sections XLV - Studies in Nonlinear Modeling-II," J. E. Belyea, J. W. Crispin, Jr., R. D. Low, D. M. Raybin, R. K. Ritt, O. G. Ruehr and F. B. Sleator, Radiation Laboratory Report 2871-6-F, December 1960, Contract AF 19(604)-4993.

3 "Studies in Nonlinear Modeling III - On the Interaction of Electromagnetic Fields with Plasmas," K-M Chen, J. Meixner, D. L. Sengupta, R. E. Kleinman and O. G. Ruehr, Radiation Laboratory Report 4134-2-F, July 1961, Contract AF 19(604)-7428.

4 "Studies in Nonlinear Modeling IV - Far Field Scattering by Simple Shapes at Low Frequencies," R. E. Kleinman, R. D. Low, and F. B. Sleator, Radiation Laboratory Report 4405-1-F, February 1962, Contract AF 19(604)-8030.

These reports are concerned with attempts to develop and apply suitable nonlinear modeling techniques to particular scattering and propagation problems in electromagnetic theory. This work revealed a need for better understanding of the mathematics of nonlinear modeling before the techniques could be successfully utilized in problems of practical importance. The present report is concerned with filling this need and answering some of the underlying mathematical questions. More remain, however, and until these problems are solved the crucial question of whether or not the promise of nonlinear modeling will be fulfilled must remain unanswered.

Ralph E. Hiatt

ABSTRACT

This paper discusses the nonlinear modeling of three types of partial differential equations in n variables, elliptic, parabolic, and hyperbolic. The modeling functions are restricted to depend only on the (measured) dependent variable and not on the coordinates. For the scalar wave equation (elliptic) and the diffusion equation (parabolic) it is found that the allowable modeling functions must satisfy a particular n'th order nonlinear ordinary differential equation. A simple counter-example shows that similar restrictions do not hold for the time-dependent wave equation (hyperbolic). The sets of allowable modeling functions corresponding to the wave and diffusion equations are shown to be identical to those obtained by modeling certain second order linear ordinary differential equations. The problem of similitude restrictions is interpreted as the study of certain polynomials generated by Bürmann series expansions. The limiting behavior of these polynomials is obtained in special cases.

I. INTRODUCTION

The determination of physical quantities by means of laboratory measurements which involve the use of scaled models is justified by the mathematical description of the physical systems. Changes are made in other parameters to compensate for the size of the model so that the desired quantities are proportional to those measured in the laboratory. In some situations, however, these changes are not feasible. This led to a proposal (Ritt[1], Belyea, Low, Siegel[2]) to investigate the use of more general, i.e.nonlinear, transformations between the systems. These transformations came to be called nonlinear modeling functions and the activity of studying them, nonlinear modeling.

If one assumes that the actual and the laboratory systems are described exactly by equations which have unique solutions, the problem of modeling becomes a purely mathematical one. The essential question is the determination of restrictions under which a single-valued modeling function exists. Ritt[1] has discussed the philosophy of modeling and proposed the term "similitude conditions" for these restrictions. He also pointed out that the computation of a modeling function may be, in general, as difficult or more difficult than a solution of the original problem. This has been found to be the case in most of the situations studied thus far and has led to considerable pessimism among investigators in this field. On the other hand, some of the modeling functions which have been determined, while they apply to systems whose solutions are well known, are simpler in form compared to these solutions. The writer takes a somewhat different view. The problem of modeling, because of its novelty, may stimulate new questions and new approaches which will be of interest in other fields as well.

Belyea, Low, and Siegel[2] discussed the modeling of Helmholtz equations

$$\nabla^2 \phi + k^2 \phi = 0 \qquad \text{(I-1)}$$

$$\nabla^2\psi + \ell^2\psi = 0 \qquad \text{(I-2)}$$

with different wave numbers k and ℓ. They assumed on physical grounds that the modeling function should depend only on the field

$$\phi = \Phi(\psi) \qquad \text{(I-3)}$$

and not on the spatial coordinates. The problem is of physical interest because it relates to the determination of radar cross sections of large objects using relatively small laboratory models at the same frequency. Linear modeling would require a higher frequency in the laboratory so that k and ℓ could be equal. The one dimensional case is particularly interesting since it yields some of the simple modeling functions mentioned above. For example, if

$$\phi = e^{ikx} \qquad \text{(I-4)}$$

and

$$\psi = e^{i\ell x}, \qquad \text{(I-5)}$$

we have the modeling function

$$\phi = \Phi(\psi) = \psi^{k/\ell} \qquad \text{(I-6)}$$

obtained by eliminating x between (I-4) and (I-5). As pointed out by Ritt[1], it is necessary to restrict k and ℓ so that k/ℓ is an integer in order that $\Phi(\psi)$ be a single valued function. Similarly, for solutions

$$\phi = \cos kx \qquad \text{(I-7)}$$

$$\psi = \cos\ell x \qquad \text{(I-8)}$$

we obtain

$$\Phi = \cos\left[k/\ell \ \cos^{-1}\psi \right]. \qquad \text{(I-9)}$$

Here again it is necessary to impose $k/\ell = m$, an integer. The result is a Tschebyscheff polynomial, $T_m(\psi)$.

In general, of course, we will not have expressions for the solutions, ϕ and ψ, and the elimination technique is not possible. One approach is to attempt to find differential equations satisfied by the modeling functions. This was done by Belyea, Low and Siegel[2] for the Helmholtz equation in three dimensions under the assumption $\phi = \Phi(\psi)$. Although their solution was incorrect because of an obscure error, (equation (61) of Ref. 2), their assumption that $\Phi(\psi)$ must satisfy an ordinary differential equation was correct. The determination of this differential equation for the Helmholtz equation in n dimensions is given in Chapter II. A change of space variables

$$\vec{x} \longrightarrow \ell \vec{x} \tag{I-10}$$

and a redefinition of k allow us to suppress ℓ. That is, we need only consider

$$\nabla^2 \phi + k^2 \phi = 0 \tag{I-11}$$

$$\nabla^2 \psi + \psi = 0 \tag{I-12}$$

$$\phi = \Phi(\psi) \tag{I-13}$$

In Theorem 1 of Chapter II we find that $\Phi(\psi)$ satisfies

$$f(\psi) \frac{d^2\Phi}{d\psi^2} - \psi \frac{d\Phi}{d\psi} + k^2 \Phi = 0 \tag{I-14}$$

$$f(\psi) = (\nabla \psi)^2 = \sum_{i=1}^{n} \left(\frac{\partial \psi}{\partial x_i} \right)^2 \tag{I-15}$$

What Eq. (I-15) says is that the square of the gradient of ψ must depend upon ψ only (and not on spatial coordinates). This apparently is a severe restriction which stems from the assumption (I-13). In the one dimensional case we easily obtain $f(\psi)$:

$$f(\psi) = \psi_x^2 \qquad \frac{df}{d\psi} = \frac{df}{dx}\Big/\frac{d\psi}{dx} = 2\psi_{xx} = -2\psi \tag{I-16}$$

The last step is from (I-12). Integrating (I-16) and substituting in (I-14) yields

$$(c-\psi^2)\frac{d^2\Phi}{d\psi^2} - \psi\frac{d\Phi}{d\psi} + k^2\Phi = 0 \tag{I-17}$$

an equation previously obtained in Ref. 2. In higher dimensions (I-12) and (I-15) are successively differentiated until all spatial variables are eliminated. The following recursively defined differential equation for $f(\psi)$ results:

$$\beta_1 = f' + 2\psi \tag{I-18}$$

$$\beta_i = 2f\beta'_{i-1} - (if' + 2\psi)\beta_{i-1} \qquad i=2, \ldots, n \tag{I-19}$$

$$\beta_n = 0 \tag{I-20}$$

A similar, but more difficult, analysis is carried out in Chapter III for the pair of diffusion equations

$$\nabla^2\phi = k^2\frac{\partial\phi}{\partial t} \tag{I-21}$$

$$\nabla^2\psi = \frac{\partial\psi}{\partial t} \tag{I-22}$$

again under the assumption

$$\phi = \Phi(\psi) \; . \tag{I-23}$$

The results parallel those above. That is, we again obtain a differential equation which the modeling function must satisfy. It is to be emphasized, of course, that the manipulations are formal in that sufficient differentiability is assumed.

The attempt to find similar restrictions for the time dependent wave equation must fail. A simple counter-example shows that solutions of

$$\nabla^2 \phi - k^2 \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad \text{(I-24)}$$

and

$$\nabla^2 \psi - \frac{\partial^2 \psi}{\partial t^2} = 0 \qquad \text{(I-25)}$$

exist for which a general set of modeling functions of the form $\phi = \Phi(\psi)$ can be obtained which satisfy no particular differential equation. Simply take

$$\phi = g(kx - t) \qquad \text{(I-26)}$$

$$\psi = kx - t \qquad \text{(I-27)}$$

for any twice differentiable function g. It is well known that (I-26) satisfies (I-24). Clearly (I-27) is a trivial solution of (I-25). We obtain immediately from (I-26) and (I-27)

$$\phi = \Phi(\psi) = g(\psi) \ . \qquad \text{(I-28)}$$

Thus we see that, because of the generality of the functions $g(\psi)$, no particular restrictions of the type found for the Helmholtz and diffusion equations can exist. This holds in n dimensions since (I-26) and (I-27) trivially satisfy (I-24) and (I-25) in any number of independent variables. The natural speculation is that the restriction to modeling functions $\phi = \Phi(\psi)$ is more reasonable for the time dependent scalar wave equation than for the other equations. In a private communication, H. Yilmaz, who has done considerable work in general relativity[3], concurred in the stipulation, $\phi = \Phi(\psi)$, but suggested a relativistic approach to the problem of modeling. In this

connection it is important to note that the restrictions obtained in Chapters II and III, while independent of coordinates, depend upon the possibility of introducing a rectangular coordinate system.

Having obtained a set of restrictive differential equations for the allowable modeling functions, we continue to investigate their properties in Chapter IV. A parametric reduction (in a sense, a uniformization) of equations (I-18) - (I-20) above is obtained which facilitates the study of their solutions. It is shown that the set of allowable modeling functions connecting Helmholtz equations (I-1) and (I-2) in n independent variables under the restriction (I-3) is the same as that obtained from the modeling of ordinary differential equations:

$$\frac{d^2\phi}{dt^2} + \frac{d\phi}{dt}\left[\frac{1}{t-c_1} + \ldots + \frac{1}{t-c_{n-1}}\right] + k^2\phi = 0 \qquad \text{(I-29)}$$

$$\frac{d^2\psi}{dt^2} + \frac{d\psi}{dt}\left[\frac{1}{t-c_1} + \ldots + \frac{1}{t-c_{n-1}}\right] + \psi = 0 \qquad \text{(I-30)}$$

where $c_1, c_2, \ldots c_{n-1}$ are arbitrary constants. Some interesting structural properties in the set of modeling functions are obtained by considering multiplicities of the singularities c_i. Similar results are found corresponding to the diffusion equation.

Thus far we have concerned ourselves entirely with restrictions in modeling which derive from the assumption of a particular form, $\phi = \Phi(\psi)$, of modeling function and are obtained from the equations being modeled without regard to boundary conditions. In Chapter V we begin the study of the particular subcase of (I-29) and (I-30) above obtained by taking the arbitrary constants c_i to be zero. Boundary conditions are assigned which specify unique solutions for the equations being modeled. Local modeling functions are then defined and expressed as power series

whose coefficients are polynomials, $P_j^{(n)}(k^2)$, in the parameter k. The study of similitude conditions thus depends on the properties of these polynomials. It is found that the zeroes, $\mu_{ij}^{(n)}$, of $P_j^{(n)}(k^2)$ have limiting values as $j \to \infty$. This interesting result leads to a determination of $\lim_{j\to\infty} P_j^{(n)}(k^2)$, which in turn confines the allowable values of k for global extension to a discrete set of real numbers, evaluated in terms of the ratios of the zeroes of the derivatives of the functions being modeled.

In the final chapter the above question is re-examined in a more general setting. The problem of modeling a pair of functions $\{y = g(kt),\ x = g(t)\}$ is interpreted as the study of the Bürmann series for g(kt) in powers of g(t). It is found that the coefficients of this series are polynomials in k. A program is proposed for determining the class of functions g for which the corresponding polynomials have limiting properties of the type discovered in Chapter V. Although this problem is not solved, it is reduced to a study of certain inequalities on the coefficients of the series development of g(t) and its inverse. It is interesting to note that this question is closely related to the study of the limiting behavior of the Faber polynomials[4] for g^{-1}. To this extent the hope expressed earlier, that problems of modeling might stimulate new approaches in other fields, has been realized.

II. THE HELMHOLTZ EQUATION

Theorem 1. A nonlinear modeling function of the form $\phi = \Phi(\psi)$ connecting solutions ϕ and ψ of the Helmholtz equations

$$\nabla^2\phi + k^2\phi = 0 \tag{II-1}$$

$$\nabla^2\psi + \psi = 0 \tag{II-2}$$

must satisfy the differential equation

$$f(\psi)\Phi'' - \psi\Phi' + k^2\Phi = 0 \tag{II-3}$$

Moreover the square of the magnitude of the gradient of ψ must be a function of ψ only (the first coefficient above).

$$(\nabla\psi)^2 = \sum_{i=1}^{n}\left(\frac{\partial\psi^2}{\partial x_i}\right) = f(\psi) \tag{II-4}$$

Proof: Since $\phi = \Phi(\psi)$, we have:

$$\nabla\phi = \frac{d\Phi}{d\psi}\nabla\psi = \Phi'\nabla\psi \tag{II-5}$$

$$\nabla^2\phi = \Phi''\nabla\psi \cdot \nabla\psi + \Phi'\nabla^2\psi = \Phi''(\nabla\psi)^2 + \Phi'\nabla^2\psi \tag{II-6}$$

Substituting from the Helmholtz equations we obtain:

$$(\nabla\psi)^2\Phi'' - \psi\Phi' + k^2\Phi = 0 \tag{II-7}$$

Since each quantity in the above equation (except perhaps $(\nabla\psi)^2$) is a function of ψ only and since Φ'' does not vanish identically (nonlinear modeling) we conclude that

$$(\nabla\psi)^2 = f(\psi) \tag{II-8}$$

To characterize the modeling functions $\Phi(\psi)$ completely it remains to determine the functions $f(\psi)$. In the one dimensional case we have

$$f(\psi) = \psi_x^2 \tag{II-9}$$

$$\frac{df}{d\psi}\psi_x = 2\psi_x\psi_{xx} = -2\psi_x\psi \qquad (\text{since } \psi_{xx}+\psi = 0) \tag{II-10}$$

$$\frac{df}{d\psi} = -2\psi \tag{II-11}$$

$$f(\psi) = c-\psi^2 \tag{II-12}$$

In two dimensions:

$$f(\psi) = \psi_x^2 + \psi_y^2 \tag{II-13}$$

$$\psi_{xx}+\psi_{yy}+\psi = 0 \tag{II-14}$$

Differentiating $f(\psi)$ with respect to x and y twice yields:

$$\frac{f'\psi_x}{2} = \psi_x\psi_{xy} + \psi_y\psi_{yx} \tag{II-15}$$

$$\frac{f''}{2}\psi_x^2 + \frac{f'}{2}\psi_{xx} = \psi_{xx}^2 + \psi_x\psi_{xxx} + \psi_{yx}^2 + \psi_y\psi_{yxx} \tag{II-16}$$

$$\frac{f'\psi_y}{2} = \psi_x\psi_{xy} + \psi_y\psi_{yy} \tag{II-17}$$

$$\frac{f''}{2}\psi_y^2 + \frac{f'}{2}\psi_{yy} = \psi_{xy}^2 + \psi_x\psi_{xyy} + \psi_{yy}^2 + \psi_y\psi_{yyy} \tag{II-18}$$

Differentiate equation (II-14) with respect to x and to y:

$$\psi_{xxx} + \psi_{yyx} + \psi_x = 0 \qquad \text{(II-19)}$$

$$\psi_{xxy} + \psi_{yyy} + \psi_y = 0 \qquad \text{(II-20)}$$

Add (II-16) and (II-18) using the above and (II-13) to get:

$$\frac{ff''}{2} - \frac{\psi f'}{2} = \psi_{xx}^2 + \psi_{yy}^2 + 2\psi_{xy}^2 - f \qquad \text{(II-21)}$$

From (II-15) and (II-17) we have the determinant vanishing:

$$\begin{vmatrix} \psi_{xx} - \frac{f'}{2} & \psi_{yx} \\ \psi_{xy} & \psi_{yy} - \frac{f'}{2} \end{vmatrix} = 0 \qquad \text{(II-22)}$$

$$\psi_{xx}\psi_{yy} + \frac{\psi}{2} f' + \left(\frac{f'}{2}\right)^2 = \psi_{xy}^2 \qquad \text{(II-23)}$$

From (21) and (23) and the square of the Helmholtz equation we have:

$$\frac{ff''}{2} - \frac{\psi f'}{2} = -f + \psi f' + \frac{(f')^2}{2} + \psi^2 \qquad \text{(II-24)}$$

which is simplified to:

$$f(f'' + 2) = (f' + \psi)(f' + 2\psi) \qquad \text{(II-25)}$$

Again $f(\psi)$ is limited to the solutions of a particular ordinary diffential equation. We obtain similar results in higher dimensions.

Theorem 2. If $\psi = \psi(x_1, x_2, \ldots, x_n)$ is a solution of the Helmholtz equation

$$\nabla^2 \psi + \psi = 0 \tag{II-26}$$

and if also the square of the magnitude of the gradient of ψ is a function of x_i through ψ only,

$$(\nabla \psi)^2 = \sum_{i=1}^{n} \left(\frac{\partial \psi}{\partial x_i} \right)^2 = f(\psi) \tag{II-27}$$

then, necessarily, $f(\psi)$ must satisfy the following nonlinear n^{th} order ordinary differential equation.

$$\overline{\beta}_n = 0 \tag{II-28}$$

where

$$\overline{\beta}_1 = f' + 2\psi \tag{II-29}$$

$$\overline{\beta}_i = 2f\overline{\beta}'_{i-1} - (if' + 2\psi)\beta_{i-1} \tag{II-30}$$

$$i = 2, \ldots, n$$

(Note: The apparently superfluous bars on the β's obviate the necessity for redefinitions as will be seen in the proof).

Before proceeding to the proof of theorem 2, we note that for $n = 1$ we obtain

$$\overline{\beta}_1 = f' + 2\psi = 0 \tag{II-31}$$

which agrees with (II-11). For $n = 2$ we compute:

$$\overline{\beta}_2 = 2f\overline{\beta}'_1 - (2f' + 2\psi)\overline{\beta}_1 = 2f(f'' + 2) - 2(f' + \psi)(f' + 2\psi) = 0 \tag{II-32}$$

Equation (II-32), when divided by 2, yields the previous result (II-25). To indicate the complexity of the equation for f, described recursively by (II-28)-(II-30), we write the result for $n = 3$

$$f^2f''' - \frac{5}{2}\,ff'f'' + \frac{11}{2}\,\psi(f')^2 - 4\psi ff'' + \frac{3}{2}\,(f')^3 - 4ff' + 6\psi^2 f' - 6\psi f + 2\psi^3 = 0 \qquad \text{(II-33)}$$

Proof: Define nxn matrices, A and B, of partial derivatives of ψ:

$$A = \frac{\partial\psi}{\partial x_i}\,\frac{\partial\psi}{\partial x_j} = \psi_i\psi_j \qquad \text{(II-34)}$$

$$B = \frac{\partial^2\psi}{\partial x_i\,\partial x_j} = \psi_{ij} \qquad \text{(II-35)}$$

Let the trace (tensor contraction) of a matrix be denoted by a bar and use the repeated index summation convention. From (II-26) and (II-27) we see that the traces of A and B are functions of ψ only:

$$\overline{A} = \psi_i\psi_i = f(\psi) \qquad \text{(II-36)}$$

$$\overline{B} = \psi_{ii} = -\psi \qquad \text{(II-37)}$$

Define an n^{th} degree polynomial form in B, $C_i(B)$, recursively:

$$C_i(B) = B^i - \sum_{j=1}^{i} \frac{(\overline{BC_{j-1}})}{j}\,B^{i-j} \qquad j = 1, \ldots, i \qquad \text{(II-38)}$$

$$C_o = I$$

Lemma 1. If B is an ixi matrix, then the form $C_i(B)$ is its characteristic polynomial. Proof: Appendix A.

Since we have assumed that B is nxn, it follows from the Cayley-Hamilton theorem and lemma 1 that

$$C_n(B) \equiv 0 \qquad \text{(II-39)}$$

In terms of $C_i(B)$ we define matrix polynomials β_i whose traces yield equations (II-28)-(II-30).

$$\beta_i = \frac{(-1)^{i+1} 2^i i!\, AC_i}{\overline{A}} \tag{II-40}$$

From lemma 1 it is clear that

$$\beta_n \equiv 0 \tag{II-41}$$

Hence, trivially

$$\overline{\beta}_n = 0 \tag{II-42}$$

This establishes (II-28). To get (II-29) we write out $\overline{\beta}_1$:

$$\beta_1 = \frac{2AC_1}{\overline{A}} = \frac{2A}{\overline{A}}\left[B - \overline{B}I\right] \tag{II-43}$$

$$\overline{\beta}_1 = \frac{2}{\overline{A}}\,\overline{(AB)} - 2\overline{B} \tag{II-44}$$

To evaluate $\overline{(AB)}$ we differentiate $f(\psi)$:

$$f = \psi_i \psi_i \tag{II-45}$$

$$f'\psi_i = 2\psi_i \psi_{ij} \tag{II-46}$$

Multiply by ψ_k and interpret using the definitions of A and B

$$f'\psi_i\psi_j = 2\psi_k \psi_i \psi_{ij} \tag{II-47}$$

$$f'A = 2AB \tag{II-48}$$

Take the trace of (II-48)

$$f'\overline{A} = 2\overline{(AB)} \tag{II-49}$$

Recalling from (II-37) that $\overline{B} = -\psi$ and using (II-49) we find from (II-44) that

$$\overline{\beta}_1 = f' + 2\psi \tag{II-50}$$

This is just equation (II-29). Equation (II-30) remains to be proved and, being essentially the inductive step, is more difficult. Introduce the linear differential operator L:

$$L \equiv \frac{\psi_i}{\overline{A}} \frac{\partial}{\partial x_i} \tag{II-51}$$

L operates on scalars and on matrices (each component). For a scalar function of ψ only, L becomes just the derivative. Lemma 2 summarizes some of the necessary properties of L.

Lemma 2.

i. $g = g(\psi) \Longrightarrow L(g) = \frac{dg}{d\psi}$ (II-52)

ii. P, Q arbitrary matrices or scalars

$$L(PQ) = P(LQ) + (LP)Q \tag{II-53}$$

iii. trace $L(P) = L(\text{trace } P)$, i.e.

$$\overline{L(P)} = L(\overline{P}) \tag{II-54}$$

iv. $L(A) = \frac{AB + BA}{\overline{A}}$ (II-55)

v. $L(B) = \frac{1}{2\overline{A}} \left[f'' A + f' B - 2B^2 \right]$ (II-56)

vi. $L(\overline{A}) = f'$ (II-57)

vii. $L(\overline{B}) = -1$ (II-58)

Proof: Appendix A

Lemmas 3 and 4 deal with the effect of L on polynomial forms related to the β's.

Lemma 3.

$$f\,L(\overline{BC_{i-1}}) = i\left[\frac{f''}{2}\,(\overline{AC_{i-1}}) + \frac{f'}{2}\,(\overline{BC_{i-1}}) - (\overline{B^2C_{i-1}})\right] \tag{II-59}$$

Proof: Appendix A.

Lemma 4.

$$2f\,L(\overline{AC_i}) = \left[(i+3)\,f' + 2\psi\,(\overline{AC_i}) - 2(i+1)\,(\overline{AC_{i+1}})\right] \tag{II-60}$$

Proof: Appendix A.

Lemma 5.

$$\overline{\beta_i} = 2f\,L(\overline{\beta_{i-1}}) - (if' + 2\psi)\,(\overline{\beta_{i-1}}) \tag{II-61}$$

Proof: By definition

$$\beta_{i-1} = \frac{(-1)^i 2^{i-1}(i-1)!\;AC_{i-1}}{\bar{A}} \tag{II-62}$$

Operate on (II-62) with L and trace:

$$L(\overline{\beta_{i-1}}) = (-1)^i 2^{i-1}(i-1)!\left[\frac{L(\overline{AC_{i-1}})}{\bar{A}} - \frac{f'}{\bar{A}^2}\,(\overline{AC_{i-1}})\right] \tag{II-63}$$

From (II-63) and the definition of β_i

$$\overline{\beta_i} - 2f\,L(\overline{\beta_{i-1}}) + \overline{\beta_{i-1}}\,(if' + 2\psi) =$$

$$\frac{(-1)^{i+1}2^i i!\,(\overline{AC_i})}{\bar{A}} + 2(-1)^{i+1}2^{i-1}(i-1)!\left[L(\overline{AC_{i-1}}) - \frac{f'}{\bar{A}}\,(\overline{AC_{i-1}})\right]$$

$$- (if' + 2\psi)(-1)^{i+1}(i-1)!\;2^{i-1}\,\frac{(\overline{AC_{i-1}})}{\bar{A}} =$$

$$\frac{(-1)^{i+1}2^{i-1}(i-1)!}{\bar{A}}\left[2i\,(\overline{AC_i}) + 2f\,L(\overline{AC_{i-1}}) - (i+2)f' + 2\psi\,(\overline{AC_{i-1}})\right] \tag{II-64}$$

The last bracket is zero by lemma 4.

Finally, we observe that from lemma 5 all the $\overline{\beta}_i$ are functions of ψ only. We have seen (II-31) that $\overline{\beta}_1$ is a function of ψ only. Hence, $L(\overline{\beta}_1)$ is $\frac{d\overline{\beta}_1}{d\psi}$ which is a function of ψ only. By (II-61) β_2 is a function of ψ only. Proceeding by induction we have that all the $\overline{\beta}_i$ are such functions. We can replace $L(\overline{B_{i-1}})$ with $\overline{\beta}'_{i-1}$ in (II-61), and equation (II-30) results. Thus we have proved theorem 2.

III. THE DIFFUSION EQUATION

Theorem 1. A nonlinear modeling function of the form $\phi = \Phi(\psi)$ connecting solutions ϕ and ψ of the diffusion equations

$$\nabla^2 \phi = k^2 \frac{\partial \phi}{\partial t} \tag{III-1}$$

$$\nabla^2 \psi = \frac{\partial \psi}{\partial t} \tag{III-2}$$

must satisfy the ordinary differential equation

$$f(\psi)\, \Phi'' = (k^2 - 1)\, \Phi' \tag{III-3}$$

where

$$f(\psi) = \frac{(\nabla \psi)^2}{\psi_t} \tag{III-4}$$

Proof: Since

$$\phi = \Phi(\psi), \tag{III-5}$$

$$\nabla \phi = \Phi' \nabla \psi \tag{III-6}$$

$$\nabla^2 \phi = \Phi''(\nabla \psi)^2 + \Phi' \nabla^2 \psi \tag{III-7}$$

$$\phi_t = \Phi' \psi_t \tag{III-8}$$

From the above and the diffusion equations (III-11) and (III-2) we obtain

$$k^2 \Phi' \psi_t = \Phi''(\nabla \psi)^2 + \Phi' \psi_t \tag{III-9}$$

$$\frac{(\nabla \psi)^2}{\psi_t}\, \Phi'' = (k^2 - 1)\, \Phi' \tag{III-10}$$

Since we have assumed that $\Phi(\psi)$ is not a linear function of ψ, $\Phi'' \not\equiv 0$.

$$\frac{(\nabla \psi)^2}{\psi_t} = \frac{(k^2 - 1)\, \Phi'}{\Phi''} \tag{III-11}$$

Since the right side of (III-11) depends on ψ only, the left side must also. We define this combination, $f(\psi)$

$$f(\psi) = \frac{(\nabla\psi)^2}{\psi_t} \tag{III-12}$$

Equations (III-10) and (III-12) give the conclusions of the theorem.

We find that $f(\psi)$ must satisfy an n^{th} order nonlinear ordinary differential equation, where n is the number of independent variables on which ψ depends.

Theorem 2. If $\psi = \psi(x_1, x_2, \ldots, x_n, t)$ is a solution of the differential equation

$$\nabla^2\psi = \psi_t \tag{III-13}$$

and if the combination $\frac{(\nabla\psi)^2}{\psi_t}$ is a function of ψ only,

$$f(\psi) = \frac{(\nabla\psi)^2}{\psi_t} \quad , \tag{III-14}$$

then $f(\psi)$ must satisfy the ordinary differential equation

$$\beta_n = 0 \tag{III-15}$$

where

$$\beta_o = f' - 1 \tag{III-16}$$

$$2i\ \beta_i = f\beta'_{i-1} - 2\beta_o\beta_{i-1} \qquad i=1, 2, \ldots, n \tag{III-17}$$

Proof: Use indices to denote partial derivatives with respect to "space" variables x_i. We reserve the subscript t for derivations with respect to t. For example

$$\psi_i = \frac{\partial\psi}{\partial x_i}$$

$$\psi_{ij} = \frac{\partial^2\psi}{\partial x_i\,\partial x_j} \quad , \qquad \psi_{it} = \frac{\partial^2\psi}{\partial x_i\partial t} \tag{III-18}$$

We also use the repeated index summation convention (for indices other than t).

Multiply equation (III-14) by ψ_t.

$$f\psi_t = \psi_i\psi_i \tag{III-19}$$

The diffusion equation is written with index notation

$$\psi_t = \psi_{ii} \tag{III-20}$$

Differentiate (III-19) with respect to x_k and with respect to t:

$$f'\psi_t\psi_k + f\psi_{tk} = 2\psi_i\psi_{ik} \tag{III-21}$$

$$f'\psi_t^2 + f\psi_{tt} = 2\psi_i\psi_{it} \tag{III-22}$$

Multiply (III-21) by ψ_j and rearrange:

$$f\psi_j\psi_{tk} = 2\psi_j\psi_i\psi_{ik} - f'\psi_t\psi_j\psi_k \tag{III-23}$$

Contract (III-23) with respect to j and k.

$$f'\psi_j\psi_{tj} = 2\psi_j\psi_i\psi_{ij} - f'\psi_t\psi_j\psi_j \tag{III-24}$$

Differentiate (III-21) with respect to x_j:

$$f''\psi_t\psi_k\psi_j + f'\psi_t\psi_{kj} + f'\psi_{tj}\psi_k + f'\psi_j\psi_{tk} + f\psi_{tkj} = 2\psi_{ij}\psi_{ik} + 2\psi_i\psi_{ikj} \tag{III-25}$$

Contract (III-25) with respect to k and j:

$$f''\psi_t\psi_j^2 + f'\psi_t\psi_{jj} + f'\psi_{tj}\psi_j + f'\psi_j\psi_{tj} + f\psi_{tjj} = 2\psi_{ij}^2 + 2\psi_i\psi_{ijj} \tag{III-26}$$

Equation (III-26) is simplified using (III-19), (III-20), (III-22), and (III-24):

$$ff''\psi_t^2 + \frac{2f'}{f}\left[2\psi_j\psi_i\psi_{ij} - f'f\psi_t^2\right] = 2\psi_{ij}^2 \tag{III-27}$$

Suppose, for a moment, that the problem is one dimensional. Then all derivatives are with respect to x in equation (III-27), and we have

$$\psi_j\psi_i\psi_{ij} = \psi_x^2\psi_{xx} = f\psi_t^2 \tag{III-28}$$

$$\psi_{ij}^2 = \psi_{xx}^2 = \psi_t^2 \tag{III-29}$$

Using (III-28) and (III-29) and dividing (III-27) by ψ_t^2, we obtain

$$ff'' + 2f'\left[2 - f'\right] = 2 \tag{III-30}$$

$$ff'' = 2(f' - 1)^2 \tag{III-31}$$

Comparing these results with the general expression (III-15), (III-16), (III-17), we get agreement:

$$\beta_o = f' - 1 \tag{III-32}$$

$$\beta_1 = \frac{1}{2}\left[ff'' - 2(f' - 1)^2\right] \tag{III-33}$$

$$\beta_1 = 0. \tag{III-34}$$

We return to the proof in n dimensions. Matrices are used to simplify the notation and facilitate the elimination of the independent variables x_i and t.

Introduce matrices A and B of partial derivatives of ψ:

$$A = \frac{\partial\psi}{\partial x_i} \quad \frac{\partial\psi}{\partial x_j} = \psi_i\psi_j \tag{III-35}$$

$$B = \frac{\partial^2\psi}{\partial x_i\, \partial x_j} = \psi_{ij} \tag{III-36}$$

Note that A and B are symmetric. Denote the trace of a matrix with a bar

$$\text{trace } C = \overline{C} = \sum_i C_{ii} = C_{ii} \ . \tag{III-37}$$

Equations (III-19), (III-20), (III-22), and (III-25), are rewritten in terms of A and B.

$$f\psi_t = \overline{A} \tag{III-38}$$

$$\psi_t = \overline{B} \tag{III-39}$$

$$f\psi_j\psi_{tk} = 2AB - f'\overline{B}A \tag{III-40}$$

$$2\psi_i\psi_{ijk} - f\psi_{tjk} = -2B^2 + f''\overline{B}A + f'\overline{B}B + \frac{2f'}{f}\,2AB - f'\overline{B}A \tag{III-41}$$

Define the linear differential operator L.

$$L = \frac{1}{\overline{A}}\left\{2\psi_k \frac{\partial}{\partial x_k} - f\frac{\partial}{\partial t}\right\} \tag{III-42}$$

L operates on scalar and matrix functions of x_i and t. The important properties of L are summarized in lemma 1.

Lemma 1.

i. g is a scalar function of ψ only $\Longrightarrow L(g) = \frac{dg}{d\psi}$ (III-43)

ii. $\overline{L(C)} = L(\overline{C})$ for any matrix C. (III-44)

iii. $L(PQ) = P(LQ) + (LP)Q$ (III-45)

P, Q are arbitrary matrices or scalars.

iv. $L(A) = \frac{2f'}{f}A$ (III-46)

v. $L(B) = \left[\frac{f''}{f} - 2\frac{f'}{f}\right]^2 A + \frac{f'}{f}B + \frac{4f'}{f}\frac{AB}{\overline{A}} - \frac{2B^2}{\overline{A}}$ (III-47)

vi. $L(\overline{B}) = \frac{f'}{f}\overline{B}$. (III-48)

Proof: Appendix B.

Define recursively the following normalized polynomial forms in A and B.

$$\delta_o = \frac{A}{\overline{A}} \tag{III-49}$$

$$\eta_o = \frac{B}{\overline{B}} \tag{III-50}$$

$$\eta_i = (-1)^i \left[\eta_o^{i+1} + \sum_{j=1}^{i} \frac{(-1)^j}{j} (\overline{\eta_{j-1}}) \eta_o^{i-j+1} \right] \quad i=1,2,\ldots \qquad \text{(III-51)}$$

$$\delta_i = (-1)^i \left[\delta_o \eta_o^i + \sum_{j=1}^{i} \frac{(-1)^j}{j} (\overline{\eta_{j-1}}) \delta_o \eta_o^{i-j} \right] \quad i=1,2,\ldots \qquad \text{(III-52)}$$

Lemma 2. If A and B are nxn matrices, then

$$\eta_n(B) \equiv 0 \qquad \delta_n(A) \equiv 0 \qquad \text{(III-53)}$$

Proof: Appendix B.

The following useful properties are derived directly from equations (III-51) and (III-52).

Lemma 3.

$$\delta_i = \frac{\delta_o(\overline{\eta_{i-1}})}{i} - \delta_o \eta_{i-1} \qquad i=1,2,\ldots \qquad \text{(III-54)}$$

$$\eta_i = \frac{\eta_o(\overline{\eta_{i-1}})}{i} - \eta_o \eta_{i-1} \qquad i=1,2,\ldots \qquad \text{(III-55)}$$

Proof: Appendix B.

We are now ready to define matrix polynomials whose traces are directly related to the ordinary differential equations we seek.

Define:

$$\beta_i = (i+1)(\overline{\beta}_o + 1)\ \delta_i - \eta_i \qquad i=0,1,\ldots \qquad \text{(III-56)}$$

$$\overline{\beta}_o = f'' - 1 \text{ (a scalar function of } \psi). \qquad \text{(III-57)}$$

It should be noted that (III-56) and (III-57) are consistent, since $\overline{\delta}_o = \overline{\eta}_o = 1$. To establish the result we have claimed it will be necessary to operate on β_i (and hence

δ_i and η_i) with L. This is the difficult part of the discussion particularly because of the length of the proofs. We break this up into a series of relatively simple lemmas whose proofs are left to Appendix B.

Lemma 4.

i. $$f L(\eta_o) = \epsilon \left[\delta_o \eta_o - \delta_o(\overline{\delta_o \eta_o})\right] + 2\delta_o(\overline{\eta_o^2}) - 2\eta_o^2 \qquad \text{(III-58)}$$

ii. $$f L(\eta_o^i) = \sum_{k=1}^{i} \eta_o^{i-k} (f L\eta_o) \eta_o^{k-1} \qquad i=1,2,\ldots \qquad \text{(III-59)}$$

$$\epsilon = 4(\overline{\beta}_o + 1) \text{ a scalar function of } \psi \qquad \text{(III-60)}$$

(This redundant notation is used to emphasize the scalar nature of ϵ.)

Proof: Appendix B. Note in (ii) the importance of the order of multiplication since η_o and $f L\eta_o$ do not commute.

Lemma 5.

$$L(\eta_j) = -(j+1)(\eta_{j-1} L\eta_o) \qquad j=1,2,\ldots \qquad \text{(III-61)}$$

Proof: Appendix B.

Lemma 6.

Let A be as defined previously and C and D be arbitrary nxn matrices. Then

$$(\overline{ACAD}) = (\overline{AC})(\overline{AD}) \qquad \text{(III-62)}$$

Proof: Appendix B.

Lemma 7.

$$f L(\overline{\delta}_i) = 2\left[\overline{\eta}_i - \overline{\delta}_i - i(\overline{\delta_o \eta_i})\right] \quad i=0,1,\ldots \qquad \text{(III-63)}$$

Proof: Appendix B. This is the most difficult and tedious of the lemmas to prove. As is discussed in the Appendix, the following lemma results rather directly from lemma 7.

Lemma 8.

$$f L(\bar{\delta}_i) = 2(i+1)\bar{\delta}_i - \frac{2}{i}(\overline{\eta_{i-1}}) - 4(\overline{\delta_o \eta_o \eta_{i-2}}) + 4(\overline{\delta_o \eta_o}) - 4\delta_i(\overline{\delta_{i-1}}) + 2(\overline{\delta_{i-1}}) \tag{III-64}$$

Proof: Appendix B.

Lemma 9.

$$f L(\bar{\beta}_o) = 2\bar{\beta}_1 + 2(\bar{\beta}_o)^2 \tag{III-65}$$

Proof: Appendix B.

Lemma 10.

$$2i\bar{\beta}_i = f L(\overline{\beta_{i-1}}) - 2\beta_o(\overline{\beta_{i-1}}) \tag{III-66}$$

Proof: Appendix B.

Lemma 10 is what we have been aiming at, the differential properties of the polynomials, β_i, whose traces yield a chain of differential forms defining the auxiliary function, f. We restate the conclusions of the theorem for convenience:

i. $\bar{\beta}_i$ is a scalar function of ψ only. $i=0, 1, \ldots$ (III-67)

ii. $$2i\overline{\beta_i} = f\frac{d\bar{\beta}_{i-1}}{d\psi} - 2\bar{\beta}_o(\overline{\beta_{i-1}}) \qquad i=1, 2, \ldots \tag{III-68}$$

iii. $\bar{\beta}_n = 0$. (III-69)

Proof: We proceed by finite induction. $i = 0$. $\bar{\beta}_o$ is a scalar function of ψ only, by definition (III-57). Assume (i) for $i = j-1$. Then by lemma 10 and lemma 1, part (i), we have

$$2j\bar{\beta}_j = f\frac{d\bar{\beta}_{j-1}}{d\psi} - 2\bar{\beta}_o(\overline{\beta_{j-1}}) \tag{III-70}$$

(L operating on a scalar function of ψ is just its derivative). By equation (III-70) $\bar{\beta}_j$ is also a scalar function of ψ, which completes the induction for part (i). Equation (III-67) also establishes part (ii). That $\bar{\beta}_n = 0$ follows immediately from the definition of β_n (equation (III-56) and lemma 2.)

IV. PARAMETRIC REDUCTION AND STRUCTURE

We have obtained, in the study of the Helmholtz and diffusion equations, certain classes of auxiliary functions $f(\psi)$, defined by n^{th} order nonlinear ordinary differential equations, which characterize the possible nonlinear modeling functions of the form $\phi = \Phi(\psi)$. In this chapter we obtain an equivalent characterization in terms of a pair of second order linear ordinary differential equations whose coefficients depend on n, the dimension of the original system. This reduction has a simple and useful interpretation: the set, S, of nonlinear modeling functions obtainable under the restriction $\phi = \Phi(\psi)$ from the partial differential equations is identical with the set of modeling functions connecting a pair of ordinary differential equations. The reduction facilitates the discovering of certain structural relations on S.

We begin by reducing the defining equations of the auxiliary function, $f(\psi)$, for the Helmholtz equation, to parametric form.

Theorem 1. The differential equation for $f(\psi)$ given inductively by the system H:

$$\beta_1 = f' + 2\psi \qquad \text{(IV-1)}$$

$$\text{(H)} \qquad \beta_i = 2f\beta'_{i-1} - (if' + 2\psi)\beta_{i-1} \qquad \text{((IV-2)}$$

$$\beta_n = 0 \qquad \text{(IV-3)}$$

has the parametric solution

$$\psi = \psi(t) \qquad \text{(IV-4)}$$

$$f = f(t) = \left(\frac{d\psi}{dt}\right)^2 \qquad \text{(IV-5)}$$

for solutions, $\psi(t)$, of

$$\frac{d^2\psi}{dt^2} + \frac{d\psi}{dt}\left[\frac{1}{t-c_1} + \ldots + \frac{1}{t-c_{n-1}}\right] + \psi = 0 \qquad n \geqslant 2 \qquad \text{(IV-6)}$$

$$\frac{d^2\psi}{dt^2} + \psi = 0 \qquad n = 1 \qquad \text{(IV-7)}$$

where $c_1, c_2, \ldots, c_{n-1}$ are arbitrary constants.

Proof: Define a polynomial, $s(t)$

$$s(t) = \prod_{i-1}^{n-1} (t - c_i) \qquad \text{(IV-8)}$$

Assume $n \geqslant 2$. Then

$$\frac{\dot{s}}{s} = \frac{1}{t - c_1} + \ldots + \frac{1}{t - c_{n-1}} \qquad \text{(IV-9)}$$

Since, by (IV-5)

$$f = \dot{\psi}^2 \qquad \text{(IV-10)}$$

$$\frac{df}{d\psi} = \frac{1}{\dot{\psi}} \frac{df}{dt} = 2\ddot{\psi} \qquad \text{(IV-11)}$$

From equations (IV-1) and (IV-6)

$$\beta_1 = f' + 2\psi = 2(\ddot{\psi} + \psi) = -2\dot{\psi} \frac{\dot{s}}{s} \qquad \text{(IV-12)}$$

Compute $\beta_2(t)$ from (IV-1) and (IV-2)

$$\beta_2 = 2f(f'' + 2) - 2(f' + \psi)(f' + 2\psi) \qquad \text{(IV-13)}$$

$$\beta_2 = -4\dot{\psi}\ddot{\psi}\frac{\dot{s}}{s} - 4\dot{\psi}^2 \frac{\ddot{s}}{s} + 4\dot{\psi}^2 \left(\frac{\dot{s}}{s}\right)^2 + 4\dot{\psi} \frac{\dot{s}}{s} \ddot{\psi} - 4\dot{\psi}^2 \left(\frac{\dot{s}}{s}\right)^2 \qquad \text{(IV-14)}$$

$$\beta_2 = -4\dot{\psi}^2 \frac{\ddot{s}}{s} \qquad \text{(IV-15)}$$

Inductively, take

$$\beta_{i-1} = -\frac{(2\dot{\psi})^{i-1}}{s} \frac{d^{i-1}s}{dt^{i-1}} \qquad \text{(IV-16)}$$

From (IV-16) and (IV-2)

$$\beta_i = 2\dot{\psi}\left[-2(i-1)(2\dot{\psi})^{i-2}\frac{\ddot{\psi}}{s}\frac{d^{i-1}s}{dt^{i-1}} - \frac{(2\dot{\psi})^{i-1}}{s}\frac{d^{i}s}{dt^{i}}\right.$$

$$\left. + \frac{(2\dot{\psi})^{i-1}}{s^2}\frac{ds}{dt}\frac{d^{i-1}s}{dt^{i-1}} + \frac{(2i\ddot{\psi}+2\psi)(2\dot{\psi})^{i-1}}{s}\frac{d^{i-1}s}{dt^{i-1}}\right] \qquad \text{(IV-17)}$$

$$\beta_i = (2\dot{\psi})^{i-1}\frac{(2\ddot{\psi}+2\psi)}{s}\frac{d^{i-1}s}{dt^{i-1}} - \frac{(2\dot{\psi})^{i}}{s}\frac{d^{i}s}{dt^{i}} + \frac{(2\dot{\psi})^{i}}{s^2}\frac{ds}{dt}\frac{d^{i-1}s}{dt^{i-1}} \qquad \text{(IV-18)}$$

By equation (IV-6) the first and third terms vanish. Thus

$$\beta_i = -\frac{(2\dot{\psi})^{i}}{s}\frac{d^{i}s}{dt^{i}} \qquad \text{(IV-19)}$$

This completes the induction on i and establishes (IV-19) for all i. In particular, since s(t) is a polynomial of degree (n-1) in t, we have

$$\frac{d^{n}s}{dt^{n}} = 0 \qquad \text{(IV-20)}$$

From (IV-19) and (IV-20)

$$\beta_n = 0 \qquad \text{(IV-21)}$$

We complete the proof by considering the case n = 1. Equations (IV-1), (IV-3), and (IV-11) yield

$$\beta_i = f' + 2\psi = 2\ddot{\psi} + 2\psi = 0 \qquad \text{(IV-22)}$$

which is satisfied for solutions $\psi(t)$ of (IV-7).

Remark: This proof has been presented for brevity. It does not indicate how the result was originally obtained from the system (IV-1) - (IV-3). Since the case n = 1 is immediately integrable, it sheds no light on the problem.

$$\beta_1 = f' + 2\psi = 0 \qquad \text{(IV-23)}$$

$$f = c - \psi^2 \qquad \text{(IV-24)}$$

For n = 2 we indicate briefly the manipulations which led to the parametric solutions. This solution was then generalized and found to be correct.

$$\beta_2 = 2f(f'' + 2) - (2f' + 2\psi)(f' + 2\psi) = 0 \qquad \text{(IV-25)}$$

$$\frac{f'' + 2}{f' + 2\psi} = \frac{f'}{f} + \frac{\psi}{f} \qquad \text{(IV-26)}$$

$$\frac{d}{d\psi} \ln\left[\frac{f' + 2\psi}{f}\right] = \frac{\psi}{f} \qquad \text{(IV-27)}$$

The equation for $\Phi(\psi)$, (see Chapter II, theorem 1), is integrated together with (IV-27):

$$f\Phi'' - \psi\Phi' + k^2\Phi = 0 \qquad \text{(IV-28)}$$

$$\frac{d}{d\psi}\left[\Phi' e^{-\int \frac{\psi}{f}}\right] + \frac{k^2\Phi}{f} e^{-\int \frac{\psi}{f}} = 0 \qquad \text{(IV-29)}$$

Let

$$u(\psi) = e^{-\int \frac{\psi}{f}} = \frac{d\psi}{d\tau} \qquad \text{(IV-30)}$$

Then

$$\frac{d^2\Phi}{d\tau^2} + \frac{k^2\Phi}{f}\left(\frac{d\psi}{d\tau}\right)^2 = 0 \qquad \text{(IV-31)}$$

Integration of (IV-27) yields

$$\frac{c}{u} = \frac{f'}{f} + \frac{2\psi}{f} \qquad \text{(IV-32)}$$

$$\tau = \int \frac{d\psi}{u} = \frac{1}{c}\left[\ln f - 2\ln u\right] \qquad \text{(IV-33)}$$

$$e^{-c\tau} = \frac{u^2}{f} \qquad \text{(IV-34)}$$

From (IV-31) and (IV-34)

$$\frac{d^2\Phi}{d\tau^2} + c^{-c\tau} k^2 \Phi = 0 \qquad \text{(IV-35)}$$

To obtain the desired equation for $\psi(\tau)$ we differentiate (IV-30)

$$\frac{d\psi}{d\tau} = u \qquad \text{(IV-36)}$$

$$\frac{d^2\psi}{d\tau^2} = \frac{du}{d\tau} = \frac{du}{d\psi}\frac{d\psi}{d\tau} = u^2\left(-\frac{\psi}{f}\right) \qquad \text{(IV-37)}$$

From equations (IV-34) and (IV-37)

$$\frac{d^2\psi}{d\tau^2} + e^{-c\tau}\psi = 0 \qquad \text{(IV-38)}$$

A change of parameter transforms (IV-38) into the form of Theorem 1.

The similarity between equations (IV-35) and (IV-38) suggests the same parametric approach for $\Phi(\psi)$.

Theorem 2. Let $f(\psi)$ be a solution of (H). Then a parametric solution of the equation

$$f(\psi)\frac{d^2\Phi}{d\psi^2} - \psi\frac{d\Phi}{d\psi} + k^2\Phi = 0 \qquad \text{(IV-39)}$$

is described by the pair, $\Phi = \Phi(t)$, $\psi = \psi(t)$, where Φ and ψ satisfy

$$\frac{d^2\Phi}{dt^2} + \frac{d\Phi}{dt}\left[\frac{1}{t-c_1} + \ldots + \frac{1}{t-c_{n-1}}\right] + k^2\Phi = 0 \qquad n \geqslant 2 \qquad \text{(IV-40)}$$

$$\frac{d^2\psi}{dt^2} + \frac{d\psi}{dt}\left[\frac{1}{t-c_1} + \ldots + \frac{1}{t-c_{n-1}}\right] + \psi = 0 \qquad n \geqslant 2 \qquad \text{(IV-41)}$$

$$\frac{d^2\Phi}{dt^2} + k^2\Phi = 0 \qquad n = 1 \qquad \text{(IV-42)}$$

$$\frac{d^2\psi}{dt^2} + \psi = 0 \qquad n = 1 \qquad \text{(IV-43)}$$

$c_1, c_2, \ldots, c_{n-1}$ are arbitrary constants.

Proof: Use the results of theorem 1. Take $f = \left(\frac{d\psi}{dt}\right)^2$

$$\frac{d\Phi}{dt} = \frac{d\phi}{d\psi}\frac{d\psi}{dt} \qquad \text{(IV-44)}$$

$$\frac{d^2\Phi}{dt^2} = \frac{d^2\phi}{d\psi^2}\left(\frac{d\psi}{dt}\right)^2 + \frac{d\phi}{d\psi}\frac{d^2\psi}{dt^2} \qquad \text{(IV-45)}$$

Substitute in equation (IV-39)

$$\frac{d^2\phi}{dt^2} - \frac{d\phi}{dt}\frac{d^2\psi/dt^2}{d\psi/dt} - \psi\frac{d\phi/dt}{d\psi/dt} + k^2\Phi = 0 \qquad \text{(IV-46)}$$

$$\frac{d^2\phi}{dt^2} - \frac{d\phi}{dt}\left[\frac{d^2\psi/dt^2 + \psi}{d\psi/dt}\right] + k^2\Phi = 0 \qquad \text{(IV-47)}$$

Comparison of (IV-47) and (IV-41) or (IV-43) yields the theorem.

Corresponding results hold for the diffusion equation. We combine these into a single theorem and again give a direct rather than a constructive proof.

Theorem 3. Let $\Phi(\psi)$ satisfy

$$f(\psi)\frac{d^2\Phi}{d\psi^2} + (1 - k^2)\frac{d\Phi}{d\psi} = 0 \qquad \text{(IV-48)}$$

where $f(\psi)$ is a solution of the system (D)

$$\beta_o = f' - 1 \tag{IV-49}$$

(D)
$$2i\beta_i = f\beta'_{i-1} - 2\beta_o\beta_{i-1} \tag{IV-50}$$

$$\beta_n = 0 \tag{IV-51}$$

Then a parametric solution of (IV-48) is given by solutions $\Phi(t)$ and $\psi(t)$ of

$$\frac{d^2\Phi}{dt^2} + \frac{d\Phi}{dt}\left[k^2 + \frac{1}{2}\left\{\frac{1}{t-c_1} + \ldots + \frac{1}{t-c_n}\right\}\right] = 0 \tag{IV-52}$$

$$\frac{d^2\psi}{dt^2} + \frac{d\psi}{dt}\left[1 + \frac{1}{2}\left\{\frac{1}{t-c_1} + \ldots + \frac{1}{t-c_n}\right\}\right] = 0 \tag{IV-53}$$

Proof: We begin by establishing equation (IV-48). Take

$$f(\psi) = -\frac{d\psi}{dt} = -\dot{\psi}\,. \tag{IV-54}$$

Since

$$\dot{\Phi} = \Phi'\dot{\psi} \tag{IV-55}$$

$$\ddot{\Phi} = \Phi''\dot{\psi}^2 + \Phi'\ddot{\psi}\,, \tag{IV-56}$$

and from (IV-52) and (IV-53)

$$\frac{\ddot{\Phi}}{\dot{\Phi}} - \frac{\ddot{\psi}}{\dot{\psi}} = 1 - k^2, \tag{IV-57}$$

we have

$$\frac{\Phi''}{\dot{\Phi}}\dot{\psi}^2 + \frac{\Phi'\psi}{\dot{\Phi}} - \frac{\ddot{\psi}}{\dot{\psi}} = 1 - k^2 \tag{IV-58}$$

Since the second and third terms on the left cancel, we obtain, using (IV-54) and (IV-58)

$$(1 - k^2) = \frac{\Phi''\dot{\psi}^2}{\dot{\Phi}} = \frac{\Phi''}{\Phi'}\dot{\psi} = -f\frac{\Phi''}{\Phi'} \tag{IV-59}$$

which is (IV-48)

To satisfy (D), we begin by defining the polynomial

$$s(t) = \prod_{i=1}^{n} (t - c_i) \tag{IV-60}$$

Then, from (IV-53) we obtain

$$\frac{\dot{\psi}}{\dot{\psi}} = -1 - \frac{1}{2}\frac{\dot{s}}{s} \tag{IV-61}$$

Since

$$\beta_o = f' - 1 - \frac{\dot{\psi}}{\dot{\psi}} - 1 , \tag{IV-62}$$

$$\beta_o = \frac{\dot{s}}{2s} \tag{IV-63}$$

$$\beta_1 = \frac{ff''}{2} - (f' - 1)^2 \tag{IV-64}$$

$$\beta_1 = \frac{\dot{\psi}}{2}\frac{d}{d\psi}\left[\frac{\dot{s}}{2s}\right] - \left(\frac{\dot{s}}{2s}\right)^2 \tag{V-65}$$

$$\beta_1 = -\frac{1}{2}\left[\frac{\ddot{s}}{2s} - \frac{(\dot{s})^2}{2s^2}\right] + \left(\frac{\dot{s}}{2s}\right)^2 = -\frac{\ddot{s}}{4s} \tag{IV-66}$$

Inductively, we assume that

$$\beta_1 = \frac{(-1)^i}{2^{i+1}\, i!}\,\frac{d^{i+1}s/dt^{i+1}}{s} \tag{IV-67}$$

From equation (IV-50)

$$2(i+1)\beta_{i+1} = f\beta_i' - 2\beta_o\beta_i \tag{IV-68}$$

$$= -\dot{\psi}\beta_i' - 2\beta_o\beta_i \tag{IV-69}$$

$$= -\frac{d}{dt}\beta_i - 2\beta_o\beta_i \tag{IV-70}$$

We compute β_{i+1} from (IV-67) and (IV-70)

$$2(i+1)\beta_{i+1} = \frac{(-1)^{i+1}}{2^{i+1}i!}\left[\frac{d^{i+2}s/dt^{i+2}}{s} - \frac{\dot{s}d^{i+1}s/dt^{i+1}}{s^2}\right] + \frac{(-1)^{i+1}\dot{s}}{2^{i+1}i!\,s^2}\frac{d^{i+1}s}{dt^{i+1}} \tag{IV-71}$$

$$\beta_{i+1} = \frac{(-1)^{i+1}d^{i+1}s/dt^{i+2}}{2^{i+2}(i+1)!} \tag{IV-72}$$

We have completed the induction. Since $s(t)$ is a polynomial of degree n,

$$\frac{d^{n+1}s}{dt^n} = 0 \tag{IV-73}$$

Hence $\beta_{n=0}$ and the theorem is established.

We now return to a discussion of the system (H) describing the auxiliary functions $f(\psi)$ obtained from the Helmholtz equation. It is apparent that as n increases we obtain successively larger classes of functions, each of which contains all the previous ones. This follows from the original definition since a solution of the Helmholtz equation in m dimensions is trivially a solution in higher dimensions. This can be seen also from the equations defining (H).

$$\beta_1 = f' + 2\psi \tag{IV-74}$$

$$\beta_i = 2f\beta'_{i-1} - (if' + 2\psi)\beta_{i-1} \tag{IV-75}$$

$$\beta_n = 0 \tag{IV-76}$$

If $\beta_m = 0$ for $m < n$, then $\beta'_m = 0$. By (IV-75) we have $\beta_{m+1} = 0$ and inductively $\beta_n = 0$. From the parametric form,

$$f = \dot{\psi}^2 \tag{IV-77}$$

$$\ddot{\psi} + \dot{\psi}\left[\frac{1}{t-c_1} + \ldots + \frac{1}{t-c_{n-1}}\right] + \psi = 0\ , \tag{IV-78}$$

we see that choosing $c_m, c_{m+1}, \ldots, c_{n-1}$ infinite shows that the set of solutions

corresponding to $\beta_m = 0$ is contained in the set corresponding to $\beta_n = 0$. A more interesting specialization is to take certain of the c's to be equal (confluence of singularities). This certainly gives rise to different types of solutions $\psi(t)$ and presumably to a great variety of functions $f(\psi)$. Supposing that n is fixed, the number of possible choices is

$$c_n = \sum_{i=0}^{n-1} p(i) \qquad \text{(IV-79)}$$

where p(i) is the partition function of the integer i. (p(o) is understood to be unity.) c_n is known to grow rapidly with n. On the other hand, since $f(\psi)$ is the solution of an n^{th} order equation, $\beta_n = 0$, we expect that the variety of possible f's is relatively limited. It would seem, then, that many relations exist among the solutions of (IV-78).

Writing equation (IV-78) in the form

$$\ddot{\psi} + \frac{\dot{s}}{s}\dot{\psi} + \psi = 0 \qquad \text{(IV-80)}$$

$$s = \prod_{i=1}^{n-1} (t - c_i)^{p_i} \qquad \text{(IV-81)}$$

We see that a sequence $\{p_i\}$ describes each situation as discussed above. Of course, we impose the restrictions

$$0 \leqslant p_i \leqslant n-1 \qquad \text{(IV-82)}$$

$$\sum_{i=1}^{n-1} p_i = d \leqslant n-1 \qquad \text{(IV-83)}$$

The set of functions $f(\psi)$ corresponding to each sequence is obtained by eliminating t between (IV-77) and (IV-80):

$$f = \dot{\psi}^2 \qquad \text{(IV-84)}$$

$$f' = 2\ddot{\psi} \tag{IV-85}$$

$$\frac{f' + 2\psi}{2\sqrt{f}} = -\frac{\dot{s}}{s} \tag{IV-86}$$

Define

$$Q_o(\psi) = -\frac{(f' + 2\psi)}{2\sqrt{f}} = \frac{\dot{s}}{s} \tag{IV-87}$$

and

$$Q_j(\psi) = \frac{(-1)^j}{j!}\left(\sqrt{f}\,\frac{d}{d\psi}\right)^j Q_o(\psi) \tag{IV-88}$$

Then

$$Q_j(\psi) = \sum_{i=1}^{n-1} \frac{p_i}{(t-c_i)^{j+1}} = \sum_{i=1}^{n-1} p_i \tau_i^{j+1} \tag{IV-89}$$

where

$$\tau_i = \frac{1}{t-c_i} \tag{IV-90}$$

Now suppose that the number of non-zero elements of $\{p_i\}$ is m. Consider the elimination of the m quantities τ_i from the (m+1) equations

$$Q_j(\psi) = \sum_{i=1}^{n-1} p_i \tau_i^{j+1} \qquad j=0, 1, \ldots, m \tag{IV-91}$$

This finally yields a differential polynomial whose solutions $f(\psi)$ correspond to the partition (sequence) $\{p_i\}$. The order (highest derivative of f) of this differential polynomial will be m+1. The formal elimination of τ_i from equations (IV-91) in the general case is apparently a difficult problem.

Introduce a partial order in the set of differential polynomials described above under the relation

$$P \supset Q \iff \text{Solutions of P contain the solutions of Q} \tag{IV-92}$$

Upper and lower universal bounds are given by

$$\beta_n \supset P \qquad \text{all P} \tag{IV-93}$$

$$P \supset \beta_1 \qquad \text{all } P \qquad \text{(IV-94)}$$

Thus for each fixed n we have a lattice L_n, of differential polynomials corresponding to the various partitions of the integers $0, \ldots, n-1$. In order to be able to determine the comparability of P and Q, given the defining sequences $\{p_i\}$ and $\{q_i\}$, we define a partition lattice L_n^*, as follows. For each n, introduce in the set of sequences $\{p_i\}$, restricted by (IV-82) and (IV-83), the operations deletion and combination. Deletion is the replacement of a non-zero element by a zero. Combination is the replacement of two non-zero elements p_i and p_k by

$$p_k \rightarrow 0$$

$$p_j \rightarrow p_k + p_j \qquad \text{(IV-95)}$$

A partial ordering is defined by

$P^* \supset Q^* \iff Q^*$ is obtained from P^* by a succession of deletions and/or combinations.

The element $\{1, 1, \ldots, 1\}$ is a universal upper bound and the element $\{0, 0, \ldots, 0\}$ is a universal lower bound. We state theorem 4 without proof.

Theorem 4. L_n and L_n^* are isomorphic.

A one-to-one correspondence between L_n and L_n^* exists by definition. The deletion operation offers little difficulty. The combination operation seems to involve the solution of equations (IV-91) which is tractable only in special cases. At any rate, theorem 4 can be used as a guide to establish relations which can be checked directly from the system (H).

V. BESSEL FUNCTIONS

In this chapter we investigate the special case obtained by taking all the singularities to be equal in the parametric form obtained from the Helmholtz equation. This corresponds to the partition $(n-1, 0, 0, \dots)$ or to $s(t)=(t-c_1)^{n-1}$. To simplify the notation and emphasize that this could be taken as a starting point we consider solutions y(t) and x(t) of:

$$\ddot{y}+\frac{(n-1)}{t}\dot{y}+k^2 y = 0 \qquad \text{(V-1)}$$

$$\ddot{x}+\frac{(n-1)}{t}\dot{x}+x=0 \quad . \qquad \text{(V-2)}$$

From the previous chapter (or by direct calculation) we find that the (modeling) functions y(x) obtained by eliminating t between (V-1) and (V-2) satisfy

$$f(x)\frac{d^2y}{dx^2} -x\frac{dy}{dx}+k^2 y=0 \quad , \qquad \text{(V-3)}$$

where f(x) is a solution of a second order nonlinear differential equation $u_n(f, f', f'', x)=0$. This follows from the discussion pertaining to the elimination of t between $f(x)=\dot{x}^2$ and the equation of which (V-2) above is a specialization. The differential polynomial u_n is easily obtained from (V-2)

$$f(x)=\dot{x}^2 \qquad \text{(V-4)}$$

$$\frac{df}{dx} = \frac{2\dot{x}}{\dot{x}}\ddot{x} = 2\ddot{x} \quad . \qquad \text{(V-5)}$$

From (V-2)

$$-\left[\frac{f'+2x}{2f}\right] = \frac{(n-1)}{t} \quad . \qquad \text{(V-6)}$$

Differentiate (V-6) with respect to t:

$$\sqrt{f}\,\frac{d}{dx}\left[\frac{f'+2x}{2\ f}\right] = \frac{n-1}{t^2} \quad . \tag{V-7}$$

Eliminate t between (V-6) and (V-7) and simplify

$$f(f'+2)=(f'+2x)^2\left[\frac{n}{2(n-1)}\right] -x(f'+2x) \tag{V-8}$$

$$u_n = f(f'+2)-\left[\frac{n}{2(n-1)}\right](f'+2x)^2+x(f'+2x)=0 \quad . \tag{V-9}$$

Now we impose boundary conditions on (V-1) and (V-2) which determine specific solutions for each n. These conditions in turn specify f(x) and $y[x]$ locally. We are interested in extending these local solutions as far as possible by particular choices of k. The boundary conditions are assumed at t=0 .

$$y(0)=x(0)=1 \tag{V-10}$$

$$\dot{y}(0)=\dot{x}(0)=0 \quad . \tag{V-11}$$

For f(x) and $y[x]$ we have, at x=1,

$$f(1)=0 \tag{V-12}$$

$$f'(1)= -\frac{2}{n} \tag{V-13}$$

$$y[1]=1 \tag{V-14}$$

$$y'[1]=k^2 \quad . \tag{V-15}$$

Attempt a series of solutions of (V-8) for f(x), near x=1, of the form

$$f(x)= \sum_{j=0}^{\infty} b_j^{(n)} (x-1)^j \quad . \tag{V-16}$$

The calculation offers no difficulty and we obtain the recurrence relation

$$\frac{(n+1)}{(n-1)}\, j b_j^{(n)} + \left[(j-1)\left(\frac{n+1}{n-1}\right) - 2\right] b_j^{(n)} = \sum_{i=1}^{j} i b_i^{(n)} b_{j-i+1}^{(n)} \left[(i-1) - \frac{n}{2}\left(\frac{j-i+1}{n-1}\right)\right]$$

$$j \geq 4, \quad n \geq 1 \qquad \text{(V-17)}$$

$$b_0^{(n)} = 0$$

$$b_1^{(n)} = -\frac{2}{n}$$

$$b_2^{(n)} = -\frac{3}{n+2}$$

$$b_3^{(n)} = \frac{-2n\,(n-1)}{3(n+2)^2(n+4)} \qquad \text{(V-18)}$$

$$b_4^{(n)} = \frac{n^2(n-1)(n+14)}{3 \cdot 2(n+2)^3(n+4)(n+6)}$$

$$b_5^{(n)} = \frac{-n^3(n-1)(n^3+31n^2+322n+888)}{3 \cdot 5(n+2)^4(n+4)^2(n+6)(n+8)}$$

.
.
.

Inserting the series representation (V-16) for f(x) in

$$f(x)y'' - xy' + k^2 y = 0 \qquad \text{(V-19)}$$

and assuming a solution for $y[x]$ in the form

$$y_n[x] = \sum_{j=0}^{\infty} p_j^{(n)} (k)(x-1)^j \qquad \text{(V-20)}$$

results in the recurrence relation

$$j p_j^{(n)} - (k^2-j+1) p_{j-1}^{(n)} = \sum_{s=2}^{j-1} s(s-1) b_{j-s+1}^{(n)} p_s^{(n)} \quad . \qquad j \geq 3 \qquad \text{(V-21)}$$

$$p_0^{(n)} = 1$$

$$p_1^{(n)} = k^2$$

$$p_2^{(n)} = \frac{k^2(k^2-1)}{2!\,(1+\frac{2}{n})}$$

$$p_3^{(n)} = \frac{k^2(k^2-1)}{3!\,(1+\frac{2}{n})(1+\frac{4}{n})} \left[k^2 - \frac{2(n+5)}{(n+2)} \right] \qquad \text{(V-22)}$$

$$p_4^{(n)} = \frac{k^2(k^2-1)}{4!\,(1+\frac{2}{n})(1+\frac{4}{n})(1+\frac{6}{n})} \left[\left(k^2 - \frac{2(n+5)}{n+2} \right) \left(k^2 - \frac{3(n+8)}{n+2} \right) - \frac{4(n-1)}{(n+2)^2} \right] .$$

There is little hope of solving the simultaneous recurrence relations (V-17) and (V-21) in the general case. For n=1 however, the coefficients $b_j^{(n)}$ simplify

$$b_0^{(1)} = 0$$

$$b_1^{(1)} = -2$$

$$b_2^{(1)} = -1$$

$$b_j^{(1)} = 0 \qquad j \geq 3 \quad . \qquad \text{(V-23)}$$

The coefficients $p_j^{(1)}$ become

$$p_0^{(1)} = 1$$

$$p_1^{(1)} = k^2 \qquad \text{(V-24)}$$

$$p_2^{(1)} = \frac{k^2(k^2-1)}{3 \cdot 2!}$$

$$j p_j^{(1)} - (k^2 - j + 1) p_{j-1}^{(1)} = -(j-1)(j-2) p_{j-1}^{(1)} \qquad \text{(V-25)}$$

$$j p_j^{(1)} = \left[k^2 - (j-1)^2 \right] p_{j-1}^{(1)} \quad . \qquad \text{(V-26)}$$

Equation (V-26) is easily solved and we have

$$p_j^{(1)}(k^2) = \prod_{i=0}^{j-1} \frac{(k^2 - i^2)}{(i+1)(2i+1)} \qquad j \geq 1 \quad . \qquad \text{(V-27)}$$

There is nothing novel about this result, of course. It could be obtained, for example, by expanding the function $\cos(k \cos^{-1} x)$ near $x=1$ or equivalently, since $f(x)$ in this case is $(1-x^2)$, by solving

$$(1-x^2)y'' - xy' + k^2 y = 0 \quad , \qquad \text{(V-28)}$$

in the neighborhood of $x=1$ using the appropriate boundary conditions. It is interesting to note, however, that the zeros of the polynomials $p_j^{(1)}(k^2)$ yield the integer values obtained by Ritt[1] as conditions under which the local solution (V-20) extends to the entire x-plane. (If k is an integer, m, then $p_j = 0$ for $j > m$ and the series (V-20) becomes a polynomial.) We expect that the zeros of the polynomials $p_j^{(n)}(k^2)$ will be significant in the attempt to extend the radius of convergence of (V-20) for larger values of n. Before embarking on a study of these zeros we note that in the limiting case $n = \infty$ we can solve the recurrence relations and obtain $p_j^{(\infty)}(k^2)$. Examination of (V-17) and (V-18) shows that

$$b_j^{(\infty)} = 0 \quad . \qquad \text{(V-29)}$$

The recurrence relation for p_j^{∞} becomes

$$
\begin{aligned}
& j p_j^{\infty} = (k^2 - j + 1) p_{j-1}^{\infty} \qquad j \geqslant 3 \\
& p_0^{\infty} = 1 \\
& p_1^{\infty} = k^2 \\
& p_2^{\infty} = \frac{k^2(k^2-1)}{2!} \quad .
\end{aligned}
\tag{V-30}
$$

The solution for $p_j^{(\infty)}$ is

$$
p_j^{(\infty)} = \prod_{i=0}^{j-1} \frac{(k^2 - i)}{i+1} \quad . \tag{V-31}
$$

Using this result we can obtain $y_{\infty}(x)$ in closed form. From (V-20)

$$
y_{\infty}(x) = \sum_{j=0}^{\infty} p_j^{(\infty)}(k) \, (x-1)^j = x^{k^2} \quad . \tag{V-32}
$$

An expansion of $y_n(x)$ near $n=\infty$ can be carried out by retaining successively terms of order $\frac{1}{n}, \frac{1}{n^2}$, etc., in the recurrence relations (V-17) and (V-21). This quickly becomes cumbersome and the results do not seem to justify the effort. We return, then, to the study of the zeros of $p_j^{(n)}(k^2)$.

Denote the zeros of $p_j^{(n)}$ considered as a function of k^2 by

$$
\begin{aligned}
& \mu_{ji}^{(n)} \qquad && i = 1, \ldots, j \\
& && j = 1, \ldots, \infty
\end{aligned}
\tag{V-33}
$$

ordered according to increasing magnitude for each fixed n and j. For n=2 a few of the zeros are computed:

j ↓	i → 1	2	3	4	5	6
1	0					
2	0	1				
3	0	1	3.5000			
4	0	1	3.4385	7.5615		
5	0	1	3.4103	7.3735	13.2162	
6	0	1	3.3944	7.2990	12.8431	20.4826
	.	.	.	.	.	.
	.	.	.	.	.	.
	.	.	.	.	.	.
Conjectured limits:	0	1	3.3523	7.0493	12.091	18.477

(V-34)

The conjectured limits indicated above are obtained as follows: Consider the derivative of $y[x]$ as it depends on t:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = k\,\frac{dx/dt\,(kt)}{dx/dt\,(t)} \quad . \tag{V-35}$$

The last expression in (V-35) is obtained by observing that $y(k, t)=x(kt)$, which follows from a substitution $t \to kt$ in (V-2). This is also compatible with the boundary conditions (V-10) and (V-11). Thus, for the equations we are discussing in this chapter we can represent $y[x]$ symbolically by

$$y_n[x]=g_n\left[kg_n^{-1}(x)\right] \tag{V-36}$$

where $g_n(t)$ is the solution of (V-2) subject to the assumed boundary conditions, i. e.

$$\ddot{g}_n+\frac{(n-1)}{t}\dot{g}_n+g_n=0 \tag{V-37}$$

$$g_n(0)=1 \qquad \dot{g}_n(0)=0 \tag{V-38}$$

$$g_n(t) = \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n-2}{2}}(t)}{\left(\frac{t}{2}\right)^{\frac{n-2}{2}}} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma\left(\frac{n}{2}\right)\left(\frac{t}{2}\right)^{2j}}{j!\,\Gamma\left(\frac{n}{2}+j\right)} \quad . \tag{V-39}$$

We have avoided the discussion of properties of the solutions of equations being modeled because in more general situations they are not available. We introduce them now in order to explain the apparent limiting properties of the zeros $\mu_{ij}^{(n)}$.

For $|t| < \eta_1^{(n)}$ where $\eta_1^{(n)}$ is the first positive zero of dg_n/dt , we expect no difficulty in defining $y[x]$ for the corresponding range of x. In (V-35) observe that dy/dx has a singularity at $t = \eta_1$. If we take $k = \eta_i^{(n)} / \eta_1^{(n)}$, however, where $\eta_i^{(n)}$ is another zero of dg_n/dt, we find that the singularity is removable. For example, if n=1, we have

$$\eta_1^{(1)} = \pi \tag{V-40}$$

$$\eta_i^{(1)} = \pi i \qquad i=0, \pm 1, \pm 2, \ldots \tag{V-41}$$

taking

$$k = \frac{\eta_i^{(1)}}{\eta_1^{(1)}} = \frac{\pi i}{\pi} = i \quad . \tag{V-42}$$

We have again the integer values for k found previously by another means. For n=2 we obtain from (V-39)

$$g_2(t) = J_0(t) \quad . \tag{V-43}$$

The quantities $\eta_i^{(2)}$ are the zeros of $J_1(t)$. Since $g_n(t)$ is even in t and $p_{ij}^{(n)}$ is even in k, we need only the non-negative zeros. These considerations lead to the conjectured limiting values indicated in (V-34).

$$\lim_{j \to \infty} \mu_{ij}^{(2)} = \left(\frac{\eta_i^{(2)}}{\eta_1^{(2)}} \right)^2 \quad . \tag{V-44}$$

Further numerical investigation of the μ's indicates that (V-44) holds for higher values of n. We formulate this and other conjectures as a theorem which is stated without proof.

Theorem 1. Let $g_n(t)$ be defined by

$$g_n(t) = \sum_{y=0}^{\infty} \frac{(-1)^j \Gamma\left(\frac{n}{2}\right)\left(\frac{t}{2}\right)^{2j}}{j!\,\Gamma\left(\frac{n}{2}+j\right)} \qquad n \geqslant 1 \quad . \qquad \text{(V-45)}$$

Then the expansion

$$y[x] = g_n\left(k g_n^{-1}(x)\right) = \sum_{y=0}^{\infty} p_j^{(n)}(k^2)(x-1)^j \quad , \qquad \text{(V-46)}$$

generates a set of polynomials, $p_j^{(n)}(k^2)$, which, when considered as functions of k^2, possess real non-negative zeros, $\mu_{ij}^{(n)}$, satisfying

$$(i-1) < \mu_{ij}^{(n)} \leqslant (i-1)^2 \qquad 1 \leqslant i \leqslant j \qquad \text{(V-47}$$

$$\mu_{ij}^{(n)} \leqslant \mu_{i,j+1}^{(n)} \qquad \text{(V-48)}$$

$$\lim_{j \to \infty} \mu_{ij}^{(n)} = \left(\frac{\eta_i^{(n)}}{\eta_1^{(n)}}\right)^2 \qquad \text{for each fixed n and i} \qquad \text{(V-49)}$$

$$\mu_{ij}^{(1)} = (i-1)^2 \qquad \text{(V-50)}$$

$$\lim_{n \to \infty} \mu_{ij}^{(n)} = i-1 \qquad i \text{ and } j \text{ fixed} \quad . \qquad \text{(V-51)}$$

Note: We have taken n to be integral. Although we do not emphasize this aspect of the problem, it seems that the above results would hold for real $n \geqslant 1$.

Since the zeros, $\mu_{ij}^{(n)}$, are closely related to the zeros of dg_n/dt, and this function is known[6] to have the infinite product representation,

$$\frac{dg_n}{dt} = \frac{-t}{n} \prod_{i=1}^{\infty} \left(1 - \frac{t^2}{\left(\eta_i^{(n)}\right)}\right) \quad , \tag{V-52}$$

we naturally expect that the polynomials $p_j^{(n)}$, when suitably normalized, might have limiting forms similar to (V-52). Let us examine the case n=1, for which an explicit representation is known (V-27).

$$p_j^{(1)}(k^2) = \prod_{i=0}^{j-1} \frac{(k^2 - i^2)}{(i+1)(2i+1)} = (-1)^{j-1} k^2 \, c_j^{(1)} \prod_{i=1}^{j-1} \left(1 - \frac{k^2}{i^2}\right) \tag{V-53}$$

$$c_j^{(1)} = \frac{\left[(j-1)!\right]^2}{j!\ 1 \cdot 3 \cdot 5 \ldots (2j-1)} = \frac{2^j (j-1)!}{(2j)!} \quad . \tag{V-54}$$

If we define a normalized set of polynomials, $R_j^{(1)}(k^2)$, as

$$R_j^{(1)}(k^2) = \frac{(-1)^j \pi \, p_j^{(1)}}{k \, c_j^{(1)}}(k^2) = -\pi k \prod_{i=1}^{j-1} \left(1 - \frac{k^2}{i^2}\right), \tag{V-55}$$

we obtain a form similar to (V-52). From the well-known infinite product representation for the sine function (or from (V-52)directly) we find that

$$\lim_{j \to \infty} R_j^{(1)}(k^2) = -\sin \pi k = \frac{dg_1}{dt}(\pi k) \quad . \tag{V-56}$$

It is not difficult to show that the coefficients $c_j^{(1)}$ are those obtained by expanding the inverse of $g_1(t)$ in a series near t=0(g=1)

$$\frac{t^2}{2} = \sum_{j=1}^{\infty} c_j^{(1)} (1-\cos t)^j \qquad \text{(V-57)}$$

These considerations generalize for finite values of n. We state them in the form of a theorem whose proof will result from a more general attack given in the next chapter.

Theorem 2. For $1 \leqslant n < \infty$ define the normalized polynomials

$$R_j^{(n)}(k^2) = \frac{(-1)^j \eta_1^{(n)} P_j^{(n)}(k^2)}{k\, c_j^{(n)}} \qquad j \geqslant 1 \ , \qquad \text{(V-58)}$$

where $c_j^{(n)}$ are given by the expansion near $t=0$

$$\frac{t^2}{2n} = \sum_{j=1}^{\infty} c_j^{(n)} \left(1-g_n(t)\right)^j \ . \qquad \text{(V-59)}$$

Then, for any finite domain of the k-plane, we have uniformly in k

$$\lim_{j \to \infty} R_j^{(n)}(k^2) = \frac{dg_n}{dt} (\eta_1^{(n)} k) \ . \qquad \text{(V-60)}$$

Consider, again, the modeling function defined locally by (V-20):

$$y_n[x] = \sum_{j=0}^{\infty} p_j^{(n)}(k) (x-1)^j \ . \qquad \text{(V-61)}$$

Using the normalized form for the polynomials we have

$$y_n[x] = 1 + \sum_{j=1}^{\infty} k \eta_1^{(n)} c_j^{(n)} R_j^{(n)} (k^2) (1-x)^j \ . \qquad \text{(V-62)}$$

Because of the uniform behavior of $R_j^{(n)}(k^2)$, the possibility of extending this local

representation exists only for those values of k for which

$$\lim_{j \to \infty} R_j^{(n)}(k^2) = 0 \quad , \tag{V-63}$$

i.e.

$$k^2 = \left[\frac{\eta_i^{(n)}}{\eta_1^{(n)}} \right]^2 \quad . \tag{V-64}$$

There remains open, of course, the problem of determining whether these choices of k^2 do in fact allow an extension and the amount of the extension. As we have seen, the cases $n = 0$ and $n = \infty$ yield a global extension, i.e. $y[x]$ is defined for the entire x-plane. From our knowledge of the functions $g_n(t)$ we would not expect a global extension for other values of n.

VI. POLYNOMIALS GENERATED BY BÜRMANN SERIES

In this chapter we begin a study of certain polynomials generated by the Burmann series[7] representation of a function y(kx) in powers of y(x). Although the approach is motivated by the results of Chapter V, we defer specialization to the functions considered there as long as possible for the sake of generality. The work is greatly facilitated by a formalism due to Jabotinsky[4] for the series representation of integral powers of a function.

Following Jabotinsky[4], we consider the family, Ω, of functions y(x) such that for $x < \rho$ with $\rho > 0$.

$$y(x) = \sum_{n=1}^{\infty} a_n x^n \qquad a_1 = 1 \quad . \tag{VI-1}$$

Raising (VI-1) to the m'th power (m integer, $-\infty < m < +\infty$), we can always find coefficients $a_{m,n}$ such that for $|x| < \rho'$, $\rho' > 0$,

$$\left[y(x)\right]^m = \sum_{n=-\infty}^{\infty} a_{m,n} x^n \quad . \tag{VI-2}$$

Since $y(x) \in \Omega$, we have

$$a_{m,n} = 0 \quad \text{for } n < m \quad . \tag{VI-3}$$

By virtue of (VI-1), y(x) possesses a unique inverse, y(x), also belonging to Ω

$$x(y) = \sum_{n=1}^{\infty} b_n y^n \quad . \tag{VI-4}$$

The powers of x(y) have series expressions

$$\left[x(y)\right]^m = \sum_{n=-\infty}^{\infty} b_{m,n} y^n \tag{VI-5}$$

$$b_{m,n} = 0 \quad \text{for } n < m \quad . \tag{VI-6}$$

We pause to note an interesting relation given in Theorem II of Reference 4, which

will be of some use later

$$b_{m,n} = \frac{m}{n} a_{-n,-m} \qquad n \neq 0 \quad . \tag{VI-7}$$

Now consider the Bürmann series for y(kx) developed in powers of y(x)

$$y(kx) = \sum_{n=1}^{\infty} S_n(k) \left[y(x)\right]^n \quad . \tag{VI-8}$$

Theorem 1. The coefficients of $S_n(k)$ are polynomials in k

$$S_n(k) = \sum_{m=1}^{n} a_m b_{m,n} k^m \quad . \tag{VI-9}$$

Proof: Expand y(kx) in powers of y and equate coefficients of like powers of y in (VI-8) using (VI-5) and (VI-6)

$$y(kx) = \sum_{m=1}^{\infty} a_m k^m x^m = \sum_{m=1}^{\infty} a_m k^m \sum_{n=m}^{\infty} b_{m,n} y^n \tag{VI-10}$$

(since $b_{m,n} = 0$ for $n < m$) .

Interchange the order of summation in (VI-10)

$$y(kx) = \sum_{n=1}^{\infty} y^n \sum_{m=1}^{n} a_m b_{m,n} k^m \quad . \tag{VI-11}$$

Comparison of (VI-8) and (VI-11) yields (VI-9).

It is of interest to note that the polynomials, $S_n(k)$, are closely related to the modified Faber polynomials, $F_n(t)$, corresponding to the inverse of y(x). These were introduced by Schiffer[8] in a study of univalent functions and expressed simply in terms of the $b_{m,n}$ by Jabotinsky (Theorem IV of Reference 4)

$$F_n(t)= \sum_{m=1}^{n} b_{m,n}\, t^{m-1} \quad . \tag{VI-12}$$

We shall see that the determination of limiting properties of $b_{m,n}$, necessary for the study of $S_n(k)$ as $n\to\infty$, has immediate application to the corresponding problem for the Faber polynomials.

Motivated by the considerations in Chapter V, we normalize $S_n(k)$. Define

$$R_n(k)=\frac{1}{b_n k}\, S_n(k)= \sum_{m=1}^{n} a_m \frac{b_{m,n}}{b_n}\, k^{m-1} \quad . \tag{VI-13}$$

In general it is necessary to assume that

$$b_n \neq 0 \quad . \tag{VI-14}$$

We now formulate sufficient conditions on $b_{m,n}$ for the evaluation of the limiting properties of $R_n(k)$. We confine our attention for the present to entire functions, $y(x)$, which are schlict in a circle $|x| < \lambda$ and satisfy $y'(\lambda)=0$.

Theorem 2. Let $y(x)$ be an entire function defined by the series

$$y(x)= \sum_{m=1}^{\infty} a_m x^m, \qquad a_1=1 \quad , \tag{VI-15}$$

which is schlict in the circle $|x| < \lambda$ and satisfies

$$y'(\lambda)=0 \, . \tag{VI-16}$$

The inverse function, $x(y)$, is defined by

$$x(y)= \sum_{n=1}^{\infty} b_n y^n \quad , \qquad b_n \neq 0 \, . \tag{VI-17}$$

The polynomials, $R_n(k)$, generated by the Bürmann series of $y(kx)$ in powers of y,

$$y(kx)= \sum_{n=1}^{\infty} k\, b_n\, R_n(k) \left[y(x)\right]^n \qquad \text{(VI-18)}$$

$$R_n(k)= \sum_{m=1}^{n} a_m \frac{b_{m,n}}{b_n} k^{m-1} , \qquad \text{(VI-19)}$$

converge uniformly in any finite domain of the k plane to the limit function

$$\lim_{n\to\infty} R_n(k)= \sum_{m=1}^{\infty} a_m (\lambda k)^{m-1} = y'(\lambda k) , \qquad \text{(VI-20)}$$

provided only that for each fixed m and for n sufficiently large

(i) $$\lim_{n\to\infty} \frac{b_{m,n}}{m b_n} = \lambda^{m-1} , \qquad \text{(VI-21)}$$

(ii) $$\left| \frac{b_{m,n}}{m b_n \lambda^{m-1}} \right| \leqslant 1 . \qquad \text{(VI-22)}$$

Proof: We follow an argument used by Hurwitz[6] in a discussion of the limiting behavior of Lommel polynomials. Conditions (i) and (ii) allow a direct application of Tannery's theorem[9] which states that

$$\lim_{n\to\infty} \sum_{m=1}^{n} v_m(n) = \sum_{m=1}^{\infty} w_m , \qquad \text{(VI-23)}$$

provided that

$$\lim_{n\to\infty} v_m(n) = w_m , \qquad m \text{ fixed} , \qquad \text{(VI-24)}$$

and

$$\left| v_m(n) \right| \leqslant M_m \qquad \text{(VI-25)}$$

where M_m is independent of n and $\sum_{m=1}^{\infty} M_m$ is convergent. Take

$$v_m(n) = a_m \frac{b_{m,n}}{b_n} k^{m-1} \quad , \qquad \text{(VI-26)}$$

and

$$w_m = M_m = m\, a_m \lambda^{m-1} k^{m-1} \quad . \qquad \text{(VI-27)}$$

Observe that the uniform convergence of

$$\sum_{m=1}^{\infty} m\, a_m\, \lambda^{m-1}\, k^{m-1} \quad , \qquad \text{(VI-28)}$$

is guaranteed since this sum represents the derivative of the entire function y(x) evaluated at $x = \lambda k$.

Corollary 1. Under the assumptions of the theorem, each zero of $y'(\lambda k)$ is either a zero of $R_m(k)$ for all sufficiently large values of n or else is a limiting point of the set of zeros of $R_n(k)$.

Proof: The uniform convergence of $R_n(k)$ allows the direct application of a theorem of Hurwitz[10].

Corollary 2. Under the assumptions of the theorem, the modified Faber polynomials $F_n^*(t)$, possess the limiting behavior

$$\lim_{n \to \infty} \frac{F_n^*(t)}{b_n} = \frac{1}{(1-\lambda t)^2} \quad , \quad |t| < \frac{1}{\lambda} \quad . \qquad \text{(VI-29)}$$

Proof: Apply the argument of the theorem to (VI-12).

$$\lim_{n\to\infty} \frac{F_n^*(t)}{b_n} = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{b_{m,n}}{b_n} t^{m-1} = \sum_{m=1}^{\infty} m(\lambda t)^{m-1} = \frac{1}{(1-\lambda t)^2} \quad \text{(VI-30)}$$

APPENDIX A

Lemma 1. Let B be an nxn matrix over the complex field. Define recursively the polynomials in B:

$$C_o = I \tag{A-1}$$

$$C_i = B^i - \sum_{j=1}^{i} \frac{\left(\overline{BC_{j-1}}\right)}{j} B^{i-j} \qquad i=1,2,\ldots,n \tag{A-2}$$

(bar denotes trace)

Then $C_n(B)$ is, apart from sign, the characteristic polynomial of B.

Proof: Consider the elementary symmetric functions σ_j of the eigenvalues, λ_i, of B:

$$\begin{aligned} \sigma_1 &= \lambda_1 + \cdots + \lambda_n \\ \sigma_1 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n \\ &\vdots \\ \sigma_n &= \lambda_1\lambda_2\cdots\lambda_n \end{aligned} \tag{A-3}$$

Since the eigenvalues are the zeros of the characteristic polynomial, $p(\lambda)$, we have:

$$p(\lambda) = (-1)^n \prod_{i=1}^{n} (\lambda - \lambda_i) = (-1)^n \left[\lambda^n - \sum_{j=1}^{n} (-i)^{j+1} \sigma_j \lambda^{n-j} \right] . \tag{A-4}$$

To show that

$$C_n(B) = (-1)^n p(B) \tag{A-5}$$

it remains only to prove that

$$\frac{\left(\overline{BC_{j-1}}\right)}{j} = (-1)^{j+1} \sigma_j \qquad j=1,2,\ldots,n . \tag{A-6}$$

Equation (A-6) clearly holds for j=1, since the trace of B is the sum of the eigenvalues, σ_1 . Proceeding by induction we expand for j=i using the defining equation (A-2):

$$(\overline{BC_{i-1}}) = (B^i) - \sum_{j=1}^{i-1} \frac{(\overline{BC_{j-1}})}{j} (\overline{B^{i-j}}) \quad . \tag{A-7}$$

The induction hypothesis yields

$$(\overline{BC_{i-1}}) = (\overline{B^i}) - \sum_{j=1}^{i-1} (-1)^{j+1} \sigma_j (\overline{B^{i-j}}) \quad . \tag{A-8}$$

To complete the induction we must show that

$$(-1)^{i+1} i\sigma_i = (\overline{B^i}) - \sum_{j=1}^{i-1} (-1)^{j+1} \sigma_j (\overline{B^{i-j}}) \quad . \tag{A-9}$$

Introduce the symmetric functions, u_j , of the powers of the eigenvalues

$$u_j = \sum_{m=1}^{n} \lambda_m^j \quad . \tag{A-10}$$

The trace of a power of the matrix B is an invariant which can be evaluated in terms of the eigenvalues. It is necessary only to put B in triangular form with the eigenvalues along the diagonal to see that

$$(\overline{B^i}) = \sum_{m=1}^{n} \lambda_m^i = u_i \quad . \tag{A-11}$$

This reduces (A-9) to Newton's formula connecting the elementary symmetric functions σ_i with the power symmetric functions u_i :

$$(-1)^{i+1} i\sigma_i = u_i - \sum_{j=1}^{i-1} (-1)^{j+1} \sigma_j u_{i-j} \quad . \tag{A-12}$$

A simple proof of Newton's formula is given in Reference 5 (pp. 38-39). Thus the induction is completed and the lemma is proved. It is interesting to note that the lemma holds without restriction for square matrices B.

Lemma 2.

(i) $g = g(\psi) \rightarrow L(g) = \frac{dg}{d\psi}$ (A-13)

(ii) For arbitrary (nxn) matrices or scalar functions of ψ and $\frac{\partial \psi}{\partial x_i}$

$$L(PQ) = P(LQ) + (LP)Q \tag{A-14}$$

(iii) trace $L(P) = L(\text{trace } P)$, i. e. $\overline{L(P)} = L(\bar{P})$ (A-15)

(iv) $L(A) = \frac{1}{\bar{A}} (BA + AB)$ (A-16)

(v) $L(B) = \frac{1}{2\bar{A}} (f' A + f' B - 2B^2)$ (A-17)

(vi) $L(\bar{A}) = f'$ (A-18)

(vii) $L(\bar{B}) = -1$. (A-19)

Proof: (i) Recall the definition of L:

(i) $L \equiv \frac{\psi_i}{\bar{A}} \frac{\partial}{\partial x_i}$. (A-20)

If $g = g(\psi)$ is a function of ψ only, we have:

$$L(g) = \frac{\psi_i}{\bar{A}} \frac{dg}{d\psi} \psi_i = \frac{\psi_i \psi_i}{\bar{A}} \frac{dg}{d\psi} = \frac{dg}{d\psi} \quad , \tag{A-21}$$

since

$$\bar{A} = \psi_i \psi_i \quad . \tag{A-22}$$

(ii) Let P and Q be matrices with components p_{ij} and q_{ij} respectively (we depart momentarily from the convention that subscripts denote derivatives and use a comma for differentiation).

$$PQ = p_{ij}\, q_{jk} \tag{A-23}$$

$$L(PQ) = \frac{\psi_\ell}{\bar{A}}\left[p_{ij}\, q_{jk,\ell} + p_{ij,\ell}\, q_{jk}\right] \tag{A-24}$$

$$= p_{ij}\, \frac{\psi_\ell}{\bar{A}}\, q_{jk,\ell} + \frac{\psi_\ell\, p_{ij,\ell}}{\bar{A}}\, q_{jk} \tag{A-25}$$

$$= P\,L(Q) + (LP)\,Q \quad . \tag{A-26}$$

(iii) Again let the components of P be p_{ij} and obtain directly:

$$L(P) = \frac{\psi_\ell}{\bar{A}}\, p_{ij,\ell} \tag{A-27}$$

$$L(\bar{P}) = \frac{\psi_\ell}{\bar{A}}\, p_{jj,\ell} = \overline{L(P)} \tag{A-28}$$

(return to the convention that subscripts denote partial derivatives).

(iv) Compute L(A) from the definition of A:

$$A = \psi_i\, \psi_j$$

$$L(A) = \frac{\psi_\ell}{\bar{A}}\left[\psi_{i\ell}\, \psi_j + \psi_i\, \psi_{j\ell}\right] \quad . \tag{A-29}$$

Recall that B is symmetric and obtain:

$$L(A)=\frac{1}{\bar{A}}\left[\psi_{i\ell}\psi_{\ell}\psi_{j}+\psi_{i}\psi_{\ell}\psi_{\ell j}\right] \tag{A-30}$$

$$L(A)=\frac{1}{\bar{A}}\,(BA+AB) \quad . \tag{A-31}$$

(v) Differentiate f with respect to ψ_j and then ψ_k:

$$f=\psi_i\psi_i \tag{A-32}$$

$$f'\psi_j=2\psi_i\psi_{ij} \tag{A-33}$$

$$f''\psi_j\psi_k+f'\psi_{jk}=2\psi_i\psi_{ijk}+2\psi_{ik}\psi_{ij} \tag{A-34}$$

$$f''A+f'B=2\psi_i\psi_{ijk}+2B^2 \quad . \tag{A-35}$$

On the other hand

$$L(B)=\frac{\psi_{\ell}}{\bar{A}}\,\psi_{ij\ell} \quad . \tag{A-36}$$

Combine (A-35) and (A-36) to obtain

$$L(B)=\frac{1}{2\bar{A}}\,(f''A+f'B-2B^2) \quad . \tag{A-37}$$

(vi) Since $\bar{A}$ = f(ψ), we have immediate from (i) above

$$L(\bar{A})=f' \quad . \tag{A-38}$$

(vii) Similarly, since $\bar{B}=-\psi$,

$$L(\bar{B})=-1 \quad . \tag{A-39}$$

Lemma 3.

$$f\,L(\overline{BC_i}) = (i+1)\left[\frac{f''}{2}(\overline{AC_i}) + \frac{f'}{2}(\overline{BC_i}) - (\overline{B^2 C_i})\right] \quad . \tag{A-40}$$

Proof: Induction on i .

For i = 0, we must show that (recall that $C_o = 1$)

$$f\,L(\overline{B}) = \frac{f''}{2}\overline{A} + \frac{f'}{2}(\overline{B}) - (\overline{B^2}) \quad . \tag{A-41}$$

From lemma 2, part (v) we have

$$L(B) = \frac{1}{2\overline{A}}\left(f''A + f'B - 2B^2 \right) \quad . \tag{A-42}$$

Take the trace of (A-42) and multiply by f to obtain (A-41). Now assume (A-40) for $i = 0, \ldots, j-1$. From the definition of C_i we have:

$$C_i = B^i - \sum_{j=1}^{i} \frac{\overline{BC_{j-1}}}{j} B^{i-j} \tag{A-43}$$

$$(\overline{BC_i}) = (\overline{B^{i+1}}) - \sum_{j=1}^{i} \frac{\overline{BC_{j-1}}}{j} (\overline{B^{i-j+1}}) \quad . \tag{A-44}$$

Operate on (A-44) with L:

$$L(\overline{BC_i}) = (i+1)\left(\overline{B^i L(B)}\right) - \sum_{j=1}^{i} \frac{L(\overline{BC_{j-1}})}{j} (\overline{B^{i-j+1}}) - \sum_{j=1}^{i} \frac{(\overline{BC_{j-1}})}{j} (i-j+1)(B^{i-j} LB) \quad . \tag{A-45}$$

Using the known (lemma 2) expression for L(B) and the induction hypothesis for $L(\overline{BC_{j-1}})$, we obtain

$$L(\overline{BC_i})=\frac{(i+1)}{2f}\left[f''(\overline{B^iA})+\frac{f'}{2}(\overline{B^{i+1}})-(\overline{B^{i+2}})\right]-\sum_{j=1}^{i}\frac{(\overline{B^{i-j+1}})}{f}\left[\frac{f''}{2}(\overline{AC_{j-1}})+\frac{f'}{2}(\overline{BC_{j-1}})-\frac{(\overline{B^2C_{j-1}})}{2}\right]$$

$$-\sum_{j=1}^{i}\frac{(\overline{BC_{j-1}})}{jf}(i-j+1)\left[\frac{f''}{2}(\overline{B^{i-j}A})+\frac{f'}{2}(\overline{B^{i-j+1}})-\frac{(\overline{B^{i-j+2}})}{2}\right] . \qquad (A-46)$$

Compare coefficients of f' and f'' and the remaining terms of the right sides of (A-40) and (A-46). To complete the induction it is sufficient to prove the following three equations.

$$(i+1)(\overline{BC_i})=(i+1)(\overline{B^{i+1}})-\sum_{j=1}^{i}(\overline{B^{i-j+1}})(\overline{BC_{j-1}})-\sum_{j=1}^{i}\frac{(i-j+1)}{j}(\overline{BC_{j-1}})(\overline{B^{i-j+1}}) \qquad (A-47)$$

$$(i+1)(\overline{AC_i})=(i+1)(\overline{B^iA})-\sum_{j=1}^{i}(\overline{AC_{j-1}})(\overline{B^{i-j+1}})-\sum_{j=1}^{i}\frac{(\overline{BC_{j-1}})(i-j+1)(\overline{B^{i-j}A})}{j} \qquad (A-48)$$

$$(i+1)(\overline{B^2C_i})=(i+1)(\overline{B^{i+2}})-\sum_{j=1}^{i}(\overline{B^{i-j+1}})(\overline{B^2C_{j-1}})-\sum_{j=1}^{i}\frac{(i-j+1)}{j}(\overline{BC_{j-1}})(\overline{B^{i-j+2}}) . \qquad (A-49)$$

Equation (A-47) follows easily with the substitution of the definition of C_i in the left side (use (A-43) above). Equation (A-49) is similar but slightly more complicated. Expanding the left side of(A-49) we have:

$$(i+1)(\overline{B^2C_i})=(i+1)(\overline{B^{i+2}})-(i+1)\sum_{j=1}^{i}\frac{(\overline{BC_{j-1}})}{j}(\overline{B^{i-j+2}}) . \qquad (A-50)$$

Compare the right sides of (A-49) and (A-50). It remains to show that

$$\sum_{j=1}^{i} (\overline{B^{i-j+1}})(\overline{B^2 C_{j-1}}) = \sum_{j=1}^{i} (\overline{B^{i-j+2}})(\overline{BC_{j-1}}) \quad . \tag{A-51}$$

The following identity is easily established from the definition of C_i (A-43):

$$(\overline{B^2 C_j}) = (\overline{B^2 C_{j-1}}) - \overline{B}\,\frac{(\overline{BC_{j-1}})}{j} \quad . \tag{A-52}$$

Change index on the right side of (A-51), $j' = j-1$, and substitute (A-52):

$$\sum_{j=1}^{i} (\overline{B^{i-j+1}})(\overline{B^2 C_{j-1}}) = \sum_{j=1}^{i-1} (\overline{B^{i-j+1}}) \left[(\overline{B^2 C_{j-1}}) - \overline{B}\,\frac{\overline{BC_{j-1}}}{j} \right] + (\overline{B^{i+1}})\,\overline{B} \tag{A-53}$$

Simplify (A-53)

$$\overline{B}(\overline{B^2 C_{i-1}}) = (\overline{B^{i+1}})(\overline{B}) - \overline{B} \sum_{j=1}^{i-1} \frac{(\overline{B^{i-j+1}})(\overline{BC_{j-1}})}{j} \quad . \tag{A-54}$$

That (A-54) is true follows, after dividing by $\overline{B}$, from the definition of C_i. Equations (A-50) - (A-54) in reverse order prove (A-49).

Finally, we attack (A-48). Multiply (A-43) by A and trace:

$$(\overline{AC_i}) = (\overline{AB^i}) - \sum_{j=1}^{i} \frac{(\overline{BC_{j-1}})}{j} (\overline{AB^{i-j}}) \quad . \tag{A-55}$$

Multiply (A-55) by (i+1) and subtract this from (A-48) (recall that $(\overline{AB^i}) = (\overline{B^i A})$). It remains to prove that:

$$\sum_{j=1}^{i} (\overline{AC_{j-1}})(\overline{B^{i-j+1}}) = \sum_{j=1}^{i} \frac{(\overline{BC_{j-1}})}{j} (\overline{AB^{i-j}}) \quad . \tag{A-56}$$

Expand both sides of (A-56) using (A-43):

$$\sum_{j=1}^{i} (\overline{B^{i-j+1}}) \left[(\overline{AB^{j-1}}) - \sum_{n=1}^{j-1} \frac{(\overline{BC_{n-1}})}{n} (\overline{AB^{j-n-1}}) \right]$$

$$= \sum_{j=1}^{i} (\overline{AB^{i-j}}) \left[(\overline{B^{j}}) - \sum_{n=1}^{i-1} \frac{(\overline{BC_{n-1}})}{n} (\overline{B^{j-n}}) \right] \quad . \qquad \text{(A-57)}$$

The single sums are equal (change index $j' = i-j+1$). Thus (A-57) reduces to:

$$\sum_{j=1}^{i} \sum_{n=1}^{j-1} \frac{(\overline{B^{i-j+1}})(\overline{AB^{j-n-1}})}{n} (\overline{BC_{n-1}}) = \sum_{j=1}^{i} \sum_{n=1}^{j-1} \frac{(\overline{AB^{i-j}})(\overline{BC_{n-1}})\,(\overline{B^{j-n}})}{n} \quad .$$

(A-58)

Equation (A-58) is an identity of the type

$$\sum_{j=1}^{i} \sum_{n=1}^{j-1} \alpha_{n-1} \beta_{j-n-1}\ \gamma_{i-j+1} = \sum_{j=1}^{i} \sum_{n=1}^{j-1} \alpha_{n-1}\, \beta_{i-j}\, \gamma_{j-n} \qquad \text{(A-59)}$$

in which α, β, γ's are independent quantities. It is easily proved by equating coefficients of α_{n-1} for fixed n. Since $j-1 \geqslant n$,

$$\sum_{j=n+1}^{i} \beta_{j-n-1}\ \gamma_{i-j+1} = \sum_{j=n+1}^{i} \beta_{i-j}\, \gamma_{j-n} \quad , \qquad \text{(A-60)}$$

let $j' = i-j+n+1$.

Equations (A-55) - (A-60)) (in reverse order) prove (A-48), which completes the induction and establishes the lemma.

Lemma 4.

$$2\,f\,L(\overline{AC_i})=(\overline{AC_i})\left[(l+3)f'+2\psi\right]-2(i+l)(\overline{AC_{i+l}}) \quad . \tag{A-61}$$

Proof: We begin by restating the inductive definition of C_i:

$$C_i=\mathbf{B}^i-\sum_{j=l}^{i}\frac{(\overline{BC_{j-l}})}{j}B^{i-j} \quad . \tag{A-62}$$

Multiply (A-62) by A and replace i by i+l:

$$AC_{i+l}=AB^{i+l}-\sum_{j=l}^{i+l}\frac{(\overline{BC_{j-l}})}{j}AB^{i-j+l} \quad . \tag{A-63}$$

From the definition of $f(\psi)$ we obtain by differentiation

$$f=\psi_i\psi_i \tag{A-64}$$

$$f'\psi_j=2\psi_i\psi_{ij} \quad . \tag{A-65}$$

Multiply by ψ_i:

$$f'\psi_k\psi_j=2\psi_k\psi_i\psi_{ij} \quad . \tag{A-66}$$

In matrix form:

$$f'A=2AB \quad . \tag{A-67}$$

Multiply (A-67) by B^i to obtain the reduction formula:

$$AB^{i+l}=\frac{f'}{2}AB^i \quad . \tag{A-68}$$

Using (A-68) we obtain from (A-63)

$$AC_{i+l}=\frac{f'}{2}\left[AB^i-\sum_{j=l}^{i}\frac{(BC_{j-l})}{j}(AB^{i-j})\right]-\frac{(\overline{BC_i})A}{i+l} \quad . \tag{A-69}$$

The quantity in brackets is just AC_i . Taking the trace of (A-69) we have:

$$(\overline{AC_{i+1}}) = \frac{f'}{2}(\overline{AC_i}) - \frac{f(\overline{BC_i})}{i+1} \quad . \tag{A-70}$$

Using (A-70), (A-61) is simplified:

$$f\,L(\overline{AC_i}) = (f'+\psi)(\overline{AC_i}) + f(\overline{BC_i}) \quad . \tag{A-71}$$

Rewrite (A-70) replacing (i+1) by i:

$$\overline{(AC_i)} = \frac{f'}{2}(\overline{AC_{i-1}}) - \frac{f(\overline{BC_{i-1}})}{i} \quad . \tag{A-72}$$

Operate on (A-72) with L:

$$L(\overline{AC_i}) = \frac{f'}{2}(\overline{AC_{i-1}}) + \frac{f'}{2}\,L(\overline{AC_{i-1}}) - \frac{f'}{i}(\overline{BC_{i-1}}) - \frac{f}{i}\,L(\overline{BC_{i-1}}) \quad . \tag{A-73}$$

That (A-71) holds for i=0 is clear:

$$f\,L(\bar{A}) = f\,f' = (f'+\psi)f - \psi\,f \quad . \tag{A-74}$$

Proceeding by induction we assume (A-71) for i=0, . . . j-1. From this induction hypothesis and from lemma 3 we simplify (A-73)

$$L(\overline{AC_i}) = \frac{f'}{2}(\overline{AC_{i-1}}) + \frac{f'}{2f}\left[(f'+\psi)(\overline{AC_{i-1}}) + f(\overline{BC_{i-1}})\right]$$
$$- \frac{f'}{i}(\overline{BC_{i-1}}) - \left[\frac{f'}{2}(\overline{AC_{i-1}}) + \frac{f'}{2}(\overline{BC_{i-1}}) - (\overline{B^2 C_{i-1}})\right] \quad . \tag{A-75}$$

$$L(\overline{AC_i}) = (\overline{B^2 C_{i-1}}) - \frac{f'}{i}(\overline{BC_{i-1}}) + \frac{f'(f'+\psi)}{2f}(\overline{AC_{i-1}}) \tag{A-76}$$

Substitute for $\overline{(AC_{i-1})}$ using (A-70):

$$L(\overline{AC_i}) = (\overline{B^2 C_{i-1}}) - \frac{f'}{i}(\overline{BC_{i-1}}) + \frac{(f'+\psi)}{f}\left[(\overline{AC_i}) + \frac{f}{2}(\overline{BC_{i-1}})\right] . \quad \text{(A-77)}$$

From (A-62) we easily obtain (see (A-57)):

$$(\overline{BC_i}) = (\overline{B^2 C_{i-1}}) + \frac{\psi}{i}(\overline{BC_{i-1}}) \quad . \quad \text{(A-78)}$$

Substitute (A-78) into (A-77), multiply by f, and simplify:

$$f\, L(\overline{AC_i}) = f(\overline{BC_i}) + (f' + \psi)(\overline{AC_i}) \quad . \quad \text{(A-79)}$$

This completes the induction proof for (A-71), which is equivalent to the lemma.

APPENDIX B

Lemma 1.

(i) g is a scalar function of ψ only $\rightarrow L(g)=\frac{dg}{d\psi}$ (B-1)

(ii) $\overline{L(C)}=L(\overline{C})$ for any matrix C. (B-2)

(iii) For arbitrary matrices P, Q,

$$L(PQ)=PL(Q)+L(P)Q \tag{B-3}$$

(iv) $L(A)=\frac{2f'}{f}A$ (B-4)

(v) $L(B)=\left[\frac{f''}{f}-2\left(\frac{f'}{f}\right)^2\right]A+\frac{f'}{f}B+\frac{4f'}{f}\frac{AB}{\overline{A}}-\frac{2B^2}{\overline{A}}$ (B-5)

(vi) $L(\overline{B})=\frac{f'}{f}\overline{B}$. (B-6)

Proof:

(i) By definition of L:

$$L(g)=\frac{1}{\overline{A}}\left(2\psi_k\frac{\partial g}{\partial x_k}-f\frac{\partial g}{\partial t}\right) . \tag{B-7}$$

Since g is a function of ψ only,

$$L(g)=\frac{1}{\overline{A}}(2\psi_k g'\psi_k-fg'\psi_t)=g'\left(\frac{2\psi_k\psi_k-f\psi_t}{\overline{A}}\right) : \tag{B-8}$$

The result follows by recalling that

$$f\psi_t=\psi_k\psi_k=\overline{A} \tag{B-9}$$

(ii) Let C have components c_{ij} :

$$L(C)=\frac{1}{\overline{\overline{A}}}\left[2\psi_k \frac{\partial c_{ij}}{\partial x_k}-f\frac{\partial c_{ij}}{\partial t}\right] \tag{B-10}$$

Hence

$$\overline{L(C)}=\frac{1}{\overline{\overline{A}}}\left[2\psi_k \frac{\partial c_{ii}}{\partial x_k}-f\frac{\partial c_{ii}}{\partial t}\right]=L(\bar{C}) \tag{B-11}$$

(iii) Let P and Q have components p_{ij} and q_{ij} respectively

$$PQ=p_{ij}q_{js}$$

$$L(PQ)=\frac{1}{\overline{A}}\left[2\psi_k\left(\frac{\partial p_{ij}}{\partial x_k}q_{js}+p_{ij}\frac{\partial q_{js}}{\partial x_k}\right)\right. \tag{B-12}$$

$$\left.-f\left(\frac{\partial p_{ij}}{\partial t}q_{js}+p_{ij}\frac{\partial q_{js}}{\partial t}\right)\right] \tag{B-13}$$

$$=L(P)Q+PLQ \quad . \tag{B-14}$$

(iv) Compute directly from $A=\psi_i\psi_j$:

$$L(A)=\frac{1}{\overline{\overline{A}}}\left[2\psi_k(2\psi_i\psi_{jk})-2f\psi_i\psi_{jt}\right] \tag{B-15}$$

$$=\frac{1}{\overline{\overline{A}}}\left[4AB-2((2AB-f'\bar{B}A)\right] \tag{B-16}$$

(substitute from (III-40) of Chapter III).

$$L(A)=\frac{2f'\bar{B}A}{\bar{A}} = \frac{2f'}{f}A \tag{B-17}$$

(v) Compute from $B = \psi_{ij}$:

$$L(B)=\frac{1}{\bar{A}}(2\psi_k\psi_{ijk}-f\psi_{ijt}) \tag{B-18}$$

$$=\frac{1}{\bar{A}}(-2B^2+f''\bar{B}A+f'\bar{B}B+\frac{4f'}{f}AB-2\frac{(f')^2}{f}\bar{B}A) \tag{B-19}$$

(substitute from (III-41) of Chapter III).

$$=\left[\frac{f''}{f}-2\left(\frac{f'}{f}\right)^2\right]A+\frac{f'}{f}B+\frac{4f'}{f}\frac{AB}{\bar{A}}-\frac{2B^2}{\bar{A}} \tag{B-20}$$

(vi) Since $\bar{B} = \psi_t$,

$$L(\bar{B})=\frac{1}{\bar{A}}(2\psi_k\psi_{tk}-f\psi_{tt}) \tag{B-21}$$

$$=\frac{f'\psi_t^2}{\bar{A}} = \frac{f'\bar{B}}{f} \tag{B-22}$$

(use (III-22) of Chapter III).

Lemma 2. $$\eta_n(B)\equiv 0 \tag{B-23}$$

$$\delta_n(B)\equiv 0 \tag{B-24}$$

Proof: From the definition of $\eta_i(B)$,

$$\eta_i=(-1)^i\left[\eta_o^{i+1}+\sum_{j=1}^{i}\frac{(-1)^j(\eta_{j-1})}{j}\eta_o^{i-j+1}\right] \qquad i=1,2,\ldots \tag{B-25}$$

$$\eta_o = \frac{B}{\bar{B}} \quad ; \tag{B-26}$$

we see that η_i is always divisible by η_o. Define

$$C_i(B)=(-1)^i(\bar{B})^i \frac{\eta_i}{\eta_o} \quad . \tag{B-27}$$

From (B-25) and (B-27) we obtain

$$C_i(B)=B^i - \sum_{j=1}^{i} \frac{\overline{(BC_{j-1})}}{j} B^{i-j} \qquad i=1,2,\ldots \tag{B-28}$$

$$C_o(B)= I \quad . \tag{B-29}$$

Since B is assumed to be nxn, it follows from lemma 1 of Chapter II (Appendix A) that

$$C_n(B)= 0 \quad . \tag{B-30}$$

Since

$$\delta_i = \frac{(-1)^i \eta_o C_i(B)}{(\bar{B})^i} \quad , \tag{B-31}$$

the lemma follows from (B-27), (B-30) and (B-31).

Lemma 3. For i=1, 2, . . .

$$\eta_i = \frac{\eta_o\overline{(\eta_{i-1})}}{i} - \eta_o\eta_{i-1} \tag{B-32}$$

$$\delta_i = \frac{o\overline{(\eta_{i-1})}}{i} - \delta_o\eta_{i-1} \tag{B-33}$$

Proof: Recall the defnition of η_i

$$\eta_i = (-1)^i \left[\eta_o^{i+1} + \sum_{j=1}^{i} \frac{(-1)^j}{j} \overline{(\eta_{j-1})}\, \eta_o^{i-j+1} \right] \tag{B-34}$$

Transpose (B-32) and expand using (B-34):

$$\eta_i - \frac{\eta_o \overline{(\eta_{i-1})}}{i} + \eta_o \eta_{i-1} = (-1)^i \left[\eta_o^{i+1} + \sum_{j=1}^{i} \frac{(-1)^j}{j} \overline{(\eta_{j-1})}\, \eta_o^{i-j+1} \right]$$

$$+(-1)^{i-1} \left[\eta_o^{i+1} + \sum_{j=1}^{i-1} \frac{(-1)^j}{j} \overline{(\eta_{j-1})}\, \eta_o^{i-j+1} \right]$$

$$+ \frac{(-1)^i}{i} \eta_o \left[\overline{(\eta_o^i)} + \sum_{j=1}^{i-1} \frac{(-1)^j}{j} \overline{(\eta_{j-1})}\, \overline{(\eta_o^{i-j})} \right]$$

$$= \frac{(-1)^i \eta_o}{i} \left[\overline{(\eta_o^i)} + \sum_{j=1}^{i-1} \frac{(-1)^j}{j} \overline{(\eta_{n-1})}\, \overline{(\eta_o^{i-j})} + (-1)^i \overline{(\eta_{i-1})} \right]$$

$$= 0 \tag{B-35}$$

Similarly:

$$\delta_i+\delta_o\eta_{i-1}-\frac{\delta_o\overline{(n_{i-1})}}{i}=(-1)^i\left[\delta_o\eta_o^i+\sum_{j=1}^{i}\frac{(-1)^j}{j}\overline{(\eta_{j-1})}\,\delta_o\eta_o^{i-j}\right]$$

$$+(-1)^{i-1}\left[\delta_o\eta_o^i+\sum_{j=1}^{i-1}\frac{(-1)^j}{j}\overline{(\eta_{j-1})}\,\delta_o\eta_o^{i-j}\right]$$

$$+\frac{(-1)^i\delta_o}{i}\left[\overline{(\eta_o^{i-1})}+\sum_{j=1}^{i-1}\frac{(-1)^j}{j}\overline{(\eta_{j-1})}\;\overline{(\eta_o^{i-j})}\right]$$

$$=\frac{(-1)^i\delta_o}{i}\left[\overline{(\eta_o^{i-1})}+\sum_{j=1}^{i-1}\frac{(-1)^j}{j}\overline{(\eta_{j-1})}\;\overline{(\eta_o^{i-j})}+(-1)^i\overline{(\eta_{i-1})}\right]=0 \qquad \text{(B-36)}$$

Lemma 4.

(i) $$fL(\eta_o)=\epsilon\left[\delta_o\eta_o-{}_o\overline{(\delta_o\eta_o)}\right]+2\delta_o\overline{(\eta_o^2)}-2\eta_o^2 \qquad \text{(B-37)}$$

(ii) $$L(\eta_o^i)=\sum_{k=1}^{i}\eta_o^{i-k}(L\eta_o)\eta_o^{k-1} \qquad \text{(B-38)}$$

where

$$\epsilon=4(\bar{\beta}_o+1)=4f' \quad . \qquad \text{(B-39)}$$

Proof: (i) Since $\eta_o=\frac{B}{\bar{B}}$, we expand by lemma 1:

$$L(\eta_o)=\frac{LB}{\bar{B}}-\frac{B\overline{LB}}{\bar{B}^2} \qquad \text{(B-40)}$$

$$L(\eta_o)=\frac{A}{\bar{B}}\left[\frac{f''}{f}-2\left(\frac{f'}{f}\right)^2\right]+\frac{f'}{f}\frac{B}{\bar{B}}+\frac{4f'}{f}\frac{AB}{\bar{A}\bar{B}}-\frac{2B^2}{\bar{A}\bar{B}}-\frac{f'}{f}\frac{B}{\bar{B}} \quad . \qquad \text{(B-41)}$$

Multiply by f and introduce δ_o and η_o :

$$fL\eta_o = \delta_o\left[ff'' - 2(f')^2\right] + \epsilon\delta_o\eta_o - 2\eta_o^2 \quad . \tag{B-42}$$

Comparing (B-37) and (B-42) it remains only to show that:

$$ff'' - 2(f')^2 = \overline{2(\eta_o^2)} - \epsilon\overline{(\delta_o\eta_o)} \quad . \tag{B-43}$$

Equation (B-43) follows directly from lemma 1, parts (v) and (vi), by equating two expressions for $fL(\bar{B})$:

$$fL(\bar{B}) = f'\bar{B} = \left[ff'' - 2(f')^2\right]\bar{B} + f'\bar{B} + \epsilon\frac{(\overline{AB})}{\bar{A}} - \frac{2(\overline{B^2})}{\bar{B}} \quad . \tag{B-44}$$

Equation (B-43) is obtained by cancelling $f'B$, dividing by B and introducing δ_o and η_o.

(ii) Equation (B-38) is clearly true for i=1. Proceed by induction. Suppose:

$$L(\eta_o^{j-1}) = \sum_{k=1}^{j-1} \eta_o^{j-k-1}(L\eta_o)\eta_o^{k-1} \quad . \tag{B-45}$$

Then by lemma 1, part (iii), since $\eta_o^j = \eta_o\eta_o^{j-1}$, we have:

$$L(\eta_o^j) = (L\eta_o)\eta_o^{j-1} + \eta_o L(\eta_o^{j-1}) \tag{B-46}$$

$$= (L\eta_o)\eta_o^{j-1} + \sum_{k=1}^{j-1} \eta_o^{i-k}(L\eta_o)\eta_o^{k-1} \tag{B-47}$$

$$= \sum_{k=1}^{j} \eta_o^{j-k}(L\eta_o)\eta_o^{k-1} \quad . \tag{B-48}$$

Equation (B-48) completes the induction.

Lemma 5.

$$L(\overline{\eta_i}) = -(i+1)(\overline{\eta_{i-1} L\eta_o}) \qquad i=1, 2, \ldots \tag{B-49}$$

Proof: From the definition of η_i and from lemma 1:

$$\eta_i = (-1)^i \left[\eta_o^{i+1} + \sum_{j=1}^{i} \frac{(-1)^j}{j} (\overline{\eta_{j-1}})\, \eta_o^{i-j+1} \right] \tag{B-50}$$

$$L(\overline{\eta_i}) = (-1)^i \left[L(\overline{\eta_o^{i+1}}) + \sum_{j=1}^{i} \frac{(-1)^j}{j} L(\overline{\eta_{j-1}})(\overline{\eta_o^{i-j+1}}) + \sum_{j=1}^{i} \frac{(-1)^j}{j} (\overline{\eta_{j-1}})\, L(\overline{\eta_o^{i-j+1}}) \right]. \tag{B-51}$$

Expand $L(\overline{\eta_o^i})$ using lemma 4:

$$L(\overline{\eta_o^i}) = \sum_{k=1}^{i} (\overline{\eta_o^{i-k} (L\eta_o) \eta_o^{k-1}}) \tag{B-52}$$

$$f\, L(\overline{\eta_o^i}) = \sum_{k=1}^{i} \left[\eta_o^{i-k} \left\{ \epsilon \left[\delta_o \eta_o - \delta_o (\overline{\delta_o \eta_o}) \right] + 2\delta_o (\overline{\eta_o^2}) - 2\eta_o^2 \right\} \eta_o^{i-1} \right] \tag{B-53}$$

$$= \epsilon \sum_{k=1}^{i} \left[(\overline{\eta_o^{i-k} \delta_o \eta_o \eta_o^{k-1}}) - (\overline{\delta_o \eta_o})(\overline{\eta_o^{i-k} \delta_o \eta_o^{k-1}}) \right]$$

$$+ 2(\overline{\eta_o^2}) \sum_{k=1}^{i} (\overline{\eta_o^{i-k} \delta_o \eta_o^{k-1}}) - 2 \sum_{k=1}^{i} (\overline{\eta_o^{i-k} \eta_o^2 \eta_o^{k-1}}) \quad . \tag{B-54}$$

Since $(\overline{B^iAB^j})=(\overline{AB^{i+j}})$, we can rearrange the factors in (B-54). Since the summands then no longer depend on k, we obtain:

$$f\,L(\overline{\eta_o^i})=i\left[\epsilon\left\{(\overline{\delta_o\,\eta_o^{i-1}})-(\overline{\delta_o\,\eta_o})(\overline{\delta_o\,\eta_o^{i-1}})\right\}+2(\overline{\eta_o^2})(\overline{\delta_o\,\eta_o^{i-1}})-2(\overline{\eta_o^{i+1}})\right] \tag{B-55}$$

$$f\,L(\overline{\eta_o^i})=i(\overline{\eta_o^{i-1}\,L\eta_o}) \quad . \tag{B-56}$$

Combining (B-56) with (B-51) we have:

$$L(\overline{\eta_i})=(-1)^i\left[(i+1)(\overline{\eta_o^i L\eta_o})+\sum_{j=1}^{i}\frac{(-1)^j}{j}\,L(\overline{\eta_{j-1}})(\overline{\eta_o^{i-j+1}})\right.$$

$$\left.+\sum_{j=1}^{i}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})(i-j+1)(\overline{\eta_o^{i-j}\,L\eta_o})\right] \quad . \tag{B-57}$$

Since $(\overline{\eta_1})=1-(\overline{\eta_o^2})$, we see that the lemma is true for i=1:

$$L(\overline{\eta_1})=-2(\overline{\eta_o L\eta_o}) \quad . \tag{B-58}$$

Make the induction hypothesis that (B-49) holds for j=1, . . . , i-1. Equation (B-57) becomes:

$$L(\overline{\eta_i})=(-1)^i\left[(i+1)(\overline{\eta_o^i L\eta_o})+\sum_{j=2}^{i}(-1)^{j-1}(\overline{\eta_{j-2}L\eta_o})(\overline{\eta_o^{i-j+1}})\right.$$

$$\left.+\sum_{j=1}^{i}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})(i-j+1)(\overline{\eta_o^{i-j}\,L\eta_o})\right] \quad . \tag{B-59}$$

Grouping coefficients of (i+1), changing index, and using the definition of η_j again

yields:

$$L(\overline{\eta_i}) = -(i+1)(\overline{\eta_{i-1} L\eta_o}) + \sum_{j=1}^{i-1} (-1)^j (\overline{\eta_{j-1} L\eta_o})(\overline{\eta_o^{i-j}}) - \sum_{j=1}^{i-1} (-1)^j (\overline{\eta_{j-1}})(\overline{\eta_o^{i-j} L\eta_o}) \qquad \text{(B-60)}$$

It remains to show that the last two terms of (B-60) vanish. Expanding them yields:

$$\sum_{j=1}^{i-1} (-1)^j (\overline{\eta_o^{i-j}}) \left[(-1)^{j-1} \left\{ \overline{(\eta_o^j L\eta_o)} + \sum_{k=1}^{j-1} \frac{(-1)^k}{k} \overline{(\eta_{k-1})(\eta_o^{j-k} L\eta_o)} \right\} \right]$$

$$- \sum_{j=1}^{i-1} (-1)^j \overline{(\eta_o^{i-j} L\eta_o)} \left[(-1)^{j-1} (\overline{\eta_o^j}) + \sum_{k=1}^{j-1} \frac{(-1)^k}{k} (\overline{\eta_{k-1}})(\overline{\eta_o^{j-k}}) \right]$$

$$= \sum_{j=1}^{i-1} \sum_{k=1}^{j-1} \frac{(-1)^k}{k} (\overline{\eta_{k-1}}) \left[\overline{(\eta_o^{i-j} L\eta_o)}(\overline{\eta_o^{j-k}}) - (\overline{\eta_o^{i-j}})\overline{(\eta_o^{j-k} L\eta_o)} \right] \qquad \text{(B-61)}$$

(The single-sum terms drop out by a change of index s=i-j.) The right side of (B-61) vanishes as required by the following algebraic identity which is proved in lemma 11:

$$\sum_{j=1}^{i-1} \sum_{k=1}^{j-1} a_k \left[b_{i-j} c_{j-k} - c_{i-j} b_{j-k} \right] = 0 \ . \qquad \text{(B-62)}$$

Lemma 6. Let A be as defined previously and let C and D be arbitrary nxn matrices. Then:

$$\overline{(ACAD)} = \overline{(AC)}\,\overline{(AD)} \ . \qquad \text{(B-63)}$$

Proof: Let C and D have components c_{jk} and d_{sm} respectively. Since by definition,

$$A = \psi_i \psi_j \ , \qquad \text{(B-64)}$$

we compute

$$ACAD=\psi_i\psi_j c_{jk}\psi_k\psi_s d_{sm} \tag{B-65}$$

$$\overline{(ACAD)}=\psi_i\psi_j c_{jk}\psi_k\psi_s d_{si}$$

$$=\psi_k\psi_j c_{jk}\psi_i\psi_s d_{si}$$

$$=\overline{(AC)}\ \overline{(AD)} \quad . \tag{B-66}$$

Lemma 7.

$$fL(\overline{\delta_i})=2\left[\overline{\eta_i}-\overline{\delta_i}-i\overline{(\delta_o\eta_i)}\right] \quad . \tag{B-67}$$

Proof: By definition

$$\delta_i=(-1)^i\left[\delta_o\eta_o^i+\sum_{j=1}^{i}\frac{(-1)^j}{j}\overline{(\eta_{j-1})}\,\delta_o\eta_o^{i-j}\right] \tag{B-68}$$

$$L\delta_i=(-1)^i\left[\delta_o\sum_{s=1}^{i}\eta_o^{s-1}(L\eta_o)\eta_o^{i-s}-\sum_{j=2}^{i}(-1)^j\overline{(\eta_{j-2}L\eta_o)}\,\delta_o\eta_o^{i-j}\right.$$

$$\left.+\sum_{j=1}^{i}\frac{(-1)^j}{j}\overline{(\eta_{j-1})}\,\delta_o\sum_{s=1}^{i-j}\eta_o^{s-1}(L\eta_o)\eta_o^{i-j-s}\right] \quad . \tag{B-69}$$

Note that

$$L\delta_o=\frac{L(A)}{\bar{A}}-A\frac{L(\bar{A})}{(\bar{A})^2}=\frac{2f'}{f}\left[\frac{A}{\bar{A}}-\frac{A}{\bar{A}}\right]=0 \tag{B-70}$$

(use also lemma 4, part ii)

Recall that by lemma 4, part (i):

$$fL(\eta_o)=\epsilon\left[\delta_o\eta_o-\delta_o(\overline{\delta_o\eta_o})\right]+2\left[\delta_o(\overline{\eta_o^2})-(\eta_o^2)\right] \quad . \tag{B-71}$$

In computing $L(\overline{\delta}_i)$ we notice that there will be terms proportional to ϵ and other terms. We expect the "ϵ-terms" to drop out. Define G_i to be the coefficient of ϵ when $fL\eta_o$ is substituted in the expansion of $L(\overline{\delta}_i)$.

$$G_i=(-1)^i\Bigg[\sum_{s=i}^{i}\overline{\Big[\delta_o\eta_o^{s-1}\Big\{\delta_o\eta_o-\delta_o(\overline{\delta_o\eta_o})\Big\}\eta_o^{i-s}\Big]}$$

$$-\sum_{j=2}^{i}(-1)^j\Big[\overline{(\eta_{j-2}\delta_o\eta_o)}-(\overline{\delta_o\eta_o})(\overline{\eta_{j-2}\delta_o})\Big](\overline{\delta_o\eta_o^{i-j}})$$

$$+\sum_{j=1}^{i}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})\sum_{s=1}^{i-j}\overline{\Big[(\delta_o\eta_o^{s-1})\ \delta_o\eta_o-\delta_o(\overline{\delta_o\eta_o})\ \eta_o^{i-j-s}\Big]}\Bigg]. \tag{B-72}$$

Using lemma 4 and changing an index, we obtain:

$$G_i=(-1)^i\Bigg[\sum_{s=1}^{i}(\overline{\delta_o\eta_o^{s-1}})\Big[\overline{(\delta_o\eta_o^{i-s+1})}-(\overline{\delta_o\eta_o})(\overline{\delta_o\eta_o^{i-s}})\Big]$$

$$+\sum_{j=1}^{i-1}(-1)^j(\overline{\delta_o\eta_o^{i-j-1}})\Big[\overline{(\eta_{j-1}\delta_o\eta_o)}-(\overline{\delta_o\eta_o})(\overline{\eta_{j-1}\delta_o})\Big]$$

$$+\sum_{j=1}^{i}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})\sum_{s=1}^{i-j}(\overline{\delta_o\eta_o^{s-1}})\Big[\overline{(\delta_o\eta_o^{i-j-s+1})}-(\overline{\delta_o\eta_o})(\overline{\delta_o\eta_o^{i-j-s}})\Big]\Bigg]. \tag{B-73}$$

We leave the proof that $G_i=0$ for a moment and return to the terms of $L(\overline{\delta}_i)$ that are not proportional to ϵ. They are:

$$2(-1)^i\left[\sum_{s=1}^{i}\overline{\left[\delta_o\eta_o^{s-1}\left\{\delta_o(\overline{\eta_o^2})-\eta_o^2\right\}\eta_o^{i-s}\right]}-\sum_{j=2}^{i}(-1)^j\overline{\left[\eta_{j-2}\delta_o(\overline{\eta_o^2})-\eta_o^2\right]}\ (\overline{\delta_o\eta_o^{i-j}})\right.$$

$$\left.+\sum_{j=1}^{i}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})\sum_{s=1}^{i-j}\overline{\left[\delta_o\eta_o^{s-1}\left\{\delta_o(\overline{\eta_o^2})-\eta_o^2\right\}\eta_o^{i-j-s}\right]}\right] \quad . \qquad \text{(B-74)}$$

It is clear that the lemma is proved if G_i is zero for all i and if the above quantity is equal to:

$$2\left\{\overline{\eta_i}-\overline{\delta}_i-i(\overline{\delta_o\eta_i})\right\} \quad . \qquad \text{(B-75)}$$

We define the difference (divided by 2) as F_i and require F_i to be zero.

$$F_i=(-1)^i\left[(\overline{\eta_o^2})\sum_{s=1}^{i}(\overline{\delta_o\eta_o^{s-1}})(\overline{\delta_o\eta_o^{i-s}})-\sum_{s=1}^{i}(\overline{\delta_o\eta_o^{i+1}})\right.$$

$$+\sum_{j=1}^{i}(-1)^j(\overline{\delta_o\eta_o^{i-j}})\left[(\overline{\eta_o^2\eta_{j-2}})-(\overline{\eta_o^2})(\overline{\delta_o\eta_{j-2}})\right]$$

$$\left.+\sum_{j=1}^{i}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})\sum_{s=1}^{i-j}\left[(\overline{\eta_o^2})(\overline{\delta_o\eta_o^{s-1}})(\overline{\delta_o\eta_o^{i-j-s}})-(\overline{\delta_o\eta_o^{i-j+1}})\right]\right]$$

$$-(\overline{\eta_i})+\overline{\delta_i}+i(\overline{\delta_o\eta_i}) \quad . \qquad \text{(B-76}$$

$$F_i=(-1)^i\left[(\overline{\eta_o^2})\sum_{s=1}^{i}(\overline{\delta_o\eta_o^{s-1}})(\overline{\delta_o\eta_o^{i-s}})\right]+\sum_{j=2}^{i}(-1)^j(\overline{\delta_o\eta_o^{i-j}})\left[\overline{(\eta_o^2\eta_{j-2})}-(\overline{\eta_o^2})(\overline{\delta_o\eta_{j-2}})\right]$$

$$+(\overline{\eta_o^2})\sum_{j=1}^{i}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})\sum_{s=1}^{i-j}(\overline{\delta_o\eta_o^{s-1}})(\overline{\delta_o\eta_o^{i-j-s}})+\sum_{j=1}^{i}(-1)^j(\overline{\eta_{j-1}})(\overline{\delta_o\eta_o^{i-j+1}})$$

$$-(\eta_o^{i+1})-\sum_{j=1}^{i}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})(\overline{\eta_o^{i-j+1}})+(\overline{\delta_o\eta_o^{i}})+\sum_{j=1}^{i}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})(\overline{\delta_o\eta_o^{i-j}}) \quad . \qquad \text{(B-77)}$$

$$F_i=(\eta_o^2)\left\{\sum_{s=1}^{i}(\overline{\delta_o\eta_o^{s-1}})(\overline{\delta_o\eta_o^{i-s}})-\sum_{j=2}^{i}(-1)^j(\overline{\delta_o\eta_o^{i-j}})(\overline{\delta_o\eta_{j-2}})\right.$$

$$\left.+\sum_{j=1}^{i-1}\frac{(-1)^j(\overline{\eta_{j-1}})}{j}\sum_{s=1}^{i-j}(\overline{\delta_o\eta_o^{s-1}})(\overline{\delta_o\eta_o^{i-j-s}})\right\}+(\overline{\delta_o\eta_o^{i}})-(\overline{\eta_o^{i+1}})$$

$$+\sum_{j=2}^{i}(-1)^j(\overline{\eta_o^2\eta_{j-2}})(\overline{\delta_o\eta_o^{i-j}})-\sum_{j=1}^{i}\frac{(-1)^j(\overline{\eta_{j-1}})}{j}(\overline{\eta_o^{i-j+1}})+(-1)^i(\overline{\eta_i})-(\overline{\delta_o\eta_o^{i}})$$

$$+\sum_{j=1}^{i}(-1)^j(\overline{\eta_o\eta_{j-1}})(\overline{\delta_o\eta_o^{i-j}}) \quad . \qquad \text{(B-78)}$$

The following expression from lemma 3 is used in obtaining (B-78) from (B-77):

$$\left[\eta_j=\frac{(\overline{\eta_{j-1}})}{j}-(\overline{\eta_o\eta_{j-1}})\right] \quad .$$

$$(-1)^i F_i = (\overline{\eta_o^2}) \left\{ \sum_{s=1}^{i} (\overline{\delta_o \eta_o^{s-1}})(\overline{\delta_o \eta_o^{i-s}}) + \sum_{j=1}^{i-1} \frac{(-1)^j (\overline{\eta_{j-1}})}{j} \sum_{s=1}^{i-j} (\overline{\delta_o \eta_o^{s-1}})(\overline{\delta_o \eta_o^{i-j-s}}) \right.$$

$$\left. - \sum_{j=2}^{i} (\overline{\delta_o \eta_o^{i-j}}) \left[(\overline{\delta_o \eta_o^{j-1}}) + \sum_{k=1}^{j-2} \frac{(-1)^k}{k} (\overline{\eta_{k-1}})(\overline{\delta_o \eta_o^{j-k-1}}) \right] \right\}$$

$$+ \sum_{j=1}^{i} (-1)^j (\overline{\eta_1 \eta_{j-1}})(\overline{\delta_o \eta_o^{i-j}}) + \sum_{j=2}^{i} (-1)^j (\overline{\eta_o^2 \eta_{j-2}})(\overline{\delta_o \eta_o^{i-j}}) \quad . \qquad \text{(B-79)}$$

We first expand the last two terms and then consider the coefficient of $(\overline{\eta_o^2})$.

$$\sum_{j=2}^{i} (\overline{\delta_o \eta_o^{i-j}}) \left[(\overline{\eta_o^{j+1}}) + \sum_{k=1}^{j-2} \frac{(-1)^k}{k} (\overline{\eta_{k-1}})(\overline{\eta_1^{j-k+1}}) \right]$$

$$- \sum_{j=1}^{i} (\overline{\delta_o \eta_o^{i-j}}) \left[(\overline{\eta_o^{j+1}}) + \sum_{k=1}^{j-1} \frac{(-1)^k}{k} (\overline{\eta_{k-1}})(\overline{\eta_o^{j-k+1}}) \right]$$

$$= - (\overline{\eta_o^2})(\overline{\delta_o \eta_o^{i-1}}) + \sum_{j=2}^{i} \frac{(-1)^j (\overline{\eta_{j-2}})}{j-1} (\overline{\eta_o^2})(\overline{\delta_o \eta_o^{i-j}}) \quad . \qquad \text{(B-80)}$$

Each term of F_i has a coefficient $(\overline{\eta_o^2})$. With some cancellation we obtain:

$$F_i=(-1)^i(\overline{\eta_o^2})\left[\sum_{j=1}^{i-1}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})\sum_{s=1}^{i-j}(\overline{\delta_o\eta_o^{s-1}})(\overline{\delta_o\eta_o^{i-j-s}})\right.$$

$$\left.-\sum_{j=3}^{i}\sum_{k=1}^{j-2}(\overline{\delta_o\eta_o^{i-j}})\frac{(-1)^k}{k}(\overline{\eta_{k-1}})(\overline{\delta_o\eta_o^{j-k-1}})+\sum_{j=2}^{i}\frac{(-1)^j(\overline{\eta_{j-2}})}{j-1}(\overline{\delta_o\eta_o^{i-j}})\right]. \tag{B-81}$$

The single sum term with a change of index is equal to:

$$-\sum_{j=1}^{i-1}\frac{(-1)^j(\overline{\eta_{j-1}})}{j}(\overline{\delta_o\eta_o^{i-j-1}}). \tag{B-82}$$

This quantity just cancels the s=1 term of the first double sum. Finally we have with a change of index:

$$F_i=(-1)^i(\overline{\eta_o^2})\left[\sum_{j=1}^{i-1}\sum_{k=2}^{i-j}\frac{(-1)^j}{j}(\overline{\eta_{j-1}})(\overline{\delta_o\eta_o^{i-1}})(\overline{\delta_o\eta_o^{i-j-k}})\right.$$

$$\left.-\sum_{j=2}^{i-1}\sum_{k=1}^{j-1}\frac{(-1)^k}{k}(\overline{\eta_{k-1}})(\overline{\delta_o\eta_o^{i-j-1}})(\overline{\delta_o\eta_o^{j-k}})\right]. \tag{B-83}$$

That F_i is zero as required follows from the purely algebraic identity (proved in lemma 11).

$$\sum_{j=1}^{i-1}\sum_{k-2}^{i-j}a_jb_{k-1}b_{i-j-k}=\sum_{j=2}^{i-1}\sum_{k=1}^{j-1}a_kb_{i-j-1}b_{j-k} \tag{B-84}$$

with obvious substitutions.

We complete the proof of the lemma by showing that $G_i=0$.

$$(-1)^i G_i = \sum_{s=1}^{i} \overline{(\delta_o \eta_o^{s-1})} \left[\overline{(\delta_o \eta_o^{i-s+1})} - \overline{(\delta_o \eta_o)}\,\overline{(\delta_o \eta_o^{i-s})} \right]$$

$$- \sum_{s=1}^{i-1} \left[\overline{(\delta_o \eta_o^{i-s-1})} \left\{ \overline{(\delta_o \eta_o^{s+1})} - \overline{(\delta_o \eta_o)}\,\overline{(\delta_o \eta_o^{s})} \right\} \right]$$

$$+ \sum_{j=1}^{i-1} \sum_{k=1}^{j-1} \frac{(-1)^k \overline{(\eta_{k-1})}}{k} \left[\overline{(\delta_o \eta_o)}\,\overline{(\delta_o \eta_o^{i-j-1})}\,\overline{(\delta_o \eta_o^{j-k})} - \overline{(\delta_o \eta_o^{i-j-1})}\,\overline{(\delta_o \eta_o^{j-k+1})} \right]$$

$$+ \sum_{j=1}^{i-1} \sum_{k=1}^{i-j} \frac{(-1)^j}{j} \overline{(\eta_{j-1})} \left[\overline{(\delta_o \eta_o^{i-1})}\,\overline{(\delta_o \eta_o^{i-j-k+1})} - \overline{(\delta_o \eta_o)}\,\overline{(\delta_o \eta_o^{i-1})}\,\overline{(\delta_o \eta_o^{i-j-k})} \right] . \qquad \text{(B-85)}$$

A change of index, $k=i-s$, yields for the first two sums:

$$\sum_{k=1}^{i} \overline{(\delta_o \eta_o^{i-1})} \left[\overline{(\delta_o \eta_o^{i-k+1})} - \overline{(\delta_o \eta_o)}\,\overline{(\delta_o \eta_o^{i-k})} \right]$$

$$- \sum_{k=1}^{i-1} \overline{(\delta_o \eta_o^{k-1})} \left[\overline{(\delta_o \eta_o^{i-k+1})} - \overline{(\delta_o \eta_o)}\,\overline{(\delta_o \eta_o^{i-k})} \right] = \overline{(\delta_o \eta_o^{i-1})} \left[\overline{(\delta_o \eta_o)} - \overline{(\delta_o \eta_o)} \right] = 0 . \qquad \text{(B-86)}$$

The following algebraic identity, proved in lemma 11;

$$\sum_{j=1}^{i-1} \sum_{k=1}^{j-1} a_k \left[b_1 b_{i-j-1} b_{j-k} - b_{i-j-1} b_{j-k+1} \right] = \sum_{j=1}^{i-1} \sum_{k=1}^{i-j} \left[a_j b_1 b_{k-1} b_{i-j-k} - b_{k-1} b_{i-j-k+1} \right],$$

$$b_o = 1 , \qquad \text{(B-87)}$$

yields the required result that $G_i=0$ when we substitute

$$a_k = \frac{(-1)^k}{k}(\overline{\eta_{k-1}}), \qquad b_j = (\overline{\delta_o \eta_o^j}) \quad . \tag{B-88}$$

Lemma 8.

$$fL(\overline{\delta_{i-1}}) = 2(i+1)\,\bar{\delta}_i - \frac{2}{i}(\overline{\eta_{i-1}}) - 4(\overline{\delta_o \eta_o \eta_{i-2}}) + 4(\overline{\delta_o \eta_o})(\overline{\delta_o \eta_{i-2}}) \tag{B-89}$$

$$-4\bar{\delta}_i(\overline{\delta_{i-1}}) + 2(\overline{\delta_{i-1}}) \qquad i=2,\ldots \tag{B-89}$$

Proof: Reduce the right side using lemma 3. Since

$$\bar{\delta}_i = \frac{(\overline{\eta_{i-1}})}{i} - (\overline{\delta_o \eta_{i-1}}) \tag{B-90}$$

and

$$\eta_{i-1} = \frac{\eta_o(\overline{\eta_{i-2}})}{i-1} - \eta_o \eta_{i-2} \quad , \tag{B-91}$$

we obtain:

$$\text{right side} = 2(i+1)\left[\frac{(\overline{\eta_{i-1}})}{i} - (\overline{\delta_o \eta_{i-1}})\right] - \frac{2}{i}(\overline{\eta_{i-1}}) - 4\left[\frac{(\overline{\delta_o \eta_o})(\overline{\eta_{i-2}})}{i-1} - (\overline{\delta_o \eta_{i-1}})\right]$$

$$+4(\overline{\delta_o \eta_o})(\overline{\delta_o \eta_{i-2}}) - 4(\overline{\delta_{i-1}})\left[1-(\overline{\delta_o \eta_o})\right] + 2(\overline{\delta_{i-1}}) \tag{B-92}$$

$$\text{right side} = 2(\overline{\eta_{i-1}}) - 2(i-1)(\overline{\delta_o \eta_{i-1}}) - 2(\overline{\delta_{i-1}}) + 4(\overline{\delta_o \eta_o})\left[\bar{\delta}_{i-1} + (\overline{\delta_o \eta_{i-2}}) - \frac{\overline{\eta_{i-2}}}{i-1}\right] \tag{B-93}$$

$$\text{right side} = 2\left[(\overline{\eta_{i-1}}) - (\overline{\delta_{i-1}}) - (i-1)(\overline{\delta_o \eta_{i-1}})\right] \tag{B-94}$$

By lemma 7, the above is equal to $fL(\delta_{i-1})$.

Lemma 9.

$$fL(\bar{\beta}_o)=2\bar{\beta}_1+2(\bar{\beta}_o)^2 \quad . \tag{B-95}$$

Proof: By definition we have

$$\bar{\beta}_1=2(\bar{\beta}_o+1)\bar{\delta}_1-\bar{\eta}_1=2f'\left[1-(\overline{\delta_o\eta_o})\right]-\left[1-(\overline{\eta_o^2})\right] \tag{B-96}$$

$$=2f'\left[1-\frac{(\overline{AB})}{\bar{A}\bar{B}}-1+\frac{(\overline{B^2})}{(\bar{B})^2}\right] \quad . \tag{B-96}$$

Write (III-27) of Chapter III in matrix notation:

$$ff'\,\psi_t^2+\frac{2f'}{f}\left[2\psi_j\psi_i\psi_{ij}-f'f\psi_t^2\right]=2\psi_{ij}^2 \tag{B-97}$$

$$ff''(\bar{B})^2+4f'\frac{(\overline{AB})}{\bar{A}}-2(f')^2(\bar{B})^2=2(\overline{B^2}) \tag{B-98}$$

$$ff''=\frac{2(\overline{B^2})}{(\bar{B})^2}+2(f')^2-4f'\frac{(\overline{AB})}{\bar{A}\bar{B}} \quad . \tag{B-99}$$

Combine (B-96) and (B-99):

$$ff'=2\bar{\beta}_1+2+2(f')^2-4f'=2\bar{\beta}_1+(\bar{\beta}_o)^2 \quad . \tag{B-100}$$

Since $L\bar{\beta}_o=f''$ by lemma 1, the lemma follows from (B-100).

Lemma 10. $$2i\bar{\beta}_i=fL(\overline{\beta_{i-1}})-2\bar{\beta}_o\,\overline{\beta_{i-1}} \quad . \tag{B-101}$$

Proof: By definition

$$\bar{\beta}_i=(i+1)(\bar{\beta}_o+1)\,\bar{\delta}_i-\bar{\eta}_i \tag{B-102}$$

$$\overline{\beta_{i-1}}=i(\bar{\beta}_o+1)\,\bar{\delta}_{i-1}-\overline{\eta_{i-1}} \quad . \tag{B-103}$$

Operate on (B-102) with L and use lemmas 5 and 9:

$$fL(\overline{\beta_{i-1}})=2i(\overline{\delta_{i-1}})\left[\overline{\beta_1}+(\overline{\beta_o^2})\right]+i(\overline{\beta_o}+1)fL(\overline{\delta_{i-1}})+i(\overline{\eta_{i-2}\,fL\eta_o}) \quad . \tag{B-104}$$

From lemma 4:

$$(\overline{\eta_{i-2}\,fL\eta_o})=4(\overline{\beta_o}+1)\left[(\overline{\delta_o\eta_o\eta_{i-2}})-(\overline{\delta_o\eta_{i-2}})(\overline{\delta_o\eta_o})\right]+2(\overline{\eta_o^2})(\overline{\delta_o\eta_{i-2}})-2(\overline{\eta_o^2\eta_{i-2}}) \quad . \tag{B-105}$$

Using (B-101), (B-102), (B-103), and (B-104) and lemma 7, we form

$$fL(\overline{\beta_{i-1}})-2\overline{\beta_o}(\overline{\beta_{i-1}})-2i\overline{\beta_i}=2i(\overline{\delta_{i-1}})\left[\overline{\beta_i}+(\overline{\beta_o^2})\right]+i(\overline{\beta_o}+1)\ X$$

$$X\left[2(i+1)\overline{\delta_i}-\frac{2}{i}\ \overline{\eta_{i-1}}-4(\overline{\delta_o\eta_o\eta_{i-2}})+4(\overline{\delta_o\eta_o})(\overline{\delta_o\eta_{i-2}})-4(\overline{\delta_{i-1}})\overline{\delta_i}+2\overline{\delta_{i-1}}\right]$$

$$+4i(\overline{\beta_o}+1)\left[(\overline{\delta_o\eta_o\eta_{i-2}})-(\overline{\delta_o\eta_{i-2}})(\overline{\delta_o\eta_o})\right]+2i(\overline{\eta_o^2})(\overline{\delta_o\eta_{i-2}})$$

$$-2i(\overline{\eta_o^2\eta_{i-2}})-2i\ \overline{\beta_o}(\overline{\beta_o}+1)(\overline{\delta_{i-1}})+2\overline{\beta_o}(\overline{\eta_{i-1}})-2i(i+1)(\overline{\beta_o}+1)\overline{\delta_i}+2i\,\overline{\eta_i} \quad . \tag{B-106}$$

After simplification the right side of (B-105) has the remainder, R

$$R=2i\left[\overline{\eta_i}+(\overline{\eta_o^2})(\overline{\delta_o\eta_{i-2}})-(\overline{\eta_o^2\eta_{i-2}})+\overline{\delta_{i-1}}(\overline{\beta_1}-\overline{\beta_o})+(1-2\overline{\delta_1})(\overline{\beta_o}+1)\overline{\delta_{i-1}}\right]-2(\overline{\eta_{i-1}}). \tag{B-107}$$

From (B-102) we have:

$$\overline{\beta_1}=2(\overline{\beta_o}+1)\overline{\delta_1}-\overline{\eta_1}=2(\overline{\beta_o}+1)\overline{\delta_1}-1+(\overline{\eta_o^2}) \quad . \tag{B-108}$$

The coefficient of $\overline{\delta_{i-1}}$ in R becomes

$$\overline{\beta_1}-\overline{\beta_o}+(1-2\overline{\delta_1})(\overline{\beta_o}+1)=(\overline{\eta_o^2}) \quad . \tag{B109}$$

Thus R reduces to

$$R=2i\left[\overline{\eta}_i+(\overline{\eta_o^2})(\overline{\delta_o\eta_{i-2}})-(\overline{\eta_o^2\eta_{i-2}})+(\overline{\eta_o^2})\overline{\delta}_{i-1}\right]-2(\overline{\eta_{i-1}}) \quad . \tag{B-110}$$

From lemma 3 we obtain the reduction formulas:

$$\delta_{i-1}=\frac{(\overline{\eta_{i-2}})}{i-1}-(\overline{\delta_o\eta_{i-2}}) \tag{B-111}$$

$$\overline{\eta}_i=\frac{(\overline{\eta_{i-1}})}{i}-(\overline{\eta_o\eta_{i-1}})=\frac{(\overline{\eta_{i-1}})}{i}-\frac{(\eta_o^2)(\overline{\eta_{i-2}})}{i-1}+(\overline{\eta_o^2\eta_{i-2}}) \quad . \tag{B-112}$$

With the aid of equations (B-111) and (B-112) we find that R vanishes, which proves the lemma.

$$R=2i\left[\frac{(\overline{\eta_{i-1}})}{i}-\frac{(\overline{\eta_o^2})(\overline{\eta_{i-2}})}{i-1}+(\overline{\eta_o^2\eta_{i-2}})+(\overline{\eta_o^2})(\overline{\delta_o\eta_{i-2}})\right.$$

$$\left.-(\overline{\eta_o^2\eta_{i-2}})+\frac{(\overline{\eta_o^2})(\overline{\eta_{i-2}})}{i-1}-(\overline{\eta_o^2})(\overline{\delta_o\eta_{i-2}})\right]-2(\overline{\eta_{i-1}}) \quad . \tag{B-113}$$

$$R=0. \tag{B-114}$$

Lemma 11. The following algebraic identities hold with the indicated restrictions:

(i) $$\sum_{j=2}^{i-1}\sum_{k=1}^{j-1}a_kb_{i-j}c_{j-k}=\sum_{j=2}^{i-1}\sum_{k=1}^{j-1}a_kb_{j-k}c_{i-j} \qquad i=3,4,\ldots \tag{B-115}$$

(ii) $$\sum_{j=1}^{i-1}\sum_{k=2}^{i-j}a_jb_{k-1}b_{i-j-k}=\sum_{j=2}^{i-1}\sum_{k=1}^{j-1}a_kb_{i-j-1}b_{j-k} \qquad i=3,4,\ldots \tag{B-116}$$

(iii) $$\sum_{j=2}^{i-1}\sum_{k=1}^{j-1} a_k(b_1 b_{i-j-1} b_{j-k} - b_{i-j-1} b_{j-k+1}) =$$

$$\sum_{j=1}^{i-1}\sum_{k=1}^{i-j} a_j(b_1 b_{k-1} b_{i-j-k} - b_{k-1} b_{i-j-k+1}) \qquad b_o = 1,$$

$$i = 3, 4, \ldots \qquad \text{(B-117)}$$

Proof: (i) Compare coefficients of a_r, $1 \leqslant r \leqslant i-2$, for fixed r.

$$\sum_{j=r+1}^{i-1} b_{i-j} c_{j-r} = \sum_{j=r+1}^{i-1} b_{j-r} c_{i-j} \quad . \qquad \text{(B-118)}$$

The above follows from the change of index $j'-r = i-j$.

(ii) Again compare coefficients of a_r for fixed r, $1 \leqslant r \leqslant i-1$. Since, on the right, $j-1 \geqslant r$, we require

$$\sum_{k=2}^{i-r} b_{k-1} b_{i-k-r} = \sum_{j=r+1}^{i-1} b_{i-j-1} b_{j-r} \quad . \qquad \text{(B-119)}$$

It suffices to change index: $j-r = k-1$.

(iii) Equate coefficients of a_r. We must show

$$\sum_{j=r+1}^{i-1} (b_1 b_{i-j-1} b_{j-k} - b_{i-j-1} b_{j-r+1})$$

$$= \sum_{k=1}^{i-r} (b_1 b_{k-1} b_{i-r-k} - b_{k-1} b_{i-r-k+1}) \quad . \qquad \text{(B-120)}$$

On the left let k=i-j; then break the right side into two terms and use the restriction $b_o = 1$:

$$\sum_{k=1}^{i-r-1} (b_1 b_{i-1} b_{i-r-k} - b_{k-1} b_{i-r-k+1}) = \sum_{k=1}^{i-r-1} (b_1 b_{k-1} b_{i-r-k} - b_{k-1} b_{i-r-k+1})$$

$$+ b_{i-r-1} (b_1 b_o - b_1) \quad . \qquad \text{(B-121)}$$

APPENDIX C

Supplement to Chapter VI

The problem of determining the limiting behavior for large n of polynomials, $R_n(k)$, generated by the Burmann expansion of an entire function, $y(kx)$, in powers of $y(x)$, was introduced in Chapter VI. Sufficient conditions involving the series coefficients of the inverse $x(y)$ were given in theorem 2 in order that $\lim R_n(k) = y'(\lambda k)$. We designate the radius of convergence of the inverse to be ρ. Then

$$y(\lambda) = \rho \tag{C-1}$$

A number of examples will be discussed which illustrate the results of theorem 2 and bear upon the question of determining the class of functions $y(x)$ having the above properties. We adopt the following notation and conventions:

$$y(x) = \sum_{n=1}^{\infty} a_n x^n \qquad a_1 = 1 \tag{C-2}$$

$$y^m(x) = \sum_{n=m}^{\infty} a_{m,n} x^n \tag{C-3}$$

$$x(y) = \sum_{n=1}^{\infty} b_n y^n \qquad |y| \leqslant \rho \tag{C-4}$$

(i.e. we assume the series converges and represents $x(y)$ on the circle of convergence, e.g. $b_n \sim \frac{\rho^{-n}}{n^{1+\epsilon}}$, $\epsilon > 0$.)

$$x^m(y) = \sum_{n=m}^{\infty} b_{m,n} y^n \tag{C-5}$$

$$y(kx) = \sum_{n=1}^{\infty} k b_n R_n(k) \left[y(x)\right]^n \tag{C-6}$$

$$R_n(k) = \sum_{m=1}^{n} a_m \frac{b_{m,n}}{b_n} K^{m-1} \qquad \text{(C-7)}$$

(This follows from theorem 1 and equation (VI-13).)

Putting (C-5) into (C-2) yields

$$y = \sum_{m=1}^{\infty} a_m x^m = \sum_{m=1}^{\infty} a_m \sum_{n=m}^{\infty} b_{m,n} y^n = \sum_{n=1}^{\infty} y^n \sum_{m=1}^{n} a_m b_{m,n} \qquad \text{(C-8)}$$

Hence

$$b_1 = 1 \qquad \text{(C-9)}$$

$$\sum_{m=1}^{n} a_m b_{m,n} = 0 \qquad n > 1 \qquad \text{(C-10)}$$

Since $x^m = x^{m-1} \cdot x$, we have

$$\sum_{n=m}^{\infty} b_{m,n} y^n = \sum_{n=m-1}^{\infty} b_{m-1,n} y^n \sum_{j=1}^{\infty} b_j y^j = \sum_{n=m}^{\infty} y^n \sum_{j=m-1}^{n-1} b_{n-j} b_{m-1,j} \qquad \text{(C-11)}$$

Hence

$$b_{m,n} = \sum_{j=m-1}^{n-1} b_{n-j} b_{m-1,j} \qquad \text{(C-12)}$$

Equations (C-9), (C-10), and (C-11) suffice to determine $b_{m,n}$. Following theorem 2, we are particularly interested in establishing conditions on a_n or b_n such that

$$\frac{b_{m,n}}{m b_n} \leqslant \lambda^{m-1} \qquad m \text{ fixed, } n \text{ sufficiently large} \qquad \text{(C-13)}$$

$$\lim_{n \to \infty} \frac{b_{m,n}}{m b_n} = \lambda^{m-1} \qquad m \text{ fixed} \qquad \text{(C-14)}$$

Example 1. Consider $y(x) = xe^{-x}$. Then

$$a_n = \frac{(-1)^{n-1}}{(n-1)!} \tag{C-15}$$

The coefficients for the non-zero powers of y are easily obtained in this case.

$$\left[y(x)\right]^m = x^m e^{-mx} = \sum_{j=0}^{\infty} \frac{(-1)^j m^j x^{m+j}}{j!} = \sum_{n=m}^{\infty} \frac{(-1)^{n-m} m^{n-m} x^n}{(n-m)!} \tag{C-16}$$

Hence

$$a_{m,n} = \frac{(-1)^{n-m} m^{n-m}}{(n-m)!} \qquad m \neq 0 \quad n \geqslant m \tag{C-17}$$

The simplest and most elegant way of determining the coefficients $b_{m,n}$ is by means of Jabotinsky's formula

$$b_{m,n} = \frac{m}{n} a_{-b,-m} \qquad n \neq 0 \tag{C-18}$$

$$b_{m,n} = \frac{m}{n} \frac{(-1)^{n-m}(-n)^{n-m}}{(n-m)!} \frac{mn^{n-m-1}}{(n-m)!} \qquad n \neq 0 \tag{C-19}$$

In particular

$$b_n = b_{1,n} = \frac{n^{n-2}}{(n-1)!} \tag{C-20}$$

From Stirling's formula

$$b_n = \frac{n^{n-2}}{\Gamma(n)} \cong \frac{1}{\sqrt{2\pi}} \frac{n^{n-2}}{n^{n-1/2}e^{-n}} = \frac{e^n}{\sqrt{2\pi}\, n^{3/2}} \tag{C-21}$$

$$\rho = 1/e$$

$$\lambda = 1$$

Since

$$y = xe^{-x} \tag{C-22}$$

$$y' = (1 - x)e^{-x} \tag{C-23}$$

$$y'(\lambda) = 0 \tag{C-24}$$

Let us examine the conditions of equations (C-13) and (C-14). In this case we have

$$\frac{b_{m,n}}{mb_n} = \frac{(n-1)!}{(n-m)!\, n^{m-1}} = \frac{(n-1)(n-2)\ldots(n-m+1)}{n^{m-1}} \tag{C-25}$$

$$= \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{(m-1)}{n}\right) \tag{C-26}$$

It is clear from equation (C-26) that

$$b_{m,n} \leqslant \lambda^{m-1} = 1 \tag{C-27}$$

and

$$\lim_{n\to\infty} \frac{b_{m,n}}{mb_n} = \lambda^{m-1} \qquad m \text{ fixed} \tag{C-28}$$

The conditions of theorem 2 are satisfied. We expect, therefore, that

$$\lim_{n\to\infty} R_n(k) = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{(-1)^{m-1} m(n-1)!\, k^{m-1}}{(m-1)!\,(n-m)!\, n^{m-1}} = y'(\lambda k) = (1-k)e^{-k} \tag{C-29}$$

The verification is facilitated by writing out a few R_n's and inducing the factored form

$$R_n(k) = (1-k)\left(1 - \frac{k}{n}\right)^{n-2}, \tag{C-30}$$

which is easily checked by a binomial expansion. From equation (C-30) the uniform convergence of $R_n(k)$ to the expected limit follows directly.

Finally, we note an interesting identity arising from a direct verification of the form of $b_{m,n}$ which was obtained indirectly from Jabotinsky's formula. From

equations (C-5) and (C-19) we must have

$$x^m = \sum_{n=m}^{\infty} \frac{mn^{n-m-1}}{(n-m)!} x^n e^{-nx} \qquad x \leqslant 1 \qquad \text{(C-31)}$$

Inserting a Taylor series for the exponential and rearranging the order of summation yields ultimately

$$\sum_{m=0}^{n} \frac{(-1)^m (m+j)^{n-1}}{m!\,(n-m)!} = 0 \qquad n \geqslant 1 \quad j \text{ an integer} \qquad \text{(C-32)}$$

That (C-32) holds for integral j follows from the preceding argument. Alternatively, we can establish (C-32) independently by showing that the coefficients of the expansion of the sum as a polynomial in j vanish identically. This requires that

$$\sum_{m=1}^{n} \frac{(-1)^m m^s}{m!\,(n-m)!} = 0 \qquad n \geqslant 2 \quad n-1 \geqslant s \geqslant 1 \qquad \text{(C-33)}$$

The left side of (C-33) may be written as

$$\lim_{x \to 1} \left(-x \frac{d}{dx}\right)^s \frac{(1-x)^n}{n!} \qquad n \geqslant 2 \quad n-1 \geqslant s \geqslant 1 \qquad \text{(C-34)}$$

which must be zero under the indicated restrictions since a factor (1 - x) appears in each term. This establishes (C-33) and (C-32). Thus, the restriction in (C-32) that j be an integer is unnecessary.

Example 2. A subclass of the entire functions of natural interest is that of the polynomials. The simplest case, $y = x$, is trivial. The polynomial of second degree

$$y = x + a_2 x^2 \qquad a_2 \neq 0 \qquad \text{(C-35)}$$

can be simplified by the transformation, $\bar{y} = -a_2 y$, $\bar{x} = -a_2 x$ to

$$y = x - x^2 \qquad \text{(C-36)}$$

Substituting the inverse series for x and x^2 yields

$$y = \sum_{n=1}^{\infty} b_n y^n - \sum_{n=2}^{\infty} b_{2,n} y^n \qquad \text{(C-37)}$$

Since $b_1 = 1$, we conclude

$$b_n = b_{2,n} \qquad n \geqslant 2 \qquad \text{(C-38)}$$

Since, from (C-36)

$$a_1 = 1 \qquad a_2 = -1 \qquad a_n = 0 \qquad n \geqslant 3 \qquad \text{(C-39)}$$

we have

$$R_n(k) = \sum_{m=1}^{n} a_m \frac{b_{m,n}}{b_n} k^{m-1} = \begin{cases} 1 & n = 1 \\ 1-k & n \geqslant 2 \end{cases} \qquad \text{(C-40)}$$

Since

$$y' = 1 - 2x, \qquad \lambda = \frac{1}{2}, \qquad y'(\lambda k) = 1 - k, \qquad \text{(C-41)}$$

$$\lim_{n \to \infty} R_n(k) = y'(\lambda k) . \qquad \text{(C-42)}$$

Thus we have a simple illustration of theorem 2. Note that

$$\frac{b_{2,n}}{2b_n} = \frac{1}{2} = \lambda . \qquad \text{(C-43)}$$

Example 3. The polynomial of degree three

$$y = x + a_2 x^2 + a_3 x^3 \qquad a_2, a_3 \neq 0 \qquad \text{(C-44)}$$

can be simplified by the transformation $\bar{y} = -a_2 y$, $\bar{x} = -a_2 x$ to

$$y = x - x^2 + \delta x^3 \qquad \left(\delta = \frac{a_3}{a_2^2}\right) \qquad \text{(C-45)}$$

In order to employ Jabotinsky's formula, we attempt to determine $a_{-n,-m}$. Consider

$$\left[\frac{1}{y}\right]^n = \frac{1}{x^n(1-x+\delta x^2)^n} \tag{C-46}$$

The well-known generating function for the Gegenbauer polynomials,

$$(1 - 2\alpha t + \alpha^2)^{-n} = \sum_{j=0}^{\infty} C_j^n(t)\alpha^j , \tag{C-47}$$

yields, upon substituting $\alpha = \sqrt{\delta x}$ and $t = \frac{1}{2\sqrt{\delta}}$,

$$\frac{1}{y^n} = \frac{1}{x^n} \sum_{j=0}^{\infty} C_j^n \left(\frac{1}{2\sqrt{\delta}}\right) \delta^{j/2} x^j \tag{C-48}$$

Let $m = n - j$

$$\frac{1}{y^n} = \sum_{m=-\infty}^{n} C_{n-m}^n \left(\frac{1}{2\sqrt{\delta}}\right) \delta^{n-m/2} x^{-m} \tag{C-49}$$

We conclude that

$$a_{-n,-m} = C_{n-m}^n \left(\frac{1}{2\sqrt{\delta}}\right) \delta^{n-m/2} \tag{C-50}$$

By Jabotinsky's formula

$$b_{m,n} = \frac{m}{n} a_{-n,-m} = \frac{m}{n} C_{n-m}^n \left(\frac{1}{2\sqrt{\delta}}\right) \delta^{n-m/2} \tag{C-51}$$

That $b_{m,n}$ must be a polynomial function of δ is evident from the more direct methods of construction. Explicitly we have

$$b_{m,n} = m \sum_{j=0}^{\left[\frac{n-m}{2}\right]} \frac{(2n-m-j-1)!\,(-\delta)^j}{j!\,n!\,(n-m-2j)!} \tag{C-52}$$

Other representations are obtained by manipulating the hypergeometric form of (C-51).

So long as δ is not real and greater than $1/3$, we can define $\lambda(\delta)$ as the smallest zero of y' in magnitude

$$\lambda(\delta) = \min \frac{1 \pm \sqrt{1-3\delta}}{3\delta} = \frac{1 - \sqrt{1-3\delta}}{3\delta} \qquad \text{(C-64)}$$

(take $1-3\delta = r^2 e^{2i\theta}$, $-\pi/2 < \theta < \pi/2$, etc.)

From (C-62)

$$y'(\lambda k) = 1 - 2\lambda k + 3\delta k^2 = (1-k)\left[1+(1-2\lambda)k\right] \qquad \text{(C-65)}$$

By comparison with equation (C-61) we see that

$$R_n(k) \longrightarrow y'(\lambda k) \iff \frac{b_{2,n}}{2b_n} = \lambda \qquad \text{(C-66)}$$

By virture of the fact that $b_{2,n}$ and b_n are polynomials in δ with real coefficients, it is clear that for real δ their ratio must approach a real limit (if the limit exists). On the other hand, either of the possible definitions for λ in (C-64) is complex-valued for $\delta > 1/3$. We shall see that the cut, $\delta > 1/3$, gives difficulty in each of the various methods of attack.

From the hypergeometric expression for $b_{m,n}$ of equation (C-54), we obtain

$$\frac{b_{2,n}}{2b_n} = \frac{(3n-3)!\,2^{n-1}(n-1)!}{(n-2)!\,2^{n-2}(3n-2)!} \; \frac{{}_2F_1\left[\frac{2-n}{2}, \frac{3-n}{2}; n+\frac{1}{2}; 1-4\delta\right]}{{}_2F_1\left[\frac{1-n}{2}, \frac{2-n}{2}; n+\frac{1}{2}; 1-4\delta\right]} \qquad \text{(C-67)}$$

which is of the form

$$\frac{b_{2,n}}{2b_n} = c_n \frac{F\left[a, b+1; c; z\right]}{F\left[a, b; c; z\right]} \qquad \text{(C-68)}$$

where

$$c_n = \frac{2(n-1)}{3n-2}, \quad a = \frac{2-n}{2}, \quad b = \frac{1-n}{2}, \quad c = n+\frac{1}{2}, \quad z = 1-4\delta. \qquad \text{(C-69)}$$

$$b_{m,n} = \frac{m\delta^{\frac{n-m}{2}}\,(3n-m-1)}{n(n-m)!\,(2n-1)}\;{}_2F_1\left[3n-m,\ m-n;\ n+\frac{1}{2};\ \frac{1-\frac{1}{2\sqrt{\delta}}}{2}\right] \tag{C-53}$$

$$b_{m,n} = \frac{m(3n-m-1)!}{n(n-m)!\,(2n-1)!\,2^{n-m}}\;{}_2F_1\left[\frac{m-n}{2},\ \frac{1+m-n}{2};\ n+\frac{1}{2};\ 1-4\delta\right] \tag{C-54}$$

$$b_{m,n} = \frac{m(3n-m-1)!}{n(n-m)!\,\left[(n-1)!\right]^2 2^{n-m}}\int_0^1 \left[t(1-t)\right]^{n-1}\left[1+(1-2t)\sqrt{1-4\delta}\right]^{n-m} dt \tag{C-55}$$

Before attempting to determine the asymptotic behavior of the coefficients for large n, we return to the function y(x) to get some idea of what might be expected.

$$y(x) = x - x^2 + \delta x^3 \tag{C-56}$$

$$y(kx) - ky(x) = k(1-k)\left[x^2 - (1+k)x^3\right] \tag{C-57}$$

$$y(kx) = ky(x) + k(1-k)\left[-kx^2 - (1+k)(y-x)\right] \tag{C-58}$$

$$y(kx) = ky + k(1-k)\sum_{n=2}^{\infty}\left[(1+k)b_n y^n - kb_{2,n}y^n\right] \tag{C-59}$$

By comparison with the defining equation for $R_n(k)$ we conclude

$$R_1(k) = 1 \tag{C-60}$$

$$R_n(k) = (1-k)\left[1 + k\left(1 - \frac{b_{2,n}}{b_n}\right)\right] \tag{C-61}$$

In order to determine the limiting behavior of $R_n(k)$, it is necessary only to examine $b_{2,n}$ and b_n. Differentiating equation (C-56) we obtain

$$y' = 1 - 2x + 3\delta x^2 \tag{C-62}$$

$$y' = 0 \implies x = \frac{1 \pm \sqrt{1-3\delta}}{3\delta} \tag{C-63}$$

Since

$$F(a, b; c; z) = F(a, b+1; c; z) - \frac{az}{c} F(a+1, b+1; c+1; z), \qquad \text{(C-70)}$$

we obtain

$$\frac{b_{2,n}}{2b_n} = c_n \frac{1}{1 + \frac{(n-2)}{2n+1} z \frac{F(a+1, b+1, c+1; z)}{F(a, b+1, c+1; z)}} = \frac{c_n}{1 + \frac{(n-2)}{2n+1} z \left[\frac{1}{1 - \frac{u_1 z}{1 - \frac{v_1 z}{1 - u_2 z}}} \right]} \qquad \text{(C-71)}$$

where

$$u_j = \frac{(j + \frac{1}{2} - \frac{n}{2})(j + \frac{3n}{2} - \frac{1}{2})}{(2j + n - \frac{1}{2})(2j + n + \frac{1}{2})} \to -\frac{3}{4}$$

$$v_j = \frac{(j + 1 - \frac{n}{2})(j + \frac{3n}{2})}{(2j + n + \frac{1}{2})(2j + n + \frac{3}{2})} \to -\frac{3}{4}$$

The last step is a special case of Gauss' continued fraction for the ratio of two hypergeometric functions. The convergence of continued fraction of this type has been investigated by Van Vleck[11]. For $\frac{3z}{4} \neq -\frac{1}{4} - c$, c real and positive, i.e. for δ not on the cut, $\delta > 1/3$, we obtain

$$\lim_{n \to \infty} \frac{b_{2,n}}{2b_n} = \frac{2/3}{1 + \frac{z}{2} \left[\frac{1}{1 + \frac{3z}{4(1 + \frac{3z}{4 \ldots}}} \right]} = \frac{2/3}{1 + \frac{1}{3} \left[-i \mp 2\sqrt{1 - 3\delta} \right]} \qquad \text{(C-72)}$$

$$= \frac{1}{1 \mp \sqrt{1 - 3\delta}} = \frac{1 \mp \sqrt{1 - 3\delta}}{3\delta} \qquad \text{(C-73)}$$

The evaluation of the continued fraction in (C-72) given in Van Vleck[11] requires that the sign be chosen so that the modulus of $\left[-1 \pm 2\sqrt{1-3\delta}\right]$ be minimum. We obtain finally

$$\lim_{n\to\infty} \frac{b_{2,n}}{2b_n} = \frac{1-\sqrt{1-3\delta}}{3\delta} \qquad \delta \notin \left[\frac{1}{3} < \delta < \infty\right] \qquad \text{(C-74)}$$

which agrees with the expected result.

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