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BOUNDARY VALUE PROBLEMS FOR NONLINEAR
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BOUNDARY VALUE PROBLEMS FOR NONLINEAR
ORDINARY DIFFERENTIAL EQUATIONS

by

Richard Kent Nagle

A dissertation submitted in partial fulfillment
of the requirements for the degree of
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CHAPTER 1

INTRODUCTION

We are concerned in this thesis with the solution of nonlinear boundary value problems. Mainly we are concerned with nonselfadjoint problems, in the sense that the underlying linear problem is nonselfadjoint. We shall be particularly interested in the difficult problems "at resonance", in the sense that the underlying linear homogeneous problem has nontrivial solutions.

We shall consider mainly two aspects of nonlinear boundary value problems.

In Chapters 2, 3, and 4 we consider boundary value problems for a system of n ordinary differential equations with homogeneous linear boundary conditions of the form

$$x' = A(t)x + f(t, x, x', \epsilon), \quad t \in [a, b], \quad (1.1)$$

$$B_1 x(a) + B_2 x(b) = 0 \quad (1.2)$$

where x is an $n \times 1$ vector function, $A(t)$ is an $n \times n$ matrix with real measurable entries, and B_1, B_2 are constant $n \times n$ matrices. We assume that $f = \text{col}(f_1, \dots, f_n)$ is an $n \times 1$ vector function mapping $[a, b] \times R^{2n+1}$ into R^n whose entries are

locally bounded and measurable in t , locally Lipschitz continuous in (x, x') , and continuous in ϵ . Thus, we assume here that the nonlinear terms in (1.1) may depend on the derivatives, and that A and f are only measurable in t .

In Chapters 5 and 6 we consider problems in the large in the abstract form

$$Ex = Nx, \quad x \in H, \quad (1.3)$$

where E is a linear operator with $D(E) \subset H$ a Hilbert space where N is an operator, not necessarily linear, with $D(E) \subset D(N) \subset H$, and E may be non-selfadjoint and may have a nontrivial null space (problems at resonance). In applications, E may be a differential operator in some domain with linear homogeneous boundary conditions. We treat (1.3) under monotonicity and analogous hypotheses on $-E$ and N in the line of Minty's and Brezis's theory of monotone and maximal monotone operators in Hilbert spaces. Mainly, we assume $-E$ to be a strongly monotone operator on the complement of a finite dimensional subspace of H . The results apply to elliptic boundary problems, in ordinary and partial differential equations.

Consistent use is made throughout the thesis of alternative methods. By this is meant the classical bifurcation process by Poincare [25], Lyapunov [20], and Schmidt [26], in terms of present day functional analysis, as initially proposed by Cesari [6,7], or any of the many subsequent modifications discussed by Locker [19], Hale [14,15], Hale, Bancroft, and Sweet [16], Cesari [8,9], Cesari and Kannan [10], Osborn and Sather [24], and others. Actually, we are using here a number of new modifications of these schemes to suit our problems. In alternative methods the given problem is usually split, by the use of projections and partial inverse maps, into a system of two equations, possibly in different spaces, one usually denoted as the auxiliary equation, the other the bifurcation equation. Whenever the auxiliary equation can be uniquely solved by general theorems of functional analysis (contraction maps, strictly monotone maps, etc.), then the remaining equation, the bifurcation, often in a finite dimensional space, is said to be an alternative problem.

And now a few details.

Concerning problem (1.1) - (1.2) we use an extension to homogeneous boundary conditions of the alternative scheme proposed by Hale [15] for handling periodic

solutions of nonselfadjoint problems. We define projection operators P , Q , and a partial inverse operator K , and show first that these operators have the same properties as Hale's operators by the same name. Using these operators we decompose (1.1) into the equivalent system of two equations (Chapter 2)

$$x = Px + K(I-Q)f(\cdot, x, x', \epsilon) \quad (1.4)$$

$$Qf(\cdot, x, x', \epsilon) = 0 \quad (1.5)$$

These are the auxiliary equation and the bifurcation equation respectively for problem (1.1) - (1.2).

In problem (1.1) - (1.2) we assume that $f \equiv 0$ for $\epsilon = 0$, that is, the problem is a perturbation of the corresponding linear problem for the system $x' = A(t)x$, and we think of ϵ as a "small" parameter. In this situation it is natural to suppose that for suitably continuous functions $M(\epsilon) \geq 0$, $L(\epsilon) \geq 0$, $0 \leq \epsilon \leq \epsilon_0$, with $M(0) = 0$, $L(0) = 0$, and constants R_1, R_2 we have

$$|f(t, x, x', \epsilon)| \leq M(\epsilon)$$

$$|f(t, x, x', \epsilon) - f(t, y, y', \epsilon)| \leq L(\epsilon) \{|x-y| + |x'-y'|\}$$

for all $a \leq t \leq b$, $|x|, |y| \leq R_1$, $|x'|, |y'| \leq R_2$.

Then we prove that, for ϵ sufficiently small, $0 \leq \epsilon \leq \epsilon_0$, the second member of (1.4) is a contraction map in a suitable (Sobolev) space, and thus (1.4), by the Banach fixed point theorem, has a unique solution $x = \zeta x^*$ for every $x^* = Px \in S_0$, the range of P . If we now substitute into the bifurcation equation (1.5), this then becomes an equation $F(x^*) = 0$ in the space S_0 (Chapter 3). This equation $F(x^*) = 0$ represents the alternative problem. If $p \geq 0$ denotes the dimension of the null space of the linear problem $x' = A(t)$, $B_1 x(a) + B_2 x(b) = 0$, and $q \geq 0$ the dimension of the null space of the adjoint problem, then the alternative problem $F(x^*) = 0$ is a system of q equations in p unknowns. If $q \leq p$, f smooth, it is enough to verify that the relevant $q \times p$ Jacobian matrix for $\epsilon = 0$ has maximum rank q , and then by the implicit function theorem, system (1.1) - (1.2) has one solution for each $\epsilon \geq 0$ small. The precise result contains as particular cases previous results of Gambill and Hale [13], Hale [14], and Mawhin [21] for periodic solutions only, which in turn had been obtained by simpler versions of the alternative method for selfadjoint problems. Many examples illustrate our results.

In Chapter 4 we show that if $f(t, x, \varepsilon)$ and B_1, B_2 satisfy certain symmetry conditions, then the q equations of the alternative problem $F(x^*) = 0$ are linearly dependent, and thus we may expect a family of solutions to the problem depending on a suitable number of parameters. The symmetry conditions extend those of Hale's work on periodic solutions. These symmetry conditions can be expressed in terms of matrices, and are easy to verify as we show by examples.

In Chapters 5 and 6 we consider problems in the large of the abstract form (1.3). By using the alternative scheme of Hale, Bancroft, and Sweet and results from the theory of monotone operators we show that, if $-E$ and N satisfy certain simple continuity and monotonicity conditions, then the auxiliary equation, and even the bifurcation equation, namely the ensuing alternative problem, can be solved. We extend thereby to nonselfadjoint problems in the results of Cesari and Kannan [10] who have handled by analogous arguments the selfadjoint case. By a more delicate argument we also weaken the monotonicity assumptions by assuming only quasimonotonicity. We give examples illustrating our results and concerning nonselfadjoint boundary value problems for elliptic ordinary differential equations. In Chapter 6 we consider the analogous

alternative scheme proposed by Osborn and Sather, and we show that again their assumption of monotonicity of N may be weakened to an assumption of quasimonotonicity. Furthermore, using the decomposition of $-E$ that Osborn and Sather use, we are able to extend the results of Chapter 5 and show that the auxiliary equation is solvable with N defined on only a dense subspace of H instead of all of H . We apply the results to elliptic ordinary differential equations.

CHAPTER 2

A METHOD FOR HANDLING NONLINEAR BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

2.1. INTRODUCTION

In this chapter we present a method for handling nonlinear boundary value problems for a real system of ordinary differential equations over a fixed interval $[a,b]$. In particular, we study the system of differential equations

$$x' = A(t)x + f(t, x, x', \epsilon) \quad t \in (a, b) \quad (2.1)$$

where $x = \text{col}(x_1, \dots, x_n)$ is a function of t , $A(t)$ is an $n \times n$ matrix whose entries are bounded measurable functions, and $f = \text{col}(f_1, \dots, f_n)$ is an $n \times 1$ vector function defined on $[a, b] \times E^{2n+1}$ whose entries are measurable in t for every (x, x', ϵ) and continuous in (x, x', ϵ) for every t . Moreover, we assume that for any given pair of constants R_1, R_2 there are two other constants L, M such that

$$\begin{aligned} |f(t, x, x', \epsilon)| &\leq M \\ |f(t, x, x', \epsilon) - f(t, y, y', \epsilon)| &\leq L\{|x-y| + |x'-y'|\} \end{aligned} \quad (2.2)$$

for all ϵ , all $a \leq t \leq b$, $|x|, |y| \leq R_1$,

$|x'|, |y'| \leq R_2$. Here ϵ is a real parameter, $x' = dx/dt$, and $||$ denotes the Euclidean norm in E^n .

On system (2.1) we impose the boundary conditions

$$B_1 x(a) + B_2 x(b) = 0 \quad (2.3)$$

where B_1 and B_2 are constant $m \times n$ matrices such that

$$W = (B_1 \ B_2) \quad (2.4)$$

is a $m \times 2n$ matrix of rank m .

Throughout this chapter we will use the following notation. Let $(AC[a,b])^n$ denote the set of n -vector functions y whose components are absolutely continuous functions on $[a,b]$ and such that $B_1 y(a) + B_2 y(b) = 0$. For y in $(AC[a,b])^n$ define $\|y\|_1$ by

$$\|y\|_1 = \sup_{a \leq t \leq b} |y(t)| + (b-a)^{-1} \int_a^b |y'(s)| ds. \quad (2.5)$$

This defines a norm on $(AC[a,b])^n$ and $(AC[a,b])^n$ is a Banach space with this norm. Let $(L_1[a,b])^n$ denote the set of equivalence classes of n -vector

functions y whose components are Lebesgue integrable over $[a,b]$. For y in $(L_1[a,b])^n$ define $\|y\|_0$ by

$$\|y\|_0 = (b-a)^{-1} \int_a^b |y(s)| ds. \quad (2.6)$$

This defines a norm on $(L_1[a,b])^n$ which makes $(L_1[a,b])^n$ a Banach space.

If T is a linear operator between two Banach spaces $(B_a, \|\cdot\|_a)$ and $(B_b, \|\cdot\|_b)$, the operator norm of T is given by $\|T\|$ where

$$\|T\| = \sup_{\substack{f \in B_a \\ \|f\|_a = 1}} \|Tf\|_b \quad (2.7)$$

2.2 LINEAR THEORY

The linear part of system (2.1) - (2.3) is

$$x' = A(t)x \quad t \in (a,b) \quad (2.8)$$

where x satisfies the boundary conditions

$$B_1 x(a) + B_2 x(b) = 0. \quad (2.9)$$

Since $A(t)$ has bounded measurable entries, system

(2.8) has n linearly independent solutions, $x_1, \dots, x_n$. Let X be the fundamental matrix for (2.8), the matrix D defined by

$$D = B_1 X(a) + B_2 X(b) \quad (2.10)$$

is called a characteristic matrix for the boundary value problem (2.8) - (2.9). Since every solution to (2.8) is given by $x = Xc$ where c is a constant $n \times 1$ vector, $x = Xc$ is a solution to the boundary conditions (2.9) if and only if

$$Dc = 0,$$

where D is the characteristic matrix defined by (2.10).

The adjoint boundary problem for (2.8) - (2.9) is

$$y' = -\tilde{y}A(t) \quad \text{or} \quad y' = -\tilde{A}(t)y \quad (2.11)$$

with adjoint boundary conditions

$$\begin{cases} \tilde{y}(a) = \tilde{\alpha}B_1 \\ \tilde{y}(b) = \tilde{\alpha}B_2 \end{cases} \quad \text{or} \quad \begin{cases} y(a) = \tilde{B}_1\alpha \\ y(b) = -\tilde{B}_2\alpha \end{cases} \quad (2.12)$$

where α is an arbitrary vector in R^m , and $\sim$ denotes the transpose. (See Cole [12]).

If X is a fundamental matrix for (2.8), then $\tilde{X}^{-1}$ is a fundamental matrix for (2.11). Every solution to (2.11) has the form

$$\tilde{z} = \tilde{c}X^{-1} \quad (2.13)$$

hence, (2.12) is satisfied by $\tilde{z} = \tilde{c}X^{-1}$ if and only if

$$\begin{aligned} \tilde{c} X^{-1}(a) &= \tilde{\alpha}B_1 \\ \tilde{c} X^{-1}(b) &= \tilde{\alpha}B_2 \end{aligned} \quad (2.14)$$

Now multiply the first equation in (2.14) by $X(a)$, the second by $X(b)$, and subtract the second from the first to get

$$\tilde{0} = \tilde{\alpha}D.$$

Thus the adjoint boundary value problem has a non-trivial solution only if there exists $\tilde{\alpha}$ such that $0 = \tilde{\alpha}D$. Moreover if there exists $\tilde{\alpha}$ such that $\tilde{\alpha}D = \tilde{0}$, then $\tilde{z} = \tilde{c} X^{-1}$ is a solution to the adjoint

boundary problem where $\tilde{c} = \tilde{\alpha}B_1X(a) = -\tilde{\alpha}B_2X(b)$.

(Since $\tilde{\alpha}D = \tilde{\alpha}B_1X(a) + \tilde{\alpha}B_2X(b) = 0$.)

$\tilde{z}$ is a solution to the adjoint boundary problem if and only if $\tilde{z}$ is given by

$$\tilde{z} = \tilde{\alpha}B_1X(a)X^{-1} \quad (2.15)$$

or what is equivalent

$$\tilde{z} = -\tilde{\alpha}B_2X(b)X^{-1}. \quad (2.16)$$

Formulas (2.15) and (2.16) follow from solving (2.14) for $\tilde{c}$ then plugging the expression for c into (2.13).

Associated with the boundary value problem (2.8) - (2.9) is the nonhomogeneous boundary problem

$$x' = A(t)x + f(t) \quad t \in (a,b) \quad (2.17)$$

with boundary conditions

$$B_1X(a) + B_2X(b) = 0 \quad (2.18)$$

where $f = \text{col}(f_1, \dots, f_n)$ is assumed to be integrable in each component over $[a,b]$. The only solution of (2.17) whose initial values at $t = a$ are $X(a)c$ is

given by

$$x(t) = X(t) \int_a^t X^{-1}(s) f(s) ds + X(t) c . \quad (2.19)$$

Indeed, by differentiation and substitution we see that this satisfies (2.17) and has initial values $X(a)c$, and by the uniqueness theorem, (2.19) is the only solution with these initial values as stated.

Substituting (2.19) into (2.18) , equation (2.18) becomes

$$B_2 X(b) \int_a^b X^{-1}(s) f(s) ds + Dc = 0 \quad (2.20)$$

where

$$D = B_1 X(a) + B_2 X(b) .$$

From (2.20) it follows that the nonhomogeneous boundary value problem (2.17) - (2.18) has a solution if and only if c can be chosen so that equation (2.20) is satisfied.

The following theorem, referred to as "Fredholm's Alternative", illustrates the connection between the nonhomogeneous boundary problem and the adjoint boundary problem.

Theorem 2.1: (Fredholm's Alternative)

The nonhomogeneous boundary problem (2.17) - (2.18) has a solution if and only if

$$\int_a^b \tilde{z}(s) f(s) ds = 0 \quad (2.21)$$

for every solution $z(s)$ to the adjoint boundary problem (2.11) - (2.12) .

Proof:

From the above discussion recall that the nonhomogeneous boundary problem (2.17) - (2.18) has a solution if and only if there exists c such that

$$D c = \beta \quad (2.22)$$

where

$$\beta = -B_2 X(b) \int_a^b X^{-1}(s) f(s) ds. \quad (2.23)$$

Since D is a $m \times n$ matrix, the range of D is the annihilator of the null space of D , (Hoffman and Kunze [17], p. 103). That is, β is in the range of D if and only if

$$\tilde{\beta}\gamma = 0 \quad (2.24)$$

for every γ such that

$$\tilde{D}\gamma = 0 \quad (2.25)$$

Taking the transpose of (2.24) and (2.25) and substituting in the expression for β it follows that the nonhomogeneous boundary problem has a solution if and only if

$$\int_a^b (-\tilde{\gamma}) B_2 X(b) X^{-1}(s) f(s) ds = 0 \quad (2.26)$$

for every γ such that

$$\tilde{\gamma}D = 0 \quad .$$

Now recall that the adjoint boundary problem (2.11) - (2.12) has a solution

$$\tilde{z} = -\tilde{\gamma}B_2X(b)X^{-1}$$

if $\tilde{\gamma}D = 0$ and conversely, if z is a solution to (2.11) - (2.12), then $\tilde{z} = -\tilde{\gamma}B_2X(b)X^{-1}$ and $\tilde{\gamma}D = 0$.

Thus equation (2.26) becomes

$$\int_a^b z(s)f(s)ds = 0 \quad (2.27)$$

where z is a solution to the adjoint boundary problem (2.11) - (2.12) and the theorem is proved.

Q.E.D.

2.3 THE PROJECTION OPERATORS P AND Q

Let the columns of U , an $n \times p$ matrix, form a basis for the solutions to the boundary problem (2.8) - (2.9), and let the rows of V , a $q \times n$ matrix, form a basis for the solutions to the adjoint boundary problem (2.11) - (2.12). Let

$$c = \int_a^b \tilde{U}(s) U(s) ds \quad (2.28)$$

and

$$d = \int_a^b V(s) \tilde{V}(s) ds \quad (2.29)$$

The $p \times p$ matrix c is a nonsingular matrix. To see this let $\phi = Ua$ where a is a $p \times 1$ matrix.

Then $ca = 0$

if and only if $\int_a^b \tilde{U}(s) U(s) a \, ds = 0$

if and only if $\int_a^b a \tilde{\tilde{U}}(s) U(s) a \, ds = 0$

if and only if $\int_a^b |\phi|^2 \, ds = 0$

if and only if $\phi = 0$.

Since U has rank p , a must be zero, and hence c is nonsingular. Similarly d is a nonsingular $q \times q$ matrix.

For y in $(AC[a,b])^n$ we define the mapping $P: (AC[a,b])^n \rightarrow (AC[a,b])^n$ by

$$Py(t) = U(t)\alpha \quad (2.30)$$

where $\alpha = c^{-1} \int_a^b \tilde{U}(s) y(s) \, ds$.

It follows from the definition of U that the range of P is contained in the subspace spanned by the solutions to the boundary value problem (2.8) - (2.9). Since c is nonsingular and U has rank p ,

then y may be chosen so that α takes on any value in $\mathbb{R}^p$. Hence the range of P is all of the subspace spanned by the solutions to (2.8) - (2.9).

Lemma 2.1:

P is a bounded linear projection.

Proof:

P is linear. Let α and β be real numbers and f and g elements of $(AC[a,b])^n$. Then

$$\begin{aligned} \alpha Pf + \beta Pg &= \alpha \{Uc^{-1} \int_a^b \tilde{U}(s) f(s) ds\} + \beta \{Uc^{-1} \int_a^b \tilde{U}(s) g(s) ds\} \\ &= Uc^{-1} \int_a^b \tilde{U}(s) \{\alpha f(s) + \beta g(s)\} ds \\ &= P(\alpha f + \beta g) . \end{aligned}$$

Thus P is linear .

P is a projection. Let $f \in (AC[a,b])^n$, then

$$\begin{aligned}
 P^2 f &= P U c^{-1} \int_a^b \tilde{U}(s) f(s) ds \\
 &= U c^{-1} \int_a^b \tilde{U}(t) U(t) c^{-1} \int_a^b \tilde{U}(s) f(s) ds dt \\
 &= U c^{-1} \left\{ \int_a^b \tilde{U}(t) U(t) dt \right\} c^{-1} \int_a^b \tilde{U}(s) f(s) ds \\
 &= U c^{-1} c c^{-1} \int_a^b \tilde{U}(s) f(s) ds \\
 &= U c^{-1} \int_a^b \tilde{U}(s) f(s) ds \\
 &= P f .
 \end{aligned}$$

Thus P is a projection.

P is bounded. Let $f \in (AC[a, b])^n$.

$$\begin{aligned}
 \|Pf\|_1 &= \sup_{a \leq t \leq b} |(Pf)(t)| + (b-a)^{-1} \int_a^b |(Pf)'(t)| dt \\
 &= \sup_{a \leq t \leq b} |U(t) c^{-1} \int_a^b \tilde{U}(s) f(s) ds| \\
 &\quad + (b-a)^{-1} \int_a^b | \{ U(t) c^{-1} \int_a^b \tilde{U}(s) f(s) ds \}' | dt \\
 &= I_1 + I_2 .
 \end{aligned}$$

We shall consider I_1 and I_2 independently.

$$I_1 \leq \sup_{a \leq t \leq b} M_u^2 M_c \int_a^b |f(s)| ds$$

where M_u is a bound for the sum of the absolute values of the entries of U and M_c is a bound for the sum of the absolute values entries of c^{-1} .

(M_u exists since the entries of U are continuous functions of t over $[a, b]$.) Thus,

$$\begin{aligned} I_1 &\leq M_u^2 M_c (b-a) \sup_{a \leq t \leq b} |f(s)| \\ &\leq M_u^2 M_c (b-a) \|f\|_1. \end{aligned}$$

Since $U\alpha$ is a solution to equation (2.8), $(U\alpha)' = A(t)U(t)\alpha$. Hence

$$\begin{aligned} I_2 &= (b-a)^{-1} \int_a^b |(U(t) c^{-1} \int_a^b \tilde{U}(s) f(s) ds)'| dt \\ &= (b-a)^{-1} \int_a^b |A(t) U(t) c^{-1} \int_a^b \tilde{U}(s) f(s) ds| dt \\ &\leq (b-a)^{-1} M_A M_u^2 M_c \int_a^b \int_a^b |f(s)| ds dt \\ &\leq M_A M_u^2 M_c (b-a) (b-a)^{-1} \int_a^b |f(s)| ds \\ &\leq M_A M_u^2 M_c (b-a) \|f\|_1 \end{aligned}$$

where M_A is a bound for the sum of the absolute values of the entries of A . Thus

$$\|Pf\|_1 \leq (1+M_A) M_u^2 M_c (b-a) \|f\|_1.$$

Hence P is bounded.

Q.E.D.

In a similar fashion we define the mapping $Q: (L_1[a,b])^n \rightarrow (L_1[a,b])^n$ by

$$(Qg)(t) = \tilde{V}(t)\beta \quad (2.31)$$

where $\beta = d^{-1} \int_a^b V(s)g(s)ds$ for all $g \in (L_1[a,b])^n$.

Using arguments similar to, but simpler than, those used with regards to the mapping P , we find Q to be a bounded linear projection onto the subspace spanned by the solutions to the adjoint boundary problem (2.11) - (2.12).

2.4 THE RIGHT INVERSE K

In the next theorem the map K is defined from the kernel of Q into $(AC[a,b])^n$. The map $K(I-Q)$

is shown to be a continuous linear transformation of $(L_1[a,b])^n$ into $(AC[a,b])^n$.

Theorem 2.2:

If h is in $(L_1[a,b])^n$, then a necessary and sufficient condition that the boundary problem

$$x' = A(t)x + h(t) \quad (2.32)$$

$$B_1x(a) + B_2x(b) = 0 \quad (2.33)$$

has a solution is that $Qh = 0$. If $Qh = 0$, then there exists a unique solution Kh of (2.32) - (2.33) such that $PKh = 0$. Furthermore, $K(I-Q)$ is a continuous linear mapping of $(L_1[a,b])^n$ into $(AC[a,b])^n$.

Proof:

By the definition of Q , we have $Qh = 0$ if and only if $\tilde{V}\alpha = 0$. Since $\tilde{V}$ has rank m , $\tilde{V} = 0$ if and only if $\alpha = 0$, i.e.

$$c^{-1} \int_a^b V(s)h(s)ds = 0. \text{ Since } c^{-1} \text{ is nonsingular}$$

$$\alpha = 0 \text{ if and only if } \int_a^b V(s)h(s)ds = 0. \text{ The rows}$$

of $V(s)$ from a basis for the solutions to the adjoint boundary problem (2.11) - (2.12) . Hence by the Fredholm Alternative, (2.32) - (2.33) has a solution if and only if $Qh = 0$.

For h in the range of Q , define $Kh = 0$. For h in the null space of Q we define Kh as follows.

If $X(a)c$ is the initial value for a solution $y(t)$ to (2.32) - (2.33) , then

$$y(t) = X(t) \int_a^t X^{-1}(s)h(s)ds + X(t)c ,$$

and moreover, since $y(t)$ must satisfy (2.33),

$$Dc = -B_2 X(b) \int_a^b X^{-1}(s)h(s)ds$$

or

$$Dc = \beta$$

where

$$\beta = -B_2 X(b) \int_a^b X^{-1}(s)h(s)ds .$$

Let D^* be a partial right inverse for D , i.e.

D^* maps the range of D into R^n such that $D D^* = I$

on the range of D . Thus D^* maps everything not in the range of D to 0. Since $Qh = 0$, β must be in the range of D , hence $D^*\beta$ corresponds to the initial value of a solution $y^*(h)$ of (2.32) - (2.33). We define K on the null space of Q by $Kf = (I-P) K_0(f) = (I-P) y^*(f)$. Since any element in $(L_1[a,b])^n$ can be written as a sum of an element in the range of Q plus an element in the null space of Q , we may define K for an arbitrary element h in $(L_1[a,b])^n$ by $Kh = Kh_0 + Kh_1$ where h_0 is in the null space of Q and h_1 is in the range of Q . It follows that K is a linear mapping of $(L_1[a,b])^n$ into $(AC[a,b])^n$. Moreover, it follows immediately from our definition of K that if $Qh = 0$, then Kh is a solution to (2.32) - (2.33) and $PKh = 0$.

Let $Qh = 0$. To show Kh is unique solution to (2.32) - (2.33) such that $PKh = 0$, assume r is also a solution to (2.32) - (2.33) such that $Pr = 0$. Let $r^* = r - Kh$, then

$$(r^*)' = r' - (Kh)' = A(t)(r-Kh) = A(t)r^*$$

Hence, r^* is a solution to homogeneous boundary problem (2.8) - (2.9), so there exists a $p \times 1$

vector α such that $r^* = U(\cdot)\alpha$. Solving for α we find that

$$\tilde{U}(s)r^*(s) = \tilde{U}(s)U(s)\alpha$$

and

$$\begin{aligned}\int_a^b \tilde{U}(s)r^*(s) &= \int_a^b \tilde{U}(s)U(s)ds\alpha \\ &= c\alpha.\end{aligned}$$

Thus

$$\alpha = c^{-1} \int_a^b \tilde{U}(s)r^*(s)ds.$$

But recall

$$Pr^* = U(\cdot) \quad \text{where} \quad \hat{\alpha} = c^{-1} \int_a^b \hat{U}(s)r^*(s)ds.$$

Thus $\hat{\alpha} = \alpha$ hence $Pr^* = U(\cdot)\hat{\alpha} = U(\cdot)\alpha = r^*$. So

$$P(Kh) = P(r+r^*) = Pr + Pr^* = r^*.$$

Since $P(Kh) = 0$, then we must have $r^* = 0$ and hence $r = Kh$. This proves the uniqueness.

It remains to prove that $K(I-Q)$ is bounded. Let $K_0 h = x^*(h)$ for h in the null space of Q . Since $Kh = (I-P)K_0(h)$ for h in the null space of Q and since $(I-P)$ is bounded, it suffices to show K_0 is a bounded mapping from the null space of Q into $(AC[a,b])^n$. Let H be in the null space of Q . We want to show $\|K_0 h\|_1 \leq$ constant $\|h\|_0$. First consider $|(K_0 h)(a)|$.

$$|(K_0 h)(a)| = |x^*(h)(a)| = |D^* \beta|$$

where $\beta = -B_2 X(b) \int_a^b X^{-1}(s) h(s) ds$.

Hence,

$$|(K_0 h)(a)| \leq \|D^*\| \|B_2\| \|X(b)\| \|X^{-1}\| (b-a) \|h\|_0 \quad (2.34)$$

where

$$\|D^*\| = \sum_{i,j} |d_{ij}^*|, \quad \|B_2\| = \sum_{i,j} |b_{ij}|,$$

$$\|X(b)\| = \sum_{i,j} |x(b)_{ij}|, \text{ and } \|X^{-1}\| \text{ is a bound}$$

for the sum of the absolute values of the entries of X^{-1} .

Hence,

$$|(K_0 h)(a)| \leq C_1 \|h\|_0 \quad (2.35)$$

where C_1 is a constant defined by (2.34) .

Since $K_0 h$ is a solution to (2.32) , integrating we have, for $t \in [a, b]$,

$$|(K_0 h)(t)| \leq |(K_0 h)(a)| + \int_a^t |h(s)| ds + \int_a^t |A(s)(K_0 h)(s)| ds.$$

Hence, by the generalized Gronwell inequality (See Hale [15], p. 36)

$$\begin{aligned} |(K_0 h)(t)| &\leq |(K_0 h)(a)| + \int_a^t |h(s)| ds \\ &\quad + \int_a^t |A(s)| \{ |(K_0 h)(a)| + \int_a^s |h(u)| du \} \\ &\quad \exp \{ \int_s^t |A(r)| dr \} ds \\ &\leq C_1 \|h\|_0 + (b-a) \|h\|_0 \\ &\quad + M_A \{ C_1 \|h\|_0 + (b-a) \|h\|_0 \} \exp \{ (b-a) M_A \} (b-a) \\ &= [C_1 + (b-a) + M_A (b-a) \{ C_1 + (b-a) \} \\ &\quad \exp \{ (b-a) M_A \}] \|h\|_0 \end{aligned} \quad (2.36)$$

Hence,

$$\sup_{a \leq t \leq b} |(K_0 h)(t)| \leq C_2 \|h\|_0, \quad (2.37)$$

where C_2 is the constant defined by equation (2.36).

Since $K_0 h$ satisfies equation (2.32), we have for $t \in [a, b]$,

$$\begin{aligned} |(K_0 h)'(t)| &= |A(t)(K_0 h)(t) + |h(t)| \\ &\leq M_A |(K_0 h)(t)| + |h(t)| \\ &\leq M_A C_2 \|h\|_0 + |h(t)|. \end{aligned}$$

Hence,

$$\begin{aligned} (b-a)^{-1} \int_a^b |(K_0 h)'(t)| dt &\leq (b-a)^{-1} \int_a^b \{M_A C_2 \|h\|_0 \\ &\quad + |h(t)|\} dt = M_A C_2 \|h\|_0 + \|h\|_0 \\ &= (M_A C_2 + 1) \|h\|_0. \end{aligned} \quad (2.38)$$

It now follows from equations (2.37) and (2.38) that

$$\begin{aligned} \|(K_0 h)\|_1 &= \sup_{a \leq t \leq b} |(K_0 h)(t)| + (b-a)^{-1} \int_a^b |(K_0 h)'(t)| dt \\ &\leq C_2 \|h\|_0 + (M_A C_2 + 1) \|h\|_0 \\ &= (C_2 + M_A C_2 + 1) \|h\|_0. \end{aligned}$$

Hence, K_0 is a bounded map from the kernel of Q into $(AC[a,b])^n$.

This completes the proof of theorem 2.2.

Q.E.D.

Corollary 2.1:

If g is in $(L_1[a,b])^n$ and α is a given p -vector, then the unique solution of the boundary problem

$$x' = A(t)x + (I-Q)g$$

$$B_1x(a) + B_2x(b) = 0$$

with $Px = U(\cdot)\alpha$ is given by $x = U\alpha + K(I-Q)g$.

Proof:

Just differentiate $U\alpha + K(I-Q)g$.

Q.E.D.

2.5 THE AUXILIARY AND BIFURCATION EQUATIONS

We shall now split the boundary value problem (2.1), (2.3) into the equivalent system of two

equations (the auxiliary and the bifurcation equations). Our splitting will be such that the auxiliary equation can be solved by the contraction mapping principle and the bifurcation equation lies in a finite dimensional space.

Theorem 2.3:

Let ϵ be fixed. Then $x(t)$ is a solution of (2.1) - (2.3) if and only if $x(t)$ is a solution for the system

$$x = Px + K(I-Q)f(\cdot, x, x', \epsilon) \quad (2.39)$$

$$Qf(\cdot, x, x', \epsilon) = 0. \quad (2.40)$$

Proof:

(only if) Let x be a solution of (2.1) - (2.3), hence by Theorem 2.2, $Qf(\cdot, x, x', \epsilon) = 0$. To show x satisfies (2.39), let $\phi = x - Px$, $P\phi = 0$ and since x and Px satisfy the boundary conditions (2.3), ϕ also satisfies (2.3). Differentiating ϕ , we find

$$\begin{aligned} \phi' &= x' - (Px)' \\ &= \{A(t)x + f(t, x, x', \epsilon)\} - \{A(t)Px\} \\ &= A(t)\phi + f(t, x, x', \epsilon). \end{aligned}$$

But, by theorem 2.2, the unique solution of

$$y' = A(t)y + f(t, x, x', \epsilon)$$

$$B_1 y(a) + B_2 y(b) = 0$$

with $Py = 0$ is given by $Kf(\cdot, x, x', \epsilon) = K(I-Q)f(\cdot, x, x', \epsilon)$. Hence $\phi = K(I-Q)f(\cdot, x, x', \epsilon)$. Thus $x = Px + K(I-Q)f(\cdot, x, x', \epsilon)$.

(if) If $Qf(\cdot, x, x', \epsilon) = 0$, then (2.39) reads

$$x = Px + K\{f(\cdot, x, x', \epsilon)\}.$$

Now Corollary 2.1 implies x is unique solution to

$$x' = A(t)x + f(\cdot, x, x', \epsilon)$$

$$B_1 x(a) + B_2 x(b) = 0.$$

That is, x is a solution to (2.1) - (2.3).

Q.E.D.

Equation (2.39) is referred to as the auxiliary equation. In later chapters we will solve the auxiliary

equation using the contraction mapping principle. Equation (2.40) is referred to as the bifurcation equation. Since the null space of Q is finite dimensional, we will use implicit function theorems or symmetry conditions to solve the bifurcation equation.

CHAPTER 3
NONLINEAR BOUNDARY VALUE PROBLEMS
WITH SMALL PARAMETERS

3.1. INTRODUCTION

In this chapter we concern ourselves with the boundary value problem

$$\begin{aligned}x' &= A(t)x + f(t, x, x', \epsilon) \\ B_1 x(a) + B_2 x(b) &= 0\end{aligned}\tag{3.1}$$

where x , A , f , B_1 , B_2 , and ϵ satisfy the assumptions made in Chapter 2, along with two additional assumptions on f . Let us assume for each pair of constants R_1 and R_2 , there exists $M(\epsilon)$ and $L(\epsilon)$ continuous functions of ϵ such that $M(0) = L(0) = 0$, and whenever $|x|, |y| \leq R_1$ and $|x'|, |y'| \leq R_2$ then

$$|f(t, x, x', \epsilon)| \leq M(\epsilon) \quad \text{all } t \in [a, b]$$

and

$$|f(t, x, x', \epsilon) - f(t, y, y', \epsilon)| \leq L(\epsilon)\{|x-y| + |x'-y'|\}.$$

In most applications f is defined in $[a,b] \times R^{2n} \times [0,\epsilon_0]$, for some given ϵ_0 , and $f(t,x,x',\epsilon) = \epsilon h(t,x,x')$, $t \in [a,b]$, $x, x' \in R^n$ and $0 \leq \epsilon \leq \epsilon_0$. Where $h: [a,b] \times R^{2n} \rightarrow R^n$ satisfies the assumptions made on f in Chapter 2. That is h is bounded by a constant M and has Lipschitz constant L . We then have $M(\epsilon) = \epsilon M$ and $L(\epsilon) = \epsilon L$.

3.2. EXISTENCE OF SOLUTIONS TO AUXILIARY EQUATION

The following theorem shows that the auxiliary equation always has a solution for ϵ small.

Theorem 3.1:

There exists $\rho > 0$ and $\epsilon_0 > 0$ such that for any constant p -vector α , $|\alpha| \leq \rho$ and ϵ such that $|\epsilon| \leq \epsilon_0$ then there exists a unique n -vector x^* , $x^* = x^*(\alpha, \epsilon)$ such that

$$x^* = U\alpha + K(I-Q)f(\cdot, x^*, x^{*'}, \epsilon).$$

Furthermore, if there is an $\alpha = \alpha(\epsilon)$ with $|\alpha(\epsilon)| \leq \rho$ for $|\epsilon| \leq \epsilon_0$, and

$$G(\alpha, \epsilon) \equiv Qf(\cdot, x^*(\alpha(\epsilon), \epsilon), x^{*'}(\alpha(\epsilon), \epsilon), \epsilon) = 0$$

then $x^*(\alpha(\epsilon), \epsilon)$ is a solution of the boundary problem (3.1) .

Proof:

The columns of U are continuous over $[a, b]$, hence bounded. Choose $\rho > 0$ so that $|\alpha| \leq \rho$, α a p -vector, implies $\|U\alpha\|_1 = \beta < \min(R_1, R_2/B)$ where B is a bound for the sum of the absolute values of the entries of A .

Fix $\rho > 0$. Let α be a p -vector with $|\alpha| \leq \rho$. Define S_α , a subset of $(AC[a, b])^n$ by

$$\begin{aligned} S_\alpha = \{y \text{ in } (AC[a, b])^n : & Py = U\alpha , \\ & |y(t)| \leq R_1 \quad t \in [a, b] , \text{ and} \\ & |y'(t)| \leq R_2 \quad \text{a.e. } t \in [a, b]\} . \end{aligned}$$

Let $S = \bigcup \{S_\alpha : |\alpha| \leq \rho, \alpha \text{ a } p\text{-vector}\}$. On S we define a family of maps $F(\alpha, \epsilon)$ as follows. For $y \in S$,

$$F(\alpha, \epsilon)y = U\alpha + K(I-Q)f(\cdot, y, y', \epsilon) .$$

For y in S , $f(t, y(t), y'(t), \epsilon)$ is in $(L_1[a, b])^n$ since f is bounded a.e., hence $K(I-Q)f(\cdot, y, y', \epsilon)$

is in $(AC[a,b])^n$. Since $U\alpha$ is in $(AC[a,b])^n$, it follows $F(\alpha, \epsilon): S \rightarrow (AC[a,b])^n$.

Claim: S is closed in $(AC[a,b])^n$, in fact S_α is closed.

Let $y_n \rightarrow y$ in $(AC[a,b])^n$, $y_n \in S$. If $y_n \in S$, then $Py_n = U\alpha_n$ where $|\alpha_n| \leq \rho$. Now by the continuity of P , $Py_n \rightarrow Py$ and $Py = U\alpha_n \rightarrow U\alpha_0$ and since U has rank p we must have $\alpha_n \rightarrow \alpha_0$. Since $|\alpha_n| \leq \rho$ we must have $|\alpha_0| \leq \rho$.

If $y_n \in S$, then $|y_n(t)| \leq R_1$ and $|y'_n(t)| \leq R_2$ a.e. t . Since $y_n \rightarrow y$ in $(AC[a,b])^n$ we must have $\sup_{a \leq t \leq b} |y_n(t) - y(t)| \rightarrow 0$ as $n \rightarrow \infty$. Hence $|y(t)| \leq R_1$.

Assume $|y'(t)| > R_2$ on a set of positive measure, then there is a $\delta > 0$ and a set W such that measure $(W) > 0$ and on W $|y'(t)| < R_2 + \delta$. Hence for every n we have

$$\begin{aligned} \|y_n - y\|_1 &\geq (b-a)^{-1} \int_a^b |y'_n(t) - y'(t)| dt \\ &\geq (b-a)^{-1} \int_W |y'_n(t) - y'(t)| dt \\ &\geq (b-a)^{-1} \int_W \delta dt, \\ &\geq \delta (b-a)^{-1} \text{measure}(W) > 0. \end{aligned}$$

This contradicts our assumption that $y_n \rightarrow y$ in $(AC[a,b])^n$. Hence $|y'(t)| \leq R_2$ a.e. $t \in [a,b]$. Thus S is closed. Since $y_n \in S_\alpha$ implies $\alpha_n = \alpha$ for all n , then $Py = U\alpha$. Hence the above proof shows that S_α is also closed.

Claim: There exists an $\epsilon_1 > 0$ such that for $|\epsilon| \leq \epsilon_1$, then $F(\alpha, \epsilon): S \rightarrow S$ (in fact, $F(\alpha, \epsilon): S_\alpha \rightarrow S_\alpha$). For $y \in S$, we have

$$\begin{aligned} P(F(\alpha, \epsilon)y) &= P(U\alpha) + PK(I-Q)f(\cdot, y, y', \epsilon) \\ &= P(U\alpha) = U\alpha. \end{aligned}$$

Now

$$\begin{aligned} |(F(\alpha, \epsilon)y)(t)| &\leq |U\alpha| + |K(I-Q)f(\cdot, y, y', \epsilon)(t)| \\ &\leq \beta + \|K(I-Q)f(\cdot, y, y', \epsilon)\|_1 \\ &\leq \beta + \|K(I-Q)\| \|f(\cdot, y, y', \epsilon)\|_0 \\ &\leq \beta + \|K(I-Q)\| M(\epsilon). \end{aligned}$$

Since $\beta < R_1$, we may choose $\epsilon_1 > 0$ such that $M(\epsilon)$ is small enough to assure that $\beta + \|K(I-Q)\| M(\epsilon) < R_1$.

Now for a.e. t we have

$$\begin{aligned}
 |(F(\alpha, \epsilon)y)'(t)| &= |A(t)F(\alpha, \epsilon)y(t) + (I-Q)f(t, y, y', \epsilon)| \\
 &\leq B|F(\alpha, \epsilon)y(t)| + |(I-Q)f(t, y, y', \epsilon)| \\
 &\leq B\beta + \|K(I-Q)\| M(\epsilon) \\
 &\quad + |f(t, y, y', \epsilon)| + |Qf(t, y, y', \epsilon)| \\
 &\leq B\beta + \|K(I-Q)\| M(\epsilon) + M(\epsilon) \\
 &\quad + |\tilde{V}(t) d^{-1} \int_a^b v(s)f(s, y, y', \epsilon) ds| \\
 &\leq B\beta + \|K(I-Q)\| M(\epsilon) + M(\epsilon) \\
 &\quad + M_V^2 M_d (b-a) M(\epsilon) \\
 &\leq B\beta + \{\|K(I-Q)\| + 1 + M_V^2 M_d (b-a)\} M(\epsilon).
 \end{aligned}$$

Since $B\beta < R_2$, then we can choose ϵ_1 small enough so that $B\beta + \{\|K(I-Q)\| + 1 + M_V^2 M_d (b-a)\} M(\epsilon) < R_2$. Hence $F(\alpha, \epsilon)y$ lies in S . Since $P(F(\alpha, \epsilon)y) = U\alpha$, it also follows that $F(\alpha, \epsilon)y$ lies in S_α .

Claim: $F(\alpha, \epsilon)$ is a contraction mapping of S into S (again as a map of S_α into S_α .)

Let $y, z \in S$, then

$$\begin{aligned}
 & \|F(\alpha, \epsilon)y - F(\alpha, \epsilon)z\|_1 \\
 &= \|K(I-Q)\{f(\cdot, y, y', \epsilon) - f(\cdot, z, z', \epsilon)\}\|_1 \\
 &\leq \|K(I-Q)\| \|f(\cdot, y, y', \epsilon) - f(\cdot, z, z', \epsilon)\|_0 \\
 &= \|K(I-Q)\| (b-a)^{-1} \int_a^b |f(t, y(t), y'(t), \epsilon) \\
 &\quad - f(t, z(t), z'(t), \epsilon)| dt \\
 &\leq \|K(I-Q)\| (b-a)^{-1} \int_a^b L(\epsilon) \{|y(t) - z(t)| \\
 &\quad + |y'(t) - z'(t)|\} dt \\
 &\leq \|K(I-Q)\| L(\epsilon) \{(b-a)^{-1} \int_a^b |y(t) - z(t)| dt \\
 &\quad + (b-a)^{-1} \int_a^b |y'(t) - z'(t)| dt\} \\
 &\leq \|K(I-Q)\| L(\epsilon) \{ \sup_{a \leq t \leq b} |y(t) - z(t)| \\
 &\quad + (b-a)^{-1} \int_a^b |y'(t) - z'(t)| dt \} \\
 &= \|K(I-Q)\| L(\epsilon) \|y - z\|_1 .
 \end{aligned}$$

If we choose ε_1 such that $L(\varepsilon) \|K(I-Q)\| < 1$, then $F(\alpha, \varepsilon)$ will be a contraction map for S into S , whenever $|\varepsilon| < \varepsilon_1$.

We have shown that there is $\varepsilon_1 > 0$ so that for $|\varepsilon| \leq \varepsilon_0$ and all α_1 $|\alpha| \leq \rho$, then $F(\alpha, \varepsilon): S \rightarrow S$ is a contraction map. Moreover, $\{F(\alpha, \varepsilon)\}$ is a uniform family of contractions on S . Hence each $F(\alpha, \varepsilon)$ has a fixed point in S . In fact the fixed point lies in S_α .

We have shown that there exists $\rho > 0$ and $\varepsilon_0 > 0$ such that for $|\alpha| \leq \rho$, $|\varepsilon| \leq \varepsilon_0$, then there is an x^* in S_α , such that

$$x^* = U\alpha + K(I-Q)f(\cdot, x^*, x^{*'}, \varepsilon).$$

If there is an $\alpha = \alpha(\varepsilon)$ with $|\alpha(\varepsilon)| \leq \rho$ for $|\varepsilon| \leq \varepsilon_0$, and $G(\alpha, \varepsilon) = 0$, then $x^* = x^*(\alpha(\varepsilon), \varepsilon)$ satisfies both

$$x^* = Px^* + K(I-Q)f(\cdot, x^*, x^{*'}, \varepsilon)$$

and

$$Qf(\cdot, x^*, x^{*'}, \varepsilon) = 0.$$

Hence in theorem 2.3 in Chapter 2, x^* is a solution to (3.1) .

Q.E.D.

Theorem 3.1 assures us that for ϵ small the auxiliary equation always has a solution. Thus the problem (3.1) is reduced to solving

$$Qf(\cdot, x^*, x^{*'}, \epsilon) = 0$$

where x^* is the solution to the auxiliary equation (2.39) in Chapter 2.

Since $Qf(\cdot, x^*, x^{*'}, \epsilon) = 0$ if and only if

$$\int_a^b V(s) f(s, x^*(s), x^{*'}(s), \epsilon) ds = 0, \text{ let}$$

$$H(\alpha, \epsilon) = \int_a^b V(s) f(s, x^*(s), x^{*'}(s), \epsilon) ds. \quad (3.4)$$

The boundary problem (3.1) has a solution if the equation $H(\alpha, \epsilon) = 0$ has a solution for $|\epsilon| \leq \epsilon_0$ and $|\alpha| \leq \rho$. Equation

$$H(\alpha, \epsilon) = 0$$

is also referred to as the bifurcation equation as is equation (2.40) .

3.3. PROPERTIES OF THE SOLUTION TO THE AUXILIARY EQUATION

The results in this section are consequences of the Uniform Contraction Principle (See Hale [15], p. 7).

Theorem 3.2: (Uniform Contraction Principle)

If F is a closed subset of a Banach space X , G a subset of a Banach space Y , $T_y: F \rightarrow F$, y in G , is a uniform contraction of F (i.e. T_y all contraction on F with the same constant), then the unique fixed point $g(y)$ of T_y , y in G , is continuous in y . Furthermore, if F, G are the closures of open sets F^0, G^0 , and T_{yx} has continuous first derivatives in y when x is fixed and in x when y is fixed, then $g(y)$ has a continuous first derivative with respect to y in G^0 .

Our first result involves the uniqueness of the solution to the auxiliary equation.

Corollary 3.1:

Let S and $F(\alpha, \epsilon)$ be as defined in the proof of Theorem 3.1. If for $|\epsilon| \leq \epsilon_0$, $\bar{x}(\epsilon) \in S$ (hence $P\bar{x} = U\alpha(\epsilon)$ with $|\alpha(\epsilon)| \leq \rho$) and $\bar{x}(\epsilon)$ is a solution to (3.1), then $\bar{x}(\epsilon) = x^*(\alpha(\epsilon), \epsilon)$ and $G(\alpha(\epsilon), \epsilon) = 0$.

Proof:

The corollary follows from the uniqueness of the fixed point of the mapping $F(\alpha(\epsilon), \epsilon)$.

Q.E.D.

Our next result involves the smoothness properties of the solution to the auxiliary equation.

Corollary 3.2:

Let $x^* = x^*(\alpha, \epsilon)$ be the solution to the auxiliary equation for $|\alpha| \leq \rho$ and $|\epsilon| \leq \epsilon_0$, then x^* depends continuously on α and ϵ . Furthermore, if $f(\cdot, x, y, \epsilon)$ has continuous partial derivatives with respect to x , y , and ϵ , then x^* is continuously differentiable with respect to α and ϵ .

Proof:

$F(\alpha, \epsilon)$ as defined in the proof of Theorem 3.1, depends continuously on α and ϵ since f is a continuous function of ϵ . Hence by the Uniform Contraction Principle, the fixed point x^* depends continuously on α and ϵ . If $f(\cdot, x, y, \epsilon)$ has a continuous derivative with respect to y when α and ϵ are fixed, and has a continuous derivative

with respect to ϵ (or α) whenever y and α (or ϵ) are fixed.

Q.E.D.

Corollary 3.2 says that $x^*(\alpha, \epsilon)$ is Fréchet differentiable with respect to α when $f(\cdot, x, y, \epsilon)$ is differentiable with respect to x, y , and ϵ . Since $x^*(\alpha, \epsilon)$ is in $(AC[a, b])^n$, $x^*(\alpha, \epsilon)$ differentiable with respect to α implies $x^*(\alpha, \epsilon)$ is differentiable with respect to α as a vector function in $(L_1[a, b])^n$, and $x^{*'}(\alpha, \epsilon)$ is differentiable with respect to α as a vector function in $(L_1[a, b])^n$.

As a consequence of the continuity of $x^*(\alpha, \epsilon)$ with respect to ϵ , it follows that $x^*(\alpha, 0) = U\alpha$.

3.4 SOLVING THE BIFURCATION EQUATION

Since we are dealing with small parameters we may use the implicit function theorem to solve the bifurcation equation.

Theorem 3.3:

Let $H(\alpha, \epsilon)$, as defined in (3.4), be continuously differentiable with respect to both α and ϵ , for

$|\alpha| \leq \rho$ and $|\epsilon| \leq \epsilon_0$. If there is an $\tilde{\alpha}$,
 $|\tilde{\alpha}| \leq \rho$ such that

$$H(\tilde{\alpha}, 0) = 0$$

and

$$\frac{\partial}{\partial \alpha} (H(\tilde{\alpha}, 0)) \neq 0$$

then there is an $\epsilon_1 > 0$ such that for every ϵ ,
 $|\epsilon| < \epsilon_1$, there is a solution $x^*(\alpha(\epsilon), \epsilon)$ of the
 boundary value problem (3.1).

Proof:

By the implicit function theorem there exists
 $\epsilon_1 > 0$, ($\epsilon_1 < \epsilon_2$) and a function $\alpha = \alpha(\epsilon)$ such
 that $H(\alpha(\epsilon), \epsilon) = 0$. Since $x^*(\alpha(\epsilon), \epsilon)$ is a
 solution to both the auxiliary and bifurcation equa-
 tions, $x^*(\alpha(\epsilon), \epsilon)$ is a solution to the boundary
 value problem (3.1).

Q.E.D.

$H(\alpha, \epsilon)$ will be continuously differentiable
 with respect to α and ϵ , if $f(t, x, y, \epsilon)$ is
 continuously differentiable with respect to x , y , and ϵ .

To prove this notice

$$\begin{aligned} \frac{\partial}{\partial \alpha} (H(\alpha, \epsilon)) = & \int_a^b V(s) \{ f_2(s, x^*, x^{*'}, \epsilon) x_a^* \\ & + f_3(s, x^*, x^{*'}, \epsilon) x^{*'} \} ds \end{aligned} \quad (3.5)$$

where $f_2 = \frac{\partial f}{\partial x}$, $f_3 = \frac{\partial f}{\partial y}$, $x_a^* = \frac{\partial x^*}{\partial a}$ as a vector

function in $(L_1[a, b])^n$, and $x^{*'} = \frac{\partial x^{*'}}{\partial x}$ as a vector

function in $(L_1[a, b])^n$. Equation (3.5) makes sense whenever the right hand side of (3.5) is defined. If we assume $f(t, x, y, \epsilon)$ is continuously differentiable with respect to x , y , and ϵ , then Corollary 3.2, implies that the right hand side of (3.5) is defined since x_a^* and $x^{*'}$ will be integrable and f is bounded.

Once one knows that the boundary value problem (3.1) has a solution for fixed α and ϵ , $|\alpha| \leq \rho$ and $|\epsilon| \leq \epsilon_0$, then the solution to (3.1) may be found by the method of successive approximation. One may begin the approximation with any element y_0 in S , then the sequence $\{y_n\}$ defined by

$$y_{n+1} = F(\alpha, \epsilon) y_n \quad n = 0, 1, 2, \dots$$

converges in $(AC[a,b])^n$ to $x^*(\alpha, \epsilon)$ the solution to (3.1). However, if we choose $y_0 = U\alpha = x^*(\alpha, 0)$, then $y_1 = F(\alpha, \epsilon) y_0$ will contain only terms of order ϵ . Moreover, y_n will contain only terms up to order n in ϵ .

3.5 EXAMPLES

In this section we demonstrate how the method developed in this chapter can be used to discuss boundary value problems. Our emphasis is on the resonance cases which are harder to handle, but since this method does give results in the non-resonance case we have included an example (Example 6) of the non-resonance case. When the bifurcation equations are linearly dependent, or identically zero, we have a family of solutions. In Example 7 we consider such a case. Finally, if the bifurcation equation is never zero, then we conclude that there is never a solution to the boundary value problem. This is demonstrated in Example 8.

For the linear theory for boundary value problems refer to Coddington and Levinson [11] for n^{th} order ordinary differential equations and Cole [12] for systems of ordinary differential equations.

Example 1:

$$x'' = \epsilon \{-e^{x''} + x^2 \sin^2 t\} \quad (3.6)$$

We are interested in 2π - periodic solutions.

Equation (3.6) may be written as the system

$$y' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y + \epsilon g(t, y, y')$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y(0) + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} y(2\pi) = 0$$

where $y = \text{col}(y_1, y_2)$, $y_1 = x$, and $y_2 = x'$ and $g(t, y, y') = \text{col}(0, -e^{x''} + x^2 \sin^2 t)$. For this system $U(t) = \text{col}(1, 0)$ and $V(t) = (0, 1)$. The solution to the auxiliary equation has the form $x(t) = \alpha + 0(\epsilon)$. Hence

$$H(\alpha, \epsilon) = \int_0^{2\pi} -e^{0(\epsilon)} + (\alpha^2 + 0(\epsilon)) \sin^2 t \, dt.$$

So $H(\alpha, 0) = \pi\alpha^2 - 2\pi$ and $\frac{\partial H(\alpha, 0)}{\partial \alpha} = 2\pi \alpha$. It

now follows from Theorem 3.3 that for ϵ small we have a 2π -periodic solution to equation (3.6) of the form

$$x(t) = \alpha + 0(\epsilon) \quad \text{where} \quad \alpha = \pm \sqrt{2}.$$

Example 2:

$$x'' = \epsilon \{ (1-x^2) (x'')^2 + \sin(t+\phi) + (1-x^2) \sin^2 t \} \quad (3.7)$$

We are interested in 2π -periodic solutions. Equation (3.7) may be written as the system

$$y' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y + \epsilon g(t, y, y')$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y(0) + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} y(2\pi) = 0$$

where $y = \text{col}(y_1, y_2)$, $y_1 = x$, and $y_2 = x'$ and
 $g(t, y, y') = \text{col}(0, (1-x^2) (x'')^2 + \sin(t+\phi) + (1-x^2) \sin^2 t)$.
 For this system $U(t) = \text{col}(1, 0)$ and $V(t) = (0, 1)$.
 The auxiliary equation has a solution of the form
 $x(t) = \alpha + o(\epsilon)$. Hence

$$H(\alpha, \epsilon) = \int_0^{2\pi} (1-\alpha^2 + o(\epsilon)) o(\epsilon) + \sin(t+\phi) + (1-\alpha^2 + o(\epsilon)) \sin^2 t \, dt$$

So $H(\alpha, 0) = \pi(1-\alpha^2)$ and $\frac{\partial H(\alpha, 0)}{\partial \alpha} = -2\pi\alpha$. It

now follows from theorem 3.3 that for ϵ small equation (3.7) has a 2π -periodic solution of the form

$$x(t) = \alpha + o(\epsilon) \quad \text{where} \quad \alpha = \pm 1.$$

Example 3:

$$x'' = \epsilon \{ c^2 - (x'')^2 + (2c^2 - x^2) \sin^2 t \} \quad (3.8)$$

Again we are interested in 2π -periodic solutions.

The linear part of this equation is the same as

Example 2 . In this case we have

$$H(\alpha, 0) = 4\pi c^2 - \pi \alpha^2$$

and

$$\frac{\partial H(\alpha, 0)}{\partial \alpha} = -2\pi \alpha.$$

Hence equation (3.8) has a 2π -periodic solution for

ϵ small of the form $x(t) = \alpha + o(\epsilon)$ where

$$\alpha = \pm 2c.$$

Example 4:

$$x'' + x = \epsilon \{ (x'')^3 + (x')^3 + x^3 + \cos t \}$$

$$x(0) = x(2\pi) = 0 \quad (3.9)$$

We may write this boundary value problem as the following system

$$y' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} y + \epsilon g(t, y, y')$$

where $y_1 = x$ and $y_2 = x'$. Here $g(t, y, y') =$

$$\begin{pmatrix} 0 \\ (x'')^3 + (x')^3 + x^3 + \cos t \end{pmatrix}. \text{ For this system}$$

$$U(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \text{ and } V(t) = (\sin t \cos t).$$

Here the solution to the auxiliary equation is of the form $x(t) = \sin t + o(\epsilon)$. Hence

$$H(\alpha, 0) = \pi \left(\frac{3}{4} \alpha^3 + 1 \right) \text{ and } \frac{\partial H(\alpha, 0)}{\partial \alpha} = \frac{9}{4} \pi \alpha^2$$

Hence the boundary value problem (3.9) has a solution of the form

$$x = \left(-\frac{4}{3} \right)^{1/3} \sin t + o(\epsilon).$$

Example 5:

$$x'' + \sigma^2 x = \epsilon \{ (1-x^2)x' + a\sigma^{-1} x'' + b\sigma(\cos(\sigma t + \alpha)) \}$$

(3.10)

We are interested in $2\pi/\sigma$ -periodic solutions. Equation (3.10) can be written as the system

$$y' = \begin{pmatrix} 0 & 1 \\ -\sigma^2 & 0 \end{pmatrix} y + g(t, y, y')$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y(0) + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} y(2\pi/\sigma) = 0$$

where y and g are defined as usual. For this

system $U(t) = \begin{pmatrix} \sin \sigma t & \cos \sigma t \\ \sigma \cos \sigma t & -\sigma \sin \sigma t \end{pmatrix}$ and we may take

$$V(t) = \begin{pmatrix} \sigma \sin(\sigma t + \theta_0) & \cos \sigma t + \theta_0 \\ \sigma \cos(\sigma t + \theta_0) & -\sin \sigma t + \theta_0 \end{pmatrix} \quad \text{where } \theta_0 \text{ will}$$

be defined later. The solution to the auxiliary

equation has the form $x(t) = c_1 \sigma^{-1} \sin(\sigma t) + c_2 \sigma^{-1} \cos(\sigma t)$ or $x(t) = \lambda(\epsilon) \sigma^{-1} \sin(\sigma t + \theta(\epsilon)) + o(\epsilon)$

where $\lambda(\epsilon) = \lambda_0 + o(\epsilon)$
 $\theta(\epsilon) = \theta_0 + o(\epsilon)$.

In this problem we have

$$H_1(\lambda, \theta, \epsilon) = \int_0^{2\pi/\sigma} \{ (1 - \lambda^2 \sigma^{-2} \sin^2(\sigma t + \theta))$$

$$\lambda \cos(\sigma t + \theta) + b \sigma \cos(\sigma t + \alpha)$$

$$- a \lambda \sin(\sigma t + \theta) \} \cos(\sigma t + \theta_0) dt$$

$$H_1(\lambda_0, \theta_0, 0) = \frac{-\pi}{4\sigma^3} \lambda_0^3 - 4\lambda_0\sigma^2 - 4b\sigma^3 \cos(\alpha - \theta_0)$$

$$H_2(\lambda, \theta, \varepsilon) = \int_0^{2\pi/\sigma} \{ \dots \} \sin(\sigma t + \theta_0) dt$$

$$H_2(\lambda_0, \theta_0, 0) = \frac{-\pi}{\sigma} [a\lambda_0 + b\sigma \sin(\alpha - \theta_0)]$$

$$\frac{\partial H(\lambda_0, \theta_0, 0)}{\partial (\lambda, \theta)} = \begin{pmatrix} 2\lambda_0^2 + 4\sigma^2 & -4b\sigma^2 \sin(\alpha - \theta) \\ a & -b\sigma \cos(\alpha - \theta) \end{pmatrix}$$

Now $H(\lambda_0, \theta_0, 0) = 0$ has a solution when $a \ll b$ and $b \ll 1$ of the form $\lambda_0^2 \approx 4\sigma^2$ and $\theta_0 \approx \alpha$, but for these value of λ_0 and θ_0 the Jacobian determinant becomes

$$\lambda_0^2 + 2\sigma^2 - 2a\sigma^2 \tan(\alpha - \theta_0) \approx 6\sigma^2 \neq 0.$$

Hence for σ different from zero, $a \ll b$ and $b \ll 1$, then it follows from theorem 3.3 that equation (3.10) has a $2\pi/\alpha$ -periodic solution of the form $x(t) = \lambda(\varepsilon) \sigma^{-1} \sin(\sigma t + \theta(\varepsilon)) + 0(\varepsilon)$ where $\lambda(\varepsilon) \approx 2\sigma + 0(\varepsilon)$ and $\theta(\varepsilon) \approx \alpha + 0(\varepsilon)$.

Example 6:

$$\begin{aligned} x'' &= \varepsilon \sin x + \varepsilon t \\ x(0) &= x(1) = 0 \end{aligned} \tag{3.11}$$

In this equation $p = q = 0$, hence the bifurcation equation is always satisfied. Hence for ϵ small by theorem 3.1 equation (3.11) has a solution of the form $x(t) = 0(\epsilon)$.

Example 7:

$$y' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} y + \epsilon \begin{pmatrix} f_1(t, y, y') \\ f_2(t, y, y') \\ f_3(t, y, y') \end{pmatrix} \quad (3.12)$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} y(0) + \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} y(1) = 0$$

For this system we have $U(t) = \text{col}(e^t, e^t, e^t)$ and $v(t) = (-1, t, 1-t)$. The bifurcation equation has the form

$$\begin{aligned} H(\alpha, \epsilon) = & \int_0^1 \{-f_1(t, x^*(\alpha, \epsilon), x^{*'}(\alpha, \epsilon)) \\ & + tf_2(t, x^*(\alpha, \epsilon), x^{*'}(\alpha, \epsilon)) \\ & + (1-t)f_3(t, x^*(\alpha, \epsilon), x^{*'}(\alpha, \epsilon))\} dt . \end{aligned}$$

If it happens that $f_1 = f_2 = f_3$ then we would have

$$H(\alpha, \epsilon) \equiv 0 .$$

Thus for each ε small we have a one parameter family of the form

$$x(t) = \text{col}(ce^t, ce^t, ce^t)$$

where $|c| \leq \rho$ and this family is a solution to equation (3.12) .

Example 8:

$$\begin{aligned} x'' + \sigma^2 x &= \varepsilon f(t, x, x', x'') \\ x'(0) &= x'(\pi/\sigma) = 0 \end{aligned} \quad (3.13)$$

For this boundary value problem $U(t) = \begin{pmatrix} \cos \sigma t \\ -\sigma \sin \sigma t \end{pmatrix}$

hence the solution to the auxiliary equation is of the form $x(t) = \alpha \cos \sigma t + 0(\varepsilon)$. Here $V(t) = (\cos \sigma t \quad -\sigma^{-1} \sin \sigma t)$ hence the bifurcation equation takes the form

$$H(\alpha, \varepsilon) = \int_0^{\pi/\sigma} -\sigma^{-1} \sin(\sigma t) f(t, x, x', x'') dt = 0$$

Now if $\sin(\sigma t) f(t, x, x', x'') \geq 0$ and $\neq 0$, then we do not have a solution to the boundary value problem

even if ε is small . For example equation (3.13) does not have a solution if

$$f(t,x,x',x'') = \sin(\sigma t) \{x^2 + x'^2 + x''^2 + 1\} .$$

CHAPTER 4
NONLINEAR BOUNDARY VALUE PROBLEMS WITH
SMALL PARAMETERS AND SYMMETRIES

4.1 INTRODUCTION

In this chapter we will study nonlinear boundary value problems which satisfy a symmetry condition called proper (E) . (See Hale [14, 15]. In section 2 we will see: it is often the case that whenever a boundary value problem satisfies property (E) , then the number of components in the bifurcation equation may be reduced, and in some cases, the bifurcation equation is identically satisfied. In section 3 we will apply the results of section 2 to two examples and give sufficient conditions for the existence of a 2-parameter family of solutions for each of the examples.

A system of differential equations $x' = g(t, x)$ where x and g are n -vectors is said to have property (E) with respect to S if there exists a symmetric, constant $n \times n$ matrix S such that $S^2 = I$ and

$$S g(-t, Sx) = -g(t, x) \quad (4.1)$$

for $t \in [-a, a]$ and all x in R^n .

If $x(t)$ is a solution to a system $x' = g(t, x)$ which has property (E) with respect to S , then $y(t) = Sx(-t)$ is also a solution for the system $x' = g(t, x)$. For

$$\begin{aligned} y' &= \frac{d}{dt} (Sx(-t)) = -Sx'(-t) = -Sg(-t, x(-t)) \\ &= -Sg(-t, S(S(x(-t)))) = g(t, Sx(-t)) = g(t, x). \end{aligned}$$

Throughout the remainder of this chapter we will be dealing with the boundary value problem

$$x' = A(t)x + f(t, x, \epsilon) \quad (4.2)$$

$$B_1 x(-a) + B_2 x(a) = 0. \quad (4.3)$$

We will make the same assumptions on x , A , f , B_1 , B_2 , and ϵ as were made in Chapter 3. The operators P , Q , and K , and the matrices U , V , c , and d are those defined in Chapter 2. We will be dealing with the bifurcation equation

$$H(\alpha, \epsilon) = 0 \quad (4.4)$$

derived in Chapter 3.

4.2 DEPENDENCE OF THE COMPONENTS OF THE VECTOR

$H(\alpha, \epsilon)$

Lemma 4.1:

Suppose S is a constant, symmetric, $n \times n$ matrix such that $S^2 = I$. Let $A(t)$ be an $n \times n$ matrix whose entries are bounded measurable functions over the interval $[-a, a]$, and let $f \in (L_1[-a, a])^n$. If

$$a) \quad SA(-t) = -A(t)S \quad (4.5)$$

$$b) \quad Sf(-t) = -f(t)$$

c) if $z(t)$ satisfies (4.3), then $Sz(-t)$ also satisfies (4.3)

d) if $z(t)$ satisfies the adjoint boundary conditions to (4.3), i.e., $\tilde{z}(-a) = \tilde{\alpha}B_1$, $\tilde{z}(a) = -\tilde{\alpha}B_2$, then so does $Sz(-t)$.

then

$$a) \quad SU(-t)\alpha = U(t)\alpha \quad \text{for } t \in [-a, a] \quad (4.6)$$

if $SU(0)\alpha = U(0)\alpha$, α is a p-vector.

$$b) \quad SQf(-t) = -Qf(t)$$

$$c) \quad S\{K(I-Q)f\}(-t) = \{K(I-Q)f\}(t)$$

Remark:

Conditions (4.5a) and (4.5b) imply the non-homogeneous system $x' = A(t)x + f(t)$ has property (E) with respect to S . Assumptions (4.5c) and (4.5d) on the boundary conditions insure $Sz(-t)$ is a solution to the boundary value problem (or its adjoint) whenever z is a solution to the boundary value problem (or its adjoint).

Proof:

Since A satisfies (4.5a), the system

$$x' = A(t)x \quad (4.7)$$

has property (E) with respect to S . Hence $SU(-t)\alpha$ is a solution to (4.7) as is $U(t)\alpha$. If $SU(0)\alpha = U(0)\alpha$, then by uniqueness we must have $SU(-t)\alpha = U(t)\alpha$ for $t \in [-a, a]$. This proves (4.6a).

The columns of the matrix $U(t)$ form a basis for the solutions to (4.7) which satisfy (4.3). The columns of $SU(-t)$ also form a basis for the solution to (4.7) which satisfy (4.3) since (4.5c) insures the columns of $S(-t)$ satisfy (4.3). Therefore there is a nonsingular matrix G such that G is a $p \times p$ matrix and

$$SU(-t) = U(t)G \quad t \in [-a, a] . \quad (4.8)$$

Similarly, there is a nonsingular $q \times q$ matrix G_0 such that

$$V(-t)S = G_0 V(t) \quad t \in [-a, a]. \quad (4.9)$$

Let f satisfy (4.5b) , then since S is self-adjoint

$$\begin{aligned} SQf(-t) &= S\tilde{V}(-t) d^{-1} \int_{-a}^a V(u) f(u) du \\ &= \tilde{V}(t)\tilde{G}_0 d^{-1} \int_{-a}^a V(-u) f(u) du \\ &= -\tilde{V}(t)\tilde{G}_0 d^{-1} \int_{-a}^a V(-u) Sf(u) du \\ &= -\tilde{V}(t)\tilde{G}_0 d^{-1} G_0 \int_{-a}^a V(u) f(u) du . \end{aligned}$$

It follows from the definition of d that $G_0 d \tilde{G}_0 = d$. Moreover, $V(-t)S = G_0 V(t)$ thus

$$V(-t) = V(-t)S^2 = G_0 V(t)S = G_0^2 V(-t)$$

and hence $G_0^2 = I$. It now follows that $\tilde{G}_0 d^{-1} G_0 = d^{-1}$.

Continuing our calculation of $SQf(-t)$ we find

$$\begin{aligned} SQf(-t) &= -\tilde{V}(t) \tilde{G}_0 d^{-1} G_0 \int_{-a}^a V(u) f(u) du \\ &= -\tilde{V}(t) d^{-1} \int_{-a}^a V(u) f(u) du \\ &= -Qf(t) . \end{aligned}$$

This proves (4.6b).

It follows from (4.5b) and (4.6b) that $S(I-Q)f(-t) = -(I-Q)f(t)$. Hence to prove (4.6c), it suffices to prove (4.6c) with $(I-Q)f$ replaced by f such that $Qf = 0$.

Let $Qf = 0$, then Kf is the unique solution to the boundary value problem

$$x' = A(t)x + f(t) \quad t \in [-a, a] \quad (4.10)$$

$$B_1 x(-a) + B_2 x(a) = 0 \quad (4.11)$$

such that $Px = 0$.

Assumptions (4.5a) and (4.5b) imply system (4.10) has property (E) with respect to S , hence $SKf(-t)$ is also a solution to (4.10). Since Kf satisfies (4.11), (4.5c) implies $SKf(-t)$ also satisfies (4.11).

Since Kf is unique solution of (4.10) - (4.11) such that $PKf = 0$, it remains only to show $P(SKf)(-\cdot) = 0$.

$$\begin{aligned} P\{(SKf)(-\cdot)\}(t) &= U(t) c^{-1} \int_{-a}^a \tilde{U}(u) SKf(-u) du \\ &= U(t) c^{-1} \int_{-a}^a \tilde{\tilde{G}}\tilde{U}(-u) Kf(-u) du \\ &= U(t) c^{-1} \tilde{G} \left\{ \int_{-a}^a \tilde{U}(u) Kf(u) du \right\} \\ &= 0 . \end{aligned}$$

Hence $SKf(-\cdot) = Kf(\cdot)$. This proves (4.6c).

Q.E.D.

Theorem 4.1:

Let $x^*(\alpha, \epsilon)$ be the function which is the solution to the auxiliary equation corresponding to a particular α and ϵ . Suppose that the system

$$x' = A(t)x + f(t, x, \epsilon) \quad (4.12)$$

has property (E) with respect to S for all $|\epsilon| \leq \epsilon_0$ and assumptions (4.5c) and (4.5d) of lemma 4.1 are satisfied, then

$$Sx^*(\alpha, \epsilon)(-t) = x^*(\alpha, \epsilon)(t) \quad (4.13)$$

for $t \in [-a, a]$ provided $SU(0)\alpha = U(0)\alpha$.

Moreover, if $H(\alpha, \epsilon)$ denotes the vector defined in Chapter 3, then

$$(I + G_0) H(\alpha, \epsilon) = 0, \quad (4.14)$$

where G_0 is the nonsingular $q \times q$ matrix such that $V(-t) = G_0 V(t)$ for $t \in [-a, a]$.

Proof:

Let $SU(0)\alpha = U(0)\alpha$ and S_α as defined in Chapter 3. Let S_α^* be the subset of S made up of those y in S_α such that $Sy(-t) = y(t)$. Recall $F(\alpha, \epsilon): S_\alpha \rightarrow S_\alpha$ for $|\alpha| \leq \rho$, $|\epsilon| \leq \epsilon_0$, was given by

$$F(\alpha, \epsilon)y = U\alpha + K(I-Q)f(\cdot, y, \epsilon).$$

Since (4.12) has property (E) with respect to S , then for fixed y in S_α^* , $x' = A(t)x + f(t, y(t), \epsilon)$ has property (E) with respect to S (note: $SF(-t, y(-t), \epsilon) = Sf(-t, Sy(t), \epsilon) = -f(t, y(t), \epsilon)$) hence

$K(I-Q)f(\cdot, y, \epsilon)$ is in S_α^* . Moreover, if $SU(0)\alpha = U(0)\alpha$, then $U(t)\alpha$ is in S_α^* hence $F(\alpha, \epsilon)y$ is in S_α^* . $F(\alpha, \epsilon): S_\alpha^* \rightarrow S_\alpha^*$, hence the fixed point $x^*(\alpha, \epsilon)$ of $F(\alpha, \epsilon)$ must lie in S_α^* . That is $Sx^*(\alpha, \epsilon)(-t) = x^*(\alpha, \epsilon)$.

Using (4.13) and the fact $V(-t)S = G_0 V(t)$ we have

$$\begin{aligned} H(\alpha, \epsilon) &= \int_{-a}^b V(u) f(u, x^*(\alpha, \epsilon)(u), \epsilon) du \\ &= \int_{-a}^a V(u) f(u, Sx^*(-t), \epsilon) du \\ &= - \int_{-a}^a V(u) Sf(-u, x^*(-u), \epsilon) du \\ &= -G_0 \int_{-a}^a V(-u) f(-u, x^*(-u), \epsilon) du \\ &= -G_0 \int_{-a}^a V(u) f(u, x^*(u), \epsilon) du \\ &= -G_0 H(\alpha, \epsilon) . \end{aligned}$$

This proves (4.14) .

Q.E.D.

Whenever the assumptions of theorem 4.1 are satisfied, then the components of $H(\alpha, \epsilon)$ are

dependent. Hence the number of components is the bifurcation equation may be reduced. For a discussion of property (E) and periodic solutions see Hale [15, pp. 267-269].

4.3. FAMILIES OF SOLUTIONS

Let us begin by applying the methods of section 2 to the boundary value problem

$$x' = A(t)x + \epsilon f(t, x) \quad t \in [-\pi, \pi] \quad (4.15)$$

$$\text{diag}(1, 1, 1)x(-\pi) + \text{diag}(-1, 1, -1)x(\pi) = 0 \quad (4.16)$$

where $x = \text{col}(x_1, x_2, x_3)$. $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ hence

A satisfies those assumptions made upon A in Chapter 3 . Let $f = \text{col}(f_1, f_2, f_3)$, where $f \in (L_1[-\pi, \pi])^3$, f lipschitzian and bounded , i.e. for $x, y \in (AC[-\pi, \pi])^3$, $|x|, |y| \leq R_1$. There are constants M and L such that

$$|f(t, x) - f(t, y)| \leq L|x - y| \quad \text{and} \quad |f(t, x)| \leq M$$

for all $t \in [-\pi, \pi]$ (4.17)

Thus f satisfies those assumptions made on f in Chapter 3. Finally, we assume f has the following

symmetry properties:

$$\begin{aligned} f_1(-t, x_1, -x_2, x_3) &= -f_1(t, x) \\ f_2(-t, x_1, -x_2, x_3) &= f_2(t, x) \\ f_3(-t, x_1, -x_2, x_3) &= -f_3(t, x) \end{aligned} \quad (4.18)$$

Let $S = \text{diag}(1, -1, 1)$, then S is a 3×3 constant symmetric matrix and $S^2 = I$. We will show (4.15) has property (E) with respect to S . We must show

$$S\{A(-t)Sx + \epsilon f(-t, Sx)\} = -\{A(t)x + \epsilon f(t, x)\}$$

$$\begin{aligned} SA(-t)S &= \text{diag}(1, -1, 1) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{diag}(1, -1, 1) \\ &= \text{diag}(1, -1, 1) \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= -A(t) \end{aligned}$$

Now $Sx = \text{diag}(1, -1, 1)x = \text{col}(x_1, -x_2, x_3)$, so we have

Now $Sx = \text{diag}(1, -1, 1)X = \text{col}(x_1, -x_2, x_3)$, so we have

$$\begin{aligned} Sf(-t, Sx) &= \text{diag}(1, -1, 1) \begin{pmatrix} f_1(-t, x_1, -x_2, x_3) \\ f_2(-t, x_1, -x_2, x_3) \\ f_3(-t, x_1, -x_2, x_3) \end{pmatrix} \\ &= \begin{pmatrix} f_1(-t, x_1, -x_2, x_3) \\ -f_2(-t, x_1, -x_2, x_3) \\ f_3(-t, x_1, -x_2, x_3) \end{pmatrix} \\ &= -f(t, x) . \end{aligned}$$

It easily follows that (4.15) has property (E) with respect to S .

We will now show that the boundary condition (4.16) is such that (4.5c) and (4.5d) hold .

Assume z satisfies (4.16) , then

$$\begin{aligned} \text{diag}(1, 1, 1)Sz(-(-\pi)) &= \text{diag}(-1, 1, -1)Sz(-\pi) \\ &= \text{diag}(1, -1, 1)z(\pi) + \text{diag}(-1, -1, -1)z(-\pi) \\ &= -\{\text{diag}(1, 1, 1)z(-\pi) + \text{diag}(-1, 1, -1)z(\pi)\} \end{aligned}$$

But the term in brackets is zero since z satisfies (4.16) . Hence $Sz(-\cdot)$ satisfies (4.16) also. This

shows (4.5c) holds . The adjoint boundary conditions are the same as (4.16) , hence (4.5d) also holds.

We will now find U and V for the boundary value problem (4.15) = (4.16) , then calculate G_0 and $H(\alpha, \epsilon)$. To find U we consider the system

$$x' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x \quad (4.19)$$

$$\text{diag}(1,1,1)x(-\pi) + \text{diag}(-1,1,-1)x(\pi) = 0 \quad (4.20)$$

A fundamental matrix solution for (4.19) is

$$\begin{pmatrix} \sin t & \cos t & 0 \\ \cos t & -\sin t & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Hence, U is given by

$$U(t) = \begin{pmatrix} \cos t & 0 \\ -\sin t & 0 \\ 0 & 1 \end{pmatrix} .$$

The adjoint is (4.19) is

$$x' = - \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{} x$$

or

$$x' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x \quad (4.21)$$

with the same boundary conditions (4.20) . In this case the problem is selfadjoint hence

$$V(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

We now calculate G_0 . Recall G_0 is the non-singular constant 2×2 matrix such that

$$V(-t)S = G_0V(t)$$

$$\begin{aligned} \text{i.e. } & \begin{pmatrix} \cos(-t) & -\sin(-t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{diag}(1, -1, 1) \\ &= \begin{pmatrix} \cos t & \sin t & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{diag}(1, -1, 1) \\ &= \begin{pmatrix} \cos t & -\sin t & 0 \\ 0 & 0 & 1 \end{pmatrix} = V(t) . \end{aligned}$$

Hence $G_0 = \text{diag}(1, 1, 1)$.

Let α be any 2-vector, then

$$\begin{aligned} SU(0)\alpha &= \text{diag}(1, -1, 1) \begin{pmatrix} \cos 0 & 0 & 0 \\ -\sin 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \alpha \\ &= \text{diag}(1, -1, 1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \alpha \end{aligned}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \alpha = U(0)\alpha .$$

Hence for any 2-vector α , $SU(0)\alpha = U(0)\alpha$. It now follows as a consequence of Theorem 4.1, that for $|\alpha| \leq \rho$ and $|\epsilon| \leq \epsilon_0$, then

$$(I + G_0) H(\alpha, \epsilon) = 0 .$$

But since $G_0 = I$, that implies $H(\alpha, \epsilon) = 0$. Hence the bifurcation equation is always satisfied. We have shown that the boundary value problem (4.15) - (4.16) has a 2-parameter family of solutions for each ϵ , $|\epsilon| \leq \epsilon_0$. The two parameters are α_1 and α_2 .

As a final example we will show that the following boundary value problem also has a 2-parameter family of solutions.

$$x' = A(t)x + \epsilon f(t, x) \tag{4.22}$$

$$\text{diag}(1, 1, 1)x(-a) + \text{diag}(-1, 1, -1)x(a) = 0 \tag{4.23}$$

where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. We assume $f \in (L_1[-a, a])^n$

satisfies the same Lipschitzian assumptions made upon f in the previous example . Finally we assume f has the following symmetry.

$$\begin{aligned} f_1(-t, x_1, -x_2, x_3) &= -f_1(t, x) \\ f_2(-t, x_1, -x_2, x_3) &= f_2(t, x) \\ f_3(-t, x_1, -x_2, x_3) &= -f_3(t, x) \end{aligned} \tag{4.24}$$

Let $S = \text{diag}(1, -1, 1)$. We will show (4.22) has property (E) with respect to S .

$$\begin{aligned} SA(-t)S &= \text{diag}(1, -1, 1) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{diag}(1, -1, 1) \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \text{diag}(1, -1, 1) \\ &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = -A(t) \end{aligned}$$

$$\begin{aligned} Sf(-t, Sf) &= \text{diag}(1, -1, 1)f(-t, x_1, -x_2, x_3) \\ &= \text{diag}(1, -1, 1) \begin{pmatrix} -f_1(t, x) \\ f_2(t, x) \\ -f_3(t, x) \end{pmatrix} \\ &= -f(t, x) . \end{aligned}$$

Hence it follows that (4.22) has property (E) with respect to S . Assume z satisfies boundary conditions (4.23). Then

$$\begin{aligned} & \text{diag}(1,1,1)Sz(a) + \text{diag}(-1,1,-1)Sz(-a) \\ &= \text{diag}(1,-1,1)z(a) + \text{diag}(-1,-1,-1)z(-a) \\ &= -\{\text{diag}(1,1,1)z(-a) + \text{diag}(-1,1,-1)z(a)\} \\ &= 0 \end{aligned}$$

Since the adjoint boundary conditions is the same as (4.23), (4.5c) and (4.5d) are satisfied.

For our system (4.22) - (4.23), one can show that

$$U(t) = \begin{pmatrix} 1 & t^2 \\ 0 & 2t \\ 0 & 2 \end{pmatrix} \text{ and } V(t) = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -2t & t^2 \end{pmatrix}.$$

In this case our system is not selfadjoint. $G_0 = \text{diag}(1,1)$ and for every 2-vector α , $SU(0)\alpha = U(0)\alpha$. Hence by theorem 4.1

$$(I + G_0) H(\alpha, \epsilon) = 0.$$

Since $G_0 = I$, it follows that

$$H(\alpha, \epsilon) = 0$$

for all $|\alpha| \leq \rho$ and $|\epsilon| \leq \epsilon_0$. Hence, for each fixed ϵ , (4.22) - (4.23) has a 2-parameter family of solutions where the two parameters are α_1 and α_2 .

CHAPTER 5

ABSTRACT PROBLEMS IN THE LARGE AND MONOTONE OPERATORS

5.1 INTRODUCTION

As mentioned in Chapter 1 we consider here problems in abstract form, and use the alternative scheme of Hale, Bancroft, and Sweet [16] to split the problem into a system of two equations, the auxiliary and the bifurcation equations. We then use monotone operator theory to show that, with suitable hypotheses, both the auxiliary and the bifurcation equations are solvable, and therefore the original problem has a solution. We extend thereby to nonselfadjoint problems the results of Cesari and Kannan who have handled by analogous arguments the selfadjoint case. By a more delicate argument we also weaken the monotonicity assumptions by assuming only quasimonotonicity.

Finally, we apply the results to nonlinear boundary value problems in the large, for which the underlying linear problems are elliptic and non-selfadjoint.

5.2 A DECOMPOSITION SCHEME

Let us consider the equation

$$Ex = Nx \quad (5.1)$$

where E is a linear operator whose domain $D(E)$ is a subspace of a real Hilbert space X and whose range $R(E)$ lies in X .

Here N is an operator, not necessarily linear, with $D(N) \subset X$, and $R(N) \subset X$. Moreover, we assume $D(E) \subset D(N)$.

Let us assume there are bounded projection operators $P: X \rightarrow X$ and $Q: X \rightarrow X$ and a linear operator $H: D(H) \rightarrow X$, $D(H) \subset X$, such that for all $x \in D(E)$

$$(A_1) \quad H(I-Q)Ex = (I-P)x$$

$$(A_2) \quad QEx = EPx$$

$$(A_3) \quad EH(I-Q)Nx = (I-Q)Nx$$

and where $R(P) \subset D(E)$, $R(H) \subset D(E)$, $R(E) = D(H)$, and $(I-Q)R(E) = D(H)$.

In view of (A_1) and (A_3) , we may think of H as a partial inverse for E . It follows from (A_1) that $\ker(E) \subset PX$. Since $D(H) \subset R(E)$, this implies

$(I-Q)R(E) \subset R(E) \cap \{(I-Q)X\}$ and since $I-Q$ is the identity on $(I-Q)X$, we must have $(I-Q)R(E) = R(E) \cap \{(I-Q)X\}$.

We will use the following notation: $X_0 = PX$ and $X_1 = (I-P)X$ hence $X = X_0 + X_1$; $Y_0 = QX$ and $Y_1 = (I-Q)X$ hence $X = Y_0 + Y_1$.

Theorem 1:

If (A_{123}) are satisfied, then $Ex = Nx$ for some $x \in D(E)$ if and only if

$$x = Px + H(I-Q)Nx \quad (5.2)$$

$$Q(Ex-Nx) = 0 \quad (5.3)$$

Proof:

(Only if). If $Ex = Nx$ for some $x \in D(E)$, then $Q(Ex-Nx) = 0$ and $(I-Q)Ex = (I-Q)Nx$. Now apply H to the second equation to get $H(I-Q)Ex = H(I-Q)Nx$. By (A_1) we have $(I-P)x = H(I-Q)Nx$ or $x = Px + H(I-Q)Nx$. Hence (5.2) and (5.3) are satisfied.

(If). If (5.2) and (5.3) are satisfied, then applying E to (5.2) we have $Ex = EPx + EH(I-Q)Nx$ or by

(A_2) $(I-Q)Ex = EH(I-Q)Nx$ and by (A_3) $(I-Q)Ex = (I-Q)Nx$. Rewritten we have $Ex - Nx = Q(Ex - Nx)$ where the right hand side is zero since (5.3) is satisfied.

Q.E.D.

We may write equation (5.2) as

$$x = x_0 + H(I-Q)Nx \quad (5.2)'$$

where $x_0 \in X_0 = PX$ since by (A_1) $PH(I-Q)Nx = 0$. Equation (5.2)' is referred to as the auxiliary equation and equation (5.3) as the bifurcation equation. It is often possible to choose P , Q , and H so that the auxiliary equation has a unique solution for each $x_0 \in X_0$, hence equation (5.3) may be viewed as an equation in the unknown x_0 over X_0 . If X_0 is finite dimensional, then we need only solve an alternative problem in a finite dimensional space X_0 . In the literature techniques involving contraction maps or monotone operators have been used to choose P , Q , and H in order to solve equation (5.2)'. In the following sections we make additional assumptions of E , N , P , and Q in order to use monotone operator theory to solve equation (5.2)' and equation (5.3).

5.3 RESULTS FROM MONOTONE OPERATOR THEORY

Let H be a real Hilbert space. An operator $T: D(T) \rightarrow 2^H$ is said to be monotone if

$$(u-v, x-y) \geq 0 \quad \text{for all } x, y \in D(T), \\ u \in Tx, v \in Ty.$$

It follows that T is monotone if and only if the graph of T is a monotone set. Since monotonicity is a property of the graph of a map, it follows that T^{-1} is monotone if and only if T is monotone.

A monotone set V is said to be maximal monotone if V is not a proper subset of another monotone set V' . A map T is maximal monotone if its graph is maximal monotone. Again we have T^{-1} is maximal monotone if and only if T is maximal monotone.

A map $T: D(T) \rightarrow 2^H$ is said to be strongly monotone if there is a constant $c > 0$ such that

$$(u-v, x-y) \geq c \|x-y\|^2 \quad \text{for all } x, y \in D(T) \\ u \in Tx, v \in Ty.$$

A map $T: D(T) \rightarrow 2^H$, $D(T) \subset H$, is said to be quasimonotone if there is a constant $c > 0$ such that

$$(u-v, x-y) \geq -c \|x-y\|^2 \quad \text{for all } x, y \in D(T) \\ u \in Tx, v \in Ty.$$

Brézis has shown (Brézis [3]) that the map $x \rightarrow \text{convex hull } Tx$ is monotone if T is monotone. Hence, if T is maximal monotone, then Tx is a closed convex set in H . We may now define T^0x as the projection of 0 onto Tx , that is, T^0x is the element of Tx with minimal norm.

A map $T: D(T) \rightarrow 2^H$ is said to be coercive if there exists $x_0 \in H$ such that

$$\lim_{\substack{x \in D(T) \\ \|x\| \rightarrow \infty}} (T^0x, x-x_0) \|x\|^{-1} = +\infty$$

If $T: D(T) \rightarrow H$ is strongly monotone, then T is coercive. To show this fix $x_0 \in D(T)$ then

$$(Tx, x-x_0) = (Tx-Tx_0, x-x_0) + (Tx_0, x-x_0) \\ \geq c \|x-x_0\|^2 - \|Tx_0\| \|x-x_0\|$$

Since x_0 is fixed, $\|x\|^{-1} \|x-x_0\|$ approaches one as $\|x\| \rightarrow \infty$. Hence T is coercive.

As a reference for the theory of monotone operators in a Hilbert space see Brézis [3]. We will need the following results:

Theorem 5.1: (Brézis[3])

Let $T: D(T) \rightarrow 2^H$ be monotone. T is maximal monotone if and only if $R(aI+T) = H$ for $a > 0$.

Theorem 5.2: (Brézis[3])

If $T: D(T) \rightarrow 2^H$ is maximal monotone and coercive, then $R(T) = H$.

Theorem 5.3: (Brézis, Crandell, and Pazy [4])

If both A and B are maximal monotone and either

$$(a) \quad \{\text{Int } D(A)\} \cap D(B) \neq \emptyset$$

or

$$(b) \quad D(A) \subset D(B) \quad \text{and}$$

$$\|Bx\| \leq k(\|x\|) \|Ax\| + C(\|x\|),$$

where $k(r)$ and $C(r)$ are non-decreasing functions of r and $k(r) < 1$ for every r . Then $A + B$ is maximal monotone.

A map $T: D(T) \rightarrow H$ is said to be hemicontinuous if T is continuous on every line segment S of $D(T)$ with respect to the strong topology on S and the weak topology in the range.

Theorem 5.4: (Browder [5])

A monotone hemicontinuous map $T: H \rightarrow H$ is maximal monotone.

We now prove the following theorem for quasimonotone maps.

Theorem 5.5:

Let $A: D(A) \rightarrow 2^H$, $D(A) \subset H$ and $B: H \rightarrow 2^H$ hemicontinuous. If A is strongly monotone with constant μ , A maximal monotone, and B quasimonotone with constant $\eta < \mu$, then $A + B$ is maximal monotone. If A and B are single-valued, then $R(A+B) = H$.

Proof:

Let $\alpha > \eta > 0$, then $B + \alpha I$ is strongly monotone. Since $B + \alpha I$ is defined over all of H and hemicontinuous over H , then $B + \alpha I$ is maximal monotone. $A + B + \alpha I$ is maximal monotone since $B + \alpha I$ is defined for all of H (see Theorem 5.3a). Hence $R((1+\alpha)I+A+B) = H$. Now $A + B$ is strongly monotone with monotonicity constant $(\mu-\eta) > 0$, hence by Theorem 5.1 $A + B$ is maximal monotone. If A and B are single-valued and strongly monotone hence coercive then $R(A+B) = H$.

Q.E.D.

The concept of monotonicity has been extended to maps $T: X \rightarrow 2^{X^*}$ where X is a real reflexive Banach space and X^* the dual of X . Let (y, x) denote $y(x)$ for $y \in X^*$, $x \in X$. For references on monotone operators in reflexive Banach space see Browder [5], Minty [22], or Brézis, Crandell, and Pazy [4].

A map $T: D(T) \rightarrow 2^{X^*}$, $D(T) \subset X$ is said to be monotone provided

$$(u-v, x-y) \geq 0 \quad \text{for all } x, y \in D(T) \\ u \in Tx, v \in Ty.$$

In this fashion all the theorems and definitions given for maps in Hilbert spaces carry over to Banach spaces except for Theorems 5.1 and 5.5 and the definition of coercive. For coerciveness we have

A map $T: D(T) \rightarrow X^*$, $D(T) \subset X$ is said to be coercive if there is a constant $R > 0$, such that for $\|x\| > R$, $x \in D(T)$

$$(u, x) > 0 \quad \text{for all } u \in Tx.$$

If $T: D(T) \rightarrow X^*$ and $D(T) \subset X$, then a sufficient condition for T to be monotone is

$$\lim_{\substack{x \in D(T) \\ \|x\| \rightarrow \infty}} \frac{(Tx, x)}{\|x\|} = +\infty$$

We will now apply monotone operator theory to solve the bifurcation and auxiliary equations.

5.4 SOLVING THE AUXILIARY EQUATION BY MONOTONE OPERATORS

Let X be a real Hilbert space. We assume E , P , Q , H , and N satisfy assumptions (A_{123}) . In addition we assume

(A₄) There is a constant $\mu > 0$ such that for $x \in D(E) \cap X_1$ $(-Ex, x) \geq \mu \|x\|^2$ or for $y \in Y_1$ $(y, -H_y) \geq \mu \|-Hy\|^2$.

(A₅) Let P and Q be chosen so that $(y, x) = 0$ for every $y \in Y_0$ and $x \in X_1$. We say Y_0 is orthogonal to X_1 , i.e., $Y_0 \perp X_1$.

With assumptions (A₁₂₃₄₅) and only quasimonotonicity assumed on N , we will show that the auxiliary equation (5.2)' is uniquely solvable for each $x_0 \in X_0$.

Theorem 5.6:

Let P , Q , and H be defined as in section 2, and satisfy conditions (A₁₂₃₄₅). Let $D(N) = X$, N hemicontinuous from X to X , and N quasimonotone with constant $\eta > 0$, $\eta < \mu$. Then, the auxiliary equation (5.2)' has a unique solution $x = x(x_0)$ for each $x_0 \in X_0$.

Proof:

We write the auxiliary equation (5.2)' in the form

$$x - x_0 - H(I-Q)Nx = 0 \quad (5.4)$$

then since $0 \in [-H(I-Q)]^{-1} 0$, equation (5.4) implies

$$\begin{aligned} [-H(I-Q)]^{-1} (x-x_0) + Nx &\ni 0 \\ x &\in x_0 + (X_1 \cap D(E)) \end{aligned} \tag{5.5}$$

Conversely, if (5.5) holds, by applying $-H(I-Q)$ to both sides we get

$$x - x_0 + -H(I-Q)Nx = -H(I-Q)0 = 0.$$

So equation (5.4) and (5.5) are equivalent.

Equation (5.5) is of the form $Ax + Bx \ni 0$ where $Ax = [-A(I-Q)]^{-1}(x-x_0)$ and $Bx = Nx$. By our assumption on N we find that B satisfies the hypothesis of Theorem 5.5. We will now show that A is strongly monotone with monotonicity constant $\mu > \eta$ and A is maximal monotone.

With assumption (A_{45}) we have for all $y \in X$,

$$\begin{aligned} (y, -H(I-Q)y) &= ((I-Q)y, -H(I-Q)y) + (Qy, -H(I-Q)y) \\ &= ((I-Q)y, -H(I-Q)y) \\ &\geq \mu \| -H(I-Q)y \|^2 \end{aligned} \tag{5.6}$$

so $-H(I-Q)$ is monotone. Since $-H(I-Q)$ is bounded

linear operator defined on all of X , $-H(I-Q)$ is continuous over X , hence $-H(I-Q)$ is maximal monotone. Since x_0 is fixed, $-H(I-Q) + x_0$ is maximal monotone. Now A is just the inverse of the map $Ky = -H(I-Q)y + x_0$ so it follows that A is maximal monotone. To show A is strongly monotone, let $x, y \in D(A) = x_0 + (X_1 \cap D(E))$. Then $x - x_0 \in X_1 \cap D(E)$ and $y - x_0 \in X_1 \cap D(E)$. For $x^* \in Ax$, $x - x_0 = -H(I-Q)x^*$ and for $y^* \in Ay$, $y - x_0 = -H(I-Q)y^*$. Using equation (5.6), we have

$$\begin{aligned} (x^* - y^*, x - y) &= (x^* - y^*, -H(I-Q)x^* + H(I-Q)y^*) \\ &= ((I-Q)x^* - (I-Q)y^*, -H(I-Q)x^* \\ &\quad + H(I-Q)y^*) \geq \mu \| -H(I-Q)x^* \\ &\quad + H(I-Q)y^* \|^2 \geq \mu \|x - y\|^2. \end{aligned}$$

So A is strongly monotone with monotonicity constant μ .

To show that $A + B$ is coercive we use the fact that $-H(I-Q)$ is a one to one map from Y_1 onto $X_1 \cap D(E)$. Thus for $y_1 \in Y_1$ there is a unique $x_1 \in X_1 \cap D(E)$ such that $x_1 = -H(I-Q)y_1$. Thus, $[-H(I-Q)]^{-1}x_1 = y_1 + y_0$. Let $x = x_0 + x_1$, $x_1 \in X_1 \cap D(E)$, then

$$\begin{aligned} Ax &= [-H(I-Q)]^{-1}(x-x_0) = [-H(I-Q)]^{-1}x_1 \\ &= y_1 + y_0 . \end{aligned}$$

Let $A^0x = y_1 + y_0^*$, $y_0^* \in Y_0$ (in fact since $Y_0 \perp X_1$ we have $A^0x = y_1$) . Using equation (5.6) and condition (A_3) we have

$$\begin{aligned} (A^0x, x-x_0) &= (A^0x, x_1) = (y_1+y_0^*, x_1) \\ &= (y_1, x_1) + (y_0^*, x_1) \\ &= (y_1, x_1) = (y_1, -H(I-Q)y_1) \\ &\geq \mu \| -H(I-Q)y_1 \|^2 = \mu \| x_1 \|^2 \\ &= \mu \| x-x_0 \|^2 \end{aligned}$$

Since x_0 is fixed, this implies A is coercive. Since B is single-valued and montone, $A + B$ is coercive.

We now apply Theorem 5.2 to find that $R(A+B) = X$. The equation $Ax + Bx \ni 0$ has a solution, hence the auxiliary equation has a solution for each $x_0 \in X_0$.

To prove uniqueness, let x_1 and x_2 both be solutions to the auxiliary equation for a fixed $x_0 \in X_0$, then $x_1 - x_0 = H(I-Q)Nx_1$; $x_2 - x_0 = H(I-Q)Nx_2$;

and $x_1 - x_2 = H(I-Q)Nx_1 - H(I-Q)Nx_2$. By assumption (A_5) , equation (5.6), and the hypothesis of N we have

$$\begin{aligned} -\eta \|x_1 - x_2\|^2 &\leq (Nx_1 - Nx_2, x_1 - x_2) \\ &= -(Nx_1 - Nx_2, -H(I-Q)Nx_1 + H(I-Q)Nx_2) \\ &\leq -\mu \|-H(I-Q)Nx_1 + H(I-Q)Nx_2\|^2 \\ &= -\mu \|x_1 - x_2\|^2 \end{aligned}$$

That is $0 \leq (\eta - \mu) \|x_1 - x_2\|^2$. Since $\mu > \eta$, we must have $x_1 = x_2$. This completes the proof of theorem.

Q.E.D.

We will now solve the bifurcation equation using monotone operators.

5.5 SOLVING THE BIFURCATION EQUATION WITH MONOTONE OPERATORS

In section 4 we gave conditions on E , P , Q , and N which allowed us to obtain a unique solution to the auxiliary equation. We may express x by

$$x = [I - H(I - Q)N]^{-1}x_0 \quad (5.7)$$

In this section we will make the following additional assumption on P and Q .

$$(A_6) \quad (y, x) = 0 \quad \text{for } y \in Y_1, \text{ and } x \in X_0.$$

Theorem 5.7:

Let X_0, Y_0 be finite dimensional subspaces of X , and assume the dimension of X_0 equals the dimension of Y_0 . Under assumptions (A_{123456}) :

(a) If $X_1 = X$, and $Y_1 = X$, i.e., $(-Ex, x) \geq \mu \|x\|^2$ For all $x \in D(E)$ and $N: X \rightarrow X$ is hemicontinuous and quasimonotone with constant $\eta \geq 0$ and $\eta < \mu$, then $Ex = Nx$ has a unique solution.

(b) If $(-Ex, x) \geq -\alpha \|x\|^2$ For all $x \in D(E)$ $\alpha > 0$, and if $(Nx - Ny, x - y) \geq \eta \|x - y\|^2$ For all $x, y \in X$, N hemicontinuous and bounded, then the equation $Ex = Nx$ has at least one solution provided $\eta > \alpha \|P\|^2$ or $\eta = \alpha = 0$ and N is coercive.

Proof:

(a) Since $Y_1 = X$, Q is the zero operator on X , hence the bifurcation equation is always satisfied. Part (a) of the theorem now follows from Theorem 5.6.

(b) Since the dimension of X_0 and Y_0 are equal and finite, we may identify X_0^* with Y_0 . The identification may be made as follows; let $y \in Y_0 \subset X$, for $x_0 \in X_0$ let $y_0(x_0) = (y_0, x_0)$, where on the right hand side we view y_0 as an element of X and x_0 as an element of X . This identification defines a continuous, linear mapping of Y_0 into X_0^* . Our identification of Y_0 with X_0^* is complete if we can show that this map is one to one and onto. Since the dimension is finite we need only show the map is one to one or the kernel of the map is just the zero element of Y_0 . For this let y_0 be an element of Y_0 . If $(y_0, x_0) = 0$ for all $x_0 \in X_0$ then by (A_5) , $(y_0, x_1) = 0$ for all $x_1 \in X_1$ hence $(y_0, x) = 0$ for all $x \in X$ hence y_0 must be the zero element of X and Y_0 .

We now reduce $Ex = Nx$ to the system of auxiliary and bifurcation equations $(5.2)'$ and (5.3) . For any $x_0 \in X_0$, the auxiliary equation has a unique solution (by Theorem 5.6) given by (5.7) . The bifurcation equation may now be written

$$QN[I-H(I-Q)N]^{-1}x_0 - QEx = 0 \quad (5.8)$$

Since $QEx = EPx = Ex_0$, equation (5.8) becomes

$$QN[I-H(I-Q)N]^{-1}x_0 - Ex_0 = 0 \quad (5.9)$$

Again since $QEx = EPx = Ex_0$, and $Y_0 = X_0^*$, $-E: X_0 \rightarrow X_0^*$, and $T = QN[I-H(I-Q)N]^{-1}: X_0 \rightarrow X_0^*$, $T-E: X_0 \rightarrow X_0^*$.

We will now show that $T-E$ is monotone, continuous, and coercive. Then, since $T-E$ is defined on all X_0 , $R(T-E) = X_0^*$ and equation (5.9) will be solved.

Let $x, y \in X_0$, and let $u = [I-H(I-Q)N]^{-1}x$, $v = [I-H(I-Q)N]^{-1}y$. Then

$$u-H(I-Q)Nu = x \quad v-H(I-Q)Nv = y$$

Since $P(u-v) = x-y$, $\|x-y\| \leq \|P\| \|u-v\|$. So

$$\begin{aligned} (Tx_0 - Ty, x-y) &= (QNu - QNv, x-y) = (Nu - Nv, x-y) \\ &= (Nu - Nv, u-v) + (Nu - Nv, -H(I-Q)Nu \\ &\quad + H(I-Q)Nv) = (Nu - Nv, u-v) \\ &\quad + ((I-Q)Nu - (I-Q)Nv, -H(I-Q)Nu \\ &\quad + H(I-Q)Nv) \geq (Nu - Nv, u-v) \geq \eta \|u-v\|^2 \\ &\geq \eta (\|P\|^{-1})^2 \|x-y\|^2. \end{aligned}$$

Now by hypothesis, $x, y \in X_0$, $(-E(x-y), x-y) \geq -\alpha \|x-y\|^2$. Thus

$$((T-E)x - (T-E)y, x-y) \geq (\eta \|P\|^{-2} - \alpha) \|x-y\|^2$$

This proves $T-E$ is monotone. If $\eta \|P\|^{-2} > \alpha$, then it follows that $T-E$ is also coercive since T and E are single-value maps.

If $\eta = \alpha = 0$, then we must show that N coercive implies that T is coercive. Let $x_0 \in X_0$ and $x = [I-H(I-Q)N]^{-1}x_0$; $Px = x_0$ thus $\|x_0\| \leq \|P\| \|x\|$, and $\|x_0\| \rightarrow +\infty$ implies $\|x\| \rightarrow +\infty$. We now have

$$\begin{aligned} (Tx_0, x_0) &= (QN[I-H(I-Q)N]^{-1}x_0, x_0) = (Nx, x_0) \\ &= (Nx, x-H(I-Q)Nx) \\ &= (Nx, x) + (Nx, -H(I-Q)Nx) \\ &\geq (Nx, x) \end{aligned}$$

So

$$\begin{aligned} \|x_0\|^{-1} (Tx_0, x_0) &\geq \|x_0\|^{-1} (Nx, x) \\ &\geq \|P\|^{-1} \|x\|^{-1} (Nx, x) . \end{aligned}$$

Since N is coercive, T must be coercive. Since $-E$ is linear and monotone, $T-E$ is coercive.

It remains to show $T-E$ is continuous. We begin by showing that $[I-H(I-Q)N]^{-1}$ is bounded. Let u, v be such that $u - H(I-Q)Nu = v$, then

$$\begin{aligned} \mu \|u-v\|^2 &= \mu \|-H(I-Q)Nu\|^2 \\ &\leq (-H(I-Q)Nu, (I-Q)Nu) \\ &= (-H(I-Q)Nu, Nu) \\ &= (v-u, Nu) \\ &= (v-u, Nu-Nv+Nv) \\ &= -(u-v, Nu-Nv) + (v-u, Nv) \\ &\leq (v-u, Nv) \\ &\leq \|v-u\| \|Nv\| \end{aligned}$$

hence

$$\|u-v\| \leq \mu^{-1} \|Nv\|$$

Thus if $\|v\| < R$, then $\|Nv\| < R'$ for some R' depending only on R since N is bounded, and

$$\begin{aligned} \|u\| &\leq \|v\| + \mu^{-1} \|Nv\| \\ &\leq R + \mu^{-1} R' \end{aligned}$$

Thus $[I-H(I-Q)N]^{-1}$ is bounded. For x, y in X we have

$$\|x+y\|^2 + \|y\|^2 \geq \frac{1}{4} \|x\|^2 \quad (5.10)$$

To prove this consider the two cases $\|x\| \geq \frac{1}{2} \|y\|$
and $\|x\| \leq \frac{1}{2} \|y\|$. To show $[I-H(I-Q)N]^{-1}$
is continuous let

$$u - H(I-Q)Nu = u^* \quad \text{and} \quad v - H(I-Q)Nv = v^*$$

then

$$u^* - v^* = u - v - H(I-Q)Nu + H(I-Q)Nv.$$

Now

$$\begin{aligned} \mu \| -H(I-Q)Nu + H(I-Q)Nv \|^2 &\leq (Nu - Nv, u - v) \\ &\quad + (Nu - Nv, -H(I-Q)Nu) \\ &\quad + H(I-Q)Nv) \\ &= (Nu - Nv, u - H(I-Q)Nu - v \\ &\quad + H(I-Q)Nv) \\ &= (Nu - Nv, u^* - v^*) \\ &\leq \|Nu - Nv\| \|u^* - v^*\| \end{aligned} \quad (5.11)$$

Now let $x = v - u$ and $y = u^* - v^*$ in (5.10), then

$$\begin{aligned} \frac{1}{4} \|u - v\|^2 - \|u^* - v^*\|^2 &\leq \|u^* - v^* - u - v\|^2 \\ &= \| -H(I-Q)Nu + H(I-Q)Nv \|^2 \end{aligned} \quad (5.12)$$

Now multiply (5.12) by μ and using (5.11) we have

$$\begin{aligned} (\mu/4) \|u-v\|^2 &\leq \mu \|u^*-v^*\|^2 \\ &+ \|Nu-Nv\| \|u^*-v^*\|. \end{aligned}$$

If $\|u^*\|, \|v^*\| \leq R$, then since $[I-H(I-Q)N]^{-1}$ is bounded, we have $\|u\|, \|v\| \leq R'$ for some R' and $\|Nu\|, \|Nv\| \leq R''$ since N is bounded. Thus

$$\begin{aligned} \|u-v\|^2 &\leq 4 \|u^*-v^*\|^2 + 4 \mu^{-1} \|Nu-Nv\| \|u^*-v^*\| \\ &\leq (8R+8\mu^{-1}R'') \|u^*-v^*\|. \end{aligned}$$

Hence $[I-H(I-Q)N]^{-1}$ is continuous.

Before proving the continuity of T , recall a map S is demicontinuous if $x_n \rightarrow x$ strongly in X implies $Sx_n \rightarrow Sx$ weakly in X^* . Kato [18] has shown that for monotone operators defined on a Banach space X with range in X^* , then hemicontinuity and demicontinuity agree. Hence, since $[I-H(I-Q)N]^{-1}$ is continuous, $N[I-H(I-Q)N]^{-1}$ is demicontinuous and $QN[I-H(I-Q)N]^{-1}$ is demicontinuous. Since the dimension of X_0^* is finite, weak convergence in X_0^* is the same as strong convergence. Hence $T = QN[I-H(I-Q)N]^{-1}$ is continuous. This completes the proof.

Q.E.D.

5.6 APPLICATIONS

In this section we demonstrate how the above theory applies to boundary value problems for non-linear ordinary differential equations.

Example 5.1:

Let us consider the existence of solutions to the differential equation

$$x'' + x' - x = f(t, x) \quad 0 \leq t \leq 1 \quad (5.13)$$

with Dirichlet boundary conditions

$$x(0) = x(1) = 0 \quad (5.14)$$

where $f(t, x)$ is a Nemitsky operator which maps $L_2[0, 1]$ into $L_2[0, 1]$ in a continuous fashion. Moreover, we assume $Nx = f(t, x)$ is quasimonotone with constant less than 1. Let $Ex = x'' + x' - x$, then E is a linear map from $D(E) = \{x \in L_2[0, 1] : x'' \in L_2[0, 1], x(0) = x(1) = 0\}$ into $L_2[0, 1]$. Let $X = L_2[0, 1]$ and $(x, y) = \int_0^1 x(t)y(t) dt$,

then X is a Hilbert space. For $x \in D(E)$ we have

$$\begin{aligned} (-Ex, x) &= (-x'' - x' + x, x) = (-x'', x) - (x', x) + (x, x) \\ &= (x', x') - \frac{1}{2} x^2 \Big|_0^1 + (x, x) \\ &= \|x'\|^2 + \|x\|^2 \geq \|x\|^2 \end{aligned} \tag{5.15}$$

Hence $-E$ is strongly monotone with constant one.

It also follows from equation (5.15) that the kernel of $E = \{0\}$ and E has a bounded right inverse H .

Let $P = Q =$ zero operator. It is easy to show that

(A_{1-6}) are all satisfied. We may now apply part a of theorem 5.7 and conclude that the boundary value problem (5.13) - (5.14) has a unique solution.

In the above example we could have replaced $f(t, x)$ by any of the following functions:

$$(i) \quad f(t, x) = \frac{x}{(1+x^2)^{\frac{1}{2}}} + p(t)$$

where $p(t)$ is a function in $L_2[0, 1]$. Since

$x/(1+x^2)^{\frac{1}{2}}$ is a monotone function, $f(t, x)$ is a monotone mapping. Since the derivative of $x/(1+x^2)^{\frac{1}{2}}$ is bounded, $f(t, x)$ is Lipschitzian. Thus $f(t, x)$ maps $L_2[0, 1]$ into $L_2[0, 1]$ in a continuous fashion.

$$(ii) \quad f(t, x) = ax + \arctan x + p(t)$$

where $p(t)$ is in $L_2[0,1]$ and $a > -1$. The fact that $\arctan x$ is increasing and Lipschitzian implies $f(t,x)$ maps $L_2[0,1]$ into $L_2[0,1]$ and is quasimonotone with constant a .

$$(iii) \quad f(t,x) = \frac{x^3}{1+x^2} + p(t)$$

where $p(t)$ is in $L_2[0,1]$. Since the derivative of $x^3/(1+x^2)$ is bounded and positive, $f(t,x)$ is Lipschitzian and monotone. Moreover, $f(t,x)$ maps $L_2[0,1]$ into $L_2[0,1]$. In this case $Nx = f(t,x)$ is actually coercive since for $x \in L_2[0,1]$ we have

$$\begin{aligned} (x,x) &= \int_0^1 \frac{x^4(t)}{1+x^2(t)} dt + \int_0^1 p(t)x(t) dt \\ &\geq \int_0^1 x^2(t) dt - \int_0^1 \frac{x^2(t)}{1+x^2(t)} dt - \|p\| \|x\| \\ &\geq \|x\|^2 - 1 - \|p\| \|x\|. \end{aligned}$$

Example 5.2:

Let us now consider the differential equation

$$x'' + x' = \frac{x^3}{1+x^2} + p(t) \quad 0 \leq t \leq \pi \quad (5.16)$$

with Neumann boundary conditions

$$x'(0) = x'(\pi) = 0 \quad (5.17)$$

where $p(t)$ is in $L_2[0, \pi]$. Let $Nx = x^3/(1+x^2) + p(t)$. Then Nx is monotone, continuous map from $L_2[0, \pi]$ into $L_2[0, \pi]$. As shown in example 5.1, Nx is coercive. Let $Ex = x'' + x'$ and $D(E) = \{x \in L_2[0, \pi] : x'' \in L_2[0, \pi] \text{ and } x'(0) = x'(\pi) = 0\}$. The kernel of E is the constant functions, so

set $Px = \pi^{-1} \int_0^\pi x(t) dt$. Now P is just the pro-

jection onto the constants. Let $Qx = Px$. On $(\ker E)^\perp$, E has a bounded right inverse H (see Nirenberg [23]). The adjoint of E is given by $E^*x = x'' - x'$ with the same domain as E . It now follows from the Fredholm Alternative that $D(H) = R(E) = (I-P)L_2[0, \pi] = (I-Q)L_2[0, \pi]$. It is obvious that (A_{12356}) are satisfied. To show that the hypothesis of theorem 5.7 on $-E$ are satisfied, let $x \in D(E)$ then

$$(-Ex, x) = (-x'' - x', x) = (-x'', x) - (x', x)$$

$$= (x', x') - \frac{1}{2} x^2 \Big|_0^\pi = \|x'\|^2$$

$$\geq 0.$$

To verify (A_4) , notice that the operator $Lx = -x''$, $x'(0) = x'(\pi) = 0$ has a complete orthonormal set of eigenvectors

$$U_n = \begin{cases} \pi^{-\frac{1}{2}} & n = 0 \\ (2/\pi)^{\frac{1}{2}} \cos(nt) & n = 1, 2, 3, \dots \end{cases}$$

with corresponding eigenvalues $0, 1^2, 2^2, 3^2, \dots$, hence for $x \in X_1$ where here $X_1 = (I-P)L_2[0, \pi]$

we have $x = \sum_{n=1}^{\infty} (x, U_n) U_n$. Thus,

$$(-Ex, x) = (-x'', x) - (x', x) = (-x'', x)$$

$$\begin{aligned} &= \left(\sum_{n=1}^{\infty} n^2 (x, U_n) U_n, \sum_{m=1}^{\infty} (x, U_m) U_m \right) \\ &= \sum_{n=1}^{\infty} n^2 |(x, U_n)|^2 \geq \sum_{n=1}^{\infty} |(x, U_n)|^2 \\ &= \|x\|^2. \end{aligned}$$

Hence (A^4) is satisfied with $\mu = 1$.

Now $Nx = x^3/(1+x^2) + p(t)$ is a continuous map from $L_2[0, \pi]$ into $L_2[0, \pi]$ and N certainly maps bounded set into bounded sets, so N is a bounded mapping. Now N is also monotone and

coercive, hence all the assumptions of part b of theorem 5.7 are satisfied with $\mu = 1, \eta = \alpha = 0$. Hence the boundary value problem (5.16) - (5.17) has at least one solution.

Example 5.3:

As a final example we shall consider the boundary value problem

$$\begin{aligned} x'' - x' - 2x &= x \arctan x + p(t) \\ 0 \leq t \leq 2\pi \end{aligned} \quad (5.18)$$

with periodic boundary conditions

$$x(0) = x(2\pi) \quad \text{and} \quad x'(0) = x'(2\pi) \quad (5.19)$$

where $p(t)$ is in $L_2[0, 2\pi]$.

Let $Ex = x'' - x' - 2x$ and the domain of E is the set

$$\{x \in L_2[0, 2\pi]: x'' \in L_2[0, 2\pi], x(0) = x(2\pi),$$

$$x'(0) = x'(2\pi)\}$$

For $x \in D(E)$ we have

$$\begin{aligned}
 (-Ex, x) &= (-x'', x) + (x', x) + 2(x, x) \\
 &= \|x'\|^2 + \frac{1}{2} x^2 \Big|_0^{2\pi} + 2 \|x\|^2 \\
 &\geq 2 \|x\|^2 .
 \end{aligned}$$

Hence $-E$ is strongly monotone with constant 2 .

As in example 5.1, let $P = Q =$ zero operator.

Again (A_{1-6}) are all satisfied. Now $Nx = x$.

$\arctan x + p(t)$ maps $L_2[0, 2\pi]$ into itself since

$|Nx| \leq \pi/2 |x|$ and since $-\pi/2 (x-y) \leq Nx - Ny \leq$

$\pi/2 (x-y)$ it follows that N is continuous and

quasimonotone with constant $\pi/2 < 2$. Hence by part a

of theorem 5.7 the boundary value problem (5.18) -

(5.19) has a unique solution.

CHAPTER 6

REMARKS ON THE WORK OF OSBORN AND SATHER

6.1 INTRODUCTION

In this chapter we consider a nonlinear equation of the form

$$Ex = Nx \quad x \in H \quad (6.1)$$

where H is a real Hilbert space, E is a linear operator, and N is a nonlinear operator. In chapter 5 when we considered such an equation we had to assume the nonlinear operator N was defined on all of H . Osborn and Sather [24] have proposed a method for handling the case where E is nonselfadjoint and the domain of E is a dense subset of H contained in a Hilbert space $H_1 \subset H$, and N , defined over H_1 , is monotone.

In this chapter we extend the results of Osborn and Sather to the case where N satisfies a condition similar to but weaker than quasimonotonicity. In section 2 we will use the bifurcation method used by Osborn and Sather to reduce (6.1) to an alternative problem which is usually finite dimensional. In section 3 we discuss the conditions which we assume

E must satisfy and show that they are satisfied when $-E$ is elliptic and E^* has a complete system of eigenfunctions whose span is dense in H . Finally in section 4 examples are given showing how certain elliptic ordinary differential equations with nonlinearities satisfying our quasimonotonicity assumption can be reduced to an alternative problem.

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ and let H_1 be another Hilbert space with inner product $(\cdot, \cdot)_1$ and norm $\|\cdot\|_1$. We assume H_1 and H satisfy

- i) H_1 is a dense subset of H
- ii) The inclusion mapping of H_1 into H is compact
- iii) $\|x\|_1 \geq \|x\|$ for $x \in H_1$.

Let H be decomposed into X_0 and X_1 where $H = X_0 + X_1$ and $X_0 \subset D(E)$, $EX_0 \subset X_0$, and $E(X_1 \cap D(E)) \subset X_1$. Such a decomposition may be obtained by setting X_0 equal to the subspace of H spanned by a finite set of eigenvalue for E and $H = X_0 \oplus X_1$. We do not assume that the above sum is

orthogonal. Let P be the orthogonal projection of H onto X_1 . For $x \in H$, x can be written uniquely as $x = x_0 + x_1$ where $x_0 \in X_0$ and $x_1 \in X_1$. Now since $E: X_0 \rightarrow X_0$ and $E: X_1 \cap D(E) \rightarrow X_1$, then equation (6.1) is equivalent to the system

$$-Ex_1 + PN(x_0+x_1) = PEx_0 \quad (6.2)$$

$$(I-P) \{-Ex_0 + N(x_0+x_1)\} = 0. \quad (6.3)$$

Thus if (6.2) is uniquely solvable for each x_0 in X_0 , i.e. (6.2) has a solution $x_1 = x_1(x_0)$ then solving (6.1) is reduced to solving the alternative problem

$$(I-P) \{-Ex_0 + N(x_0+x_1(x_0))\} = 0 \quad (6.4)$$

over X_0 .

6.2 REDUCTION TO THE ALTERNATIVE PROBLEM

We assume E satisfies the following assumptions.

(E₁) There is a constant $\mu > 0$ such that

$$(-Ex, x) \geq \mu \|x\|_1^2 \text{ for } x \in D(E) \cap X_1$$

(E₂) There is a decomposition of $-E$ into $-E = S(I+A)S$ where A is a bounded antisymmetric linear operator on X_1 and S is a positive self-adjoint operator on X_1 with $D(S) = H_1 \cap X_1$ and $R(S) = X_1$.

When S is viewed as a linear map from $H_1 \cap X_1$ into X_1 it is bounded, one to one, and onto, thus for $w \in H_1 \cap X_1$ there is a z in X_1 such that $z = Sw$ and

$$\|z\| \leq \|S\| \|w\|_1 \quad (6.5)$$

We also have for $x \in H_1 \cap X_1$

$$\begin{aligned} \|Sx\|^2 &= (Sx, Sx) = ((I+A)Sx, Sx) = (S(I+A)Sx, x) \\ &= (-Ex, x) \geq \mu \|x\|_1^2. \end{aligned} \quad (6.6)$$

We assume N satisfies the following assumptions

(N₁) The restriction of N to H_1 is hemi-continuous from H_1 into H .

(N₂) There is a constant $\eta \geq 0$ such that $(Nx - Ny, x - y) \geq -\eta \|x - y\|_1^2$ for $x, y \in H_1$.

The following theorem shows that equation (6.2) has a unique solution $x_1 = x_1(x_0)$ for each $x_0 \in X_0$ hence equation (6.1) is reduced to the alternative problem (6.4).

Theorem 6.1:

Let E satisfy (E_1) and (E_2) and N satisfy (N_1) and (N_2) . If $\eta < \mu$, then equation (6.2) has a unique solution for each $x_0 \in X_0$.

Proof:

Solving equation (6.2) is equivalent to solving

$$(I+A)Sx_1 + S^{-1}PN(x_0+x_1) = -S^{-1}PEx_0. \quad (6.7)$$

Let $y = Sx$, for $x \in H_1 \cap X_1$, then solving (6.7) is equivalent to solving

$$My = -S^{-1}PEx_0$$

where $M: X_1 \rightarrow X_1$ is given by

$$My \equiv (I+A)y + S^{-1}PN(x_0+S^{-1}y).$$

It suffices to show that $R(M) = X_1$. Since N is hemicontinuous, it follows that M is hemicontinuous, hence we need only show that M is strongly monotone over X_1 . Thus it then follows that M is coercive and maximal monotone since it is defined for all of X_1 and hemicontinuous. It then follows from a standard theorem from monotone operator theory that $R(M) = X_1$.

Let $y_1, y_2 \in X_1$ and $x_1 = S^{-1}y_1$ and $x_2 = S^{-1}y_2$. Using (N_2) , equation (6.5), equation (6.6), and noticing that $S^{-1}(y_1 - y_2) \in H_1 \cap X_1$ hence in the range of P , we have

$$\begin{aligned}
 (My_1 - My_2, y_1 - y_2) &= ((I+A)y_1 - y_2, y_1 - y_2) \\
 &\quad + (S^{-1}PN(x_0 + S^{-1}y_1) - S^{-1}PN(x_0 + S^{-1}y_2), \\
 &\quad \quad \quad y_1 - y_2) \\
 &= \|y_1 - y_2\|^2 + (N(x_0 + S^{-1}y_1) \\
 &\quad - N(x_0 + S^{-1}y_2), S^{-1}y_1 - S^{-1}y_2) \\
 &\geq \|Sx_1 - Sx_2\|^2 - \eta \|S^{-1}y_1 - S^{-1}y_2\|_1^2 \\
 &\geq \mu \|x_1 - x_2\|_1^2 - \eta \|x_1 - x_2\|_1^2 \\
 &= (\mu - \eta) \|x_1 - x_2\|_1^2 \\
 &\geq (\mu - \eta) \|S\|^{-2} \|y_1 - y_2\|^2
 \end{aligned}$$

Since strong monotonicity implies the operator M is one to one, the uniqueness follows.

This completes the proof of theorem 6.1.

Q.E.D.

In the next section we discuss the assumptions made on E and the existence of the projection operator P .

6.3 REMARKS CONCERNING THE ASSUMPTIONS ON E

In the paper by Osborn and Sather [24] they show that if E satisfies the following assumptions, then P may be chosen so that condition (E_1) is satisfied. They assume that $-E$ is generated by a bilinear form B on H_1 which satisfies

(B_1) There is a constant β such that

$$|B(w, z)| \leq \beta \|w\|_1 \|z\|_1 \quad w, z \in H_1$$

(B_2) There are constants $\gamma > 0$, $\sigma \geq 0$ such that

$$B(w, w) \geq \gamma \|w\|_1^2 - \sigma \|w\|^2 \quad w \in H_1$$

There are general classes of boundary value problems for differential equations (ordinary and partial) which may be studied via bilinear forms (see Agmon [2]). For example, elliptic boundary value problems can often be studied in this fashion.

Also Osborn and Sather assume that the generalized eigenvectors of $(E^+)^*$ are complete in H^+ . Here E^+ and H^+ are the complexifications of E and H and $(E^+)^*$ is the adjoint of E^+ . For a discussion of boundary value problems for which this assumption holds see Agmon [1].

To define P , let U^+ be the spectral projection associated with $-E^+$ and the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ i.e.

$$U^+ = (2\pi i)^{-1} \int_r (\mu + E^+)^{-1} d\mu$$

where r is a contour lying in the resolvent set of $-E^+$ and contains only $\{\lambda_1, \dots, \lambda_n\}$ in its interior. Let $V^+ = I - U^+$. Now if V is the image of the range of V^+ when projected onto H , then let P be the orthogonal projection of H onto V . We now have

Lemma 6.1:

Let $0 < \theta < 1$. Then there is an integer J such that whenever $n > J$

$$(-Ex, x) \geq \gamma(1-\theta) \|x\|_1^2$$

for all x in V where V is defined above for the eigenvalues $\lambda_1, \dots, \lambda_n$.

Proof:

The proof is a slight modification of Lemma 1 in [24] (see also Remark 3.2 in [24]).

Q.E.D.

The second assumption on E concerning the decomposition of $-E$ restricted to V as $S(I+A)S$ will also be satisfied when $-E$ is generated by a bilinear form as mentioned above. Osborn and Sather have shown

Lemma 6.2:

The restriction of $-E$ to V can be written $-E = S(I+A)S$ where A is a bounded antisymmetric

operator on V and S is a positive operator on V with $D(S) = H_1 \cap V$ and $R(S) = V$.

Proof:

See Osborn and Sather [24] Lemma 2 .

Q.E.D.

As mentioned before we have used the above results of Osborn and Sather to decompose the linear operator $-E$ in order to handle nonlinear operators which are not defined on all of H . We have extended the results of Osborn and Sather to nonlinear operators which satisfy an assumption somewhat weaker than quasimonotonicity.

In the next section we will apply the results of section 2 to boundary value problems for ordinary differential equations.

6.4 EXAMPLES

In this section we demonstrate how the method developed in this chapter is used to reduce nonlinear boundary value problems to a finite dimensional alternative problem.

Example 6.1:

Let us consider the differential equation

$$x'' + \sin(x') = f(t) \quad t \in [0, \pi] \quad (6.8)$$

with Neumann boundary conditions

$$x'(0) = x'(\pi) = 0 \quad (6.9)$$

where $f(t) \in L_2[0, \pi]$.

Let $H = L_2[0, \pi]$ with inner product $(w, z) = \int_0^\pi w(t)z(t) dt$. Let $H_1 = \{w \in H: w' \in H\}$ with inner product $(w, z)_1 = (w, z) + (w', z')$. Let $I-P$ be the projection of H onto the constants, i.e.

$$(I-P)w = (\pi)^{-1} \int_0^\pi w(t) dt.$$

P is the projection onto the subspace of H spanned by the nonconstant eigenvectors of the boundary value problem $-x'' = 0, x'(0) = x'(\pi) = 0$.

The operator $-E$ is generated by the bilinear form B , where

$$B(w, z) = (w', z').$$

Thus B certainly satisfies (B_1) and $B(w, w) =$

$$\|w\|_1^2 - \|w\|^2. \text{ Now for } w \text{ in the range of } P,$$

$$w = \sum_{n=1}^{\infty} (w, U_n) U_n \text{ where } U_n \text{'s are the nonconstant}$$

eigenvectors. Since for all the eigenvalues λ_n

corresponding to U_n , $\lambda_n \geq 1$ then

$$B(w, w) = (-w'', w) = \sum_{n=1}^{\infty} \lambda_n |(w, U_n)|^2 \geq \|w\|^2.$$

So

$$B(w, w) \geq \frac{1}{2} \|w'\|^2 + \frac{1}{2} \|w\|^2 = \frac{1}{2} \|w\|_1^2.$$

Now $Nw = \sin w' - f(t)$. For $w \in H_1$, $|\sin w'| \leq 1$,

hence $Nw \in H$. Since $\sin w'$ is Lipschitz in

w' , Nw is a continuous map from $H_1 \cap X_1$ into

X_1 .

For w, z in H_1 , we have

$$(Nw - Nz, w - z) = \int_0^\pi (\sin w' - \sin z') (w - z)$$

$$\geq \int_0^\pi |w' - z'| |w - z|$$

$$\geq - \|w' - z'\| \|w - z\| \geq - \frac{1}{2} \|w - z\|_1^2.$$

Now since $1 = \mu > \eta = \frac{1}{2}$ problem (6.8) - (6.9) can be reduced to the alternative problem

$$(I-P)(-Ea+N(a+x(a))) = 0$$

Let a be a constant and let $x(a)$ be the solution to (6.2), then the alternative problem becomes

$$\int_0^\pi [\sin x'(a)(t) - f(t)] dt = 0.$$

In this example the nonlinearity is not quasimonotone, but does satisfy assumption (N_2) .

Example 6.2:

Let us consider the differential equation

$$x'' + x e^x = f(t) \quad t \in [0, \pi] \quad (6.10)$$

with Neumann boundary conditions

$$x'(0) = x'(\pi) = 0 \quad (6.11)$$

where $f \in L_2[0, \pi]$. Let P , E , H , and H_1 be the same as in Example 6.1. Let $Nx = x e^x$. For

$x \in H_1$, $Nx \in H$ and Nx is continuous map from H_1 into H . For $x, y \in X_1$.

$$\begin{aligned} (Nx - Ny, x - y) &= \int_0^\pi (xe^x - ye^y) (x - y) \\ &= \int_0^\pi (re^r + e^r) (x - y)^2 \\ &\geq -1/e^2 \int_0^\pi |x - y|^2 \\ &= -1/e^2 \|x - y\|^2 \\ &\geq -1/e^2 \|x - y\|_1^2. \end{aligned}$$

Now since $1/e^2 < \frac{1}{2}$, the boundary value problem (6.10) - (6.11) is reduced to the alternative problem

$$\int_0^\pi \{ (a + x(a)(t)) e^{a + x(a)(t)} - f(t) \} dt = 0$$

where a is a constant and $x(a)$ is the solution to (6.2). In this example the nonlinearity is not Lipschitzian, but satisfies a one-sided Lipschitz property.

Example 6.3:

Let us consider the differential equation

$$x'' - ax = \frac{d}{dt} (f(t,x)) + p(t) \quad t \in [0, 2\pi] \quad (6.12)$$

with periodic boundary conditions

$$x(0) = x(2\pi) \quad \text{and} \quad x'(0) = x'(2\pi) \quad (6.13)$$

where a is a positive constant, $p(t) \in L_2[0, 2\pi]$ and $p(t)$ has period 2π . We assume $f(t,x)$ has continuous first and second partial derivatives hence f is Lipschitz in x with a Lipschitz constant independent of t . Moreover, we assume f is 2π -periodic in t for each x and $\frac{d}{dt} (f(t,x))$ is also 2π -periodic in t .

Let $H = \{x \in L_2[0, 2\pi] : x(0) = x(2\pi)\}$ and $H_1 = \{x \in H : x' \in H\}$. On H we take the inner product $(x, y) = \int_0^{2\pi} x(t)y(t) dt$ and on H_1 the inner product $(x, y)_1 = (x, y) + (x', y')$. In this example we take $P = I$, i.e. $X_1 = H$.

Let $Ex = x'' - ax$, then $-E$ is generated by the bilinear form

$$B(w, z) = (w', z') + (aw, z)$$

Now B obviously satisfies (B_1) . For $z \in H_1$,

$$\begin{aligned} B(z, z) &= (z', z') + a(z, z) \\ &= \|z'\|^2 + a \|z\|^2 \geq \mu \|z\|_1^2 \end{aligned}$$

$$\text{where } \mu = \sup_{z \neq 0} \frac{\|z'\|^2 + a \|z\|^2}{\|z'\|^2 + \|z\|^2} \geq \min(a, 1)$$

Let $Nx = \frac{d}{dt} (f(t, x)) + p(t)$. Then

$$\left| \frac{d}{dt} (f(t, x)) \right| = |f_1(t, x) + f_2(t, x)x'|$$

and since f_1 and f_2 are bounded, it follows that Nx maps H_1 into H . Since f is Lipschitz in x , one can show that

$$\left| \frac{d}{dt} \{f(t, x) - f(t, y)\} \right| \leq L|x' - y'|.$$

Hence N is a continuous map from H_1 into H .

For x, y in H_1 , we have

$$\begin{aligned} (Nx - Ny, x - y) &= \left(\frac{d}{dt} [f(\cdot, x) - f(\cdot, y)], x - y \right) \\ &= -(f(\cdot, x) - f(\cdot, y), x' - y') \\ &\quad + [f(t, x(t)) - f(t, y(t))] [x(t) - y(t)] \Big|_0^{2\pi} \end{aligned}$$

$$\begin{aligned}
 &= -(f(\cdot, x) - f(\cdot, y), x' - y') \\
 &\geq -L(|x - y|, |x' - y'|) \\
 &\geq -L \|x - y\| \|x' - y'\| \\
 &\geq -\frac{1}{2} L \{ \|x - y\|^2 + \|x' - y'\|^2 \} \\
 &= -\frac{1}{2} L \|x - y\|_1^2.
 \end{aligned}$$

Now if $\frac{1}{2} L < \mu$, then (6.12) - (6.13) has a unique solution since $P = I$ and thus there is no alternative problem.

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