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TIME DOMAIN METHOD OF DESIGN FOR A CLASS OF NONLINEAR SYSTEMS

by

Anthony John Gregory

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Doctoral Committee:

Associate Professor Edward O. Gilbert, Chairman
Associate Professor Elmer G. Gilbert
Professor Robert M. Howe
Professor Lawrence L. Rauch

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LIST OF SYMBOLS

a_u - u^{th} coefficient in output-element differential equation

A - $(K-1)x(K-1)$ matrix defined by $A'A = -2\left[Re\omega_{K1} U_2 + D_\omega \right]$

b_{nm} - n^{th} coefficient in $Z^1_m(f_\epsilon)$ $(m = 1, ---, N)$

B - column matrix with elements θ_k $(k = 1, ---, K-1)$

c_n - constant equal to $f_{en}(\lambda)$ $(n = 1, ---, N)$

C - K x K matrix equal to diag $\left[A^{-1} 1\right]$

C_o - gain coefficient of the function $F(s)$

C_m - K x K diagonal matrix with elements ρ_{km}

C_λ - K x K diagonal matrix with elements λ_k

d_k - k^{th} diagonal element of D_P

D_1 - K x K diagonal matrix with elements τ_k

D_m - K x K diagonal matrix defined by $\psi^{-1} C_m U_1 = D_m U_1$ $(m = 2, ---, N)$

D_p - K x K diagonal matrix with elements d_k

D_s - K x K diagonal transformation matrix with elements κ_k

D_{sm} - K x K diagonal matrix equal to $D_m (\bar{D}_s D_s)^{-1}$ $(m = 1, ---, N)$

D_ψ - K x K diagonal matrix defined by $U_1 \psi = U_1 D_\psi$

D_ω - $(K-1)x(K-1)$ diagonal matrix with elements $Re(\omega_{k1})$

$e_n(t)$ - approximation error equal to $f_{cn}(t) - f^a_{cn}(t)$ $(n=1, ---, N)$

$f_c(t), F_c(s)$ - general compensator response

$f_{cn}(t), F_{cn}(s)$ - n^{th} prescribed compensator response

$f^a_{cn}(t), F^a_{cn}(s)$ - compensator response in approximation problem

$f_e(t), F_e(s)$ - general system error response

$f_{en}(t), F_{en}(s)$ - n^{th} prescribed error response

$f_f(t), F_f(s)$ - general feedback element response

$f_{fn}(t), F_{fn}(s)$ - n^{th} prescribed feedback element response

$f_i(t)$, $F_i(s)$ - general system input function

$f_{in}(t)$, $F_{in}(s)$ - n^{th} prescribed system input function

$f_o(t)$, $F_o(s)$ - general system response

$f_{on}(t)$, $F_{on}(s)$ - n^{th} prescribed system response

$F(s)$ - real, rational function of s

$G_n(s)$ - compensator function $F_{cn}(s)/F_{en}(s)$ $(n = 1, ---, N)$

$G^s(s)$ - small-signal compensator function

$h_f(t)$, $H_f(s)$ - feedback element transfer function

$H_m(s)$ - m^{th} linear element in compensator

$H_m^a(s)$ - m^{th} linear element in approximate compensator

$H_m^c(s)$ - m^{th} linear element in constrained compensator

$H_o(s)$ - output element transfer function

$H_o^s(s)$ - small-signal approximation to nonlinear output element

$\overline{H}(s)$ - transfer function with zero at $s = 0$

I - Identity matrix

I_n - n^{th} integral-square error

j - imaginary unit

K - order of nonlinear system

$M(s)$ - N x N coefficient matrix with elements $\Gamma_{nm}(s)$

$M^*(s)$ - N x N coefficient matrix with elements $\Gamma^*_{nm}(s)$

$M_t(t)$ - inverse Laplace image of the matrix M

$M_t^*(t)$ - inverse Laplace image of the matrix $M^*(s)$

$M_{nm}(s)$ - submatrix of M(s) obtained by deleting rows 1 and n and columns 1 and m

$M^c(s)$ - coefficient matrix with elements $\Gamma_{nm}(s)$ $(n, m=2, ---, N)$

$M_m(\xi_F)$ - matrix constructed by replacing column m in M(s) by ξ_F

M_R - N x N matrix with (n, m) element $R_n^{(2m-2)}$

N - number of prescribed input-response pairs

p_i - coefficients in Laurent series expansion of $|M(s)|$

P_n - difference in order of denominator and numerator of $F_{cn}(s)$

$q(\phi)$ - vector with K components q_i

$Q_J(\phi)$ - K x K Jacobian matrix of vector $q(\phi)$

$\hat{Q}(\phi)$ - K x K Hermitian matrix $\overline{Q}_J'(\phi) + Q_J(\phi)$

$r(t)$ - Laplace image of $1/(a_U s^U + - - - + a_o)$

R - order of fixed output element

R_n - n^{th} system operating point in small-signal design

s - Laplace variable

t - time variable

T - K x K transformation matrix defined by Equation (4.51)

T_1, T_2 - limits on time interval of approximation

U - number of derivatives of $f_c(t)$ in the fixed output element

U_1 - K x K matrix with unity elements

U_2 - (K-1) x (K-1) matrix with unity elements

$V(\phi)$ - Lyapunov function of autonomous system

$V(\phi, t)$ - Lyapunov function employed in stability analysis of system forced trajectories

$V_a(\delta, y)$ - Lyapunov function of the form $\overline{\delta}' \hat{\Upsilon}(y) \delta$

$V_f(\delta, y_o, y_1)$ - Lyapunov function of the form $\overline{\delta}' \hat{\Upsilon}(y_o, y_1) \delta$

y, y_o, y_1 - substitution variables

$Y_s(s)$ - prescribed small-signal transfer function

$Y_{on}(s)$ - open-loop transfer function about operating point R_n

$Y_{sn}(s)$ - closed-loop transfer function corresponding to $Y_{on}(s)$

$Z_m(f_e)$ - m^{th} compensator nonlinear element

$Z_m^1(f_e)$ - m^{th} nonlinear element $\sum_{n=1}^{N} b_{nm} f_e^{2n-1}$

$Z_m^2(f_e)$ - m^{th} nonlinear element f_e^{2m-1}

$Z_m^3(f_e)$ - m^{th} nonlinear element f_e^m

$Z_m^4(f_e)$ - m^{th} nonlinear element f_e^{2m}

a_i - i^{th} zero of F(s)

β_j - j^{th} pole of F(s)

$\gamma_m(t)$, $\Gamma_m(s)$ - general response of the nonlinear element Z_m

$\gamma_{nm}(t)$, $\Gamma_{nm}(s)$ - n^{th} response of Z_m

$\Delta_{nm}(s)$ - coefactor of $\Gamma_{nm}(s)$ in M(s)

$\Delta^c_{nm}(s)$ - function equal to $\Gamma_{11}(s)\left|M_{nm}(s)\right|$

ϵ - positive constant

$\eta(f_o)$ - multinomial in f_o and is first R derivatives

$\eta_n(t)$ - n^{th} function $\eta(f_{on})$

θ_k - k^{th} function of f_e^2 defined by equations (4.59) and (4.60)

θ^*_k - k^{th} element of column matrix $(A^{-1})' B$

κ_k - k^{th} diagonal element of D_s

λ - positive constant

λ_k - k^{th} characteristic root of open-loop system

$\mu_o(t)$ - unit step function

μ_{kj} - k, j element of ψ

$\xi_F(s)$ - vector with components $F_{cn}(s)$ (n = 1, ---, N)

$\xi_H(s)$ - vector with components $H_m(s)$ (m = 1, ---, N)

$\xi^c_F(s)$ - vector with components $\left[F_{cn}(s) - G^s(s)\right]$ (n = 2, ---, N)

$\xi^c_H(s)$ - vector with components $H^c_m(s)$ (m = 2, ---, N)

$\xi_Y(s)$ - vector with components $Y_{on}(s)/H_o(s)$ (n = 1, ---, N)

ρ_{km} - residue of pole at $s = \lambda_k$ in $H_m(s)\ H_o(s)\ H_f(s)$

τ - variable of integration

τ_k - k^{th} characteristic root of system first approximation

$\Upsilon(\phi)$ - K x K matrix $D_1 - \sum_{m=2}^{N} f_e^{2m-2} D_m U_1$

$\hat{\Upsilon}(\phi)$ - K x K Hermitian matrix $\overline{\Upsilon}'(\phi) + \Upsilon(\phi)$

$\hat{\Upsilon}_1(\phi)$ - K x K matrix $\overline{D}_1 + D_1$

$\hat{\Upsilon}_m(\phi)$ - K x K matrix $\overline{D}_s U_1 \overline{D}_m \overline{D}_s^{-1} + D_s^{-1} D_m U_1 D_s$ (m = 2, ---,N)

$\hat{\Upsilon}_s(f_e)$ - matrix identical to $\hat{\Upsilon}(\phi_s)$

$\hat{\Upsilon}_T(f_e)$ - K x K matrix $T' \overline{D}_s^{-1} \hat{\Upsilon}_s(f_e) D_s^{-1} T$

$\Upsilon(f_e^0, f_e^1)$ - K x K matrix defined by Equation (4.73)

ϕ - vector with components ϕ_k (k = 1, ---, K)

ϕ_k - k^{th} state variable of nonlinear system

ϕ_ϵ - equilibrium point of nonlinear system

ϕ_s - vector with components ϕ_{sk} (k = 1, ---, K)

ϕ_T - vector equal to $T^{-1} \phi$

ψ - K x K transformation matrix

ω_{km} - k^{th} diagonal element of D_{sm} (m = 1, --- , N)

CHAPTER I
INTRODUCTION

The remarkable growth of applications of nonlinear systems in such fields as automatic control, data transmission, and noise filtering is well known. This growth is a reflection of the demand for system performance, weight, and reliability characteristics beyond those obtainable from linear devices. However, techniques for the design of nonlinear systems are still in the early stages of development with only a limited range of problems susceptible to satisfactory treatment. In particular, relatively few results have been obtained in the intentional use of nonlinearities to improve system performance characteristics or to extend the tasks that can be accomplished.

The aim of this investigation is to develop a systematic design technique for a special class of nonlinear systems. The system specifications are assumed to consist of a set of deterministic forcing functions, each with a corresponding prescribed system response. The resulting nonlinear systems are those that can be described by a set of ordinary nonlinear differential equations in which each nonlinear term consists of some product of the dependent variables and their derivatives. Both inherent and intentional nonlinearities are considered. Before discussing the results of this investigation, the present state of nonlinear design theory will be reviewed briefly.

Published results in nonlinear-control design theory are numerous and only a few typical examples can be mentioned here. The greater part of past work can be classified as approximate.

The most general method of this type is certainly the describing function technique which has been the subject of numerous investigations (10, 12, 17, 20, 21). From the viewpoint of the system engineer, the primary significance of the describing function is the possibility of controlling, at least roughly, the relative stability of a nonlinear system by conventional methods (gain adjustment and linear compensation). Although the engineer has found such sinusoidal techniques as the describing function invaluable for avoiding unwanted oscillations, he is perhaps more often interested in obtaining acceptable forced or transient response from a nonlinear system. In general, it appears that satisfactory control over response can only be obtained directly in the time domain. A number of such methods have been proposed (3, 4, 14, 23, 24). In most of these, results have been obtained by placing rather severe restrictions on the problem. For example, Caldwell and Rideout (4) have devised a method of improving the forced response of a second-order linear system by adding a nonlinear damping term. The function of this term is to reduce the transient response time by reducing the system damping for large error signals. This technique, as with many similar methods, is based on intuitive reasoning and the validity must be checked by experimentation.

In the area of exact theories, many of the present results are concerned with contactor systems and related optimal controls (2, 7, 8, 11, 13, 22, 25, 27). These systems have proven amenable to analysis; and, with respect to certain response criteria, provide the ultimate in performance. Unfortunately, optimal controls tend to be complex, and realization has been limited to the simpler low-order systems.

An important source of difficulty in the development of general nonlinear design theories is the diversity of nonlinear phenomena and

the consequent lack of formal methods of classification. Several attempts at classification have been made. Zadeh (29) has suggested one based on an integral representation in which the first class includes the linear system as a special case and each class contains the lower-order classes. The significance of such schemes has not been determined, and so far, practical applications have not evolved.

With the exception of some general results for second- and third-order nonlinear systems, the most powerful tool for nonlinear stability investigations is Lyapunov's "second method" (5, 6, 15, 18). The Lyapunov method uses a generalization of Lagrange's theorem of minimum potential energy to establish conditions sufficient for a stable system equilibrium. While there is no formal process for applying this method, many problems that could be treated in no other way have been resolved by a fortunate combination of intuition and manipulation.

In summary, nonlinear design theory at present consists of a wide range of partial results. As yet, general procedures that are practical have not been developed. Thus, most successful designs are based on a combination of techniques aided by the experience and insight of the designer.

In this investigation, a systematic procedure is developed for a special class of design problems by placing restrictions on the form of the system specifications and the nonlinear fixed elements. The method does not require approximations and the resulting systems satisfy the specifications exactly. Chapter II outlines the problem and the proposed design technique. Following this, the required preliminary assumptions and the limitations of the method are discussed. The development of a general class of nonlinear compensators is presented in Chapter III. Included here are discussions of compensator

characteristics and the question of realizability. Lyapunov's method is used in Chapter IV for a stability analysis of the system in which the compensator is the only nonlinear element. The stability in-the-large of both the free system and the forced responses of the system are studied. Chapter V examines methods of reducing the proposed technique to practice, summarizes the design procedure, and presents an approximate extension of the method to a wider range of problems. Several example problems are presented at the end of Chapter V to illustrate some of the important points brought out in the text.

CHAPTER II

PRELIMINARY ASSUMPTIONS

Design theories, both for linear and nonlinear systems, are often based on methods that were originally developed for analysis. This approach has resulted in a large number of practical theories for linear systems--primarily as a result of the powerful analytical tools available. Conversely, the lack of general methods for the analysis of nonlinear differential equations has limited the development of nonlinear design theories. Since this situation is not likely to change in the immediate future, it seems pertinent to consider the development of methods that do not require the solution of nonlinear differential equations. A design technique of this type is the subject of the present investigation.

In many applications it would be desirable to have a system whose performance changes in a prescribed nonlinear fashion with the form and amplitude of the input signal. The ideal design technique would completely control the relationship between system input and response over some continuous range. At present, it seems improbable that a general technique of this type can be developed. One method of reducing the problem to a manageable form is to consider only a finite set of input and response functions. This approach is considered here and should be adequate for a large class of system design problems. The system considered is a closed-loop configuration enclosing output and feedback elements, and a nonlinear compensator. The compensator is required, in general, to be nonlinear since the transfer operators

defined by corresponding input-response pairs can be distinct, and the output element can be nonlinear.

The method proposed here yields an exact solution to this design problem. That is to say, if it is stable, the final system produces exactly the specified response for each input. This approach has several important advantages and disadvantages. On the plus side, an exact technique can be reduced to a systematic procedure that does not require trial-and-error computation. In addition, it avoids the difficult if not impossible problem of establishing approximation errors. Of the disadvantages, perhaps the greatest is the loss of flexibility: the designer has little control over the complexity of the resulting system. A second disadvantage is that there are several important types of system design requirements, such as threshold and resolution, not easily controlled by selecting an appropriate system input-response pair. This problem is characteristic of exact or analytical methods of design. In general, such methods must be followed by empirical techniques of adjusting the system parameters to obtain requirements that cannot be stated within the allowed form of the specifications.

In the development of the present design technique, the form of the input and response functions and the class of fixed elements must be appropriately restricted. The final system must also satisfy certain general requirements. The following sections discuss the design technique and the necessary assumptions and limitations.

Design Technique and Assumptions

In the present design problem it will be assumed that specifications are given in the form of N prescribed system input functions and corresponding response functions. It will be further assumed that system output and feedback elements have been selected and are, therefore, fixed and part of the system specifications.

The closed-loop system and notation used is shown in Figure 2.1. The system inputs are $f_{in}(t)$ (n = 1, - - -, N) and the corresponding prescribed responses are $f_{on}(t)$ (n = 1, - - -, N). In the following work, all time functions considered will be deterministic and zero for $t < 0$. It will be presumed that these time functions have Laplace transforms, if required. The theory in following chapters considers the single-loop system shown in Figure 2.1. The methods presented, however, are applicable to multiloop systems as long as the restrictions indicated below are satisfied.

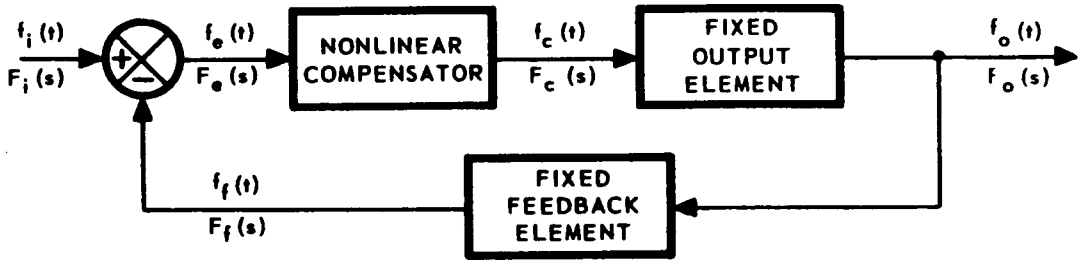


Figure 2.1 - Design Problem Notation

A major obstacle in the design of nonlinear systems is the nature of the closed-loop design problem. If approximate methods are used, the designer is forced to work inside the loop and splice things together so that the over-all system will do the required job. The method proposed here avoids this difficulty since an exact design is obtained. Thus, the first step of the design procedure[1] is to reduce the closed-loop problem to an equivalent open-loop problem. In this first step, for each pair of functions $f_{in}(t)$ and $f_{on}(t)$, the corresponding system error $f_{en}(t)$ and compensator response $f_{cn}(t)$ are computed. The sets

---

[1]The method of design used here is similar in several respects to Guillemin's method which considers the design of linear systems. A detailed discussion of Guillemin's method is presented by Truxal (28), Chapter V.

of functions $[f_{en}(t)]$ and $[f_{cn}(t)]$ define the required nonlinear compensator. The determination of a suitable compensator is therefore reduced to a relatively simple open-loop design problem.

Since an exact solution to the design problem is sought, the fixed elements and each of the $f_{in}(t)$ and $f_{on}(t)$ must be such that the corresponding $f_{en}(t)$ and $f_{cn}(t)$ can be determined in closed form. In addition, the sets $[f_{en}(t)]$ and $[f_{cn}(t)]$ must be such that a suitable nonlinear compensator can be found. The following preliminary assumptions restrict the problem to satisfy these requirements:

1. Let each of the $f_{in}(t)$ and $f_{on}(t)$ be a member of the class of time functions that can be generated as the impulse response of a linear system of the form

$$F(s) = C_o \frac{(s - a_1)(s - a_2) \text{ --- } (s - a_I)}{(s - \beta_1)(s - \beta_2) \text{ --- } (s - \beta_J)}, \qquad (2.1)$$

where a_i and β_j are arbitrary and appear in conjugate complex pairs when complex. The poles and zeros of F(s) are not required to be simple.

Additional restrictions are imposed on the $f_{in}(t)$ and $f_{on}(t)$ by the requirement that the system defined by the specifications must be physically realizable.[1] Since, in a physical system, response is uniquely related to the input, a distinct input $f_{in}(t)$ must be assigned to each distinct response $f_{on}(t)$, assuming the initial state of the system remains the same. The converse, of course, is not true. In fact, the f_{o1}, ---, f_{oN} can all be identical, even if the corresponding f_{i1}, ---, f_{iN} are distinct. A further limitation on the $f_{in}(t)$ and $f_{on}(t)$ is discussed in Chapter III.

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[1] The mathematical conditions for realizability require that a system must be non-predicting. Since this limitation is satisfied as a result of these preliminary assumptions, only the practical requirements for realizability are of concern here.

2. Let the fixed feedback element be a real, rational, linear function of s denoted by $H_f(s)$. To avoid the stability problem which may arise as a result of the cancellation of a right-half-plane pole, it will be assumed that $H_f(s)$ is minimum phase. This problem is discussed in Chapter IV. While it is not essential to assume that the feedback element is linear--it could for example, consist of a polynomial $g(x) = \Sigma\, b_i\, x^i$ in series with $H_f(s)$--most feedback devices of practical interest are linear over a reasonable range of inputs.

3. Let the fixed output element be a nonlinear device capable of being represented by a single differential equation of the form

$$\eta\left(f_o^{(R)},\ f_o^{(R-1)},\ - - -,\ f_o\right) = \sum_{u=o}^{U} a_u \frac{d^u f_c}{d t^u} \quad . \qquad (2.2)$$

The nonlinear function $\eta(f_o)$ is a multinomial in which each term is an arbitrary product of the variable f_o and its first R derivatives. For example, a forced Van der Pol equation $(\ddot{x} - \epsilon (1 - x^2)\ \dot{x} + x = y)$ is of the type defined by equation (2.2). The fixed output element could also be represented as a set of nonlinear differential equations. However, for each $f_o(t) = f_{on}(t)$, it must be possible to compute the corresponding $f_c(t) = f_{cn}(t)$. This can be accomplished if the representation of the output element can be reduced to a single differential equation. When the function $\eta(f_o)$ is linear, the fixed output element will be denoted as the transfer function

$$H_o(s) = \frac{F_o(s)}{F_c(s)} \quad .$$

It will be assumed that the functions $H_o(s)$ and $a_U s^U + \text{---} + a_o$ are minimum phase as above. Unfortunately, no similar restriction can be imposed on $\eta(f_o)$, since the cancellation problem for nonlinear functions is not obviously defined.

The above preliminary assumptions satisfy the requirements of the design problem and lead to a suitable class of nonlinear compensators. However, they do not assure that a design can be successfully completed. Within the restrictions imposed by the preliminary assumptions and a lesser limitation discussed later, it is always possible to determine a nonlinear compensator which will produce the required response $f_{cn}(t)$ for each forcing function $f_{en}(t)$. Unfortunately, there is no assurance that the resulting system will have acceptable stability characteristics.

Several attempts have been made by the author to employ alternate sets of preliminary assumptions. However, in each case, unless very restrictive assumptions were placed on the problem, no suitable class of nonlinear compensators could be found. Fortunately, the time functions defined in assumption (1) are those commonly used in system design, and many output elements can be represented adequately by the type of nonlinear differential equation defined in assumption (3).

The $f_{in}(t)$ and $f_{on}(t)$ have been restricted by assumption (1) to a class $\mathcal{R}$ of functions closed under linear operations and multiplication. The result of the remaining assumptions is to restrict the $f_{en}(t)$ and $f_{cn}(t)$ to class $\mathcal{R}$. Thus, one general requirement for the nonlinear compensator is that when its input is an element of $\mathcal{R}$, its response must also be an element of $\mathcal{R}$.

The $f_{en}(t)$ and $f_{cn}(t)$ required for the evaluation of the nonlinear compensator can be computed as follows: using the superposition

integral, $f_{fn}(t)$ can be expressed as (assuming the system is initially at rest)

$$f_{fn}(t) = \int_o^\infty f_{on}(\tau)\, h_f(t-\tau)\, d\tau, \qquad n = 1, - - -, N. \tag{2.3}$$

Thus, $f_{en}(t)$ is given by

$$f_{en}(t) = f_{in}(t) - \int_o^\infty f_{on}(\tau)\, h_f(t-\tau)\, d\tau, \quad n = 1, - - -, N. \tag{2.4}$$

The required compensator response $f_{cn}(t)$ can be computed by substituting $f_o(t) = f_{on}(t)$ into Equation (2.2) and solving for the corresponding $f_c(t) = f_{cn}(t)$. From Equation (2.2),

$$\sum_{u=o}^{U} a_u \frac{d^u f_{cn}}{d t^u} = \eta_n(t), \qquad n = 1, - - -, N \tag{2.5}$$

where

$$\eta_n(t) = \eta(f_{on}^{(R)}, f_{on}^{(R-1)}, - - -, f_{on}), \quad n = 1, - - -, N. \tag{2.6}$$

Now, let the inverse Laplace transform define the relationship

$$r(t) = L^{-1}\left[\frac{1}{a_U s^U + - - - + a_o}\right], \tag{2.7}$$

then

$$f_{cn}(t) = \int_o^\infty \eta_n(\tau)\, r(t-\tau)\, d\tau, \qquad n = 1, - - -, N. \tag{2.8}$$

Before turning to the investigation of nonlinear compensators in Chapter III, several further requirements, necessary to achieve a satisfactory solution to the system design problem, will be considered.

General Design Requirements

In addition to the limitations imposed by the preliminary assumptions, there are four general requirements that the nonlinear system must satisfy. While it would be difficult to establish conditions sufficient to assure successful completion of the design problem, it appears that a satisfactory design has been accomplished when the following conditions are satisfied:

1. Nonlinear Compensator. The system compensator defined by the sets $[f_{en}(t)]$ and $[f_{cn}(t)]$ must exist and be physically realizable. While a general investigation of existence and realizability would be very difficult, these questions can be answered for the class of compensators developed here. Conditions sufficient to assure the existence of a realizable nonlinear compensator are discussed in Chapter III.

2. Autonomous System. The autonomous system must have acceptable stability characteristics. Ideally, it should be stable in-the-large. In most cases, system stability characteristics cannot be established rigorously; however, an accurate indication can often be obtained by experimentation. In Chapter IV, the second method of Lyapunov is employed to examine the case in which the compensator is the only nonlinear element in the system. The conditions established are sufficient for the existence of a Lyapunov function and the asymptotic stability in-the-large of the nonlinear system.

3. Forced System Trajectories. As a minimum requirement, each of the N forced system trajectories $f_{on}(t)$ should be stable relative to neighboring trajectories. While this would be adequate to meet the specifications, a system would be highly unsatisfactory if any possible trajectory was unstable relative to a neighboring

trajectory--the imperfections and noise always present in physical systems might cause the actual system response to depart indefinitely from the desired response.

As with the autonomous system, very little can be said in general about the stability of forced trajectories, except for the case in which the compensator is the only nonlinear element in the system. In this case, it will be shown that each forced trajectory of the system is asymptotically stable relative to the class of all trajectories whenever the autonomous system has a Lyapunov function of the type considered.

4. System Equilibrium Point. While not stated explicitly in the specifications, all systems are, of course, required to have a unique equilibrium point. This is essential for stability in-the-large. Since the general class of compensators considered here can occasionally produce a system with more than one point of equilibrium, a special configuration which avoids this difficulty is presented in Chapter III.

The above list summarizes several of the major problem areas in the design of nonlinear systems. While certain conditions sufficient to avoid these difficulties are established in the following investigations, the conditions cannot, in general, be interpreted in terms of appropriate limitations on the system specifications. It is apparent, therefore, that no assurance can be given that a particular design will be successfully completed. The final system must always be tested to determine if the design goals have been achieved.

CHAPTER III

NONLINEAR COMPENSATION

The previous chapter has attempted to formulate a design problem which spans a significant range of application and is capable of solution. To meet these goals, it is necessary to forego placing restrictions on the final system, and thus the compensator, except for those implied by the preliminary assumptions. Other restrictions, for example a limit on the order of the nonlinear system, might be desirable; however, in this case a solution of the design problem becomes difficult, if not impossible. Since the limitations imposed on the nonlinear compensator are relatively weak, it might be expected there are numerous compensators that will meet the requirements of a given problem. As will be seen, this is actually the case.

The present chapter examines the configurations, existence and realizability of the nonlinear compensator. This is followed by a proof that a linear system is generated when the specifications can be satisfied by a linear function. The final section presents a method of constraining the compensator to obtain a prescribed small-signal transfer function for the nonlinear system.

Compensator Configurations

The type of nonlinear compensator proposed here and the notation used is shown in Figure 3.1. As shown, the nonlinear compensator is divided into N parallel paths which are summed to form

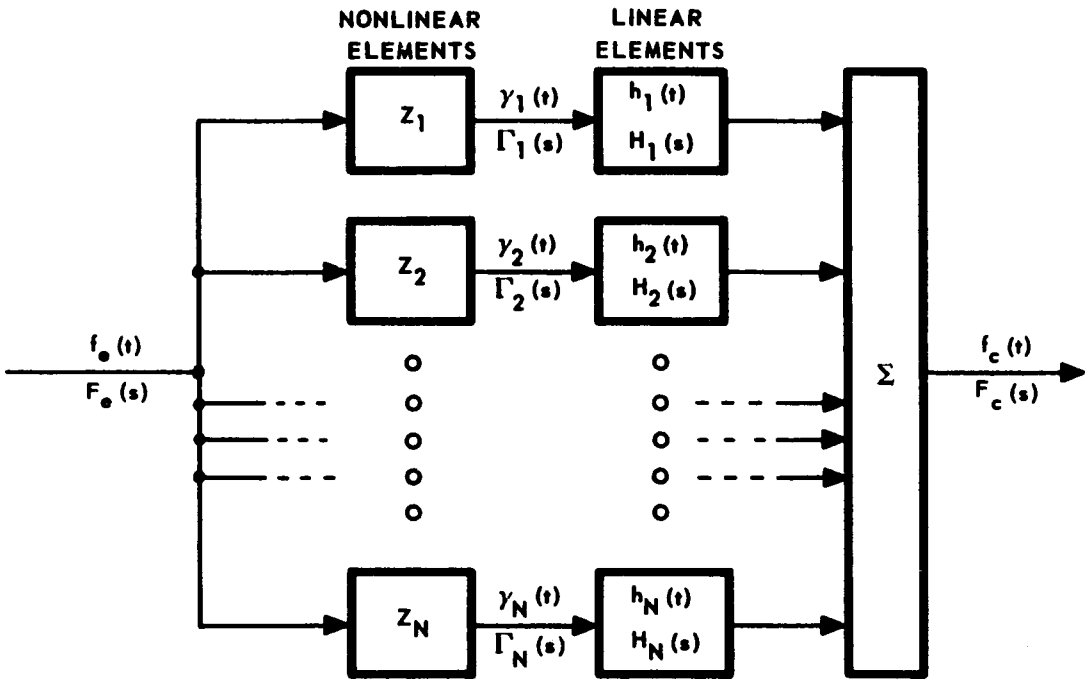


Figure 3.1 - Nonlinear Compensator Notation

the compensator output $f_c(t)$. Each path consists of a nonlinear and a linear element in series. Many other suitable configurations can be found. This compensator, however, appears to be the simplest which will meet the requirements of the design problem, and will therefore be employed in most of the following work. Alternate configurations of special interest are considered below. The compensator schematics discussed in this section are convenient for computation, but are not necessarily the form in which the compensator should be realized. A configuration which requires realization of the minimum number of dynamic elements is presented in Chapter V.

For the N prescribed system input-response pairs, the equations relating the output of the nonlinear elements to the compensator output are

$$\Gamma_{n1} H_1 + \Gamma_{n2} H_2 + \text{----} + \Gamma_{nN} H_N = F_{cn}, \quad n = 1, \text{----}, N. \qquad (3.1)$$

In the symbols $\Gamma_{nm}(s)$, the subscripts have been inverted so that standard matrix notation can be employed below. The set of equations (3.1) can be expressed in matrix notation as

$$M\xi_H = \xi_F \tag{3.2}$$

where the NxN coefficient matrix is

$$M(s) = \begin{bmatrix} \Gamma_{11}(s) & \Gamma_{12}(s) & \cdots & \Gamma_{1N}(s) \\ \Gamma_{21}(s) & \Gamma_{22}(s) & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ \Gamma_{N1}(s) & \cdots & & \Gamma_{NN}(s) \end{bmatrix}, \tag{3.3}$$

and

$$\xi_H(s) = \begin{bmatrix} H_1(s) \\ \vdots \\ H_N(s) \end{bmatrix}, \quad \xi_F(s) = \begin{bmatrix} F_{c1}(s) \\ \vdots \\ F_{cN}(s) \end{bmatrix}. \tag{3.4}$$

The requirements the nonlinear elements Z_m must satisfy are now seen as: (1) for the specified $f_{en}(t)$ and $f_{cn}(t)$, the set $[Z_m]$ must be such that equation (3.2) has a solution for ξ_H; (2) each of elements Z_m must be such that its response is an element of $\mathcal{R}$ when its input is an element of $\mathcal{R}$. Conditions sufficient for (1) are established in the next section. One set of nonlinear elements which satisfies (2) is

$$Z_m^1(f_e) = b_{1m} f_e + b_{2m} f_e^3 + \cdots + b_{Nm} f_e^{2N-1}, \quad m = 1, \text{---}, N. \tag{3.5}$$

Other suitable elements are easily found. For example, each nonlinear element could consist of some linear transfer function followed by the above function $Z_m^1(f_e)$.

With any set of the coefficients b_{nm} for which the equations (3.5) are linearly independent, the matrix (b_{nm}) is nonsingular. Thus, defining $Z_m^2(f_e) = (b_{nm})^{-1} Z_m^1$, the elements

$$Z_m^2(f_e) = (f_e)^{2m-1}, \quad m = 1, ---, N \tag{3.6}$$

will yield effectively the same nonlinear compensator as the $Z_m^1(f_e)$. Most of the following theory will be concerned with the elements Z_m^2. While alternate sets of nonlinear elements could be considered, these offer the least difficulty in analysis of both the compensator and the over-all system.

With the N nonlinear elements Z_m^2, the final system can have as many as 2N-1 points of equilibrium, assuming the output element has a unique point of rest. The most direct method of avoiding this difficulty is to devise a compensator which cannot produce extraneous points of equilibrium. The penalty is a slight increase in complexity of the compensator. A suitable configuration is shown in Figure 3.2.

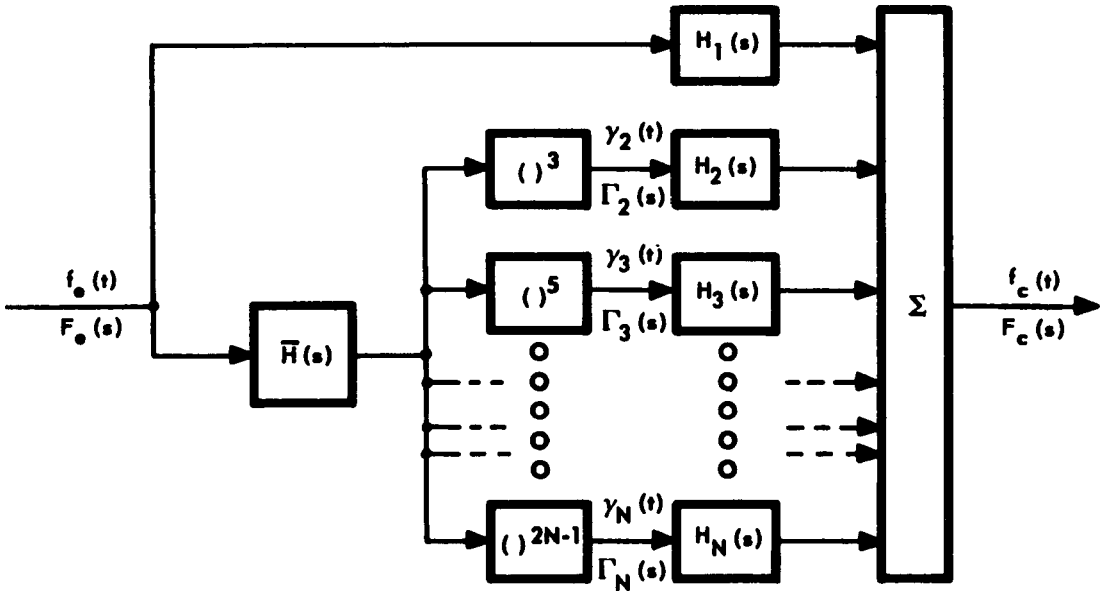


Figure 3.2 - Compensator which Yields a Unique System Equilibrium Point

In this compensator, the transfer function $\overline{H}(s)$ can be any real, rational function of s which has a zero at the origin, $\overline{H}(0) = 0$. Realizability requires, in addition, that $\overline{H}(s)$ be bounded at infinite frequency. When the system is at rest, the output of the element $\overline{H}(s)$ is zero, and consequently, the output of each compensator path 2 through N is independent of the steady state value of the system error. The compensator, therefore, cannot produce extraneous equilibrium points in the resulting system.

At first glance it might seem that this conclusion is incorrect when some of the $H_m(s)$ have a pole at $s = 0$, since the output of these elements may not be zero at equilibrium. The following shows that this is not the case. Suppose some of the $H_m(s)$ contain a simple pole at $s = 0$. Let $H_m(s) = \left[\frac{s}{s-a} H_m(s)\right]\left[\frac{s-a}{s}\right]$ where $a < 0$ is arbitrary. Now extract $\left[\frac{s-a}{s}\right]$ from each of the linear elements and place it in series with the compensator after the summation. Since at equilibrium, the output of paths 2 through N is now zero, the compensator cannot be a source of unwanted system equilibrium points.

In most design problems, the system is required to have characteristics such that if $f_{ii}(t) = -f_{ij}(t)$, then $f_{oi}(t) = -f_{oj}(t)$. Thus, only odd power terms have been used for the above nonlinear elements. There are problems, however, where it would be desirable to include even power terms in the $Z_m(fe)$, as when a system with an unsymmetrical characteristic is desired. A second possibility occurs in design problems where the output element has some unsymmetrical characteristic. Here a suitable nonlinear compensator could be employed to produce an over-all system with the desired odd characteristic, at least for specific pairs of input functions. As will be seen, this result cannot

be obtained with the nonlinear elements Z_m^2, since if $f_{ei} = -f_{ej} (i \neq j)$ Equation (3.2) does not have a solution.

When the fixed output element is nonlinear, it may be possible to eliminate some or all of the nonlinear compensator elements. This requires a different compensator configuration, but the above method of computing the $H_m(s)$ is easily modified for this design problem. As an illustration, consider a fixed output element defined by

$$\ddot{f}_o + d_1 \left[(\dot{f}_o)^2 + 1\right] \dot{f}_o + d_2 f_o = f_c$$

Assuming $\left[(\dot{f}_o)^2 + 1\right] \dot{f}_o$ can be sensed as an explicit function in the output element, the method of compensation shown in Figure 3.3 might be employed for a problem with N = 2.

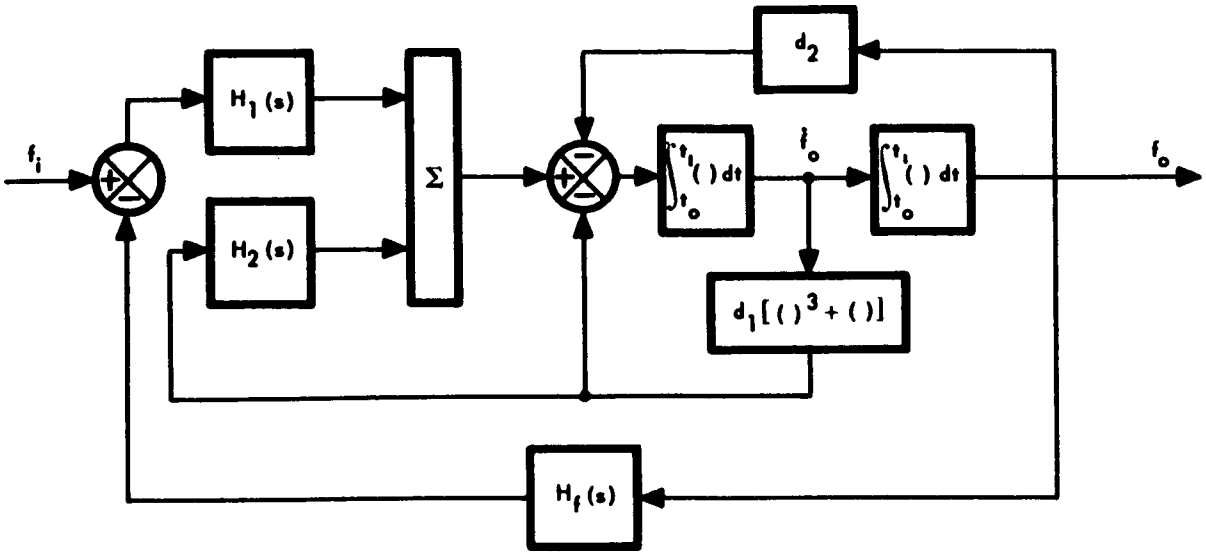


Figure 3.3 - Method of Implementing Compensator without Nonlinear Elements

This method of compensation is particularly attractive since it reduces the number of components required to implement the compensator. The concept of modifying the characteristics of a nonlinear output element by sensing some internal variable and "feeding it back" is well known. For example, the "pressure-feedback" commonly used in valve-controlled hydraulic systems tends

to "linearize" the system by countering the nonlinear pressure-flow relationship at the valve metering orifices. The method compensation developed here supplements this idea by providing a logical technique for determining suitable compensator elements.

It is apparent that the general method of compensation developed here leaves a great deal of design freedom to the discretion of the designer, since there are numerous compensator configurations which could be employed in a given problem. At present no systematic method of exploiting this freedom has been found. However, it can always be used, if only by trial-and-error methods, to obtain some additional control over the nonlinear transition in system performance. Similarly, if an initial design proves unstable, a different choice of nonlinear elements may give more satisfactory results.

Existence and Physical Realizability

Two conditions are sufficient to assure the existence and realizability of the nonlinear compensator defined by the $[f_{en}(t)]$ and $[f_{cn}(t)]$ when the elements Z_m^2 are considered. The Z_m^2 are realizable, of course, since they involve only multiplication. The two conditions are: (1) equation (3.2) must have a solution for ξ_H, (2) the linear elements $H_m(s)$ must be physically realizable.

Equation (3.2) has a solution if and only if the coefficient matrix M and the augmented matrix (M, ξ_F) have the same rank.[1] Neglecting the trivial case of identical system input-response pairs, the following argument shows that the two matrices have the same rank if and only if M(s) is nonsingular, i.e., $|M(s)|$ is not identically zero.

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[1] Perlis [26]; Theorem 3.6, page 45.

First, it will be proven that M(s) is nonsingular if and only if the $|f_{en}(t)|$ (n = 1, ---, N) are distinct, which can be written

$$|f_{ei}(t)| \neq |f_{ej}(t)| \text{ for some } t,\ i \neq j.$$

The first part of the proof is obtained by showing that |M(s)| cannot be identically zero when the condition holds.

Consider the NxN matrix $M_t(t)$ whose elements $\gamma_{nm}(t)$ are the inverse Laplace images of the elements $\Gamma_{nm}(s)$ of M(s). Now form a new matrix $M_t^*(t)$ such that

$$M_t^*(t) = M_t(t+\lambda),\ t > 0$$

and

$$M_t^*(t) = 0,\ t < 0$$

where λ is a positive constant. Employing the shift rule of Laplace Transform Theory, the matrix M*(s), whose elements $\Gamma_{nm}^*(s)$ are the Laplace images of the elements of $M_t^*(t)$, can be expressed as

$$M^*(s) = e^{\lambda s} M(s) \tag{3.7}$$

where the corresponding elements of M*(s) and M(s) have the same strip of convergence. Equation (3.7) shows that M(s) is nonsingular if and only if M*(s) is nonsingular.

Let $f_{en}(\lambda) = c_n$; then $\gamma_{nm}(\lambda) = c_n^{(2m-1)}$, and the Initial Value Theorem yields

$$\lim_{s \to \infty} \left[s\Gamma_{nm}^*(s) \right] = c_n^{(2m-1)} \tag{3.8}$$

Since M*(s) is an NxN matrix, its determinant can be expanded in a Laurent series as

$$\left| M^*(s) \right| = \frac{p_1}{s^N} + \frac{p_2}{s^{N-1}} + \text{----} \tag{3.9}$$

where the series converges outside of any circle enclosing the poles of |M*(s)| and having its center at the origin s = 0. The constant p_1 can be expressed as

$$p_1 = \begin{vmatrix} c_1 & c_1^3 & \cdots & c_1^{(2N-1)} \\ c_2 & c_2^3 & \cdots & \\ \vdots & \vdots & & \\ c_N & \cdots & & c_N^{(2N-1)} \end{vmatrix}, \tag{3.10}$$

or

$$p_1 = c_1 c_2 \cdots c_N \begin{vmatrix} 1 & c_1^2 & \cdots & c_1^{(2N-2)} \\ 1 & c_2^2 & \cdots & \\ \vdots & \vdots & & \\ 1 & \cdots & & c_1^{(2N-2)} \end{vmatrix}. \tag{3.11}$$

The determinant in (3.11) is the determinant of Vandermonde's matrix which is nonsingular if $c_i^2 \neq c_j^2$ $(i \neq j)$.

Suppose the $|f_{e_n}(t)|$ $(n = 1, \text{---}, N)$ are distinct; then under preliminary assumption (1) the constant λ can be chosen such that the c_n are non-zero and $c_i^2 \neq c_j^2$ $(i \neq j)$. Hence, $p_1 \neq 0$ and the matrices $M^*(s)$ and $M(s)$ are nonsingular.

When the $|f_{e_n}(t)|$ are not distinct, $M(s)$ has at least two rows that are either equal or equal with opposite sign, and therefore $|M(s)| \equiv 0$, which completes the proof.

The method employed above can be used to show conditions under which the coefficient matrix is nonsingular for other sets of nonlinear elements. For example, when $Z_m^3(f_e) = (f_e)^m$, the coefficient matrix $M(s)$ is nonsingular if the $f_{e1}(t), \text{---}, f_{eN}(t)$ are distinct.

To complete the argument, it will be proven next that equation (3.2) fails to have a solution when $|M(s)| \equiv 0$, with the exception of the

trivial case occurring with identical system input-response pairs. The above analysis shows that $|M(s)| \equiv 0$ can only result if at least two of the $f_{en}(t)$ are equal or equal with opposite sign. For convenience, suppose $f_{e1}(t) = f_{e2}(t)$ are the only functions of this type. The rank of M(s) is then N-1, since it has two equal rows and just one of its rows can be made zero by row operations. Assume the system input-response pairs are distinct; then $F_{c1}(s)$ cannot be identical to $F_{c2}(s)$. Hence, the augumented matrix (M, ξ_F) must be of rank N, since none of its rows can be reduced to zero by row operations. This argument generalizes, which proves that the matrices M and (M, ξ_F) cannot have the same rank if any of the error functions $f_{en}(t)$ are equal. A similar proof shows that M and (M, ξ_F) will fail to have the same rank if any of the $f_{en}(t)$ are equal with opposite sign. In summary, equation (3.2) has a solution for ξ_H if and only if M(s) is nonsingular.

The requirement that the $|f_{en}(t)|$ (n=1, ---, N) must be distinct places a corresponding limitation on the set of input-response pairs $[f_{in}(t), f_{on}(t)]$. This is an addition to the requirement that a distinct $f_{in}(t)$ must be assigned to each distinct $f_{on}(t)$, as imposed in the preliminary assumptions.

Mathematical conditions for the physical realizability of linear operators are well known.[1] In practice, a linear operation which is not physically realizable may still be utilized in system design if it can be adequately approximated with a physical device. In this case, engineering judgment must be used to restrict the admissible class of linear functions. As a result of the preliminary assumptions, the $H_m(s)$ are real, rational functions of s. For most practical

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[1] A detailed discussion of these conditions is presented by James, Nichols and Phillips [16].

purposes, no further restrictions are required, except perhaps that $H_m(s)$ should not have a pole at infinity. An accurate synthesis is difficult to achieve unless the $H_m(s)$ have finite gain at infinite frequency. While it is possible to formulate a general constraint on the system specification sufficient to assure that $H_m(s)$ is bounded at infinite frequency, such a constraint is complex beyond practical value. The following example, considering the special case in which the $|f_{en}(o)|$ (n=1, ---, N) are distinct non-zero constants, indicates that the $H_m(s)$ will often be realizable.

Applying Cramer's rule to Equation (3.2), $H_m(s)$ is given by

$$H_m = \frac{|M_m(\xi_F)|}{|M|} \quad , \quad m = 1, \text{---}, N \tag{3.12}$$

where $M_m(\xi_F)$ is constructed from M by replacing column m in M by the vector ξ_F. With the above assumption, the previous results show that M(s) is nonsingular and $\lim_{s \to \infty} \left[s^N |M(s)| \right]$ is a non-zero constant. In other words, the difference in order of the denominator and numerator of $|M(s)|$ is N. Now let $0 < \left| \lim_{s \to \infty} \left[s^{P_n} F_{cn}(s) \right] \right| < \infty$ (n = 1, ---, N), then the difference in order of the denominator and numerator of $|M_m(\xi_F)|$ is at least $\left[N-1 + (P_n)_{min} \right]$, where $(P_n)_{min}$ is the smallest of the P_n. Hence, the $H_m(s)$ will have finite gain at infinite frequency if $\left[N-1 + (P_n)_{min} \right] \geq N$ or $(P_n)_{min} \geq 1$. Clearly, for this special case, the $H_m(s)$ will be realizable in any practical problem.

A Characteristic of the Method

When the specifications can be satisfied with a linear system, a nonlinear system should not be generated. This is a desirable feature for any method of compensation. The following shows that it is a characteristic of the present method. In this proof it is sufficient to

assume that among the $Z_m(f_e)$ there is one linear element, say $Z_1(f_e) = f_e$.

Let $G_n(s)$ be the individual linear compensator defined by the pair $f_{en}(t)$ and $f_{cn}(t)$.

$$G_n(s) = \frac{F_{cn}(s)}{F_{en}(s)}, \quad n = 1, ---, N. \tag{3.13}$$

The vector ξ_F can now be expressed as

$$\xi_F = \begin{bmatrix} F_{c1} \\ \vdots \\ F_{cN} \end{bmatrix} = \begin{bmatrix} G_1 F_{e1} \\ \vdots \\ G_N F_{eN} \end{bmatrix} \tag{3.14}$$

Let Δ_{nm} be the coefactor of Γ_{nm} in M, then (3.12) can be rewritten as

$$H_m = \frac{\sum_{n=1}^{N} G_n F_{en} \Delta_{nm}}{|M|}, \quad m = 1, ---, N. \tag{3.15}$$

From an expansion $|M|$ in terms of coefactors,

$$O = |M| - \sum_{n=1}^{N} \Gamma_{n1} \Delta_{n1} \tag{3.16}$$

Similarly, since $F_{en} = \Gamma_{n1}$,

$$O = \sum_{n=1}^{N} \Gamma_{n1} \Delta_{nm}, \quad m=2, ---, N. \tag{3.17}$$

The right side of this last equation can be expressed as a determinant with two equal columns. The determinant is therefore equal to zero.

Multiplying equation (3.16) by $\frac{G_1}{|M|}$ and adding the result to the equation for $H_1(s)$ yields

$$H_1 = G_1 + \frac{\sum_{n=1}^{N} (G_n - G_1)\Gamma_{n1}\Delta_{n1}}{|M|} \tag{3.18}$$

Similarly, multiplying equation (3.17) by $\frac{G_1}{|M|}$ and subtracting the result from $H_m(s)$ (m=2, ---, N) gives

$$H_m = \frac{\sum_{n=1}^{N} (G_n - G_1)\Gamma_{n1}\Delta_{nm}}{|M|}, \quad m = 2, ---, N. \tag{3.19}$$

When $G_n(s) = G_1(s)$ (n = 1, ---, N), equations (3.18) and (3.19) show that the compensator reduces to a single linear element.

$$H_1(s) = G_1(s),$$
$$H_m(s) = 0, \; m = 2 \; ---, N. \tag{3.20}$$

Thus, when the output element is linear, and all of the prescribed input-response pairs define the same linear operator Y(s), the resulting system is Y(s). With the present method of compensation, this result is obtained with any set of nonlinear elements that contains at least one linear term.

Small-Signal Response

In any design, the system small-signal response characteristics are of primary concern. While the design technique, as considered to this point, will often yield a satisfactory system in this respect, a better approach would be to constrain the design to obtain a prescribed small-signal transfer function.

The following analysis considers the nonlinear elements Z_m^2. In this case, the small-signal transfer function for the compensator is

simply $H_1(s)$. It should be noted that the small-signal function does not necessarily correspond to the linearized compensator, since any modes of $H_m(s)$ (m=2, ---, N) which are not common to $H_1(s)$ are uncontrollable by infinitesimal f_e. These modes, which will be called small-signal uncontrollable, must be stable. Otherwise, the system equilibrium point will be unstable.

Intuitively, it would seem that one method of obtaining required small-signal performance characteristics would be simply to include an appropriate low-amplitude input-response pair in the specifications. This approach can be justified. Assume that the system output element responds as a linear device for the low amplitude input-response pair (f_{i1}, f_{o1}). Replacing this pair with (ϵf_{i1}, ϵf_{o1}), the coefficient matrix M becomes

$$M = \begin{bmatrix} \epsilon\Gamma_{11} & \epsilon^3\Gamma_{12} & \cdots\cdots\cdots & \epsilon^{2N-1}\Gamma_{1N} \\ \Gamma_{21} & \Gamma_{22} & \cdots\cdots\cdots & \vdots \\ \vdots & \vdots & & \vdots \\ \Gamma_{N1} & \cdots\cdots\cdots & & \Gamma_{NN} \end{bmatrix} \tag{3.21}$$

Inspection of (3.21) shows

$$\begin{aligned} &\lim_{\epsilon \to o} \left[\Delta_{1m}\right] = \Delta_{1m}, \quad m = 1, ---, N \\ &\lim_{\epsilon \to o} \left[\frac{1}{\epsilon}\Delta_{n1}\right] = O, \quad n = 2, ---, N \\ &\lim_{\epsilon \to o} \left[\frac{1}{\epsilon}\Delta_{nm}\right] = \Delta^{c}_{nm}, \quad n, m = 2, ---, N \end{aligned} \tag{3.22}$$

where $\Delta^{c}_{nm} = \Gamma_{11} |M_{nm}|$. Here M_{nm} is a submatrix of M obtained by deleting rows 1 and n, and columns 1 and m.

The $H_m(s)$ are defined by a set of equations similar to (3.18) and (3.19).

$$H_1 = G_1 + \frac{\sum_{n=2}^{N} (G_n - G_1)\Gamma_{n1}\Delta_{n1}}{\epsilon\Gamma_{11}\Delta_{11} + \sum_{n=2}^{N} \Gamma_{n1}\Delta_{n1}} \tag{3.23}$$

$$H_m = \frac{\sum_{n=2}^{N} (G_n - G_1)\Gamma_{n1}\Delta_{nm}}{\epsilon\Gamma_{11}\Delta_{11} + \sum_{n=2}^{N} \Gamma_{n1}\Delta_{n1}}, \quad m = 2, ---, N. \tag{3.24}$$

Now define

$$H^{c}_{1} = \lim_{\epsilon \to o} \left[H_1\right] = G_1 \tag{3.25}$$

$$H^{c}_{m} = \lim_{\epsilon \to o} \left[H_m\right] = \frac{\sum_{n=2}^{N} (G_n - G_1)\Gamma_{n1}\Delta^{c}_{nm}}{\Gamma_{11}\Delta_{11}} \tag{3.26}$$

$$= \frac{\sum_{n=2}^{N} (G_n - G_1)\Gamma_{n1}|M_{nm}|}{\Delta_{11}}, \quad m = 2, ---, N.$$

Equation (3.25) shows as expected that in the limit a small amplitude input-response pair will yield exactly the required small-signal transfer function for the compensator and thus for the over-all system. In fact, the compensator can be constrained to yield the required small-signal transfer function using equations (3.25) and (3.26). The

following method is more direct and shows the condition necessary for a solution to the constrained design problem.

It will be assumed that the system specifications now include a transfer function $Y_s(s)$ for the small-signal system and N-1 input-response pairs ($f_{i2}(t)$, ---, $f_{iN}(t)$; $f_{o2}(t)$, ---, $f_{oN}(t)$). Using the theory of linear systems, $Y_s(s)$ can be expressed as

$$Y_s(s) = \frac{G^s(s)\, H_o^s(s)}{1 + H_f(s)\, G^s(s)\, H_o^s(s)} \tag{3.27}$$

where $H_o^s(s)$ and $G^s(s)$ correspond to the small-signal output-element and compensator functions respectively. It will be assumed that the fixed output element has a first approximation ($H_o^s(s) \neq 0$), otherwise $Y_s(s)$ must be zero. Solving equation (3.27) for $G^s(s)$,

$$G^s(s) = \frac{Y_s(s)}{H_o^s(s)\left[1 - Y_s(s)\, H_f(s)\right]}. \tag{3.28}$$

The equations required for evaluation of the $H_m^c(s)$ (m = 2, ---, N) are

$$\Gamma_{n1} H_1^c + \text{-----} + \Gamma_{mn} H_N^c = F_{cn}, \quad m = 2, ---, N. \tag{3.29}$$

Setting $H_1^c(s) = G^s(s)$, these N-1 equation can be expressed as

$$M^c\, \xi_H^c = \xi_F^c, \tag{3.30}$$

where M^c is a coefficient matrix with elements Γ_{nm} (n, m = 2, ---, N) and

$$\xi_H^c = \begin{bmatrix} H_2^c \\ \vdots \\ H_N^c \end{bmatrix}, \quad \xi_F^c = \begin{bmatrix} F_{c2} - \Gamma_{21} G^s \\ \vdots \\ F_{cN} - \Gamma_{N1} G^s \end{bmatrix}. \tag{3.31}$$

If M^c is nonsingular, the H_m^c (m = 2, ---, N) can be computed with

$$\xi_H^c = (M^c)^{-1} \xi_F^c \tag{3.32}$$

The previous results concerning the singularity of the matrix M can be directly modified for M^c. Thus, M^c is nonsingular if the $|f_{e_n}(t)|$ (n = 2, ---, N) are distinct.

As should be expected, this last method of obtaining the specified $Y_s(s)$ yields the same linear elements $H_m^c(s)$ as the earlier method. Equation (3.32) can readily be rewritten in a form identical to equation (3.26).

All of the results of this section apply directly to the compensators in Figures 3.1 and 3.2.

CHAPTER IV
SYSTEM STABILITY

The system specifications do not contain an explicit statement concerning the stability characteristics required of the nonlinear system. However, most nonlinear control or information processing devices have the twin requirements that the autonomous system be stable in-the-large and that the forced responses of the system be stable. These are ideal goals. In practice, weaker types of stability are often accepted as an expediency.

Unfortunately, the method of compensation outlined in Chapter III does not provide any form of control over system stability. The designer, consequently, has no recourse other than to check the final system to see if an acceptable design has been achieved. As with most design techniques for nonlinear systems, a direct test of the actual system or a computer simulation is usually required. With this approach, system stability characteristics can be checked for only a few initial states and forcing functions. This empirical method may lead to the wrong conclusion and has in the past produced some spectacular and costly failures.

In some cases, it is possible to establish system stability characteristics with precise techniques. Lyapunov's second method is one of the principal tools available for tackling nonlinear stability problems. In this chapter, the "second method" is applied to the problem in which the fixed output element is a linear device $H_o(s)$. For designs in which the output element is nonlinear, no method of attack has been found; each output element produces a different and difficult problem.

In the following, the first sections examine the problem of pole cancellation, develop an appropriate form for the system equations, and discuss some possible Lyapunov functions. Following this, the "second method" is employed to establish certain conditions under which the free system has asymptotic stability in-the-large. The final section considers the stability of forced responses of the system.

Pole Cancellation

In the present method of design, the compensator often is required to cancel characteristics of the fixed elements. Since in the physical implementation of the system perfect cancellation cannot be achieved, a stability problem may arise, identical to the difficulty which occurs in a linear system with the cancellation of a right-half-plane pole. For completeness, the problem will be reviewed here to show that the limitations imposed in the preliminary assumptions are sufficient to avoid the creation of an unstable system mode due to the cancellation of a pole or zero in one of the fixed elements.

All of the investigations in this chapter will be concerned with the system illustrated in Figure 4.1, where the fixed elements are lumped with the compensator elements $H_m(s)$ to simplify the form of the system equations.

It will be assumed that among the N + 2 linear elements $H_m(s)$ ($m = 1, - - -, N$), $H_o(s)$ and $H_f(s)$, considered separately, there are K distinct, simple poles. Thus,

$$H_m(s)\, H_o(s)\, H_f(s) = \sum_{k=1}^{K} \frac{\rho_{km}}{s - \lambda_k}, \quad m = 1, - - -, N . \qquad (4.1)$$

Multiple poles can be handled in a similar manner. However, since the design process does not tend to generate poles of higher order, the assumption of simple poles will often be valid. The nonlinear

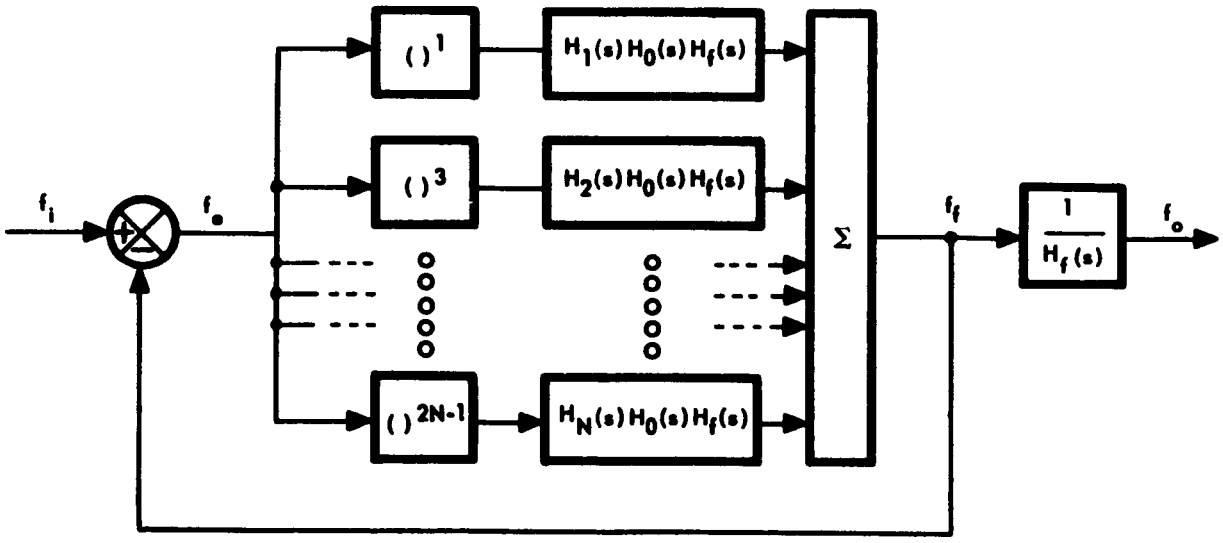


Figure 4.1 - System with Lumped Linear Elements

system can now be expressed in terms of K first-order differential equations as

$$\dot{\phi}_k = \lambda_k \phi_k + \rho_{k1} f_e + \text{- - -} + \rho_{kN} (f_e)^{2N-1}, \quad k = 1, \text{- - -}, K, \tag{4.2}$$

where

$$f_e = f_i - f_f, \tag{4.3}$$

and

$$f_f = \sum_{k=1}^{K} \phi_k . \tag{4.4}$$

The equations (4.2) account for all modes of the nonlinear system and will therefore lead to a correct evaluation of system stability.

Suppose for the open-loop system mode corresponding to λ_a, the coefficient ρ_{a1} is zero. This mode is then small-signal uncon - trollable, as mentioned earlier. Since it is unaffected by the feedback, λ_a will also appear as a characteristic root of the closed-loop linearized system. Hence, if $\text{Re}(\lambda_a) > 0$, the system equilibrium point will be unstable. The condition $\rho_{a1} = 0$ will occur most often when a perfect

pole cancellation takes place in the product $H_1(s)\, H_o(s)\, H_f(s)$. Since no special relationship exists between $H_o(s)$ and $H_f(s)$, a cancellation will usually occur when $H_1(s)$ is required by the specifications to cancel a pole or zero of $H_o(s)\, H_f(s)$. In general, when such a cancellation occurs in $H_1(s)\, H_o(s)\, H_f(s)$, the same cancellation will take place in the $H_m(s)\, H_o(s)\, H_f(s)$ $(m = 2, ---, N)$. In this case, $\rho_{am} = 0 (m = 1, ---, N)$, and this mode of the system is uncontrollable.

While the design process will often require pole cancellations in $H_1(s)\, H_o(s)\, H_f(s)$, such cancellations cannot be the cause of an unstable equilibrium point. The preliminary assumptions have restricted $H_o(s)$ and $H_f(s)$ to the class of minimum phase functions, which excludes the possibility of cancelling a right-half-plane pole. The restriction of $H_o(s)$ to a minimum phase function is not as severe as it might seem. If the output element is unstable, a stable $H_o(s)$ can be produced by enclosing the original element with an appropriate feedback loop. Unfortunately, no similar procedure is available for eliminating right-half-plane zeros of $H_o(s)$. Turning to the feedback element, it is often possible to relax the restriction on $H_f(s)$ to allow right-half-plane zeros, since the zeros of $H_f(s)$ will rarely appear as poles of $H_1(s)$. This occurs because the pole formulation in $H_1(s)$ involves the addition of certain rational functions of s. In any case, to avoid the creation of an unstable equilibrium point as a result of pole cancellation, it is necessary and sufficient to assume that right-half-plane pole cancellation does not occur in the product $H_1(s)\, H_o(s)\, H_f(s)$. Thus, the fixed elements can be allowed right-half-plane poles and zeros as long as the prescribed system input-response pairs do not require their cancellation.

System Equations

Before a stability analysis can be undertaken, the system equations must be placed in a suitable form. The set of equations (4.2) can be divided into two groups, the controllable and uncontrollable. The uncontrollable modes are linear and assumed to be stable. These can be omitted from the stability analysis, since they cannot influence the stability of the controllable part of the system. While perfect pole cancellation cannot be achieved, and thus the ρ_{km} involved will be small but non-zero, this simplifying assumption is still permissible. The stability analyses to follow, hinge on the proof that a function $\dot{V}(\phi)$ (i.e., the time derivative of a Lyapunov function) is negative definite. The sign of $\dot{V}(\phi)$ will remain unchanged for sufficiently small variations in the ρ_{km} corresponding to the uncontrollable modes of the system. Hence, a correct indication of stability is obtained even though perfect pole cancellation cannot be achieved. For simplicity, it will be assumed that the set of equations (4.2) includes only the controllable modes of the system.

With the exception of the last section, this chapter is concerned with the free system ($f_i = 0$). Thus, from Equation (4.3)

$$f_e = - \sum_{k=1}^{K} \phi_k . \tag{4.5}$$

In Equation (4.2), for each term that contains the variable f_e, it is convenient to make the following substitution:

$$(f_e)^{2m-1} = -(f_e)^{2m-2} \sum_{j=1}^{K} \phi_j . \tag{4.6}$$

Thus, Equation (4.2) can be written

$$\dot{\phi}_k = \lambda_k \phi_k - \sum_{m=1}^{N} (f_e)^{2m-2} \rho_{km} \sum_{j=1}^{K} \phi_j, \quad k = 1, ---, K. \tag{4.7}$$

The error function f_e will be retained in the closed-loop equations, as shown above, for simplicity. The system (4.7) can now be expressed as the vector differential equation

$$\dot{\phi} = \left[C_\lambda - \left(\sum_{m=1}^{N} (f_e)^{2m-2} C_m \right) U_1 \right] \phi, \tag{4.8}$$

where

$$\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_K \end{bmatrix}, \tag{4.9}$$

$$C_\lambda = \text{diag}\,(\lambda_1, \text{----}, \lambda_K), \tag{4.10}$$

$$C_m = \text{diag}\,(\rho_{1m}, \text{----}, \rho_{Km}), \qquad m = 1, \text{---}, N \tag{4.11}$$

and U_1 is the singular K x K matrix

$$U_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & \cdots & \cdots & 1 \end{bmatrix}. \tag{4.12}$$

For later stability studies it is necessary to diagonalize the matrix $\left[C_\lambda - C_1 U_1 \right]$ of the system first approximation. First, some of the characteristics of this transformation will be examined. Let ψ be a constant nonsingular matrix such that

$$\psi^{-1} \left[C_\lambda - C_1 U_1 \right] \psi = D_1, \tag{4.13}$$

where

$$D_1 = \text{diag}\,(\tau_1, \ldots, \tau_K). \tag{4.14}$$

It is assumed that the characteristic roots τ_k of the closed-loop linearized system are distinct.

For any K x K matrix A, the product AU_1 can be rewritten $D_A U_1$ where D_A is a diagonal matrix. A similar result is obtained if the order of the matrices is inverted. Consequently, the matrix $C_m U_1$ is similar to the matrix

$$\psi^{-1} C_m U_1 \psi = D_m U_1 D_\psi, \quad m = 2, ---, N \tag{4.15}$$

where D_ψ and D_m (m = 2, --- N) are defined by the relationships

$$U_1 \psi = U_1 D_\psi \tag{4.16}$$

and

$$\psi^{-1} C_m U_1 = D_m U_1, \quad m = 2, ---, N. \tag{4.17}$$

The analysis in Appendix I shows that the diagonal matrix D_ψ is nonsingular and arbitrary. In the following work it will be convenient to let D_ψ equal the identity matrix, $D_\psi = I$. In Appendix II, it is proven that the elements μ_{kj} of ψ ($D_\psi = I$) are of the form $\mu_{kj} = \rho_{kl}/(\lambda_k - \tau_j)$, assuming $\lambda_k \neq \tau_j$. Knowledge of the form of ψ aids in the numerical computation required in a stability analysis, but is not necessary for the following investigations.

Applying the transformation

$$\phi = \psi \phi^* \tag{4.18}$$

to the system (4.8) yields

$$\dot{\phi}^* = \left[D_1 - \left(\sum_{m=2}^{N} (f_e)^{2m-2} D_m \right) U_1 \right] \phi^* , \tag{4.19}$$

where

$$\begin{aligned} f_e &= - \boxed{1\ 0\ ---\ 0}\, U_1 \psi \phi^* \\ &= - \boxed{1\ 0\ ---\ 0}\, U_1 D_\psi \phi^* \\ &= - \sum_{k=1}^{K} \phi_k^* . \end{aligned} \tag{4.20}$$

Equation (4.19) is a nonlinear system of the type (dropping the asterisks),

$$\dot{\phi} = \Upsilon(\phi)\ \phi. \tag{4.21}$$

For the present system,

$$\Upsilon(\phi) = D_1 - \left(\sum_{m=2}^{N} (f_e)^{2m-2} D_m \right) U_1 , \tag{4.22}$$

where

$$f_e = - \sum_{k=1}^{K} \phi_k . \tag{4.23}$$

In the following investigations it will be understood that all matrices symbolized by D are diagonal. Before undertaking a stability analysis of the system (4.21), the formulation of two potential Lyapunov functions will be considered.

Lyapunov Functions

Lyapunov's second method provides a general and powerful tool for the stability analysis of nonlinear systems. One basic difficulty limits the application of this method: there is no systematic way of testing the existence of a Lyapunov function. While the discovery of a suitable function is sufficient to establish the in-the-large stability of a nonlinear system, failure to find one provides no information. The purpose of this section is to define a function $V(\phi)$ and to show conditions for which $V(\phi)$ is a Lyapunov function of the nonlinear system (4.21). The properties of $V(\phi)$ determine the type of stability shown. Asymptotic stability in-the-large will be considered here. Following is one of the basic theorems[1] of the "second method" which shows conditions sufficient for this type of stability.

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[1] This theorem is proven by Kalman and Bertram (18), page 378. The definitions and background required here are presented in this paper.

THEOREM 1 - Consider the continuous-time, autonomous dynamic system

$$\dot{\phi} = q(\phi) \quad (q(0) = 0) \tag{4.24}$$

Suppose there exists a scaler function $V(\phi)$ with continuous first partial derivatives with respect to ϕ such that $V(0) = 0$ and,

(i) $V(\phi) > 0$ for all $\phi \neq 0$,

(ii) $\dot{V}(\phi) < 0$ for all $\phi \neq 0$,

(iii) $V(\phi) \to \infty$ with $\| \phi \| \to \infty$.

Then the equilibrium state $\phi_\epsilon = 0$ is asymptotically stable in-the-large and $V(\phi)$ is a Lyapunov function of the system (4.24).

Using this theorem conditions can be established for which the system (4.21) has a Lyapunov function. These conditions are expressed as limitations on the complex matrix $\Upsilon(\phi)$. The next theorem, while a straightforward application of the "second method," provides a strong result for the system (4.21). It is not restricted by dimensionality.

THEOREM 2 - Consider the continuous-time, autonomous, dynamic system

$$\dot{\phi} = \Upsilon(\phi)\,\phi \quad (\phi_\epsilon = 0). \tag{4.21}$$

Let $\Upsilon(\phi)$ be such that the Hermitian matrix

$$\hat{\Upsilon}(\phi) = \overline{\Upsilon}'(\phi) + \Upsilon(\phi) \tag{4.25}$$

is negative definite for all ϕ. Then the equilibrium state $\phi_\epsilon = 0$ of system (4.21) is asymptotically stable in-the-large and

$$V(\phi) = \overline{\phi}'\,\phi \tag{4.26}$$

is one of its Lyapunov functions.

PROOF - The function $\overline{\phi}'\,\phi$ is a positive definite Hermitian form. Thus $V(0) = 0$ and $V(\phi) > 0$ for all $\phi \neq 0$. Computing $\dot{V}(\phi)$ yields:

$$\dot{V}(\phi) = \dot{\bar{\phi}}' \phi + \bar{\phi}' \dot{\phi} \tag{4.27}$$

$$= \bar{\phi}' \left[\bar{\Upsilon}'(\phi) + \Upsilon(\phi) \right] \phi .$$

By the above hypothesis $\bar{\phi}' \left[\bar{\Upsilon}' + \Upsilon \right] \phi$ is a negative definite Hermitian form. Hence $\dot{V}(\phi) < 0$ for all $\phi \neq 0$. Finally, $V(\phi)$ tends to infinity with $\| \phi \| = (\bar{\phi}' \phi)^{1/2} \longrightarrow \infty$, which completes the proof.

Several of the results presented in the literature can also be applied to this problem. In particular, Krasovskii's Theorem[1] provides another method which is not restricted by dimensionality. A modified form of Krasovskii's Theorem is presented below. The modification provides a convenient extension of the theorem to complex systems.

THEOREM 3 - Consider the continuous-time, autonomous, dynamic system

$$\dot{\phi} = q(\phi) \tag{4.28}$$

which has a unique singular point $\phi_\epsilon = 0$. Let the system (4.28) be such that $\| q(\phi) \| \longrightarrow \infty$ with $\| \phi \| \longrightarrow \infty$. Now assume that $q(\phi)$ has continuous first partial derivatives and that its Jacobian matrix $Q_J(\phi) = \left[\partial q_i / \partial \phi_j \right]$ is such that the Hermitian matrix

$$\hat{Q}(\phi) = \bar{Q}'_J(\phi) + Q_J(\phi) \tag{4.29}$$

is negative definite for all ϕ. Then the equilibrium state of the system (4.28) is asymptotically stable in-the-large and

$$V(\phi) = \dot{\bar{\phi}}' \dot{\phi} = \bar{q}'(\phi)\, q(\phi) \tag{4.30}$$

is one of its Lyapunov functions.

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[1] Ibid., 385.

PROOF - From Equation (4.30), $V(0) = 0$ and $V(\phi) > 0$ for $\phi \neq 0$. Computing $\dot{V}(\phi)$ gives:

$$\dot{V}(\phi) = \overline{\dot{\phi}'}\ \dot{\phi} + \overline{\phi}'\ \ddot{\phi} \tag{4.31}$$

$$= \overline{\dot{\phi}}'\left[\overline{Q}'_J(\phi) + Q_J(\phi)\right]\dot{\phi}.$$

By the above hypothesis $\dot{V}(\phi)$ is negative definite and $\dot{V}(\phi) < 0$ for $\phi \neq 0$. Finally, since

$$\|q(\phi)\| \longrightarrow \infty \text{ with } \|\phi\| \longrightarrow \infty,$$

then

$$V(\phi) \longrightarrow \infty \text{ with } \|\phi\| \longrightarrow \infty$$

which completes the proof.

Theorems 2 and 3 appear to be the strongest results available for the system (4.21). The application of these theorems is examined in the next section.

Stability of the Autonomous System

If either $\hat{T}(\phi)$ or $\hat{Q}(\phi)$ is negative definite, the autonomous system is asymptotically stable in-the-large. While the matrices $\hat{T}(\phi)$ and $\hat{Q}(\phi)$ can be directly tested for negative definiteness, the results can be extended further when the system defined by the matrix (4.22) is considered. The purpose of the present section is to show some rather general conditions for which $\hat{T}(\phi)$ and $\hat{Q}(\phi)$ are negative definite. First, a special case which sheds some light on the general problem is considered. Following this, it is shown that Theorem 2 provides a stronger result than Theorem 3. Finally, a set of conditions sufficient for the asymptotic stability in-the-large of the general system are developed.

The analyses presented below are concerned with the system equation in which the first approximation is diagonalized. Other analyses could consider different forms of the system equation. The author

has investigated several of these alternate cases; however, the rather complete result presented below has not been duplicated.

Stability Analysis of Special Case -- The application of Theorem 2 will be considered first. In the following special case of the system (4.21), it will be assumed that $\Upsilon(\phi)$ is of the form:

$$\Upsilon(\phi) = D_1 - (f_e)^{2P-2} D_P U_1, \tag{4.32}$$

where $P > 1$ is an interger. The system must be properly scaled to obtain results. While suitable scaling could be obtained by an appropriate choice of D_ψ, it is convenient to rescale the system. When the system is rescaled with the transformation $\phi = D_s \phi_s$, the matrix $\hat{\Upsilon}(\phi)$ becomes

$$\hat{\Upsilon}(\phi_s) = \bar{D}_s \bar{\Upsilon}'(\phi_s) \bar{D}_s^{-1} + D_s^{-1} \Upsilon(\phi_s) D_s \tag{4.33}$$

$$= \hat{\Upsilon}_1 - (f_e)^{2P-2} \hat{\Upsilon}_P$$

where

$$\hat{\Upsilon}_1 = \bar{D}_1 + D_1, \tag{4.34}$$

$$\hat{\Upsilon}_P = \bar{D}_s U_1 \bar{D}_P \bar{D}_s^{-1} + D_s^{-1} D_P U_1 D_s . \tag{4.35}$$

Denoting the diagonal elements of D_s as κ_k, the system error is now

$$f_e = - \sum_{k=1}^{K} \kappa_k \phi_{sk} .$$

The following analysis shows that $\hat{\Upsilon}(\phi_s)$ is negative definite if and only if: (1) $Re(\tau_k) < 0$ $(k = 1, ---, K)$, (2) the matrix D_P is positive definite, and (3) $\bar{D}_s D_s = D_P$.

Suppose $\hat{\Upsilon}(\phi_s)$ is negative definite for all ϕ_s. Then, since $\hat{\Upsilon}(\phi_s) \rightarrow \hat{\Upsilon}_1$ with $f_e \rightarrow 0$, the matrix $\hat{\Upsilon}_1$ must be negative definite.

In addition, since $\hat{T}(\phi_s)/(f_e)^{2P-2} \longrightarrow -\hat{T}_P$ with $f_e \longrightarrow \infty$, the matrix $\hat{T}_P$ must be positive semi-definite. These conditions are necessary and sufficient for a negative definite $\hat{T}(\phi_s)$. The limitations they place on D_1 and D_P will be examined next.

The diagonal elements of $\hat{T}_1 = \bar{D}_1 + D_1$ are 2 $Re(\tau_k)$, and thus $\hat{T}_1$ is negative definite only if $Re(\tau_k) < 0$. This requirement is obvious since any stable system must have a stable equilibrium point.

Several steps are necessary to establish the conditions for which $\hat{T}_P$ is positive semi-definite. First, it will be shown sufficient to consider a real scaling matrix D_s. Following this, a proof will be presented which shows that $\hat{T}_P$ cannot be positive semi-definite for any real D_s if the elements d_k of D_P are complex or nonpositive, $d_k \leqq 0$. Finally, it will be proven that if $d_k > 0$, then $\hat{T}_P$ is positive semi-definite for $D_s^2 = D_P$ (or $\bar{D}_s D_s = D_P$) and only for D_s which satisfy this condition.

The matrix $\hat{T}_P$ is Hermitely congruent to

$$\bar{D}_s^{-1} \hat{T}_P D_s^{-1} = U_1 \bar{D}_P (\bar{D}_s D_s)^{-1} + (\bar{D}_s D_s)^{-1} D_P U_1. \quad (4.36)$$

Since $(\bar{D}_s D_s)^{-1}$ scales only the magnitude of the elements d_k of D_P in Equation (4.36), the range of values of the Hermitian form $V_P(\phi_s) = \bar{\phi}_s' \hat{T}_P \phi_s$ is independent of the angle of the elements κ_k of D_s. Therefore, it can be assumed that the κ_k are real.

Next it will be shown that $\hat{T}_P$ is not positive semi-definite for any real D_s if D_P is complex. Note that complex diagonal elements of D_P occur in conjugate complex pairs. The following proof employs the fact that all principle submatrices of a positive semi-definite matrix must be positive semi-definite. Consider any 2 x 2 principal submatrix of D_P with elements $d_i = \alpha + j\beta$ and $d_j = \alpha - j\beta$. The form $V_P(\phi_{si}, \phi_{sj})$ can be written:

$$V_P(\phi_{si}, \phi_{sj}) = \begin{bmatrix} \bar{\phi}_{si} \\ \bar{\phi}_{sj} \end{bmatrix}' \begin{bmatrix} 2a & (a+j\beta)\left(\frac{\kappa_i}{\kappa_j} + \frac{\kappa_j}{\kappa_i}\right) \\ (a-j\beta)\left(\frac{\kappa_i}{\kappa_j} + \frac{\kappa_j}{\kappa_i}\right) & 2a \end{bmatrix} \begin{bmatrix} \phi_{si} \\ \phi_{sj} \end{bmatrix}.$$

Suppose $\hat{T}_P$ is positive semi-definite, then for any real constant $\epsilon > 0$, the matrix

$$\begin{bmatrix} 2a + \epsilon & (a+j\beta)\left(\frac{\kappa_i}{\kappa_j} + \frac{\kappa_j}{\kappa_i}\right) \\ (a-j\beta)\left(\frac{\kappa_i}{\kappa_j} + \frac{\kappa_j}{\kappa_i}\right) & 2a + \epsilon \end{bmatrix},$$

is positive definite and its determinant must be positive.

$$(2a+\epsilon)^2 - (a^2+\beta^2)\left(\frac{\kappa_i}{\kappa_j} + \frac{\kappa_j}{\kappa_i}\right) > 0 .$$

The left side of this inequality is maximum when $\kappa_i = \kappa_j$. Thus,

$$4a\epsilon + \epsilon^2 - 4\beta^2 > 0 ,$$

which is a contradiction for ϵ sufficiently small unless $\beta = 0$. Since this argument is valid for all 2 x 2 principal submatrices, the matrix $\hat{T}_P$ cannot be positive semi-definite if any of the d_k are complex.

A similar argument proves that $\hat{T}_P$ cannot be positive semi-definite if $d_k \leq 0$. First, note that the d_k cannot be negative; otherwise, some of the principal submatrices of $\hat{T}_P$ will fail to be positive semi-definite. Thus, only the case in which some of the d_k are zero must be investigated. Suppose some of the d_k are zero. Then, for any 2 x 2 principal submatrix of D_P in which $d_i = 0$ and $d_j \neq 0$, the form $V_P(\phi_{si}, \phi_{sj})$ can be written:

$$V_P(\phi_{si}, \phi_{sj}) = \begin{bmatrix} \bar{\phi}_{si} \\ \bar{\phi}_{sj} \end{bmatrix}' \begin{bmatrix} 0 & d_j \frac{\kappa_i}{\kappa_j} \\ d_j \frac{\kappa_i}{\kappa_j} & 2 d_j \end{bmatrix} \begin{bmatrix} \phi_{si} \\ \phi_{sj} \end{bmatrix}.$$

If this form is positive semi-definite, for any real $\epsilon > 0$ the matrix

$$\begin{bmatrix} \epsilon & d_j \frac{\kappa_i}{\kappa_j} \\ d_j \frac{\kappa_i}{\kappa_j} & 2 d_j + \epsilon \end{bmatrix},$$

is positive definite and its determinant must be positive

$$2 d_j \epsilon + \epsilon^2 - \left(d_j \frac{\kappa_i}{\kappa_j} \right)^2 > 0 .$$

For any fixed D_s, the constant κ_i/κ_j is non-zero. Thus, for ϵ sufficiently small the above inequality is a contradiction. This result shows that with the exception of the trivial case $D_P = 0$, the matrix $\hat{T}_P$ cannot be positive semi-definite if $d_k \leq 0$.

It remains to be proven that $\hat{T}_P$ is positive semi-definite if $d_k > 0$ and then only if $\bar{D}_s D_s = D_P$. Suppose $d_k > 0$, then Equation (4.36) shows that for $\bar{D}_s D_s = D_P$, the matrix $\hat{T}_P$ is positive semi-definite since it is Hermitely congruent to the positive semi-definite matrix $2 U_1$. To prove that $\bar{D}_s D_s = D_P$ is a necessary condition, it is sufficient to show at least one 2×2 principal submatrix of $D_s^{-1} \hat{T}_P D_s^{-1}$ fails to be positive semi-definite unless $D_s^2 = D_P$. This argument will be omitted since it is similar to those presented above.

While the above results were derived for a system that has one nonlinear element, they also establish two necessary conditions

for the negative definiteness of $\hat{T}(\phi_s)$ when $N > 2$ -- $\left[\bar{D}_1 + D_1\right]$ must be negative definite and D_N must be positive definite. These will be required later when the general system is considered.

Theorem 3 yields the same result as Theorem 2 for the system defined by the matrix (4.32). The proof is obtained by a straightforward computation of the matrix $\hat{Q}(\phi_s)$. For this special case the vector $q(\phi)$ is given by

$$q(\phi) = T(\phi)\,\phi$$

$$= D_1\,\phi - (f_e)^{2P-2}\,D_P\,U_1\,\phi\,. \qquad (4.37)$$

Substituting $f_e = -\sum_{k=1}^{K} \phi_k$ into this equation, the components of the vector $q(\phi)$ can be written

$$q_i(\phi) = \tau_i\,\phi_i + \left(-\sum_{k=1}^{K} \phi_k\right)^{2P-1} d_i\,. \qquad (4.38)$$

Computing the Jacobian matrix $Q_J(\phi)$ of the vector $q(\phi)$,

$$Q_J(\phi) = \left[\partial q_i / \partial \phi_j\right]$$

$$= D_1 - (2P-1)\,(f_e)^{2P-2}\,D_P U_1\,. \qquad (4.39)$$

If the system is rescaled as before with $\phi = D_s\,\phi_s$, the new Jacobian matrix $Q_J(\phi_s)$ is

$$Q_J(\phi_s) = D_s^{-1}\,D_1\,D_s - (2P-1)\,(f_e)^{2P-2}\,D_s^{-1}\,D_P U_1 D_s, \qquad (4.40)$$

and therefore,

$$\hat{Q}(\phi_s) = \bar{Q}'_J(\phi_s) + Q_J(\phi_s)$$

$$= \hat{T}_1 - (2P-1)\,(f_e)^{2P-2}\,\hat{T}_P\,. \qquad (4.41)$$

Noting the similarity between the matrices $\hat{T}(\phi_s)$ and $\hat{Q}(\phi_s)$, it is apparent that Theorems 2 and 3 yield the same result for this special case. Thus, the $\hat{Q}(\phi_s)$ given by Equation (4.41) is negative definite only if: (1) $\left[\overline{D}_1 + D_1\right]$ is negative definite, (2) D_P is positive definite, and (3) $\overline{D}_s D_s = D_P$.

Comparison of Theorems 2 and 3 -- As indicated by the above results, Theorems 2 and 3 are closely related. For the general system defined by the matrix (4.22), this relationship is especially simple. The general matrix $\hat{T}(\phi_s)$ can be written

$$\hat{T}(\phi_s) = \hat{T}_1 - \sum_{m=2}^{N} (f_e)^{2m-2} \hat{T}_m , \tag{4.42}$$

where, using the above form of notation,

$$\hat{T}_m = \overline{D}_s U_1 \overline{D}_m \overline{D}_s^{-1} + D_s^{-1} D_m U_1 D_s , \quad m = 2, ---, N. \tag{4.43}$$

The corresponding $\hat{Q}(\phi_s)$ is

$$\hat{Q}(\phi_s) = \hat{T}_1 - \sum_{m=2}^{N} (2m - 1) (f_e)^{2m-2} \hat{T}_m . \tag{4.44}$$

For purposes of the following discussion, let

$$\hat{T}(\phi_s) = \hat{T}_s (f_e) \qquad \text{and} \qquad \hat{Q}(\phi_s) = \hat{Q}_s (f_e) .$$

The relationship between these matrices can be stated:

$$\hat{T}_s(f_e) = \frac{1}{f_e} \int_o^{f_e} \hat{Q}_s(x)\, dx, \tag{4.45}$$

and therefore,

$$\overline{\phi}'_s \hat{T}_s(f_e)\, \phi_s = \frac{1}{f_e} \int_o^{f_e} \overline{\phi}'_s \hat{Q}_s(x)\, \phi_s\, dx . \tag{4.46}$$

Since $\hat{Q}_s(f_e) = \hat{Q}_s(-f_e)$, the right side of Equation (4.46), is an even function of f_e. If $\hat{Q}_s(f_e)$ is negative definite, then the right side of (4.46) is negative for all f_e and $\hat{T}_s(f_e)$ is necessarily negative definite. The same result is obtained if an arbitrary transformation is applied to the system, and therefore, the function $\bar{\phi}' \phi$ will be a Lyapunov function of the system whenever $\dot{\bar{\phi}}' \dot{\phi}$ is a Lyapunov function. Since the converse is not true, Theorem 2 provides the stronger result. The remaining sections of this chapter will be concerned with Theorem 2.

Stability Analysis of General System -- To initiate the analysis of the general system it will be assumed the two necessary conditions established earlier, $\bar{D}_1 + D_1$ negative definite and D_N positive definite, are satisfied. In addition, since $\hat{T}_N(\phi_s)$ must be positive semi-definite, it will be necessary to assume D_s is constrained such that $\bar{D}_s D_s = D_N$. As the first step in this analysis, it will be proven that under these assumptions the matrix $\hat{T}(\phi_s)$ is negative definite if and only if, for at least one D_s, $\left|-\hat{T}(\phi_s)\right| > 0$ for all ϕ_s --a surprisingly simple result for a K^{th} order system with N-1 nonlinear elements. In the last part of the analysis, a method of expressing the determinant $\left|-\hat{T}(\phi_s)\right|$ as a polynominal in f_e^2 is developed. The condition $\left|-\hat{T}(\phi_s)\right| > 0$ can then be restated as a requirement that this polynomial must have no real roots for at least one D_s.

From Theorem 2, the derivative of the function $V(\phi_s)$ is

$$\dot{V}(\phi_s) = \bar{\phi}_s' \hat{T}(\phi_s) \phi_s , \tag{4.47}$$

where $\hat{T}(\phi_s) = \hat{T}_s(f_e)$ is defined by Equation (4.42). Applying the linear substitution $\phi_s = D_s^{-1} \phi$ to $\dot{V}(\phi_s)$ gives

$$\dot{V}(\phi) = \bar{\phi}' \bar{D}_s^{-1} \hat{T}_s(f_e) D_s^{-1} \phi \tag{4.48}$$

$$= \bar{\phi}' \left[\bar{D}_{s1} + D_{s1}\right] \phi - \sum_{m=2}^{N-1} (f_e)^{2m-2} \bar{\phi}' \left[U_1 \bar{D}_{sm} + D_{sm} U_1\right] \phi - 2(f_e)^{2N-2} \bar{\phi}' U_1 \phi$$

where

$$D_{sm} = D_m \left(\overline{D}_s \, D_s \right)^{-1} = D_m \, D_N^{-1} , \quad m = 1, \text{---}, N. \tag{4.49}$$

The diagonal elements of the D_{sm} will be denoted as ω_{km}.

Now consider the nonsingular linear substitution

$$\phi = T \, \phi_T , \tag{4.50}$$

where

$$T = \begin{bmatrix} 1 & 0 & 0 \ldots\ldots & 0 \\ 0 & 1 & 0 \ldots\ldots & 0 \\ 0 & 0 & 1 \ldots & . \\ . & . & . & . \\ . & . & . & . \\ . & . & & . \\ 0 & 0 & \ldots\ldots & 1 \quad 0 \\ -1 & -1 & \ldots\ldots & -1 \quad 1 \end{bmatrix} . \tag{4.51}$$

Applying this substitution to (4.48) gives

$$\dot{V}(\phi_T) = \overline{\phi}_T' \, \hat{T}_T(f_e) \, \phi_T , \tag{4.52}$$

where

$$\hat{T}_T(f_e) = T' \, \overline{D}_s^{-1} \, \hat{T}_s(f_e) \, D_s^{-1} \, T. \tag{4.53}$$

Since this equation is rather cumbersome, the matrix in each of its terms will be examined separately before they are combined.

To simplify the following work, it is convenient to formulate the matrix $T' \left[\overline{D}_{s1} + D_{s1} \right] T$ as follows:

(4.54)

$$T' \left[\overline{D}_{s1} + D_{s1} \right] T = 2 \, \mathrm{Re} \, \omega_{K1} \begin{bmatrix} 1 \ldots\ldots 1 & -1 \\ . \; 1 \quad . & . \\ . \quad . & . \\ . \quad . & . \\ . \quad . & . \\ . \quad . & . \\ 1 \ldots\ldots 1 & -1 \\ -1 \ldots\ldots -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 \, \mathrm{Re} \, \omega_{11} & & & 0 \\ & \ddots & & \vdots \\ & & 0 & \\ & 0 & \ddots & \\ & & & 2 \, \mathrm{Re} \, \omega_{(K-1)1} \\ 0 & \ldots\ldots\ldots & & 0 \end{bmatrix} .$$

Let U_2 be a $(K-1) \times (K-1)$ submatrix of U_1 and $D_\omega = \text{diag}(\text{Re}\,\omega_{11}, \cdots, \text{Re}\,\omega_{(K-1)1})$, then as a result of the initial assumptions the matrix $\left[\text{Re}(\omega_{K1} U_2) + D_\omega\right]$ is negative definite. Therefore, let

$$A' A = -2\left[\text{Re}\ (\omega_{K1} U_2) + D_\omega\right]. \tag{4.55}$$

Now the matrix (4.54) can be rewritten

$$T'\left[\bar{D}_{s1} + D_{s1}\right] T = \begin{bmatrix} & & -2\,\text{Re}\,\omega_{K1} \\ & -A' A & \vdots \\ & & -2\,\text{Re}\,\omega_{K1} \\ -2\,\text{Re}\,\omega_{K1} \cdots -2\,\text{Re}\,\omega_{K1} & & +2\,\text{Re}\,\omega_{K1} \end{bmatrix} \tag{4.56}$$

The $(N-2)$ matrices $T'\left[U_1 \bar{D}_{sm} + D_{sm} U_1\right] T$ are (4.57)

$$T'\left[U_1 \bar{D}_{sm} + D_{sm} U_1\right] T = \begin{bmatrix} & & (\omega_{1m} - \omega_{Km}) \\ & 0 & \vdots \\ & & (\omega_{(K-1)m} - \omega_{Km}) \\ (\bar{\omega}_{1m} - \bar{\omega}_{Km}) \cdots (\bar{\omega}_{(K-1)m} - \bar{\omega}_{Km}) & & (\bar{\omega}_{Km} + \omega_{Km}) \end{bmatrix}.$$

Finally, the matrix $T'\left[2 U_1\right] T$ is

$$T'\, 2 U_1\, T = \begin{bmatrix} & & 0 \\ & 0 & \vdots \\ & & 0 \\ 0 \cdots\cdots 0 & & 2 \end{bmatrix}. \tag{4.58}$$

To simplify the notation for $\hat{T}_T(f_e)$, let:

$$\theta_k = +2 \text{ Re } \omega_{K1} + \sum_{m=2}^{N-1} (f_e)^{2m-2} (\omega_{km} - \omega_{Km}), \; k = 1, ---, K-1 \qquad (4.59)$$

and

$$\theta_K = -2 \text{ Re } \omega_{K1} + \sum_{m=2}^{N-1} (f_e)^{2m-2} (\bar{\omega}_{Km} + \omega_{Km}) + 2(f_e)^{2N-2}, \qquad (4.60)$$

then

$$-\hat{T}_T(f_e) = \begin{bmatrix} & & \theta_1 \\ & A'A & \vdots \\ & & \theta_{K-1} \\ \bar{\theta}_1 \cdots\cdots \bar{\theta}_{K-1} & & \theta_K \end{bmatrix}. \qquad (4.61)$$

The following criterion[1] for positive definiteness will be required to progress further.

THEOREM 4 - Let H be a K x K Hermitian matrix and H^k denote the submatrices of H taken from the first k rows and columns. Then H is positive definite if and only if all K submatrices H^k have positive determinants.

The matrix A'A is positive definite, and therefore, $\left|-\hat{T}_T^k(f_e)\right| > 0$ $(k = 1, ---, K-1)$. By Theorem 4, $\hat{T}_T(f_e)$ is then negative definite if and only if $\left|-\hat{T}_T(f_e)\right| > 0$. However, $\hat{T}_T(f_e)$ is congruent to $\hat{T}(\phi_s)$, and hence $\hat{T}(\phi_s)$ is negative definite if and only if $\left|-\hat{T}(\phi_s)\right| > 0$. In summary, the nonlinear system defined by the matrix $\hat{T}(\phi_s)$ is asymptotically stable in-the-large if the three conditions,

(i) $\text{Re } (\tau_k) < 0$,

(ii) D_N positive definite,

(iii) $\left|-\hat{T}(\phi_s)\right| > 0$ for at least one D_s,

are satisfied.

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[1] Perlis (26), page 103.

Condition (iii) can be further reduced as shown next. Let

$$B = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_{K-1} \end{bmatrix}, \tag{4.62}$$

then

$$-\hat{\dot{T}}_T(f_e) = \begin{bmatrix} A'A & B \\ \bar{B}' & \theta_K \end{bmatrix}. \tag{4.63}$$

This matrix is congruent to

$$-C'\hat{\dot{T}}_T(f_e)\,C = \begin{bmatrix} I & (A')^{-1}B \\ \bar{B}'(A)^{-1} & \theta_K \end{bmatrix}, \tag{4.64}$$

where $C = \mathrm{diag}\left[A^{-1}, 1\right]$. For computational purposes, this last step can be performed readily by row and column operations on the matrix (4.63). If the elements of the column matrix $(A')^{-1}B$ are denoted θ_k^* $(k = 1, \text{---}, K-1)$, expanding the determinant $\left|-C\,\hat{\dot{T}}_T(f_e)\,C\right|$ gives

$$\left|-C\,\hat{\dot{T}}_T(f_e)\,C\right| = \theta_K(f_e) - \sum_{k=1}^{K-1} \bar{\theta}_k^*(f_e)\,\theta_k^*(f_e). \tag{4.65}$$

Hence, condition (iii) becomes

$$\theta_K(f_e) - \sum_{k=1}^{K-1} \bar{\theta}_k^*(f_e)\,\theta_k^*(f_e) > 0. \tag{4.66}$$

The left side of (4.66) is a polynomial in f_e^2. Therefore, the above inequality is satisfied if $\theta_K - \sum_{k=1}^{K-1} \bar{\theta}_k^*\,\theta_k^*$ has no real roots. It can be shown that this requirement is automatically satisfied for N = 2 since

the initial assumptions are sufficient for the negative definiteness of $\hat{T}(\phi_s)$ in this case.

In summary, the procedure for testing the negative definiteness of the matrix $\hat{T}(\phi_s)$ consists of the following sequence:

1. The first step is to evaluate the transformation matrix $\psi = \left(\frac{\rho_{kl}}{\lambda_k - \tau_j}\right)$ and its inverse ψ^{-1}.

2. Using ψ, the characteristic roots τ_k of the linearized system matrix D_l can be computed with the relationship $D_l = \psi^{-1}\left[C_\lambda - C_l U_l\right]\psi$. If any $\mathrm{Re}(\tau_k)>0$ there is no need to proceed further.

3. The third step is to evaluate the matrix D_N which is defined by $D_N U_l = \psi^{-1} C_N U_l$. If D_N is positive definite, the final step can be undertaken.

4. Computation of the function $\theta_K - \sum_{k=1}^{K-1} \bar{\theta}_k^* \theta_k^*$ can be carried out by the method presented above. If this function has no real roots for at least one D_s, the system is asymptotically stable in-the-large.

Stability of Forced Trajectories

In nonlinear systems very little is known about the relationship between the stability of the free system and the stability of forced trajectories. However, as with the free system, the stability of system trajectories can be investigated with Lyapunov's second method, since for a fixed $f_i(t)$ deviations from the corresponding $f_o(t)$ can be expressed in terms of a free dynamic system with an equilibrium state at the origin. For the present case, it can be proven that this new system is asymptotically stable in-the-large for any $f_i(t)$ when $\bar{\phi}' \phi$ is a Lyapunov function of the autonomous system. In this case, each

forced trajectory is said to be asymptotically stable relative to the class of all trajectories.

The following theorem, extracted from Corollary 1.1 (Theorem 1) of the paper by Kalman and Bertram (18), will be required.

THEOREM 5 - Consider the continuous-time, free dynamic system

$$\dot{\phi} = q(\phi, t) \tag{4.67}$$

where $q(0, t) = 0$ for all t. Suppose there exists a scalar function $V(\phi, t)$ with continuous first partial derivatives with respect to ϕ and t such that $V(0, t) = 0$ and,

(i) $V(\phi, t)$ is positive definite,

(ii) $\dot{V}(\phi, t)$ is negative definite,

(iii) $V(\phi, t) \rightarrow \infty$ with $||\phi|| \rightarrow \infty$.

Then the equilibrium state is asymptotically stable in-the-large and $V(\phi, t)$ is a Lyapunov function of the system (4.67).

The analysis in this section is general and does not require the system first approximation to be diagonalized. However, it is convenient to use the previous notation and express the forced nonlinear system as

$$\dot{\phi}^0 = D_1 \phi^0 + \sum_{m=2}^{N} (f_e^0)^{2m-1} D_m, \tag{4.68}$$

where

$$f_e^0 (\phi^0, t) = f_i(t) - \sum_{k=1}^{K} \phi_k^0 (t), \tag{4.69}$$

and $f_i(t)$ is fixed. In (4.68), replacing ϕ^0 with $\phi^0 + \phi^1$ and subtracting (4.68) yields

$$\dot{\phi}^1 = D_1 \phi^1 + \sum_{m=2}^{N} \left[(f_e^0 + f_e^1)^{2m-1} - (f_e^0)^{2m-1} \right] D_m, \tag{4.70}$$

where

$$f_e^1 = - \sum_{k=1}^{K} \phi_k^1 . \tag{4.71}$$

The system (4.70) can be rewritten as

$$\dot{\phi}^1 = \Upsilon (f_e^0, f_e^1)\ \phi^1 , \tag{4.72}$$

where

$$\Upsilon(f_e^0, f_e^1) = D_1 - \sum_{m=2}^{N} \left[\frac{(f_e^0 + f_e^1)^{2m-1} - (f_e^0)^{2m-1}}{f_e^1} \right] D_m U_1 . \tag{4.73}$$

Selecting

$$V(\phi^1, t) = (\bar{\phi}^1)'\ \phi^1 \tag{4.74}$$

as a potential Lyapunov function for the system (4.72), then

$$\dot{V}(\phi^1, t) = (\bar{\phi}^1)'\ \hat{\Upsilon}\ (f_e^0, f_e^1)\ \phi^1 , \tag{4.75}$$

where

$$\hat{\Upsilon}(f_e^0, f_e^1) = \bar{\Upsilon}' (f_e^0, f_e^1) + \Upsilon(f_e^0, f_e^1) . \tag{4.76}$$

By Theorem 5, the system is asymptotically stable in-the-large if $\dot{V}(\phi^1, t)$ is negative definite.

Now a new function $\dot{V}_f(\delta, y_o, y_1)$ will be defined as

$$\dot{V}_f(\delta, y_o, y_1) = \bar{\delta}'\ \hat{\Upsilon}\ (y_o, y_1)\ \delta, \tag{4.77}$$

where y_o and y_1 are independent of δ. If the function $\dot{V}_f(\delta, y_o, y_1)$ is negative definite then clearly $\dot{V}(\phi^1, t)$ is negative definite. Thus, the system (4.72) is asymptotically stable in-the-large if $\dot{V}_f(\delta, y_o, y_1) < 0$ for $\delta \neq 0$.

$$\dot{V}_f(\delta, y_o, y_1) = \bar{\delta}'\hat{\Upsilon}_1\ \delta - \bar{\delta}' \left\{ \sum_{m=2}^{N} \left[\frac{(y_o + y_1)^{2m-1} - (y_o)^{2m-1}}{y_1} \right] \hat{\Upsilon}_m \right\} \delta . \tag{4.78}$$

Expanding the bracketed term in the last equation:

$$\frac{(y_o+y_1)^{2m-1}-(y_o)^{2m-1}}{y_1} = (y_o+y_1)^{2m-2} + \frac{y_o}{y_1}(y_o+y_1)^{2m-2} - \frac{y_o}{y_1}(y_o)^{2m-2} . \tag{4.79}$$

Then Equation (4.78) can be written

$$\dot{V}_f(\delta, y_o, y_1) = \overline{\delta}' \left[\hat{T}_1 - \sum_{m=2}^{N} (y_o+y_1)^{2m-2}\hat{T}_m\right]\delta - \overline{\delta}' \left[\frac{y_o}{y_1} \sum_{m=2}^{N} (y_o+y_1)^{2m-2}\hat{T}_m\right]\delta$$

$$+ \overline{\delta}' \left[\frac{y_o}{y_1} \sum_{m=2}^{N} (y_o)^{2m-2}\hat{T}_m\right]\delta. \tag{4.80}$$

Adding the functions

$$\overline{\delta}' \frac{y_o}{y_1}\hat{T}_1\delta \quad \text{and} \quad -\overline{\delta}'\frac{y_o}{y_1}\hat{T}_1\delta$$

respectively to the last two terms of (4.80) gives

$$\dot{V}_f(\delta, y_o, y_1) = \overline{\delta}' \left[\hat{T}_1 - \sum_{m=2}^{N} (y_o+y_1)^{2m-2}\hat{T}_m\right]\delta + \overline{\delta}' \frac{y_o}{y_1}\left[\hat{T}_1 - \sum_{m=2}^{N} (y_o+y_1)^{2m-2}\hat{T}_m\right]\delta$$

$$- \overline{\delta}' \frac{y_o}{y_1}\left[\hat{T}_1 - \sum_{m=2}^{N} (y_o)^{2m-2}\hat{T}_m\right]\delta . \tag{4.81}$$

Now let

$$\dot{V}_a(\delta, y) = \overline{\delta}' \left[\hat{T}_1 - \sum_{m=2}^{N} (y)^{2m-2}\hat{T}_m\right]\delta . \tag{4.82}$$

If $\dot{V}_a(\delta, y)$ is negative definite $\overline{\phi}'\phi$ is a Lyapunov function of the autonomous system. Using Equation (4.82), the $\dot{V}_f(\delta, y_o, y_1)$ given by (4.81) can be written

$$\dot{V}_f(\delta, y_o, y_1) = \overline{\delta}' \frac{1}{y_1}\left[(y_o+y_1)\dot{V}_a(\delta, y_o+y_1) - y_o \dot{V}_a(\delta, y_o)\right]\delta. \tag{4.83}$$

Suppose $\dot{V}_a(\delta, y)$ is negative definite, then the following argument shows that $\dot{V}_f(\delta, y_o, y_1)$ is necessarily negative definite. The first step is to prove that $y\dot{V}_a(\delta, y)$ is monotonic and decreasying in y.

$$y\dot{V}_a(\delta, y) = \bar{\delta}'\left[\hat{T}_1 y - \sum_{m=2}^{N} (y)^{2m-1}\hat{T}_m\right]\delta. \tag{4.84}$$

The first partial derivative of $y\dot{V}_a(\delta, y)$ with respect to y is

$$\frac{\partial\left[y\dot{V}_a\right]}{\partial y} = \bar{\delta}'\left[\hat{T}_1 - \sum_{m=2}^{N} (2m-1)(y)^{2m-2}\hat{T}_m\right]\delta. \tag{4.85}$$

From the argument presented in the last section, the bracketed matrix in Equation (4.85) is negative definite. Thus $y\dot{V}_a(\delta, y)$ is monotonic and decreasing in y, and for any $\delta \neq 0$,

$$\begin{aligned} y_o\dot{V}_a(\delta, y_o) &> (y_o+y_1)\dot{V}_a(\delta, y_o+y_1), \quad y_1 > 0, \\ y_o\dot{V}_a(\delta, y_o) &< (y_o+y_1)\dot{V}_a(\delta, y_o+y_1), \quad y_1 < 0. \end{aligned} \tag{4.86}$$

This result proves that the right side of Equation (4.83) is negative for $y_1 \neq 0$, and thus $\dot{V}_f(\delta, y_o, y_1)$, is negative for all $y_o, y_1 \neq 0$, and $\delta \neq 0$.

It remains to be shown that $\dot{V}_f(\delta, y_o, 0) < 0$ for all y_o, $\delta \neq 0$. This can be accomplished by proving that the matrix $\hat{T}(y_o, 0)$ is negative definite.

$$\hat{T}(y_o, y_1) = \hat{T}_1 - \sum_{m=2}^{N} \frac{(y_o+y_1)^{2m-1} - (y_o)^{2m-1}}{y_1}\hat{T}_m. \tag{4.87}$$

Expanding the binomial $(y_o+y_1)^{2m-1}$ in this equation yields

(4.88)

$$\hat{T}(y_o, y_1) = \hat{T}_1 - \sum_{m=2}^{N}\left[\binom{2m-1}{1} y_o^{2m-2} + \binom{2m-1}{2} y_o^{2m-3} y_1 + \cdots + \binom{2m-1}{2m-1} y_1^{2m-1}\right]\hat{T}_m.$$

Thus,

$$\hat{T}(y_o, 0) = \hat{T}_1 - \sum_{m=2}^{N} (2m-1) y_o^{2m-2}\hat{T}_m. \tag{4.89}$$

As before, the argument in the last section shows that $\hat{T}(y_o, 0)$ is negative definite.

Consequently,

$$\dot{V}_f(\delta, y_o, y_1) < 0, \quad \delta \neq 0 \tag{4.90}$$

and therefore, each forced trajectory of the system, defined by equations (4.21) and (4.22), is asymptotically stable relative to the class of all trajectories if $\bar{\phi}' \phi$ is a Lyapunov function of the autonomous system.

Summary of Results

The above investigations are concerned with the nonlinear system

$$\dot{\phi} = \left[C_\lambda - \left(\sum_{m=1}^{N} (f_e)^{2m-2} C_m\right) U_1\right] \phi \tag{4.8}$$

where

$$f_e = f_i - f_f \tag{4.3}$$

and

$$f_f = \sum_{k=1}^{K} \phi_k \tag{4.4}$$

Employing the transformation $\phi = \psi\phi^*$ ($U_1 \psi = U_1$) to diagonalize the matrix $\left[C_\lambda - C_1 U_1\right]$ corresponding to the system first approximation,

$$\dot{\phi}^* = \left[D_1 - \left(\sum_{m=2}^{N} (f_e)^{2m-2} D_m\right) U_1\right] \phi^* \tag{4.19}$$

where, for the autonomous system,

$$f_e = -\sum_{k=1}^{K} \phi_k^*. \tag{4.20}$$

The matrices D_m are defined by the relationships

$$D_1 = \psi^{-1} \left[C_\lambda - C_1 U_1 \right] \psi \tag{4.13}$$

and

$$D_m U_1 = \psi^{-1} C_m U_1, \; m = 2, \text{---}, N \tag{4.17}$$

Assuming the diagonal elements λ_k of C_λ are distinct from the diagonal elements τ_k of D_1, the (k, j) element of ψ is of the form $\mu_{kj} = \rho_{kl} / (\lambda_k - \tau_j)$. If $\lambda_k = \tau_j$, the corresponding μ_{kj} is indeterminant. Any indeterminant μ_{kj} can be evaluated by the method presented in Appendix II.

The principal stability results obtained in this chapter are summarized in the following theorem, which considers the system (4.19) after a scaling transformation $\phi^* = D_s \phi_s$.

THEOREM 6 - The autonomous nonlinear system

$$\dot{\phi}_s = \left[D_1 - \sum_{m=2}^{N} (f_e)^{2m-2} D_s^{-1} D_m U_1 D_s \right] \phi_s \tag{4.91}$$

has a Lyapunov function of the form $\overline{\phi}_s' \phi_s$ if and only if

(i) $\left[\overline{D}_1 + D_1 \right]$ is negative definite.

(ii) D_N is positive definite.

(iii) $\left| -\hat{T}_s(f_e) \right| = \left| -\overline{D}_1 - D_1 + \sum_{m=2}^{N} (f_e)^{2m-2} \left(\overline{D}_s U_1 \overline{D}_m \overline{D}_s^{-1} + D_s^{-1} D_m U_1 D_s \right) \right| > 0$

for all f_e and for at least one D_s where $\overline{D}_s D_s = D_N$.

Furthermore, if the system (4.91) has a Lyapunov function $\overline{\phi}_s' \phi_s$, then (1) the system equilibrium point is asymptotically stable in-the-large, and (2) each forced system trajectory is asymptotically stable relative to the class of all trajectories.

For computational purposes, condition (iii) can be re-expressed as

$$\theta_K(f_e) - \sum_{k=1}^{K-1} \bar{\theta}_k^*(f_e)\,\theta_k^*(f_e) > 0 \qquad (4.66)$$

where the left side of (4.66) is a polynomial in f_e^2. The inequality is satisfied, therefore, if this polynomial has no real roots. Computational steps which simplify evaluation the polynomial are presented in the text.

CHAPTER V

SYSTEM DESIGN

The previous chapters have presented the formal steps required to establish a nonlinear system which yields the specified forced responses. There are many other factors, not accounted for by the formal procedure, which must be considered to achieve a practical design. For example, a compensator with several parallel paths can have a form of instability resulting from drift. The compensator must, therefore, be realized in a configuration which avoids this problem.

The purpose of this chapter is to summarize the design procedure and examine some of the additional factors necessary to reduce the method to a practical tool. Since no single design procedure is suitable for all problems, an attempt is made in the following to point out some of the possible variations of the present technique. The first two sections of this chapter consider the exact design procedure and an approximate extension of the method. Following this, an alternate approach to the design problem, in which the nonlinear system is specified in terms of a small-signal transfer function about each of N operating points, is discussed. The chapter concludes with several example problems.

Design Procedure

The design procedure starts with the system specifications which consist of the fixed elements, the sets $\left[f_{in}(t)\right]$ and $\left[f_{on}(t)\right]$ and possibly a transfer function $Y_s(s)$ for small-signal performance. The

following procedure applies to any of the compensators discussed in the preceding work.

1. The first step is to compute the compensator inputs $f_{en}(t)$ and corresponding responses $f_{cn}(t)$, using equations (2.4) and (2.8). In practice, there is often some freedom to vary the specifications. This freedom can be utilized in many cases to reduce the complexity of the final system. Minor variations in the $f_{in}(t)$ and $f_{on}(t)$ may reduce the number of terms in the $f_{en}(t)$ and $f_{cn}(t)$, with a consequent simplification of the compensator. For example, when the output element is linear, a simplification of $f_{cn}(t)$ is obtained by utilizing at least some of the poles of $H_o(s)$ in the poles of the $F_{on}(s)$.

2. Next, a suitable compensator configuration and a set of nonlinear elements $Z_m(f_e)$ must be selected. The elements $Z_m^2(f_e) = (f_e)^{2m-1}$ tend to yield the lowest order linear elements $H_m(s)$. While exceptions can be found, the added complexity of the $Z_m(f_e)$ tends to offset any simplification obtained. On the other hand, the selection made for the $Z_m(f_e)$ will strongly influence the over-all characteristics of the nonlinear system. In a given problem, some trial-and-error changes may be required to obtain a satisfactory design. The compensator in Figure 3.2 represents a possible alternate choice.

3. If a transfer function $Y_s(s)$ is included in the system specifications, the small-signal compensator function $G^s(s)$ can be computed with Equation (3.28). For the nonlinear elements $Z_m^2(f_e)$, the constraint $H_1^c(s) = G^s(s)$ is placed on the compensator.

4. For each $f_{en}(t)$, the corresponding responses $\gamma_{nm}(t)$ of the elements $Z_m(f_e)$ must be computed; for example $\gamma_{nm}(t) = f_{en}(t)^{2m-1}$ $(n, m = 1, ---, N)$.

5. At this point, the Laplace image $\Gamma_{nm}(s)$ for each $\gamma_{nm}(t)$ and the image $F_{cn}(s)$ for each $f_{cn}(t)$ can be evaluated. These are always straightforward transformations since each time function is an element of class $\mathcal{R}$.

6. Using the $\Gamma_{nm}(s)$ and $F_{cn}(s)$, the set of equations (3.1) (or 3.29), can be formulated as the vector equation $M\xi_H = \xi_F$ $(M^c\ \xi_H^c = \xi_F^c)$. The unknown linear elements can then be computed by determining the inverse of M (M^c) and solving $\xi_H = M^{-1}\ \xi_F$ $(\xi_H^c = (M^c)^{-1}\ \xi_F^c)$.

7. Since there is no assurance that this technique will yield a successful design, the final step of the procedure is to test the resulting system to determine if all specified and implied design goals have been achieved. In some cases, the stability of the system can be examined by the methods presented in Chapter IV.

The above approach to design is basically a method of coercing the system to meet the specifications. The coercion, furthermore, is not in the least subtle --the compensator is required to cancel any undesired characteristics of the fixed elements. In this respect the present method is similar to Giullimen's approach to linear system design. In fact, when the fixed elements are linear and N = 1, it reduces to Guillimen's method, except that computations are made in the time domain.

Several objections can be raised to the coercion or cancellation approach to system design. The primary objection is that it is impossible to cancel certain characteristics of the fixed elements and then expect the final system to yield anything like the predicted performance. This argument rests on the supposition that the original description of the fixed elements is not precise, having neglected various time lags and nonlinearities. On the whole, this is not a valid objection, as long as the attempted cancellations do not cause a stability problem. It simply implies that either the description of the fixed elements must be refined or that the elements chosen at the outset are not adequate to do the job. Regardless of the method, suitable fixed elements must be employed and accurate models are required for design.

A second objection to the cancellation method of compensation is the complex compensator networks that often result. Actually, this is not as strong an objection as it might appear. In the first place, the synthesis of complex networks is not especially troublesome. In addition, in many cases it will be possible to reduce the order of the linear elements $H_m(s)$, for example, by cancelling poles and zeros which lie in close proximity. Such cancellations are only permissible of course, if they do not introduce new stability problems. In any case, the fact that a complex compensator is required is more an indication of the difficulty of the problem being solved than a fault of the method.

An important advantage of the present time domain method of design is that it provides a means of determining when the fixed output element is not compatible with the specified responses. The problem which usually arises is some form of saturation within the fixed element. With a time domain approach, the signal at each point within the fixed element is computed as part of the procedure. Thus, if the magnitude of any signal exceeds the known saturation level, a new fixed element must be sought (or the requirements reduced).

As mentioned earlier, a compensator which employs several parallel paths can have a form of instability. This, of course, depends on the configuration employed in the compensator realization. As an illustration, suppose the compensator is realized in the form shown in Figure 3.1. If two or more of the functions $H_m(s)$ contain a pole λ_a with $\text{Re}(\lambda_a) \geq 0$, the "unstable paths" can drift simultaneously without producing a net effect on f_c. The system, therefore, cannot generate a corrective action for this drift, and these paths will eventually saturate, causing a change in performance characteristics. Fortunately, the compensator can be realized in a number of forms which avoid this problem. The configuration discussed next is of this type.

In synthesizing the compensator, it is desirable to utilize the minimum number of dynamic elements. The schematics used previously for illustration would be an extravagant way to realize a compensator, since the $H_m(s)$ will, in general, have many common poles. This can be seen by inspection of the equations $H_m(s) = \left| M_m(\xi_F) \right| / \left| M \right|$. If the N linear elements have Q distinct poles, only one set of these is required. The remaining operations can be accomplished by introducing coefficient gains and summing junctions. For example, the compensator configuration illustrated in Figure 5.1 can be used with $Z_m^2(f_e) = (f_e)^{2m-1}$.

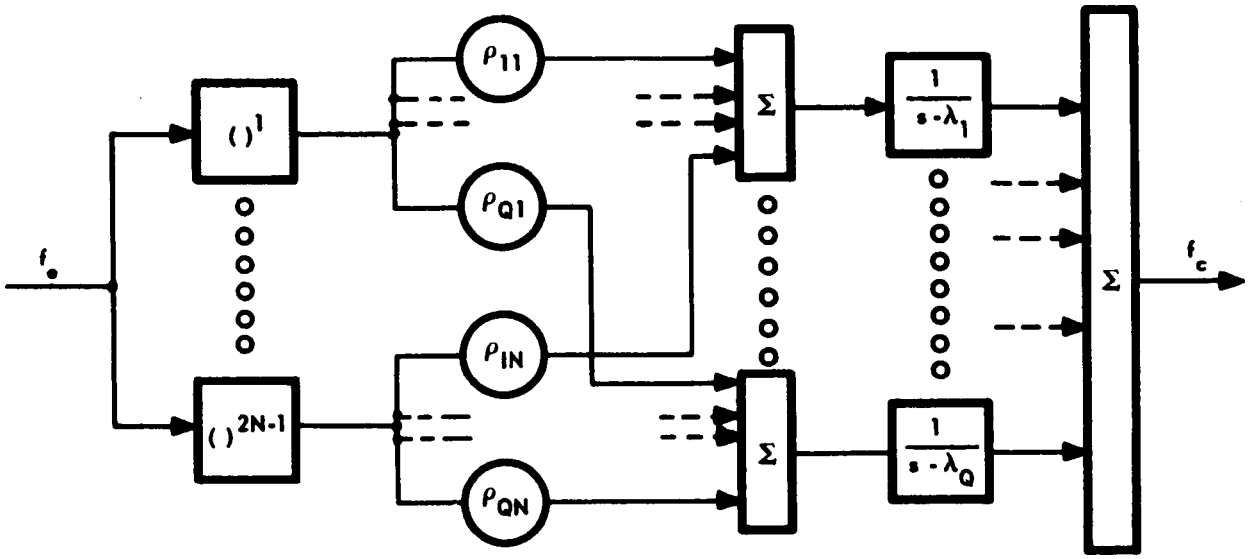


Figure 5.1 - Diagram for Realization of Nonlinear Compensator

Approximate Method of Design

In many important design problems, the system specifications cannot be placed in a form which satisfies the preliminary assumptions stated in Chapter II. Most often, the fixed output element cannot be adequately represented by a single nonlinear differential equation of the type defined in assumption (3). If the restriction on the output element is removed, it still may be possible to obtain an approximate design. With the method discussed here there is no way of evaluating the approximation errors; however, the results appear to be sufficiently accurate for many purposes. Even when the system does not meet all specifications, this approach will often provide an initial design on which to base refinements.

While no direct restrictions will be placed on the system output element, there is one important requirement --an $f_{cn}(t)$ must exist which will produce the prescribed response $f_{on}(t)$. This limits the admissible class of responses for certain nonlinear output elements. For example, an on-off element can only produce a pulse-wave type

of response; and the response of a simple saturation element must be less than the level of saturation. If this requirement is not satisfied, the specifications are inconsistent and must be changed.

Steps (1) through (4) of the previous design procedure also apply here. In this problem, however, computer techniques may be required to evaluate the $f_{cn}(t)$, since the solution of forced nonlinear differential equations may be involved. If a $Y_s(s)$ is specified, the required small-signal compensator function can again be calculated exactly. The exception, of course, occurs when the output element does not have a linear approximation about equilibrium, as for example, when it contains coulomb friction. Evaluation of the $\gamma_{nm}(t)$ proceeds as before since each $f_{en}(t)$ is still a time function of class $\mathcal{R}$.

To this point, the design procedure is essentially unchanged. Now, however, the $H_m(s)$ defined by the $\Gamma_{nm}(s)$ and $F_{cn}(s)$ are not, in general, rational functions of finite order. The next step is, therefore, to approximate each $H_m(s)$ with a rational function $H_m^a(s)$ of arbitrary order. It would be desirable if this approximation could be carried out by minimizing some measure of the error between the $f_{on}(t)$ and the actual responses which result when the $H_m^a(s)$ are employed. In a closed-loop nonlinear system, this approach is not feasible; some empirical method must be used. Several suitable methods can be found. The most direct approach is simply to approximate each $f_{cn}(t)$ with an element of the class $\mathcal{R}$. Since the $\Gamma_{nm}(s)$ are rational functions of s, the design can then proceed as before. One method of accomplishing the required approximation is to employ the integral-square error criterion. The error $e_n(t)$ is defined as

$$e_n(t) = f_{cn}(t) - f_{cn}^a(t) . \tag{5.1}$$

The approximate function $f_{cn}^a(t)$ is chosen to minimize the integral I_n.

$$I_n = \int_{T_1}^{T_2} e_n^2(t)\, dt$$
$$= \int_{T_1}^{T_2} \left[f_{cn}(t) - f_{cn}^a(t)\right]^2 dt \,, \tag{5.2}$$

where $T_1 \leq t \leq T_2$ is the interval of approximation.

The integral-square error method of approximation is well known. Gilbert (9) has investigated the accuracy and limitations of the method and has developed analytic and analog computer techniques for the above case, and for more general approximation problems. Therefore, the steps of the approximation procedure will not be recounted here.

When N is large, the lengthy computations required will become onerous, and the accumulated approximation errors may lead to an unacceptable design. However, for one or two input-response pairs, this method appears to yield designs suitable for most engineering purposes. In many cases, no alternate approach is available, even for the single-input problem.

Alternate Methods of Design

In all of the above investigations, the specifications have defined the required nonlinear system in terms of a set of input-response pairs. This section considers briefly a second method in which the specifications are stated in terms of a required small-signal performance about each of N operating points. Compensators similar to those developed in Chapter III can again be utilized to obtain the desired system. This type of design problem is most readily explained using the system configuration illustrated in Figure 5.2.

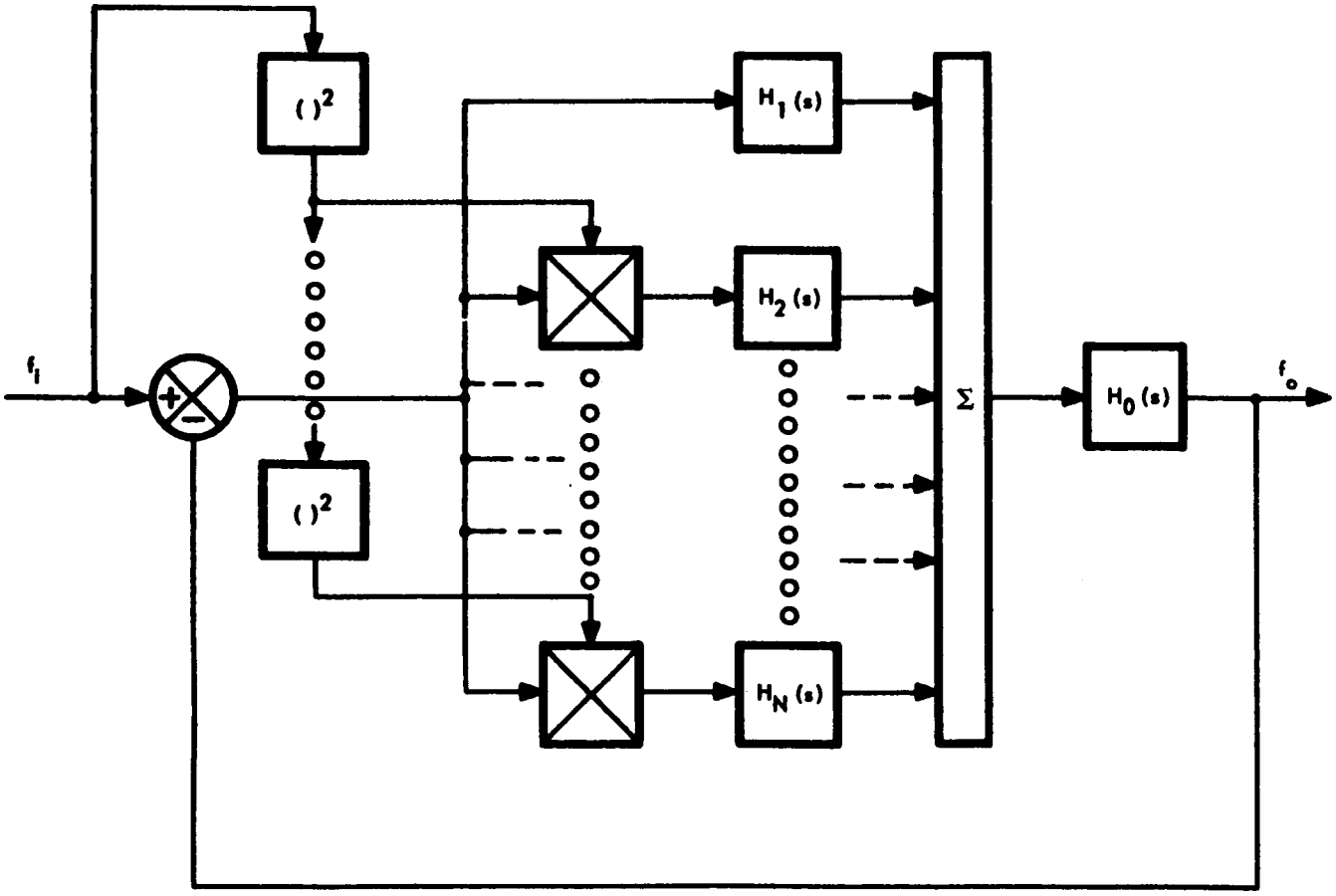


Figure 5.2 - System Configuration for Small-Signal Design

In this system, the compensator again consists of N parallel paths, each containing a nonlinear element of the form $(f_i)^{2m-2} f_e$. Thus, the path "gain" is now a function of the amplitude of the system input. The goal of this design problem is to choose the linear elements $H_m(s)$ such that for small perturbations about the operating point $f_i = R_n$ $(n = 1, ---, N)$, the over-all system has a small-signal transfer function $Y_{sn}(s)$.

Let $Y_{on}(s)$ be the open-loop transfer function corresponding to $Y_{sn}(s)$.

$$Y_{on}(s) = \frac{Y_{sn}(s)}{1 - Y_{sn}(s)}, \quad n = 1, ---, N. \tag{5.3}$$

Assuming $\lim_{s \to 0} \left[s H_o(s) \right] = 0$ (i.e., the system steady-state error is zero at each operating point), the transfer function $Y_{on}(s)$ is given by

$$H_o(s)\left[(R_n)^{2N-2} H_N(s) + \ldots + H_1(s)\right] = Y_{on}(s), \quad n = 1, \ldots, N. \tag{5.4}$$

These equations can be expressed in matrix notation as

$$M_R \, \xi_H = \xi_Y \tag{5.5}$$

where $M_R = \left[(R_n)^{2m-2} \right]$ and

$$\xi_H = \begin{bmatrix} H_1 \\ \vdots \\ H_N \end{bmatrix}, \qquad \xi_Y = \begin{bmatrix} Y_{o1}/H_o \\ \vdots \\ Y_{oN}/H_o \end{bmatrix}. \tag{5.6}$$

From previous results, the matrix M_R is nonsingular since the R_n are distinct. Hence, the required $H_m(s)$ can be computed with

$$\xi_H = M_R^{-1} \, \xi_Y . \tag{5.7}$$

For a constant f_i the above system is linear and stability can, therefore, be investigated with any of the techniques for analysis of linear systems. The earlier results on stability can be applied with little modification to the investigation of the forced system. However, the forced system probably offers few practical stability problems of the free system is stable for all constant values of f_i.

Several variations on the above method of design are possible. For example, the operating point could be defined in terms of system output level. Suitable compensator configurations can easily be devised for the alternate cases.

Examples

The methods developed in the preceding work are illustrated here by a group of simple examples. Each example shows some characteristic of this approach to the design of nonlinear systems. The example specifications are summarized in Table I at the end of the section. Table II lists the elements of the resulting nonlinear compensators. All of the following examples consider unity feedback systems.

Example 1 - In this problem the specifications call for a 10:1 reduction in transient response time when the amplitude of a step function input is increased from unity to four. The transition in system step response is shown in Figure 5.3. Since the system error is zero at rest, the same result is obtained about any initial output level. This system is characterized by an increase in responsiveness with error amplitude.

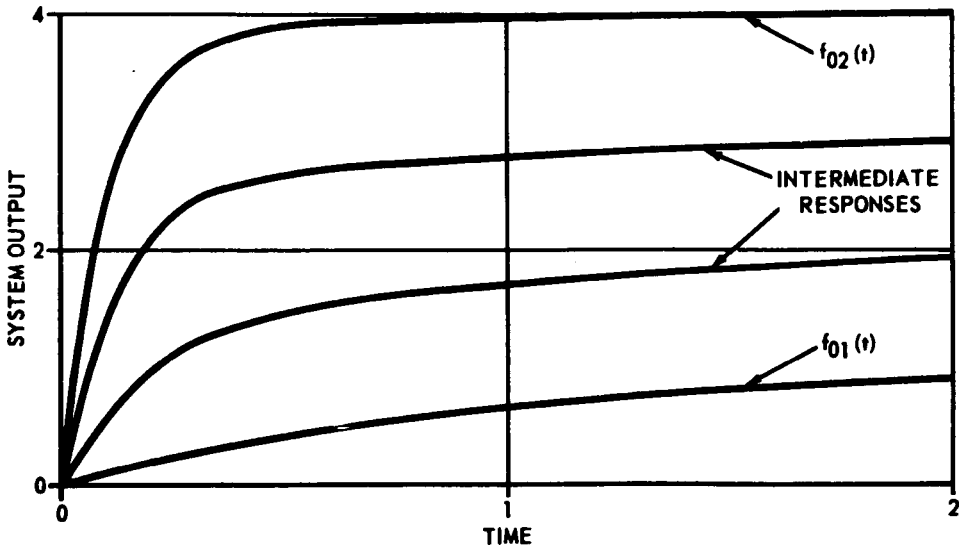


Figure 5.3 - Transition in System Step Response

The system does not have a Lyapunov function of the type developed in Chapter IV. However, experimentation with an analog

simulation failed to show any tendency toward instability, within the range of the simulation. The linear elements $H_m(s)$ in this system, as in all of the following examples, are physically realizable.

Examples 2 and 3 - Example 2 presents a problem in which computation shows the system equilibrium point to be unstable. When this situation arises, the design must be modified, either by selecting a different set of nonlinear elements or by altering the specifications. In this problem, the specifications require a marked change in system performance for a relatively small change of the input signal. As the demand for an abrupt transition in system performance is reduced, for example by reducing the amplitude of the smaller input-response pair, the equilibrium point will eventually become stable. Of course, this approach will not work for all problems --in each case the trends must be established.

Example 3 considers the limiting condition in which the amplitude is reduced to zero, and thus $F_{i1}(s)/F_{o1}(s) = Y_s(s)$. The resulting linear elements are

$$H_1^c(s) = 1,$$

$$H_2^c(s) = \frac{0.5\,(s + 11)\,(s^2 + 18\,s + 97)\,(s^2 + 18\,s + 225)}{(s^2 + 6\,s + 25)\,(s + 9)\,(s^2 + 18\,s + 194)} .$$

The close proximity of certain poles and zeros of $H_2^c(s)$ indicates that considerable simplification is possible. Figure 5.4 illustrates the over-all approximation error that results with

$$H_2^c(s) \cong 0.708\,\frac{s^2 + 18\,s + 97}{s^2 + 6\,s + 25} .$$

The simplified system is clearly adequate for most purposes. Neither the exact nor the simplified system has a Lyapunov function of the

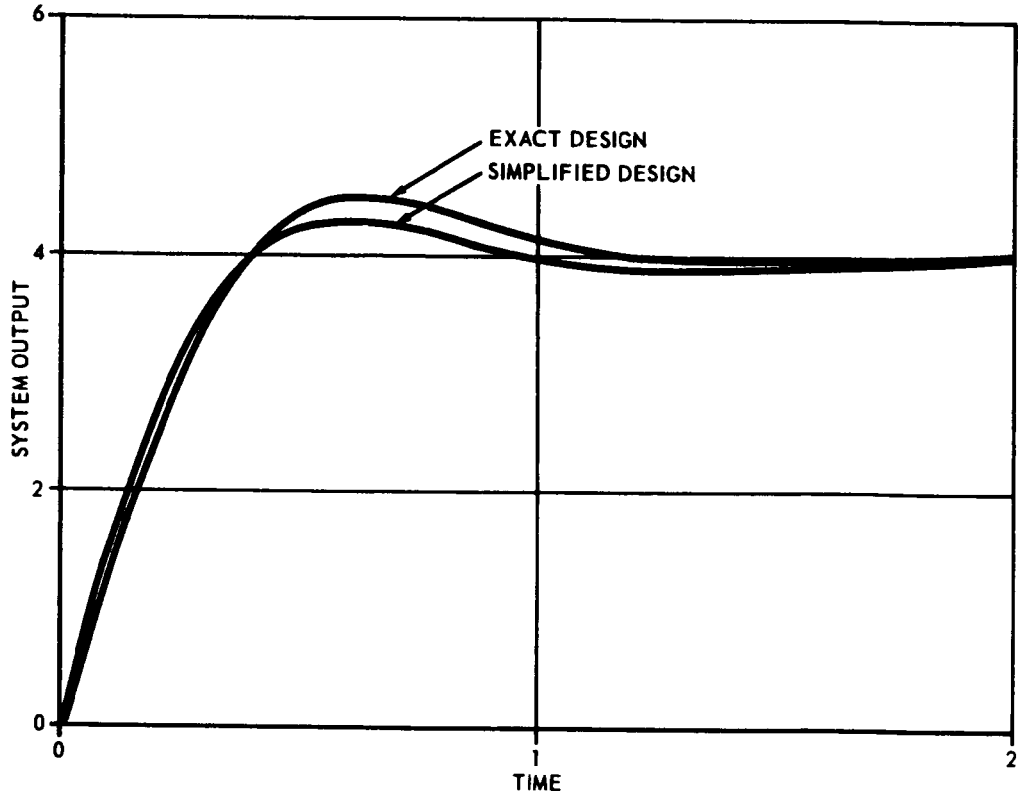


Figure 5.4 - Effect of Compensator Simplification

type discussed. However, again no stability problems were observed in the analog simulation.

Example 4 - Examples in which the final system is asymptotically stable in-the-large can easily be found. It is interesting to examine a general problem to show the bounds within which the system will have this stability. In this example, the linear elements are

$$H_o(s)\, H_1^c(s) = \frac{as + b}{s(s + c)},$$

$$H_o(s)\, H_2^c(s) = \frac{(s + 3a)\left[(a - a)\, s + ac - b\right]}{d^2\, s\, (s + a)\, (s + c)}.$$

Applying the methods of Chapter IV, it can be shown that the third-order system resulting with these elements fails to have the desired

Lyapunov function. However, if $a(a - a) = a\,c - b$, then $H_o(s)\,H_2^c(s)$ reduces to

$$H_o(s)\,H_2^c(s) = \frac{(a\,c - b)\,(s + 3\,a)}{a^2 d^2 s(s + c)}.$$

The system is now second order and stable for some range of the parameters. To establish this range, first note that with the above constraint the transfer function for the linearized system becomes

$$Y(s) = \frac{a\,s + b}{s^2 + (a + c)\,s + b},$$

$$= \frac{a\,s + b}{(s + a)\,(s + \frac{b}{a})}.$$

The equilibrium point is stable, therefore, if

$$a > 0,$$

$$b > 0.$$

The remaining requirements are found by establishing the conditions for which the matrix D_2 is positive definite. This matrix is defined by the relationship

$$\psi^{-1}\,C_2\,U_1 = D_2\,U_1.$$

For the present case,

$$\psi = \begin{bmatrix} \frac{b}{a\,c} & \frac{a}{c} \\ \frac{a\,c - b}{a\,c} & \frac{c - a}{c} \end{bmatrix}, \quad C_2 = \begin{bmatrix} \frac{3(a\,c - b)}{d^2(b - a^2)} & 0 \\ 0 & \frac{(c - 3\,a)(a\,c - b)}{a^2 d^2 (b^2 - a^2)} \end{bmatrix}.$$

Therefore,

$$D_2 = \begin{bmatrix} \frac{2(ac - b)}{d^2(b - a^2)} & 0 \\ 0 & \frac{(ac - b)(b - 3a^2)}{a^2 d^2 (b^2 - a^2)} \end{bmatrix}.$$

The matrix D_2 is positive definite if

$$b > 3a^2,$$

$$ac > b \quad .$$

In summary, the system is asymptotically stable in-the-large if

$$a > 0,$$

$$b > 3a^2,$$

$$c > \frac{b}{a},$$

$$a = a - c + \frac{b}{a}.$$

The parameter d is arbitrary. With knowledge of these conditions, a design procedure could be devised which assures the stability of the final system. The task becomes one of finding an acceptable design within the four constraints. It would be desirable if similar results could be obtained for higher values of N and K. Unfortunately, with the present method the constraints become enormously complex.

Examples 5 and 6 - As mentioned in Chapter III, the nonlinear compensator can produce extraneous equilibrium points. The system in Example 5 shows this conditions: it has equilibrium points at $f_e = 0, \pm \sqrt{250/18}$. If the unwanted points of equilibrium were located far beyond the saturation levels in the compensation network, it is conceivable that the design would be acceptable. In the present case, however, they are relatively close to the origin and must be eliminated; either by altering the specifications or the compensator configuration.

Example 6 repeats the design utilizing the compensator illustrated in Figure 3.2 to produce a unique equilibrium point. An analog simulation of the system showed satisfactory performance and no indication of stability problems, within the range of the simulation. However, the system will likely enter an unstable mode of operation when the error amplitude is not appropriately bounded by saturation, since the gain coefficient of $H_2(s)$ is negative.

Example 7 - This example considers a design in which the output element contains a nonlinear term $\dot{f}_o f_o$, and the specified input-response pairs are of the form $f_{i1}(t) = -f_{i2}(t)$ and $f_{o1}(t) = -f_{o2}(t)$. With the compensator listed in Table II, the system and its forced trajectories appear to have acceptable stability characteristics.

Systems of this type have an interesting characteristic -- the small signal transfer function is automatically of the form $Y_s(s) = F_{o1}(s)/F_{i1}(s) = F_{o2}(s)/F_{o2}(s)$. This result can always be obtained if each of the nonlinear terms in the output element is a product of even order, for example $\ddot{f}_o(\dot{f}_o)^2 f_o$. The remaining requirements are that the input-response pairs must appear in odd sets, as above, and that the nonlinear compensator elements are chosen as $Z^4_m(f_e) = (f_e)^{2m}$. The reasoning can be shown as follows: for the prescribed output $F_{on}(s)$, let $F_{cn}(s)$ be divided into two parts --$F^1_{cn}(s)$ and $F^2_{cn}(s)$, corresponding respectively to the quantities resulting from the linear and nonlinear terms of the output element. For any two odd input-response pairs say (f_{i1}, f_{o1}) and (f_{i2}, f_{o2}), the corresponding equations of the set (3.1) are then

$$\Gamma_{11} H_1 + \Gamma_{12} H_2 + \ldots\ldots + \Gamma_{1N} H_N = F^1_{c1} + F^2_{c1},$$

$$-\Gamma_{11} H_1 + \Gamma_{12} H_2 + \ldots\ldots + \Gamma_{1N} H_N = -F_{c1} + F^2_{c1}.$$

Subtracting these equations yields:

$$H_1 = \frac{F_{c1}^{1}}{\Gamma_{11}} = \frac{F_{c1}^{1}}{F_{e1}} \quad .$$

Since $H_1(s)$ is the small-signal transfer function for the compensator, the over-all system will have a small-signal transfer function

$$Y_s = \frac{F_{o1}}{F_{i1}} = \frac{F_{o2}}{F_{i2}} \quad .$$

Example 8 - The approximate method of design discussed earlier is employed here to solve a single input problem. The differential equation of the output element, which represents a valve-controlled hydraulic motor, is such that $f_{c1}(t)$ is not an element of $\mathcal{R}$. Therefore, an approximation f_{c1}^{a} was computed using the orthonormal-function method of approximation described by Gilbert (9). In this approach, $f_{c1}^{a}(t)$ is expressed as

$$f_{c1}^{a}(t) = a_1 \omega_1(t) + a_2 \omega_2(t) + a_3 \omega_3(t) \quad ,$$

where the functions $\omega_i(t)$ are sums of exponentials. For this problem, the $\omega_i(t)$ were selected to have the Laplace images

$$\Omega_1(s) = \frac{11.32}{s + 3.20} \quad ,$$

$$\Omega_2(s) = \frac{12.24\,(s - 3.20)}{(s + 3.20)(s + 3.74)} \quad .$$

$$\Omega_3(s) = \frac{4.24\,(s - 3.20)(s - 3.74)}{(s + 3.20)(s + 3.74)(s + 7.92)} \quad .$$

In the computer solution of the approximation problem, the time scale was reduced by a factor of 20, in which case the $\omega_i(t/20)$ are orthonormal. The approximation problem is solved by computing the a_i which minimize the integral-square error. The resulting linear compensator $H_1^{a}(s) = F_c^{a}(s)/F_e(s)$ is listed in Table II.

Figure 5.5 compares the specified and actual response of system. Since care was taken to insure that the nonlinearities in the output element have a significant effect, the result indicates that this approximate method of design can be quite accurate.

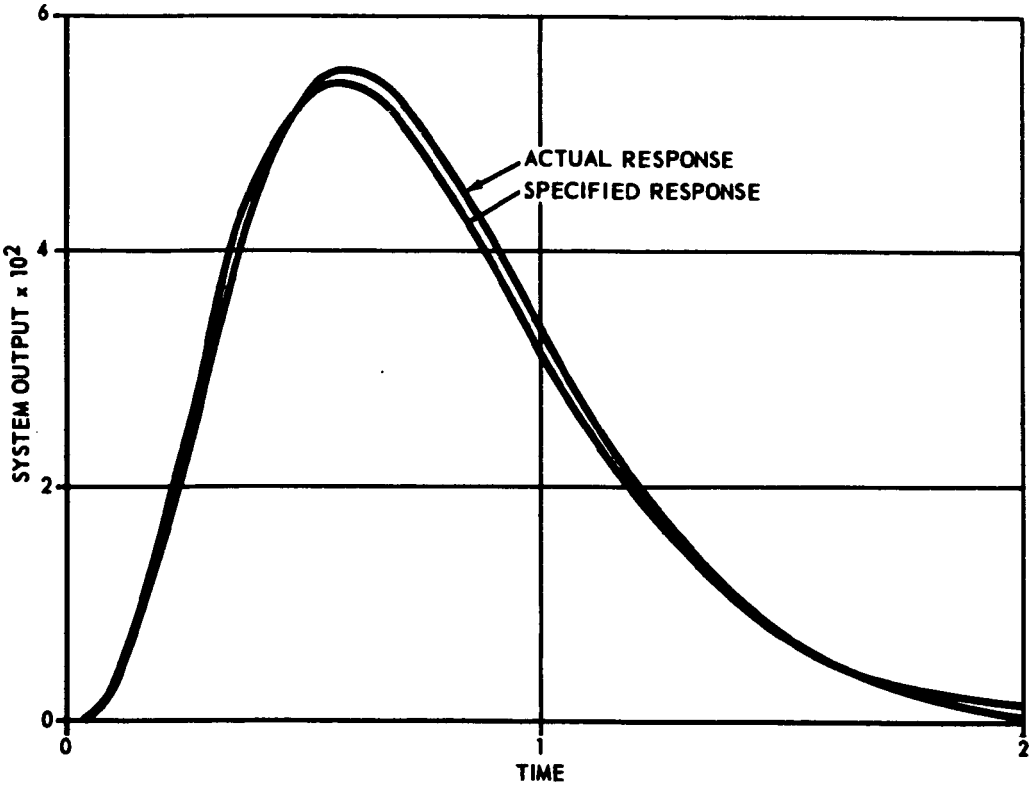


Figure 5.5 - System Response Obtained with Approximate Design

Table 1 - Example Specifications

EXAMPLE	$Y_s(s)$	$f_{i1}(t)$, $f_{o1}(t)$	$f_{i2}(t)$, $f_{o2}(t)$	FIXED OUTPUT ELEMENT
1	———	$f_{i1}(t) = \mu_o(t)$ $f_{o1}(t) = \mu_o(1-e^{-t})$	$f_{i2}(t) = 4\,\mu_o(t)$ $f_{o2}(t) = \mu_o(4-e^{-10t})$	$H_o(S) = \frac{1}{s}$
2	———	$f_{i1}(t) = \mu_o(t)$ $f_{o1}(t) = \mu_o(1-e^{-t})$	$f_{i2}(t) = 2\,\mu_o(t)$ $f_{o2}(t) = \mu_o(2-2e^{-3t}\cos 4t)$	$H_o(S) = \frac{1}{s}$
3	$\frac{1}{s+1}$	———	$f_{i2} = 2\,\mu_o(t)$ $f_{o2}(t) = \mu_o(2-2e^{-3t}\cos 4t)$	$H_o(S) = \frac{1}{s}$
4	$\frac{as+b}{s^2+(a+c)s+b}$	———	$f_{i2}(t) = d\,\mu_o(t)$ $f_{o2}(t) = \mu_o(d-de^{-at})$	$H_o(S) = \frac{1}{s}$
5	$\frac{5}{s+5}$	———	$f_{i2}(t) = 5\,\mu_o(t)$ $f_{o2}(t) = \mu_o(5-5e^{-2t})$	$H_o(S) = \frac{1}{s}$
6	$\frac{5}{s+5}$	———	$f_{i2}(t) = 5\,\mu_o(t)$ $f_{o2}(t) = \mu_o(5-5e^{-2t})$	$H_o(S) = \frac{1}{s}$
7	———	$f_{i1}(t) = \mu_o(t)$ $f_{o1}(t) = \mu_o(1-2e^{-t}+e^{-2t})$	$f_{i2}(t) = -\mu_o(t)$ $f_{o2}(t) = \mu_o(-1+2e^{-t}-e^{-2t})$	$\ddot{f}_o + \dot{f}_o + 4\,\dot{f}_o\,f_o = f_c$
8	———	$f_{i1}(t) = \mu_o(22.20\,e^{-4t}-22.18)$ $f_{o1}(t) = \mu_o(208.3e^{-4t}-621.9e^{-4.8t}$ $+621.9e^{-5.6t}-208.3e^{-6.4t})$	———	$\dddot{f}_o + 80(1-X)\sqrt{2+\ddot{f}_o} - 80(1+X)\sqrt{2-\ddot{f}_o} + 400\,\dot{f}_o = 0$ $f_c = X + 4X^3$

Table II - Nonlinear-Compensator Elements

EXAMPLE	ARBITRARY ELEMENTS	$H_1(s)$, $H_1^c(s)$, $H_1^a(s)$	$H_2(s)$, $H_2^c(s)$
1	$Z_1(f_e) = f_e$ $Z_2(f_e) = f_e^3$	$H_1(s) = \dfrac{0.4(s-2)(s-15)}{(s+3.708)(s+8.092)}$	$H_2(s) = \dfrac{0.6(s+3)(s+30)}{(s+3.708)(s+8.092)}$
2	$Z_1(f_e) = f_e$ $Z_2(f_e) = f_e^3$	$H_1(s) = \dfrac{s^6 + 8s^5 - 403s^4 - 11632s^3 - 112725s^2 - 350248s - 21825}{3s^6 + 104s^5 + 1175s^4 + 16256s^3 + 85241s^2 + 302424s + 458325}$	$H_2(s) = \dfrac{s^6 + 48s^5 + 1089s^4 + 13949s^3 + 98474s^2 + 325557s + 240075}{3s^6 + 104s^5 + 1175s^4 + 16256s^3 + 85241s^2 + 302424s + 458325}$
3	$Z_1(f_e) = f_e$ $Z_2(f_e) = f_e^3$	$H_1^c(s) = 1$	$H_2^c(s) = \dfrac{0.5(s+11)(s^2+18s+225)(s^2+18s+97)}{(s^2+6s+25)(s+9)(s^2+18s+194)}$
4	$Z_1(f_e) = f_e$ $Z_2(f_e) = f_e^3$	$H_1^c(s) = \dfrac{as+b}{s+c}$	$H_2^c(s) = \dfrac{(s+3a)[(a-a)s+(ac-b)]}{d^2(s+a)(s+c)}$
5	$Z_1(f_e) = f_e$ $Z_2(f_e) = f_e^3$	$H_1^c(s) = 5$	$H_2^c(s) = -\dfrac{3}{25}\dfrac{s+6}{s+2}$
6	$Z_1(f_e) = f_e$ $Z_2(f_e) = f_e^3$ $\bar{H}(s) = \dfrac{s}{s+3}$	$H_1^c(s) = 5$	$H_2^c(s) = -\dfrac{3}{25}\dfrac{(s+6)(s+7)(s+8)(s+9)}{(s+2)(s+3.42)(s^2+11.58s+52.56)}$
7	$Z_1(f_e) = f_e$ $Z_2(f_e) = f_e^2$	$H_1(s) = 2\dfrac{s+1}{s+3}$	$H_2(s) = \dfrac{48}{(s+1)(s^2+9s+22)}$
8	———	$H^a(s) = \dfrac{2.12(s+0.268)(s+4)(s+4.8)(s+5.6)(s+6.4)(s+59.8)}{s(s+3.2)(s+3.74)(s+7.92)(s+12)(s+892)}$	———

CHAPTER VI

CONCLUSION

The goal of the preceding work has been to establish new and practical methods for the design of nonlinear systems. Effort was concentrated on the development of time-domain techniques. For nonlinear systems, this seems the logical approach, since transient or forced response is, at present, the only acceptable measure of performance. Furthermore, the time-domain approach provides a means of determining if the specifications require performance beyond the capability of the output element, since the signal and saturation levels can be compared. The success that has been achieved in this investigation can be attributed primarily to the emphasis placed on the development of exact design tools. Many of the pitfalls inherent in approximate methods are avoided in this way. First, the difficult problem of determining approximation errors is avoided. In addition, the design procedure can be placed on a more or less systematic basis, reducing reliance on trial-and-error.

For the most part, attention has been focused on the general aspects of system design rather than on specific problem areas. Of the general results, the more important can be summarized as:

1. The development of a class of nonlinear compensators for the multiple input-response design problem.
2. The investigation of compensator characteristics and conditions sufficient for existence and realizability.

3. The formulation of a potential Lyapunov function for the system with linear fixed elements. A precise statement of conditions sufficient for the existence of this Lyapunov function.

4. A proof that the forced system trajectories are asymptotically stable relative to the class of all trajectories when the autonomous system has the above Lyapunov function.

5. The formulation of a systematic procedure for exact system design. An approximate extension of this technique to a broader class of problems.

While several limitations were necessary in the formulation of the above design tools, they do not seriously restrict the usefulness of the results. Many practical problems fall directly within the scope of the exact methods, and many others can be brought within range by suitable preliminary approximations. Should this fail, the approximate method may prove applicable.

The above results provide the nucleus of a design theory for nonlinear systems. However, many important and difficult problems remain. The most serious shortcoming of the method is that stability is not assured as part of the design process. No design tool which leaves stability unresolved can ever be completely satisfactory. In this respect, virtually all present methods for nonlinear design are deficient. Another limitation of the present method is the restriction to deterministic time functions. Knowledge of the instantaneous system error is essential to the design process. While the use of stochastic input signals is a more realistic approach to design, countless successful systems have been produced with the test-signal method. Finally, the required calculations become lengthy and the compensator tends to become complex when N is large, or when the specified inputs and responses contain a large number of terms.

These difficulties are primarily an indication of the complexity of the design problem. Moreover, as pointed out in the text, some simplification can often be obtained through judicious manipulation of the specifications or by approximations within the compensator.

There are numerous possibilities for further work in this area. Practically all of the present results could be extended in one direction or another. One attractive possibility is that of developing a more general class of potential Lyapunov functions. Work in this direction has shown considerable promise. Hopefully, this might lead to a method of design which assures system stability, if only for a small range of useful problems. Undoubtedly there are numerous refinements which would improve the practicality of the present results.

APPENDIX I

INVESTIGATION OF THE MATRIX D_ψ

The purpose of this appendix is to show that the diagonal matrix D_ψ is nonsingular and arbitrary. Equation (4.13) states:

$$\begin{aligned} D_1 &= \psi^{-1} C_\lambda \psi - \psi^{-1} C_1 U_1 \psi \\ &= \psi^{-1} C_\lambda \psi - \psi^{-1} C_1 U_1 D_\psi , \end{aligned} \tag{I.1}$$

where D_1 is a diagonal matrix with characteristic roots τ_k and C_λ is a diagonal matrix with characteristic roots λ_k.

First it will be shown that D_ψ is nonsingular. Suppose $\left|D_\psi\right| = 0$. It will be sufficient to assume that D_ψ has one zero element along the diagonal, say the first element. Then for any C_1, the first column of the matrix $\psi^{-1} C_1 U_1 D_\psi$ has zero elements, and the vector ξ_1 corresponding to the first column of $\psi^{-1} C_\lambda \psi$ must be, according to Equation (I.1),

$$\xi_1 = \begin{bmatrix} \tau_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} . \tag{I.2}$$

Now if the elements of ψ are denoted μ_{kj}, the vector ξ_2 corresponding to the first column $C_\lambda \psi$ is

$$\xi_2 = \begin{bmatrix} \lambda_1 \mu_{11} \\ \vdots \\ \lambda_K \mu_{K1} \end{bmatrix} . \tag{I.3}$$

A second expression for ξ_2 can be obtained by noting

$$\xi_1 = \psi^{-1} \xi_2 , \tag{I.4}$$

and thus:

$$\xi_2 = \psi \xi_1 = \begin{bmatrix} \tau_1 \mu_{11} \\ \vdots \\ \tau_1 \mu_{K1} \end{bmatrix} . \qquad (I.5)$$

Since the λ_k were assumed to be distinct, Equations (I.3) and (I.5) can both be valid only if (K - 1) of the μ_{k1} are zero. However, in this event the first element in D_ψ cannot be zero as assumed. The above result is therefore a contradiction. This argument can be repeated for any diagonal element of D_ψ, and therefore $\left|D_\psi\right| \neq 0$.

To show that D_ψ is arbitrary, it is sufficient to note that the column lengths in the nonsingular matrix ψ are arbitrary. Thus, the diagonal elements $\sum_{j=1}^{K} \mu_{kj} \neq 0$ (k = 1, - - -, K) of D_ψ are arbitrary.

APPENDIX II

EVALUATION OF THE MATRIX ψ

Knowledge of the form of the matrix ψ simplifies the numerical computations required in a stability analysis. The following shows that the transformation matrix ψ is of the form ($D_\psi = I$)

$$\psi = \begin{bmatrix} \frac{\rho_{11}}{\lambda_1 - \tau_1} & \frac{\rho_{11}}{\lambda_1 - \tau_2} & \cdots & \frac{\rho_{11}}{\lambda_1 - \tau_K} \\ \frac{\rho_{21}}{\lambda_2 - \tau_1} & \frac{\rho_{21}}{\lambda_2 - \tau_2} & & \vdots \\ \vdots & & & \vdots \\ \frac{\rho_{K1}}{\lambda_K - \tau_1} & \cdots & & \frac{\rho_{K1}}{\lambda_K - \tau_K} \end{bmatrix} . \qquad \text{(II.1)}$$

Letting $D_\psi = I$, Equation (I.1) from Appendix I can be written

$$D_1 = \psi^{-1} C_\lambda \psi - \psi^{-1} C_1 U_1 . \qquad \text{(II.2)}$$

Multiplying this equation by ψ yields

$$\psi D_1 = C_\lambda \psi - C_1 U_1 . \qquad \text{(II.3)}$$

If the elements of ψ are again denoted μ_{kj}, performing the indicated matrix multiplications in (II.3) gives

$$\psi D_1 = (\mu_{kj} \tau_j), \qquad \text{(II.4)}$$

$$C_\lambda \psi = (\lambda_k \mu_{kj}), \qquad \text{(II.5)}$$

$$C_1 U_1 = (\rho_{k1}) . \qquad \text{(II.6)}$$

Equating the (k, j) elements of the matrices in Equation (II.3),

$$\mu_{kj} \tau_j = \lambda_k \mu_{kj} - \rho_{k1} . \qquad \text{(II.7)}$$

Suppose $\rho_{kl} \neq 0$ $(k = 1, - - - , K)$, then

$$\mu_{kj} = \frac{\rho_{kl}}{\lambda_k - \tau_j}, \quad k, j = 1, - - -, K. \tag{II.8}$$

If some of the ρ_{kl} are zero, the corresponding modes of the system are small-signal uncontrollable. Suppose $\lambda_a = \tau_b$ and $\rho_{al} = 0$, then $\mu_{aj} = 0$ $(j \neq b)$ and μ_{ab} is indeterminant as defined by Equation (II.8). Since the λ_k are assumed to be distinct each column of ψ can have at most, one such indeterminant element. Each of the indeterminate μ_{kj} can be computed with one of the K equations

$$\sum_{j=1}^{K} \mu_{kj} = 1, \quad k = 1, - - -, K. \tag{II.9}$$

These equations are a restatement of the relationship

$$U_1 \psi = U_1 . \tag{II.10}$$

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