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STABILITY OF STEADY AND PERIODIC NATURAL CONVECTION IN A VERTICAL SLOT

by

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of the requirements for the degree of
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NOMENCLATURE

a	thermal diffusivity
a_o	amplitude of oscillations
A	acceleration ratio,
B	the coefficient matrix, equation (81)
C	function defined by equation (84)
c	wave speed of disturbance
D	operator, d/dy
d	width of slot
f	frequency of oscillation of the slot
F_x	body force in the X-direction
F	Froude number
G	amplitude function, equation (42)
g	gravitational acceleration
g'	infinitesimal disturbance, equation (42)
Gr	Grashof number, $g\gamma\Delta T\, d^3/\nu^2$
H	height of slot
h	aspect ratio, H/d
k	thermal conductivity
L	linear operator
p	pressure
P	amplitude function, equation (42)
Pr	Prandtl number, ν/a
R	vibrational Reynolds number, $\omega d^2/\nu$
Ra	Rayleigh number, $Gr\,Pr$
S	function defined by equation (85)

t	time
T	temperature
U	amplitude function, equation (42)
u	velocity component in X-direction
V	amplitude function, equation (42)
v	velocity component in Y-direction
W	amplitude function, equation (42)
w	velocity component in Z-direction
α	wave number of disturbance, X-direction
β	wave number of disturbance, Y-direction
γ	coefficient of thermal expansion
ϵ	perturbation parameter
θ	dimensionless temperature, $(T - T_0)/\Delta T$
κ	eigenvalue, equation (93)
λ	wavelength of disturbance, or eigenvalue, equation (99)
μ	eigenvalue,equation (87)
ν	kinematic viscosity
ρ	mass density; or eigenvalue, equation (92)
φ	amplitude function,equation (42)
ψ	stream function
ϑ	amplitude function, equation (42)
σ	eigenvalue
ω	frequency of oscillation , $2\pi f$
τ	period, $2\pi/\omega$

Overlines

$\sim$ quantities defined in equation (43)

$-$ base flow parameters

Superscripts

$+$ dimensional quantities

$'$ disturbance parameters

o hydrostatic condition

Brackets

( ) equation numbers

[ ] references numbers

CHAPTER I

INTRODUCTION

In this dissertation the linear stability of steady and periodic natural convection in a vertical slot is investigated analytically. The model consists of a slot of large aspect ratio (height to width ratio), with isothermal vertical walls and closed at both ends. For small temperature differences the flow in the slot is antisymmetric and parallel. When the temperature difference exceeds a certain critical value the parallel flow becomes unstable and a secondary motion appears in the slot. It is the purpose of this analysis to find this critical limit in terms of non-dimensional parameters involved.

A slot of height H, width d and a temperature difference ΔT, (Figure 1) oscillating harmonically with an amplitude a_o, and frequency ω, is considered. The non-dimensional parameters which describe this system are the Grashof number Gr, the Prandtl number Pr, the aspect ratio $h=H/d$, the vibrational Reynolds number $R=\omega d^2/\nu$, and the acceleration ratio $A=a_o\omega^2/g$.

In the case of a stationary slot of sufficiently large aspect ratio three regimes of laminar convective motion have been observed [11]. These are the boundary layer, transition and conduction regimes. When $\frac{GrxPr}{h}$ is greater than a number of order 10^4 and $h \gtrsim 1$, a boundary-layer regime is observed with variation of temperature in horizontal and vertical directions [12,21]. The regime $300 \lesssim \frac{GrxPr}{h} \lesssim 10^4$ is a transition regime between the boundary layer (also referred to as finite slot case) and the conduction regime. For $\frac{GrxPr}{h} < 300$ the end effects disappear in the central portion of the slot, the velocity profile becomes a cubic

function of y and the temperature profile is linear in the horizontal direction. The velocity profile for the boundary layer regime is obtained by using the experimentally observed value of the vertical temperature gradient [40]. However, in case of the conduction regime the velocity profile is an exact solution of the governing equations for parallel flow in the central portion of the slot. In this study the analysis is limited to the conduction regime with steady and periodic body forces because in both cases exact solutions to the base flow can be obtained.

The stability of steady natural convection for the conduction regime has received considerable attention in the last 20 years. Gershuni [16] was the first to consider this problem analytically. He assumed a stationary disturbance and solved the resulting eigenvalue problem by the Galerkin [14,15,31] method. The results were approximate because the expansions used for the disturbance stream function and temperature were in terms of polynomials truncated after two terms. Accurate results for zero Prandtl number are given by Birikh [2]. The Galerkin method was applied using an expansion of up to 18 orthogonal functions. It was concluded that the critical disturbance is stationary. Rudakov [36,37] has discussed this problem in two separate papers. In the first paper [36] the disturbance stream function and the temperature were expanded in powers of the wave number-Grashof number product in an analysis limited to small Prandtl ($Pr < 1$) number. In the second paper by Rudakov [37] the Galerkin method was applied using a set of orthogonal functions as approximating functions and the Prandtl number range 10^{-2} to 10 is covered. In both these works Rudakov [36,37] found that the

critical disturbance is stationary. Vest and Arpaci [41] have considered the stability of natural convection in a vertical slot with respect to stationary disturbances. The Galerkin method was applied using orthogonal functions tabulated by Harris and Reid [29]. The critical Grashof number was found to be nearly independent of the Prandtl number for stationary disturbances. They verified their theoretical result for Pr=0.7 by experiments with air. In a recent paper Gill and Kirkham [23] have discussed the stability of natural convection in a rectangular slot in the limit of large Prandtl number. They used a numerical integration procedure to find that in this limit the system is unstable with respect to traveling wave disturbances. For large Prandtl number they gave the following relation between the critical Grashof number Gr_c and the Prandtl number Pr,

$$Gr_c = \frac{(9.4)10^3}{Pr^{1/2}}$$

The critical wave number is given as 2.5, and the dimensionless wave speed equal to 8.5×10^{-3}, which is greater than the maximum velocity of the base flow, $(\bar{u})_{max}$ ($\cong 8. \times 10^{-3}$). Gotoh and Sotoh [24] have investigated the problem for large Grashof number and small wave number by an asymptotic method. They report a critical Grashof number of 7.7×10^5, two orders higher than reported in references [36,37,41]. It was found that for the Prandtl number of order one, at small wave numbers the system is unstable to the traveling wave disturbance with wave speed smaller than the maximum velocity of the base flow. This result is similar to the result obtained for the stability of plane Poiseulle flow [8,26,27,33]. Lin [33] has discussed the singularity

associated with the inviscid Orr-Sommerfeld equation when the wave speed is smaller than the maximum base flow velocity.

Due to the limitation that the conduction regime exists only for $\frac{GrxPr}{h} < 300$ [22], experiments to observe the stability of natural convection, with a slot of reasonable height are possible only for small Prandtl number, if ΔT is to remain small in magnitude. Such experiments [41] have shown that the instability is of the stationary type. For air (Pr=0.7) Gr_c=8700 at wave number 2.74 was obtained by Vest and Arpaci [41].

Compared to the vast body of literature [33] on the stability of steady flows, very little work has been done on the stability of unsteady flows. One of the most common (and simple) problems in this class is the effect of periodically varying parameters on the onset of convection in a horizontal layer of fluid heated from below (the Bénard problem). The first of these is by Gershuni and Zhukhovitskii [18] who analyzed the case of free boundaries with periodically varying temperature gradient. In this case the stability can be found from a Mathieu equation [32]. It is concluded that stability limits are significantly affected by the oscillating temperature gradient. Gresho and Sani [25] solved the above problem for free and rigid boundaries when the body force is periodic. Their conclusions were the same as Gershuni and Zukhovitskii [18]. However, this agreement is expected since periodic body force and periodic temperature gradient have identical effects on the stability equations in this problem. In fact, the equations governing the stability of the Benard problem with free boundaries and oscillations of either type discussed above are analogous to the stability of an inverted pendulum with gravity modulation [6]. Some finite amplitude (non-linear) effects are also con-

sidered by Gresho and Sani [25]. Venezian [39] has studied the effect of small sinusoidal fluctuations of the wall temperature. He solved the problem as a perturbation on the classical Bénard problem and concluded that, depending on frequency and amplitude of the perturbation, onset of convection can either by delayed or advanced by the modulation. In a paper published recently, Davis [5] considered the finite amplitude instability of Bénard convection, oscillating in the vertical direction.

Gunness and Gebhart [28] discussed the effect of boundary temperatures exponentially varying in time on the stability of fully developed natural convection in a vertical channel. They found that an accelerating base flow is more stable than the steady base flow. This tendency is found to increase rapidly at first and then to slowly decrease with increasing acceleration of the base flow.

A large portion of the literature devoted to stability of periodic flows deals with isothermal flows. In one of the earliest works on stability of unsteady flows, Gershuni and Zukhovitskii [18] discussed the effect of oscillating angular velocity on the stability of rigid rotation of a fluid. They found that although the stationary rotation of the fluid as a rigid body is stable for all values of the angular velocity, modulation of the angular velocity has a destabilizing effect on the rigid rotation of a fluid. Yih [43] has discussed the stability of a viscous fluid layer on a horizontal plate oscillating in its own plane. His analysis is limited to long waves. Grosh and Salven [26] used the Galerkin method to study the stability of plane Poiseuille flow with oscillating pressure gradient. The effects of amplitude and frequency of oscillations on the maximum eigenvalue (growth factor) is

studied to conclude that at small amplitudes the effect is in general stabilizing, and at large amplitudes it is destabilizing.

Due to the difficulties associated with experiments with unsteady flows, there are only a few experimental studies of stability of unsteady flows. For a modulated Couette flow between rotating cylinders, Donnelly et al [8] and Donnelly [9] report that in all cases the onset of instability was delayed by oscillations. Gilbrech and Combs [18] conducted experiments for periodic Poiseuille flow in a pipe to find that the critical Reynolds number is considerably increased (up to double the steady value) by modulation.

In this analysis the Galerkin method is used to reduce the partial differential equations to a set of ordinary differential equations. This is one of the "weighted residual techniques" reviewed by Finlayson and Scriven [14]. In reference [14] applications of many such methods are discussed. The conclusion is reached that selection of the approximating function is more important than the selection of any particular method of weighted residuals. Kantorovich and Krylov [31] discuss the variational and Galerkin methods for reducing the order of differential equations. Among others, Gershuni [16], Gershuni and Zhukhovitskii [17], Birikh [2], Rudakov [37], and Vest and Arpaci [41] have used the Galerkin method for solution of stability problems.

In the present analysis a set of orthogonal functions satisfying four homogeneous boundary conditions is required for the application of the Galerkin method. The functions tabulated by Harris and Reid [29] are used for this purpose. They are also discussed by Chandrasekhar [3]. Some of the integrals which arise in the application of the Galerkin

method to stability problems are evaluated by Reid and Harris [35] and Vest [40].

A survey of the literature indicates that stability of steady natural convection in the vertical slot has been investigated for a limited range of the Prandtl number. In this study stability of steady convection with respect to traveling wave disturbance is studied for the range of Prandtl number 0 to 1000. In addition, stability of unsteady convection is also investigated. Specifically, the case of a slot oscillating harmonically in the vertical direction is considered. The slot is chosen sufficiently long so that a conduction regime exists and exact solutions for the base flows are obtainable.

CHAPTER II

THEORETICAL ANALYSIS

2.1 Introduction

In this chapter the base flow solution is obtained. A complete derivation of the linearized disturbance equations is presented. It is proved that two dimensional disturbances are less stable than three dimensional disturbances. The partial differential equations which govern two dimensional disturbances are reduced to a set of ordinary differential equations by the Galerkin method. The method of finding the eigenvalues of this set is discussed.

2.2 The Base Flow

The following assumptions are made in the derivation of the equations governing the natural convective motion of a viscous fluid in the vertical slot of figure 1:

(1) The specific heat, thermal conductivity and kinematic viscosity are constant.

(2) The pressure, temperature and density deviate from their hydrostatic (i.e. pressure gradient solely due to the weight of the fluid) values by only small amounts.

(3) Viscous dissipation and work of compression are neglected.

(4) The density is considered to be a function of temperature only in the body force term; otherwise it is considered to be a constant (Boussinesq approximation).

(5) The velocity of oscillations of the flow, a_o, is of a lower order of magnitude than the velocity of sound.

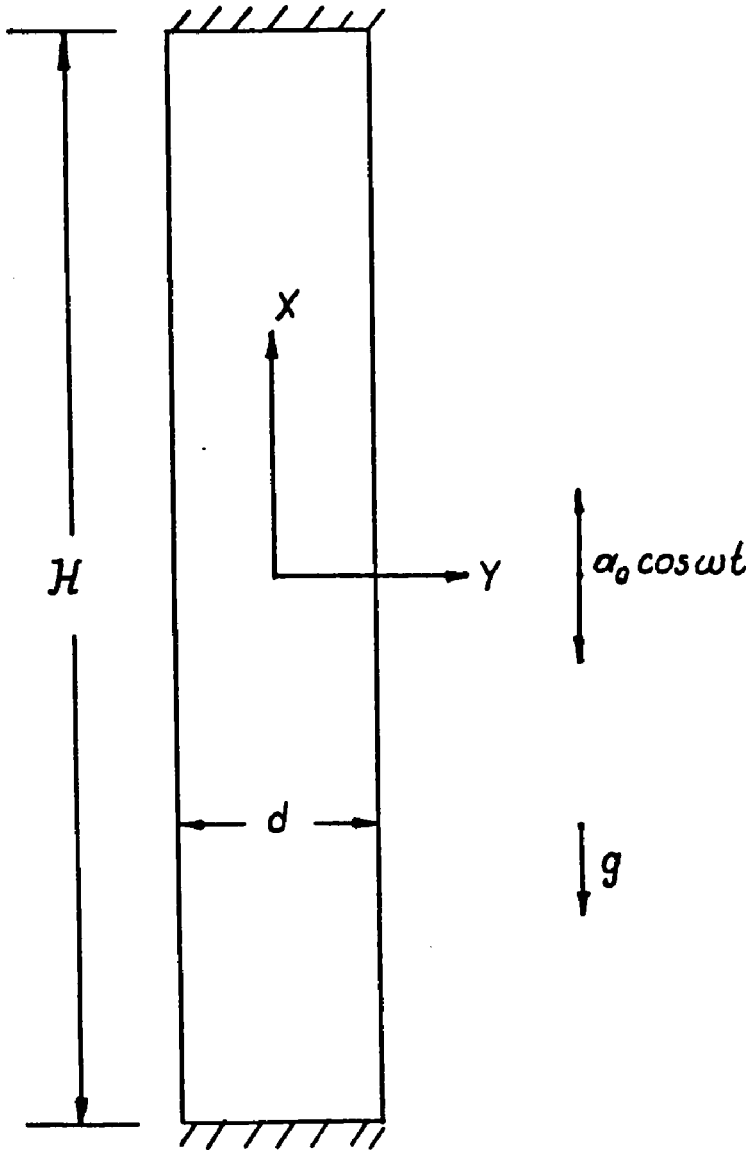


Figure 1. Vertical Slot

For a slot of large aspect ratio h, at small Grashof number Gr, the flow is parallel in the central portion of the slot and the temperature distribution is linear (conduction regime),

$$\begin{aligned} \bar{u}^+ &= \bar{u}^+(y,t) \\ \bar{v}^+ &\equiv \bar{w}^+ \equiv 0 \\ \bar{T}^+ &= \bar{T}^+(y) \end{aligned} \tag{1}$$

where all quantities with a + superscript are dimensional. In a frame of reference attached to the oscillating slot, the equations governing the base flow are given below.

Momentum:

$$\frac{\partial \bar{u}^+}{\partial t^+} = -\frac{1}{\rho}\frac{\partial \bar{p}^+}{\partial x^+} + F_x + \nu \frac{\partial^2 \bar{u}^+}{\partial y^{+2}} \tag{2}$$

$$0 = \frac{\partial \bar{p}^+}{\partial y^+} \tag{3}$$

Energy:

$$0 = \frac{\partial^2 \bar{T}^+}{\partial y^{+2}} \tag{4}$$

and continuity of mass:

$$\int_{\text{boundaries}} \bar{u}^+ dy^+ = 0 \tag{5}$$

where F_x is the body force acting in the x-direction.

$$F_x = -g\left[1 + \frac{a_o \omega^2}{g}\cos\omega t^+\right]$$

For rigid walls the boundary conditions are

$$\bar{u}^+\left(\pm\frac{d}{2},t\right)=0$$
$$\bar{T}^+\left(-\frac{d}{2}\right)=T_1^+ \qquad (6)$$
$$\bar{T}^+\left(+\frac{d}{2}\right)=T_2^+=T_1^+-\Delta T$$

where $\Delta T^+=T_1^+-T_2^+$

The Boussinesq approximation reduces the first momentum equation to,

$$\frac{\partial\bar{u}^+}{\partial t^+}=-F_x\gamma\left(\bar{T}^+-T_o^+\right)+\nu\frac{\partial^2\bar{u}^+}{\partial x^{+2}} \qquad (7)$$

where T_o^+ is the temperature under hydrostatic conditions and γ is the coefficient of expansion defined by

$$\gamma=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p=\frac{\rho_o-\rho}{\rho(T-T_o)}$$

and ρ_o is the density at temperature T_o^+.

The following non-dimensional quantities are now defined.

Gr = Grashof number = $\dfrac{g\gamma\Delta T^+d^3}{\nu^2}$

Pr = Prandtl number = $\dfrac{\mu c_p}{k}$

A = Acceleration ratio = $\dfrac{\omega^2 a_o}{g}$

R = Vibrational Reynolds number = $\dfrac{\omega d^2}{\nu}$

h = Aspect ratio = $\dfrac{H}{d}$

$$x=\frac{x^+}{d},\quad y=\frac{y^+}{d},\quad z=\frac{z^+}{d},\quad t=\frac{t^+}{d^2/\nu}$$
$$u=\frac{u^+}{Gr(\nu/d)},\quad v=\frac{v^+}{Gr(\nu/d)},\quad w=\frac{w^+}{Gr(\nu/d)}$$
$$\theta=\frac{T^+-T_o^+}{\Delta T^+},\quad p=\frac{p^+}{\rho Gr^2(\nu/d)^2} \qquad (8)$$

In terms of these dimensionless variables, the equations governing the base flow become

$$\frac{\partial \bar{u}}{\partial t} = \bar{\theta}\,\frac{F_x}{g} + \frac{\partial^2 \bar{u}}{\partial y^2} \tag{9}$$

$$0 = \frac{\partial^2 \bar{\theta}}{\partial y^2} \tag{10}$$

$$\int_{-1/2}^{+1/2} \bar{u}\, dy = 0 \tag{11}$$

Subjected to the boundary conditions

$$\bar{u}\left(\pm \frac{1}{2}\right) = 0 \quad ,$$

$$\bar{\theta}\left(-\frac{1}{2}\right) = \theta_1 = \frac{T_1^+ - T_0^+}{\Delta T^+} \quad , \quad \bar{\theta}\left(\frac{1}{2}\right) = \theta_1 - 1 \tag{12}$$

Equation (10) is readily integrated to give

$$\bar{\theta} = -y + \left(\theta_1 - \frac{1}{2}\right) \tag{13}$$

Equation (9) shows that the base flow consists of two parts

$$\bar{u} = \bar{u}_1(y) + \bar{u}_2(y,t) \tag{14}$$

Substituting this into equation (13) and separating steady and unsteady terms

$$\frac{\partial \bar{u}_1}{\partial y^2} + y - \left(\theta_1 - \frac{1}{2}\right) = 0 \tag{15}$$

$$\frac{\partial \bar{u}_2}{\partial t} = \frac{\partial^2 \bar{u}_2}{\partial y^2} + A\left(-y + \theta_1 - \frac{1}{2}\right)\cos Rt \tag{16}$$

The solution of equation (15) with the boundary conditions

$$\bar{u}_1\left(\pm \frac{1}{2}\right) = 0 \tag{17}$$

yields

$$\bar{u}_1 = +\frac{1}{6}\left(Y^3 - \frac{Y}{4}\right) - \left(\theta_1 - \frac{1}{2}\right)\left(\frac{Y^2}{2} - \frac{1}{8}\right) \tag{18}$$

and $\bar{u}_2$ is the real part of U_2. Where U_2 is the solution of

$$\frac{\partial \bar{U}_2}{\partial t} = \frac{\partial^2 \bar{U}_2}{\partial Y^2} + A\left(-Y + \theta_1 - \frac{1}{2}\right) e^{iRt} \tag{19}$$

$$\bar{U}_2\left(\pm\frac{1}{2}\right) = 0 \tag{20}$$

The steady periodic solution (solution for large time) of this system is

$$\bar{U}_2 = \frac{A}{m^2}\left[\bar{\theta} + \frac{\sinh mY}{2\sinh(m/2)} - \left(\theta_1 - \frac{1}{2}\right)\frac{\cosh mY}{\cosh(m/2)}\right] e^{iRt} \tag{21}$$

where

$$m^2 = iR$$

The continuity equation (11) must now be satisfied by $\bar{u}_1$ and $\bar{u}_2$ together.

$$\int_{-1/2}^{+1/2} \left[\bar{u}_1 + \text{Real}\left(\bar{U}_2\right)\right] dY = 0 \tag{22}$$

After some algebra this leads to the condition $(\theta_1 - 1/2) = 0$ and the velocity and temperature profiles are now given by

$$\bar{u} = \frac{1}{6}\left(Y^3 - \frac{Y}{4}\right) + \frac{A}{R}\left\{\text{Real}\left(-Y + \frac{\sinh mY}{2\sinh(m/2)}\right)\right\} \sin Rt +$$

$$+ \frac{A}{R}\left\{\text{Imag}\left(\frac{\sinh mY}{2\sinh(m/2)}\right)\right\} \cos Rt, \tag{23}$$

$$\bar{\theta} = -Y \quad . \tag{24}$$

For the special case when the slot is stationary the above profiles reduce to the classical case investigated by Vest and Arpaci [41], Rudakov [37] and Gill and Kirkham [23]. Experimental evidence obtained by Edler [12] for large Prandtl number fluids (Pr=100-2500) and by Eckert and Carlson [11] for Prandtl number 0.73 shows that this regime - the conduction regime - exists when $Gr \cdot Pr < 2 \times 10^4$ for moderately large aspect ratio (h=20-100). Gill and Davey [22] give $Gr.Pr/h \leq 300$ as the limit for this regime.

The linear temperature profile indicates that the heat transfer between the plates takes place solely by conduction. The velocity profile consists of a steady cubic profile on which is imposed an unsteady periodic profile. The oscillating part of the velocity has the same frequency of oscillation as the slot and an amplitude proportional to A/R. The velocity profile is shown in Figure 2 for A=1, R=100 at t=0. In order to observe the variation of the amplitude of $\bar{u}_2$ with frequency, the ratio, $\bar{u}_2/\bar{u}_1$ was numerically calculated at various frequencies R, Y=0.3, t=0.0 and A=1.0. Figure 3 shows that the amplitude of the unsteady part of the base flow approaches zero as the vibrational Reynolds number, R becomes large.

2.3 Formulation of the Stability Problem

In this section the equations governing infinitesimal disturbances are developed. It is proved that the two-dimensional disturbances are less stable than the three dimensional disturbances.

With the Boussinesq approximation, the three-dimensional momentum, energy and continuity equations are

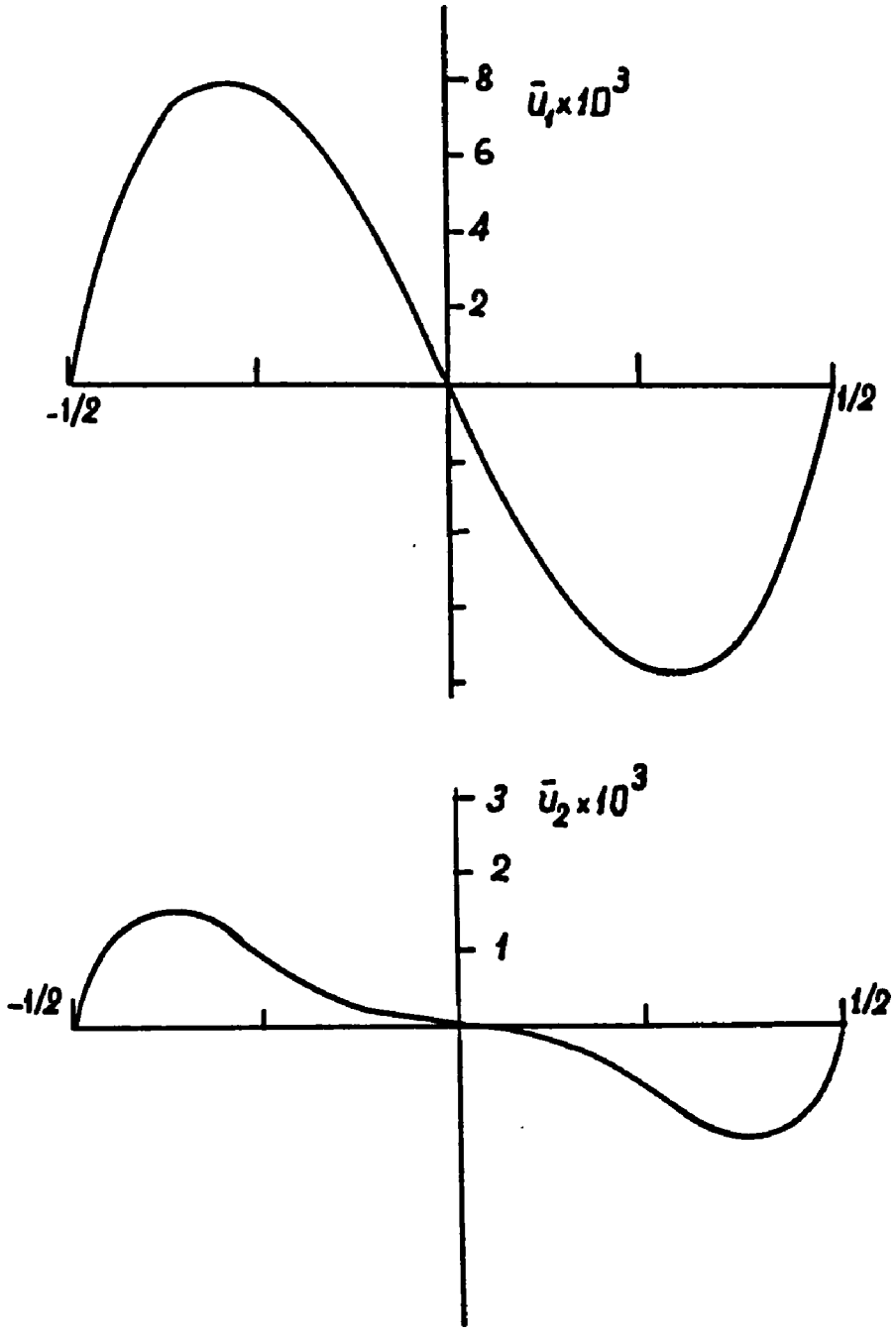


Figure 2. Velocity Profiles. $\bar{u}_2$ is for t=0.

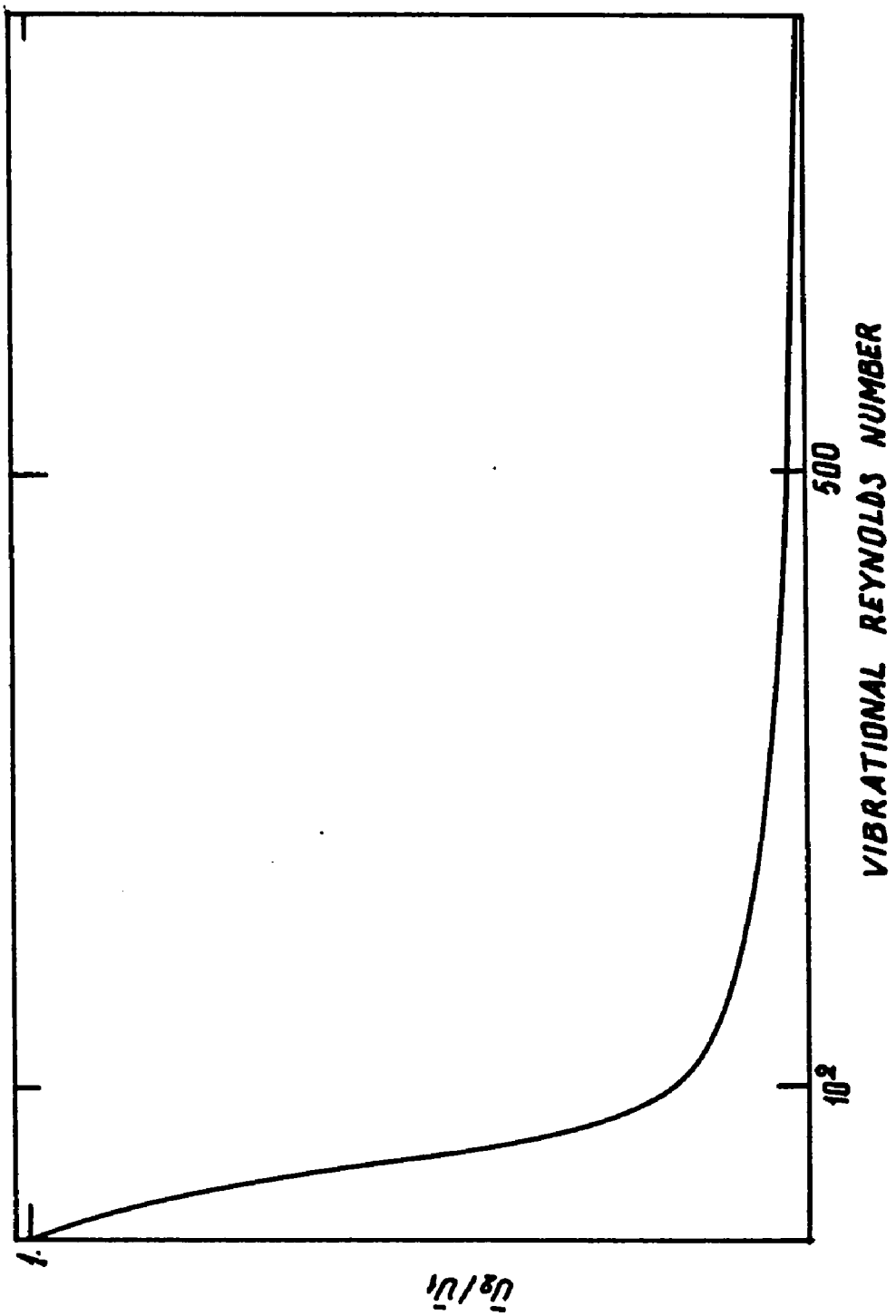


Figure 3. Ratio of unsteady to steady velocity as a function of vibrational Reynolds number at A=0, Y=.3 at t=0.

$$\frac{\partial u}{\partial t} + Gr\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) =$$

$$= -Gr\frac{\partial p}{\partial x} + F_x\,\theta + \nabla^2 u \qquad (25)$$

$$\frac{\partial v}{\partial t} + Gr\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -Gr\frac{\partial p}{\partial y} + \nabla^2 v \qquad (26)$$

$$\frac{\partial w}{\partial t} + Gr\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -Gr\frac{\partial p}{\partial z} + \nabla^2 w \qquad (27)$$

$$\frac{\partial \theta}{\partial t} + Gr\left(u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z}\right) = \frac{1}{Pr}\nabla^2\theta \qquad (28)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (29)$$

for rigid walls of high thermal conductivity the boundary conditions are

$$u\left(\pm\tfrac{1}{2}\right) = v\left(\pm\tfrac{1}{2}\right) = w\left(\pm\tfrac{1}{2}\right) = 0$$

$$\theta\left(\pm\tfrac{1}{2}\right) = \mp\frac{1}{2} \qquad (30)$$

To study small disturbances of the base flow, we seek solution of equations (25) through (30) in the form

$$u = \bar{u} + \epsilon u', \quad v = \bar{v} + \epsilon v', \quad w = \bar{w} + \epsilon w',$$

$$p = \bar{p} + \epsilon p', \quad \theta = \bar{\theta} + \epsilon \theta'$$

$\bar{u}, \bar{p}, \bar{\theta}$ represent the base flow quantities, ϵ is an unspecified perturbation parameter which is small compared to unity and u', v',....represent the three dimensional disturbance. Introducing the above pertur-

bation expansions in the equations (25) to (30) gives a sequence of sets of equations, each set corresponding to a definite power of ϵ as follows

ϵ^0 :

$$\frac{\partial \bar{u}}{\partial t} = F_x \bar{\theta} + \frac{\partial^2 \bar{u}}{\partial y^2} \tag{31}$$

$$0 = \frac{\partial \bar{P}}{\partial y} = \frac{\partial \bar{P}}{\partial z} \tag{32}$$

$$0 = \frac{\partial^2 \bar{\theta}}{\partial y^2} \tag{33}$$

$$\frac{\partial \bar{u}}{\partial x} = 0 \tag{34}$$

with the boundary conditions

$$\bar{u}\left(\pm \tfrac{1}{2}\right) = 0 \quad , \quad \bar{\theta}\left(\pm \tfrac{1}{2}\right) = \mp \tfrac{1}{2} \quad . \tag{35}$$

ϵ^1 :

$$\frac{\partial u'}{\partial t} + Gr\left(\bar{u}\frac{\partial u'}{\partial x} + v'\frac{\partial \bar{u}}{\partial y}\right) = $$

$$= -Gr\frac{\partial p'}{\partial x} + F_x \theta' + \nabla^2 u' \quad . \tag{36}$$

$$\frac{\partial v'}{\partial t} + Gr\left(\bar{u}\frac{\partial v'}{\partial x}\right) = -\frac{\partial p'}{\partial y} Gr + \nabla^2 v' \tag{37}$$

$$\frac{\partial w'}{\partial t} + Gr\left(\bar{u}\frac{\partial w'}{\partial x}\right) = -\frac{\partial p'}{\partial z} Gr + \nabla^2 w' \tag{38}$$

$$\frac{\partial \theta'}{\partial t} + Gr\left(\bar{u}\frac{\partial \theta'}{\partial x} + v'\frac{\partial \bar{\theta}}{\partial y}\right) = \frac{1}{Pr}\nabla^2 \theta' \tag{39}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{40}$$

with the boundary conditions

$$u'(\pm\tfrac{1}{2}) = v'(\pm\tfrac{1}{2}) = w'(\pm\tfrac{1}{2}) = \theta'(\pm\tfrac{1}{2}) = 0 \qquad (41)$$

Equations (31) through (34) govern the base flow and equations (35) through (40) are the linearized disturbance equations.

Because the coefficients in the equations (36) through (41) are functions of y and t only, the set accepts solutions which are exponential functions of x and z. For solutions to be bounded at infinity the exponents must be purely imaginary. Thus each of the disturbance parameters is the real part of an expression of the form

$$g' = G(y,t)\exp(i\alpha x + i\beta z) \qquad (42)$$

where α and β are real wave numbers. The disturbance parameters u', v', w', p' and θ' have a wave character in the infinite directions x and z with wave lengths $\frac{2\pi}{\alpha}$ and $\frac{2\pi}{\beta}$ respectively. This dependence on y and t is described by the parameters U, V, W, P and ϑ respectively.

For functions with the form of equation (42)

$$\frac{\partial}{\partial x} = i\alpha \;,\quad \frac{\partial^2}{\partial x^2} = -\alpha^2 \;,\quad \frac{\partial}{\partial y} \equiv D \;,$$

$$\frac{\partial^2}{\partial y^2} \equiv D^2 \;,\quad \frac{\partial}{\partial z} = i\beta \;,\quad \frac{\partial^2}{\partial z^2} = -\beta^2 \;,$$

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = D^2 - \alpha^2 - \beta^2$$

equations (36) through (41) then become

$$\frac{\partial \bar{u}}{\partial t} + Gr\left(i\alpha U\bar{u} + V\frac{\partial \bar{u}}{\partial y}\right) = -i\alpha PGr + F_x\vartheta + \nabla^2 U \qquad (43)$$

$$\frac{\partial V}{\partial t} + Gr\,(i\alpha\bar{u}V) = Gr\,DP + \nabla^2 V \tag{44}$$

$$\frac{\partial W}{\partial t} + Gr\,(i\alpha\bar{u}W) = -i\beta Gr\,P + \nabla^2 W \tag{45}$$

$$\frac{\partial \vartheta}{\partial t} + Gr\left(i\alpha\bar{u}\vartheta + V\frac{\partial\bar{\theta}}{\partial y}\right) = \frac{1}{Pr}\nabla^2\vartheta \tag{46}$$

$$i\alpha U + i\beta V + DW = 0 \tag{47}$$

$$U(\pm 1/2) = V(\pm 1/2) = W(\pm 1/2) = \vartheta(\pm 1/2) = 0. \tag{48}$$

Squire [38] has proved that the problem of a three-dimensional disturbance in a steady, incompressible parallel flow is equivalent to a two-dimensional disturbance in a similar flow at a lower Reynolds number. We will now prove that a similar result is true for the present problem also. If we introduce the following transformations

$$\begin{aligned} \alpha^2 + \beta^2 &= \tilde{\alpha}^2 \quad , \quad \alpha U + \beta W = \tilde{\alpha}\tilde{U} \\ \alpha\, Gr &= \tilde{\alpha}\tilde{Gr} \quad , \quad \vartheta\alpha = \tilde{\vartheta}\tilde{\alpha} \\ P\, Gr &= \tilde{P}\tilde{Gr} \quad , \quad V = \tilde{V} \quad , \end{aligned} \tag{49}$$

after combining equations (43) and (45), equations (43) through (48) may be written as

$$\frac{\partial\tilde{U}}{\partial t} + \tilde{Gr}\left(i\tilde{\alpha}\bar{u}\tilde{U} + \tilde{V}\frac{\partial\bar{u}}{\partial y}\right) = -i\tilde{\alpha}\tilde{P}\tilde{Gr} + \\ + \bar{\vartheta}\,F_x + (D^2 - \tilde{\alpha}^2)\tilde{U} \tag{50}$$

$$\frac{\partial\tilde{V}}{\partial t} + \tilde{Gr}\,(i\tilde{\alpha}\bar{u}\tilde{V}) = -\tilde{Gr}\,D\tilde{P} + (D^2 - \tilde{\alpha}^2)\tilde{V} \tag{51}$$

$$\frac{\partial\tilde{\vartheta}}{\partial t} + \left(i\tilde{\alpha}\bar{u}\tilde{\vartheta} + \tilde{U}\frac{\partial\bar{\theta}}{\partial y}\right)\tilde{Gr} = \frac{1}{Pr}(D^2 - \tilde{\alpha}^2)\tilde{\vartheta}\,. \tag{52}$$

$$i\tilde{\alpha}\tilde{U} + D\tilde{V} = 0 \qquad (53)$$

$$\tilde{U}(\pm 1/2) = \tilde{V}(\pm 1/2) = \tilde{\vartheta}(\pm 1/2) = 0. \qquad (54)$$

Equations (50) to (54) have the same structure as equations (43) to (48) with $W=\beta=0$. Hence it is sufficient to solve the problem of the two dimensional disturbance described by equations (50) through (54) and then use the conversions (49). In fact, since $Gr \leq \tilde{Gr}$ the critical Grashof number associated with the two-dimensional disturbance is lower than the critical Grashof number associated with the three dimensional disturbance.

For convenience, we define stream function, ψ' as follows

$$\frac{\partial \psi'}{\partial y} = u' \quad \text{and} \quad -\frac{\partial \psi'}{\partial x} = v' \qquad (55)$$

where the stream function ψ' can have the form

$$\psi'(x,y,t) = \phi(y,t)\, e^{i\alpha x} \qquad (56)$$

thus

$$u' = D\phi\, e^{i\alpha x}, \quad v' = -i\alpha\phi\, e^{i\alpha x}$$

Introducing this in the two-dimensional disturbance equations, after some algebra, the following system of equations is obtained

$$\frac{\partial}{\partial t}(D^2-\alpha^2)\phi + i\alpha Gr\left[\bar{u}(D^2-\alpha^2) - \bar{u}''\right]\phi = \qquad (57)$$

$$= \frac{F_x}{g} D\vartheta + (D^2-\alpha^2)^2\phi$$

$$\frac{\partial \vartheta}{\partial t} + i\alpha Gr\,(\bar{u}\vartheta - \phi) = \frac{1}{Pr}(D^2-\alpha^2)\vartheta \qquad (58)$$

$$\phi = \phi' = \vartheta = 0 \qquad \text{at} \qquad y = \pm 1/2 \tag{59}$$

where

$$u'' = A\,\text{Real}\left(\frac{\sinh my}{2\sinh(m/2)}\right)\cos Rt - A\,\text{Imag}\left(\frac{\sinh my}{2\sinh(m/2)}\right)\sin Rt + y.$$

2.4 Limiting Cases for Very Small and Very Large Prandtl Number

A. $Pr \to 0$.

As $Pr \to 0$ with Gr remaining finite the disturbance energy equation (58)

$$\frac{\partial \vartheta}{\partial t} + i\alpha Gr\left(\bar{u}\vartheta - \phi\right) = \frac{1}{Pr}\left(D^2 - \alpha^2\right)\vartheta \tag{60}$$

with the boundary conditions

$$\vartheta = 0 \qquad \text{at} \qquad y = \pm 1/2 \tag{61}$$

has only a trivial solution $=0$. This reduces the disturbance momentum equation to

$$\frac{\partial}{\partial t}\left(D^2 - \alpha^2\right)\phi + i\alpha Gr\left[\bar{u}\left(D^2 - \alpha^2\right) - \bar{u}''\right]\phi = \left(D^2 - \alpha^2\right)^2\phi \tag{62}$$

with

$$\phi = \phi' = 0 \qquad \text{at} \qquad y = \pm 1/2 \tag{63}$$

B. $Pr \to \infty$.

For $Pr \to \infty$, with Gr x Pr remaining finite, equations (57) and (58) become

$$\frac{\partial}{\partial t}(D^2-\alpha^2)\phi = \frac{F_x}{g} D\vartheta + (D^2-\alpha^2)^2\phi \tag{64}$$

$$\frac{\partial \vartheta}{\partial t} = \frac{1}{Pr}(D^2-\alpha^2)\vartheta. \tag{65}$$

Subjected to the boundary conditions

$$\phi = \phi' = \vartheta = 0 \quad \text{at} \quad y = \pm 1/2 . \tag{66}$$

2.5 Solution of the Eigenvalue Problem

The linear, homogeneous, partial differential equations with homogeneous boundary conditions, governing the disturbance, have a stable solution only for certain range of the parameters Gr, Pr, α, A and R. Stability of these equations can be found by first reducing them to a set of linear ordinary differential equations and then applying the known stability criterion to this set.

Various numerical schemes are available for reducing partial differential equations to a set of ordinary differential equations. Any of the weighted residual methods discussed by Finlayson and Scriven [15] can be followed. One of these, the Galerkin method, is used as this has proved successful in earlier applications to problems in hydrodynamic stability [17, 18, 26, 27, 40, 41].

A. The Galerkin Method

Representing the equations (57) and (58) by

$$\mathcal{L}_1(\phi, \vartheta) = 0 \tag{67}$$

$$\mathcal{L}_2(\phi, \vartheta) = 0 \tag{68}$$

the unknown functions $\varphi(y,t)$ and $\vartheta(y,t)$ are expanded in a set of real orthogonal functions E, O, e and o of y as follows

$$\phi(y,t) = \sum_n \left[a_n(t)E_n(y) + i b_n(t) O_n(y)\right] \quad (69)$$

$$\vartheta(y,t) = \sum_n \left[c_n(t)\, o_n(y) + i d_n(t)\, e_n(y)\right] \quad (70)$$

where 1) E_n and O_n satisfy the boundary conditions for ϕ and e_n and o_n satisfy the boundary conditions for ϑ ; 2) E_n and e_n are even functions of y and O_n and o_n are odd functions of y and 3) a_n, b_n, c_n and d_n are in general complex and are the unknown functions of time.

The expansions for ϕ and ϑ may also be written in the form

$$\phi(y,t) = \sum_n \left[a_n(t)\, O_n(y) + i\, b_n(t) E_n(y)\right] \quad (71)$$

$$\vartheta(y,t) = \sum_n \left[c_n(t) e_n(y) + i\, d_n(t)\, o_n(y)\right] . \quad (72)$$

It is immaterial whether the form (69) and (70), or (71) and (72) is chosen. The resulting eigenvalues will be identical. Expansions of this type (69 and 70 or 71 and 72) are chosen for mathematical convenience only and do not put any restriction on the generality of the solution. It will be seen later that this type of expansion makes the coefficients real in the resulting set of ordinary differential equations.

Equations for the unknown coefficients a_n, b_n, c_n and d_n are obtained by taking the inner products of the equations (67) and (68) with each of the approximating functions as follows

$$\langle \mathcal{L}_1(\phi,\vartheta), E_n \rangle = 0 \quad (73)$$

$$\langle \mathcal{L}_2(\phi,\vartheta), e_n \rangle = 0 \quad (74)$$

$$\langle \mathcal{L}_1(\phi,\vartheta), O_n \rangle = 0 \tag{75}$$

$$\langle \mathcal{L}_2(\phi,\vartheta), o_n \rangle = 0 \tag{76}$$

where

$$\langle f, g \rangle \equiv \int_{-1/2}^{+1/2} (f \cdot g)\, dy$$

In this analysis expansions of the form (69) and (70) are chosen. So that after integrating by parts and using the boundary conditions, equations (73) through (76) become

$$\sum_{n=1}^{N} \langle (E_n'' - \alpha^2 E_n), E_m \rangle \frac{da_n}{dt} = \sum_{n=1}^{N} \{ \langle (E_n^{IV} - 2\alpha^2 E_n'' + \alpha^4 E_n), E_m \rangle a_n + \alpha Gr \langle (\bar{u} O_n'' - \bar{u}\alpha^2 O_n - \bar{u}'' O_n), E_m \rangle b_n + (Fx/g) \langle o_n', E_m \rangle c_n \} \tag{77}$$

$$\sum_{n=1}^{N} \langle (O_n'' - \alpha^2 O_n), O_m \rangle \frac{db_n}{dt} = \sum_{n=1}^{N} \{ -\alpha Gr \langle (\bar{u} E_n'' - \bar{u}\alpha^2 E_n - \bar{u}'' E_n), O_m \rangle a_n + \langle (O_n^{IV} - 2\alpha^2 O_n'' + \alpha^4 O_n), O_m \rangle b_n + (Fx/g) \langle e_n', O_m \rangle d_n \} . \tag{78}$$

$$\sum_{n=1}^{N} \langle o_n, o_m \rangle \frac{dc_n}{dt} = \sum_{n=1}^{N} \{ \alpha Gr \langle O_n, o_m \rangle b_n + \frac{1}{Pr} \langle (O_n'' - \alpha^2 o_n), o_m \rangle c_n + \alpha Gr \langle \bar{u} e_n, o_m \rangle d_n \} . \tag{79}$$

$$\sum_{n=1}^{N} \langle e_n, e_m \rangle \frac{d}{dt} d_n = \sum_{n=1}^{N} \{ -\alpha Gr \langle e_n, e_m \rangle a_n - \tag{80}$$

$$-\alpha Gr \langle \bar{u} o_n, e_m \rangle c_n + (1/Pr) \langle (e_n'' - \alpha^2 e_n), e_m \rangle d_n \} .$$

Thus if the expansions (69) and (70) are truncated after N terms, equations (77) through (80) represent a set of 4N first order ordinary differential equations in 4N unknowns a_n, b_n, c_n and d_n (n=1,2,....N). This set may be written in matrix form as

$$\frac{dX}{dt} = BX \tag{81}$$

where X is the 4Nth order vector, a_1, a_2,..a_N, b_1, b_2,.....b_N, c_1, c_2, ...c_N, d_1, d_2,....d_N and B is the 4N x 4N coefficient matrix.

B. The Approximating Functions

The approximating functions O_n and E_n for ϕ must vanish along with their first derivatives on the boundary. The simplest system satisfying this condition are the eigenfunctions of

$$d^4f/dy^4 - a^4 f = 0 \tag{82}$$

$$f = f' = 0 \quad \text{at} \quad y = \pm 1/2 . \tag{83}$$

These fall into non-combining even and odd groups as

$$C_m(y) = \frac{\cosh(\lambda_m y)}{\cosh(\lambda_m/2)} - \frac{\cos(\lambda_m y)}{\cos(\lambda_m/2)} \tag{84}$$

$$S_m(y) = \frac{\sinh(\mu_m y)}{\sinh(\mu_m/2)} - \frac{\sin(\mu_m y)}{\sin(\mu_m/2)} , \tag{85}$$

where λ_m and μ_m are the roots of

$$\tanh(\lambda/2) + \tan(\lambda/2) = 0 \tag{86}$$

$$\coth(\mu/2) + \cot(\mu/2) = 0 . \tag{87}$$

These orthogonal functions have been extensively used by Chandrasekhar and others [34,41]. They are discussed by Harris and Reid [29,35] and Chandrasekhar [3].

The function o and e required to expand ϑ must vanish at the boundaries. The eigenfunctions of

$$\frac{d^2f}{dy^2} + a^2 f = 0 \tag{88}$$

$$f = 0 \quad \text{at} \quad y = \pm 1/2 \tag{89}$$

are

$$\sin_m(y) = \sin(\kappa_m y) \tag{90}$$

$$\cos_m(y) = \cos(\varsigma_m y) \tag{91}$$

where

$$\varsigma_m = (2m-1)\pi\,, \tag{92}$$

$$\kappa_m = 2m\pi\,, \quad m = 1,2,\ldots \tag{93}$$

With this in the expansions (69) and (70) we have

$$E_m = C_m\,, \quad O_m = S_m\,,$$

$$e_m = \cos m\,, \quad o_m = \sin m\,. \tag{94}$$

Most of the inner products required in the equations (77) to (80) associated with the above approximating functions have been calculated by Reid and Harris [35] and Vest [40]. Additional inner products -those associated with the unsteady part of the base flow - were calculated by exact integration and are listed in the Appendix I.

2.6 The Stability Criterion

In the steady case the coefficient matrix B is a constant matrix and the eigenvalues of the matrix determine the stability of the system. The eigenvalues of B are found by first reducing it to upper almost triangular form (Hessenberg) by a similarity transform and then using the QR algorithm [42]. Subroutines for both these methods exist in the "Scientific Subroutine Package" (IBM) available on the Michigan Terminal System. The accuracy of the eigenvalue subroutine can be tested by computing the difference between the trace (sum of the diagonal elements of B) and the sum of the eigenvalues. In all cases this difference was less than .01%.

When an N-term approximation is used in the expansions (54), B is of the order 4N and has 4N complex eigenvalues, $\sigma_i (i=1,2,\ldots 4N)$. If the eigenvalues are arranged so that

$$\text{Real}(\sigma_1) < \text{Real}(\sigma_2) < \ldots < \text{Real}(\sigma_{4N})$$

the system of equations (81) is neutrally stable when

$$\text{Real}(\sigma_1) = 0$$

If $\text{Real}(\sigma_1) > 0$

the system is unstable and if

$$\text{Real}(\sigma_1) < 0$$

the system is stable. In fact the magnitude Real (σ_1) is the growth rate (if positive) or the damping rate (if negative) of the disturbance.

When the Imag (σ_1)$\neq$0, the unstable disturbance grows as an oscillatory wave and the system is said to be overstable. When Imag (σ_1)=0, the unstable disturbance grows into stationary cells. The instability is said to be stationary in this case.

When the neutral disturbance is stationary, the condition that determinant of [B]=0 is a necessary and sufficient condition for neutral stability. The determinant of [B] was always calculated during the calculations of the steady flow as a check.

For the periodic base flow the coefficient matrix B is periodic. According to Floquet theorem [4] there exists a fundamental matrix such that

$$X(t) = P(t)\, e^{Qt} \tag{95}$$

Where P is a periodic non-singular matrix with the same period as the matrix B and Q is a constant matrix. Hence,

$$X(0) = P(0) \tag{96}$$

and

$$X(\tau) = P(\tau)\, e^{Q\tau} \tag{97}$$

Therefore

$$X(\tau) = P(0)\, e^{Q\tau}$$

$$= X(0)\, e^{Q\tau} \quad .$$

If the initial conditions are chosen, such that

$$X(0) = I$$
$$X(\tau) = e^{Q\tau}. \tag{98}$$

If λ_i, i=1,...M are the eigenvalues of $X(\tau)$ and σ_i, i=1,...M are the eigenvalues of Q, we have the following relations.

$$\text{Real}(\sigma_j) = \frac{1}{\tau} \ln(|\lambda_j|) \tag{99}$$

$$\text{Imag}(\sigma_j) = \frac{1}{\tau} \tan^{-1}\left(\frac{\text{Real}(\lambda_j)}{\text{Imag}(\lambda_j)}\right) \tag{100}$$

The flow is neutrally stable when Real $(\sigma_1)=0$, i.e., when $|\lambda_1|=1$. A neutrally stable disturbance oscillates at the frequency of the base flow when Imag $(\sigma_1)=0$. When Imag $(\sigma_1)\neq 0$, the disturbance oscillates at a frequency

$$\omega^* = f\left|\frac{1}{\tau}, \frac{\text{Imag}(\sigma_1)}{2\pi}\right| \tag{101}$$

Accuracy of the eigenvalues can be checked by the following [32],

$$\sum \sigma_i = \frac{1}{\tau}\int_0^{\tau} (\text{trace})\, dt \,. \tag{102}$$

Since all the diagonal elements of the matrix B are constants, above condition reduces to

$$\sum \sigma_i = \text{trace} \,. \tag{103}$$

CHAPTER III

RESULTS AND CONCLUSIONS

3.1 Steady Base Flow

When the slot is stationary, $\bar{u}_2 = 0$ and the base flow is steady. The equations governing the amplitude of disturbance are a set of ordinary differential equations with constant coefficients i.e. in the equation (81) B is a constant matrix.

As a check on the method used here to study stability of the steady flow it was applied in an anlogous manner to the Bénard problem, which possesses an exact solution [3]. The first approximation gave the critical Grashof number to within 1.1% and the second approximation to within .02% of the exact solution.

In order to find the stability limits of the steady natural convection in the vertical slot the stability criteria of section 2.6 was applied to the equation (81). The procedure followed was: a) for specified values of the Prandtl number and wave number, the elements of the matrix B were calculated. b) Eigenvalues σ_J(J=1,2,...4N) of the matrix B were evaluated for various values of the Grashof number and the Grashof number for neutral stability was bracketed by a sign change of Real (σ_1). The value of the Grashof number was then calculated to within .05%. Figure 4 shows the Real (σ_1) for various values of Grashof number with wave number as the parameter for Pr=1. c) The neutral stability curve was plotted from this information. Figure 6 shows this for Pr=1. The minimum value of the Grashof number on this curve is the critical Grashof number. Curves of constant Real (σ_1) are also plotted in this figure. d) The convergence of the method is examined by ob-

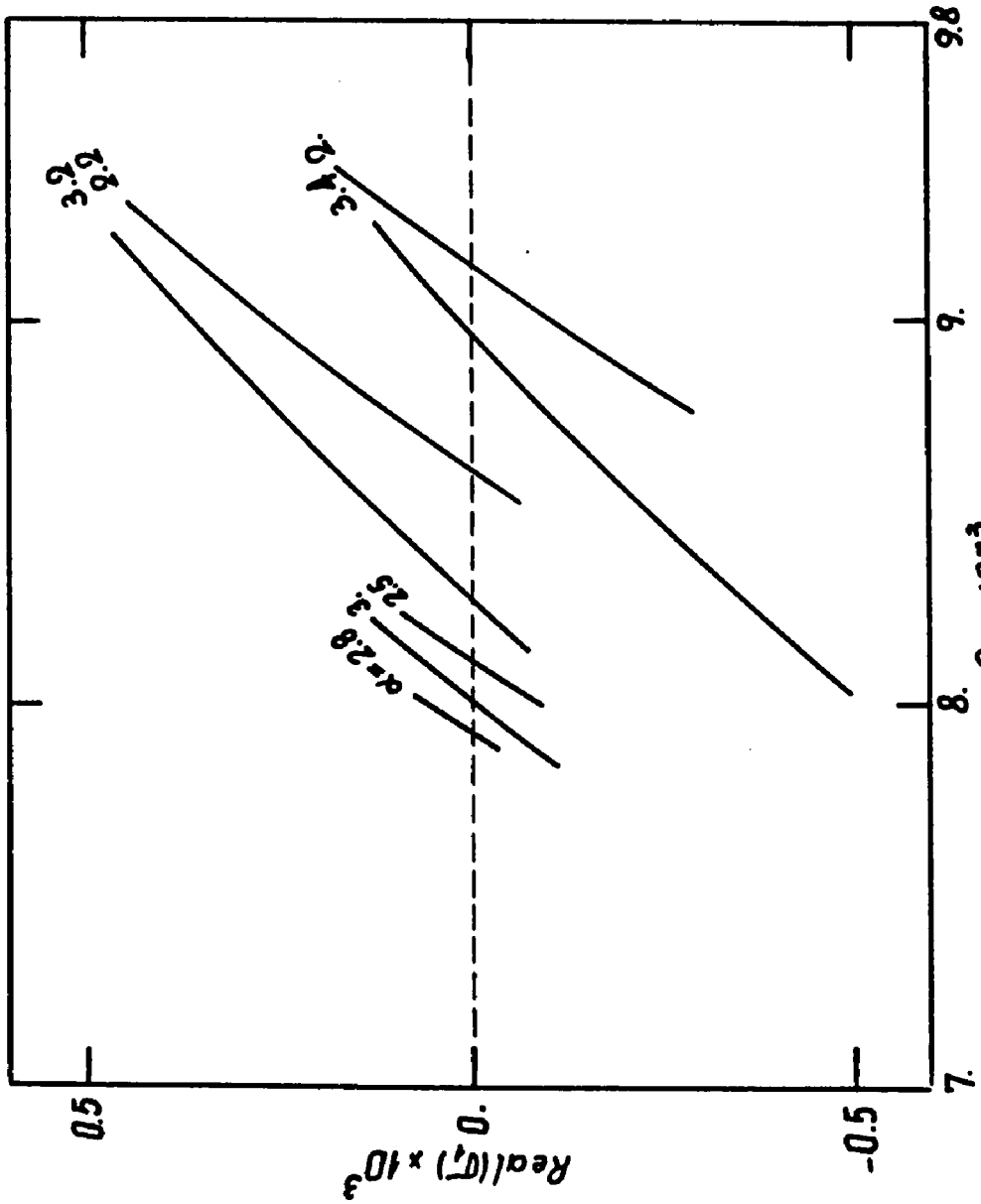


Figure 4. Real parts of the first Eigenvalue, Pr=1.

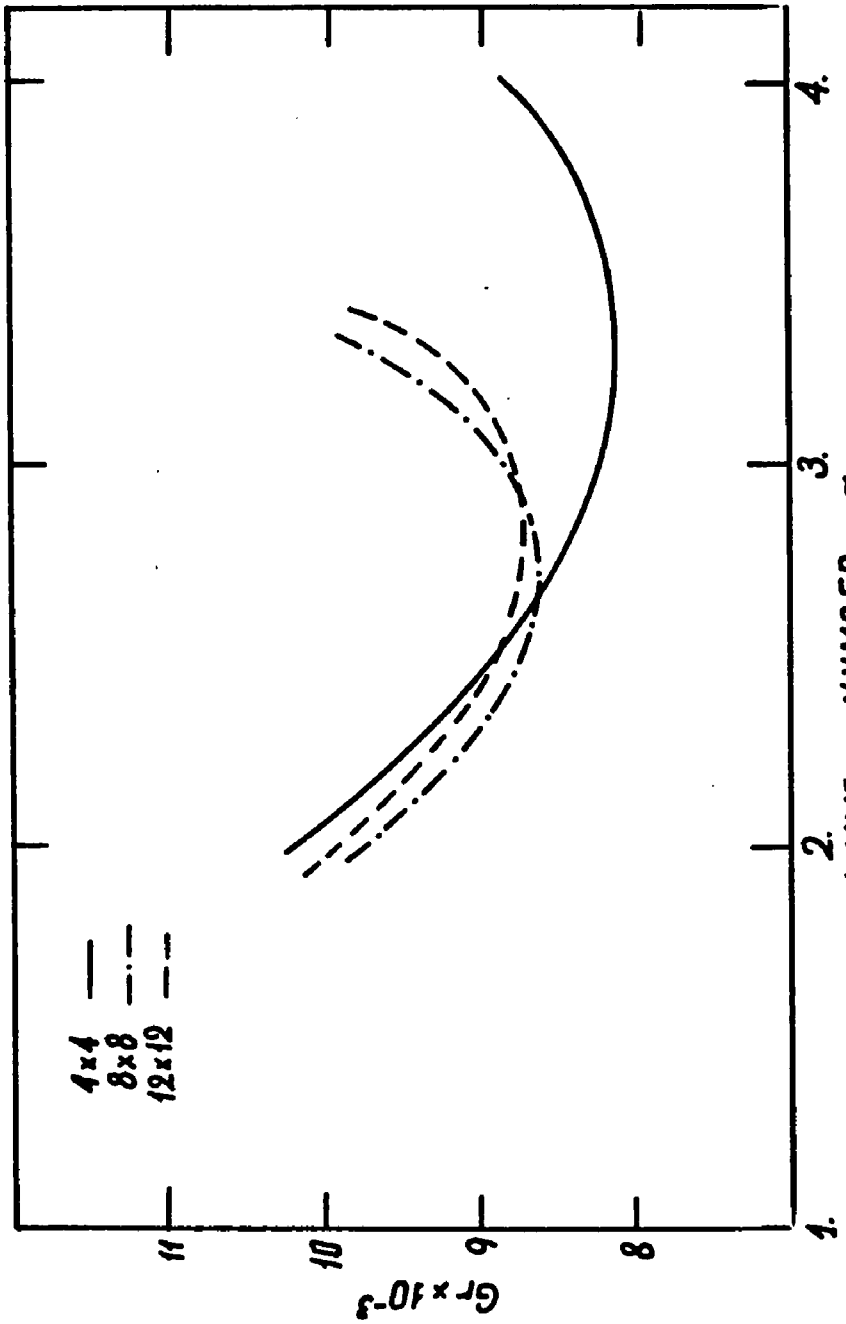


Figure 5. Convergence study for the steady problem, Pr=1.

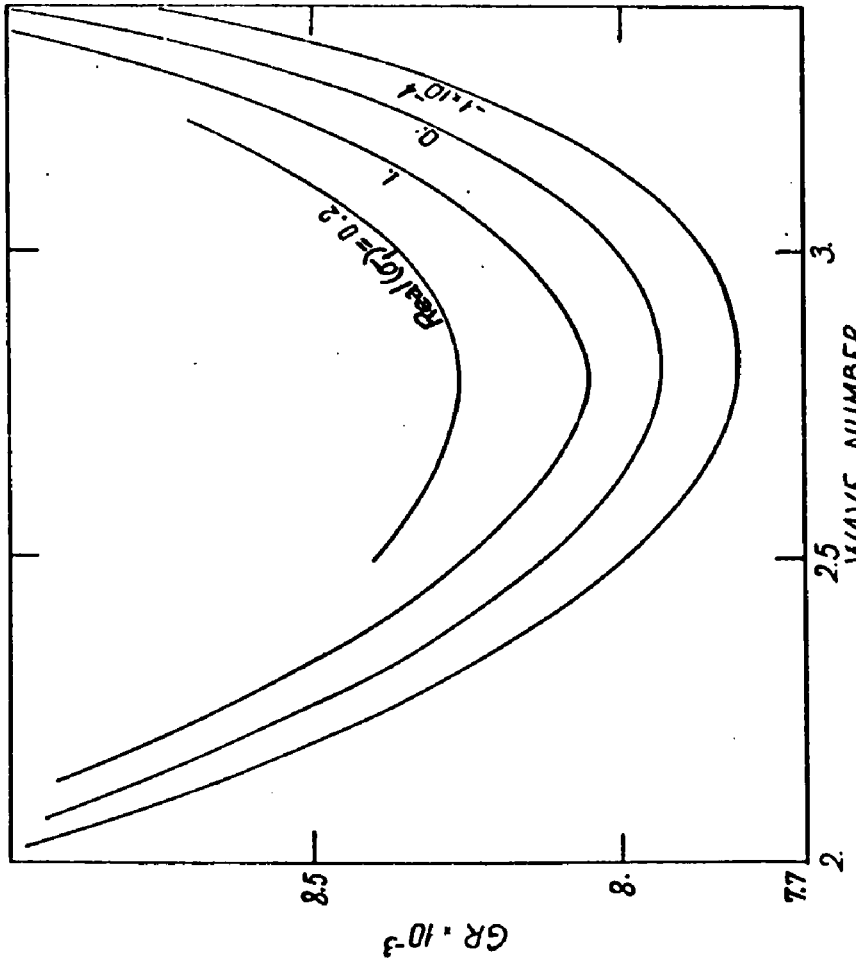


Figure 6. Neutral stability curve and curves of constant amplification, Pr=1.

taining the neutral stability curves for successively higher approximations. Figure 5 shows this study for Pr=1. In this case the change in the critical Grashof number between 3rd and 4th approximation is less than 0.3%. e) Examination of the Imag (σ_1) at the critical point reveals the nature of the critical disturbance. In fact, c=Imag(σ_1)/α Gr is the wave speed of the disturbance. If c=0 the disturbance is stationary and if c$\neq$0, it is a traveling wave. Figure 8 shows that c=0 for Pr$<$5 and c$\neq$0 for Pr$>$ 25. f) By repeating the procedure (a) to (d) for various Prandtl numbers, variation of the critical Grashof number with wave number is obtained. This is shown in Figure 7. The results in the stationary range are compared with those given by Rudakov [37] and in the overstable range they are compared with the curve suggested by Gill and Kirkham [23] for large Prandtl numbers.

Calculations for Pr=0 were done by using the simplified equation (62). The critical value of Grashof number obtained is 7923, at α =2.75. This agrees with the results of Vest and Arpaci [41] within .01%. Gill and Davey [22] have tried to use the simplified equations (64) and (65) to obtain results for Pr$\rightarrow\infty$ and found that flow is always unstable according to this equation. Hence no attempt was made to use these equations.

The value obtained for the critical Grashof number at Pr=.7 agrees with other analytical results within .5%. However, it is 10% less than the experimentally observed value given by Vest and Arpaci [41].

The critical wave number was a very weak function of the Prandtl number. It varied between 27 and 28 in the Prandtl number range 0 to 1000.

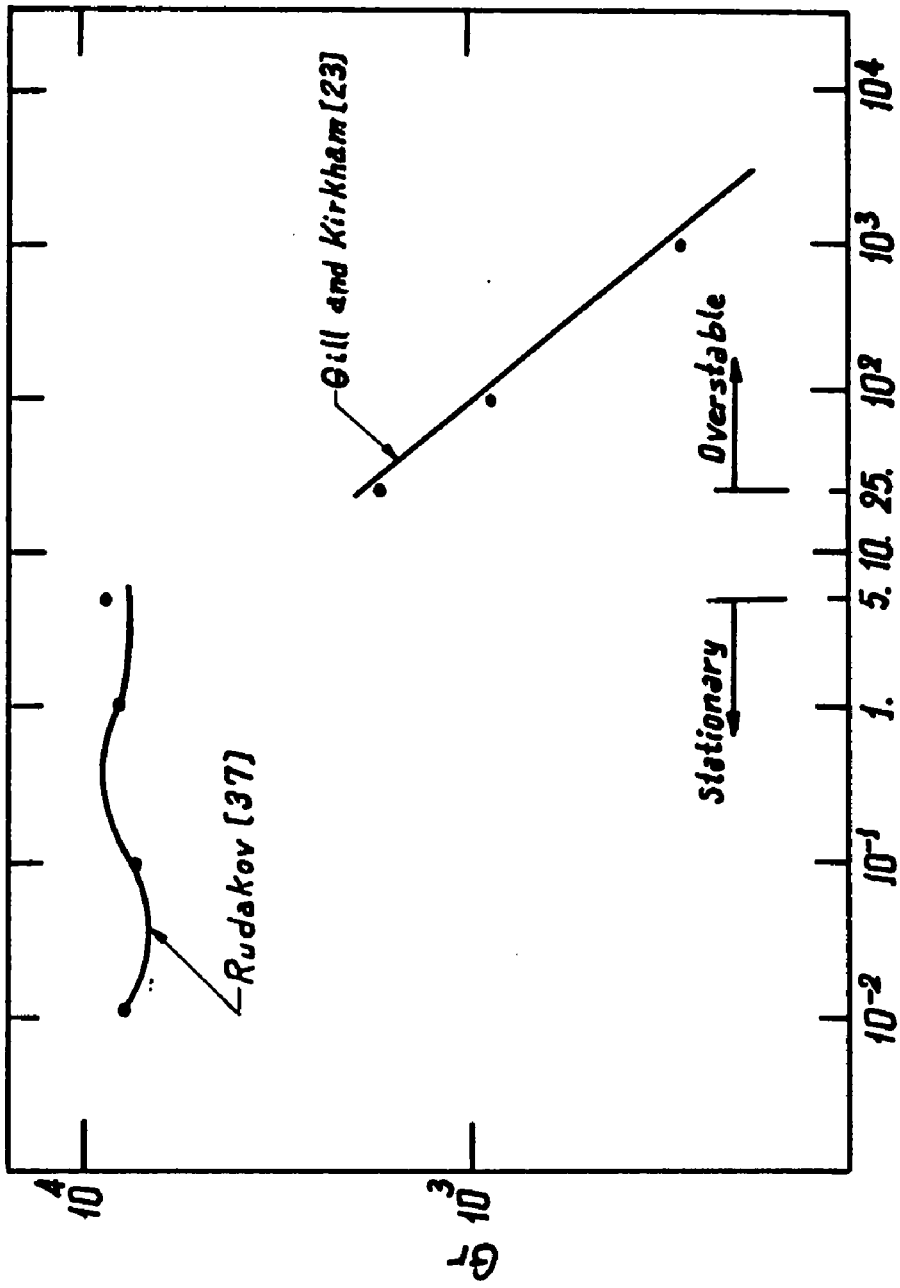


Figure 7. Variation of the critical Grashof number with the Prandtl number for the steady problem.

In the range $Pr < 5$, the wave speed was zero. Physically this means that the critical disturbance is stationary and it will presumably grow into stationary cells. This has been observed by Vest and Arpaci [41] in experiments with air. The variation of the critical Grashof number with the Prandtl number is a smooth but irregular curve. The results obtained here agree with that obtained by Rudakov [37] within .5%.

For $Pr > 25$ the wave speed was non-zero and it increased from 8.3×10^{-3} to 8.4×10^{-3} as Prandtl number varied from 25 to 1000. This means that the critical disturbance has a wave character with a wave speed larger than the maximum velocity of flow. The critical Grashof number obtained in this range were within 3% of the following equation

$$Gr = (9.4)(10^3)(Pr)^{-1/2}$$

obtained by Gill and Kirkham [23] for large Prandtl numbers.

All the above results were obtained by using the fourth approximation (i.e. B is 16 x 16 matrix). In the range of Prandtl number 5 to 25 higher approximations were required to obtain the results. Figure 8 shows that the wave speed changes from 0 to 8.3×10^{-3} in a narrow region near this range of Prandtl numbers. Since the maximum value of the base flow is $\simeq 8 \times 10^{-3}$, the wave speed is less than $(\bar{u}_1)$ max in this region i.e. the disturbance travels at a speed less than the base flow velocity. This is a situation analogous to plane Poiseuille flow where a large number of approximating functions is required to represent the disturbance adequately [8]. For the plane Poiseuille flow the neutral disturbance is a traveling wave with a wave speed smaller than the maximum velocity of the flow [8, 26, 33]. While using a method similar to the

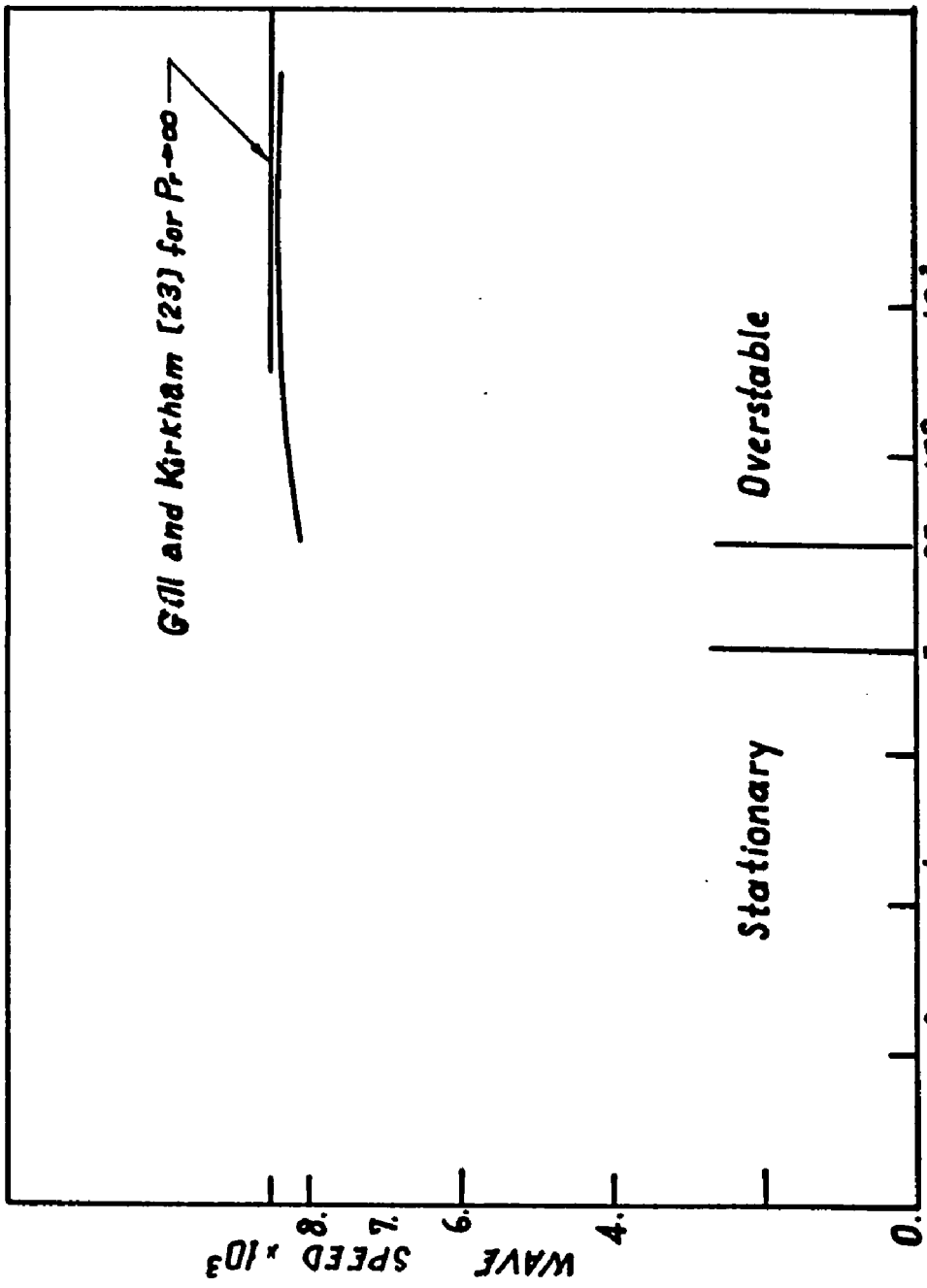


Figure 8. Variation of the wave speed with the Prandtl number.

one used in the present analysis for the plane Poiseuille flow Dolph and Lewis [8] found that 20 approximating functions were required to obtain the correct neutral curve. The reason for this bad convergence seems to be the rapid variation of the eigenfunctions for a disturbance with a wave speed smaller than the maximum flow velocity.

The following conclusions are reached regarding the stability of steady natural convection in a vertical slot in the conduction regime.

1) The natural convection in a vertical slot is unstable with respect to stationary disturbances for $Pr < 5$.

2) For $Pr > 25$, the critical disturbance is oscillatory with the wave speed greater than the velocity of the base flow.

3) The disturbance has a simple functional dependence on y in the region, $25 < Pr$, $Pr < 5$ but this dependence is complicated in the region $5 < Pr < 25$.

3.2 Periodic Base Flow

When the slot is oscillating harmonically in the vertical direction, the base flow is steady periodic. The equations governing the amplitude of the disturbance are (81), where in this case B is a periodic matrix. The procedure followed to find the stability limits was: a) for specified values of Pr, A, R and α the elements of the matrix B are calculated. b) The set of differential equations

$$\frac{dX}{dt} = BX$$

is first order equations with periodic coefficients. This set is numerically integrated M times from t=0 to $t=\tau$ by using a predictor-corrector method. The initial conditions for the first integration are

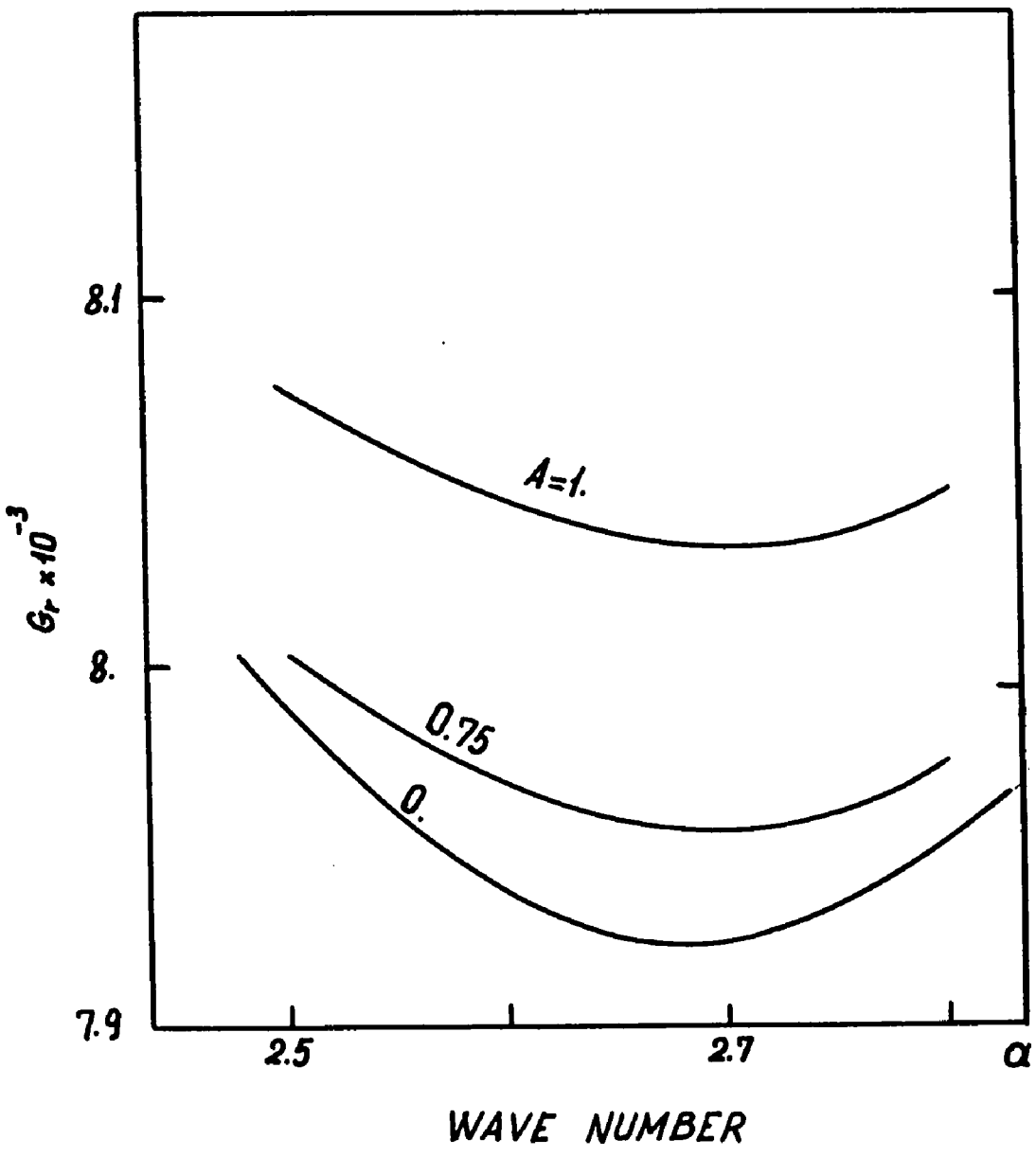


Figure 9. Neutral stability curves for the unsteady base flow, Pr=0, R=100.

$x_1(0)=1$, $x_j(0)=0$ $(j=1,2,..M)$; for the second integration $x_1(0)=0$, $x_2(0)=1$, $x_j(0)=0$ $(J=3,4,..M)$; etc. This gives $X(\tau)$ in the equation (98).

c) The eigenvalues $\lambda_J(J=1,2,..M)$ of $X(\tau)$ are related to $\sigma_J(J=1,2,..M)$ by the relation

$$\sigma_j = \frac{1}{\tau} \ln \lambda_j$$

d) A check on the entire numerical procedure is obtained by computing the difference between $\sum_{J=1}^{N} \sigma_j$ and the trace of B. e) Values of Grashof number for the neutral stability were obtained by procedures similar to the steady base flow.

The results for the periodic base flow are presented in figures 9 to 12. The first of these, figure 9 shows the neutral stability curves for Pr=0, R=100, and various amplitudes A. The curve for A=0 is identical to the neutral stability curve for the steady base flow. This figure also shows that the critical wave number is not changed much by the oscillations. In figures 10, 11 and 12 ΔGr is defined as

$$\Delta Gr = \left[\frac{Gr_{unsteady\ flow} - Gr_{steady\ flows}}{Gr_{Steady\ Flow}}\right](100)$$

so that ΔGr is the percent change in the Grashof number from the steady value, due to oscillations. Although no experimental verification of the analysis is obtained in this study, the trends exhibited by figures 10 and 11 agree with other experimental results [9, 10, 13, 20] for other periodic flows. The ΔGr is positive for small values of the amplitude and moderate frequencies. As the amplitude of oscillations is increased ΔGr increases from zero to a maximum and then decreases. At large values of the amplitude the flow is destabilized i.e. ΔGr is

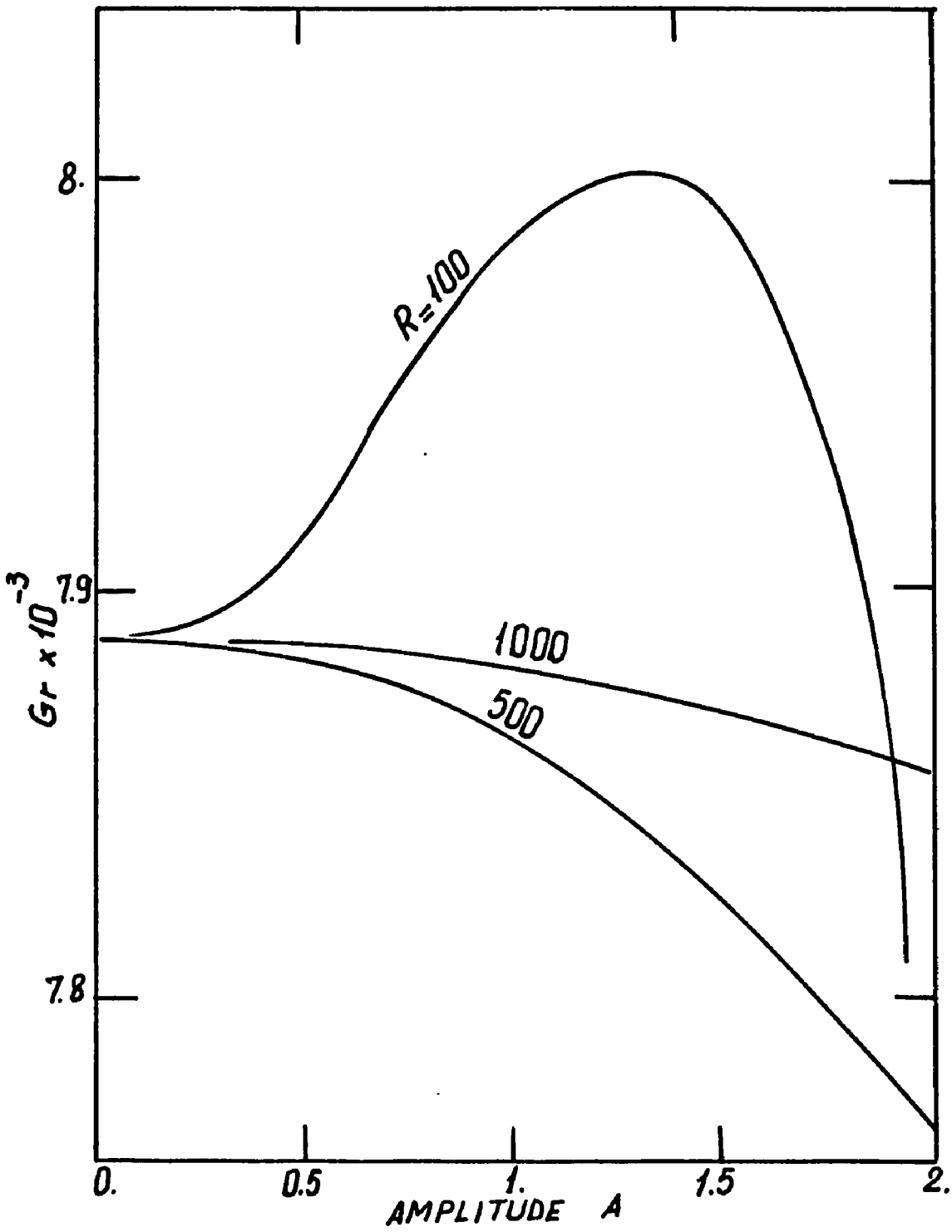


Figure 10. Effect of amplitude of oscillation on the critical Grashof number, Pr=1.

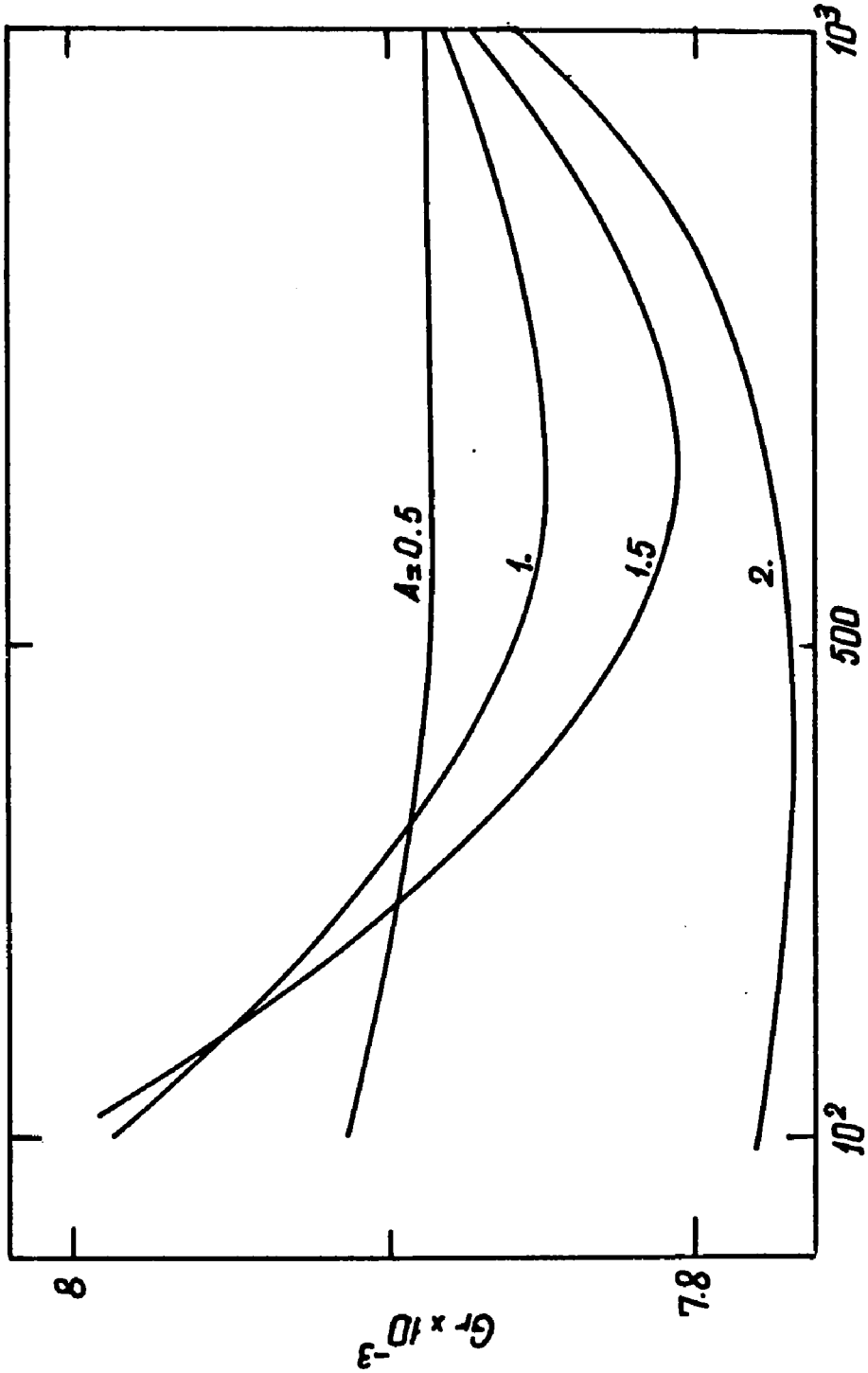


Figure 11. Effect of frequency of oscillation on the critical Grashof number, Pr=1

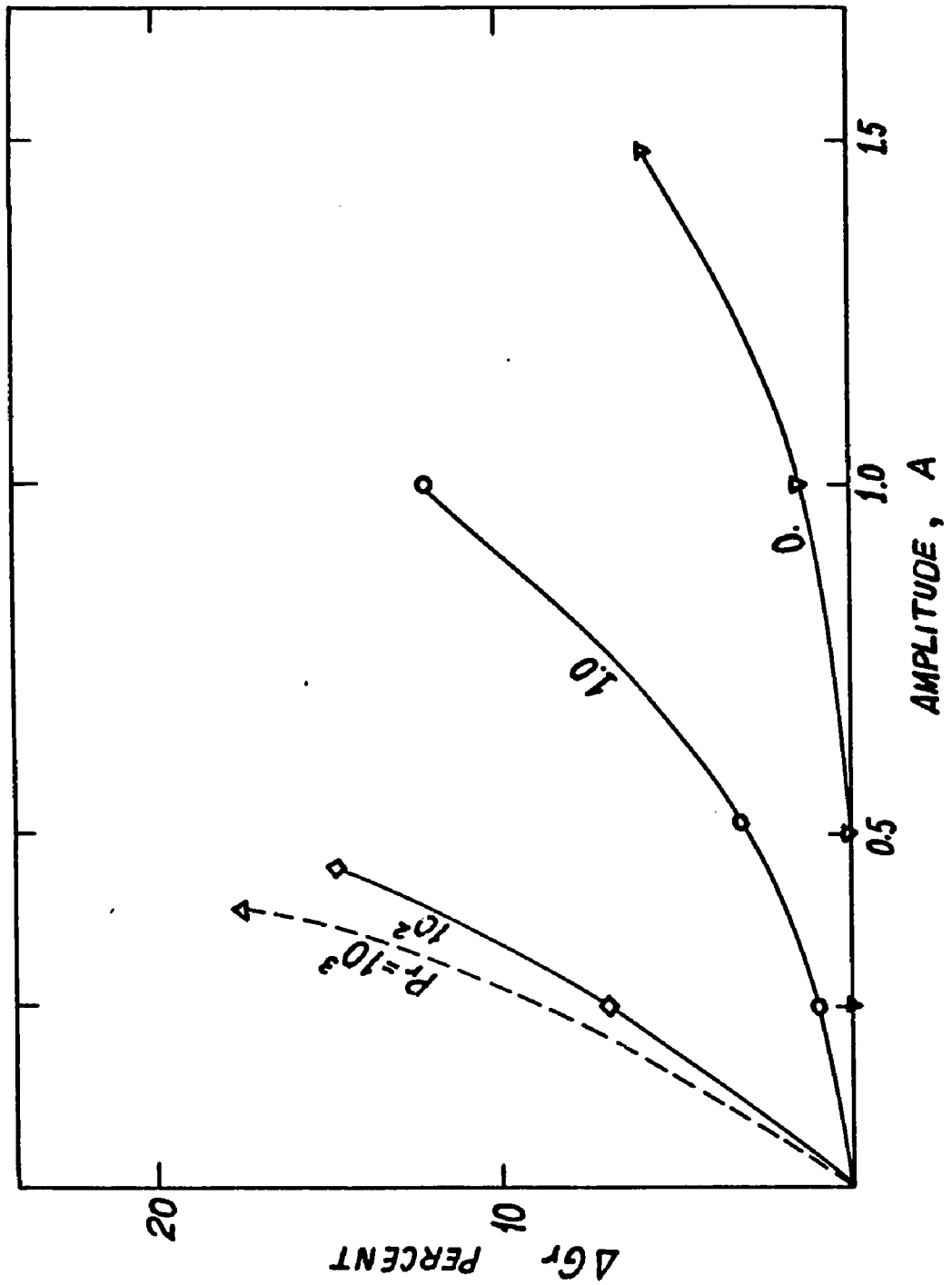


Figure 12. Effect of frequency of oscillation at various Prandtl number.

negative. The effect of oscillations disappears asymptotically as $A \to 0$ and $R \to \infty$. This is expected as in both cases, the unsteady part of the base flow $u_2 \to 0$. Finally, figure 12 shows the effect of oscillations at various Prandtl numbers. It is seen that oscillations have more effect on the stability limits at higher Prandtl number. This result is analogous to that found by Venezian [39] for Benard problem with boundary temperature modulation. Venezian [39] found that the Prandtl number was an important parameter in problems with oscillations, although in the steady problem the critical Grashof number is independent of the Prandtl number. In the present problem this effect is magnified because for the large Prandtl numbers the steady flow itself is unstable to oscillatory disturbance.

Numerical difficulty was encountered for small frequencies and also for large Prandtl numbers. In this region, in order to satisfy the condition that $\sum_{j=1}^{N} \sigma_j$ = trace of matrix B, very small steps were required in the numerical integration of the equation (81). For Pr=1000 at R=100., A=0.5, five minutes were required on the IBM 360/67 to extract one set of eigenvalues. Detailed parametric study was not undertaken as it was felt that additional expenditure of machine time was not justified in terms of further understanding of the phenomena. The following conclusions are reached about the stability of steady periodic convection

1) Oscillations enhance the stability at small amplitude and moderate frequencies.

2) Large amplitude oscillations destabilize the flow.

3) Effect of oscillations is large for large values of the Prandtl number.

4) Effect of oscillations disappears for $R \to \infty$ and $A \to 0$.

APPENDIX I

THE INNER PRODUCTS

In the application of the Galerkin method, inner products of the form

$$\langle f_1 | y^m | f_2 \rangle \equiv \int_{-1/2}^{+1/2} y^m f_1 f_2 \, dy$$

are required. Reid and Harris [35] and Vest [40] have calculated many of the inner products required in this study. There are errors in two integrals listed by Vest [40]. The correct integrals are

1. $$\langle S_m | y | C_n'' \rangle = 2(\mu_m^4 - \lambda_n^4) \{ (1/2) S_m'''(1/2) . S_n''(1/2) - (1/2) S_m''(1/2) . C_n''(1/2) - S_m''(1/2) . C_n''(1/2) - 4\lambda_n^4 (\mu_m^4 - \lambda_n^4)^{-1} S_m''(1/2) . C_n''(1/2) \}$$

2. $$\langle \cos m | y^3 | \sin n \rangle = 48 \rho_m K_n (\rho_m^2 - K_n^2)^{-3} \cdot \left[1 + 2 K_n^2 (\rho_m^2 - K_n^2)^{-1} \right] \cdot \sin(\rho_m/2) \cos(K_n/2) - 3 \rho_m K_n (\rho_m^2 - K_n^2)^{-2} \cdot \sin(\rho_m/2) \cdot \cos(K_n/2) \, .$$

The following additional inner products were required in connection with the unsteady part of the base flow.

3. $$\langle (\sinh qy) . C_n . S_m \rangle = I_1 - I_2 - I_3 + I_4$$

4. $$\langle (\sinh qy) . C_n'' . S_m \rangle = \lambda_n^2 \left[I_1 - I_2 + I_3 - I_4 \right]$$

5. $$\langle (\sinh qy) . C_n . S_m'' \rangle = \mu_m^2 \left[I_1 + I_2 - I_3 - I_4 \right]$$

6. $$\langle \sinh qy \cdot \sin_m \cos_n \rangle =$$

$$= -i\left[-(K_m+\rho_n)^2 - q^2\right]^{-1} \cdot \left\{(K_m+\rho_n)\cos\left[\frac{K_m+\rho_n}{2}\right]\cdot \right.$$

$$\left. \sin(iq/2) - iq\cdot\cos(iq/2)\cdot\sin\left[\frac{K_m+\rho_n}{2}\right]\right\} -$$

$$- i\left[-(K_m-\rho_n)^2 - q^2\right]^{-1} \cdot \left\{(K_m-\rho_n)\cos\left[\frac{K_m-\rho_n}{2}\right]\cdot \right.$$

$$\left. \sin(iq/2) - iq\cos(iq/2)\cdot\sin\left[\frac{K_m-\rho_n}{2}\right]\right\}$$

where

$$I_1 = \langle \sinh qy \cdot \sinh_m \cdot \cosh_n \rangle (1/K_1K_3),$$

$$I_2 = \langle \sinh qy \cdot \cosh_n \cdot \sin_m \rangle (1/K_1K_4),$$

$$I_3 = \langle \sinh qy \cdot \sinh_m \cdot \cos_n \rangle (1/K_2K_3),$$

$$I_4 = \langle \sinh qy \cdot \cos_n \cdot \sin_m \rangle (1/K_2K_4).$$

With

$$\sinh_m = \sinh \mu_m y$$

$$\sin_m = \sin \mu_m y$$

$$\cosh_m = \cosh \lambda_m y$$

$$\cos_m = \cos \lambda_m y$$

$$K_1 = \cosh(\lambda n/2)$$

$$K_2 = \cos(\lambda n/2)$$

$$K_3 = \sinh(\mu m/2)$$

$$K_4 = \sin(\mu m/2)$$

Each of the integrals I_1, I_2, I_3 and I_4 are similar to the integral 6 above.

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