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EXISTENCE THEOREMS FOR A CLASS
OF OPTIMAL CONTROL PROBLEMS
WITH DELAY.

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EXISTENCE THEOREMS FOR A CLASS OF OPTIMAL CONTROL PROBLEMS WITH DELAY

by
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ABSTRACT

EXISTENCE THEOREMS FOR A CLASS OF

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by

Thomas Strong Angell

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Let $[t_1,t_2]$ be any finite interval of the real line, τ a real number, and let x be an element of $C[t_1-\tau,t_2]$, the continuous functions mapping $[t_1 - \tau,t_2]$ into E^n. For each element t of the interval $[t_1,t_2]$ we define an element of the space $C[-\tau,0]$ by the equation

$$x_t(\theta) = x(t + \theta),\ -\tau \leq \theta \leq 0.$$

The purpose of this thesis is to discuss aspects of the existence problem for optimal solutions of Lagrange problems with delay. Thus, we are interested in determining, for given $t_1,t_2 \in E^1$, $t_1 < t_2$, a measurable vector function $u: [t_1,t_2] \to E^m$ and a corresponding continuous vector function $x: [t_1 - \tau,t_2] \to E^n$, which is absolutely continuous on $[t_1,t_2]$, and which satisfies, almost everywhere, the differential system

$$dx/dt = F(t,x_t,u(t)) \quad (t_1 \leq t \leq t_2),$$

subject to boundary conditions

$$(t_1,x_{t_1},t_2,x_{t_2}) \in B,$$

and constraints

$$(t,x_t) \in A(t_1 \leq t \leq t_2),$$

$$u(t) \in U(t,x_t) \text{ almost everywhere in } [t_1,t_2],$$

and for which the integral

$$I[x,u] = \int_{t_1}^{t_2} F_o(s,x_s,u(s))ds$$

takes on its minimum value. It should be noted that the viewpoint taken throughout this thesis is that both the boundary conditions and the state constraints properly should be written in terms of the state space which we have taken to be $E^1 \times C[-\tau,0]$.

The first goal will be to establish conditions for the local existence and uniqueness of solutions to differential equations of the form

$$\dot{x}(t) = F(t,x_t).$$

Chapter 1 is devoted to this task. In Chapter 2, we turn to existence theorems for the Lagrange problem. Under the single additional restriction that the initial conditions lie in a compact set, we are able to establish existence theorems for the Lagrange problem with delay which are analogous to those established by L. Cesari.

The third chapter utilizes a new technique to discuss controllability for the Lagrange problem. The theorems of Chapter 2 assume that admissible pairs exist. By using an extension of a fixed point theorem due to Kakutani, we are able to present a sufficient condition for the existence of admissible pairs under hypotheses closely related to those of the existence theorems of the previous chapter.

Finally, Chapter 4 contains a discussion of linear systems; first an existence theorem for systems which are linear in the control variable, and second, a discussion of some aspects of the controllability problem for autonomous linear systems. In particular, using the results of Hale for linear functional differential equations, we are able to reduce the problem of complete null controllability of the infinite dimensional system to the analysis of an associated finite dimensional system.

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INTRODUCTION

The ordinary differential equations used to describe physical systems are usually taken in the form

$$\dot{x}(t) = f(t,x);\ x(t_o) = x_o \tag{0.1}$$

where $f : E^1 \times E^n \to E^n$, $n \geq 1$. Implicit in this formulation is the assumption that the velocity of the system at any time does not depend on the past states but only on the present state of the system. In recent years, particularly in the Soviet Union, much interest has developed in problems in which the behavior of the system depends in some way on the past history of the process. In such cases, the value of the function f in (0.1) will depend, explicitly, not only on the instantaneous value of the state variable, but also on the past values of the state variable. Equations of this type have become particularly important in the theory of automatic control. For example, equations which reflect a dependence on past history arise in the study of technological processes which involve the transport of some kind of signal with a velocity which is small compared to the dimensions of the system [13],[6].

These technological problems are not, however, the only area in which equations with past history have arisen. Since the work of Volterra [43],[44], it has been recognized that these equations have many applications to biological

processes, particularly in the description of ecological and physiological systems. Recently suggested models for population growth [45],[46], the spread of infectious diseases [47], as well as studies of the circulatory system [4], have involved special forms of these equations. In the area of physics, Driver [15],[17], has shown that equations of motion in the two body problem of classical electrodynamics contain time lags depending on the unknown, while Coleman [11] and Gurtin [23] have discussed the thermodynamics of hysteresis phenomina using equations with varying time lags.

From the mathematical point of view, the most thoroughly studied class of equations which reflects hereditary dependence, is the class of differential equations with retarded argument or differential-difference equations. In normal form, these equations may be written as

$$\dot{x}(t) = f(t,x(t),x(t-h_1),\ldots,x(t-h_k)), \tag{0.2}$$

$$0 < h_1 \leq h_2 \leq \ldots \leq h_k.$$

The study of differential-difference equations has a long history dating back to Euler [20] who, in 1790, investigated methods of solution of the linear equation

$$\sum_{i=0}^{m} \sum_{k=1}^{k_i} a_{k,i} y^{(i)}(x + h_{k,i}) = 0 \tag{0.3}$$

More recently, linear equations have been studied by Soviet authors including Tychonoff [42], Myskis [36], Elsgoltz [19] and Krasovskii [31] and the Rumanian Halanay [24]. In this country, Bellman and Cooke [3] and Pinney [39] have published treatises on differential-difference equations.

There are many other types of equations which take into account past history and these include differential-difference equations with variable lags, that is, equations of the form

$$\dot{x}(t) = f(t,x(\alpha_0(t)),\ldots,x(\alpha_m(t))),\ t > t_0, \tag{0.4}$$

where $\alpha_i(t)$ are continuous functions such that $\alpha \leq \alpha_i(t) \leq t$ for $t \geq \alpha$, and integro-differential equations of Volterra type

$$\dot{x}(t) = F(t,x(t), \int_{\alpha}^{t} f(t,\tau,x(\tau))d\tau). \tag{0.5}$$

In this thesis we will be concerned with yet another type of equation introduced by Krasovskii in 1962. Krasovskii suggests that differential-difference equations can be considered as a special case of the more general class of functional differential equations whose right hand sides are operators on the function space $C[-\tau,0]$. This view has proven to be very fruitful as it admits an analysis of these equations which closely parallels the theory of ordinary differential equations. Recently Hale [25],[26],[27], and Hale and Perelló [28], by adopting this

point of view, have been able to develop the geometric theory of a class of functional differential equations along lines very similar to the theory for ordinary differential equations.

To be more precise, let $[t_1,t_2]$ be any finite interval of the real line, τ a real number, and let x be an element of $C[t_1 - \tau,t_2]$, the continuous functions mapping $[t_1 - \tau,t_2]$ into E^n. (For any interval I, we shall assume that C(I) is equipped with the norm of uniform convergence.) For each element t of the interval $[t_1,t_2]$ we define an element of the space $C[-\tau,0]$ by the equation

$$x_t(\theta) = x(t + \theta),\ -\tau \leq \theta \leq 0. \tag{0.6}$$

Let D be some open connected set in $E^1 \times C[-\tau,0]$ and let $F : D \to E^n$. We consider the functional differential equation of finitely retarded type

$$\dot{x}(t) = F(t,x_t). \tag{0.7}$$

In this setting it is natural to take initial conditions of the form $x_{t_o} = \phi,\ \phi \in C[-\tau,0]$, and to consider the "state" of the system to be given, not by a vector in E^n, but by a function in the Banach space $C[-\tau,0]$. Trajectories of the system will, however, be considered as paths in E^n, that is, as vector-valued functions $x(t) = (x^1(t),\ldots,x^n(t))$.

In the case that the operator F is a bounded linear operator on $E^1 \times C[-\tau,0]$, we may apply the Riesz

representation theorem and write

$$(0.8) \qquad \dot{x}(t) = \int_{-\tau}^{0} [d\eta(t,\theta)]\, x_t(\theta)$$

where $\eta(t,\theta)$ is an $n \times n$ matrix whose entries are functions of bounded variation and the integral is taken in the sense of Stieltjes. It is easy to see that a linear differential-difference equation can be written in this more general form. The representation of linear functional differential equations in the form (0.8) will be very important in our discussion of the linear control system.

In light of the late appearance of a systematic treatment of differential equations with delay, it is not surprising that relatively little work has been done on problems of optimal control of systems described by such equations. Elsgoltz [19] has obtained results for classical problems of the calculus of variations which involve time lags (for example, he has derived the analogs of Euler's equations for linear differential-difference equations). Oguztoreli [37],[38] has done considerable work on the time-optimal control problem and Chyung and Lee [10] have studied the linear time-delay control problem for a variety of cost functionals including, in particular, the quadratic case. A study of necessary conditions for systems described by differential-difference equations has been made by Pontryagin et. al. [40], Chyung and Lee [10] and by Oguztoreli [38]. Finally, we should mention the

recent work of Banks [2]. In his thesis, Banks studied necessary conditions for a class of control problems with delay and offered, in the final chapter, a generalization of an existence theorem due to Cesari [8].

We will now give a precise statement of the control problem with which we will be concerned.

Let $x = (x^1,\ldots,x^n)$ be a vector valued function mapping a closed interval of E^1 into E^n and let $u = (u^1,\ldots,u^m)$ be a vector valued function defined on the same interval and having values in E^m. Let $t \in E^1$ be an independent variable. Denote by A an arbitrary subset of $E^1 \times C[-\tau,0]$. Let U be a set-valued map $U : A \to 2^{E^m}$, and for each $(t,\phi) \in A$, denote its image under U by $U(t,\phi)$. Let F_i, $i = 0,1,\ldots,n$, be given real valued functionals defined for all $(t,\phi) \in A$ and $u \in U(t,\phi)$ and let $F = (F_1,F_2,\ldots,F_n)$. Let B be a given subset of $E^1 \times C[-\tau,0] \times E^1 \times C[-\tau,0]$.

We are interested in studying the following problem:

Determine, for given $t_1,t_2 \in E^1$, $t_1 < t_2$, a measurable vector function $u : [t_1,t_2] \to E^m$ and a corresponding continuous vector function $x : [t_1 - \tau,t_2] \to E^n$, which is absolutely continuous on $[t_1,t_2]$, and which satisfies, almost everywhere, the differential system

$$\frac{dx}{dt} = F(t,x_t,u) \qquad (t_1 \leq t \leq t_2),$$

subject to boundary conditions

$$(t_1, x_{t_1}, t_2, x_{t_2}) \in B,$$

and constraints

$$(t, x_t) \in A \qquad (t_1 \leq t \leq t_2),$$

$$u(t) \in U(t, x_t) \text{ almost everywhere in } [t_1, t_2],$$

and for which the integral

$$I[x,u] = \int_{t_1}^{t_2} F_o(s, x_s, u(s))ds$$

takes on its minimum value.

In order to discuss this problem, we will make some preliminary assumptions. We assume that the set A is closed and bounded and that $U(t,\phi)$ is closed for each $(t,\phi) \in A$. Let M be the set of all (t,ϕ,u) where $(t,\phi) \in A$ and $u \in U(t,\phi)$. Thus $M \subset E^1 \times C[-\tau,0] \times E^m$. We will assume that F is continuous on the set M. When any measurable function u is substituted in the differential equation, we obtain

$$\frac{dx}{dt} = F(t, x_t, u(t))$$

which is an ordinary differential equation in normal form. We assume in what follows that, given such a control function and initial conditions $(t_1, \phi) \in A$, this equation will have a solution $x(t_1,\phi)$ such that $x_{t_1} = \phi$. If

(t_1, ϕ) is an interior point of A, and F, in addition to being continuous on M, satisfies a local Lipschitz condition in its second argument, then we can expect that a given control function **u** will generate a unique solution x of the differential equation. In this generality, the problem is the natural extension of the Lagrange form of the optimal control problem with unilateral constraints on both the state variables and the control functions.

Our object in what follows is to consider various aspects of the existence problem for optimal solutions of the Lagrange problem with delay. The first goal will be to establish conditions for the local existence and uniqueness of solutions to functional differential equations of finitely retarded type. Chapter 1 will be devoted to this task. The theorems presented there are not altogether new, but clear statements and proofs of these results are difficult to find in the literature. The author has been unable to find any reference in the literature to the theorems of local existence and uniqueness of solutions in the sense of Carathéodory.

In Chapter 2 we turn to existence theorems for the Lagrange problem. Under the single additional restriction that the initial conditions lie in a compact set, we are able to establish existence theorems for the Lagrange problem with delay which are analogous to those established by Cesari [9]. The results here are more general than

those of Banks [2 ; Chapter II] in that both the problem, which has boundary conditions and state constraints written in terms of the function space $C[-\tau,0]$, and the existence theorems which we establish, subsume his results. It should be noted that the viewpoint taken throughout this thesis is that both the boundary conditions and the state constraints properly should be written in terms of the state space which we have taken to be $E^1 \times C[-\tau,0]$.

The third chapter utilizes a new technique, first suggested by Tarnove [41], to discuss the controllability for the Lagrange problem. The question of controllability is, essentially, the question of whether the control problem is well posed, that is whether there is any admissible control function which generates a trajectory satisfying the boundary conditions and constraints. Such a trajectory, together with the control generating it, is called an admissible pair and the existence theorems of Chapter 2 always assume that such pairs exist. By using an extension of a fixed point theorem due to Kakutani [29], we are able to present a sufficient condition for an admissible pair under hypotheses closely related to the hypotheses of the existence theorems.

Finally, Chapter 4 contains a discussion of linear systems; first an existence theorem for systems which are linear in the control variable and second a discussion of some aspects of the controllability problem for

autonomous linear systems. In particular, using the results of Hale [25] for linear functional differential equations, we are able to reduce the problem of complete null controllability of the infinite dimensional system to the analysis of an associated finite dimensional system.

CHAPTER I

EXISTENCE AND UNIQUENESS THEOREMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS OF FINITELY RETARDED TYPE

Our first task is to establish conditions for the existence and uniqueness for solutions of functional differential equations of finitely retarded type. We adopt here the viewpoint of Krasovskii [31] in that we consider the right hand side of the differential equation to be defined in an open subset of $E^1 \times C[-\tau,0]$. This point of view enables us to state and prove theorems whose form and proof closely follow the familiar case of ordinary differential equations.

The results presented here are not altogether new. Myskis [36] and Elsgoltz [19] give existence theorems for differential equations with variable time lags. Myskis also refers to a paper of Tychonoff [42] for the proof of local existence and uniqueness theorems for very general Volterra functional equations of the form $y(t) = V(t,y(s);\ t_0 \leq s \leq t)$. Driver [16] has proven existence theorems for equations of the form $x(t) = f(t,x(t),x(g(t)))$ where $g(t) < t$, and where he considers f as defined on a domain D of E^{mn+1}.

Banks [2] has presented a condition for global existence of solutions to equations of the form

$$x(t) = F(t,x(\cdot)), x(\cdot) \in C[\alpha_0,t] \quad \text{for some } \alpha_0.$$

While he proves the existence of solutions in the sense of Carathéodory or, as we will refer to them, weak solutions, he does not establish uniqueness results.

In Section 1 of this chapter we prove theorems for usual solutions to functional differential equations of finitely retarded type. In Section 2, we define solutions in the sense of Caratheodory, and give conditions for local existence and uniqueness. In Section 3, we show that the systems studied by Banks and Driver can be transformed into systems of finitely retarded type and hence, the theorems presented here guarantee existence and uniqueness for the case of variable time lags.

1. Usual Solutions.

Let E^n denote the n-dimensional Euclidean space with any suitable norm. We will denote the norm by $|\cdot|$. For any interval $I \subset E^1$, let $C(I,E^n)$ denote the set of all continuous functions defined on I with range in E^n. When no confusion can arise, we will denote this class by $C(I)$. In the case that I is a closed interval, we will consider $C(I)$ as a Banach space with norm given by

$||\psi|| = \sup \{|\psi(t)| : t \in I\}$ for any $\psi \in C(I)$.

Let t_o be any real number and α and τ positive real numbers. Consider the space $C[t_o - \tau, t_o + \alpha]$. For any $x \in C[t_o - \tau, t_o + \alpha]$ and for any $t \in [t_o, t_o + \alpha]$ we define an element x_t of the space $C[-\tau, 0]$ as follows:

$$x_t(\theta) = x(t + \theta), \quad -\tau \leq \theta \leq 0. \tag{1.1.1}$$

Let D be any open connected set in $E^1 \times C[-\tau,0]$ and let $F : D \to E^n$. We consider the functional differential equation of finitely retarded type:

$$\dot{x}(t) = F(t,x_t). \tag{1.1.2}$$

Definition (1.1.1)

Let $(t_o,\phi) \in D$. By a solution of (1.1.2) assuming initial value ϕ at t_o we mean a continuous function $x \in C[t_o - \tau, t_o + k)$, $0 < k \leq \infty$, such that

(a) $(t,x_t) \in D$ for all $t \in [t_o,t_o + k)$.

(b) $x_{t_o} = \phi$.

(c) x has a derivative $\dot{x}(t)$ at each $t \in (t_o,t_o+k)$ and a right-hand derivative $\dot{x}^+(t_o)$ at t_o.

(d) $\dot{x}(t) = F(t,x_t)$, $t_o < t < t_o + k$, and $\dot{x}^+(t_o) = F(t_o,x_{t_o})$.

Theorem (1.1.1)

If F in (1.1.2) is continuous on D, then, for each $(t_o,\phi) \in D$, there exists a solution x of (1) having initial value ϕ at t_o.

Proof: Let $(t_o,\phi) \in D$. Then, since D is open, there is a $\delta > 0$ such that $|t - t_o| \leq \delta$ and $||\psi - \phi|| \leq \delta$ implies that $(t,\psi) \in D$ and $|F(t,\psi) - F(t_o,\phi)| \leq 1$. Hence, in this δ-neighborhood of (t_o,ϕ) we have $|F(t,\psi)| \leq M$ for some $M > 0$. Now, for any function $\chi \in C[-\tau,0]$, we define

a function $\bar{x} \in C[t_o - \tau, t_o + \delta]$ as follows:

$$(1.1.3) \qquad \bar{x}(t) = \begin{cases} x(t - t_o) & \text{for } t \in [t_o - \tau, t_o] \\ x(0) & \text{for } t \in [t_o, t_o + \delta]. \end{cases}$$

Consider the function $\bar{\phi}$. Since $\bar{\phi}$ is continuous, we may choose $h > 0$ such that $h < \min\{\delta, \delta/M\}$ and such that $\sup\{||\phi - \bar{\phi}_t|| : t \in [t_o, t_o + h]\} \leq \delta/2$. Henceforth, we take $[t_o - \tau, t_o + h]$ as the domain of $\bar{\phi}$ and consider $\bar{\phi}$ as an element of $C[t_o - \tau, t_o + h]$. We now define a set $S \subset C[t_o - \tau, t_o + h]$ by saying that x is an element of S if and only if the following conditions are satisfied:

(a) $x(t) = \bar{\phi}(t)$ for $t \in [t_o - \tau, t_o]$

(b) $||x - \bar{\phi}|| \leq \delta/2$

and (c) $|x(t_1) - x(t_2)| \leq M|t_1 - t_2|$.

Clearly, S is a collection of equicontinuous, equibounded functions and is closed in the topology of $C[t_o - \tau, t_o + h)$. Hence, by the Ascoli Selection Theorem, S is compact in $C[t_o - \tau, t_o + h]$.

We may now define a map T on S by

$$T\,x(t) = \begin{cases} \bar{\phi}(t) & \text{for } t \in [t_o - \tau, t_o] \\ \bar{\phi}(t_o) + \int_{t_o}^{t} F(s, x_s)ds & \text{for } t \in [t_o, t_o + h]. \end{cases}$$

We will now show that $T : S \to S$ and that T is continuous. Clearly, for any element of T(S), condition (a) is

satisfied and we need only check (b) and (c). Condition (b) holds by virtue of the following inequalities:

$$|T\,x(t) - \bar{\phi}(t)| \leq \int_{t_o}^{t} |F(s,x_s)|ds \leq Mh \leq \delta, (t_o \leq t \leq t_o + h),$$

the estimate on F being a consequence of

$$||x_s - \phi|| \leq ||x_s - \bar{\phi}_s|| + ||\bar{\phi}_s - \phi|| \leq \delta/2 + \delta/2 = \delta.$$

Condition (c) follows similarly. Hence we see that $T(S) \subset S$.

The fact that T is continuous can be proven in the following manner. Consider first the set

$$S' = \{\psi \in C[-\tau,0] : \psi = x_t \text{ for some } x \in S, \text{ and } t \in [t_o, t_o + h]\}.$$

S' inherits the properties of equicontinuity and equiboundedness from S and it is easily verified that S' is itself compact in the topology of $C[-\tau,0]$. Since F is assumed to be continuous on D, F is certainly uniformly continuous on the smaller set $[t_o, t_o + h] \times S'$. So, given $\eta > 0$, there exists an $r > 0$ such that $||\psi^{(1)} - \psi^{(2)}|| \leq r$ implies that $|F(s,\psi^{(1)}) - F(s,\psi^{(2)})| \leq \eta$ for all $\psi^{(1)}, \psi^{(2)}$ in S'. Now, for any $\epsilon > 0$, choose $\eta = \epsilon/h$. Then for $x^{(1)}, x^{(2)} \in S$ such that $||x^{(1)} - x^{(2)}|| \leq r$, we have $||x_s^{(1)} - x_s^{(2)}|| \leq r$, for all $s \in [t_o, t_o + h]$, which implies that $|F(s,x_s^{(1)}) - F(s,x_s^{(2)})| \leq \eta = \epsilon/h$, and therefore

$$|Tx^{(1)}(t) - Tx^{(2)}(t)| \leq \int_{t_o}^{t} |F(s,x_s^{(1)}) - F(s,x_s^{(2)})|\, ds \leq \varepsilon$$

and so T is continuous.

The Schauder fixed point theorem now guarantees that the map T has a fixed point in S, that is, there exists a point $x \in S$ such that

$$(1.1.4) \qquad x(t) = \begin{cases} \bar{\phi}(t) \text{ for } t \in [t_o - \tau, t_o] \\ \bar{\phi}(t_o) + \int_{t_o}^{t} F(s,x_s)ds \text{ for } t \in [t_o, t_o + h]. \end{cases}$$

It is clear from this formula that the function x, restricted to the interval $[t_o, t_o + h)$ is indeed a solution of (1.1.2) on $[t_o - \tau, t_o + h)$ satisfying the given initial condition.

Definition (1.1.2)

$F : D \to E^n$ is said to be locally Lipschitzian with respect to ϕ on D if, for all $(t_o,\phi) \in D$, there exists a $\delta > 0$ and a constant L such that if $|t - t_o| \leq \delta$, $||\phi^{(1)} - \phi|| \leq \delta$, $||\phi^{(2)} - \phi|| \leq \delta$ then $(t,\phi^{(1)}),(t,\phi^{(2)}) \in D$ and $|F(t,\phi^{(1)}) - F(t,\phi^{(2)})| \leq L||\phi^{(1)} - \phi^{(2)}||$.

Theorem 1.1.2

Let $F : D \to E^n$ be continuous and locally Lipschitzian with respect to ϕ on D. Given $(t_o,\phi) \in D$ there exists a number h and a unique solution $x \in C[t_o - \tau, t_o + h)$ of equations (1.1.2) having initial value ϕ at t_o.

Proof: We note that, since F is continuous on D then, clearly, for $\bar{\phi}$ defined on $[t_o - \tau, t_o + \delta]$ where δ is chosen as in the previous proof, $F(t,\bar{\phi}_t)$ is a continuous function on a closed interval $[t_o, t_o + \delta]$ and hence is bounded there. So, there exists a constant $M_1 < \infty$ such that $|F(t,\bar{\phi}_t)| \leq M_1$ for all $t \in [t_o, t_o + \delta]$ where $\bar{\phi}$ is the function defined by (1.1.3). Moreover, since F is locally Lipschitzian with respect to ϕ on D, it follows that $|F(t,\phi_1)| \leq |F(t,\phi)| + |F(t,\phi_1) - F(t,\phi)| \leq M_1 + L\delta$, where L is the given Lipschitz constant for F. Now, choose h such that $h < \min\{\delta, \delta/3(M_1 + L\delta)\}$ and $\sup\{||\phi - \bar{\phi}_t|| : t \in [t_o, t_o + h)\} \leq \delta/2$.

We define a class of functions $S \subset C[t_o - \tau, t_o + h)$ by the rule $x \in S$ if and only if (a) $x(t) = \bar{\phi}(t)$ for $t \in [t_o - \tau, t_o]$ and (b) $||x - \bar{\phi}|| \leq \delta/2$. Clearly, S is closed and, since it is a subset of a complete metric space, S itself is a complete metric space. We can now define a map T on S by

$$T\,x(t) = \begin{cases} \bar{\phi}(t) \text{ for } t \in [t_o - \tau, t_o] \\ \bar{\phi}(t_o) + \int_{t_o}^{t} F(s,x_s)ds \text{ for } t \in [t_o, t_o + h). \end{cases}$$

We wish to apply the Banach contraction mapping principle to show the existence of a unique fixed point of the map T. To do this we must show that $T : S \to S$ and that T is a contraction. To show that $T(S) \subset S$ we need only prove that for $x \in S$, $||Tx - \bar{\phi}|| \leq \delta$. This may be seen from the inequalities

$$|Tx(t) - \bar{\phi}(t)| = |Tx(t) - \bar{\phi}(t_o)| \leq \int_{t_o}^{t} |F(s,x_s)|ds$$

$$\leq \int_{t_o}^{t} \{|F(s,\bar{\phi}_s)| + |F(s,x_s) - F(s,\bar{\phi}_s)|\}ds$$

$$\leq \int_{t_o}^{t} \{M_1 + L||x_s - \bar{\phi}_s||\}ds \leq \int_{t_o}^{t} \{M_1 + L||x-\bar{\phi}||\}ds$$

$$\leq (M_1 + L\delta)h < \delta/3.$$

Hence $T(S) \subset S$. To see that T is a contraction, let $x^{(1)}, x^{(2)} \in S$ and note that, in a manner analogous to the preceding reasoning,

$$||Tx^{(1)} - Tx^{(2)}|| \leq Lh||x^{(1)}-x^{(2)}|| \leq [L\delta/3(M_1-L\delta)]||x^{(1)}-x^{(2)}||$$

$$< 1/3\ ||x^{(1)} - x^{(2)}||.$$

Hence T is a contraction mapping and we are assured of the existence of a unique fixed point $x \in S$ of the mapping T. As before it is clear that the function x, restricted to the interval $[t_o,t_o + h)$, is a solution of (1.2.1) on $[t_o,t_o + h)$ and that it satisfies the required initial condition.

Finally, we must show that, if $\hat{x} \in C[t_o - \tau,t_o + h)$ satisfies conditions

(a) $\hat{x}(t) = \bar{\phi}(t)$ for $t \in [t_o - \tau,t_o]$

and

(b) $\hat{x}(t) = \bar{\phi}(t_o) + \int_{t_o}^{t} F(s,\hat{x}_s)ds$ for $t \in [t_o,t_o + h)$.

Then $\hat{x} \in S$. Clearly it is sufficient to show that $||\hat{x} - \bar{\phi}|| \leq \delta/2$. Thus, let

$$I = \{t\in[t_o,t_o+h): |\hat{x}(s) - \bar{\phi}(s)| \leq \delta/2 (t_o - \tau \leq s \leq t)\}$$

and let $t^* = \sup I$. Evidently, $t_o \in I$ and so $I \neq \emptyset$. We show that $t^* = t_o + h$. Suppose not; then

$$|\hat{x}(t^*) - \bar{\phi}(t^*)| = |\hat{x}(t^*) - \bar{\phi}(t_o)| \leq \int_{t_o}^{t^*} |F(s,\hat{x}_s)|ds$$

$$\leq \int_{t_o}^{t^*} \{|F(s,\bar{\phi}_s)| + |F(s,\hat{x}_s) - F(s,\bar{\phi}_s)|\}ds$$

$$\leq \int_{t_o}^{t^*} \{M_1 + L||\hat{x}_s - \bar{\phi}_s||\}ds \leq [M_1 + LS]h < \delta/3.$$

By the continuity of $\hat{x}$, there exists a t_1, $t^* < t_1 \leq t_o + h$, such that $|\hat{x}(s) - \bar{\phi}(s)| \leq \delta/2$ for $t_o - \tau \leq s \leq t$, which is impossible in light of the choice of t^*. Hence $I = [t_o, t_o + h)$ and so $||\hat{x} - \bar{\phi}|| \leq \delta/2$. Thus $\hat{x} \in S$ and the proof is complete.

2. Solutions in the Sense of Carathéodory.

We call attention to the fact that, in the first two theorems. The continuity of F guarantees that the solution of (1.1.2) is continuously differentiable. As in the case of ordinary differential equations, the integral on the right hand side of expression (1.1.4) can be defined for a much larger class than that of continuously differentiable functions x. Thus, if we consider functions which are not differentiable but are only absolutely continuous on an interval, we can define solutions "in the sense of Carathéodory" of (1.1.2).

For the sake of convenience, we will refer to such solutions as weak solutions of (1.1.2). In this section we will see, again, that the choice of the domain of F as a subset of the space $E^1 \times C[-\tau,0]$ allows us to state and prove theorems similar to the familiar theorems of existence and uniqueness of weak solutions for ordinary differential equations.

Definition 1.2.1

Let $(t_o,\phi) \in D$. By a weak solution of (1.1.2) assuming initial value ϕ at t_o, we mean a function $x \in C[t_o - \tau, t_o + k]$, $0 < k \leq \infty$, such that

(a) $(t,x_t) \in D$ for all $t \in [t_o,t_o + k]$

(b) $x_{t_o} = \phi$

(c) x is absolutely continuous on $[t_o,t_o + k]$

(d) x satisfies $\dot{x}(t) = F(t,x_t)$ a.e. in $[t_o,t_o+k]$.

Note that this definition requires that x be absolutely continuous only on the interval $[t_o,t_o + k]$ and not on its entire domain of definition $[t_o - \tau,t_o + k]$.

In the proof of the next theorems we will have need of a lemma whose proof follows the proof of a similar result given by McShane [34 ;Ch. 9].

Lemma

Let $F : D \to E^n$ be measurable in t for each fixed ϕ and continuous in ϕ for almost all fixed t. Then, if

u is a step function or a continuous function mapping an interval of the form [a,b] into $C[-\tau,0]$ such that $(s,u(s)) \in D$ for all $s \in [a,b]$, the function $F(\cdot,u(\cdot))$ mapping [a,b] into E^n is measurable.

Proof. Let u be a step function satisfying the hypotheses of the theorem. Then there is a finite partition $I_1,I_2,\ldots,I_k$ of [a,b] such that $u(s) = \phi_j$ for $s \in I_j$, $\phi_j \in C[-\tau,0]$, $j = 1,2,\ldots,k$. On each I_j, $F(s,u(s))=F(s,\phi_j)$ which is measurable by hypothesis. Now $F(s,u(s))$ can be written as

$$F(s,u(s)) = \sum_{j=1}^{k} F(s,\phi_j)\chi(I_j)$$

where $\chi(I_j)$ is the characteristic function of the set I_j. Thus, F being a finite sum of measurable functions, is itself measurable. Now, if u is continuous, it can be approximated uniformly by a sequence of step functions u_n. We may assume that $(s,u_n(s)) \in D$ for all n, and all $s \in [a,b]$ since the convergence is uniform. By hypothesis, F is continuous in its second argument for almost all fixed t. So $F(s,u_n(s)) \to F(s,u(s))$ almost everywhere in [a,b] and so $F(\cdot,u(\cdot))$ is measurable.

We can now state the basic existence theorem.

Theorem 1.2.1

Let $F : D \to E^n$ be measurable in t for each fixed ϕ and continuous in ϕ for almost all fixed t. Suppose that, for each $(t_o,\phi) \in D$ there exist a number $\delta > 0$

and a Lebesgue integrable function $m(\cdot;t_o,\phi)$ mapping $E^1 \to E^1$ such that if $(t,\psi) \in E^1 \times C[-\tau,0]$ satisfies $|t - t_o| < \delta$ and $||\psi - \phi|| < \delta$, then $(t,\psi) \in D$ and $|F(t,\psi)| \leq m(t;t_o,\phi)$. Then, given $(t_o,\phi) \in D$ there exists a number $h > 0$ and a weak solution of equation (1.1.2) on $[t_o - \tau, t_o + h]$ having initial value ϕ at t_o.

Proof: Choose $(t_o,\phi) \in D$. Then, by hypothesis, there exists a number $\delta > 0$ and a Lebesgue integrable function $m(\cdot;t_o,\phi)$ such that, for $(t,\psi) \in E^1 \times C[-\tau,0]$ satisfying $|t - t_o| < \delta$, $||\psi - \phi|| < \delta$, we have $(t,\psi) \in D$ and $|F(t,\psi)| \leq m(t;t_o,\phi)$. Since $m(\cdot;t_o,\phi)$ is Lebesgue integrable and positive, it is possible to choose h so that $0 \leq \int_{t_o}^{t_o+h} m(s;t_o,\phi)ds \leq \delta$.

Define a set $S \subset C[t_o - \tau, t_o + h]$ by the rule $x \in S$ if

(a) $x(t) = \overline{\phi}(t)$ for $t \in [t_o - \tau, t_o]$

and (b) $||x - \overline{\phi}|| \leq \delta$.

Clearly S is both closed and convex. Define a map T on S by

$$Tx(t) = \begin{cases} \overline{\phi}(t) \text{ for } t \in [t_o - \tau, t_o] \\ \overline{\phi}(t_o) + \int_{t_o}^{t} F(s,x_s)ds \text{ for } t \in [t_o, t_o+h]. \end{cases}$$

We claim that $T: S \to S$, that T is continuous and that T(S) is compact.

The inequalities

$$|Tx(t) - \bar{\phi}(t)| \leq \int_{t_o}^{t} |F(s,x_s)|\, ds \leq \int_{t_o}^{t} m(s;t_o,\phi)ds \leq \delta$$

show that $T(S) \subset S$. To show that T is continuous we apply the Lebesgue dominated convergence theorem. Thus, let $\{x^{(n)}\}$ be any sequence in S converging to an element $x \in S$. We certainly have that

$$|Tx(t) - Tx^{(n)}(t)| \leq \int_{t_o}^{t} |F(s,x_s) - F(s,x_s^{(n)})|\, ds.$$

According to the preceding lemma, the functions $F(s,x_s) - F(s,x_s^{(n)})$ are measurable on $[t_o,t_o + h]$ for all $n = 1,2,\ldots$ and since they are also bounded above in absolute value by $2m(s;t_o,\phi)$ we have, by Lebesgue's theorem, that

$$\lim_{n\to\infty} |Tx(t) - Tx^{(n)}(t)| \leq \lim_{n\to\infty} \int_{t_o}^{t} |F(s,x_s) - F(s,x_s^{(n)})|\, ds$$

$$= \int_{t_o}^{t} \lim_{n\to\infty} |F(s,x_s) - F(s,x_s^{(n)})|\, ds.$$

But, since F is continuous in its second argument, $\lim_{n\to\infty} |F(s,x_s) - F(s,x_s^{(n)})| = 0$. Hence $\lim_{n\to\infty} |Tx(t) - Tx^{(n)}(t)| = 0$ and so T is continuous.

Finally we show that T(S) is a collection of equibounded, equicontinuous functions and therefore is compact. The equiboundedness of T(S) follows from the inequalities

$$|Tx(t)| \leq \int_{t_o}^{t} |F(s,x_s)|ds + |\bar{\phi}(t_o)| \leq \int_{t_o}^{t_o+h} m(s;t_o,\phi)ds + |\bar{\phi}(t_o)|$$

$$\leq \delta + |\bar{\phi}(t_o)| = K, \quad K \text{ a constant.}$$

T(S) is also equicontinuous for, given $\varepsilon > 0$ it is possible to choose $\eta > 0$ such that if $|t_2 - t_1| < \eta$, then

$\int_{t_1}^{t_2} m(s;t_o,\phi)ds < \varepsilon$ and therefore we have, for

$t_1,t_2 \in [t_o,t_o + h]$, $t_1 < t_2$,

$$|Tx(t_2) - Tx(t_1)| \leq \int_{t_1}^{t_2} |F(s,x_s)|ds \leq \int_{t_1}^{t_2} m(s;t_o,\phi)ds < \varepsilon.$$

Since S is closed, convex, $T(S) \subset S$ is compact and T is continuous, we may apply the Schauder theorem and conclude that T has a fixed point $x \in S$. By definition of T, x satisfies the relation

$$x(t) = \begin{cases} \bar{\phi}(t) \text{ for } t \in [t_o - \tau, t_o] \\ \bar{\phi}(t_o) + \int_{t_o}^{t} F(s,x_s)ds \text{ for } t \in [t_o, t_o+h] \end{cases}$$

Clearly, x is absolutely continuous on $[t_o,t_o + h]$ and has a derivative almost everywhere on $[t_o,t_o + h]$ given by $\dot{x}(t) = F(t,x_t)$. Furthermore, the initial condition $x_{t_o} = \phi$ is satisfied and $(t,x_t) \in D$ for all $t \in [t_o,t_o + h]$ since $x \in S$. Thus, x is the required solution.

Theorem 1.2.2

Let $F: D \to E^n$ be measurable in t for each fixed ϕ and

continuous in ϕ for almost all fixed t. Suppose that, for each $(t_o,\phi) \in D$, there exists a number $\delta > 0$ and a Lebesgue integrable function $m(\cdot;t_o,\phi)$ mapping $E^1 \to E^1$ such that

(a) if $(t,\psi) \in E^1 \times C[-\tau,0]$ satisfies $|t - t_o| < \delta$ and $||\psi - \phi|| < \delta$ then $(t,\psi) \in D$ and $|F(t,\psi)| \leq m(t;t_o,\phi)$

and (b) if $(t,\psi_1),(t,\psi_2) \in E^1 \times C[-\tau,0]$ satisfy $|t - t_o| < \delta$ and $||\psi_i - \phi|| < \delta$, $i = 1,2$, then $|F(t,\psi_1) - F(t,\psi_2)| \leq m(t;t_o,\phi)||\psi_1 - \psi_2||$.

Then, given any $(t_o,\phi) \in D$, there exists a number $h > 0$ and a unique weak solution of (1.1.2) on $[t_o - \tau, t_o + h]$ having initial value ϕ at t_o.

Proof. Choose $(t_o,\phi) \in D$. Without loss of generality, we may assume that $\delta < 1/2$. Then, since $m(\cdot;t_o,\phi)$ is Lebesgue integrable, we may choose $h > 0$ so that

$$\int_{t_o}^{t_o+h} m(s;t_o,\phi)ds < \delta < 1/2.$$

Now, define a class $S \subset C[t_o - \tau, t_o + h]$ by the rule that $x \in S$ if

(a) $x(t) = \bar{\phi}(t)$ for $t \in [t_o - \tau, t_o]$

and (b) $||x - \bar{\phi}|| < \delta$.

Clearly S is closed and therefore is a complete metric space. Define a map T on S as in the preceding

proof. Then, exactly as before, $T(S) \subset S$.

Furthermore, T is a contraction for, if $x^{(1)}, x^{(2)} \in S$ then, for any $t \in [t_o, t_o + h]$, $||x_s^{(1)} - \phi|| \leq ||x^{(1)} - \overline{\phi}|| < \delta$ and so

$$|Tx^{(1)}(t) - Tx^{(2)}(t)| \leq \int_{t_o}^{t} |F(s, x_s^{(1)}) - F(s, x_s^{(2)})| ds$$

$$\leq \int_{t_o}^{t} m(s; t_o, \phi) \, ||x_s^{(1)} - x_s^{(2)}|| \, ds$$

$$\leq ||x^{(1)} - x^{(2)}|| \int_{t_o}^{t} m(s; t_o, \phi) ds \leq \delta/2 \, ||x^{(1)} - x^{(2)}||$$

$$< 1/2 \, ||x^{(1)} - x^{(2)}|| .$$

So, by the Banach contraction mapping principle, T has a unique fixed point x which is clearly a weak solution of (1.1.2) having initial value ϕ at t_o. Finally we show, precisely as in Theorem 1.1.2, that if $\hat{x}$ is any other element of $C[t_o - \tau, t_o + h]$ such that

$$\hat{x}(t) = \begin{cases} \overline{\phi}(t) & \text{for } t \in [t_o - \tau, t_o] \\ \overline{\phi}(t_o) + \int_{t_o}^{t} F(s, \hat{x}_s) ds & \text{for } t \in [t_o, t_o + h] \end{cases}$$

then $\hat{x} = x$.

3. Remarks on Equivalent Forms.

In this section we wish to show that the differential equations studied by Banks in [2] are equivalent to equations of the form (1.1.2) in the sense that given an equation of the type studied in [2] it is possible to

construct an equation of the form (1.1.2) whose solution gives rise, in a natural way, to solutions of the original equation. Moreover, any solution of the original equation will correspond to a solution of the constructed equation.

The equations studied in [2] are delay equations which may be described in the following manner. Let $\alpha_o, t_1, t_2 \in E^1$ with $\alpha_o \leq t_1 \leq t_2$ and let x be any function continuously mapping an interval of the form $[\alpha_o, t_o + \sigma]$, $0 < \sigma < +\infty$, into E^n. For any $t \in [t_1, t_1 + \sigma]$ we define a function $x_{(t)} \in C[\alpha_o, t]$ by means of the equation

$$x_{(t)}(s) = x(s) \quad (\alpha_o \leq s \leq t) \tag{1.3.1}$$

Let O be an open connected subset of $[t_1, t_2] \times C[\alpha_o, t_2]$ and let $G: O \to E^n$ have the property that, at any $(t,x) \in O$, the values of the function G depend only on $(t, x_{(t)})$. That is, if (t,x) and $(t,\tilde{x})$ are elements of O such that $x_{(t)} = \tilde{x}_{(t)}$, then $G(t, x_{(t)}) = G(t, \tilde{x}_{(t)})$.

Given such a function, we may consider the functional differential equation

$$\dot{x}(t) = G(t,x). \tag{1.3.2}$$

Definition 1.3.1.

A solution of (3.1.2) is a function $x \in C[\alpha_o, t_2]$ such that $(t,x) \in O$ for all $t \in [t_1, t_2]$ and such that, on some interval of the form $[t_1, t_1 + \sigma)$, $0 < \sigma \leq t_2 - t_1$, x is differentiable and satisfies the equation

$$\dot{x}(t) = G(t,x) \qquad (t_1 \leq t \leq t_1 + \sigma).$$

Lemma 1.3.1.

Let $O \subset [t_1,t_2] \times C[\alpha_o,t_2]$ be open and connected and $G: O \to E^n$ have the property described above. Let O^* be the subset of $[t_1,t_2] \times C[\alpha_o,t_2]$ consisting of all $(t,x^*) \in [t_1,t_2] \times C[\alpha_o,t_2]$ for which there exists an element $x \in C[\alpha_o,t_2]$ such that $(t,x) \in O$ and $x^*_{(t)} = x_{(t)}$. Then O^* is open and connected.

Proof. The fact that O^* is open follows easily from the definition of O^* and the fact that O is open.

To see that O^* is connected, we proceed as follows. Suppose O^* is not connected. Then there are two separated sets A and $B \subset [t_1,t_2] \times C[\alpha_o,t_2]$ such that $A \cap O^* \neq \emptyset$ and $B \cap O^* \neq \emptyset$. Since $O \subset O^*$ and O is, by hypothesis, connected, we have either $O \subset A$ or $O \subset B$. Without loss of generality, we may assume that $O \subset A$. Choose $(t,x^*) \in B \cap O^*$. By definition of O^*, there exists an $x \in [t_1,t_2] \times C[\alpha_o,t_2]$ such that $(t,x) \in O$ and $x_{(t)} = x^*_{(t)}$. Now consider the connected set $S = \{t\} \times \{\lambda x + (1 - \lambda)x^* \mid 0 \leq \lambda \leq 1\}$. Clearly, $S \subset O^*$ and since S is connected either $S \subset A$ or $S \subset B$. But $(t,x) \in S \cap A$ and $(t,x^*) \in S \cap B$ which is a contradiction. We conclude that O^* is connected.

We may now construct a map $G^* : O^* \to E^n$ which is an extension of our original map G. For each $(t,x^*) \in O^*$ let

$$G^*(t,x^*) = G(t,x) \tag{1.3.3}$$

where x is any function in $C[\alpha_o,t_2]$ such that $(t,x) \in O$ and $x^*_{(t)} = x_{(t)}$. Such an x exists by definition of O^*. It is clear that G^* is well-defined since G depends only on the value $x(s)$, $\alpha_o \leq s \leq t$.

Let $\tau = t_2 - \alpha_o$ and for any $(t,\phi) \in [t_1,t_2] \times C[-\tau,0]$ let $\phi_{[t]}$ be the element of $C[\alpha_o,t_2]$ defined by the equation

$$\phi_{[t]}(s) = \begin{cases} \phi(s - t), & \alpha_o \leq s \leq t \\ \phi(0), & t \leq s \leq t_2 . \end{cases} \tag{1.3.4}$$

Lemma 1.3.2.

Let D be the subset of all $(t,\phi) \in [t_1,t_2] \times C[-\tau,0]$ such that $(t,\phi_{[t]}) \in O^*$. Then D is open and connected.

Proof. The fact that D is open follows from the continuity of the map $(t,\phi) \to (t,\phi_{[t]})$.

To show that D is connected, it is sufficient to construct a continuous map of O^* onto D. For any $t \in [t_1,t_2]$ and $x \in C[\alpha_o,t_2]$ let $x_{\{t\}}$ be the element of $C[-\tau,0]$ defined by

$$x_{\{t\}}(\theta) = \begin{cases} x(t + \theta) & -\tau \leq \theta \leq t - t_2 \\ x(t) & t - t_2 \leq \theta \leq 0 \end{cases} \tag{1.3.6}$$

Then the map $\psi: O^* \to D$ given by

$$\psi(t,x) = (t, x_{\{t_2\}})$$

is a continuous map of O^* onto D. To see that $\psi(O^*) \subset D$ we note that, if $(t,x) \in O^*$ then $(t, x_{\{t_2\}[t]}) \in O^*$ and so $(t, x_{\{t_2\}})$ is an element of D. It is obvious that ψ is continuous. To see that ψ is onto, let $(t,\phi) \in D$. Then, by definition, $(t,\phi_{[t]}) \in O^*$ and so $(t,\phi_{[t_2]}) \in O^*$. It follows from the definition of ψ that $\psi(t,\phi_{[t_2]}) = (t,\phi)$.

We may now define a mapping $F : D \to E^n$ by the equation

$$F(t,\phi) = G^*(t,\phi_{[t]}), \quad (t,\phi) \in D. \tag{1.3.7}$$

We wish to consider the functional differential equation

$$\dot{x}(t) = F(t,x_t). \tag{1.3.8}$$

Equations (1.3.2) and (1.3.8) are equivalent in the sense of the following theorem.

<u>Theorem 1.3.1</u>.

For given $(t_o,\phi) \in D$, let x be a solution of (1.3.8) having initial function ϕ at time t_o, and let $[t_o - \tau, t_o + \sigma), 0 < \sigma \leq t_2 - t_o$, be the domain of definition of x. Let $\tilde{x}$ be the function defined on $[\alpha_o, t_2]$ by the equations

$$\tilde{x}(t) = \begin{cases} x(t), & \alpha_o \leq t \leq t_o + \sigma \\ x(t_o+\sigma), & t_o + \sigma < t \leq t_2. \end{cases}$$

Then $\tilde{x}$ is a solution of (1.3.2) with G replaced by G^*.

Conversely, let x be a solution of (1.3.2) defined on $[\alpha_o, t_2]$ and let σ, $0 < \sigma \leq t_2 - t_1$, be the number for which $\dot{x}(t)$ satisfies (1.3.2) when $t \in [t_1, t_1 + \sigma)$. Let $x : [t_1 - \tau, t_1 + \sigma) \to E^n$ be the function defined by the equation

$$x(t) = \begin{cases} \tilde{x}(\alpha_o), & t_1 - \tau \leq t \leq \alpha_o \\ \tilde{x}(t), & \alpha_o \leq t < t_1 + \sigma. \end{cases}$$

Then x is a solution of (1.3.8) having initial function x_{t_1} at time t_1.

The proof of this theorem is an immediate consequence of the definitions of O, O^*, G^* and F.

While the set D may not be open in $E^1 \times C[-\tau, 0]$, it is open in $[t_1, t_2] \times C[-\tau, 0]$. It is clear that, suitable modifications of the theorems of Sections 1 and 2 will guarantee the existence and uniqueness of solutions to equation (1.3.2).

CHAPTER II

STATEMENT OF THE OPTIMAL CONTROL PROBLEM, CLOSURE THEOREMS, AND EXISTENCE THEOREMS

Having established conditions for the existence and uniqueness of solutions of functional differential equations of finitely retarded type, we may turn to the study of optimal control problems for systems whose dynamics are described by equations of this kind. The goal of this chapter is to extend an existence theorem of Cesari [9] for Lagrange problems to the case of functional differential equations. We present an existence theorem of the type of Filippov [21], before the main existence theorem to facilitate comparison of the two results. Of course, strictly speaking, the Filippov theorem can be deduced as a corollary of the more general existence theorem.

The proof of the main theorem differs in two ways from the proof given by Cesari for ordinary differential equations. Unlike his proof, we are compelled to require that the set of initial conditions be compact. This restriction becomes necessary at the point where Ascoli's theorem is applied to the minimizing sequence in order to obtain a subsequence $\{x_k\}$ defined on $[t_{1_k} - \tau, t_{2_k}]$ each of which is absolutely continuous on $[t_{1_k}, t_{2_k}]$ and which converges uniformly to an absolutely

continuous function x defined on an interval $[t_1 - \tau, t_2]$. The growth condition is enough to insure that the elements of the minimizing sequence are equiabsolutely continuous on $[t_{1_k}, t_{2_k}]$ but gives no information on the behavior of the functions x_k on the intervals $[t_{1_k} - \tau, t_{2_k}]$. In order to apply the Ascoli theorem it is necessary to make an additional assumption on the functions $\{x_{t_{1_k}}\}$ in order that they also form an equicontinuous family. By taking the set of initial conditions to be compact, we insure equicontinuity on the initial intervals and hence equicontinuity on the intervals $[t_{1_k} - \tau, t_{2_k}]$. Finally, the assumption that A is bounded, insures that the x_k are equibounded and we may apply the Ascoli theorem.

The second difference is more indirect and more interesting. Once the proper subsequences have been extracted, it is necessary to invoke a closure theorem to insure that the limit is a trajectory of the system. The proof of this closure theorem depends on a general implicit function theorem due to McShane and Warfield [35]. In their paper they present two forms of the theorem, one which depends on the continuous hypothesis and a less general form which does not. In the case in which the state space can be expressed as the countable union of compact metrizable sets, the less general form of the implicit function theorem may be used. For

the case with which we are concerned the space is only a separable metric space and the form which depends on the validity of the continuum hypothesis must be invoked.

We should also remark here that Banks [2] has proven an existence theorem of the Filippov type. Our theorems differ from his in that we do not make the restriction that the terminal boundary conditions be given in terms of points in $E^1 \times E^n$. As in the previous chapter, we consider the state space to be $E^1 \times C[-\tau,0]$ and the boundary conditions, as well as constraints on the trajectories are given in terms of this space. It is clear, on the other hand, that problems of the type considered by Banks are included as special cases of those which we have chosen to study. Any desire to control the system to a specified set $\{t_1\} \times S \subset E^1 \times E^n$ is certainly achieved only by controlling it to the set $\{t_1\} \times \hat{S} \subset E^1 \times C[-\tau,0]$ where $\hat{S}$ consists of all functions $\phi \in C[-\tau,0]$ such that $\phi(0) \in \hat{S}$. The requirement of theorems of later chapters that the set of final states be convex is certainly fulfilled in this case. In a similar manner problems for which constraints on the trajectories are given in terms of a set in $E^1 \times E^n$ can be seen to be special cases of our form of the control problem.

In Section 1 of this chapter we will state the Lagrange Problem, state the necessary definitions, and prove an implicit function theorem. In Section 2 we present Closure Theorem 1 and prove an existence theorem

of the Filippov type. In Section 3, we prove Closure Theorem II and the main existence theorem for Lagrange Problems with delay.

1. Statement of The Lagrange Problem, Basic Definitions and an Implicit Function Theorem.

Let $x = (x^1,\ldots,x^n)$ be a vector valued function mapping a closed interval of E^1 into E^n and let $u = (u^1,\ldots,u^m)$ be a vector valued function defined on the same interval and having values in E^m. Let $t \in E^1$ be an independent variable. Denote by A an arbitrary subset of $E^1 \times C[-\tau,0]$. Let U be a set-valued map, $U : A \to 2^{E^m}$, and for each $(t,\phi) \in A$, denote its image under U by $U(t,\phi)$. Let F_i, $i = 0,1,\ldots,n$, be given real valued functionals defined for all $(t,\phi) \in A$ and $u \in U(t,\phi)$ and let $F = (F_1,F_2,\ldots,F_n)$. Let B be a given subset of $E^1 \times C[-\tau,0] \times E^1 \times C[-\tau,0]$.

We are interested in studying the following problem:

Determine, for given $t_1,t_2 \in E^1$, $t_1 < t_2$, a measurable vector function $u : [t_1,t_2] \to E^m$ and a corresponding continuous vector function $x : [t_1 - \tau,t_2] \to E^n$, which is absolutely continuous on $[t_1,t_2]$, and which satisfies, almost everywhere, the differential system

$$\frac{dx}{dt} = F(t,x_t,u) \qquad (t_1 \le t \le t_2),$$

subject to boundary conditions

$$(t_1, x_{t_1}, t_2, x_{t_2}) \in B,$$

and constraints

$$(t, x_t) \in A \quad (t_1 \leq t \leq t_2),$$

$$u(t) \in U(t, x_t) \text{ almost everywhere in } [t_1, t_2],$$

and for which the integral

$$I[x,u] = \int_{t_1}^{t_2} F_0(s, x_s, u(s))ds$$

takes on its minimum value.

In order to discuss this problem, we will make some preliminary assumptions. We assume that the set A is closed and bounded and that $U(t,\phi)$ is closed for each $(t,\phi) \in A$. Let M be the set of all (t,ϕ,u) where $(t,\phi) \in A$ and $u \in U(t,\phi)$. Thus $M \subset E^1 \times C[-\tau,0] \times E^m$. We will assume that F is continuous on the set M. When any measurable function u is substituted in the differential equation, we obtain

$$\frac{dx}{dt} = F(t, x_t, u(t))$$

which is an ordinary differential equation in normal form. We assume in what follows that, given such a control function and initial conditions $(t_1,\phi) \in A$, this equation will have a solution $x(t_1,\phi)$ such that $x_{t_1} = \phi$. If (t_1,ϕ) is an interior point of A, and F, in

addition to being continuous on M, satisfies a local Lipschitz condition in its second argument, then we can expect that a given control function u will generate a unique solution x of the differential equation.

We shall refer to this problem as a Lagrange problem with unilateral constraints. In the classical form of the problem, $U = E^m$ and the side conditions correspond to the conditions expressed here as differential equations in normal form. If we take the sets $U(t,\phi)$ to be compact, the resulting problem is usually referred to as a Pontryagin problem and if F_o is taken as $F_o \equiv 1$, the resulting problem is one of time-optimal control.

Definition 2.1.1.

A vector-valued function $u : [t_1,t_2] \to E^m$ and a continuous vector-valued function $x : [t_1 - \tau,t_2] \to E^n$ is said to form an **admissible** pair provided:

(a) u is measurable in $[t_1,t_2]$,

(b) x is absolutely continuous in $[t_1,t_2]$,

(c) $(t,x_t) \in A$ for all $t \in [t_1,t_2]$,

(d) $u(t) \in U(t,x_t)$ for almost all $t \in [t_1,t_2]$,

(e) $F_o(t,x_t,u(t))$ is Lebesgue-integrable in $[t_1,t_2]$,

(f) $dx/dt = F(t,x_t,u(t))$ almost everywhere in $[t_1,t_2]$,

(g) $(t_1,x_{t_1},t_2,x_{t_2}) \in B$.

For each $(t,\phi) \in A$, we define a map $Q : A \to 2^{E^n}$, by

$$Q(t,\phi) = \{z \in E^n \mid z = F(t,\phi,u),\ u \in U(t,\phi)\}.$$

The proof of our first result below requires a theorem due to McShane and Warfield [35 ; p. 46, Theorem 4].

Theorem.

Let $(X,\mu,\mathcal{M})$ be a measure space, Y a Hausdorff space and Z a separable metric space. Let $k : Z \to Y$ be continuous and $y : X \to Y$ be a measurable map such that $y(X) \subseteq k(Z)$. Then there exists a measurable function $u : X \to Z$ such that $k(u(x)) = y(x)$ for all $x \in X$.

Using this result, we may prove the following implicit function theorem which is due, in its original form, to Filippov [21].

Theorem 2.1.1.

Let A be a closed subset of $E^1 \times C[-\tau,0]$ and let M be the set of all (t,ϕ,u) where $(t,\phi) \in A$, $u \in U(t,\phi)$. Let F be continuous on M and, for each $(t,\phi) \in A$, let $Q(t,\phi) = F(t,\phi,U(t,\phi))$.

Let $x \in C[t_1 - \tau, t_2]$ be absolutely continuous on $[t_1,t_2]$ and be such that $(t,x_t) \in A$ for $t_1 \leq t \leq t_2$ and $\dot{x}(t) \in Q(t,x_t)$ almost everywhere in $[t_1,t_2]$. Finally assume that M is closed.

Then there exists a measurable function $u: [t_1,t_2] \to E^m$ such that $u(t) \in U(t,x_t)$ and $\dot{x}(t) = F(t,x_t,u(t))$ almost everywhere in $[t_1,t_2]$.

Proof. Following the notation of the previous theorem we let

$$Z = M = \{(t,\phi,u) \mid (t,\phi) \in A,\ u \in U(t,\phi)\}$$

$$X = I = [t_1,t_2]$$

and $Y = E^1 \times C[-\tau,0] \times E^n$.

The polynomials with rational coefficients being countably dense in $C[-\tau,0]$, it is clear that M is separable. If we define maps $k: M \to E^1 \times C[-\tau,0] \times E^n$ and $y: I \to E^1 \times C[-\tau,0] \times E^n$ by

$$k(t,\phi,u) = (t,\phi,z),\ z = F(t,\phi,u) \in E^n,$$

and

$$y(t) = (t,x_t,\dot{x}(t)),$$

then the continuity of F on M assures us that k is continuous. It is also clear that, since x is absolutely continuous on $[t_1,t_2]$, y is measurable.

The McShane-Warfield Theorem then tells us that there is a measurable function $v : I \to Z$ such that $k(v(t)) = y(t)$ for all $t \in I$. Since $Z \subset E^1 \times C[-\tau,0] \times E^m$ we can express v uniquely in the form

$$v(t) = (t,w(t),u(t)),\quad t \in I,$$

where $w: I \to C[-\tau,0]$ and $u: I \to E^n$. Note that the constraints $(t,w(t)) \in A$, for $t \in I$, and $u(t) \in U(t,w(t))$, for almost all $t \in I$, are satisfied.

Recalling the definitions of the maps k and y, we see that, almost everywhere in I,

$$(t,w(t),F(t,w(t),u(t)) = (t,x_t,\dot{x}(t))$$

and it follows immediately that $w(t) = x_t$ and $F(t,w(t),u(t)) = \dot{x}(t)$ almost everywhere. Hence, clearly $u(t) \in U(t,x_t)$ and $\dot{x}(t) = F(t,x_t,u(t))$ almost everywhere in I.

Definition 2.1.2.

A vector function $x: [t_1 - \tau, t_2] \to E^n$ will be called a trajectory if there is a vector function $u: [t_1,t_2] \to E^m$ such that the pair $\{x,u\}$ is admissible.

In what follows, it will be convenient to consider the space X of all continuous n-dimensional vector functions x defined on arbitrary intervals of the t-axis,

$$x: [a,b] \to E^n.$$

We now introduce the structure of a metric space on X by introducing a metric function $\rho : X \times X \to \mathbb{R}^+$. For this purpose, let $x,y \in X$, x defined on an interval [a,b] and y defined on [c,d]. Extend x and y to all of E^1 by $x(t) = x(a)$ for all $t < a$ and $x(t) = x(b)$ for all $t > b$ and similarly for y. We then define

$$\rho(x,y) = |a - c| + |b - d| + \sup |x(t) - y(t)|$$

where the supremum is taken over all $t \in E^1$. With this metric structure the space $\{X,\rho\}$ is complete and convergence of a sequence $\{x^{(n)}\}$ to an element x in the metric topology when all the $x^{(n)}$ are defined on a fixed interval,

is just uniform convergence on that same interval.

Definition 2.1.3.

A class Ω of admissible pairs $\{x,u\}$ is said to be complete if, given a sequence $\{x^{(k)},u^{(k)}\}$ of elements of Ω and an admissible pair $\{x,u\}$ with the property that $x^{(k)} \to x$ in the ρ-metric for the trajectory x, then $\{x,u\} \in \Omega$.

Clearly, the class of all admissible pairs is complete as is any class which consists of a finite number of admissible pairs.

Given any point $(t_o,\phi_o) \in A$ and numbers $\varepsilon > 0$ and $\delta > 0$, we denote by $N_\delta(t_o,\phi_o)$ the set of all (t,ϕ) such that $|t - t_o| \leq \delta$ and $||\phi - \phi_o|| \leq \delta$, and by $[U(t_o,\phi_o)]_\varepsilon$ the closed ε-neighborhood of $U(t_o,\phi_o)$ in E^m.

Definition 2.1.4.

A function $U: A \to 2^{E^m}$ is said to be a metrically upper semicontinuous function of (t,ϕ) at the point $(\bar{t},\bar{\phi})$ if, given $\varepsilon > 0$ there is a $\delta = \delta(\bar{t},\bar{\phi},t) > 0$ such that if $(t,\phi) \in N_\delta(\bar{t},\bar{\phi})$ then $U(t,\phi) \subset [U(\bar{t},\bar{\phi})]_\varepsilon$. The function U is said to be metrically upper semicontinuous in A if it is metrically upper semicontinuous at each point of A.

Since the sets $U(t,\phi)$ and $Q(t,\phi)$ with which we will be dealing will be closed but not necessarily compact, we will need a concept of upper semicontinuity

which is more general than the one described in Definition 2.1.4. To this end we introduce the following two definitions which are restatements of the definitions due to Cesari [9 ; pp. 372,375].

Definition 2.1.5.

A function $U: A \to 2^{E^m}$ is said to have property (U) at a point $(\bar{t},\bar{\phi}) \in A$ if

$$U(\bar{t},\bar{\phi}) = \bigcap_{\delta>0} c\ell \left[\bigcup_{(t,\phi)\in N_\delta(\bar{t},\bar{\phi})} U(t,\phi) \right]$$

$$= \bigcap_{\delta>0} c\ell\, U(\bar{t},\bar{\phi};\delta).$$

If U has property (U) at all points of A, then U is said to have property (U) in A.

Definition 2.1.6.

A function $Q: A \to 2^{E^n}$ is said to have property (Q) at a point $(\bar{t},\bar{\phi}) \in A$ if

$$Q(\bar{t},\bar{\phi}) = \bigcap_{\delta>0} c\ell\ co\ Q(\bar{t},\bar{\phi};\delta)$$

and to have property (Q) in A if it has property (Q) at each point of A.

Properties (U) and (Q) can be shown to be generalizations of the concept of metric upper semicontinuity for closed sets and for closed and convex sets respectively. Thus, it is possible to prove, for example, that if U is metrically upper semicontinuous then it satisfies

property (U), while examples show that a map U with property (U) may not have the property of metric upper semicontinuity [9 ,p. 374].

In Section 4 of [9], Cesari has thoroughly investigated properties (U) and (Q). We state here several of his results which will be needed in the sequel. Although we have reformulated these lemmas in terms of set-valued mappings defined on a subset A of $E^1 \times C[-\tau,0]$, rather than on a Euclidean space, the proofs appearing in [9 , pp. 373-377] remain valid.

(i) If the sets $U(t,\phi)$ are closed for every $(t,\phi) \in A$ and if U is a metrically upper semicontinuous function of (t,ϕ) in A, then U has property (U) in A.

(ii) If A is compact, if $U(t,\phi)$ is compact for every $(t,\phi) \in A$ and if U is metrically upper semicontinuous in A, then M is compact.

(iii) If A is compact, if the sets $U(t,\phi)$ are compact for every $(t,\phi) \in A$ and if U has property (Q) in A then M is compact.

(iv) If, for every $(t,\phi) \in A$, the set $U(t,\phi)$ is closed and convex, and if U is metrically upper semicontinuous in A, then U has property (Q) in A.

2. The First Closure Theorem and an Existence Theorem of Filippov Type

The following theorem, as well as Theorem 2.3.1, is referred to as a closure theorem. It gives conditions

under which we can assert that a function which is the limit in the ρ-metric of a convergent sequence of trajectories is itself a trajectory.

Theorem 2.2.1 (Closure Theorem I)

Let A be a closed subset of $E^1 \times C[-\tau,0]$, let $U(t,\phi)$ be closed in E^m for all $(t,\phi) \in A$ and let $F: M \to E^n$ be continuous. Define a map $Q: A \to 2^{E^n}$ by $Q(t,\phi) = F(t,\phi,U(t,\phi))$ and assume that, for each $(t,\phi) \in A$, $Q(t,\phi)$ is closed and convex. Let U have property (U) and Q have property (Q) in A. Then if $\{x^{(k)}\}$ is a sequence of trajectories, each defined on an interval $[t_{1_k} - \tau, t_{2_k}]$, which converges in the ρ-metric to a continuous function $x: [t_1 - \tau, t_2] \to E^n$ which is absolutely continuous on $[t_1,t_2]$, then x is a trajectory.

Proof. The functions $\dot{x}: [t_1 - \tau, t_2] \to E^n$ and $\dot{x}^{(k)}: [t_{1_k} - \tau, t_{2_k}] \to E^n$ are defined almost everywhere and are Lebesgue integrable. We must prove (1) that $(t,x_t) \in A$ for all $t \in [t_1,t_2]$ and (2) that there is a measurable function $u: [t_1,t_2] \to E^m$ such that $\dot{x}(t) = F(t,x_t,u(t))$ and $u(t) \in U(t,x_t)$ almost everywhere in $[t_1,t_2]$.

Since we assume that $\rho(x^{(k)},x) \to 0$ as $k \to \infty$ we have that $t_{1_k} - \tau \to t_1 - \tau$ and $t_{2_k} \to t_2$ as $k \to \infty$. For $t \in (t_1,t_2)$ and k sufficiently large, $t \in (t_{1_k},t_{2_k})$ and $(t,x_t^{(k)}) \in A$. But, A is closed and $x_t^{(k)} \to x_t$ is $(t,x_t) \in A$ for all $t \in (t_1,t_2)$. Since x(t) is continuous,

we may conclude that $(t,x_t) \in A$ for all $t \in [t_1,t_2]$.

In order to prove (2), choose $t_o \in (t_1,t_2)$ to be a point at which $\dot{x}$ exists. Then there is a number $\sigma > 0$ such that $t_1 < t_o - \sigma < t_o + \sigma < t_2$. For sufficiently large k, say $k \geq k_o$, $t_{1_k} < t_o - \sigma < t_o + \sigma < t_{2_k}$. Then $x^{(k)}(t) \to x(t)$ uniformly on $[t_o - (\sigma + \tau), t_o + \sigma]$ and $\{x, x^{(k)}\}$ are equicontinuous in $[t_o - (\sigma + \tau), t_o + \sigma]$. Hence, given $\varepsilon > 0$ there is a $\delta > 0$ such that if t and t' are elements of $[t_o - (\sigma + \tau), t_o + \sigma]$ and if $|t - t'| \leq \delta$, $k \geq k_o$, then we have

$$|x(t) - x(t')| \leq \varepsilon/2 \text{ and } |x^{(k)}(t) - x^{(k)}(t')| \leq \varepsilon/2.$$

We may take $\delta < \min\{\sigma,\varepsilon\}$ and consider, for any h, $0 < h \leq \delta$, the averages

$$m_h = h^{-1} \int_o^h \dot{x}(t_o + s)ds = h^{-1}[x(t_o + h) - x(t_o)]$$

$$m_{hk} = h^{-1} \int_o^h F(t_o + s, x^{(k)}_{t_o+s}, u^{(k)}(t_o + s))ds$$

$$= h^{-1}[x^{(k)}(t_o + h) - x^{(k)}(t_o)].$$

Now, given $\eta > 0$ we may fix h, $0 < h \leq \delta < \sigma$ so small that $|m_h - \dot{x}(t_o)| \leq \eta$. Take $k_1 \geq k_o$ so large that, for all $k \geq k_1$, $|m_{hk} - m_h| \leq \eta$ and $|x^{(k)}(t) - x(t)| \leq \varepsilon/2$ for all $t \in [t_o - (\sigma + \tau), t_o + \sigma]$. Such a choice is possible since the convergence is uniform on $[t_o - (\sigma + \tau), t_o + \sigma]$. With these choices of h and k we have that, for $0 \leq s \leq h$,

$$|x^{(k)}(t_o+s)-x(t_o)| \leq |x^{(k)}(t_o+s)-x^{(k)}(t_o)|+|x^{(k)}(t_o)-x(t_o)| \leq \varepsilon$$

and $|(t_o + s) - t_o| \leq h \leq \varepsilon$. Setting

$Q(t_o,x_{t_o},2\varepsilon) = \bigcup_{(t,\phi)\in N_{2\varepsilon}(t_o,x_{t_o})} Q(t,\phi)$ and noting that

$F(t_o + s, x^{(k)}_{t_o+s}, u^{(k)}(t_o+s)) \in Q(t_o + s, x^{(k)}_{t_o+s})$, the

preceding inequalities show that

$$F(t_o + s, x^{(k)}_{t_o+s}, u^{(k)}(t_o+s)) \in Q(t_o,x_{t_o};2\varepsilon)$$

since $(t_o + s, x^{(k)}_{t_o+s}) \in N_{2\varepsilon}(t_o,x_{t_o})$. By definition of

$m_{h,k}$, we have $m_{h,k} \in c\ell\ co\ Q(t_o,x_{t_o};2\varepsilon)$ since this set is

closed and convex. Finally we have the estimates

$$|\dot{x}(t_o) - m_{h,k}| \leq |\dot{x}(t_o) - m_h| + |m_h - m_{hk}| \leq 2\eta$$

and so $\dot{x}(t_o) \in [c\ell\ co\ Q(t_o,x_{t_o};2\varepsilon)]_{2\eta}$. Since η is

arbitrary we can conclude that

$\dot{x}(t_o) \in \bigcap_{\eta>0} [c\ell\ co\ Q(t_o,x_{t_o};2\varepsilon)]_{2\eta}$ and so

$\dot{x}(t_o) \in c\ell\ co\ Q(t_o,x_{t_o};2\varepsilon)$. But $\varepsilon > 0$ is arbitrary and

Q has property (Q) in A and so

$$\dot{x}(t_o) \in \bigcap_{\varepsilon>0} c\ell\ co\ Q(t_o,x_{t_o};2\varepsilon) = Q(t_o,x_{t_o}).$$

The point t_o was chosen as any point of $[t_1,t_2]$ at which

$\dot{x}$ exists. So this last relation holds almost everywhere

in $[t_1,t_2]$. Theorem 2.1.1 insures the existence of a

measurable map $u: [t_1,t_2] \to E^m$ such that

$$\dot{x}(t) = F(t,x_t,u(t)),\ u(t) \in U(t,x_t)$$

hold almost everywhere in $[t_1,t_2]$.

Closure Theorem I allows us to prove an existence theorem for the case in which A is compact, and $U(t,\phi)$ is compact and U is metrically upper semicontinuous in A. The result, in the case of ordinary differential equations, is due to Filippov [21].

Theorem 2.2.2.

Let A be compact in $E^1 \times C[-\tau,0]$ and let U be metrically upper semicontinuous in A. Suppose that $U(t,\phi)$ is compact for each $(t,\phi) \in A$. Let M be closed and $Q(t,\phi) = F(t,\phi,U(t,\phi))$ be convex for all $(t,\phi) \in A$. Assume that F is continuous on M, that $B \subset E^1 \times C[-\tau,0] \times E^1 \times C[-\tau,0]$ is closed and that $I[x,u] = g(t_1,x_{t_1},t_2,x_{t_2})$ is continuous on B. Then, if Ω is a non-empty complete class of admissible pairs, there exists a pair $\{x^*,u^*\} \in \Omega$ such that $I[x^*,u^*] \leq I[x,u]$ for all $\{x,u\} \in \Omega$.

Proof. Since A is compact and $U(t,\phi)$ is compact and U is metrically upper semicontinuous, the set M is compact [9, p. 375]. Hence, F is uniformly continuous and bounded on M, say $|F| \leq k_o$ on M. Since A is compact and B is closed, the set $B \cap A \times A$ is compact and $(t_1,x_{t_1},t_2,x_{t_2}) \in B \cap A \times A$.

But g is assumed to be continuous on B and so it is bounded on $B \cap (A \times A)$. Thus, there exists a constant m_o such that $g \geq m_o$ and so if $i = \inf g$ we have that i is finite. Therefore, there is a sequence of pairs $\{x^{(k)},u^{(k)}\}$ with $x^{(k)}$ defined on $[t_{1_k} - \tau, t_{2_k}]$ and $u^{(k)}$ defined on $[t_{1_k}, t_{2_k}]$ such that $I[x^{(k)},u^{(k)}] \to i$.

Since the $x^{(k)}$ are trajectories and the set A is taken to be compact, the functions $\{x^{(k)}_{t_{1_k}}\}$ are equi-bounded and equicontinuous. Further, $\dot{x}^{(k)}(t) = F(t,x^{(k)}_t,u^{(k)}(t))$ almost everywhere in $[t_1,t_2]$ and so $|\dot{x}^{(k)}(t)| \leq k_o$ on $[t_{1_k}, t_{2_k}]$ for some constant k_o. It follows that the $x^{(k)}$ are equi-Lipschitzian with constant k_o on $[t_{1_k}, t_{2_k}]$ and so form an equibounded, equicontinuous family. Thus we may extract a convergent subsequence, call it again $\{x^{(k)}\}$, which converges, uniformly, to a continuous function on some interval $[t_1 - \tau, t_2]$. Call this limit function x^*.

We must show that x^* is a trajectory, that is

(i) for all $t \in [t_1,t_2]$, $(t,x^*_t) \in A$,

(ii) x^* is absolutely continuous on $[t_1,t_2]$,

and (iii) there exists a measurable function u^* such that $\dot{x}^*(t) = F(t,x^*_t,u^*(t)), u^*(t) \in U(t,x^*_t)$ almost everywhere in $[t_1,t_2]$.

To see (i), let $t \in (t_1,t_2)$. Then, there is a constant $\bar{k}$ such that, for $k \geq \bar{k}$, $t_{1_k} \leq t \leq t_{2_k}$ and so $(t,x_t^{(k)}) \in A$. Since A is compact, $(t,x_t^*) \in A$ and, by continuity of x, $(t_1,x_{t_1}^*)$ and $(t_2,x_{t_2}^*)$ both belong to A.

To see (ii), let $t',t'' \in [t_1,t_2]$ and choose $\varepsilon > 0$. Then

$$|x^*(t')-x^*(t'')| \leq |x^*(t')-x^{(k)}(t')| + |x^{(k)}(t')-x^{(k)}(t'')| + |x^{(k)}(t'') - x^*(t'')|$$

$$\leq \varepsilon/2 + k_o|t' - t''| + \varepsilon/2$$

for $|t' - t''|$ small enough. But, since ε is arbitrary, we have

$$|x^*(t') - x^*(t'')| \leq k_o|t' - t''|$$

and so x^* is Lipschitzian and hence is absolutely continuous in $[t_1,t_2]$.

Finally, (iii) follows directly from the Closure Theorem, so x^* is indeed a trajectory and x^*, together with the generating control u^*, forms an optimal pair $\{x^*,u^*\}$.

3. The Second Closure Theorem and The Main Existence Theorem.

Denote by $y = (x^1,\ldots,x^s)$ the s-vector made up of components $x^1,\ldots,x^s$ $(0 \leq s \leq n)$ of the vector $x = (x^1,\ldots,x^n)$ and by z the complementary (n-s)-vector

$z = (x^{s+1},\ldots,x^n)$. Thus we may write $x = (y,z)$. Assume that the domain of F is contained in $E^1 \times C([-\tau,0],E^s) \times E^m$, rather than $E^1 \times C([-\tau,0],E^n) \times E^m$, and let A_o be a closed subset of $E^1 \times C([-\tau,0],E^s)$. Set $A = A_o \times C([-\tau,0],E^{n-s})$.

Theorem 2.3.1. (Closure Theorem II)

Denote by A_o a closed subset of $E^1 \times C([-\tau,0],E^s)$ and let $A = A_o \times C([-\tau,0],E^{n-s})$. Thus A is closed in $E^1 \times C([-\tau,0],E^n)$. Let $U(t,\phi)$ be a closed subset of E^m for each $(t,\phi) \in A_o$ and let $M_o = \{(t,\phi,u) \mid (t,\phi) \in A_o, u \in U(t,\phi)\}$. We take $F = (F_1,\ldots,F_n)$ to be a continuous map of M_o into E^n and define the map $Q: A_o \to E^n$ by

$$Q(t,\phi) = F(t,\phi,U(t,\phi)).$$

Finally, we assume that the maps U and Q have properties (U) and (Q) respectively in A_o.

Under these assumptions, suppose that $\{x^{(k)}\}$ is a sequence of trajectories, $x^{(k)} = (y^{(k)},z^{(k)})$, defined on $[t_{1_k} - \tau, t_{2_k}]$ and are such that: (i) the $y^{(k)}$ converge in the ρ-metric to a function y defined on $[t_1 - \tau, t_2]$ and absolutely continuous on $[t_1,t_2]$; (ii) the $z^{(k)}$ converge pointwise to a function z almost everywhere on $[t_1 - \tau, t_2]$ and z admits a decomposition $z = Z + S$ where Z is absolutely continuous on $[t_1,t_2]$ and $S' = 0$ almost everywhere in $[t_1,t_2]$.

Then, the absolutely continuous vector function $X = (y,Z)$, defined on $[t_1 - \tau, t_2]$ is a trajectory.

Proof. The functions $X':[t_1,t_2] \to E^n$ and $x^{(k)'}:[t_{1_k},t_{2_k}] \to E^n$ are defined almost everywhere and are L-integrable. We must prove that

(i) $(t,y_t,Z_t) \in A$ for all $t \in [t_1,t_2]$,

and

(ii) There is a measurable function $u:[t_1,t_2] \to E^n$ such that

$X'(t) = (y'(t),Z'(t)) = F(t,y_t u(t))$

and $u(t) \in U(t,y_t)$ almost everywhere in $[t_1,t_2]$.

Since we assume $\rho(y^{(k)},y) \to 0$ as $k \to \infty$ we have $t_{1_k} \to t_1$ and $t_{2_k} \to t_2$. For $t \in (t_1,t_2)$ and k sufficiently large, $t \in (t_{1_k},t_{2_k})$, and $(t,y_t^{(k)}) \in A_o$. Since $y_t^{(k)} \to y_t$ as $k \to \infty$ and A_o is closed we must have $(t,y_t) \in A_o$ and $(t,y_t,Z_t) \in A$ for all $t \in (t_1,t_2))$. By continuity of y and Z, $(t,y_t,Z_t) \in A$ for all $t \in [t_1,t_2])$.

Now, $X' = (y',Z')$ exists almost everywhere in $[t_1,t_2]$ and by hypothesis, S' exists and $S' = 0$ almost everywhere. Let $t_o \in (t_1,t_2)$ be a point at which S' exists and equals zero and $z^{(k)}(t_o) \to z(t_o)$. Then there exists a $\sigma > 0$ with $t_1 < t_o - \sigma < t_o + \sigma < t_2$ and a constant k_o such that, for $k > k_o$, $t_{1k} < t_o - \sigma < t_o + \sigma < t_{2k}$. Let

$X_o = X(t_o) = (y_o, Z_o)$, $z_o = z(t_o)$, $S_o = S(t_o)$. $S'(t_o) = 0$ so $z'(t_o) = Z'(t_o)$.

Since $y^{(k)} \to y$ uniformly in $[t_o - (\sigma + \tau), t_o + \sigma]$ and the functions y and $y^{(k)}$ are continuous, they are equicontinuous in $[t_o - (\sigma + \tau), t_o + \sigma]$. Thus, given $\varepsilon > 0$ there exists $\delta > 0$ such that if $t, t' \in [t_o - (\sigma+\tau), t_o + \sigma]$, $|t - t'| < \delta$ and $k > k_o$, we have

$$|y(t) - y(t')| \leq \varepsilon/2, \quad |y^{(k)}(t) - y^{(k)}(t')| \leq \varepsilon/2.$$

We may take $\delta < \sigma$ and $\delta \leq \varepsilon$. For any h, $0 < h \leq \delta$, consider the averages

$$m_h = h^{-1} \int_o^h X'(t_o + s)ds = h^{-1}[X(t_o + h) - X(t_o)]$$

and

$$m_{hk} = h^{-1} \int_o^h F(t_o+s, y^{(k)}_{t_o+s}, u^{(k)}(t_o+s))ds = h^{-1}[x^{(k)}(t_o+h) - x^{(k)}(t_o)].$$

Now given $\eta > 0$ arbitrary, fix h, $0 \leq h < \delta < \sigma$, so small that

$$|m_h - X'(t_o)| \leq \eta \text{ and } |S(t_o + h) - S(t_o)| < \eta h/4.$$

This choice is possible since

$$m_h = h^{-1}[X(t_o + h) - X(t_o)] \to X'(t_o) \text{ as } h \to 0$$

and

$$h^{-1}[S(t_o + h) - S(t_o)] \to S'(t_o) = 0 \text{ as } h \to 0.$$

We may also choose this h so that

$$z^{(k)}(t_o + h) \to z(t_o + h) \text{ as } k \to \infty .$$

For this choice of h, take $k_1 > k_o$ so large that the following inequalities are satisfied:

(i) $|y^{(k)}(t)-y(t)| \leq \min\{\eta h/4, \varepsilon/2\}$ for all $t \in [t_1-(\sigma+\tau), t_o+\sigma]$,

(ii) $|z^{(k)}(t_o) - z(t_o)| \leq \eta h/8$,

(iii) $|z^{(k)}(t_o + h) - z(t_o + h)| \leq \eta h/8$.

We have then

$$|h^{-1}[y^{(k)}(t_o+h)-y^{(k)}(t_o)]-h^{-1}[y(t_o+h)-y(t_o)]|$$

$$\leq |h^{-1}[y^{(k)}(t_o + h) - y(t_o + h)|$$

$$+ |h^{-1}[y^{(k)}(t_o) - y(t_o)]|$$

$$\leq h^{-1}\ \eta h/4 + h^{-1}\ \eta h/4 = \eta/2.$$

Also since $z = Z + S$,

$$|h^{-1}[z^{(k)}(t_o+h)-z^{(k)}(t_o) - h^{-1}[Z(t_o+h) - Z(t_o)]|$$

$$= |h^{-1}[z^{(k)}(t_o + h) - z^{(k)}(t_o)]$$

$$- h^{-1}[z(t_o+k) - z(t_o)] + h^{-1}[S(t_o+h) - S(t_o)]|$$

$$\leq h^{-1}\ \eta h/8 + h^{-1}\ \eta h/8 + h^{-1}\ \eta h/4 = \eta/2.$$

Finally

$$|m_{hk}-m_h| = |h^{-1}[x^{(k)}(t_o+h)-x^{(k)}(t_o)]-h^{-1}[X(t_o+h)-X(t_o)]|$$

$$\leq |h^{-1}[y^{(k)}(t_o+h)-y^{(k)}(t_o)]-h^{-1}[y(t_o+h)-y(t_o)]|$$

$$+ |h^{-1}[z^{(k)}(t_o+h)-z^{(k)}(t_o)]-h^{-1}[Z(t_o+h)-Z(t_o)]|$$

$$\leq \eta/2 + \eta/2 = \eta$$

So, for this value of h we have

$$(i) \quad |m_h - X'(t_o)| \leq \eta,$$

$$(ii) \quad |m_h - m_{hk}| \leq \eta,$$

$$(iii) \quad |y^{(k)}(t_o) - y(t_o)| \leq \varepsilon/2,$$

and for $0 \leq s \leq h$

$$|y^{(k)}(t_o+s)-y(t_o)| \leq |y^{(k)}(t_o+s)-y^{(k)}(t_o)|+|y^{(k)}(t_o)-y(t_o)| \leq$$

$$\varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$|(t_o + s) - t_o| \leq h < s \leq \varepsilon.$$

By definition of $Q(t,\phi)$ we see that

$$F(t_o + s, y^{(k)}_{t_o+s}, u^{(k)}(t_o + s)) \in Q(t_o + s, y^{(k)}_{t_o+s})$$

and setting

$$Q(t_o, y_{t_o}; 2\varepsilon) = \bigcap_{(t,\phi)\in N_{2\varepsilon}(t_o, y_{t_o})} Q(t,\phi)$$

we have

$$F(t_o+s, y^{(k)}_{t_o+s}, u^{(k)}(t_o+s)) \in Q(t_o, y_{t_o}; 2\varepsilon)$$

since $||y^{(k)}_{t_o+s} - y_{t_o}|| \leq ||y^{(k)} - y||_{[t_o-\sigma, t_o+k]}$.

By definition of m_{hk},

$$m_{hk} \in c\ell\ co\ Q(t_o, y_{t_o}; 2\varepsilon)$$

since this set is closed and convex.

Finally,

$$|m_{hk} - X'(t_o)| \leq |m_{hk} - m_h| + |m_h - X'(t_o)| \leq 2\eta$$

implies that

$$X'(t_o) \in [c\ell \text{ co } Q(t_o, y_{t_o}; 2\varepsilon)]_{2\eta}$$

and since η is arbitrary

$$X'(t_o) \in \bigcap_{\eta>0} [c\ell \text{ co } Q(t_o, y_{t_o}; 2\varepsilon)]_{2\eta}$$

from which we get

$$X'(t_o) \in c\ell \text{ co } Q(t_o, y_{t_o}; 2\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary and Q has property (Q) in A_o,

$$X'(t_o) \in \bigcap_{\varepsilon>0} c\ell \text{ co } Q(t_o, y_{t_o}; 2\varepsilon) = Q(t_o, y_{t_o})$$

and this relation holds almost everywhere in $[t_1, t_2]$.

By our implicit function theorem, there is a measurable function $n: [t_1, t_2] \to E^m$, such that

$$X'(t) = F(t, y_t, u(t)),\ u(t) \in U(t, y_t)$$

holds almost everywhere in $[t_1, t_2]$.

We will now find it convenient to rewrite the optimal control problem of Section 2.1 in the form of Mayer rather than in the form of Lagrange. To this end, we introduce an auxiliary variable x^o, defined on $[t_1 - \tau, t_2]$ and satisfying

$$\frac{dx^o}{dt} = F_o(t, x_t, u(t)),\ x^o_{t_1} \equiv 0$$

x^o absolutely continuous in $[t_1,t_2]$. Then

$$x^o(t_2) = \int_{t_1}^{t_2} F_o(t,x_t,u(t))dt = I[x,u].$$

Denote by $\tilde{x}$ the (n+1)-vector $\tilde{x} = (x^o,x^1,\ldots,x^n)$ and by $\tilde{F}$ the (n+1)-vector function

$$\tilde{F} = (F_o,F_1,\ldots,F_n).$$

The optimal control problem is, then, to find a pair $\{\tilde{x},u\}$, $\tilde{x}$ defined on $[t_1 - \tau,t_2]$ and absolutely continuous on $[t_1,t_2]$ and a measurable function u, defined on $[t_1,t_2]$, satisfying the functional differential equation

$$\frac{d\tilde{x}}{dt} = \tilde{F}(t,x_t,u(t)) \quad \text{almost everywhere in } [t_1,t_2],$$

subject to the boundary conditions

$$(t_1,x_{t_1},t_2,x_{t_2}) \in B,\ x^o_{t_1} \equiv 0\ ,$$

and the constraints

$$(t,x_t) \in A \text{ for all } t \in [t_1,t_2],$$
$$u(t) \in U(t,x_t) \text{ almost everywhere in } [t_1,t_2],$$

and for which $x^o(t_2)$ has its minimum value.

In order to state and prove the main result we must introduce two new maps $\tilde{Q}$ and $\tilde{\tilde{Q}}$ which map $A \subset E^1 \times C[-\tau,0]$ into subsets of E^n:

$$Q(t,\phi) = \{z \in E^n \mid z = F(t,\phi,u),\ u \in U(t,\phi)\}$$

$$\tilde{Q}(t,\phi) = \{\tilde{z} = (z_o,z) \in E^{n+1} \mid z^o = F_o(t,\phi,u),\ z = F(t,\phi,u),\ u \in U(t,\phi)\}$$

$$\tilde{\tilde{Q}}(t,\phi) = \{\tilde{z} = (z_o,z) \in E^{n+1} \mid z^o \geq F_o(t,\phi,u),\ z = F(t,\phi,u),\ u \in U(t,\phi)\}.$$

Theorem 2.3.2. (Main Existence Theorem)

Let $t_1, t_2 \in E^1$, $t_1 < t_2$, be given and let $A \subset E^1 \times C[-\tau,0]$ be closed and bounded and for every $(t,\phi) \in A$. Let $U(t,\phi)$ be closed in E^m. Define a set $M \subset E^1 \times C[-\tau,0] \times E^m$ by $M = \{(t,\phi,u) \mid (t,\phi) \in A, u \in U(t,\phi)\}$ and assume that

(a) $\tilde{F} = (F_o,F_1,\ldots,F_n)$ is continuous on M,

(b) $\tilde{\tilde{Q}}(t,\phi)$ is convex and $\tilde{\tilde{Q}}$ satisfies property (Q) in A,

(c) U satisfies property (U) in A,

(d) the set B is of the form $B = B_1 \times B_2$ where B_1 is compact in $E^1 \times C[-\tau,0]$.

Then, if for any $\varepsilon > 0$ there is a non-negative Lebesgue integrable function ψ_ε such that

$$|F(t,\phi,u)| \leq \psi_\varepsilon(t) + \varepsilon F_o(t,\phi,u)$$

for all $(t,\phi,u) \in M$, then the cost functional

$$I[x,u] = \int_{t_1}^{t_2} F_o(t,x_t,u(t))dt$$

has an absolute minimum in any non-empty complete class Ω of admissible pairs.

Proof. For $\varepsilon = 1$ we have that

$$0 \leq \psi_1(t) + F_o(t,\phi,u)$$

or

$$-\psi_1(t) \leq F_o(t,\phi,u) \text{ for all } (t,\phi,u) \in M.$$

Now the set A is bounded so that its projection into E^1 is contained in some finite interval $[t_1^*,t_2^*]$ and $[t_1,t_2] \subset [t_1^*,t_2^*]$. Thus

$$-\int_{t_1}^{t_2} \psi_1(t)dt \geq -\int_{t_1^*}^{t_2^*} \psi_1(t)dt$$

and so

$$I[x,u] = \int_{t_1}^{t_2} F_o(t,x_t,u(t))dt \geq -\int_{t_1^*}^{t_2^*} \psi_1(t)dt = -K_1 > -\infty.$$

So if $i = \inf_{\Omega} I[x,u]$, then $i > -\infty$ and there exists a sequence $\{x^{(k)},u^{(k)}\} \subset \Omega$ such that $I[x^{(k)},u^{(k)}] \to i$ as $k \to \infty$, each $x^{(k)}$ being defined on $[t_{1_k} - \tau, t_{2_k}]$, absolutely continuous on $[t_{1_k},t_{2_k}]$ and each $u^{(k)}$ being defined on $[t_{1_k},t_{2_k}]$. Furthermore, we may assume that

$$i \leq I[x^{(k)},u^{(k)}] = \int_{t_{1_k}}^{t_{2_k}} F_o(t,x_t^{(k)},u^{(k)}(t))dt \leq i+1/k \leq i+1.$$

The next step is to show that the vector functions $\{x^{(k)}\}$ which are absolutely continuous on $[t_{1_k},t_{2_k}]$, are equiabsolutely continuous there. To this end, let $\varepsilon > 0$ and choose $\sigma = \frac{1}{2}\varepsilon(|i| + 1)^{-1}$. Then, for every measurable subset of E of $[t_{1_k},t_{2_k}]$ we have

$$\int_E |\dot{x}^{(k)}(t)|dt = \int_E |F(t,x_t^{(k)},u(t))|dt$$

$$\leq \int_E \psi_\sigma(t)dt + \sigma \int_E F_o(t,x_t^{(k)},u^{(k)}(t))dt$$

$$\leq \int_E \psi_\sigma(t)dt + \sigma \int_{t_{1_k}}^{t_{2_k}} F_o(t,x_t^{(k)},u^{(k)}(t))dt$$

$$\leq \int_E \psi_\sigma(t)dt + \sigma(|i| + 1)$$

$$\leq \int_E \psi_\sigma(t)dt + \varepsilon/2.$$

By the integrability of ψ_σ we may conclude that there is a $\delta > 0$ such that if meas $E < \delta$ then $\int_E \psi_\sigma(t)dt < \varepsilon/2$. Hence the functions $\{\dot{x}^{(k)}\}$ are equiabsolutely integrable from which it follows that the functions $\{x^{(k)}\}$ are equiabsolutely continuous on $[t_{1_k},t_{2_k}]$. Furthermore, since the initial functions $\{x_{t_{1_k}}^{(k)}\}$ are elements of the compact set B_1, the functions are in fact equicontinuous on $[t_{1_k} - \tau,t_{2_k}]$.

Consider now the sequence of absolutely continuous functions defined by

$$x^{o(k)}(t) = \int_{t_{1_k}}^{t} F_o(t,x_t^{(k)},u^{(k)}(t))dt \quad (t_{1_k} \leq t \leq t_{2_k})$$

$$x^{o(k)}(t) \equiv 0 \qquad (t_{1_k} - \tau \leq t \leq 0).$$

Note that $x^{o(k)}(t_{2_k}) = I[x^{(k)},u^{(k)}] \to i$ as $k \to \infty$ and that $i \le x^{o(k)}(t_{2_k}) \le i + 1/k < i + 1$.

Define functions v_k^o, v_k^+ and v_k^- on $[t_{1_k},t_{2_k}]$ by

$$v_k^o(t) = F_o(t,x_t^{(k)},u^{(k)}(t)),$$

$$v_k^+(t) = F_o(t,x_t^{(k)},u^{(k)}(t)) + \psi_1(t) \ge 0,$$

$$v_k^-(t) = -\psi_1(t).$$

Let

$$y^{(k)^-}(t) = \begin{cases} \int_{t_{1_k}}^{t} v_k^-(t)dt, & t_{1_k} \le t \le t_{2_k} \\ 0, & t_{1_k} - \tau \le t \le t_{1_k} \end{cases}$$

and

$$y^{(k)^+}(t) = \begin{cases} \int_{t_{1_k}}^{t} v_k^+(t)dt, & t_{1_k} \le t \le t_{2_k} \\ 0, & t_{1_k} - \tau \le t \le t_{1_k} \end{cases}$$

The functions $y^{(k)^-}$ are equicontinuous since

$$|y^{(k)^-}(t) - y^{(k)^-}(t')| \le \int_{t'}^{t} \psi_1(s)ds$$

and so, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|t - t'| < \delta$, $\int_{t'}^{t} \psi_1(s)ds < \varepsilon$.

Since the sequences $\{x^{(k)}\}$ and $\{y^{(k)^-}\}$ are equicontinuous families on $[t_{1_k} - \tau, t_{2_k}]$ we may use

Ascoli's Theorem to extract a convergent subsequence which we again call $\{x^{(k)},y^{(k)^-}\}$. Let $(x^{(k)},y^{(k)^-}) \to (x,Y^-)$ where x is defined on $[t_1-\tau,t_2]$. Since the $\{x^{(k)}\}$ are equiabsolutely continuous on $[t_1,t_2]$, x will be absolutely continuous on this interval. The function Y^- is likewise defined on $[t_1 - \tau,t_2]$.

Now consider the function $y^{(k)^+}$. Certainly these functions are non-negative, monotone non-decreasing and are equibounded since

$$0 \leq y^{(k)^+}(t_{2_k}) = [y^{(k)^+}(t_{2_k})+y^{(k)^-}(t_{2_k})]-y^{(k)^-}(t_{2_k})$$
$$= x^{o(k)}(t_{2_k})-y^{(k)^-}(t_{2_k}) \leq |i|+1 + K_1,$$

where K_1 is independent of k. So we may apply Helly's Theorem to the subsequence above and extract a further subsequence, which we again call $\{y^{(k)^+}\}$, and which converges pointwise in (t_1,t_2) to a function Y^{o+} which is non-negative, non-decreasing but is not necessarily continuous. Let $Y^{o+}_{t_1} = 0$ and set

$$Y^{o+}(t_2) = \lim_{t \to t_2^-} Y^{o+}(t).$$

The limit on the right exists because Y^{o+} is monotone non-decreasing on $(t_1 - \tau,t_2)$ and is bounded there by $|i| + 1 + K_1$. Since Y^{o+} is of bounded variation, it may be written as

$$Y^{o+} = Y^{+} + Z, \quad Y^{+}_{t_1} \equiv 0, \; Z_{t_1} \equiv 0$$

where Z and Y^{+} are non-negative, monotone, and non-decreasing, Y^{+} is absolutely continuous on $[t_1,t_2]$ and $Z' = 0$ almost everywhere in $[t_1,t_2]$.

Let $Y = Y^{-} + Y^{+}$. Then $x^{o(k)} \to Y + Z$ for all $t \in [t_1 - \tau, t_2]$ where Y satisfies the condition $Y_{t_1} \equiv 0$ and

$$-K_1 \leq Y \leq |i| + 1 + K_1.$$

For this final subsequence we have $t_{2_k} \to t_2$, $x^{o(k)}(t_{2_k}) \to i$ and $x^{o(k)}(t_{2_k}) = y^{(k)^+}(t_{2_k}) + y^{(k)^-}(t_{2_k})$. Since the $\{y^{(k)^-}\}$ are equicontinuous, for each $\varepsilon > 0$ we can find a $\delta > 0$ so that if $t,t' \in [t_{1_k}, t_{2_k}]$ with $|t - t'| < \delta$ then $|y^{(k)^-}(t) - y^{(k)^-}(t')| < \varepsilon$ for all k.

Choose $\varepsilon > 0$ and let $\bar{t}_2 \in (t_1,t_2)$ be such that $|t_2 - \bar{t}_2| < \delta/2$. Then, taking k so large that $|t_2 - t_{2_k}| < \delta/2$ and $\bar{t}_2 < t_{2_k}$ we have that

$$|\bar{t}_2 - t_{2_k}| \leq |\bar{t}_2 - t_2| + |t_2 - t_{2_k}| < \delta/2 + \delta/2 = \delta$$

and so

$$|y^{(k)^-}(\bar{t}_2) - y^{(k)^-}(t_{2_k})| < \varepsilon.$$

Since the functions $\{y^{(k)^+}\}$ are non-decreasing, we have that $y^{(k)^+}(\bar{t}_2) \leq y^{(k)^+}(t_{2_k})$ and so

$$y^{(k)^-}(\bar{t}_2) + y^{(k)^+}(\bar{t}_2) \leq y^{(k)^-}(\bar{t}_2) + y^{(k)^+}(t_{2_k})$$

$$\leq y^{(k)^-}(t_{2_k}) + y^{(k)^+}(t_{2_k}) + |y^{(k)^-}(\bar{t}_2) - y^{(k)^-}(t_{2_k})|$$

$$\leq x^{o(k)}(t_{2_k}) + \varepsilon .$$

But

$$x^{o(k)}(t_{2_k}) = \int_{t_{1_k}}^{t_{2_k}} F_o(t, x_t^{(k)}, u^{(k)}(t))dt \leq 1 + 1/k$$

and so $y^{(k)^-}(\bar{t}_2) + y^{(k)^+}(\bar{t}_2) \leq 1 + \varepsilon + 1/k$. Taking the limit along the final subsequence as $k \to \infty$ we have

$$Y^-(\bar{t}_2) + Y^{o+}(\bar{t}_2) \leq 1 + \varepsilon$$

or $$Y^-(\bar{t}_2) + Y^+(\bar{t}_2) + Z(\bar{t}_2) \leq 1 + \varepsilon$$ where

$Z(\bar{t}_2) \geq 0$.

Hence

$$Y(\bar{t}_2) = Y^-(\bar{t}_2) + Y^+(\bar{t}_2) \leq 1 + \varepsilon.$$

Thus we have shown that, for any $\varepsilon > 0$ if $\bar{t}_2$ is sufficiently close to t_2 then $Y(\bar{t}_2) \leq 1 + \varepsilon$. Hence $Y(\bar{t}_2) \leq 1$ and since Y is continuous at t_2, $Y(t_2) \leq 1$.

We will now rewrite the problem in a form to which we may apply the second closure theorem. Let $\tilde{u} = (u^o, u) = (u^o, u^1, \ldots, u^m)$ and $\tilde{U}(t,\phi)$ be the set of all $\tilde{u} \in E^{m+1}$ with $u \in U(t,\phi)$ and $u^o \geq F_o(t,\phi,u)$.

Let $\tilde{x} = (x^o, x) = (x^o, x^1, \ldots, x^n)$ and $\tilde{\tilde{F}} = (\tilde{F}_o, F_1, \ldots, F_n)$ where $\tilde{F}_o = u^o$. Note that the domain of $\tilde{\tilde{F}}$ is a subset of $A \times E^{m+1}$ and $\tilde{\tilde{F}}$ does not depend on the component x^o. U is likewise independent of x^o. We now consider the differential equation

$$\frac{d\tilde{x}}{dt} = \tilde{\tilde{F}}(t, x_t, u(t)) \quad (t_1 \leq t \leq t_2)$$

with constraints

$$\tilde{u}(t) \in \tilde{U}(t, x_t) \text{ almost everywhere in } [t_1, t_2],$$

$$(t, x_t) \in A \quad \text{for all } t \in [t_1, t_2],$$

and subject to the boundary conditions

$$(t_1, x_{t_1}, t_2, x_{t_2}) \in B_1 \times B_2, \; B_1 \text{ compact}$$

$$x^o_{t_1} \equiv 0.$$

We seek to minimize the functional

$$J[\tilde{x}, \tilde{u}] = \int_{t_1}^{t_2} \tilde{F}_o \, dt = \int_{t_1}^{t_2} u^o(t) dt = x^o(t_2).$$

The set

$$\tilde{\tilde{Q}}(t, \phi) = \tilde{\tilde{F}}(t, \phi, \tilde{U}(t, \phi)) = \{\tilde{z} = (z^o, z) \in E^{n+1} \mid z^o = u^o$$

$$\geq F_o(t, \phi, u), z = F(t, \phi, u), \; u \in U(t, \phi)\}$$

is, by hypothesis, convex and $\tilde{\tilde{Q}}$ satisfies property (Q).

Consider now the sequence of trajectories $\tilde{x}^{(k)} = (x^{o(k)}, x^{(k)})$ defined on $[t_{1_k} - \tau, t_{2_k}]$ each corresponding to a control function $\tilde{u}^{(k)} = (u^{o(k)}, u^{(k)})$

defined on $[t_{1_k}, t_{2_k}]$ where $u^{o(k)} = F_o(t, x_t^{(k)}, u^{(k)}(t))$, $u^{(k)}(t) \in U(t, x_t^{(k)})$ and hence $\tilde{u}^{(k)} \in \tilde{U}(t, x_t^{(k)})$. The sequence $\{x^{(k)}\}$ converges in the ρ-metric to an absolutely continuous vector function x while $x^{o(k)} \to x^o$ for all $t \in (t_1 - \tau, t_2)$ and $x^o = Y + Z$ where Y is absolutely continuous and $Z' = 0$ almost everywhere in $[t_1, t_2]$.

By the second closure theorem (Theorem 2.3.1) we conclude that $X = (Y, x)$ is a trajectory for the problem. That is, there exists a control function $\tilde{u} = (u^o, u)$ defined on $[t_1, t_2]$ with

$$\frac{dY}{dt} = u^o(t) \geq F_o(t, x_t, u(t))$$

almost everywhere in $[t_1, t_2]$ and

$$i \geq Y_2 = J[\tilde{x}, \tilde{u}] = \int_{t_1}^{t_2} u^o(t)dt.$$

Furthermore, $\{x, u\}$ is admissible for the original problem and hence $\{x, u\} \in \Omega$ since Ω is assumed to be complete.

Thus

$$i \leq I[x, u] = \int_{t_1}^{t_2} F_o(t, x_t, u(t))dt \leq \int_{t_1}^{t_2} u^o(t)dt = Y(t_2) \leq i$$

and so $u^o(t) = F_o(t, x_t, u(t))$ almost everywhere in $[t_1, t_2]$ and $\{x, u\} \in \Omega$ is the required optimal pair.

To conclude this section we present an additional theorem which we will apply in Chapter 4.

Theorem 2.3.3.

Let $A \subset E^1 \times C[-\tau,0]$ be closed and bounded and let $U(t,\phi)$ be closed for every $(t,\phi) \in A$. Let $M = \{(t,\phi,u) \mid (t,\phi) \in A, u \in U(t,\phi)\}$ and assume that

(a) $\tilde{F} = (F_0,F_1,\ldots,F_n)$ is continuous on M,

(b) $\tilde{Q}(t,\phi)$ is convex for each $(t,\phi) \in A$ and the map $\tilde{Q}$ satisfies property (Q) in A,

(c) The map U satisfies property (U) in A,

(d) The set B has the form $B = B_1 \times B_2$, where B_1 is compact in $E^1 \times C[-\tau,0]$.

Then, if there exists a Lebesgue-integrable function ψ such that $F_0(t,\phi,u) \geq \psi(t)$ for all $(t,\phi,u) \in M$ and if Ω is a non-empty complete class of admissible pairs such that

$$\int_{t_1}^{t_2} |dx^i/dt|^p \leq N_i \quad (i = 1,2,\ldots,n)$$

for some constants $N_i \geq 0$ and $p > 1$, the cost functional $I[x,u]$ has an absolute minimum in Ω.

Proof. Since A is bounded, its projection into E^1 is contained in some finite interval $[t_1^*,t_2^*]$ and $[t_1,t_2] \subset [t_1^*,t_2^*]$. Thus

$$I[x,u] = \int_{t_1}^{t_2} F_0(t,x_t,u(t))dt \geq -\int_{t_1^*}^{t_2^*} |\phi(t)|dt$$

and so $-\infty < i = \inf_{\Omega} I[x,u]$. Let $\{x^{(k)},u^{(k)}\} \subset \Omega$ be a minimizing sequence. We may assume that

$$i \leq I[x^{(k)},u^{(k)}] = \int_{t_{1_k}}^{t_{2_k}} F_o(t,x_t^{(k)},u^{(k)}(t))dt \leq i+1/k \leq i+1.$$

By hypothesis, we have that

$$\int_{t_{1_k}}^{t_{2_k}} |dx_i^{(k)}/dt|^p \, dt \leq N_i \quad (i = 1,2,\ldots,n, \text{ and } k=1,2,\ldots)$$

and we use this condition to prove the existence of a convergent subsequence.

By the weak compactness of L^p, we conclude that there exists a subsequence, which we again call $\{x^{(k)},u^{(k)}\}$ and a continuous vector function x, defined on $[t_1 - \tau,t_2]$ and absolutely continuous on $[t_1,t_2]$ for which

$$dx_i^{(k)}/dt \rightharpoonup dx_i/dt$$

weakly in L^p on $[t_1,t_2]$. Since the initial functions $x_{t_{1_k}}$ all lie in a compact subset of $C[-\tau,0]$, we can extract a further subsequence which converges to x in the ρ-metric on all of $[t_1 - \tau,t_2]$. The proof may now be completed using the same arguments as for the preceding theorem.

CHAPTER III

NON-LINEAR CONTROLLABILITY

The existence theorems of the preceding chapter assume, as do most of the existence theorems in the literature, that if at least one admissible pair exists, then there is an optimal solution in any non-empty complete class of admissible pairs. The question of the existence of an admissible pair has come to be known as the controllability problem. There is an extensive literature dealing with the controllability of finite dimensional linear systems.

Roughly speaking, the controllability of a system is associated with the ability to drive the system from some initial state to some target in a finite amount of time. Many different types of controllability have been defined in the literature, most of them differing in the nature of the initial conditions and the target set. For example, a system is said to be completely null controllable if it can be steered from any point in the state space to the origin in finite time. It is clear that the main point at issue is whether, given the differential equations governing the system and a class of admissible controls, the boundary conditions and state constraints can be satisfied. Thus the controllability problem is, essentially, a boundary-value problem.

The concept of controllability was first introduced by Caratheodory [7], in his work on thermodynamics. Kalman[30] and Pontryagin [40] independently derived the same algebraic condition for complete null controllability of linear systems. More recently Markus and Lee [32] have studied the problem of controllability for non-linear systems and have discussed complete null controllability for non-linear systems under the hypothesis of global asymptotic stability. A complete survey of this theory is given in [32 , Chapters 2 and 6]. Finally Antosiewicz [1], Conti [12] and others have studied controllability using properties of the set of attainability.

Recently Tarnove [41] has suggested a more general approach to this problem which permits not only the discussion of non-linear systems, but also the introduction of phase constraints and very general boundary conditions. The result of his work is to reduce the problem of controllability, by means of a simple extension of the Kakutani fixed point theorem, to a problem of showing certain sets are non-void. Using these techniques, Tarnove is able to derive a sufficient condition for controllability in the case that the control set is a fixed compact set in E^m. He also suggests that his results extend to the case of control sets which are compact and metrically upper semicontinuous. However, the extension is not made in [41].

Section 1 of this Chapter will describe the background of the method. In Section 2, we will extend the results of Tarnove to functional differential equations for Pontryagin problems. The section will conclude with an example of the application of these results to a system described by a differential-difference equation.

The fixed point technique can be extended further to the case in which the control sets are closed and have property (U). This extension will be made in Section 3 with the help of a simple modification of the Closure Theorem which appears in Chapter II.

1. Statement of the Problem.

We begin by defining a very general form of controllability.

Definition 3.1.1.

Let $\mathbb{A}$ be a given subset of $C[t_1 - \tau, t_2]$. We say that the control system

$$\dot{x}(t) = F(t, x_t, u(t)), \quad u(t) \in U(t, x_t) \text{ almost everywhere}$$

is $\mathbb{A}$-controllable if there exists an admissible pair $\{x,u\}$ such that x is an element of $\mathbb{A}$.

It is easy to see how this definition fits into the context of the Lagrange problem described on p. 35. We need

only take the set $\mathbb{A}$ to be defined as the set of all elements of $C[t_1 - \tau, t_2]$ which satisfy the constraints $(t, x_t) \in A$ and the boundary conditions $(t_1, x_{t_1}, t_2, x_{t_2}) \subset B_1 \times B_2$ and are absolutely continuous on $[t_1, t_2]$.

Now, assume $\mathbb{A}$ is non-empty and let $\mathbb{B}$ be any non-void subset of $\mathbb{A}$. For each $y \in \mathbb{B}$ we define a set $\mathbb{B}(y)$ by

$$\mathbb{B}(y) = \{ z \in \mathbb{B} \mid \dot{z}(t) \in Q(t, y_t) \text{ almost everywhere} \} \tag{3.2.1}$$

This defines a map $\Phi: \mathbb{B} \to 2^{\mathbb{B}}$.

Definition 3.1.2.

Let S be any subset of a topological space and let $\Psi: S \to 2^S$. We say that the map Ψ has a fixed point if there is an element $x \in S$ such that $x \in \Psi(x)$.

Now, suppose that the map Φ has a fixed point $x \in \mathbb{B}$. Then, by definition of $\mathbb{B}(y)$, we have $x \in \mathbb{B}(x)$ and so

$$\dot{x}(t) \in Q(t, x_t)$$

almost everywhere in $[t_1, t_2]$. The implicit function theorem (Theorem 2.1.1) assures us that, under these conditions, there is an admissible control u for which the pair $\{x, u\}$ is admissible and hence the system is $\mathbb{A}$-controllable.

In this manner, the question of controllability has been reduced to the problem of finding a fixed point for a set-valued map. Kakutani [29] has given sufficient conditions under which a set-valued map, defined on a finite dimensional space, has a fixed point. Bohenblust and Karlin [5 ; p. 159] have extended the Kakutani theorem to Banach spaces. We will use their result which may be stated in the following form.

Theorem. Let S be a closed convex set in a Banach space X. If $\Psi : S \to 2^S$ is such that

(a) if $\{x_n\}$ and $\{y_n\}$ are sequences in S such that $x_n \to x$, $y_n \to y$ and $y_n \in \Psi(x_n)$, then $y \in \Psi(x)$

and

(b) $\bigcup_{x \in S} \Psi(x)$ is contained in some sequentially compact set T.

Then the map Ψ has a fixed point.

In his original paper, Kakutani referred to condition (a) as upper semicontinuity. To avoid ambiguity in this exposition, we will refer to (a) as property (K). It should be noted that property (K) is weaker than metric upper-semicontinuity for it can easily be shown that set-valued maps with property (U) also enjoy property (K) while examples can be given of maps having property (U) which are not metrically upper-semicontinuous (Cesari [9 ; p. 374]).

Our object in the following sections is to develop conditions under which the map Φ given by equation (3.2.1) has a fixed point.

2. Controllability for Pontryagin Problems.

Let A be a convex subset of $E^1 \times C[-\tau,0]$ and let $U : A \to 2^{E^m}$ be metrically upper semicontinuous. Assume that, for each $(t,\phi) \in A$, $U(t,\phi)$ is compact. Let $F : E^1 \times C[-\tau,0] \times E^m \to E^n$ be continuous on the set $M = \{(t,\phi,u) \mid (t,\phi) \in A, u \in U(t,\phi)\}$ and let $Q(t,\phi) = F(t,\phi,U(t,\phi))$ be convex. Let B be a convex subset of $E^1 \times C[-\tau,0] \times E^1 \times C[-\tau,0]$. We consider the system

$$\dot{x}(t) = F(t,x_t,u(t)), \quad u(t) \in U(t,x_t). \tag{3.2.1}$$

Let $\mathbb{A}$ denote the set of all continuous vector functions $x : [t_1 - \tau, t_2] \to E^n$ such that

(a) x is absolutely continuous on $[t_1,t_2]$,

(b) $(t,x_t) \in A$ for $t \in [t_1,t_2]$,

(c) $(t_1,x_{t_1},t_2,x_{t_2}) \in B$.

We will assume that $\mathbb{A}$ is non-void and we wish to determine sufficient conditions for $\mathbb{A}$-controllability of (3.2.1). Let $\mathbb{B}$ be a non-empty, bounded, closed and convex subset of $\mathbb{A}$. For each $y \in \mathbb{B}$, we form the subset $\mathbb{B}(y) \subset \mathbb{B}$ defined by

$$\mathbb{B}(y) = \{z \in \mathbb{B} \mid \dot{z}(t) \in Q(t, y_t) \text{ a.e. in } [t_1, t_2]\}.$$

We will assume that, for all $y \in \mathbb{B}$, $\mathbb{B}(y) \neq \emptyset$.

Lemma 3.2.1.

Let $\mathbb{B}$ be defined as above and assume that for each $y \in \mathbb{B}$, $\mathbb{B}(y) \neq \emptyset$. Then

$$\bigcup_{y \in \mathbb{B}} \mathbb{B}(y) \subset \mathbb{B}$$

is relatively compact.

Proof. Since $\mathbb{B}$ is assumed to be bounded, the set of functions $\bigcup_{y \in \mathbb{B}} \mathbb{B}(y)$ is equibounded. It remains only to show that this set is equicontinuous.

Since A is compact, U metrically upper-semicontinuous in A and since $U(t, \phi)$ is compact for each $(t, \phi) \in A$, it follows (p. 43) that the set M is compact. Since F is continuous on M, there is a constant $L \geq 0$ such that $|F(t, \phi, u)| \leq L$ for all $(t, \phi, u) \in M$. Hence, for every $x \in \bigcup_{y \in \mathbb{B}} \mathbb{B}(y)$ we have $\dot{x}(t) = F(t, y_t, u(t))$ for some $y \in \mathbb{B}$ and so

$$|x(t_2) - x(t_1)| \leq L|t_2 - t_1|.$$

Hence, the elements of $\bigcup_{y \in \mathbb{B}} \mathbb{B}(y)$ are equicontinuous on $[t_1, t_2]$. Since $(t_1, x_{t_1}) \in A$ and A is compact, the $x_{t_1} \in C[-\tau, 0]$ are equicontinuous and so the functions

$x \in \bigcup_{y \in \mathbb{B}} \mathbb{B}(y)$ are equicontinuous on the whole interval $[t_1 - \tau, t_2]$. The set $\bigcup_{y \in \mathbb{B}} \mathbb{B}(y)$ is, therefore, relatively compact.

Lemma 3.2.2.

For every $y \in \mathbb{B}$ the set $\mathbb{B}(y)$ is convex.

Proof. Let $x^{(1)}, x^{(2)} \in \mathbb{B}(y)$ for some y and let $x = \lambda x^{(1)} + (1 - \lambda)x^{(2)}$ for fixed λ, $0 \leq \lambda \leq 1$. Since

$$\dot{x}^{(i)}(t) = F(t, y_t, u^{(i)}(t)) \quad (i = 1,2)$$

we have that

$$(3.2.2) \quad \dot{x}(t) = \lambda F(t, y_t, u^{(1)}(t)) + (1 - \lambda) F(t, y_t, u^{(2)}(t))$$

almost everywhere in $[t_1, t_2]$. But the sets $Q(t, y_t)$ are assumed to be convex and so the right-hand side of (3.2.2) is an element of $Q(t, y_t)$. Since A, B and $\mathbb{B}$ are convex sets, this shows that $x \in \mathbb{B}$. Then, by definition of $\mathbb{B}(y)$ and (3.2.2) we may conclude that $x \in \mathbb{B}(y)$. This shows that $\mathbb{B}(y)$ is convex.

Lemma 3.2.3.

The map $\Phi: \mathbb{B} \to 2^{\mathbb{B}}$, given by $\Phi(y) = \mathbb{B}(y)$, enjoys property (K).

Proof. Let $\{x^{(n)}\}$ and $\{y^{(n)}\}$ be two convergent sequences in $\mathbb{B}$ such that $x^{(n)} \to x$, $y^{(n)} \to y$ and $x^{(n)} \in \mathbb{B}(y^{(n)})$.

We wish to show that $x \in \mathbb{B}(y)$ or, using the definition of $\mathbb{B}(y)$, that $\dot{x}(t) \in Q(t,y_t)$ almost everywhere in $[t_1,t_2]$.

Since M is compact and F is continuous on M the functions

$$\phi_n(t) = F(t,y_t^{(n)},u^{(n)}(t)), \ (t_1 \leq t \leq t_2),$$

are equibounded, say $||\phi_n|| \leq L$ for all n. Since $|t_2 - t_1| < \infty$ we can consider ϕ_n as a sequence in the set

$$N = \{z \in \mathcal{L}[t_1,t_2] \mid |z(t)| \leq L \text{ almost everywhere}\}.$$

For any measurable set $E \subset [t_1,t_2]$ and $z \in N$ we have

$$\left|\int_E z \, d\mu\right| \leq L\mu(E)$$

and so $\lim_{\mu(E)\to 0} \int_E z d\mu = 0$. Thus N is weakly sequentially compact [18; Corollary 8.11, p. 294]. Therefore we may extract a convergent subsequence which we again call $\{\phi_n\}$ such that $\phi_n \to \phi \in N$ weakly i.e.

$$\lim_{n\to\infty} \int_{t_1}^{t_2} \phi_n(t) - \psi(t)dt = \int_{t_1}^{t_2} \phi(t) \cdot \psi(t)dt$$

where ψ is measurable and essentially bounded on $[t_1,t_2]$. So, for any measurable set $E \subset [t_1,t_2]$ and $\omega \in E^n$, we have

$$\int_E \liminf_{n\to\infty} [\omega \cdot \phi_n(s)]ds \le \liminf_{n\to\infty} \int_E \omega \cdot \phi_n(s)ds$$

$$\le \int_E \omega \cdot \phi(s)ds \le \limsup_{n\to\infty} \int_E \omega \cdot \phi_n(s)ds$$

$$\le \int_E \limsup_{n\to\infty} [\omega \cdot \phi_n(s)]ds.$$

Therefore, for all $\omega \in E^m$,

$$\liminf_{n\to\infty} [\omega \cdot \phi_n(t)] \le \omega \cdot \phi(t) \le \limsup_{n\to\infty} [\omega \cdot \phi_n(t)]$$

holds almost everywhere. By definition of the ϕ_n, we have

$$\liminf_{n\to\infty} [\omega \cdot F(t, y_t^{(n)}, u^{(n)}(t)] \le \omega \cdot \phi \le \limsup_{n\to\infty} [\omega \cdot F(t, y_t^{(n)}, u^{(n)}(t)]$$

and so

$$\liminf_{n\to\infty} \left(\inf_{v \in U(t, y_t^{(n)})} [\omega \cdot F(t, y_t^{(n)}, v)]\right) \le \omega \cdot \phi$$

$$\le \limsup_{n\to\infty} \left(\sup_{v \in U(t, y_t^{(n)})} [\omega \cdot F(t, y_t^{(n)}, u^{(n)}(t)]\right).$$

Now M is compact and F is assumed to be continuous on M, hence F is uniformly continuous there. Thus, given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\inf_{v \in U(t, y_t)} [\omega \cdot F(t, y_t, v)] \le \inf_{v \in [U(t, y_t)]_\delta} [\omega \cdot F(t, y_t, v)] + \varepsilon.$$

The map U is metrically upper semicontinuous in A and so there is a number N_o such that if $n > N_o$,

$$\inf_{v\in[U(t,y_t)]_\delta} [\omega\cdot F(t,y_t,v)]+\varepsilon \le \inf_{v\in U(t,y_t^{(n)})} [\omega\cdot F(t,y_t,v)] + \varepsilon.$$

Combining these inequalities we get

$$\inf_{v\in U(t,y_t)} [\omega\cdot F(t,y_t,v)] \le \inf_{n>N_o} \left(\inf_{v\in U(t,y_t^{(n)})} [\omega\cdot F(t,y_t,v)]\right) + \varepsilon$$

and since ε is arbitrary we have that

$$\inf_{v\in U(t,y_t)} [\omega\cdot F(t,y_t,v)] \le \lim_{n\to\infty} \inf \left(\inf_{v\in U(t,y_t^{(n)})} [\omega\cdot F(t,y_t^{(n)},v)]\right)$$

and so we can conclude that

$$\inf_{v\in U(t,y_t)} [\omega \cdot F(t,y_t,v)] \le \omega \cdot \phi(t).$$

An identical argument shows that

$$\omega \cdot \phi(t) \le \sup_{v\in U(t,y_t)} [\omega \cdot F(t,y_t,v)].$$

We have then that

$$\inf_{v\in U(t,y_t)} [\omega \cdot F(t,y_t,v)] \le \omega\cdot \phi(t) \le \sup_{v\in U(t,y_t)} [\omega\cdot F(t,y_t,v)].$$

Since this inequality holds for all $\omega \in E^n$ and since $Q(t,y_t)$ is both compact and convex, we may conclude that $\phi(t) \in Q(t,y_t)$ almost everywhere. So, by the implicit function theorem, there is a measurable function u such that

$$\phi(t) = F(t,y_t,u(t)) \text{ and } u(t) \in U(t,y_t)$$

almost everywhere in $[t_1,t_2]$. So we have

$$x(t)=\lim_{n\to\infty} x_n(t_1)+\lim_{n\to\infty}\int_{t_1}^{t}\phi_n(s)ds=x(t_1)+\int_{t_1}^{t_2}F(s,y_s,u(s)]ds$$

and so

$$\dot{x}(t)\ \varepsilon\ Q(t,y_t)\ \text{almost everywhere in}\ [t_1,t_2]$$

which is what we wished to prove.

The conclusions of these three lemmas may be used to prove the following theorem.

Theorem 3.2.1.

Let $\mathbb{B}$ be a non-empty, closed, bounded, and convex subset of $\mathbb{A}$. Then a sufficient condition that the system be $\mathbb{B}$-controllable, and hence $\mathbb{A}$-controllable, is that, for all $y \varepsilon \mathbb{B}$, $\mathbb{B}(y) \neq \emptyset$.

Proof. The proof is an immediate consequence of the fixed point theorem, for if for all $y \varepsilon \mathbb{B}$, $\mathbb{B}(y) \neq \emptyset$ then by Lemma 1, the set $\bigcup_{y\varepsilon\mathbb{B}} \Phi(y)$ is contained in a sequentially compact set and Lemma 3 guarantees that Φ has property (K). The fixed point theorem then ensures the existence of a point $x_o \varepsilon \mathbb{B}$ such that $x_o \varepsilon \mathbb{B}(x_o)$. So, by the implicit function theorem, x_o satisfies

$$\dot{x}_o(t) = F(t,x_{o_t},u(t)),\ u(t)\ \varepsilon\ U(t,x_{o_t})$$

almost everywhere in $[t_1,t_2]$ for some measurable function u. Hence the system is $\mathbb{B}$-controllable.

We will now show how this theorem can be used in a simple situation.

Let $[t_1,t_2] = [0,1]$, $n = 1$ and suppose that the set $B = B_1 \times B_2$ is convex. We consider the linear control system given by the differential-difference equation

$$\dot{x}(t) = ax(t - \tau) + bu(t) \tag{3.2.3}$$

Assume that $U \subset E^1$ is a ball of radius $\rho > 0$. Then, if the set

$$\mathbb{A} = \{x \in C[-\tau,1] \mid (0,x_0) \in B_1,\ (1,x_1) \in B_2\}$$

contains two functions of the form $x(t) = e^{\alpha t}$, $y(t) = e^{\beta t}$, $\alpha,\beta \geq 0$, if B_1 contains the constant functions $e^{-\alpha\tau}$ and $e^{-\beta\tau}$ if B_2 contains the functions $\phi(\theta) = e^{\alpha(1+\theta)}$, $\psi(\theta) = e^{\beta(1+\theta)}$, $-2\tau \leq \theta \leq \tau$ and if $\rho > K$ where K is a constant depending on a,b,α and β, then the system is controllable.

To prove this statement, we must produce a non-empty, closed, bounded and convex set $\mathbb{B} \subset \mathbb{A}$ such that, for all $y \in \mathbb{B}$, $\mathbb{B}(y) \neq \emptyset$. Consider the set

$$\mathbb{B} = \{z \in C[-\tau,1] \mid z_0 = \lambda e^{-\alpha\tau} + (1-\lambda)e^{-\beta\tau},\ z(t) = \lambda x(t-\tau) + (1-\lambda)y(t-\tau),$$
$$t \in [0,1],\ 0 \leq \lambda \leq 1\}.$$

Since B_1 and B_2 are convex it is clear that $\mathbb{B} \subset \mathbb{A}$. By definition $\mathbb{B}$ is convex and it is certainly both closed and bounded.

Now, let $z \in \mathbb{B}$. We must show that the set $\mathbb{B}(z) \neq \emptyset$. To do this we must find a function $v \in \mathbb{B}$ such that $\dot{v}(t) = az(t - \tau) + bu$ for some admissible control u. By definition of $\mathbb{B}$, there is $a\lambda$, $0 \leq \lambda \leq 1$ such that

$$z(t) = \lambda x(t - \tau) + (1 - \lambda)y(t - \tau) \quad t \in [0,1]$$

$$z_o = \lambda e^{-\alpha\tau} + (1 - \lambda)e^{-\beta\tau} .$$

Choose a control u, such that

$$u(t) = (R/b)x(t - \tau) + (T/b)y(t - \tau), \quad t \in [0,1]$$

where the constants R and T satisfy the relations

$$-a\lambda \leq R \leq \alpha - a\lambda$$

and

$$(a\lambda + R)/\alpha = 1 - \left[\frac{a(1-\lambda)+T}{\beta} \right].$$

Clearly, if ρ is large enough, u will be admissible. Note also that $T \leq a(1 - \lambda)$ and set

$$\mu = (a\lambda + R)/\alpha.$$

Then $0 \leq \mu \leq 1$ since

$$0 \leq \frac{a\lambda - a\lambda}{\alpha} \leq \frac{a\lambda + R}{\alpha} = \mu = \frac{a\lambda + R}{\alpha} \leq \frac{a\lambda + \alpha - a\lambda}{\alpha} = 1.$$

Let v be defined by the equations

$$v(t) = \mu x(t - \tau) + (1 - \mu)y(t - \tau) \quad t \in [0,1]$$

$$v_o = \mu e^{-\alpha\tau} + (1 - \mu)e^{-\beta\tau} .$$

So $v \in \mathbb{B}$ and, substituting the expression for μ into the definition of v, we have, for $0 \leq t \leq 1$,

$$v(t) = \frac{a\lambda+R}{\alpha} e^{\alpha(t-\tau)} + [1 - (1 - \frac{a(1-\lambda)+T}{\beta})]e^{\beta(t-\tau)}$$

$$= \frac{a\lambda+R}{\alpha} e^{\alpha(t-\tau)} + \frac{a(1-\lambda)+T}{\beta} e^{\beta(t-\tau)}.$$

Differentiating both sides, we get

$$\dot{v}(t) = \frac{a\lambda+R}{\alpha} \alpha e^{\alpha(t-\tau)} + \frac{a(1-\lambda)+T}{\beta} \beta e^{\beta(t-\tau)}$$

$$= (a\lambda + R)e^{\alpha(t-\tau)} + (a(1-\lambda)+T)e^{\beta(t-\tau)}$$

$$= a(\lambda e^{\alpha(t-\tau)} + (1 - \lambda)e^{\beta(t-\tau)}) + Re^{\alpha(t-\tau)} + Te^{\beta(t-\tau)}.$$

And so v satisfies the equation

$$\dot{v}(t) = az(t - \tau) + bu(t).$$

Hence $v \in \mathbb{B}(z)$, which proves controllability.

3. Extension to Unbounded Control Sets.

We can now extend the results of the previous section to the case in which the map U has property (U) and the sets $U(t,\phi)$ are closed rather than compact.

Lemma 3.3.1.

Let A be closed and bounded in $E^1 \times C[-\tau,0]$ and suppose that the maps U and Q have properties (U) and (Q) respectively. Assume that, for each $(t,\phi) \in A$, $U(t,\phi)$ is closed and $Q(t,\phi)$ is closed and convex.

Let $\mathbb{B}$ be a non-empty compact and convex subset of $\mathbb{A}$ and assume that, for each $y \in \mathbb{B}$, $\mathbb{B}(y) \neq \emptyset$. Then the map Φ has property (K).

Proof. Suppose $y^{(k)} \to y$ and $x^{(k)} \to x$ in the ρ-metric and for each k, $x^{(k)} \in \mathbb{B}(y^{(k)})$. By the implicit function theorem, there exist measurable functions $u^{(k)}$ such that $u^{(k)}(t) \in U(t, y_t^{(k)})$ almost everywhere and

$$\phi_k(t) = \dot{x}^{(k)}(t) = F(t, y_t^{(k)}, u^{(k)}(t)) \text{ almost everywhere in } [t_{1_k}, t_{2_k}].$$

Let $\phi(t) = \dot{x}(t)$. We wish to show that $\phi(t) \in Q(t, y_t)$ almost everywhere in $[t_1, t_2]$.

For almost all $t \in (t_1, t_2)$, $\dot{x}(t)$ exists and is finite. Let t_o be such a point and choose $\sigma > 0$ such that $t_1 < t_o - \sigma < t_o + \sigma < t_2$.

We note that the condition $x^{(k)} \in \mathbb{B}(y^{(k)})$ implies that $x^{(k)}$ and $y^{(k)}$ are both defined on the same interval $[t_{1_k} - \tau, t_{2_k}]$. For some k_o we then have that

$k \geq k_o$ implies $[t_o - \sigma, t_o + \sigma] \subset [t_{1_k}, t_{2_k}]$ and so $[t_o - (\sigma+\tau), t_o + \sigma] \subset [t_{1_k} - \tau, t_{2_k}]$. Now $y^{(k)} \to y$ uniformly in $[t_o - (\sigma+\tau), t_o + \sigma]$ so given $\varepsilon > 0$ there is a $\delta > 0$ such that $t, t' \in [t_o - \sigma, t_o + \sigma]$, $|t - t'| < \delta$, $k \geq k_o$, implies that

$$||y_t - y_{t'}|| \leq \varepsilon/2, \quad ||y_t^{(k)} - y_{t'}^{(k)}|| \leq \varepsilon/2.$$

Take $0 < \delta < \sigma$, $\delta \leq \varepsilon$. For any h, $0 < h \leq \delta$ consider the averages

$$m_h = h^{-1} \int_o^h \phi(t_o+s)ds = h^{-1}[x(t_o+h)-x(t_o)]$$

$$m_{hk} = h^{-1} \int_o^h \phi_k(t_o+s)ds = h^{-1}[x^{(k)}(t_o+h)-x^{(k)}(t_o)].$$

For any $\eta > 0$, fix h, $0 < h \leq \delta < \sigma$ so small that

$$|m_h - \phi(t_o)| \leq \eta.$$

Now, take $k_1 \geq k_o$ so large that

$$|m_{hk} - m_h| \leq \eta \text{ and } ||y_{t_o}^{(k)} - y_{t_o}|| \leq \varepsilon/2.$$

This can be done since $x^{(k)} \to x$ and $y^{(k)} \to y$ uniformly in $[t_o - (\sigma + \tau), t_o + \sigma]$.

For $0 \leq \delta \leq h$, $k \geq k_1$,

$$||y_{t_o+s}^{(k)} - y_{t_o}|| \leq ||y_{t_o+s}^{(k)} - y_{t_o}^{(k)}|| + ||y_{t_o}^{(k)} - y_{t_o}|| \leq \varepsilon/2+\varepsilon/2 = \varepsilon.$$

By definition $F(t_o+s, y_{t_o+s}^{(k)}, u^{(k)}(t_o+s)) \in Q(t_o+s, y_{t_o+s}^{(k)})$

and since $||y_{t_o+s}^{(k)} - y_{t_o}|| < \varepsilon$ we have

$$\phi_k(t_o+s) = F(t_o+s, y_{t_o+s}^{(k)}, u_{t_o+s}^{(k)}) \in Q(t_o, y_{t_o}; 2\varepsilon).$$

The definition of m_{hk} shows that

$$m_{hk} \in c\ell\ co\ Q(t_o, y_{t_o}; 2\varepsilon)$$

since the latter is a closed convex set.

Finally, we have the inequalities

$$|\phi(t_o) - m_{hk}| \leq |\phi(t_o) - m_h| + |m_h - m_{hk}| \leq 2\eta .$$

Hence

$$\phi(t_o) \in [c\ell\ co\ Q(t_o, y_{t_o}; 2\varepsilon)]_{2\eta}$$

and since η is arbitrary and $Q(t,\phi)$ has property (Q) in A

$$\phi(t_o) \in \bigcap_{\varepsilon>0} c\ell\ co\ Q(t_o, y_{t_o}; 2\varepsilon) = Q(t_o, y_{t_o}).$$

Thus the relation $\phi(t) \in Q(t, y_t)$ holds almost everywhere in $[t_1, t_2]$.

We can now state a controllability result under hypotheses similar to the hypotheses of the main existence theorem (Theorem 2.3.2).

Theorem 3.3.1.

Under the hypotheses of Lemma 3.3.1, a sufficient condition for the system

$$\dot{x}(t) = F(t, x_t, u(t)) \quad u(t) \in U(t, x_t)$$

to be $\mathbb{B}$-controllable, and hence $\mathbb{A}$-controllable is that, for all $y \in \mathbb{B}$, $\mathbb{B}(y) \neq \emptyset$.

Proof. Since we have assumed that $\mathbb{B}$ was compact, it is easy to see that $\bigcup_{y \in \mathbb{B}} \mathbb{B}(y)$ is relatively compact. The convexity of $\mathbb{B}(y)$ is guaranteed by Lemma 3.2.2 and Lemma 3.3.1 shows that Φ has property (K). Thus, we may apply the fixed point theorem to obtain a fixed point of Φ and, as before, this fixed point gives rise to an admissible pair.

CHAPTER IV

LINEAR SYSTEMS

In this chapter we will consider linear control systems. Our first object will be to obtain an existence theorem for the Lagrange problem associated with the control system

$$(4.0.1)\quad \dot{x}(t) = F(t,x_t) + B(t)u(t),\ u(t) \in U$$

where $U \subset E^m$ is a fixed, closed set, F is bounded and B is a continuous $n \times m$ matrix-valued function.

We will then turn to a discussion of the controllability of the autonomous linear system

$$(4.0.2)\quad \dot{x}(t) = L(x_t) + Bu(t),\ u(t) \in U.$$

Our object will be to derive controllability results which are well-known for ordinary differential equations under an assumption which is the natural generalization of the usual algebraic criterion for controllability. The work of Hale [25],[26],[27] on linear functional differential equations will be the main tool used in Section 2. We take this opportunity to describe very briefly the results which we will need. A full discussion of these results may be found in the references just cited.

In [25] Hale considers the homogeneous equation

$$(4.0.3)\quad \dot{x}(t) = L(x_t) = \int_{-\tau}^{0} [d\eta(\theta)]x(t+\theta)$$

where η is an $n \times n$ matrix whose elements are functions

of bounded variation. If ϕ is any given function in $C[-\tau,0]$ and $x(\phi)$ is the solution of (4.0.3) with initial function ϕ at $t = 0$, we define the operator $J(t) : C[-\tau,0] \to C[-\tau,0]$ by

$$x_t(\phi) = J(t)\phi, \quad t \geq 0.$$

The operators $J(t)$, $t \geq 0$, form a strongly continuous semigroup of bounded linear operators. Moreover, for $t \geq \tau$, the operators $J(t)$ are completely continuous.

The characteristic equation associated with (4.0.3) is the equation

$$(4.0.4) \quad \det \{\lambda I - \int_{-\tau}^{0} [d\eta(\theta)]e^{\lambda\theta}\} = 0$$

which has only a finite number of roots λ satisfying $\operatorname{Re} \lambda \geq \gamma$ for any $\gamma \in E^1$. These roots are called characteristic roots of (4.0.3) and each has a finite multiplicity. It has been recently shown by Levinger [33] that if a characteristic root λ_0 has multiplicity k, then (4.0.3) has precisely k independent solutions of the form

$$q_j(t) = e^{\lambda_0 t} p_j(t), \quad j = 1,2,\ldots,k$$

where the p_j are polynomials in t having degree less than k and having coefficients in E^n.

Let $Q = (q_1,\ldots,q_k)$. Then, since the $\dot{q}_j$ also satisfy (4.0.3) and are expressible as linear combinations of the q_j, there exists a $k \times k$ matrix D such that

$$\dot{Q}(t) = Q(t)D$$

and so

$$Q(t) = Q(0)e^{Dt}.$$

Note that the functions q_j are defined on all of E^1. Define functions $\phi_j \in C[-\tau,0]$, $j = 1,2,\ldots,k$, by

$$\phi_j(\theta) = q_j(\theta),\ -\tau \leq \theta \leq 0,\ j = 1,2,\ldots,k.$$

If we let $\Phi = (\phi_1,\ldots,\phi_k)$ and $Q_t = (q_{1_t},\ldots,q_{k_t})$. Then we can write

$$Q_t(\Phi) = J(t)\Phi = \Phi e^{Dt},\ \Phi(\theta) = \Phi(0)\ e^{D\theta},\ -\tau \leq \theta \leq 0.$$

Thus, Φ is the basis of a finite dimensional subspace $M(\lambda) \subset C[-\tau,0]$ such that $J(t)M(\lambda) \subset M(\lambda)$.

If λ_o is a characteristic root of (4.0.3), $M(\lambda)$ is called the generalized eigenspace associated with λ_o provided every solution of (4.0.3) of the form $p(t)e^{\lambda_o t}$, where p is a polynomial in t, corresponds to an initial condition $\phi \in M(\lambda)$ at $t = 0$. If $\Lambda = \{\lambda_1,\ldots,\lambda_m\}$ is a finite set of characteristic roots, $M(\Lambda) \subset C[-\tau,0]$ is the generalized eigenspace associated with Λ if, for every solution of (4.0.3) which has the form $\sum_{j=1}^{m} p_j(t)e^{\lambda_j t}$, where the p_j are polynomials in t, corresponds to an initial function $\phi \in M(\Lambda)$ at $t = 0$.

In order to characterize the subspace of $C[-\tau,0]$ which is complementary to $M(\Lambda)$, Hale introduces the "adjoint" equation

$$(4.0.5)\quad \dot{y}(s) = -\int_{-\tau}^{0} [d\eta(\theta)]^T y(s-\theta),\ s \leq 0,$$

and the corresponding characteristic equation

$$(4.0.6)\quad \det\{\lambda T - \int_{-\tau}^{0} [d\eta(\theta)]^T y(s-\theta)\} = 0.$$

A solution of (4.0.5) is obtained by specifying an initial function $\psi \in C[0,\tau]$ and integrating (4.0.5) for $s \leq 0$. Notice also that the solutions of (4.0.6) are the same as those of (4.0.4).

For any $\psi \in C[0,\tau]$ and $\phi \in C[-\tau,0]$ we define a bilinear form

$$(4.0.7)\quad (\psi,\phi) = \psi^T(0)\phi(0) - \int_{-\tau}^{0}\int_{0}^{\theta} \psi^T(\xi-\theta)[d\eta(\theta)]\phi(\xi)d\xi$$

If Λ is a finite set of characteristic roots of (4.0.3) (and, therefore, also of (4.0.5)) and $M(\Lambda) \subset C[-\tau,0]$ and $M^*(\Lambda) \subset C[0,\tau]$ are the corresponding generalized eigenspaces of (4.0.3) and (4.0.5) respectively, then $M(\Lambda)$ and $M^*(\Lambda)$ have the same dimension. If $\Phi = (\phi_1,\ldots,\phi_m)$ and $\Psi = (\psi_1,\ldots,\psi_m)$ are bases for $M(\Lambda)$ and $M^*(\Lambda)$ respectively then the matrix $(\Psi,\Phi) = ((\psi_i,\phi_j))$ is non-singular and, by proper choice of Ψ and Φ, we can make $(\Psi,\Phi) = I$.

Let $S(\Lambda) = \{\phi \in C[-\tau,0] \mid (\Psi,\phi) = 0\}$ where (Ψ,ϕ) is the m-vector whose components are (ψ_i,ϕ). Then if $\phi \in S(\Lambda)$, $x_t(\phi) = J(t)\phi$ belongs to $S(\Lambda)$. Hence, for any $\phi \in C[-\tau,0]$,

there is a unique representation

$$(4.0.8)\quad \phi = \phi^M + \phi^S,\ \phi^M = \Phi(\Psi,\phi) \in M(\Lambda),\ \phi^S \in S(\Lambda).$$

Since any $\phi \in C[-\tau,0]$ can be represented in this manner we say that the space $C[-\tau,0]$ is decomposed by Λ.

We now wish to consider the autonomous linear control system (4.0.2). Suppose that $C[-\tau,0]$ is decomposed by Λ, a finite set of characteristic roots of the associated homogeneous equation. If $x(\phi,u)$ is the solution of (4.0.2) with initial condition ϕ at $t = 0$ and control function u, then there is a unique decomposition of $x_t = x_t(\phi,u)$ in the form

$$x_t = x_t^M + x_t^S\ ,\ x_t^M = \Phi(\Psi,x_t),\ (\Psi,x_t^S) = 0.$$

We wish to obtain a representation for the projections x_t^M x_t^S.

Let $X_o(\theta)$, $-\tau \leq \theta \leq 0$, be the $n \times n$ matrix defined by the equation

$$(4.0.9)\quad X_o(\theta) = \begin{cases} 0, & -\tau \leq \theta < 0 \\ I, & \theta = 0. \end{cases}$$

Then, the solution of (4.0.2) can be represented by the variation of parameters formula

$$(4.0.10)\quad x_t(\theta) = [J(t)\phi](\theta) + \int_o^t [J(t-s)X_o](\theta)Bu(s)ds.$$

If we define projection operators p^M and p^S by $p^M\phi = \phi^M$ and $p^S\phi = \phi^S$, then p^M and p^S commute with $J(t)$.

Applying these projection operators to equation (4.0.10), we get

$$(4.0.11)\quad x_t^M(\theta) = [J(t)\phi^M](\theta) + \int_o^t [J(t-s)X_o^M](\theta)Bu(s)ds$$

and

$$(4.0.12)\quad x_t^S(\theta) = [J(t)\phi^S](\theta) + \int_o^t [J(t-s)X_o^S](\theta)Bu(s)ds.$$

Direct computation using the definition of the bilinear form (4.0.7) shows that $J(t)X_o^M = \Phi e^{Dt}\Psi^T(0)$. If we define a function y by the equation

$$(\Psi, x_t) = y(t).$$

Then $x_t^M = \Phi y(t)$ and it is easily verified that y satisfies the differential equation

$$\dot{y}(t) = Dy + \Psi^T(0)Bu(t).$$

We may summarize this result in the following manner:

Theorem 4.0.1 (Hale).

Let Λ be a finite set of characteristic roots of the homogeneous equation associated with (4.0.2), and let $C[-\tau,0]$ be decomposed by Λ. Then the decomposition given by (4.0.8), applied to the control equation

$$\dot{x}(t) = L(x_t) + Bu(t)$$

yields the equivalent system

$$\dot{y}(t) = Dy + \Psi^T(0)Bu(t)$$

$$x_t^S(\theta) + [J(t)\phi^S](\theta) + \int_o^t [J(t-s)X_o^S](\theta)Bu(s)ds.$$

1. An Existence Theorem for the Non-Autonomous Problem.

We will consider the following Lagrange problem. Determine, for given $t_1, t_2 \in E^1$, $t_1 < t_2$, a measurable vector function $u: [t_1,t_2] \to E^m$ and a corresponding continuous vector function $x: [t_1 - \tau, t_2] \to E^n$ which is absolutely continuous on $[t_1,t_2]$ and which satisfies, almost everywhere, the differential system

$$(4.1.1) \qquad dx/dt = F(t,x_t) + B(t)u(t), \quad t_1 \leq t \leq t_2,$$

subject to boundary conditions

$$(t_1, x_{t_1}, t_2, x_{t_2}) \in B_1 \times B_2 ,$$

and unilateral constraints

$$(t,x_t) \in A, \quad t_1 \leq t \leq t_2 ,$$

$$u(t) \in U \text{ almost everywhere in } [t_1,t_2],$$

and for which the integral

$$I[x,u] = \int_{t_1}^{t_2} F_o(s,x_s,u(s))ds$$

takes on its minimum value.

We require that F is bounded, measurable in its first argument, continuous in its second argument and that B is a continuous $n \times m$ matrix-valued function. In

addition, we will assume that A is closed and bounded, that U is a fixed closed and bounded convex subset of E^m and that F_o is a convex function of its third argument.

The mappings Q and $\tilde{\tilde{Q}}$ for this problem are defined by

$$Q(t,\phi) = \{z \in E^n \mid z = F(t,\phi) + B(t)u,\ u \in U\}$$

$$\tilde{\tilde{Q}}(t,\phi) = \{\tilde{z} = (z^o,z) \mid z^o \geq F_o(t,\phi,u), z = F(t,\phi)+B(t)u, u \in U\}.$$

Lemma 4.1.1.

Let A be a fixed closed subset of $E^1 \times C[-\tau,0]$ and U a fixed, closed, bounded and convex subset of E^m. Let F_o be continuous on $M = A \times U$ and convex in its third argument. Then the sets $\tilde{\tilde{Q}}(t,\phi)$ and $Q(t,\phi)$ defined by (4.1.2) are convex for each $(t,\phi) \in A$.

Proof. It is immediate that $Q(t,\phi)$ is convex for each $(t,\phi) \in A$. Let us consider the set $\tilde{\tilde{Q}}(t,\phi)$ and let $\tilde{x} = (x^o,x)$ and $\tilde{y} = (y^o,y)$ be any two points of $\tilde{\tilde{Q}}(t,\phi)$. Then, for any λ, $0 \leq \lambda \leq 1$, consider $\tilde{z} = \lambda\tilde{x} + (1-\lambda)\tilde{y}$. Since $\tilde{x},\tilde{y} \in \tilde{\tilde{Q}}(t,\phi)$, there exist vectors u and v such that

$$x^o \geq F_o(t,\phi,u),\ x = F(t,\phi) + B(t)u$$

and

$$y^o \geq F_o(t,\phi,v),\ y = F(t,\phi) + B(t)v.$$

The set U is assumed to be convex and so the vector $w = \lambda u + (1-\lambda)v$ is in U. In addition, we have

$$z = \lambda x + (1-\lambda)y = \lambda[F(t,x_t)+B(t)u]+(1-\lambda)[F(t,x_t)+B(t)v]$$
$$= F(t,x_t)+B(t)[\lambda u+(1-\lambda)v] = F(t,x_t)+B(t)w$$

and, by the convexity of F_o in its third argument, we have

$$z^o = \lambda x^o + (1-\lambda)y^o \geq \lambda F_o(t,\phi,u) + (1-\lambda)F_o(t,\phi,v)$$

$$\geq F_o(t,\phi,\lambda u+(1-\lambda)v) = F_o(t,\phi,w).$$

Thus $\tilde{z} = (z^o,z) \in \tilde{\tilde{Q}}(t,\phi)$ and so $\tilde{\tilde{Q}}(t,\phi)$ is a convex set.

Lemma 4.1.2.

Let the set A be a fixed closed subset of $E^1 \times C[-\tau,0]$, and U be a fixed, closed, convex subset of E^m. Let F_o be a continuous scalar-valued function on $M = A \times U$ which is convex in its third argument. For each closed and bounded set $A_o \subset A$, let Φ_o be a continuous scalar function on the set $Z = \{z \in E^1 | z = |u|, u \in U\}$ such that $F_o(t,\phi,u) \geq \Phi_o(|u|)$ for each $(t,\phi,u) \in A_o \times U$ and $\Phi_o(z) \to +\infty$ as $z \to +\infty$. Then the set $\tilde{\tilde{Q}}(t,\phi)$ defined by (4.1.2) satisfies property (Q) in A.

The proof of this lemma is identical with the proof given by Cesari [9 ; p. 405] and we refer the reader to that paper.

We may now state the following existence theorem.

Theorem 4.1.1.

Let $A \subset E^1 \times C[-\tau,0]$ be closed and bounded and let U be a fixed compact set in E^m. Let $M = \{(t,\phi,u) | (t,\phi) \in A, u \in U\}$ and consider the functional differential equation

$$\dot{x}(t) = F(t,x_t) + B(t)u(t)$$

where F is bounded and continuous and B is a continuous

$m \times n$ matrix valued function. Let F_o be a continuous scalar-valued function on M which is convex in its third argument and let the set of initial conditions B_1 be compact in A. Then, if there exists a Lebesgue-integrable function ϕ such that $F_o(t,\phi,u) \geq \phi(|u|)$ for each $(t,\phi,u) \in M$ and $\phi(\xi)/\xi \to \infty$ as $\xi \to \infty$. Then $I[x,u]$ has an absolute minimum in any non-empty complete class of admissible pairs.

Proof. By the two preceding lemmas, the sets $\tilde{Q}(t,\phi)$ are convex and $\tilde{\tilde{Q}}$ satisfies property (Q) in A. In addition, since F and B are bounded, the set A is assumed bounded and U is compact, we have $\int_{t_1}^{t_2} |\dot{x}(t)|^2 dt \leq K$, for some constant $K < \infty$. Then, the present theorem is a corollary of Theorem 2.3.3 whose hypotheses are satisfied.

2. Controllability for Autonomous Linear Systems.

We will now consider the control system

$$\dot{x}(t) = L(x_t) + Bu(t),\ u(t) \in U \tag{4.2.1}$$

where $L: C[-\tau,0] \to E^n$ and $B: E^m \to E^n$ are bounded linear operators. We take U to be a fixed closed subset of E^m containing the zero vector in its interior.

Definition 4.2.1.

A function $\phi \in C[-\tau,0]$ is called null controllable with respect to the set U if there exists an admissible control u such that the corresponding solution of

(4.2.1), $x(\cdot;\phi,u)$, having initial function ϕ at $t = 0$, satisfies $x_{t_1}(\theta) = 0$, $-\tau \leq \theta \leq 0$, for some $t_1 < \infty$. The set of all null controllable states will be denoted by G_U. The system (4.2.1) is called completely null controllable if $G_U = C[-\tau,0]$.

The variation of parameters formula (4.0.10) shows us that the solutions of (4.2.1) may be written in the form

$$(4.2.2)\quad x_t(\theta) = J(t)x_o(\theta) + \int_o^t [J(t-s)X_o](\theta)Bu(s)ds.$$

Let $W(t) = J(t)X_o$. In the remainder of this section we will assume that the following controllability hypothesis is satisfied.

Hypothesis (C):

There exists a vector $v \in U$ and a number $\varepsilon > 0$, ε sufficiently small, so that, if $b = Bv$, the vectors

$$(4.2.3)\quad \{[W(0)](0)b,[W(\varepsilon)](0)b,\ldots,[W(n-1)\varepsilon)](0)b\}$$

are linearly independent.

This assumption is enough to insure that, for some $\bar{t} > n\varepsilon$, the vectors

$$\{[W(t^*-\varepsilon)](0)b,\ [W(t^*-2\varepsilon)](0)b,\ldots,[W(t^*-n\varepsilon)](0)b\}$$

are linearly independent for all $t^* \in [n\varepsilon,\bar{t} + n\varepsilon]$. To see this, note that the matrix $[W(t)](0)$ is continuous by virtue of the strong continuity of $J(t)$. Hence, for $\bar{t}$ sufficiently small, Hypothesis (C) guarantees that the

vectors

$$\{[W(t)](0)b,\ldots,[W(t + (n - 1)\varepsilon](0)b\}$$

are linearly independent for all $t \in [0,\bar{t}]$. Choosing $t^* \in [n\varepsilon,\bar{t} + n\varepsilon]$, we then have that

$$[W(t^*-i\ )](0)b = [W(t-n\varepsilon-i\varepsilon)](0)b = [W(t+(n-i)\varepsilon)](0)b$$

$$(i = 1,2,\ldots,n)$$

for some $t \in [0,\bar{t}]$. Since the vectors

$$\{[W(t)](0)b,\ldots,[W(t + (n - 1)\varepsilon](0)b\}$$

are linearly independent for such a t, the vectors

$$\{[W(t^* - \varepsilon)](0)b,\ldots,[W(t^* - n\varepsilon)](0)b\}$$

are linearly independent which is what we wished to show.

In the case of ordinary linear control systems

$$\dot{x}(t) = Lx + Bu,\ u \in U,$$

the semigroup J(t) of linear operators has the form e^{Lt}. In this particular case, the vectors (4.2.3) are just $\{e^{(n-1)\varepsilon L}b,\ldots,e^{\varepsilon L}b,b\}$ and this set is linearly independent if and only if the vectors

(4.2.4) $$\{e^{-\varepsilon L}b,\ldots,e^{-n\varepsilon L}b\}$$

are linearly independent. We remark here that the hypothesis of generality of position introduced by Pontryagin [40; p. 112] is used to show that the vectors (4.2.4) are linearly independent, a fact which is used in [40] to prove controllability results similar to those which are presented below.

We now use Hypothesis (C) to establish some properties of the set G_U.

Lemma 4.2.1.

Suppose that Hypothesis (C) holds for the control system

$$\dot{x}(t) = L(x_t) + Bu(t), \quad u(t) \in U. \tag{4.2.5}$$

Then there exists a neighborhood V of the zero function in $C[-\tau,0]$ such that, if $\phi \in V$ then there exists a control u for which the solution $x(\cdot;\phi,u)$ of (4.2.5), corresponding to the control u and the initial condition $x_o = \phi$, satisfies $x(t^*) = 0$ for some t^*, $0 < t^* < \infty$.

Proof. Solutions of (4.2.5) are given by the variation of parameters formula

$$x_t(\theta;\phi,u) = [J(t)\phi](\theta) + \int_o^t [J(t-s)X_o](\theta)Bu(s)ds. \tag{4.2.6}$$

By Hypothesis (C), there exists a vector $v \in U$ and a number $\varepsilon > 0$, so that the vectors

$$\{[W(0)](0)b,\ldots,[W((n-1)\varepsilon)](0)b\}, \quad b = Bv$$

form a linearly independent set. For this choice of v and ε, consider the control

$$(4.2.7)\quad u(t,\xi) = v \cdot \sum_{k=1}^{n} \sigma(t,k\varepsilon,\xi^k), \qquad \xi = (\xi^1,\ldots,\xi^n) \in E^n$$

where

$$\sigma(t,k\varepsilon,\xi^k) = \begin{cases} \operatorname{sgn} \xi^k, & \text{for } t \in [k\varepsilon, k\varepsilon + \xi^k] \\ 0, & \text{otherwise.} \end{cases}$$

Having fixed $v \in U$, we can consider the solution (4.2.6) to be a function of ξ, and we write $x_t(\theta;\phi,\xi)$. Now let $\Gamma \subset E^n$ be an open neighborhood of the origin in E^n and let S be an open subset of $C[-\tau,0] \times \Gamma$ such that $(0,0) \in S$. Fix $t^* \in [n\varepsilon, \bar{t} + n\varepsilon]$ and define a map $g: S \to E^n$ by

$$(4.2.8)\qquad g(\phi,\xi) = x_{t^*}(0;\phi,\xi).$$

Note that $g(0,0) = x_{t^*}(0;0,0) = 0$. We wish to solve the equation

$$(4.2.9)\qquad g(\phi,\xi) = 0$$

for ξ in terms of ϕ.

Since

$$x_{t^*}(\theta;\phi,\xi) = J(t^*)\phi(\theta) + \int_0^{t^*} [W(t^*-s)](\theta)Bu(s)ds$$

we have that

$$g(\phi,\xi)=J(t^*)\phi(0)+\int_0^{t^*}[W(t^*-s)](0)Bu(s)ds$$

$$= J(t^*)\phi(0)+\int_0^{t^*}[W(t^*-s)](0)b(\sum_{k=1}^{n}\sigma(s,k\varepsilon,\xi^k))ds$$

$$= J(t^*)\phi(0)+\sum_{k=1}^{n}\int_0^{t^*}[W(t^*-s)](0)b\sigma(s,k\varepsilon,\xi^k)ds$$

$$= J(t^*)\phi(0)+\sum_{k=1}^{n}\int_{k\varepsilon}^{k\varepsilon+\xi^k}[W(t^*-s)](0)b\ ds.$$

Note that, by definition of σ, the integral

$$\int_{k\varepsilon}^{k\varepsilon+\xi^k}[W(t^*-s)](0)b\ ds$$

is always positive, for if $\xi^k < 0$, $\sigma(t,k\varepsilon,\xi^k) = -1$ and

$$\int_{k\varepsilon}^{k\varepsilon+\xi^k}[W(t^*-s)](0)b\sigma(s,k\varepsilon,\xi^k)ds$$

$$= -1\int_{k\varepsilon}^{k\varepsilon+\xi^k}[W(t^*-s)](0)b\ ds = \int_{k\varepsilon+\xi^k}^{k\varepsilon}[W(t^*-s)](0)b\ ds.$$

By the fundamental theorem of calculus we can write

$$\frac{\partial g}{\partial \xi^k}(0,\xi) = \frac{\partial}{\partial \xi^{(k)}}\left[\sum_{k=1}^{n}\left(\int_{k\varepsilon}^{k\varepsilon+\xi^k}[W(t^*-s)](0)b\ ds\right)\right]$$

$$= \frac{\partial}{\partial \xi^{(k)}}\left[\int_{k\varepsilon}^{k\varepsilon+\xi^k}[W(t^*-s)](0)b\ ds\right]$$

$$= [W(t^* - k\varepsilon - \xi^k)](0)b.$$

Hence $\frac{\partial g}{\partial \xi^{(k)}}(0,0) = [W(t^* - k\varepsilon)](0)b$, $(k = 1,2,\ldots,n)$.

We have seen that Hypothesis (C) implies that these vectors are linearly independent and so $D_2g(0,0)$ is a homeomorphism of E^n onto E^n. Hence by the implicit function theorem [14; p. 265], there exists a neighborhood

V of the zero element of $C[-\tau,0]$ and a continuous map π: $V \to \mathbb{R}^n$ such that, if $\phi \in V$, then

$$g(\phi,\pi(\phi)) = x_{t*}(0;\phi,\pi(\phi)) = 0.$$

Lemma 4.2.2.

Let $L: C[-\tau,0] \to E^n$ and $B: E^m \to E^n$ be bounded linear maps such that the range of L is contained in the range of B, $R(L) \subset R(B)$. Let w be a continuous map of an interval of the form [a,b] into $C[-\tau,0]$. Then there exists a measurable function $u : [a,b] \to E^m$ such that $-Bu(t) = L[w(t)]$.

Proof. Consider the span $C[-\tau,0]$ together with the σ-algebra of Borel sets as a measure space. Clearly E^m is the countable union of compact metrizable sets and E^n is Hausdorff. Hence we may apply the McShane-Warfield lemma (see p. 38) to get a Borel measurable function $F : C[-\tau,0] \to E^m$ such that, for any $\phi \in C[-\tau,0]$, $-B(F(\phi)) = L\phi$.

Since the map $w : [a,b] \to C(-\tau,0]$ is continuous, certainly the map $u : [a,b] \to E^m$ defined by $u(t) = F(w(t))$ is measurable and by definition of F,

$$-Bu(t) = -B\ F(w(t)) = L(w(t)).$$

We can now use these two lemmas to show that G_U is open in $C[-\tau,0]$. It is easy to see that G_U is open if and only if G_U contains a neighborhood of the origin in $C[-\tau,0]$. This fact is a consequence of the continuous

dependence of solutions on initial data. The following theorem shows that G_U does contain such a neighborhood.

Theorem 4.2.1.

Let $L : C[-\tau,0] \to E^n$ and $B : E^m \to E^n$ be linear and continuous and suppose that $R(L) \subset R(B)$. Then, if Hypothesis (C) is satisfied, the domain of null controllability G_U of the system

$$\dot{x}(t) = L(x_t) + Bu(t),\ u(t) \in U$$

is open.

Proof. By the remarks preceding the theorem we need only show that there is a neighborhood V of $0 \in C[-\tau,0]$ such that $V \subset G_U$. Since Hypothesis (C) is satisfied, Lemma 4.2.1 assures the existence of a neighborhood V and a finite time $t^* > n\varepsilon$ such that if $x_o \in V$ then $x_{t^*}(0) = x(t^*) = 0$. Now the map $w : [t^*,t^* + \tau] \to C[-\tau,0]$, given by $w(t) = x_t$ is continuous. Since $R(L) \subset R(B)$, Lemma 4.2.2 guarantees the existence of a measurable function $\bar{u} : [t^*,t^* + \tau] \to E^m$ such that $-B\bar{u}(t) = L(x_t)$ for all $t \in [t^*,t^* + \tau]$. For this choice of control over the interval $[t^*,t^* + \tau]$, we have

$$\dot{x}(t) = L(x_t) + B\bar{u}(t) = 0$$

and so $x_{t^*+\tau} = 0$. Thus, if $x_o \in V$ the control u defined by

$$u(t) = \begin{cases} u(t,\pi(x_o)), & 0 \le t \le t^* \\ \bar{u}(t), & t^* \le t \le t^* + \tau \end{cases}$$

will drive x_o to 0 in time $t^* + \tau < \infty$.

Consider the functional differential equation

(4.2.10) $$\dot{x}(t) = F(t,x_t)$$

where F is continuous on an open subset $D \subset E^1 \times C[-\tau,0]$ and $F(t,0) = 0$.

Definition 4.2.2.

The solution $x(t) \equiv 0$ of (4.2.10) is stable if, for any $\varepsilon > 0$ and any $t_o \in E^1$ there is a $\delta = \delta(t_o,\varepsilon) > 0$ such that if $||x_{t_o}|| \le \delta$, then $||x_t(\cdot;t_o,x_{t_o})|| < \varepsilon$ for all $t \ge t_o$. The zero solution is asymptotically stable in the large if it is stable and if $x_t \to 0$ as $t \to \infty$.

With this definition we can state the following theorem on complete null-controllability.

Theorem 4.2.2.

Let $L: C[-\tau,0] \to E^n$ and $B : E^m \to E^n$ be linear, continuous, and suppose that $R(L) \subset P(B)$. Then, if Hypothesis (C) is satisfied and if the zero solution of

(4.2.11) $$\dot{x}(t) = L(x_t)$$

is asymptotically stable in the large, the system

(4.2.12) $$\dot{x}(t) = L(x_t) + Bu(t),\ u(t) \in U$$

is completely null controllable, that is $G_U = C[-\tau,0]$.

Proof. Choose any $\phi \in C[-\tau,0]$. The response of the system (4.2.12) to the control $u(t) = 0$ is just $J(t)\phi$ and since the zero solution of (4.2.11) is asymptotically stable in the large, we have that $||J(t)\phi|| \to 0$ as $t \to \infty$. Since G_U contains a neighborhood, V, of the origin we know that, for t sufficiently large, say $t \geq t^*$, $J(t)\phi \in V$. But there exists a control u_1 such that the response of the system (4.2.11) with initial condition $x_{t^*} \in V$ reaches the origin in finite time t_1. Hence the control u defined by

$$u(t) = \begin{cases} 0, & 0 \leq t < t^* \\ u_1(t), & t^* \leq t \leq t_1 \end{cases}$$

will take the point $\phi \in C[-\tau,0]$ to the origin in time $t_1 < \infty$.

We now want to show that the decomposition of $C[-\tau,0]$ given by Hale (see Theorem 4.0.1. p. 91) allows us to reduce the problem of complete null controllability of the infinite dimensional system, $\dot{x} = L(x_t) + Bu(t)$, to the problem of complete null controllability of a finite dimensional system.

Let γ be a real number such that no characteristic root, λ, of (4.2.11) has $\mathrm{Re}\ \lambda = \gamma$ and let Λ be the finite set of characteristic roots satisfying $\mathrm{Re}\ \lambda > \gamma$. Then, if $\phi \in S(\Lambda)$, we have the estimate

(4.2.13) $$||J(t)\phi|| \leq K\, e^{(\gamma-\alpha)t}||\phi||,\ \phi \in S(\Lambda)$$

for some positive constants $K \leq 1$ and α [28]. If we take $\gamma < 0$, then (4.2.13) is sufficient to guarantee that $||J(t)\phi|| \to 0$ as $t \to \infty$ and that the zero solution is asymptotically stable with respect to initial conditions lying in $S(\Lambda)$. Thus it is clear that, if all characteristic roots λ satisfy Re $\lambda < 0$, then the control system, $\dot{x}(t) = L(x_t) + Bu(t)$, is completely null controllable. In general, however, there will be characteristic roots whose real parts are non-negative.

In such a case we may use the results of Theorem 4.0.1 to reduce the infinite dimensional problem to one involving an ordinary differential equation. Thus, let $\gamma < 0$ be such that Re $\lambda \neq \gamma$ for all characteristic roots, λ, of (4.2.11), and let Λ be the finite set of all characteristic roots satisfying Re $\lambda > \gamma$. We consider the system, equivalent to $\dot{x}(t) = L(x_t) + Bu(t)$, described in Theorem 4.0.1, namely

$$\dot{y}(t) = Dy + \Psi^T(0)Bu(t)$$

$$x_t^S(\theta) = J(t)x_o^S(\theta) + \int_o^t [J(t-s)X_o^S](\theta)Bu(s)ds.$$

We can now state the following theorem.

Theorem 4.2.3.

The control system

(4.2.15) $$\dot{x}(t) = L(x_t) + Bu(t),\ u(t) \in U$$

is completely null controllable if, and only if, the finite dimensional system

$$\dot{y}(t) = Dy + \Psi^T(0)Bu(t) \tag{4.2.16}$$

is completely null controllable.

Proof. It is obvious that, if (4.2.15) is completely null controllable, then (4.2.16) is also. Conversely, suppose that (4.2.16) is completely null controllable. Let $\phi \in C[-\tau,0]$ and let $y_o = (\Psi,\phi)$. Then, there is a time t_1 and a control $u_1 : [0,t_1] \to U$ such that $y(t_1;y_o,u_1) = 0$. Now, let

$$x^S_{t_1}(\theta) = J(t_1)\phi^S(\theta) + \int_o^{t_1} [J(t_1 - s)X^S_o](\theta)Bu_1(s)ds,$$

where $\phi^S = \phi - \phi^M$. Now, consider $z_t \in C[-\tau,0]$ given by $z_t = J(t - t_1)x^S_{t_1}$. By (4.2.13), $||z_t|| \to 0$ as $t \to \infty$ and so z_t will eventually reach the set $V \subset S$ where $V \subset G_U$ is the neighborhood of the zero element of $C[-\tau,0]$ whose existence is guaranteed by Theorem 4.2.1. Hence, for some time t_2, $x_{t_2} = 0 + z_{t_2} \in V$ and we know that there exists a control $u_2 : [t_2,t_3] \to U$, for some finite time $t_3 > t_2$, such that the solution of

$$\dot{x}(t) = L(x_t) + Bu_2(t)$$

with initial condition (t_2,x_{t_2}), will satisfy $x_{t_3} = 0$. Thus, the controller that will take ϕ to the origin, that is, the control function u such that $x_{t_3}(\theta;\phi,u) = 0$, $-\tau \le \theta \le 0$, is

$$u(t) = \begin{cases} u_1(t), & 0 \le t \le t_1 \\ 0, & t_1 < t \le t_2 \\ u_2(t), & t_2 < t \le t_3. \end{cases}$$

This proves that (4.2.15) is completely null controllable.

Finally, we wish to remark that the complete controllability of (4.2.16) is sufficient to guarantee complete null-controllability of (4.2.16) and hence of (4.2.15). Moreover, it is well known (see, for example [32]) that in the case $U = E^m$, the finite dimensional system (4.2.16) is completely controllable if and only if the matrix

$$[\Psi^T(0)B,\ D\Psi^T(0)B,\ldots,D^{k-1}\Psi^T(0)B]$$

has rank k where $k = \dim M(\Lambda)$. Thus, we have the following corollary to Theorem 4.2.3 :

Corollary.

The control system

(4.2.17) $\dot{x}(t) = L(x_t) + Bu(t),\ u(t) \in E^m$

is completely null controllable if

$$\text{rank } [\Psi^T(0)B,\ D\Psi^T(0)B,\ldots,D^{k-1}\Psi^T(0)B] = k.$$

CHAPTER V

CONCLUDING REMARKS

It is hoped that the preceding chapters have convinced the reader that the function space $C[-\tau,0]$ is the natural setting in which to discuss the optimal control problems with which we have been concerned. As Krasovskii [31] and Hale [25] have shown in their work on functional differential equations, the adoption of the space $C[-\tau,0]$ as state space enables one to formulate and prove results which closely parallel results for ordinary differential equations. In this thesis, we have shown first, that the concept of weak solutions to functional differential equations is meaningful. Using this setting, we have then been able to prove existence theorems for Lagrange problems with delay which are analogous, both in statement and in technique of proof, to those established by Cesari in [9].

These existence theorems assume that admissible pairs, that is pairs of functions $\{x,u\}$ which satisfy the state and boundary value constraints and the control constraints respectively, do in fact exist. Thus, we have chosen to study, in addition, the problem of the existence of admissible pairs. This question is usually known as the controllability problem.

We have investigated the controllability problem from two points of view. The material of Chapter IV

shows that we can prove theorems for controllability of linear autonomous systems with delay under hypotheses similar to those used in proving similar theorems for ordinary linear autonomous systems. On the other hand, Chapter III discusses the general controllability problem for the Lagrange problem and deduces a sufficient condition for controllability using tools introduced in the preliminaries to the existence theorems.

These considerations, of course, leave many questions unanswered. Perhaps the most interesting line for further research has its origin in the content of Chapter III. Additional work should obviously be done on the applicability of the fixed point approach to systems arising in applications. The key questions suggested by the present theorem is: Under what circumstances can a map Φ: $\mathbb{B} \to 2^{\mathbb{B}}$ be constructed which satisfies the proper continuity conditions and has the added property that $\Phi(y) \neq \phi$ for all $y \in \mathbb{B}$? Hopefully, one would find some relationship between property (Q) of the map Q and the existence of such a map Φ . Also, the author is unaware of any theorems concerning the existence of fixed points for set-valued mapping, save for [29] and [5]. If less restrictive conditions for the existence of a fixed point of such a map could be established, they too could be used to find weaker conditions for controllability.

The controllability results contained in Chapter IV should be amenable to extension. There, we have treated only the linear autonomous case. Results concerning linear non-autonomous systems can probably be established along the lines of those results which exist for ordinary non-autonomous linear systems. The approach of Lee and Marcus [32] to the controllability problem for non-linear systems could probably also be extended to time-delay systems.

There are many important aspects of optimal control theory which have not been touched upon in this thesis. Among these, the outstanding problem is clearly the development of necessary conditions. In his thesis [2], Banks has used the approach of Gamkrelidze [22] to prove a maximum principle for problems with variable time dependent lags and boundary conditions in Euclidean space. The problem of establishing necessary conditions when the state constraints and boundary conditions are written in terms of the function space $C[-\tau,0]$ has not, to the author's knowledge, been treated in the literature.

Finally, extensions of the material in this thesis could be accomplished by considering more general types of delay equations. For example, one could consider systems described by delay equations of neutral type or equations containing time delay in the control variable as well as the state variable.

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