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ON THE SECOND-ORDER PROPERTIES OF THE OUTPUT OF CERTAIN FIRST- AND SECOND-ORDER NONLINEAR SYSTEMS WITH RANDOM INPUTS

by
Alfred Christian Robinson

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
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1962

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CHAPTER 1

INTRODUCTION

The general objective of the studies reported here is finding methods for calculating the power spectrum and autocorrelation of random processes which have been produced by random forcing of nonlinear differential equations (or nonlinear physical systems).

The general problem of random forcing of differential equations or physical systems is an old one, and has arisen in a number of contexts in mathematics, science and engineering. Beginning with the work of Einstein and Smoluchowski, the theory of Brownian motion stimulated the first extensive investigations in the field. Wang and Uhlenbeck[51] [1] and Chandrasekhar [8] give summaries of results of these investigations.

The largely unrecognized work of Andronov, Pontriagin and Witt [1] inaugurated the study of the effect of noise on dynamical systems more general than those of interest in Brownian motion. At about the same time, the mathematical investigations of Kolmogoroff [29] laid the basis for rigorous treatment of output random processes.

Kramers [30] used random forcing of a dynamical system to explain rates of chemical reactions. Within the past fifteen years a great amount of work has appeared in random-

---

1 Numbers in square brackets [ ] refer to entries in the list of references at the end of the dissertation.

input analysis of automatic control systems and electric or electronic circuits.

The vast majority of all these studies have dealt with linear systems. The theory of noise in linear systems has been carried quite far, and many problems have been solved, both in analysis and synthesis. Many aspects of this theory have been reduced to engineering practice in such fields as aircraft design, circuit synthesis and automatic control.

By comparison, the work on nonlinear problems has been rather meager. Most of what has been done lies in four categories: 1) open-loop problems involving a zero-memory nonlinear element preceded and/or followed by linear filters, 2) closed-loop problems involving low-pass linear elements in series with a zero-memory nonlinearity,3) perturbation methods where the "zeroth" approximation is deterministic (nonrandom), and subsequent approximations are stochastic but linear, 4) study of the first-order properties of the output, i.e. the first distribution or characteristic function.

For background, we give a brief indication of the scope of each of these categories, and cite some of the more significant work.

The zero-memory theory was developed for study of radio detectors, and is most closely associated with the names of Rice [41] and Middleton [40] , though several other authors have made contributions. The basic problem of this theory is the following: given a certain stationary random process $x(t)$, what is the spectrum of $y(t) = g[x(t)]$, where $y(t)$

depends only on x(t) and not on prior values of x. This is not a differential equation or a dynamical system, but once this problem has been solved, it is frequently easy to add linear filters in series with the nonlinear element, which does make it a nonlinear dynamical problem. While many useful and interesting results have been obtained for problems of this type, they are not of concern here.

The second class of problem, often treated by Booton's [5] approximation, concerns the type of feedback loop shown below:

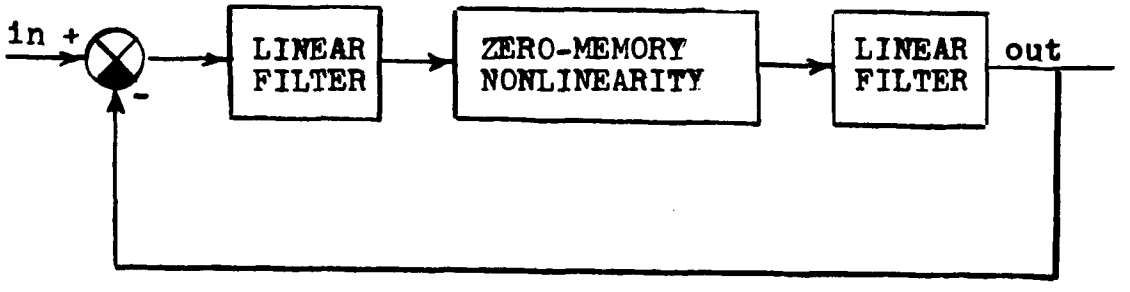


Figure 1-1 Block Diagram of Nonlinear Feedback System

The linear filters are low-pass. The input process must be stationary and Gaussian. It is then assumed that the signals are approximately Gaussian at all points in the loop, and that the nonlinearity may be replaced by that linear gain which most nearly approximates it in a certain sense. This method has been used with apparent success in a number of applications, but unfortunately there is no useful method for setting bounds on the errors introduced in the approximation. Subject to this uncertainty it does, however, permit

determination of the spectrum at any point in the loop. The technique has been widely applied, e.g. West [54] , Sawaragi [43] , Leland [33] and Elgerd [16] .

The third, or perturbational method is useful in problems where the random component is small in some sense compared with the deterministic part. It is closely analogous to the classical perturbation method in differential equations, and has many of the same benefits and limitations. Noise in oscillators has been considered in this way by Garstens [20] , Caughey [6] , and Tang [47] , but the method has not been thoroughly explored, particularly as to convergence questions.

The fourth area of investigation (first-order properties as solutions of partial differential equations) is closely related to the present study. In certain cases it is possible to derive partial differential equations whose solution is either the distribution function (Kolmogoroff [29]) or the characteristic function (Darling and Siegert [12] , Siegert [45]) of the output random process. In either case it is the first-order property which is considered. These methods have been applied to various problems (Kramers [30] , Chuang and Kazda [9] , McFadden [39] , Wishner [55] ,[56]) with apparent success.

In many cases, this output distribution is of considerable interest. It permits determination of the mean and variance (power) of the output. Frequently, however, it is desirable to know the power spectrum of the output, and

determination of this power spectrum is the principal objective of this study. It proves possible to calculate the spectrum by appropriate use of the Kolmogoroff equations.

There are apparently only two published papers dealing with exact (as opposed to approximate) computation of second-order properties of the outputs of nonlinear systems. These are the papers of Tortrat [49] and of Caughey and Dienes [7]. Tortrat's method, which will be considered in more detail later, is far from practical. It requires an amount and type of information which will rarely be available, and it is ill-suited for numerical computation.

The recent work of Caughey and Dienes [7] , on the other hand, is quite pertinent, and forms the starting point for the present study. They calculate the spectrum of the output of a first-order nonlinear dynamical system which is forced by white Gaussian noise. This problem is discussed at some length in Chapter 2, and their work is extended in a number of directions.

In this study, we will consider the spectra and auto-correlations of the outputs of certain classes of first- and second-order nonlinear systems which are forced by stationary random noise. The nature of the nonlinear systems will be restricted in various ways. The objective of these restrictions will be to assure that all dependent variables tend toward stationary, second-order (finite variance) random processes. The second-order requirement is necessary if correlations are to exist, and at least second-order station-

arity is required for the power spectrum to have meaning. Confining our attention in this way is somewhat arbitrary, but it still leaves us a useful and interesting class of problems.

One restriction made at the outset is that we will consider equations such that explicit dependence on time is confined to the noise input. This is neither necessary nor sufficient to assure output stationarity, but it might be called "almost necessary". Time-dependent systems which produce stationary outputs in response to stationary inputs are certainly possible, but they are more or less special cases. In any event, we exclude them, principally for the sake of simplicity.

It is easily seen that the above time-independency restriction is not sufficient to assure stationarity. Consider for example the equation $\dot{x} - \beta^2 x = n(t)$, $\beta = \text{const.}$ This equation is unstable in almost any sense of the word, and the random forcing will cause the output to "diverge" in some statistical sense, and be non-stationary.

We consider, then, time-invariant systems which are "stable" in some sense. In particular, we want systems in which the variances of all the dependent variables tend to finite limits as time increases, regardless of the initial conditions. This type of stability we call "non-dispersiveness", and some effort has been devoted to finding conditions sufficient to assure it.

In considering the question of dispersiveness, one takes

the point of view that at $t = 0$, the variables all had certain initial distributions, and one considers how these change with time. For calculating the autocorrelation and spectrum, on the other hand, it is assumed that the system is non-dispersive, and that the output random processes have reached their stationary condition. This means that the problem is envisioned as having started at $t = -\infty$. Both time origins are used in this study, and in any given section, the origin used will depend on the subject matter.

CHAPTER 2

FIRST-ORDER SYSTEMS

2.1 Statement of the First-Order Problem

The first-order problems considered here are related to the differential equation which may be written

$$\dot{x} + b(x) = \eta(t) \tag{2.1-1}$$

where x is the dependent variable (output), and the forcing function (input) is white Gaussian noise. This equation was first studied by Andronow, Pontriagin and Witt [1] and will be referred to as the APW equation.

There are numerous difficulties in interpreting this equation as it is written. Without the stochastic right member, of course, it presents no difficulty. If b(x) satisfies a Lipschitz condition, the solution exists, is unique, and in fact may be immediately reduced to quadratures. The meaning of the equation with the stochastic input has been the subject of a considerable amount of investigation, much of which will be cited below. For the present, suffice it to say that the conceptual and mathematical difficulties are such that for rigorous investigation, it has been found necessary to write it as an integral equation as in Doob's ([15] p 227) treatment.

$$x(t) - x(t_o) = \int_{t_o}^{t} b[x(t)]\,dt + \int_{t_o}^{t} d\eta \tag{2.1-2}$$

where $\eta(t)$ is the Wiener process. The second integral is of a special type known as the "stochastic integral" introduced

by Wiener [52] . Doob's discussion is more accessible [15] .

We do not, however, consider the many interesting problems of interpreting this equation other than to state some of the results which have been obtained for it. Our concern will rather be with certain partial differential equations associated with (2.1-1) or (2.1-2). Accordingly we will refer to (2.1-1) as the differential equation being considered despite the lack of rigorous interpretation of some of the terms.

In block diagram form, equation (2.1-1) is:

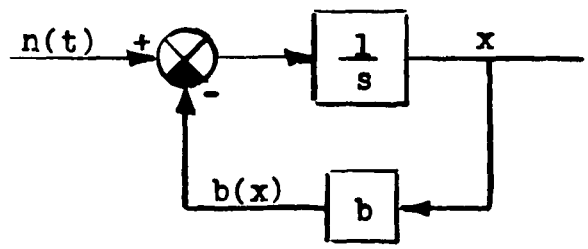


Figure 2.1-1 Block Diagram of the APW Equation

s is the differential operator, so that 1/s indicates integration. Stout [46] shows that if b(x) is uniquely invertible, then this is formally equivalent to the system:

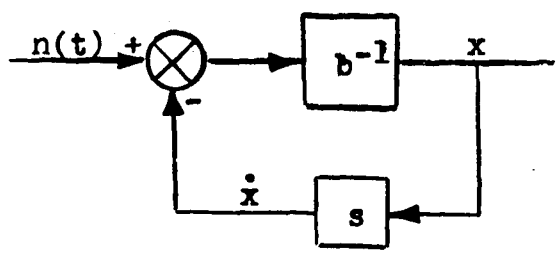


Figure 2.1-2 Alternate Block Diagram of the APW Equation

It is possible, without loss of generality, to normalize x in such a way that the noise input is of unit magnitude, i.e. the autocorrelation of the transformed noise $n_1(t)$ would be unity times the δ-function, $\Gamma_{n_1 n_1}(t) = \delta(t)$. We do not take this step, however, as we will later wish to analyze the effect of noise amplitude on the spectrum and autocorrelation. This may be more easily done if the noise magnitude is retained as an explicit parameter in the problem. Accordingly we use $\Gamma_{nn}(t) = N^2\delta(t)$.

It will be seen that with small additional labor, the following class of problems may be treated:

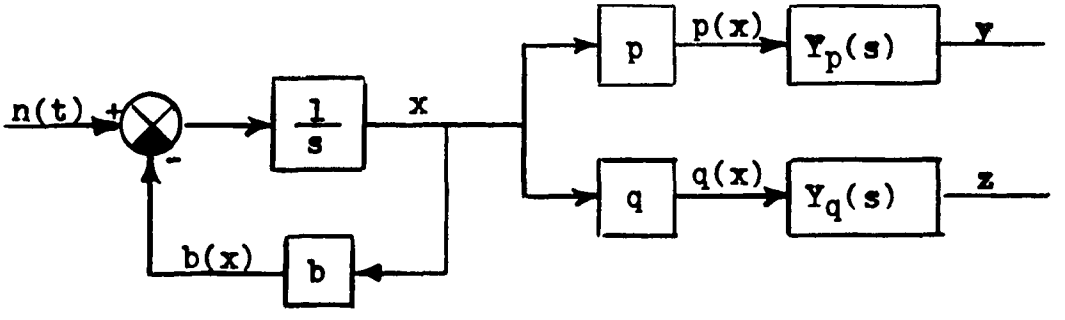


Figure 2.1-3 Augmented First-Order Problem

p, q and b are all functions of x, that is to say the quantities p(x), q(x), b(x) all depend only on the current value of x. The time-invariant linear filters of transfer functions Y_p, Y_q add nothing new. Once the cross-correlation between p and q is known, the effect of subsequent linear filtering may be calculated by known methods. The final problem, then, is to calculate the cross-correlation between y and z, $\Gamma_{yz}(t)$. Also it will be necessary to consider the conditions under which the various quantities will exist.

We now point out some of the feedback problems which can be reduced to the problem of Figure 2.1-3. The first of these is given in Figure 2.1-4.

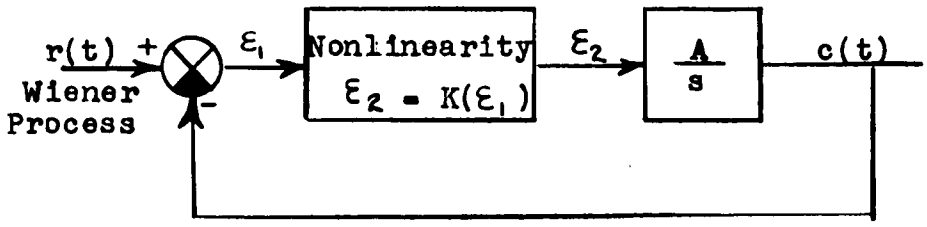


Figure 2.1-4 First-Order Servo with Wiener Process Input

If the input r(t) is a Wiener process of variance $\eta^2 t$, (r(0)= 0), then this block diagram may be rearranged as shown in Figure 2.1-5, provided appropriate initial conditions are used.

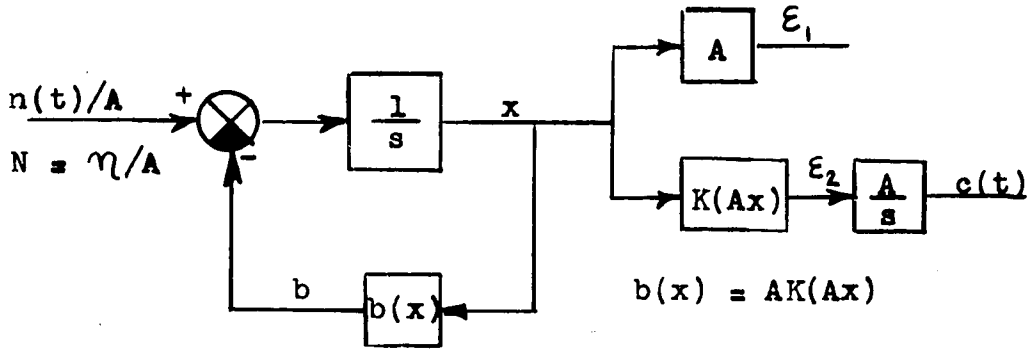


Figure 2.1-5 Rearranged Block Diagram for Wiener Process Input

This is a problem of the type of Figure 2.1-3, and all pertitnent correlations may be determined.

The input r(t) and output c(t) are non-stationary, though the error signal ε_1 is stationary if K has the proper characteristics, and if the Wiener process is assumed to have started in the distant past.

Another problem is shown in Figure 2.1-6.

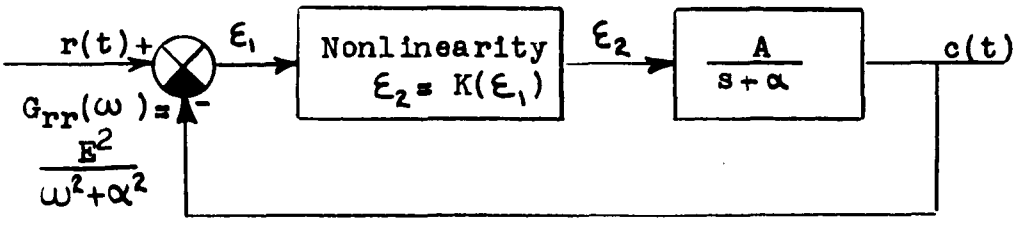


Figure 2.1-6 First-Order Servo with O.U. Process Input

The input in this case is an Ornstein-Uhlenbeck process. This is a Gaussian process with spectral density $G_{rr}(\omega) = E/\alpha^2+\omega^2$. The block diagram may be rearranged as shown below.

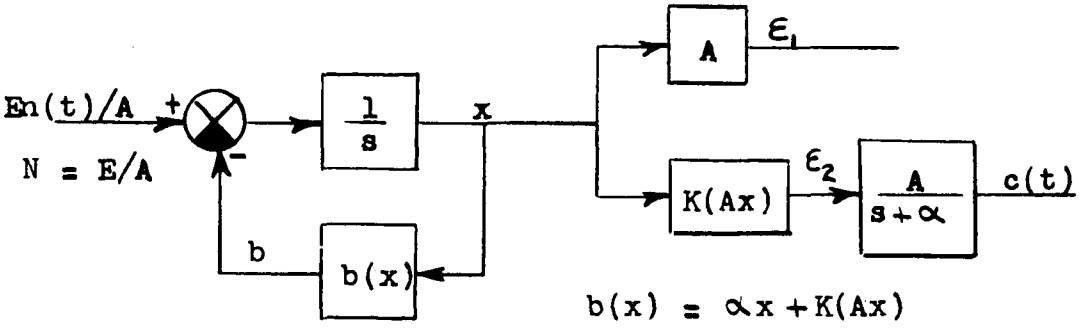


Figure 2.1-7 Rearranged Block Diagram for O.U. Input

Here the input is stationary, and the output and error will also be stationary for some $K(\epsilon_1)$. The system of Figure 2.1-4 may be viewed as the limit of the above as $\alpha \to 0$.

Next, consider the system of Figure 2.1-8.

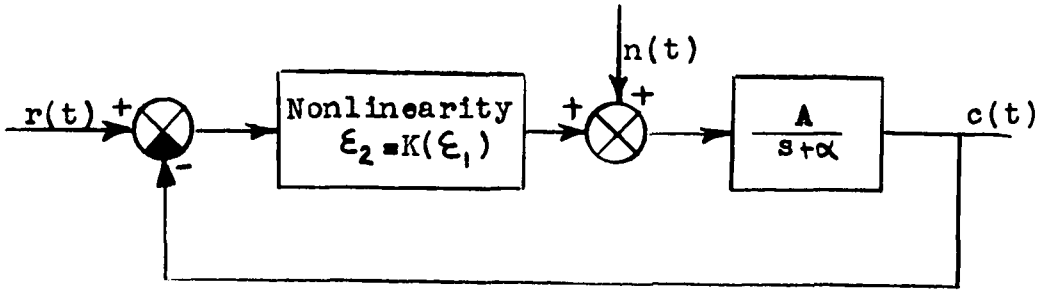


Figure 2.1-8 White Gaussian Disturbance Input

Here, white Gaussian noise of magnitude N enters the system as a disturbance. This is the system studied by Wishner [55] , [56] . If r = 0, this may be rearranged as follows:

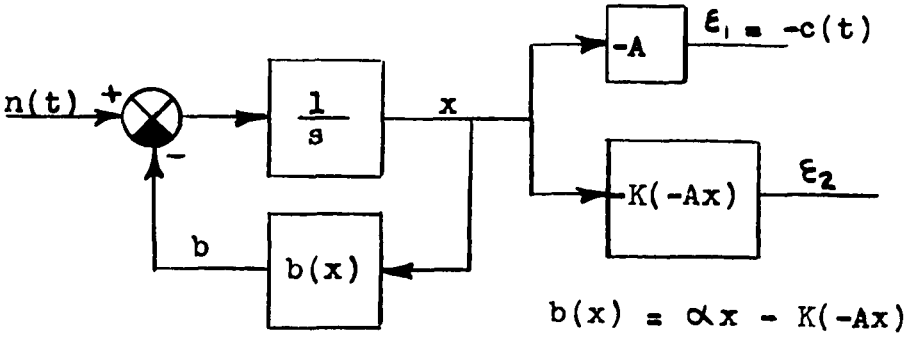


Figure 2.1-9 Rearranged Block-Diagram, Disturbance Input

This does not permit analysis of the response to simultaneous signal and noise inputs.

These examples should suffice to show that the class of problems studied here is somewhat larger than might be thought from examination of Figure 2.1-3.

2.2 Known Results on the Stochastic Differential Equation

Various generalizations and specializations of equation (2.1-1) have been studied. Probably the most important results are those of Ito [27], [28] which are given here in Doob's ([15], p 273 ff) formulation. Simplifications have been made to apply to the case at hand.

If there exists an L independent of x and t such that

2.2-1) $$|b(x_1) - b(x_2)| \leq L |x_1 - x_2| \quad 0 \leq L < +\infty$$

it follows that

i) almost all sample functions are continuous

ii) the process x(t) is uniquely determined

iii) x(t) is a Markoff process

This Lipschitz condition is sufficient to assure the conclusions, but it seems certain that it is not necessary. For example, it excludes functions with any discontinuities, and functions which grow faster than linearly with x. It seems probable that weaker conditions on b(x) can be found which permit inclusion of these classes of functions.

In a later section, we will consider a discontinuous b(x), and we will assume that x(t) is still uniquely determined and Markoffian.

Two special cases of this equation have also received particular study: $b(x) = 0$ (the Wiener process) and $b(x) = \beta x$ (the Ornstein-Uhlenbeck process). If $b(x) = 0$, the resulting x(t) is the original Einstein-Smoluchowski Brownian motion, which was rigorously described by Wiener [53] and is usually known by his name. If $b(x) = \beta x$, then (2.1-1) becomes the

Smoluchowski equation of Brownian motion theory, and the resulting process was studied by Uhlenbeck and Ornstein [50] and more rigorously and completely by Doob [14]. Doob obtained the autocorrelation of x(t) for the Ornstein-Uhlenbeck process, by a method which requires linearity of the equation.

For both processes, it has been shown that if they are "separable" (see, e.g. Doob [14] pp 51-52) then almost all sample functions are continuous, and that both are Markoffian. These are special cases of Ito's result cited above.

The sample functions of the Ornstein-Uhlenbeck process have unusual properties. They are continuous almost everywhere, but differentiable almost nowhere. The expected number of zero-crossings per unit time is infinite (Rice[40], Sect. 3.3) but the zero-crossings are not everywhere dense. The probability of finding finite intervals of time containing no zero-crossings is not zero (Siegert[44]). These sample functions may justly be described by the phrase Paul Levy applies to the closely related Wiener process. He says it is "tres zigzagante". It is to be expected that most of the qualitative features of the Ornstein-Uhlenbeck sample functions will carry over to processes generated by other b(x).

2.3 The Kolmogoroff Equations

With Kolmogoroff [29], we define $f_T(x_1,t_1;x_2,t_2)$ as the transition density function, i.e. $f_T(x_1,t_1;x_2,t_2)dx_1dx_2$ is the probability that $x_1 < x \leq x_1 + dx_1$ at time t_1 and $x_2 < x \leq x_2 + dx_2$ at time t_2, where t_1 is earlier than t_2. Kolmogoroff showed that if the process x(t) is a simple

Markoff process, then f_T satisfies a pair of partial differential equations which, when applied to the APW equation (see, e.g. Doob [15] p 275) yield

2.3-1) $$\frac{\partial f_T}{\partial t_1} = -\frac{N^2}{2}\frac{\partial^2 f_T}{\partial x_1^2} + b(x_1)\frac{\partial f_T}{\partial x_1}$$

2.3-2) $$\frac{\partial f_T}{\partial t_2} = \frac{N^2}{2}\frac{\partial^2 f_T}{\partial x_2^2} + \frac{\partial}{\partial x_2}\left[b(x_2) f_T\right]$$

where N is the noise magnitude.

Equation (2.3-1) is known as Kolmogoroff's first equation or the "backward" equation, as it deals with derivatives with respect to the earlier variables. Equation (2.3-2) is called Kolmogoroff's second equation or the "forward" equation. It was originally discovered by physicists studying Brownian motion, and is referred to in that context as the Fokker-Planck equation. We will use the latter terminology here.

For stationary processes, f_T will be a function of $t = t_2 - t_1$ rather than of both t_1 and t_2. In this case the Kolmogoroff equations become

2.3-3) $$\frac{\partial f_T}{\partial t} = \frac{N^2}{2}\frac{\partial^2 f_T}{\partial x_1^2} - b(x_1)\frac{\partial f_T}{\partial x_1}$$

2.3-4) $$\frac{\partial f_T}{\partial t} = \frac{N^2}{2}\frac{\partial^2 f_T}{\partial x_2^2} + \frac{\partial}{\partial x_2}\left[b(x_2) f_T\right]$$

Corresponding to these two equations, we define two operators

2.3-5) $$\Omega^* = \frac{N^2}{2}\frac{\partial^2}{\partial x^2} - b(x)\frac{\partial}{\partial x}$$

2.3-6) $$\Omega = \frac{N^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\left[\, b(x) \,\right]$$

which will be useful subsequently.

2.4 Known Results on the Fokker-Planck Equation

The properties of the solutions of the Fokker-Planck equation have been studied by a number of investigators, and a moderately complete theory exists. Possibly the most significant work has been that of Feller [17] , [18] , [19] , particularly in his 1952 paper [18] . Hille [24] , [25] and Yosida [57] have also contributed to the problem, and a recent convergence theorem of Mandl [37] will be most useful. Certain needed results from these investigations will be enumerated.

In order to simplify the statement of these results, it will be advisable to make use of certain semi-group ideas. We denote by $u(x,t;g)$ the solution of $\frac{\partial u}{\partial t} = \Omega^* u$ which has the boundary condition $\lim_{t\to 0} u(x,t;g) = g(x)$ $-\infty < x < +\infty$. The function $g(x)$ is the boundary condition in a Cauchy problem. Similarly, we define $v(x,t;g)$ as the solution of $\frac{\partial v}{\partial t} = \Omega v$ subject to the boundary condition $\lim_{t\to 0} v(x,t;g) = g(x)$. The functions $u(x,t;g)$ and $v(x,t;g)$ are in effect transformations of the initial function $g(x)$. In particular they are positivity-preserving, strongly continuous semi-groups of transformations, and the operators Ω^* and Ω are their infinitesmal generators (Feller [18]), if $b(x)$ is continuous $-\infty < x < +\infty$.

If $g(x) \geq 0, -\infty < x < +\infty$, then $u(x,t;g) \geq 0$, $t \geq 0$, $-\infty < x < +\infty$ and $v(x,t;g) \geq 0$, $t \geq 0$, $-\infty < x < +\infty$. (Feller [18], Mandl [37]). Ω is norm-preserving, i.e.

$$2.4\text{-}1) \qquad \int_{-\infty}^{+\infty} v(x,t;g)\,dx = \int_{-\infty}^{+\infty} g(x)\,dx \quad t \geq 0$$

if $\int_{-\infty}^{+\infty} |g(x)|dx < \infty$ (Mandl [37]).

The solutions u and v exist and are uniquely determined by g. This has been shown under various conditions by several of the above-cited authors, and may be seen also as a consequence of Hille and Phillips [24] Theorem 23.8.1, p 621 as a special case of the "abstract Cauchy problem".

If $B(x) \equiv \frac{2}{N^2}\int_0^x b(s)ds$ and if $\int_{-\infty}^{+\infty} e^{-B(x)}\,dx < \infty$ then

$$2.4\text{-}2) \qquad \lim_{t\to\infty} v(x,t;g) = \frac{e^{-B(x)}\int_{-\infty}^{+\infty} g(s)ds}{\int_{-\infty}^{+\infty} e^{-B(s)}\,ds}$$

for any g such that $\int_{-\infty}^{+\infty} |g(s)|ds < \infty$, and b(x) continuous. Convergence is in the sense that $h(x,t) \to h_s(x)$ means $\int_{-\infty}^{+\infty} |h(x,t) - h_s(x)|\,dx \to 0$. This is Mandl's Theorem 5.

2.5 The Steady-State Distribution

The above theorem makes possible a determination of the behavior of the density function as $t \to \infty$. If at $t = 0$, the density function of the random process is $g(x)$, $g(x) \geq 0$ $-\infty < x < +\infty$, $\int_{-\infty}^{+\infty} g(x)dx = 1$, then the density function at some later time may be written in terms of the transition density.

2.5-1) $$f(x_2, t; g) = \int_{-\infty}^{+\infty} g(x_1)\, f_T(x_1, x_2, t)\, dx_1$$

The absolute density function f(x,t;g) also satisfies the Fokker-Planck equation. This may be seen by noting that

$$\frac{\partial f}{\partial t} - \frac{N^2}{2}\frac{\partial^2 f}{\partial x_2^2} - \frac{\partial}{\partial x_2}\left[b(x_2) f\right] =$$

2.5-2) $$\int_{-\infty}^{+\infty} g(x_1)\left\{ \frac{\partial f_T}{\partial t} - \frac{N^2}{2}\frac{\partial^2 f_T}{\partial x_2^2} - \frac{\partial}{\partial x_2}\left[b(x_2) f_T\right]\right\} dx_1$$

By equation (2.2-4), the $\{\ \}$ expression in the integrand is zero, so the left side of (2.5-2) equals zero or

2.5-3) $$\frac{\partial f}{\partial t} = \frac{N^2}{2}\frac{\partial^2 f}{\partial x_2^2} + \frac{\partial}{\partial x_2}\left[b(x_2) f\right]$$

We may then apply the results of Section 2.4 to f(x,t;g), and it follows that, for continuous b,

2.5-4) $$f(x, t; g) \geq 0, \quad t \geq 0, -\infty < x < +\infty$$

2.5-5) $$\int_{-\infty}^{+\infty} f(x, t; g)\, dx = 1 \text{ all } g, \quad \int_{-\infty}^{+\infty} g(x)\, dx = 1, \ g(x) \geq 0$$

2.5-6) $$\lim_{t \to \infty} f(x, t; g) = \frac{e^{-B(x)}}{\int_{-\infty}^{+\infty} e^{-B(x)}\, dx} \equiv f_S(x)$$

This defines the steady-state distribution of the random process x(t). It also shows that any initial distribution will converge toward this unique steady-state.

It may be shown directly from (2.5-6) that

$$2.5\text{-}7) \qquad \frac{\partial f_s}{\partial t} = \frac{N^2}{2}\frac{\partial^2 f_s}{\partial x^2} + \frac{\partial}{\partial x}\left[b(x) f_s\right] = 0$$

as would be expected, and further that

$$2.5\text{-}8) \qquad \frac{N^2}{2}\frac{\partial f_s}{\partial x} + b(x) f_s = 0$$

In fact, equation (2.5-7) may be used as a starting point in finding f_s.

An alternative approach to demonstration of convergence to a steady-state is given in Appendix D. While not completely rigorous, it may have some interest. The steady-state distribution is shown to be the solution to a certain minimization problem.

With convergence to the steady-state established, we consider the nature of the steady-state distribution in more detail. From the foregoing it may be seen that the steady-state distribution will exist if

$$2.5\text{-}9) \qquad \int_{-\infty}^{+\infty} e^{-B(x)}\,dx < \infty$$

If this condition is not met, the Fokker-Planck equation still has a unique solution, but that solution represents a process that is not stationary.

Since we are to be concerned with stationary processes, we will restrict our interest to those b(x) which satisfy condition (2.5-9). There are, however, additional requirements. Since we will be calculating second-order properties, we must have second-order processes, i.e.

2.5-10) $$\int_{-\infty}^{+\infty} x^2 e^{-B(x)}\,dx < \infty$$

2.5-11) $$\int_{-\infty}^{+\infty} b^2(x)\, e^{-B(x)}\,dx < \infty$$

If these conditions are satisfied, $\Gamma_{xx}(t)$, $\Gamma_{bb}(t)$ and $\Gamma_{xb}(t)$ will all exist. (See, e.g. Davenport and Root [13], p 61.) Any b(x) which satisfies both these conditions is called non-dispersive. Otherwise it is called dispersive.

Under various conditions, one of these requirements may be stronger than the other, but in general, both are necessary and sufficient for existence of the correlations. While this is an accurate statement of the requirements on b(x), it is one which does not readily lend itself to testing a given b(x). For this reason, we develop several sufficient conditions for b(x) to be non-dispersive. These sufficient conditions are of such nature as to permit rapid testing of many b(x), and also to give an intuitive idea of the general nature b(x) must have.

Theorem 1 Conditions under which $E[x^2] < \infty$ implies $E[b^2] < \infty$

Let b(x) be a measurable function such that

2.5-12) $$|b(x)| \leq C_1 + C_2|x| \ , \quad 0 \leq C_1 < \infty \ , \quad 0 \leq C_2 < \infty$$

Then existence of $\int_{-\infty}^{+\infty} x^2 e^{-B}\,dx$ implies existence of $\int_{-\infty}^{+\infty} b^2 e^{-B}\,dx$.

Proof: Using the assumed inequality,

2.5-13) $$\int_{-\infty}^{+\infty} b^2 e^{-B}\,dx \leq \int_{-\infty}^{+\infty} [C_1 + C_2|x|]\, e^{-B}\,dx$$

Now consider two sets of points on the real x axis: A consists of all x such that $C_1 \geq C_2|x|$, A' consists of all x

such that $C_1 < C_2|x|$. The union of A and A' is the entire real axis, and the two sets are disjoint. If $x \in A$, then $C_1 + C_2|x| \leq 2C_1$, and $[C_1 + C_2|x|]^2 \leq 4C_1^2$. If $x \in A'$, then $C_1 + C_2|x| < 2C_2|x|$ and $[C_1 + C_2|x|]^2 < 4C_2^2x^2$. Combining the two,

2.5-14) $$[C_1 + C_2|x|]^2 \leq 4C_1^2 + 4C_2^2 x^2 \quad , \text{all } x$$

Substituting this result into (2.5-13),

2.5-15) $$\int_{-\infty}^{+\infty} b^2 e^{-B(x)}\,dx \leq 4C_1^2 \int_{-\infty}^{+\infty} e^{-B}\,dx + 4C_2^2 \int_{-\infty}^{+\infty} x^2 e^{-B}\,dx$$

The second integral is finite, from the hypothesis of the theorem. Consider the first integral. It is easily shown that

2.5-16) $$\int_{-\infty}^{+\infty} e^{-B}\,dx \leq \int_{-\infty}^{+\infty} x^2 e^{-B}\,dx + \int_{-1}^{+1} (1 - x^2) e^{-B}\,dx$$

The first integral exists by hypothesis. Under the conditions of the theorem, B(x) must be finite for finite x. Therefore, the second integral exists. This shows the existence of $\int_{-\infty}^{+\infty} e^{-B}\,dx$ which, in turn completes proof of the existence of $\int_{-\infty}^{+\infty} b^2 e^{-B}\,dx$ in equation (2.5-15)

Corollary 1 If b(x) is a bounded measurable function, then $\int_{-\infty}^{+\infty} x^2 e^{-B}\,dx < \infty$ implies $\int_{-\infty}^{+\infty} b^2 e^{-B}\,dx < \infty$.

Proof: If b(x) is bounded, then $|b(x)| \leq C_1$, which is a special case of the theorem, with C_2 equal to zero.

Corollary 2 If b(x) satisfies a Lipschitz condition

2.5-17) $$|b(x_1) - b(x_2)| \leq L|x_1 - x_2| \quad , \; 0 \leq L < +\infty, \; |x_1| < \infty, \; |x_2| < \infty$$

then existence of $\int_{-\infty}^{+\infty} x^2 e^{-B}\,dx$ implies existence of $\int_{-\infty}^{+\infty} b^2 e^{-B}\,dx$.

Proof: Since the Lipschitz condition holds for arbitrary x_1 and x_2, let $x_1 = x$, $x_2 = 0$.

2.5-18) $$|b(x)| - |b(0)| \leq |b(x) - b(0)| \leq L|x|$$

2.5-19) $$|b(x)| \leq |b(0)| + L|x|$$

which is a constraint of the type used in the theorem. Thus, if b(x) satisfies a Lipschitz condition, it satisfies the requirements of the theorem, and the conclusion follows.

Theorem 2 Sufficient conditions that $E[x^2] < \infty$.

Let b(x) be a measurable function such that

i. b(x) is integrable in some interval $-a \leq x \leq +a$, $a > 0$.

ii. $\int_a^{+\infty} \left[\frac{2}{N^2} b(\xi) - \frac{3+\delta}{\xi} \right] d\xi$ converges either to a finite value or $+\infty$, $0 < a < +\infty$, some $\delta > 0$.

iii. $\int_{-\infty}^{-a} \left[\frac{2}{N^2} b(\xi) - \frac{3+\delta}{\xi} \right] d\xi$ converges either to a finite value or $-\infty$, $0 < a < +\infty$, some $\delta > 0$.

Then $\int_{-\infty}^{+\infty} x^2 e^{-B(x)}\,dx < \infty$ where $B(x) = \frac{2}{N^2} \int_0^x b(s)\,ds$.

Proof:

2.5-20) $$\int_{-\infty}^{+\infty} x^2 e^{-B}\,dx = \int_{-\infty}^{-a} x^2 e^{-B}\,dx + \int_{-a}^{+a} x^2 e^{-B}\,dx + \int_{+a}^{+\infty} x^2 e^{-B}\,dx$$

By hypothesis (i), B(x) is bounded for all x, $-a \leq x \leq +a$. It follows that the second integral above is finite. Now define

2.5-21) $$I(x) = \int_a^x \left[\frac{2}{N^2} b(\xi) - \frac{3+\delta}{\xi}\right] d\xi \quad x \geq a$$

If the integral converges to any value greater than $-\infty$, then I(x) must be finite, for finite x, and bounded from below.

2.5-22) $$I(x) \geq M_1 > -\infty \quad x \geq a$$

From equation (2.5-21),

2.5-23) $$\int_a^x \left[\frac{2}{N^2} b(\xi) - \frac{3+\delta}{\xi}\right] d\xi \geq M_1 \quad x \geq a$$

2.5-24) $$B(x) - B(a) - (3+\delta)(\log x - \log a) \geq M_1$$

2.5-25) $$B(x) \geq (3+\delta)\log x + M_2$$

where $M_2 = M_1 + B(a) - (3+\delta)\log a$. From this, it follows that

2.5-26) $$e^{-B(x)} \leq \frac{e^{-M_2}}{x^{3+\delta}}$$

2.5-27) $$x^2 e^{-B(x)} \leq \frac{e^{-M_2}}{x^{1+\delta}} \quad x \geq a > 0$$

This proves convergence of the third integral on the right of equation (2.5-20). Convergence of the first integral may be shown in a precisely analogous way. All three integrals on the right of equation (2.5-20) exist, and the assertion is proved.

Theorem 3 Sufficient Conditions for Non-Dispersive b(x) With Linear Bound

Let b(x) satisfy the conditions of Theorem 2, and in

addition, one or both of the following:

i. $|b(x)| \leq C_1 + C_2|x| \quad C_1 \geq 0, \; C_2 \geq 0, \; -\infty < x < +\infty$

ii. $|b(x_1) - b(x_2)| \leq L|x_1 - x_2| \quad 0 \leq L < \infty, \; |x_1| < \infty, \; |x_2| < \infty$

Then b(x) is non-dispersive.

Proof: If b(x) satisfies the conditions of Theorem 2, then $\int_{-\infty}^{+\infty} x^2 e^{-B(x)} dx < \infty$. It was shown in Theorem 1 and its corollaries that the two conditions above are sufficient to assure $\int_{-\infty}^{+\infty} b^2(x) e^{-B} dx < \infty$, which completes the requirements for non-dispersive b(x).

As a special case of restrictions on b(x) which meet the above conditions, consider measurable b(x) such that

2.5-28) $$|b(x)| \leq C_1 + C_2|x| \quad -\infty < x < +\infty$$

2.5-29) $$b(x) \geq \frac{\mu}{x^{1-\varepsilon}} \quad \mu > 0, \; \varepsilon > 0, \; x \geq a > 0$$

2.5-30) $$b(x) \leq -\frac{\mu}{|x|^{1-\varepsilon}}, \quad x \leq -a < 0$$

It may easily be shown that any b(x) meeting these conditions, meets the requirements of Theorem 3, and is non-dispersive. From equation (2.5-29),

2.5-31) $$\int_a^x \left[\frac{2}{N^2} b(\xi) - \frac{3+\delta}{\xi}\right] d\xi \geq \int_a^x \left[\frac{2}{N^2} \frac{\mu}{\xi^{1-\varepsilon}} - \frac{3+\delta}{\xi}\right] d\xi$$
$$\geq x^{\varepsilon} \left\{ \frac{2\mu}{\varepsilon N^2} - (3+\delta) \frac{\log x}{x^{\varepsilon}} \right\} + M_1$$

where $M_1 = (3+\delta)\log a - \frac{2\mu}{\varepsilon N^2}[a]^{\varepsilon}$. It follows that the limit of the right side of equation (2.5-31) as $x \to \infty$ is $+\infty$. This shows that

2.5-32) $$\lim_{x \to \infty} \int_a^x \left[\frac{2}{N^2} b(\xi) - \frac{3+\delta}{\xi}\right] d\xi = +\infty$$

regardless of the size of δ . An exactly analogous pro-

cedure may be used to show that

$$2.5\text{-}33) \quad \lim_{x\to-\infty} \int_{x}^{-a} \left[\frac{2}{N^2} b(\xi) - \frac{3+\delta}{\xi}\right] d\xi = -\infty$$

The conditions of Theorem 3 are satisfied, hence b(x) is non-dispersive.

The conditions stated in this example may be shown graphically. Any measurable b(x) which lies entirely in an area such as the shaded area of Figure 2.5-1 will be non-dispersive.

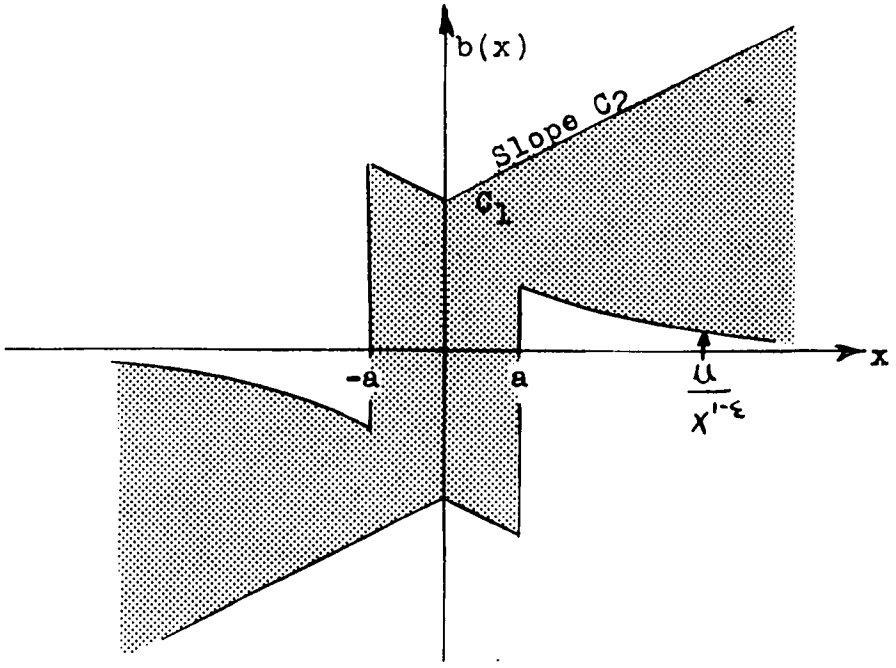


Figure 2.5-1 Sufficient Conditions for Non-Dispersive b(x)

In this case, we see that $|b(x)|$ can decrease to zero as $|x| \to \infty$, provided the decrease is sufficiently slow.

Theorem 4 Sufficient Conditions for Dispersive b(x).

Let b(x) be a measurable function such that

i. b(x) is integrable, $-a \leq x \leq +a$, $a > 0$

ii. $$\lim_{x\to+\infty}\int_a^x\left[\frac{2}{N^2}b(\xi)-\frac{3}{\xi}\right]d\xi = J_1 < +\infty$$

or

$$\lim_{x\to-\infty}\int_x^{-a}\left[\frac{2}{N^2}b(\xi)-\frac{3}{\xi}\right]d\xi = J_2 > -\infty$$

or both.

Then b(x) is dispersive.

Proof:

2.5-34) $$\int_{-\infty}^{+\infty}x^2e^{-B}\,dx = \int_{-\infty}^{-a}x^2e^{-B}dx + \int_{-a}^{+a}x^2e^{-B}dx + \int_a^{+\infty}x^2e^{-B}dx$$

Define I(x) as

2.5-35) $$I(x) = \int_a^x\left[\frac{2}{N^2}b(\xi)-\frac{3}{\xi}\right]d\xi,\quad x\geq a$$

By the same argument as in Theorem 2, I(x) must be continuous, finite for finite x, and bounded from above.

2.5-36) $$\int_a^x\left[\frac{2}{N^2}b(\xi)-\frac{3}{\xi}\right]d\xi \leq M_1 < +\infty$$

Integrating,

2.5-37) $$B(x) \leq 3\log x + M_2$$

where $M_2 = M_1 + B(a) - 3\log a$. From this it follows that

2.5-38) $$x^2e^{-B(x)} \geq \frac{e^{-M_2}}{x} \qquad x\geq a$$

so the third integral of equation (2.5-34) diverges and $\int_{-\infty}^{+\infty}x^2e^{-B(x)}\,dx$ does not exist. Thus b(x) is dispersive.

By an exactly analogous process, it may be shown that if

2.5-39) $$\lim_{x\to-\infty}\int_x^{-a}\left[\frac{2}{N^2}b(\xi)-\frac{3}{\xi}\right]d\xi = \bar{J}_2 > -\infty$$

Then the first integral of equation (2.5-34) is infinite, and b(x) is dispersive.

As an example of a restriction of the above type, let

2.5-40) $$b(x) \leq \frac{u}{x} \quad x \geq a \,, \quad u < \frac{3}{2} N^2$$

or

2.5-41) $$b(x) \geq \frac{u}{x} \quad x \leq -a \,, \quad u < \frac{3}{2} N^2$$

or both. Any b(x) satisfying these requirements will meet the requirements of Theorem 4. This may be seen from the following. Using equation (2.5-40)

2.5-42) $$\int_a^X \left[\frac{2}{N^2} b(\xi) - \frac{3}{\xi}\right] d\xi \leq \int_a^X \left[\frac{2}{N^2}\frac{u}{\xi} - \frac{3}{\xi}\right] d\xi = \left(\frac{2u}{N^2} - 3\right) \log \frac{X}{a}$$

The limit of the right side as $x \to +\infty$ is $-\infty$, so the same holds true for the left side, satisfying hypothesis (ii) of Theorem 4. By an exactly similar argument, it is possible to show that hypothesis (iii) of Theorem 4 follows from equation (2.5-41).

These requirements may also be presented graphically. Any measurable b(x), integrable on $-a \leq x \leq +a$, lying in the shaded area of Figure 2.5-2 for $x < 0$ or $x > 0$ is dispersive.

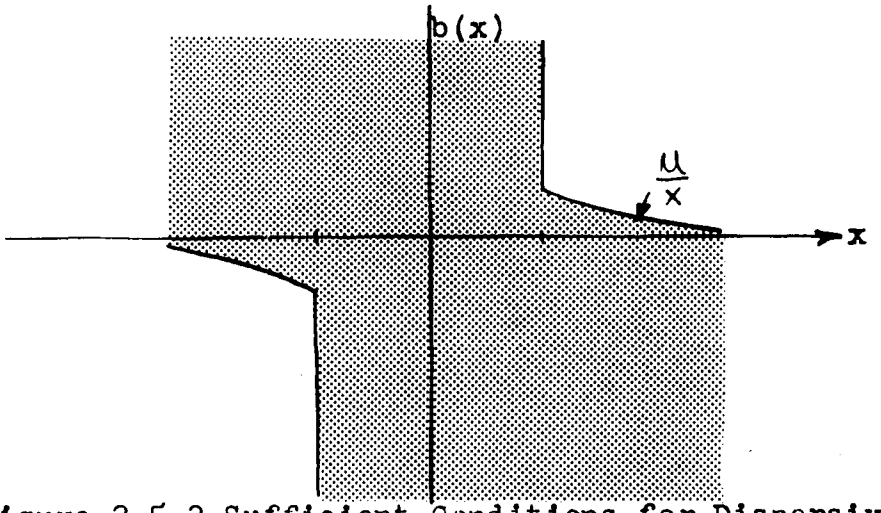


Figure 2.5-2 Sufficient Conditions for Dispersive b(x)

As an example of a borderline case with interesting properties, we consider

2.5-43) $$b(x) = \frac{Ax}{a^2+x^2}$$

The general form of this function is given in Figure 2.5-3.

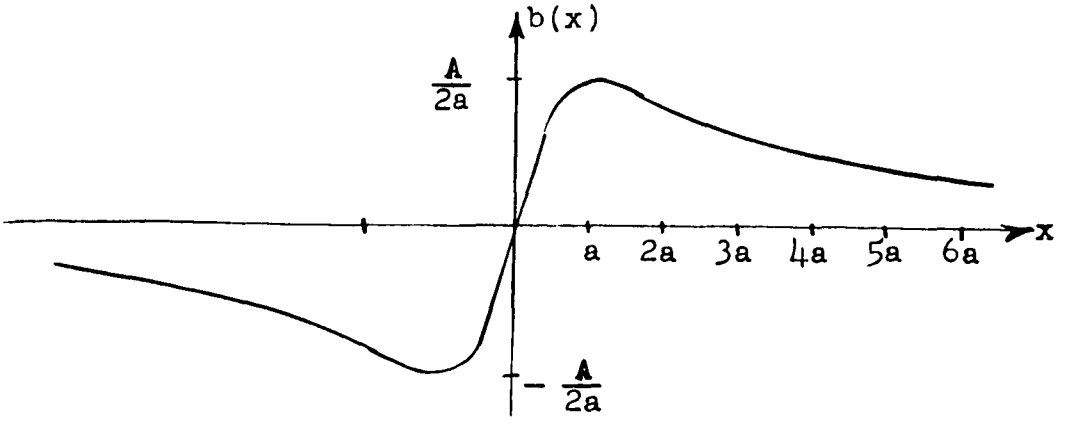


Figure 2.5-3 Plot of $b(x)= \frac{Ax}{a^2+x^2}$

For this function,

2.5-44) $$B(x) = \frac{2}{N^2}\int_0^x b(s)ds = \frac{2}{N^2}\int_0^x \frac{As}{a^2+s^2}ds = \frac{A}{N^2}\log\left[\frac{a^2+x^2}{a^2}\right]$$

2.5-45) $$e^{-B(x)} = \left(\frac{a^2}{a^2+x^2}\right)^{A/N^2}$$

2.5-46) $$f_s = \frac{C}{(a^2+x^2)^{A/N^2}} \quad ; \quad \frac{1}{C} = \int_{-\infty}^{+\infty}\frac{dx}{(a^2+x^2)^{A/N^2}}$$

Now consider the moments of this distribution.

2.5-47) $$\mu_k = \int_{-\infty}^{+\infty} x^k f_s\, dx$$

Inspection of the form of f_s shows that the number of moments which will exist depends on the magnitude of A/N^2. The larger the noise, the fewer moments will exist. We can see that if $\frac{A}{N^2} > \frac{1}{2}$, μ_0 will exist. If $\frac{A}{N^2} > \frac{3}{2}$, μ_2 will

exist, etc. In general, if $\frac{A}{N^2} > \frac{1+k}{2}$, the kth moment will exist.

Since this particular b(x) is bounded, existence of μ_2 implies non-dispersiveness. We would expect, then, that $\frac{A}{N^2} > \frac{3}{2}$ would be non-dispersive, $\frac{A}{N^2} \leq \frac{3}{2}$ would be dispersive. This may be checked by application of the foregoing theorems. First, we use Theorem 3. The function b(x) is bounded, so it remains to show that the conditions of Theorem 2 are satisfied.

The function b(x) is bounded, continuous and measurable. Now consider

$$2.5\text{-}48) \quad \int_a^{+\infty} \left[\frac{2}{N^2} b(\xi) - \frac{3+\delta}{\xi} \right] d\xi = \int_a^{+\infty} \left\{ \frac{\left[\frac{2A}{N^2} - (3+\delta) \right] \xi^2 - (3+\delta)}{\xi (a^2 + \xi^2)} \right\} d\xi$$

This integral will be finite if $\frac{2A}{N^2} = 3+\delta$. It will be $+\infty$ if $\frac{2A}{N^2} > 3+\delta$. This shows that for non-dispersiveness, $\frac{2A}{N^2} > 3$, which agrees with the condition deduced above. Since b(x) is odd-symmetric, the condition for negative x need not be investigated separately.

Similarly, applying Theorem 4, we note that

$$2.5\text{-}49) \quad \int_a^{+\infty} \left[\frac{2}{N^2} b(\xi) - \frac{3}{\xi} \right] d\xi = \int_a^{+\infty} \left[\frac{\left(\frac{2A}{N^2} - 3 \right) \xi^2 - 3}{\xi (a^2 + \xi^2)} \right] d\xi$$

This will be $-\infty$ if $\frac{2A}{N^2} < 3$, and hence b(x) will be dispersive under the same condition.

If $\frac{2A}{N^2} = 3$, the integral of equation (2.5-49) is finite, meeting the conditions of Theorem 4, and indicating dispersiveness. Thus the theorems give the same results as those derived from inspection of the form of f_s.

This phenomenon of the very existence of moments depending on the input amplitude is an unusual one. It is not observed in linear systems. Once the effect has been demonstrated, an heuristic explanation is not hard to see. This $b(x)$ becomes small for large $|x|$. It has appreciable values only in some vicinity of the origin. Accordingly, if N is very large, it seems reasonable that x spends most of the time in regions where $b(x)$ is negligible, so the general effect is that of $b(x) = 0$, which is dispersive. Conversely, for N very small, x would spend most of the time near the origin where $b(x) \simeq \frac{A}{a^2} x$. If $b(x) = \alpha x$, $\alpha > 0$, all moments exist, so small N should approximate this case.

Theorem 2 gives no information on the existence of the variance of $b(x)$. Theorem 1 and its corollaries give three different conditions which may be adjoined to the conditions of Theorem 2 to assure $E[b^2(x)] < +\infty$. It may be seen from the following example that <u>some</u> additional condition must be used. The example meets the conditions of Theorem 2, so $E[x^2] < \infty$, but $E[b^2(x)]$ turns out to be infinite. Let

$$b(x) = \frac{x}{\sqrt{|1-x^2|}} \tag{2.5-50}$$

A plot of this function is given in Figure 2.5-4. The function becomes infinite at $x = \pm 1$. $B(x)$, however, is well-behaved. It may easily be shown that

$$B(x) = \begin{cases} \frac{2}{N^2}\left[1 - \sqrt{1-x^2}\right] & |x| \le 1 \\ \frac{2}{N^2}\left[1 - \sqrt{x^2-1}\right] & |x| > 1 \end{cases} \tag{2.5-51}$$

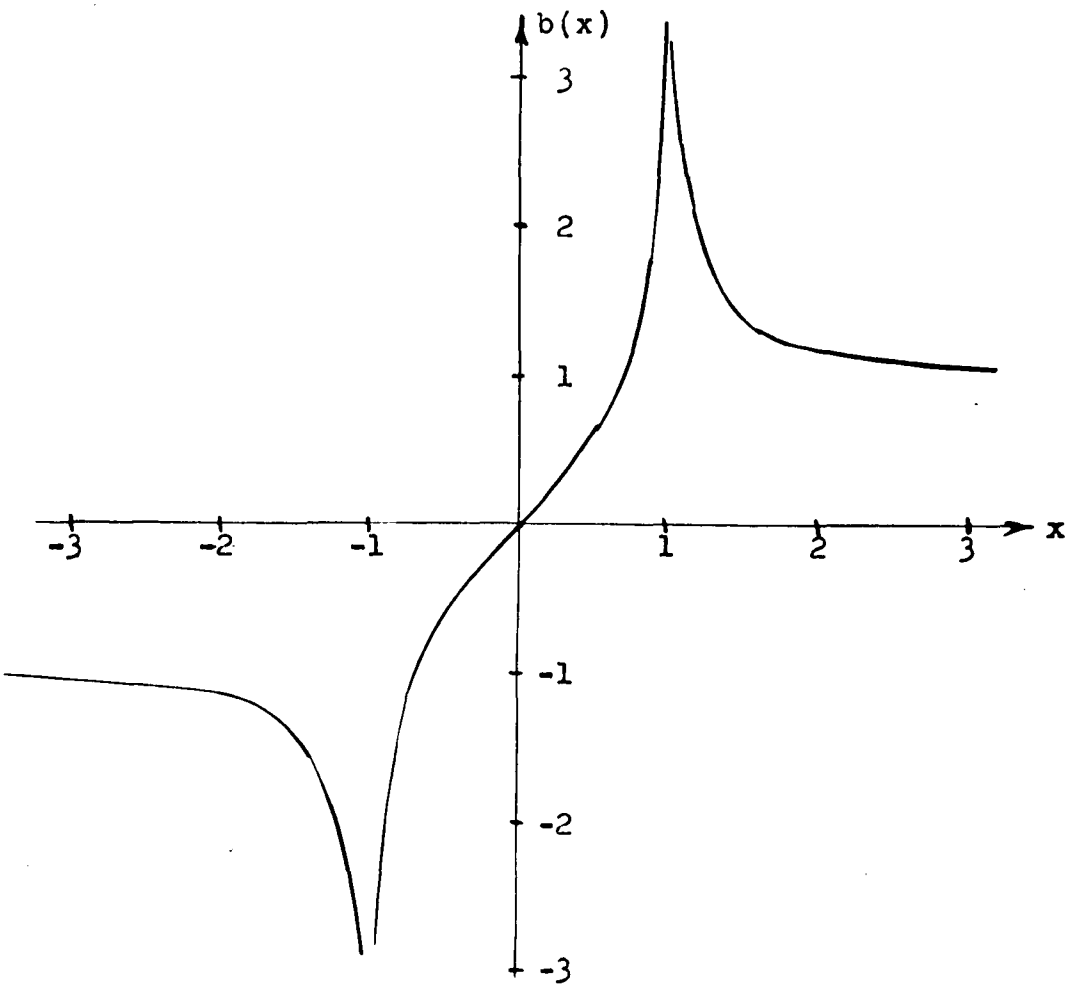


Figure 2.5-4 The Function $b(x)=\frac{x}{\sqrt{|1-x^2|}}$

Further, it may be shown that with this b(x),

2.5-52) $$\lim_{x\to\infty}\int_a^x\left[\frac{2}{N^2}b(\xi)-\frac{3+\delta}{\xi}\right]d\xi=+\infty$$

2.5-53) $$\lim_{x\to-\infty}\int_x^{-a}\left[\frac{2}{N^2}b(\xi)-\frac{3+\delta}{\xi}\right]d\xi=-\infty$$

so all the requirements of Theorem 2 are satisfied, and it follows that $E[x^2]<\infty$.

The variance of b(x) is given by

2.5-54) $$\int_{-\infty}^{+\infty}b^2(x)f_s(x)dx=2\int_0^1\frac{xf_s(x)}{1-x^2}dx+2\int_1^{\infty}\frac{xf_s(x)}{x^2-1}dx$$

These integrals diverge to $+\infty$, so $E[b^2(x)]$ does not exist.

This b(x) is exemplary of an interesting class of functions which we exclude from this study: functions which go to infinity for finite values of x. These functions are of both theoretic and practical interest, but they give rise to singular Fokker-Planck equations, and hence must be handled with special care. Because of this difficulty, we do not study them here, but they are undoubtedly worthy of attention.

Observe that with the function of Figure 2.5-4, if x initially lies on the range $-1 < x < +1$, then it will remain on that range for all time. If the input noise attempts to move x outside these "barriers", it appears that the infinitely-increasing restoring force would return it within the range. The output process x(t) would then be bounded. Almost all physical random processes are in fact bounded. This is usually caused by saturation in amplifiers or analogous phenomena, rather than by a feedback b(x) which goes to infinity. The effect on the output process is much the same, however, and it appears that saturation in the forward path could be represented by an infinite increase in b(x), and thus studied by much the same methods as are used here.

Returning to the problem of assuring the existence of the variance of b(x), there are no doubt several ways to proceed. One way is embodied in the following theorem.

Theorem 5 Sufficient Conditions for Non-Dispersive b(x).

Let b(x) be a measurable function such that

i. $|b(x)| \leq e^{M_1 + M_2|x|}$, $M_2 \geq 0$

ii. $\int_a^{+\infty} \left[\frac{2}{N^2} b(\xi) - 2M_2 - \delta\right] d\xi$ converges to a finite value or to $+\infty$, $\delta > 0$, $a > 0$.

iii. $\int_{-\infty}^{-a} \left[\frac{2}{N^2} b(\xi) - 2M_2 - \delta\right] d\xi$ converges to a finite value or to $-\infty$, $\delta > 0$, $a > 0$.

Then b(x) is non-dispersive.

Proof: First, we show that under the stated conditions, $E[b^2(x)] < \infty$. Obviously,

2.5-55) $$\int_{-\infty}^{+\infty} b^2(x) e^{-B} dx = \int_{-\infty}^{-a} b^2 e^{-B} dx + \int_{-a}^{+a} b^2 e^{-B} dx + \int_{a}^{+\infty} b^2 e^{-B} dx$$

Since b(x) is bounded for finite x, B(x) is finite for finite x, and the second integral is finite. We will show that convergence of the third integral follows from hypothesis (ii). An analogous argument would show that convergence of the first integral follows from hypothesis (iii).

By the same reasoning which has been used before, if $\lim_{x \to \infty} \int_a^x \left[\frac{2}{N^2} b(\xi) - 2M_2 - \delta\right] d\xi$ converges to a finite value or to $+\infty$, then the integral must have a lower bound.

2.5-56) $$\int_a^x \left[\frac{2}{N^2} b(\xi) - 2M_2 - \delta\right] d\xi \geq M_1 > -\infty$$

From this, it follows directly that

2.5-57) $$B(x) \geq M_2 + (2M_2 + \delta) x$$

where $M_2 = M_1 + B(a) - (2M_2 + \delta)a$. By hypothesis (i),

$$|b(x)| \leq e^{M_1 + M_2|x|}$$

2.5-58)
$$\log b^2(x) - 2M_1 \leq 2M_2 x$$

The function logx/x converges to zero for large x. Also, for $x > e^{-1}$, the function if monotonically decreasing. Accordingly, it is possible to select a sufficiently large so that

2.5-59) $$\frac{\log x}{x} \leq \frac{\delta}{1+\varepsilon} \qquad x \geq a, \ \varepsilon > 0$$

From this it follows that

2.5-60) $$(1+\varepsilon)\log x \leq \delta x$$

Substituting (2.5-60) and (2.5-58) into (2.5-57),

2.5-61)
$$\begin{aligned} B(x) &\geq M_2 + \log b^2(x) - 2M_1 + (1+\varepsilon)\log x \\ &\geq M_3 + \log b^2(x) + (1+\varepsilon)\log x \end{aligned}$$

where $M_3 = M_2 - 2M_1$. From this, it follows that

2.5-62) $$b^2(x)\, e^{-B(x)} \leq \frac{e^{-M_3}}{x^{(1+\varepsilon)}} \qquad x \geq a$$

which shows convergence of the third integral of equation (2.5-53). An exactly analogous argument shows the existence of the first integral of the same equation, so the variance of b(x) is finite.

It remains to show that $E[x^2] < \infty$. The postulates of this theorem are stronger than those of Theorem 2. Any b(x) satisfying the requirements of this theorem will satisfy those of Theorem 2. From this it follows that $E[x^2] < \infty$. This completes proof of the theorem.

Figure 2.5-5 shows an example of boundaries for b(x)

which assure that the above theorem is satisfied. Any b(x) which lies completely within the shaded area will be non-dispersive.

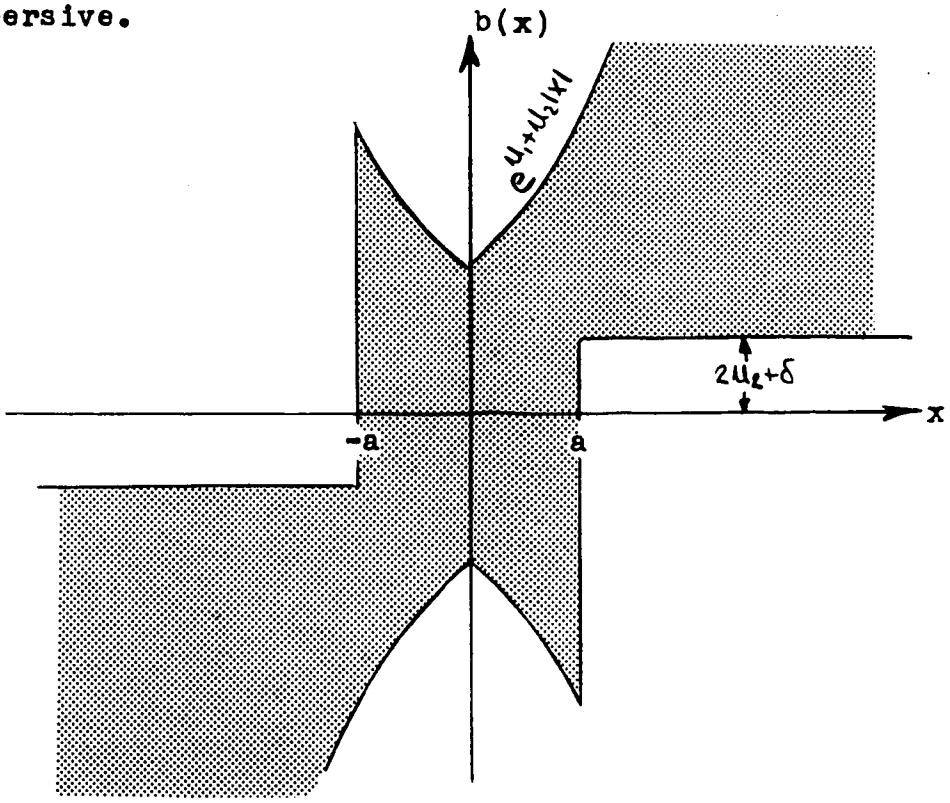


Figure 2.5-5 Sufficient Conditions for Non-Dispersive b(x)

It may be seen from the foregoing analyses, that for non-dispersiveness, b(x) must tend to be positive for large $x > 0$, and negative for large negative x. b(x) may approach zero as |x| increases, provided the approach is sufficiently slow.

Slightly weaker sufficient conditions could no doubt be derived, but those given here should cover most cases of practical interest. Throughout this study we will consider only non-dispersive functions b(x) which are bounded on every finite interval.

Non-dispersiveness clearly has some relation to the older ideas of stability. It seems that it is not exactly equivalent to any of the well-known stability concepts, but the question requires further study.

It would be valuable, for instance, if it were possible to relate properties of the solutions of $\dot{x} + b(x) = 0$ to non-dispersiveness. It seems possible that asymptotic stability in the large for the unforced equation would be sufficient (though not necessary) to assure non-dispersiveness.

2.6 Output Correlations

2.6.1 The General Case

We now consider the problem of determining the second-order properties of the output. Consider the system

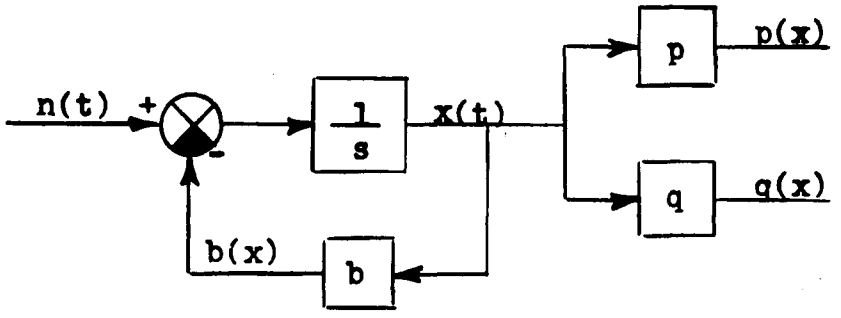


Figure 2.6.1-1 Block Diagram of First-Order System

We wish to determine the cross-correlation between p(x) and q(x) where b(x) is non-dispersive so that x(t) is a stationary second-order random process. If p and q are real processes, the cross-correlation is

$$\Gamma_{pq}(t_1,t_2) = E\left\{ p[x(t_1)]\, q[x(t_2)] \right\}$$

2.6.1-1)

$$= \int_{-\infty}^{+\infty}\int p(x_1)\, q(x_2)\, \rho(x_1,t_1;x_2,t_2)\, dx_1\, dx_2$$

where ρ is the joint density function for $x(t_1)$ and $x(t_2)$. By the Bayes formula, this may be written

2.6.1-2) $$\Gamma_{pq}(t_1,t_2) = \int_{-\infty}^{+\infty}\int p(x_1)\, q(x_2)\, f_s(x_1)\, \rho(x_2,t_2 \mid x_1,t_1)\, dx_1\, dx_2 \qquad t_2 \geq t_1$$

where $\rho(x_2,t_2 \mid x_1,t_1)$ is the conditional density at t_2 under the condition that $x(t_1) = x_1$. But this is exactly the same as the transition density introduced in Section 2.3, so

2.6.1-3) $$\Gamma_{pq}(t_1,t_2)=\iint_{-\infty}^{+\infty} p(x_1)\,q_f(x_2)\,f_s(x_1)\,f_T(x_1,t_1;x_2t_2)\,dx_1dx_2 \qquad t_2\geq t_1$$

where f_T satisfies the Kolmogoroff equations. If, as we now require, the process x(t) is stationary, (has reached its steady-state condition) then this may be written

2.6.1-4) $$\Gamma_{pq}(t)=\iint_{-\infty}^{+\infty} p(x_1)\,q_f(x_2)\,f_s(x_1)\,f_T(x_1,x_2,t)\,dx_1dx_2 \qquad t\geq 0$$

Similarly, we would have

2.6.1-5) $$\Gamma_{qp}=\iint_{-\infty}^{+\infty} q_f(x_1)\,f_s(x_1)\,p(x_2)\,f_T(x_1,x_2,t)\,dx_1dx_2 \qquad t\geq 0$$

Since $\Gamma_{pq}(t)=\Gamma_{qp}(-t)$, the two relations above permit determination of the cross-correlation for $-\infty<t<+\infty$.

If we define

2.6.1-6) $$u(x_1,t;q_f)\equiv\int_{-\infty}^{+\infty} q_f(x_2)\,f_T(x_1,x_2,t)\,dx_2 \qquad t\geq 0$$

then u satisfies the first Kolmogoroff equation and is identical with the u introduced in Section 2.4. We see that:

2.6.1-7) $$\frac{\partial u}{\partial t}-\frac{N^2}{2}\frac{\partial^2 u}{\partial x_1^2}+b(x_1)\frac{\partial u}{\partial x_1}=\int_{-\infty}^{+\infty} q_f(x_2)\left\{\frac{\partial f_T}{\partial t}-\frac{N^2}{2}\frac{\partial^2 f_T}{\partial x_1^2}+b(x_1)\frac{\partial f}{\partial x_1}\right\}dx_2$$

Since f_T satisfies the first Kolmogoroff equation, the $\{\ \}$ expression in the integrand is identically zero. It follows then, that

2.6.1-8) $$\frac{\partial u}{\partial t}=\frac{N^2}{2}\frac{\partial^2 u}{\partial x_1^2}-b(x_1)\frac{\partial u}{\partial x_1}$$

as was asserted. From the definition (2.6.1-6) it may also be seen that

2.6.1-9) $$\lim_{t\to 0} u(x_1, t; q) = q(x_1)$$

so u(x,t;q) is a one-parameter semi-group of transformations. The infinitesmal generator is the operator Ω^* defined in equation (2.3-5). The cross-correlation may now be written

2.6.1-10) $$\Gamma_{pq}(t) = \int_{-\infty}^{+\infty} p(x_1) f_s(x_1) u(x_1, t; q)\, dx_1 \quad t \geq 0$$

In an exactly analogous way, we may define

2.6.1-11) $$v(x_2, t; pf_s) \equiv \int_{-\infty}^{+\infty} p(x_1) f_s(x_1) f_T(x_1, x_2, t)\, dx_1 \quad t \geq 0$$

and show that v satisfies the second Kolmogoroff or Fokker-Planck equation, and that

2.6.1-12) $$\lim_{t\to 0} v(x_2, t; pf_s) = p(x_2) f_s(x_2)$$

v is also a one-parameter semi-group of transformations with infinitesmal generator Ω as defined in equation (2.3-6).

The cross-correlation may equally well be written

2.6.1-13) $$\Gamma_{pq}(t) = \int_{-\infty}^{+\infty} q(x_2) v(x_2, t; pf_s)\, dx_2 \quad t \geq 0$$

Similarly, Γ_{qp} may be written as

2.6.1-14) $$\Gamma_{qp}(t) = \int_{-\infty}^{+\infty} q(x_1) f_s(x_1) u(x_1, t; p)\, dx_1 \quad t \geq 0$$

2.6.1-15) $$\Gamma_{qp}(t) = \int_{-\infty}^{+\infty} p(x_2) v(x_2, t; qf_s)\, dx_2 \quad t \geq 0$$

These equations provide the basis for computation of the second-order properties. In most of what follows, use will be made of the pair (2.6.1-13) and (2.6.1-15). These

are based on the operator v and hence on the Fokker-Planck equation.

In any case, the procedure would be to solve the appropriate partial differential equation, subject to the appropriate initial condition. The solution is then weighted and integrated according to one of the above formulas. So long as this entire process is carried out analytically, there is perhaps little to choose between the two operators. Note, however, that when using the semigroup u, the initial condition is either p(x) or q(x) which are typically unbounded functions,e.g. p(x) = x. With this sort of boundary condition, it will be difficult either to approximate or truncate the problem conveniently.

The semigroup v, on the other hand, uses boundary conditions pf_s or qf_s which are usually bounded, and vanishing for large $|x|$. This will be much more convenient for numerical approximation, so in this work we concentrate on v and on the Fokker-Planck equation.

Extension of the problem to that of Figure 2.1-3 is quite straightforward. The formulae are given below for the sake of completeness, but will not be used in the sequel. We use a method closely similar to that of Laning and Battin [31] p 196, and also adopt their definition of the power- and cross-spectral densities. Consider the system of Figure 2.6.1-2.

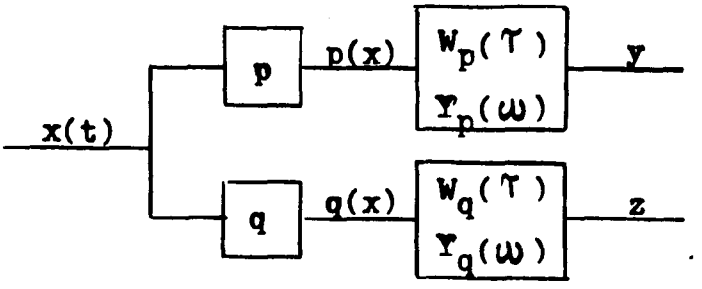


Figure 2.6.1-2 Block Diagram of Linear Filters

The W's are weighting functions, and Y's transfer functions of the time-invariant linear filters. The convolution integral gives

2.6.1-16) $$y(t) = \int_0^\infty W_p(\tau_1)\, p[x(t-\tau_1)]\, d\tau_1$$

2.6.1-17) $$z(t) = \int_0^\infty W_q(\tau_2)\, q[x(t-\tau_2)]\, d\tau_2$$

The cross-correlation between y and z is

$$\Gamma_{yz}(t) = E\{y(t_1)\, z(t_1+t)\}$$

2.6.1-18) $$= \int_0^\infty W_p(\tau_1)\, d\tau_1 \int_0^\infty W_q(\tau_2)\, E\{p[x(t_1-\tau_1)]\, q[x(t_1-t-\tau_2)]\}\, d\tau_2$$

2.6.1-19) $$\Gamma_{yz}(t) = \int_0^\infty W_p(\tau_1)\, d\tau_1 \int_0^\infty W_q(\tau_2)\, \Gamma_{pq}(t+\tau_1-\tau_2)\, d\tau_1 d\tau_2$$

This enables us to calculate Γ_{yz} once Γ_{pq} is known.

It is sometimes more convenient to use the cross-spectrum

$$G_{yz}(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \Gamma_{yz}(t)\, e^{-j\omega t}\, dt$$

2.6.1-20 $$= \frac{1}{\pi} \int_0^\infty W_p(\tau_1)\, d\tau_1 \int_0^\infty W_q(\tau_2)\, d\tau_2 \int_{-\infty}^{+\infty} \Gamma_{pq}(t+\tau_1-\tau_2)\, e^{-j\omega t}\, dt$$

$$\int_0^\infty W_p(\tau_1) e^{+j\omega\tau_1} d\tau_1 \int_0^\infty W_q(\tau_2) e^{-j\omega\tau_2} d\tau_2 \frac{1}{\pi}\int_{-\infty}^{+\infty} \Gamma_{pq}(\xi) e^{-j\omega\xi} d\xi$$

2.6.1-20)
$$= \left[\int_0^\infty W_p(\tau_1) e^{+j\omega\tau_1} d\tau_1\right]\left[\int_0^\infty W_q(\tau_2) e^{-j\omega\tau_2} d\tau_2\right]\left[\frac{1}{\pi}\int_{-\infty}^{+\infty} \Gamma_{pq}(\xi) e^{-j\omega\xi} d\xi\right]$$
$$= \left[\int_0^\infty W_p(\tau_1) e^{+j\omega\tau_1} d\tau_1\right]\left[\int_0^\infty W_q(\tau_2) e^{-j\omega\tau_2} d\tau_2\right]\left[\frac{1}{\pi}\int_{-\infty}^{+\infty} \Gamma_{pq}(\xi) e^{-j\omega\xi} d\xi\right]$$

From this it follows that

2.6.1-21) $$G_{yz}(\omega) = Y_p^*(j\omega)\, Y_q(j\omega)\, G_{pq}(\omega)$$

where the * denotes complex conjugate. This the transfer relation for cross-spectra.

2.6.2 Special Cases

There are several important special cases of the foregoing, and for convenience, we collect the simplified formulae below:

Case 1; p = q - - This occurs whenever we wish to compute an autocorrelation. Equations (2.6.1-13) and (2.6.1-15) become identical, so only one need be considered.

2.6.2-1) $$\Gamma_{pp}(t) = \int_{-\infty}^{+\infty} p(x)\, v(x,t;\, pf_s)\, dx \qquad t \geq 0$$

2.6.2-2) $$\Gamma_{pp}(-t) = \Gamma_{pp}(t)$$

2.6.2-3) $$\Gamma_{yy}(t) = \int_0^\infty W_p(\tau_1) d\tau_1 \int_0^\infty W_p(\tau_2)\, \Gamma_{pp}(t+\tau_1-\tau_2) d\tau_2$$

2.6.2-4) $$G_{yy}(\omega) = |Y_p(j\omega)|^2\, G_{pp}(\omega)$$

Case 2; Autocorrelation of x(t). Here p(x) = q(x) = x. This is a further specialization of Case 1.

2.6.2-5) $$\Gamma_{xx}(t) = \int_{-\infty}^{+\infty} x\, v(x,t;\, xf_s)\, dx \qquad t \geq 0$$

2.6.2-6) $$\Gamma_{xx}(-t) = \Gamma_{xx}(t)$$

2.6.2-7) $$G_{xx}(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \Gamma_{xx}(t) e^{-j\omega t}\, dt = \frac{2}{\pi}\int_{0}^{\infty} \Gamma_{xx}(t)\cos\omega t\, dt$$

Case 3; Autocorrelation of b(x). Let p(x) = q(x) = b(x).

2.6.2-8) $$\Gamma_{bb}(t) = \int_{-\infty}^{+\infty} b(x)\, v(x,t; bf_s)\, dx \qquad t \geq 0$$

2.6.2-9) $$\Gamma_{bb}(-t) = \Gamma_{bb}(t)$$

2.6.2-10) $$G_{bb}(\omega) = \frac{2}{\pi}\int_{0}^{\infty} \Gamma_{bb}(t)\cos\omega t\, dt$$

2.6.3 Derivative Relationships Among the Correlations

There are several interesting relationships among the correlations appearing in the first-order problem. We state these relationships in the form of a theorem, and then proceed to develop them.

Theorem 6 Relations Among the Correlations

If b(x) satisfies the following:

i. b(x), b'(x) and b"(x) are bounded

ii. $$\lim_{x\to+\infty} \int_{a}^{x} \left[\frac{2}{N^2} b(\xi) - \delta\right] d\xi = L_1 \qquad -\infty < L_1 \leq +\infty\,,\ \delta > 0,\ a > 0$$

iii. $$\lim_{x\to-\infty} \int_{x}^{-a} \left[\frac{2}{N^2} b(\xi) - \delta\right] d\xi = L_2 \qquad -\infty \leq L_1 < +\infty \qquad \delta > 0, a > 0$$

then if follows that:

a. b(x) is non-dispersive

b. $$\Gamma_{bx}(t) = \Gamma_{xb}(t) \qquad -\infty < t < +\infty$$

c. $$\Gamma'_{bx}(t) = \Gamma'_{xb} = \begin{cases} -\Gamma_{bb}(t) & 0 \leq t < +\infty \\ \Gamma_{bb}(t) & -\infty < t < 0 \end{cases}$$

d. $$\Gamma'_{xx}(t) = \begin{cases} -\Gamma_{xb}(t) = -\Gamma_{bx}(t) & 0 \le t < \infty \\ \Gamma_{xb}(t) = \Gamma_{bx}(t) & -\infty < t < 0 \end{cases}$$

e. $$\Gamma''_{xx}(t) = \Gamma_{bb}(t) \quad , -\infty < t < +\infty$$

Derivatives in these formulas are understood to be derivatives on the right. All derivatives here exist in the ordinary sense everywhere except possibly at the origin. At the origin, both right- and left-hand derivatives exist, but they may be unequal.

Proof: We begin with statement of a result of Mandl [37].

Lemma: If $h(x)$ is any once-differentiable function defined on $-\infty < x < +\infty$, such that $h(x)f_s(x) \in L_1(-\infty, \infty)$, then $v(x,t;hf_s) = f_s(x)u(x,t;h)$, where u and v are the semi-groups generated by the first and second Kolmogoroff equations respectively. The proof of this lemma will not be given here. It is quite elementary, and is given by Mandl [37].

Turning now to proof of the theorem, it follows from hypothesis (ii) that

2.6.3-1) $$\int_a^x \left[\frac{2}{N^2} b(\xi) - \delta \right] d\xi \ge M_1 > -\infty \quad , \quad x \ge a$$

from which it follows that

2.6.3-2) $$B(x) \ge M_2 + \delta x$$

where $M_2 = M_1 + B(a) - \delta a$, and

2.6.3-3) $$e^{-B(x)} \le e^{-M_2} e^{-\delta x} \qquad x \ge a$$

A corresponding result for negative x follows from hypothesis (iii). The two may be combined to give

2.6.3-4) $$e^{-B(x)} \le M_3 e^{-\delta|x|} \qquad |x| \ge a \ge 0$$

$$f_s(x) = e^{-B(x)} \Big/ \int_{-\infty}^{+\infty} e^{-B(s)} ds \le M_4 e^{-\delta|x|} \qquad |x| \ge a$$

where $M_4 = M_3 / \int_{-\infty}^{+\infty} e^{-B(s)} ds$

For any function h such that $h(x)f_s(x) \in L_1(-\infty, +\infty)$, the semi-group $v(x,t;hf_s)$ may be written in terms of the transition probability density (Green's function) in accordance with equation (2.6.1-11) as

2.6.3-5) $$v(x,t;hf_s) = \int_{-\infty}^{+\infty} h(\xi) f_s(\xi) f_T(\xi, x, t) d\xi \quad t \geq 0$$

Since f_s and f_T are non-negative, if $|h| \leq H$,

2.6.3-6) $$|v(x,t;hf_s)| \leq \int_{-\infty}^{+\infty} |h(\xi)| f_s(\xi) f_T(\xi, x, t) d\xi \leq H \int_{-\infty}^{+\infty} f_s(\xi) f_T(\xi, x, t) d\xi$$

The last integral is simply $v(x,t;f_s)$. Since f_s is the steady-state distribution, it is invariant under transformation by v for any t, i.e. $v(x,t;f_s) = f_s(x)$, $t \geq 0$. Thus (2.6.3-6) may be written

2.6.3-7) $$|v(x,t;hf_s)| \leq Hf_s(x) \leq M_4 e^{-\delta|x|}$$

for any bounded h.

Now consider the cross-correlation $\Gamma_{bx}(t)$, which, by application of equation (2.6.1-13) may be written

2.6.3-8) $$\Gamma_{bx}(t) = \int_{-\infty}^{+\infty} x\, v(x,t;bf_s)\, dx \quad t \geq 0$$

Differentiating with respect to t,

2.6.3-9) $$\Gamma'_{bx}(t) = \int_{-\infty}^{+\infty} x \frac{\partial v(x,t;bf_s)}{\partial t} dx = \int_{-\infty}^{+\infty} x \Omega\, v(x,t;bf_s) dx$$

Before proceeding, we show that differentiation under the integral sign is permissible. Clearly the integrand xv is continuous in x and t. It is also easily seen that $\int_{-\infty}^{+\infty} xv(x,t;bf_s)dx$ converges. Since b is bounded,

$|v(x,t;bf_s)| \leq M_4 e^{-\delta|x|}$, $|x| \geq a$, and $|x||v(x,t;bf_s)| \leq M_4 |x| e^{-\delta|x|}$, $|x| \geq a$, which is sufficient to assure convergence of the integral, uniformly in t.

It remains only to show that $\int_{-\infty}^{+\infty} x \Omega v dx$ converges uniformly in t. This is done in a closely similar way. Since v is a strongly continuous semi-group, $\frac{\partial}{\partial t} v(x,t;bf_s) = \Omega v(x,t;bf_s) = v(x,t;\Omega bf_s)$ (see, e.g. Hille and Phillips[24], Theorem 10.3.3). Accordingly, equation (2.6.3-9) becomes

2.6.3-10) $$\Gamma_{bx}'(t) = \int_{-\infty}^{+\infty} x\, v(x,t;\Omega bf_s)\, dx \quad t \geq 0$$

It may easily be shown that

2.6.3-11) $$\Omega bf_s = N^2 b' f_s' + \frac{N^2}{2} b'' f_s + bb' f_s = \left(\frac{N^2}{2} b'' - bb'\right) f_s$$

f_s is always bounded. By hypothesis (i), b, b' and b'' are bounded. Thus Ωbf_s is bounded also. It then follows from (2.6.3-7) that $v(x,t;\Omega bf_s)$ has an exponentially decreasing bound which is independent of t. This assures that the integral (2.6.3-10) converges uniformly in t, $0 \leq t < \infty$. This completes the proof that differentiation under the integral is valid.

Now integrate equation (2.6.3-9) by parts.

2.6.3-12)
$$\Gamma_{bx}'(t) = \int_{-\infty}^{+\infty} x \frac{\partial}{\partial x}\left\{\frac{N^2}{2}\frac{\partial v}{\partial x} + b(x)v\right\} dx$$
$$= x\left[\frac{N^2}{2}\frac{\partial v}{\partial x} + bv\right]_{-\infty}^{+\infty} - \frac{N^2}{2} v\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} b(x) v(x,t;bf_s) dx$$

Since b is bounded, $v(x,t;bf_s)$ decreases exponentially with x for large $|x|$, and the second term is zero.

Now consider the integral

2.6.3-13) $$w(x,t;bf_s) = -\int_x^{+\infty} \Omega v(\xi,t;bf_s) d\xi = -\int_x^{+\infty} v(\xi,t;\Omega bf_s) d\xi$$

If $x \geq a$,

2.6.3-14) $$|w(x,t;bf_s)| \leq \int_x^{+\infty} |v(\xi,t;\Omega bf_s)| d\xi \leq \int_x^{+\infty} M_4 e^{-\delta\xi} d\xi$$

2.6.3-15) $$|w(x,t;bf_s)| \leq M_5 e^{-\delta x} \quad x \geq a$$

where $M_5 = M_4/\delta$. Equation (2.6.3-13) may be written

2.6.3-16) $$w(x,t;bf_s) = -\int_x^{+\infty} \frac{\partial}{\partial\xi}\left\{\frac{N^2}{2}\frac{\partial v}{\partial\xi} + b(\xi)v\right\} d\xi$$
$$= \frac{N^2}{2}\frac{\partial v(x,t;bf_s)}{\partial x} + b(x)v(x,t;bf_s)$$

This quantity goes to zero exponentially in x for large x, so $\lim_{x\to\infty} x\left[\frac{N^2}{2}\frac{\partial v}{\partial x} + bv\right] = 0$, also. The same can be shown as $x \to -\infty$, so the first term of equation (2.6.3-12) vanishes and

2.6.3-17) $$\Gamma'_{bx}(t) = -\int_{-\infty}^{+\infty} b(x)\, v(x,t;bf_s) dx \quad t \geq 0$$

By use of equation (2.6.1-13), the integral on the right may be interpreted as $\Gamma_{bb}(t)$.

2.6.3-18) $$\Gamma'_{bx}(t) = -\Gamma_{bb}(t) \quad t \geq 0$$

From equation (2.6.1-10), we have

2.6.3-19) $$\Gamma_{xb}(t) = \int_{-\infty}^{+\infty} x f_s\, u(x,t,b)\, dx$$

From the lemma, and equation (2.6.3-8),

2.6.3-20) $$\Gamma_{xb}(t) = \int_{-\infty}^{+\infty} x f_s\, u(x,t;b) = \int_{-\infty}^{+\infty} x\, v(x,t;bf_s) dx = \Gamma_{bx}(t), \; t \geq 0$$

Since $\Gamma_{xb}(t) = \Gamma_{bx}(-t)$, this shows that $\Gamma_{bx}(t)$ and $\Gamma_{xb}(t)$ are identical $-\infty < t < +\infty$, and both are even functions of t. This is not true of cross-correlations in general. This

proves result (b) of the theorem.

The time derivatives $\Gamma_{xb}'(t)$, $\Gamma_{bx}'(t)$ are then identical, odd functions of time. From equation (2.6.3-18),

2.6.3-21) $$\Gamma_{bx}'(t) = -\Gamma_{bx}'(-t) = -\Gamma_{bb}(t) = -\Gamma_{bb}(-t), \quad t \geq 0$$

from which

2.6.3-22) $$\Gamma_{bx}'(t) = \Gamma_{bb}(t) \quad t < 0$$

which shows result (c) of the theorem.

To prove result (d), we start from the expression

2.6.3-23) $$\Gamma_{xx}(t) = \int_{-\infty}^{+\infty} x\, v(x,t;x f_s)\, dx \quad t \geq 0$$

which comes from equation (2.6.2-5). We follow a procedure exactly analogous to that immediately above, and obtain the results

2.6.3-24) $$\Gamma_{xx}'(t) = -\Gamma_{xb}(t) \quad t \geq 0$$

2.6.3-25) $$\Gamma_{xx}'(t) = \Gamma_{xb}(t) \quad t < 0$$

Differentiating these equations,

2.6.3-26) $$\Gamma_{xx}''(t) = -\Gamma_{xb}'(t) \quad t \geq 0$$

2.6.3-27) $$\Gamma_{xx}''(t) = \Gamma_{xb}'(t) \quad t < 0$$

Substituting equations (2.6.3-18) and (2.6.3-22) into the above,

2.6.3-28) $$\Gamma_{xx}''(t) = \Gamma_{bb}(t) \quad -\infty < t < +\infty$$

which is result (d) of the theorem.

Result (a) is obvious, as the conditions of this theorem contain those of theorem 5. This completes the proof of Theorem 6.

It seems certain that stronger theorems along these lines could be proved. Postulate (i), requiring that b(x)

be bounded with two bounded derivatives, appears to be much too strong a requirement. It is not necessary. In a later section, an example will be considered where b(x) is discontinuous, but the conclusions of the theorem are valid.

There are several implications of the theorem, which we now consider. The first of these is the fact that $\Gamma'_{xx}(t)$ does not exist at the origin. To see this, we first calculate $\Gamma_{bx}(0)$ from equations (2.6.3-8) and (2.5-8).

$$\Gamma_{bx}(0^+) = \int_{-\infty}^{+\infty} x\, b(x) f_s(x)\, dx = -\frac{N^2}{2}\int_{-\infty}^{+\infty} x \frac{\partial f_s}{\partial x}\, dx$$

2.6.3-29)

$$= -\frac{N^2}{2}\left\{ x f_s \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f_s(x)\, dx \right\} = -\frac{N^2}{2}$$

This result is independent of b(x). It was shown, following equation (2.6.3-20) that $\Gamma_{bx}(t)$ is an even function of t. Therefore, $\Gamma_{bx}(0^+) = \Gamma_{bx}(0^-) = \Gamma_{bx}(0) = \frac{N^2}{2}$. From conclusion (d) of the theorem, then,

2.6.3-30) $$\Gamma'_{xx}(0^+) = -\frac{N^2}{2} \quad ; \quad \Gamma'_{xx}(0^-) = +\frac{N^2}{2}$$

The right- and left-hand derivatives exist, but are unequal, so $\Gamma'_{xx}(t)$ does not exist at t = 0. From this, it follows (Loeve [36] p 470) that x(t) is not differentiable in mean square, which is hardly surprising.

It may also be shown that, under the conditions of the theorem, $\lim_{t\to\infty} \Gamma_{bx}(t) = 0$. To do this, we start from equation (2.6.3-8). Making use of Schwartz' inequality,

2.6.3-31) $$\Gamma_{bx}^2(t) \le \left[\int_{-\infty}^{+\infty} |x|\,|v(x,t;bf_s)|\,dx\right]^2 \le \int_{-\infty}^{+\infty} x^2 |v(x,t;bf_s)|\,dx \int_{-\infty}^{+\infty} |v(x,t;bf_s)|\,dx$$

Under the conditions of the theorem, it is easily shown that the first integral on the right is bounded, uniformly in t. If the bound is called M, then

2.6.3-32) $$\lim_{t\to\infty} \Gamma_{bx}^2(t) \le M \lim_{t\to\infty} \int_{-\infty}^{+\infty} |U(x,t;bfs)|\,dx$$

Mandl's [37] Theorem 5 shows that the right side is zero. In view of this, it follows further, by result (d) of the theorem,

2.6.3-33) $$\lim_{t\to\infty} \Gamma'_{xx}(t) = 0$$

An exactly analogous procedure shows that

2.6.3-34) $$\lim_{t\to\infty} \Gamma_{bb}(t) = \lim_{t\to\infty} \Gamma''_{xx}(t) = 0$$

Since b(x) is non-dispersive, the random process b[x(t)] will be second-order, i.e. $|\Gamma_{bb}(t)| < \infty$, $-\infty < t < +\infty$. From the theorem, it follows that Γ'_{xb}, Γ'_{bx}, Γ'_{xx}, and Γ''_{xx} will be finite.

The relationship $\Gamma''_{xx}(t) = \Gamma_{bb}(t)$ implies a corresponding relationship between the spectrum of b and the continuous portion of the spectrum of x. Since Γ_{xx} is bounded, the fact that $\lim_{t\to\infty} \Gamma'_{xx}(t) = 0$ implies that $\Gamma_{xx}(t)$ approaches a limit as $t \to \infty$. Therefore, the spectrum of x can have no discontinuities except possibly at zero frequency. Accordingly, if we define $\bar{x} = x - E[x]$, the spectrum of $\bar{x}$ will have no discontinuities. We assume that the spectral density of $\bar{x}$ exists everywhere, and is denoted by $G_{xx}(\omega)$. The spectral density of b is given by

2.6.3-35) $$G_{bb}(\omega) = \frac{2}{\pi}\int_0^\infty \Gamma_{bb}(t)\cos\omega t\,dt = \frac{2}{\pi}\int_0^\infty \Gamma''_{xx}(t)\cos\omega t\,dt$$

Integrating by parts twice, and using the facts that

$$\lim_{t\to\infty} \Gamma'_{xx}(t) = 0 \quad , \quad \Gamma'_{xx}(0^+) = -\frac{N^2}{2} \quad , \quad \lim_{t\to\infty} \Gamma_{\bar{x}\bar{x}}(t) = 0 \quad ,$$

2.6-36) $$G_{bb}(\omega) = \frac{N^2}{\pi} - \omega^2 G_{xx}(\omega)$$

which is the desired relationship.

If it happens that $\Gamma_{bb}(t) \geq 0$, $-\infty < t < +\infty$, then certain further results follow..

2.6.3-37) $\Gamma_{xx}''(t) \geq 0 \quad , t>0, t<0$

2.6.3-38) $\Gamma_{bx}'(t) \leq 0 \quad , t \geq 0$

2.6.3-39) $\Gamma_{bx}'(t) \geq 0 \quad , t<0$

2.6.3-40) $\Gamma_{bx}(t) \geq 0 \quad -\infty < t < +\infty$

2.6.3-41) $\Gamma_{xx}'(t) \leq 0 \quad t \geq 0$

2.6.3-42) $\Gamma_{xx}'(t) \geq 0 \quad t<0$

The derivative relations among the various correlations are of considerable practical use. Once $\Gamma_{xx}(t)$ has been determined, all other correlations may be found from it by differentiation.

2.6.4 Tortrat's Result

By using methods similar to the foregoing, it is possible to derive Tortrat's [49] result. He used a different procedure, but the derivation given here shows the adjoint relationship between his method and ours. In equations (2.6.1-10) and (2.6.1-14) we specialize to p = q.

2.6.4-1) $$\Gamma_{pp}(t) = \int_{-\infty}^{+\infty} p(x) f_s(x) u(x,t;p)dx \quad t \geq 0$$

2.6.4-2) $$\Gamma_{pp}(-t) = \Gamma_{pp}(t)$$

We choose to treat the spectral density as follows:

2.6.4-3) $$G_{pp}(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\Gamma_{pp}(t)e^{-j\omega t}dt = \frac{1}{\pi}\int_{-\infty}^{0}\Gamma_{pp}e^{-j\omega t}dt + \frac{1}{\pi}\int_{0}^{+\infty}\Gamma_{pp}e^{-j\omega t}dt$$

Now substitute -t for t in the first integral.

2.6.4-4) $$G_{pp}(\omega) = \frac{1}{\pi}\int_{0}^{\infty}\Gamma_{pp}(t)e^{j\omega t} + \frac{1}{\pi}\int_{0}^{\infty}\Gamma_{pp}(t)e^{-j\omega t}dt$$

These two integrals are clearly complex conjugates, so the

sum is twice the real part of either.

$$2.6.4\text{-}5) \quad G_{pp}(\omega) = 2\,\mathrm{Re}\left\{\frac{1}{\pi}\int_0^{\infty} T_{pp}(t)e^{-j\omega t}dt\right\}$$

This is Caughey and Dienes' ([7] equation 14) equation for calculation of the spectrum.

Now the <u>resolvent</u> of a semigroup is its Laplace transformation. (See, e.g. Hille and Phillips [26] Lemma 15.4.3, p 442)

$$2.6.4\text{-}6) \quad R(x,s;\Omega^*)q = \int_0^{\infty} e^{-st}u(x,t;q)\,dt$$

Taking the Laplace transform of both sides of equation (2.6.4-1), we have

$$2.6.4\text{-}7) \quad \int_0^{\infty} T_{pp}(t)e^{-st}dt = \int_{-\infty}^{+\infty} p(x)f_s(x)R(x,s;\Omega^*)p\,dx$$

Setting $s = j\omega$ gives the expression necessary for (2.6.4-5) which then becomes

$$2.6.4\text{-}8) \quad G_{pp}(\omega) = 2\,\mathrm{Re}\left\{\frac{1}{\pi}\int_{-\infty}^{+\infty} p(x)f_s(x)R(x,j\omega;\Omega^*)p\,dx\right\}$$

which is Tortrat's result, aside from certain normalizing factors. Tortrat states that $R(x,j\omega;\Omega^*)$ will exist except for those frequencies ω_c at which discontinuities occur in the spectrum, i.e. x(t) contains periodic terms of frequency ω_c. In accordance with the argument following equation (3.6.3-34), x(t) can have no such periodic terms except possibly for $\omega_c = 0$. With this exception, then, the resolvent transformation should be well-defined.

It may be seen, then, that Tortrat's method is an

adjoint of ours. His formula follows from the first Kolmogoroff equation, ours from the second. In principle the two procedures are equivalent, but, as mentioned above, practical considerations favor the second equation.

2.6.5 Example 1: The Ornstein-Uhlenbeck Process

As an example of how the procedure may be applied, we now consider the case where $b(x) = \beta x$, $\beta > 0$, and we wish to calculate the autocorrelation of x. As mentioned earlier this is the Ornstein-Uhlenbeck process, for which a number of results are known, including the autocorrelation (Doob [14] p 353). The objective here is to show that our method gives the same result as the earlier linear theory. The Ornstein-Uhlenbeck case is the only problem of the class now being studied, for which the autocorrelation is well-known. It seems advisable, then, to use it as a check.

This particular b(x) satisfies all the foregoing conditions for existence, uniqueness and non-dispersive character of the output. Putting it into equation (2.5-6), the steady-state distribution is

2.6.5-1) $$f_s(x) = \frac{\exp\left\{-\frac{x^2}{2\left(\frac{N^2}{2\beta}\right)}\right\}}{\sqrt{2\pi}\sqrt{\frac{N^2}{2\beta}}}$$

which is a Gaussian distribution of zero mean, variance $N^2/2\beta$, a result which may be easily verified by other methods. The next step is to solve the partial differential equation

2.6.5-2) $$\frac{\partial v}{\partial t} = \frac{N^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x}\left[\beta x v\right]$$

subject to the boundary condition

2.6.5-3) $$v(x,0;xf_s) = \frac{x \exp\left\{-\frac{x^2}{2\left(\frac{N^2}{2\beta}\right)}\right\}}{\sqrt{2\pi}\sqrt{\frac{N^2}{2\beta}}}$$

This is a relatively elementary problem in partial differential equations, and the details of solution are given in Appendix A. The result (equation (A-11)) is

2.6.5-4) $$v(x,t;xf_s) = \sqrt{\frac{\beta}{N^2\pi}}\, x e^{-\beta t} \exp\left\{-\frac{\beta x^2}{N^2}\right\}$$

Substituting this into equation (2.6.2-5), the autocorrelation is

2.6.5-5) $$\Gamma_{xx}(t) = \sqrt{\frac{\beta}{N^2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-\beta t} \exp\left\{-\frac{\beta x^2}{N^2}\right\} dx \quad t \geq 0$$

2.6.5-6) $$\Gamma_{xx}(t) = \frac{N^2}{2\beta} e^{-\beta t} \quad t \geq 0$$

2.6.5-7) $$\Gamma_{xx}(-t) = \Gamma_{xy}(t)$$

2.6.5-8) $$\Gamma_{xy}(t) = \frac{N^2}{2\beta} e^{-\beta|t|} \quad -\infty < t < +\infty$$

This agrees with the result obtained by conventional linear methods.

2.6.6 Example 2: A Square-Law Detector Problem

Consider the equation of the preceeding example followed by a square-law detector.

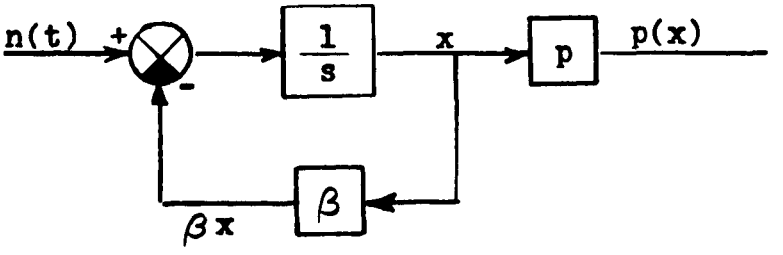


Figure 2.6.6-1 Block Diagram of Detector Problem

where $p(x) = ax^2$. To calculate the autocorrelation of p(x), the output of the detector, equation (2.6.2-1) will be used. To this end, it is necessary to solve

2.6.6-1) $$\frac{\partial v}{\partial t} = \frac{N^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x}[\beta x v]$$

subject to the boundary condition

2.6.6-2) $$v(x,0;pf_s) = \frac{ax^2}{\sqrt{2\pi}\sqrt{\frac{N^2}{2\beta}}}\exp\left\{-\frac{x^2}{2\left(\frac{N^2}{2\beta}\right)}\right\}$$

The partial differential equation here is identical with that of Example 1, but the boundary condition is different. Details of the solution are given in Appendix A, and the result is (equation (A-15)):

2.6.6-3) $$v(x,t;ax^2 f_s) = \frac{aN}{2\sqrt{\pi\beta}}e^{-\frac{\beta x^2}{N^2}}\left\{1 - e^{-2\beta t} + 2\frac{\beta}{N^2}x^2 e^{-2\beta t}\right\}$$

From equation (2.6.2-1) we have

2.6.6-4) $$\Gamma_{pp}(t) = \int_{-\infty}^{+\infty} a x^2 v(x,t; a x^2 f_s)\,dx$$

Substituting (2.6.6-3) into this, and performing the nececessary quadratures,

2.6.6-5) $$\Gamma_{pp}(t) = a^2\left(\frac{N^2}{2\beta}\right)^2 + 2a^2\left(\frac{N^2}{2\beta}\right)^2 e^{-2\beta t} \qquad t \geq 0$$

As mentioned earlier, the variance of x is $\sigma_x^2 = \frac{N^2}{2\beta}$, so the equation may be written

2.6.6-6) $$\Gamma_{pp}(t) = a^2\sigma_x^4 + 2a^2\Gamma_{xx}^2(t)$$

which agrees with the result calculated by other methods. See, e.g. Davenport and Root [13] p 255, equation (12-24b).

2.7 The Fokker-Planck Equation for Discontinuous b(x).

We now consider functions b(x) which are bounded on every closed interval of x, and continuous and differentiable except possibly at a finite number of points x_i. We allow finite discontinuities in b(x), i.e. if x_c is a point of discontinuity, then

2.7-1) $$\left.\begin{aligned} \lim_{\delta \to 0} b(x_c + \delta) &= b(x_c^+) \\ \lim_{\delta \to 0} b(x_c - \delta) &= b(x_c^-) \end{aligned}\right\} \quad \delta > 0$$

where $b(x_c^+)$ and $b(x_c^-)$ are both finite but unequal.

No rigorous theory now exists for this situation. Even existence and uniqueness of solutions of the original APW equation have not been shown, as discontinuous b's do not satisfy a Lipschitz condition. Furthermore, the meaning

of the Fokker-Planck equation is obscure in this case because some of the terms are undefined at x_c.

No rigorous discussion of these questions is included here, either, but a possible treatment of the Fokker-Planck equation is proposed on the basis of an heuristic derivation. This treatment is a reasonable one, and appears to fit the known facts.

In the following, we will assume there is only one point of discontinuity, i.e. b(x) is continuous and differentiable everywhere except at x_c. The extension to any finite number of points of discontinuity will be obvious.

In traditional derivations of the Fokker-Planck equation, continuity of f and $\frac{\partial f}{\partial x}$ are assumed, with $\frac{\partial^2 f}{\partial x^2}$ finite, though not necessarily continuous. For discontinuous b(x) we retain the assumption that f will be continuous, but assume that $\frac{\partial f}{\partial x}$ will be finite but not necessarily continuous. At present, this can be justified only on the basis of physical intuition. If f is discontinuous, this would mean that it is distinctly more probable that the process lies on one side of the discontinuity than on the other. It seems inevitable that the action of the Gaussian forcing function would be to smooth out such irregularities, even if they existed initially, so that only on the boundary could such a discontinuity occur.

We ignore the problem of existence and uniqueness of solutions to the APW equation, or to be more exact, we assume it solved.

In order to obtain the desired result, we make use of a modification of the Bernstein derivation (see Levy [32] p 64 ff) of the Fokker-Planck equation. Consider a class of functions $\varphi(x)$ which have: a) three finite derivatives, b) φ''' uniformly bounded, c) identically zero outside some finite interval. Define

$$2.7\text{-}2) \qquad M(t) = E\{\varphi(x)\} = \int_{-\infty}^{+\infty} \varphi(x) f(x,t)\, dx$$

If the time derivative and expectation commute,

$$2.7\text{-}3) \qquad M'(t) = \frac{d}{dt}\left\{E[\varphi(x)]\right\} = E\frac{d}{dt}\varphi(x) = \int_{-\infty}^{+\infty} \varphi(x) \frac{\partial f}{\partial t}\, dx$$

The time derivative of M may be computed another way. By definition,

$$2.7\text{-}4) \qquad \frac{d\varphi}{dt} = \lim_{\Delta t \to 0} \frac{\varphi[x(t+\Delta t)] - \varphi[x(t)]}{\Delta t}$$

We define $x(t + \Delta t) = x(t) + \Delta x$, where Δx is the change in x during the time increment Δt. If three derivatives of φ exist, then $\varphi[x(t)+\Delta x]$ may be expanded in a Taylor's series.

$$2.7\text{-}5) \qquad \varphi[x(t)+\Delta x] = \varphi[x(t)] + \varphi'\Delta x + \tfrac{1}{2}\varphi''\Delta x^2 + \frac{\theta K}{6}(\Delta x)^3, \quad |\theta| \leq 1$$

where K is the bound on φ''', i.e. $|\varphi'''| \leq K$. From equation (2.1-2),

$$2.7\text{-}6) \qquad \Delta x = \int_t^{t+\Delta t} b[x(t)]dt + \int_t^{t+\Delta t} d\eta$$

The second integral is a Gaussian random variable with zero mean and variance $N^2\,\Delta t$. Take the expectation of (2.7-4) and interchange the order of the limit and the expectation.

$$2.7\text{-}7)\quad E\left\{\frac{d\varphi}{dt}\right\} = \lim_{\Delta t \to 0} E\left\{ \frac{\varphi[x(t)+\Delta x] - \varphi[x(t)]}{\Delta t} \right\}$$

Making use of (2.7-5) and (2.7-6), it may be shown that

$$2.7\text{-}8)\quad E\left\{ \frac{d\varphi}{dt} \right\} = \int_{-\infty}^{+\infty} \left\{ -b(x)\varphi' + \tfrac{1}{2}\varphi'' N^2 \right\} f(x,t)\,dx$$

Up to this point, the development is the same as Bernstein's. He proceeds at this point to integrate (2.7-8) by parts. If b'(x) is not finite, however, this is not possible. Accordingly we proceed as follows. Recall that we have assumed a single point of discontinuity, at x_c. Equation (2.7-8) may clearly be written

$$2.7\text{-}9)\quad E\left\{ \frac{d\varphi}{dt} \right\} = \int_{-\infty}^{x_c} \left\{ -b(x)\varphi' + \frac{N^2}{2}\varphi'' \right\} f(x,t)\,dx + \int_{x_c}^{+\infty} \left\{ -b(x)\varphi' + \frac{N^2}{2}\varphi'' \right\} f\,dx$$

In each of these ranges of integration, b'(x) is finite, so each integral may be integrated by parts with the result

$$\begin{aligned} E\left\{ \frac{d\varphi}{dt} \right\} &= \varphi(x_c)\left[b(x_c^+) f(x_c^+, t) - b(x_c^-) f(x_c^-, t) \right] \\ &\quad + \int_{-\infty}^{x_c} \varphi(x) \frac{\partial}{\partial x}[b(x) f]\,dx + \int_{x_c}^{+\infty} \varphi(x) \frac{\partial}{\partial x}[b(x) f]\,dx \\ &\quad + \varphi(x_c) \frac{N^2}{2}\left[\frac{\partial f}{\partial x}(x_c^+, t) - \frac{\partial f}{\partial x}(x_c^-, t) \right] \\ &\quad + \int_{-\infty}^{x_c} \varphi \frac{N^2}{2} \frac{\partial^2 f}{\partial x^2}\,dx + \int_{x_c}^{+\infty} \varphi \frac{N^2}{2} \frac{\partial^2 f}{\partial x^2}\,dx \end{aligned} \qquad 2.7\text{-}10)$$

Equating this to (2.7-3),

$$\int_{-\infty}^{x_c} Q(x)\left\{\frac{N^2}{2}\frac{\partial^2 f}{\partial x^2}+\frac{\partial}{\partial x}[bf]-\frac{\partial f}{\partial t}\right\}dx+\int_{x_c}^{+\infty} Q\left\{\frac{N^2}{2}\frac{\partial^2 f}{\partial x^2}+\frac{\partial}{\partial x}[bf]-\frac{\partial f}{\partial t}\right\}dx$$

$$+Q(x_c)\left\{b(x_c^+)f(x_c^+,t)-b(x_c^-)f(x_c^-,t)\right.$$

2.7-11)

$$\left.+\frac{N^2}{2}\frac{\partial f}{\partial x}(x_c^+,t)-\frac{N^2}{2}\frac{\partial f}{\partial x}(x_c^-,t)\right\}=0$$

In order for this equation to be satisfied for all admissible $Q(x)$, it is necessary that

2.7-12) $$\frac{\partial f}{\partial t}=\frac{N^2}{2}\frac{\partial^2 f}{\partial x^2}+\frac{\partial}{\partial x}[bf] \qquad x<x_c \text{ and } x>x_c$$

2.7-13) $$\frac{N^2}{2}\frac{\partial f}{\partial x}(x_c^+,t)+b(x_c^+)f(x_c^+,t)=\frac{N^2}{2}\frac{\partial f}{\partial x}(x_c^-,t)+b(x_c^-)f(x_c^-,t)$$

Thus the Fokker-Planck equation holds everywhere except for $x = x_c$. At that point, (2.7-13) must be satisfied. Clearly the quantity $\frac{N^2}{2}\frac{\partial f}{\partial x}+b(x)f$ must be continuous everywhere. If, as we assumed, f is continuous everywhere, then discontinuity of b(x) requires discontinuity of $\frac{\partial f}{\partial x}$.

Equation (2.7-13) is precisely Wishner's [55], [56] "conservation of probability" condition which he derived by an entirely different procedure which, in fact, amounts to little more than a plausibility argument.

This density continuity condition may be extended to the functions v. From equation (2.6.1-11), we may see that

2.7-14) $$\frac{N^2}{2}\frac{\partial v}{\partial x_2}+b(x_2)v=\int_{-\infty}^{+\infty} p(x_1)f_5(x_1)\left\{\frac{N^2}{2}\frac{\partial f_1}{\partial x_2}+b(x_2)f_1\right\}dx_1 dx_1$$

The integrand $\{\}$ is continuous everywhere, and it fo.t follows that $\frac{N^2}{2}\frac{\partial v}{\partial x_2}+b(x_2)v$ is also continuous everywhere.

The same approach may clearly be extended to b(x) with any finite number of discontinuities.

All the differential relations among the correlations which were developed in Section 2.6.3 are valid in the discontinuous case. In particular, this implies $|\Gamma''_{xx}(t)| < \infty$, $t < 0$, $t > 0$.

2.8 Example 3: A Discontinuous b(x)

As an example of the theory of the foregoing section, we now consider the "infinite-clipping" b(x) defined by

2.8-1) $$b(x) = \begin{cases} +1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$

This is the first of the examples treating a system which is nonlinear and hence lies outside the scope of older theories. The distribution of x has been found by Wishner [56] , [55] and the autocorrelation of x by Caughey and Dienes [7] .

2.8.1 The discontinuous case as a Limit of a Sequence of Analytic b(x)

It was postulated above that the density f would be continuous even when b(x) is discontinuous. For the particular discontinuous b(x) considered here, it is possible to indicate the continuity of f more strongly, at least in the steady-state. Consider a b(x) of the form

2.8.1-1) $$b(x) = \frac{2}{\pi}\tan^{-1}ax$$

where a is a positive constant. The general form of this function is given in Figure 2.8.1-1.

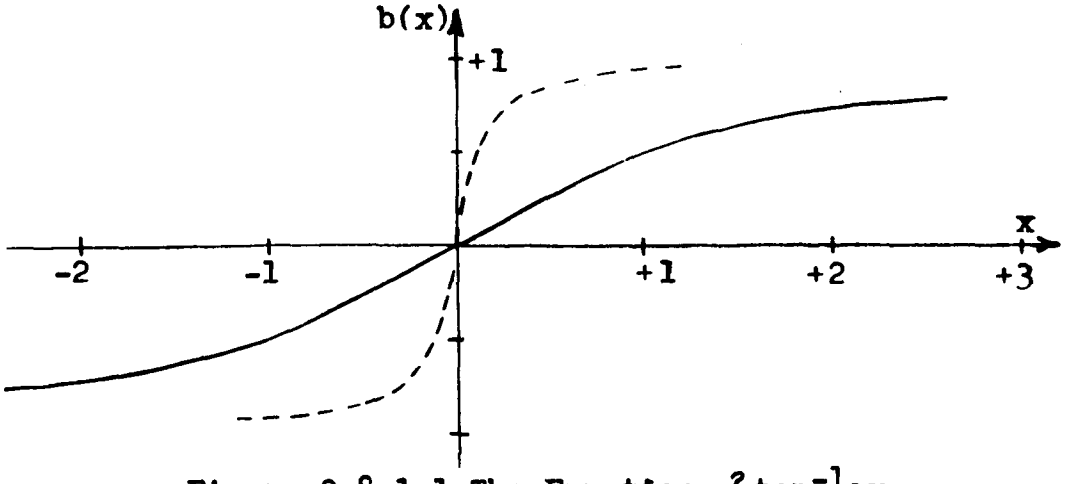


Figure 2.8.1-1 The Function $\frac{2}{\pi}\tan^{-1}ax$

The solid curve represents a = 1, and the dashed curve a = 10. For any a, the function approaches +1 for large positive x, -1 for large negative x. We may, then, select a sequence of functions of this form, with increasing a, and this sequence will approach the discontinuous function of equation (2.8-1).

For given a, the steady-state distribution may be computed from equation (2.5-6). The result is

$$2.8.1\text{-}2)\quad f_s(x,a)=\frac{\exp\left\{-\frac{4x}{\pi N^2}\tan^{-1}ax+\frac{2}{N^2\pi a}\log(1+a^2x^2)\right\}}{2\int_0^\infty \exp\left\{-\frac{4x}{\pi N^2}\tan^{-1}ax+\frac{2}{N^2\pi a}\log(1+a^2x^2)\right\}dx}$$

For any a, f_s is continuous everywhere. As $a\to\infty$, it may be shown that

$$2.8.1\text{-}3)\quad \lim_{a\to\infty} f_s(x,a)=\frac{1}{N^2}\exp\left\{-\frac{2|x|}{N^2}\right\}$$

which is also continuous everywhere. This distribution also satisfies the density continuity condition of equation (2.7-13). Each member of the sequence is continuous everywhere and so is the limit function. The continuity of f seems well-established in this particular case.

2.8.2 The Autocorrelation and Spectrum of x

As mentioned above, the spectrum was calculated by Caughey and Dienes [7] . We use a method considerably different from theirs, however. We will make use of equations (2.6.2-5) and(2.6.2-7). The first task is to determine $v(x,t;xf_s)$ which is the solution of

2.8.2-1) $$\frac{\partial v}{\partial t} = \frac{N^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x}[\operatorname{sgn} x\, v]$$

subject to the boundary condition

2.8.2-2) $$v(x,0;xf_s) = \frac{1}{N^2} x e^{-\frac{2}{N^2}|x|}$$

and the continuity condition

2.8.2-3) $$\frac{N^2}{2}\frac{\partial v}{\partial x}(0^-,t;xf_s) + v(0,t;xf_s) = \frac{N^2}{2}\frac{\partial v}{\partial x}(0^-,t;xf_s) + v(0,t;xf_s) \qquad t\geq 0$$

From the symmetry of the problem, $v(0,t;xf_s) = 0$, $t \geq 0$, so it follows that $\frac{\partial v}{\partial x}$ is continuous everywhere in this case.

The solution of this problem is given in Appendix B. the result is

2.8.2-4) $$v(x,t;xf_s) = \operatorname{sgn} x\, \frac{4N^4}{\pi} e^{-\frac{|x|}{N^2} - \frac{t}{2N^2}} \int_0^\infty \frac{w e^{-\frac{N^2 t}{2} w^2}}{(1+N^4 w^2)^2} \sin x w\, dw$$

The integral cannot be expressed in closed form in terms of tabulated functions, but it is nonetheless possible to proceed. Substituting this result into equation (2.6.2-5),

2.8.2-5) $$\Gamma_{xx}(t) = \frac{8N^4}{\pi} e^{-\frac{t}{2N^2}} \int_0^\infty dx\, x e^{-\frac{x}{N^2}} \int_0^\infty \frac{w e^{-\frac{N^2 t}{2} w^2}}{(1+N^4 w^2)^2} \sin x w\, dw$$

Interchanging orders and doing the x integration,

2.8 .2-6) $$\Gamma_{xx}(t) = \frac{16N^{10}}{\pi} e^{-\frac{t}{2N^2}} \int_0^\infty \frac{w^2 e^{-\frac{N^2 t}{2} w^2}}{(1+N^4 w^2)^4} dw \qquad t \geq 0$$

Taking advantage of equation (2.6.2-6), we may include all t by writing

2.8.2-7) $$\Gamma_{xx}(t) = \frac{16N^{10}}{\pi} e^{-\frac{|t|}{2N^2}} \int_0^{\infty} \frac{w^2 e^{-\frac{N^2|t|}{2}w^2}}{(1+N^4w^2)^4}\, dw \qquad -\infty<t<+\infty$$

This integral cannot be evaluated either, but its characteristics may be found. Making a change of variable $N^2w = z$,

2.8.2-8) $$\Gamma_{xx}(t) = \frac{16N^4}{\pi} \int_0^{\infty} \frac{z^2 e^{-\frac{|t|}{2N^2}(1+z^2)}}{(1+z^2)^4}\, dz$$

Now define a function

2.8.2-9) $$\rho_x(\xi) = \frac{32}{\pi} \int_0^{\infty} \frac{z^2 e^{-|\xi|(1+z^2)}}{(1+z^2)^4}\, dz$$

so that Γ_{xx} may be expressed as

2.8.2-10) $$\Gamma_{xx}(t) = \frac{N^4}{2} \rho_x\left(\frac{|t|}{2N^2}\right)$$

It may easily be shown that $\rho_x(0) = 1$, so ρ_x is the normalized version of the function Γ_{xx}. Clearly, then, the average power is

2.8.2-11) $$\Gamma_{xx}(0) = \frac{N^4}{2}$$

which agrees with the result calculated directly from the steady-state distribution.

2.8.2-12) $$E\{x^2\} = \int_{-\infty}^{+\infty} x^2 f_5(x)dx = \frac{2}{N^2}\int_0^{\infty} x^2 e^{-\frac{2}{N^2}x} dx = \frac{N^4}{2}$$

The function $\rho_x(\xi)$ contains the shape of the autocorrelation. Accordingly, the shape is a function only of the parameter $\xi = \frac{|t|}{2N^2}$. This means that for large N, the correlation times are longer than for small N.

From the foregoing, we can see that $\Gamma_{xx}(t)$ is continuous

at t = 0. From this it follows that the sample functions of x(t) are continuous in mean square (Loeve [35] p 315, [36] p 470, Sales Valles [42]). Furthermore, the derivative $\frac{d\Gamma_{xx}}{dt}$ does not exist at t=0, and from this it follows that the sample functions are not differentiable in mean square (Loeve [36] p 470). Both these results would certainly be expected for continuous b(x). They apparently carry over to this discontinuous example.

The derivative of $\rho_x(\xi)$, $\xi \geq 0$ is

2.8.2-13) $$\rho_x'(\xi) = -\frac{32}{\pi}\int_0^\infty \frac{z^2 e^{-\xi(1+z^2)}}{(1+z^2)^3}\, dz \qquad \xi \geq 0$$

2.8.2-14) $$\rho_x'(0^+) = -2$$

Similarly, $\rho_x'(0^-) = +2$.

Since $\rho_x(\xi)$ cannot be evaluated analytically, it was calculated numerically. The values obtained are given in Table 2.8.2-1.

TABLE 2.8.2-1

THE FUNCTION $\rho_x(\xi)$

ξ	$\rho_x(\xi)$
0.0	1.000
0.1	0.826
0.2	0.693
0.4	0.495
0.8	0.268
1.2	0.150

These values were calculated by graphical integration, and the last figure given in the table probably has little significance. The error should not exceed 0.01, however.

A plot of this function is given in Figure 2.8.2-1.

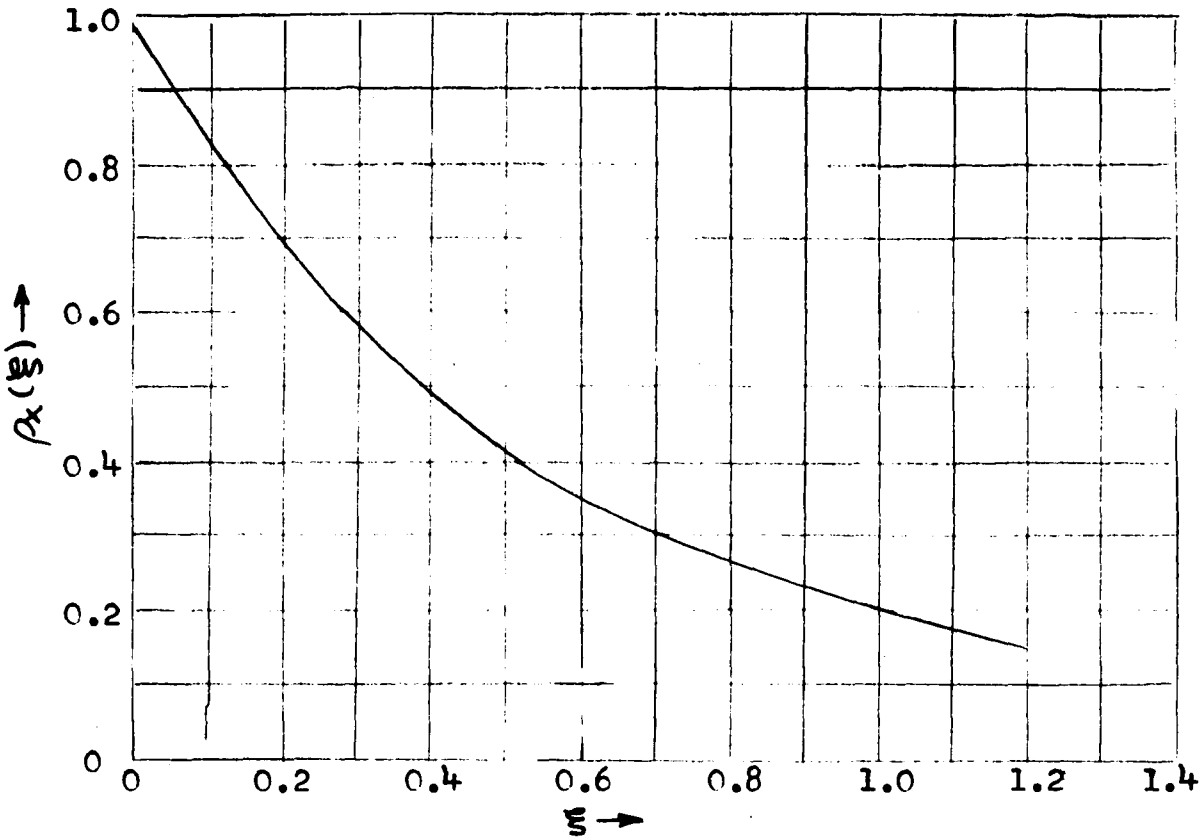


Figure 2.8.2-1 The Function $\rho_x(\xi)$

Now we calculate the spectrum from equation (2.6.2-7).

2.8.2-15) $$G_{xx}(\omega)=\frac{2}{\pi}\int_0^\infty dt\cos\omega t\,\frac{16}{\pi}N^4\int_0^\infty \frac{z^2 e^{-\frac{t}{2N^2}(1+z^2)}}{(1+z^2)^4}\,dz$$

Interchanging the order and performing the t integration (Bierens de Haan [4] Table 261, 2) p 383) gives

2.8.2-16) $$G_{xx}(\omega)=\frac{32}{\pi^2}N^6\int_{-\infty}^{+\infty}\frac{z^2dz}{(1+z^2)^3\left[(1+z^2)^2+4N^4\omega^2\right]}$$

We choose to express this as

2.8.2-17) $$G_{xx}(\omega)=\frac{5N^6}{4\pi}\varphi_x(2N^2\omega)$$

where the function φ_x is defined by

2.8.2-18) $$\varphi_x(u)=\frac{128}{5\pi}\int_{-\infty}^{+\infty}\frac{z^2dz}{(1+z^2)^3\left[(1+z^2)^2+u^2\right]}$$

This integrand is a rational function of z, and may be treated by straightforward contour integration, with the result

2.8.2-19) $$\varphi_x(u) = \frac{16}{5u^4}\left\{ u^2 + 8 - 8\sqrt{\frac{1+\sqrt{1+u^2}}{2}} \right\}$$

which agrees with Caughey and Dienes ([7] equation 45) except for the overall normalizing constant. When the different normalization is accounted for, their result appears too small by the factor 2.

We now consider the characteristics of this function, which expresses the shape of the spectral density. It may be shown that $\lim_{u\to 0} \varphi(u) = 1$, which was the reason for picking the normalizing factor. Also, for large u, $\varphi(u) \simeq \frac{16}{5u^2}$. This is the high-frequency asymptote. Calculated values of $\varphi(u)$ are given in Table 2.8.2-2.

TABLE 2.8.2-2

THE FUNCTION $\varphi_x(u)$

u	$\varphi_x(u)$	u	$\varphi_x(u)$
0.0	1.0000	7	5.455×10^{-2}
0.5	0.8864	8	4.295×10^{-2}
0.7	0.8028	9	3.466×10^{-2}
1.0	0.6737	10	2.845×10^{-2}
2.0	0.3648	12	2.030×10^{-2}
3.0	0.2157	14	1.517×10^{-2}
4.0	0.1400	16	1.175×10^{-2}
5.0	9.743×10^{-2}	18	9.368×10^{-3}
6.0	7.147×10^{-2}	20	7.641×10^{-3}

In the analysis of linear systems, it has become standard practice to plot power spectra on logarithmic scales. This is done in Figure 2.8.2-2 for the nonlinear result, to facilitate comparison with linear systems. On this plot, the high-frequency asymptote becomes a straight line, which is shown. The dashed curve is the spectrum of that Ornstein-Uhlenbeck process which has the same low- and high-frequency asymptotes. It is included to show the difference in shape between the linear and nonlinear spectra.

It may be seen from the figure, that the concept of "break frequency" or "corner frequency" will be a useful one for this spectrum. If the "break frequency" is defined as the frequency at which the asymptotes intersect, then $\frac{16}{5\mu_b^2} = 1$ gives $\mu_b = 4/\sqrt{5} = 1.788\ 854$. At the break frequency, the nonlinear spectrum is below unity by the factor 2.175 (6.75 db). The linear spectrum is down by the factor 2.00 (6 db). This nonlinear spectrum does not have quite so sharp a corner as does the linear one.

The above remarks are concerned with the shape of the spectral density curve. Since $\mu = 2N^2\omega$, it is clear that for the nonlinear system, the larger N, the lower the break frequency ω_b, and conversely. The smaller N, the wider bandpass the nonlinear system shows.

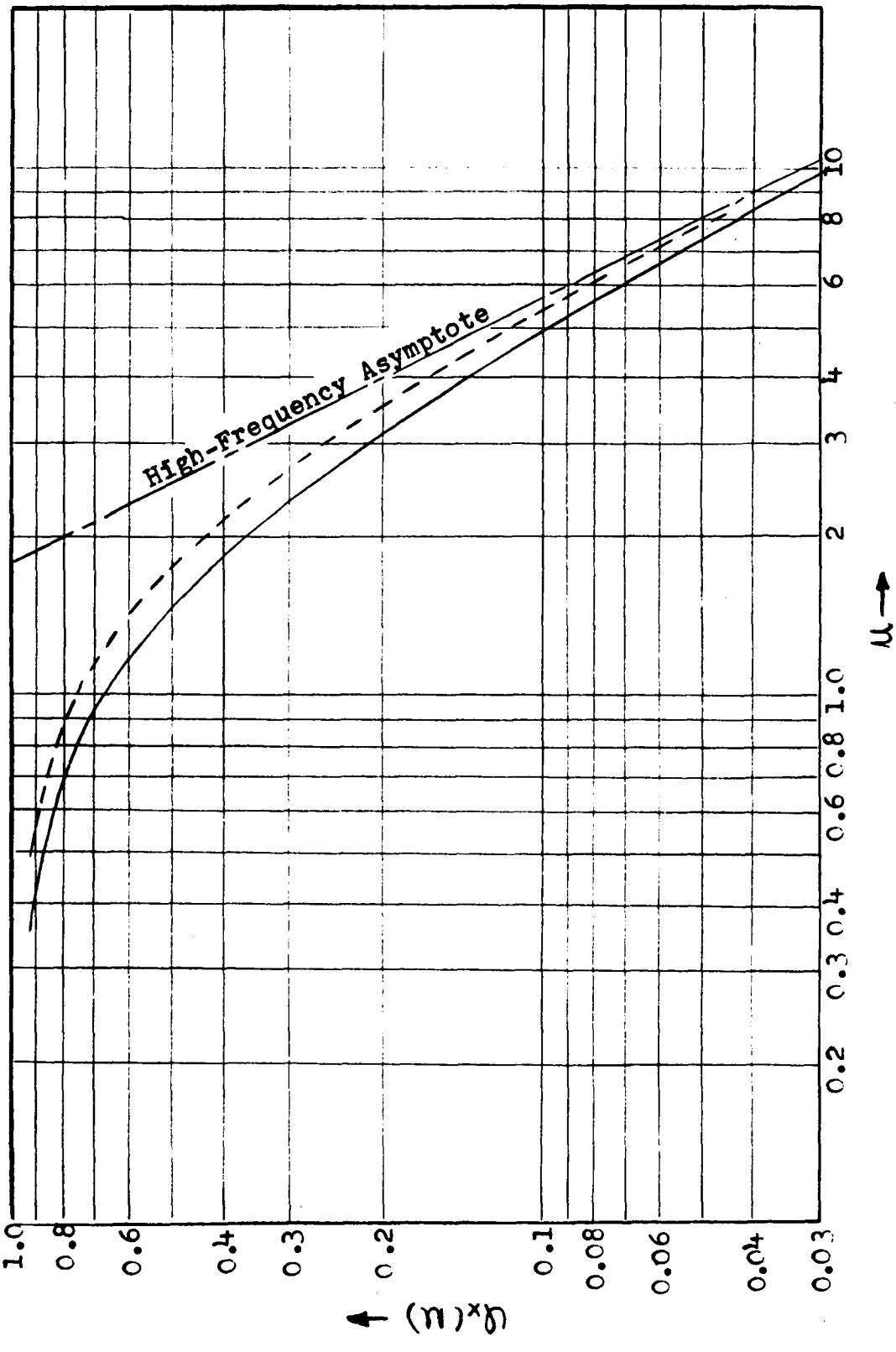


Figure 2.8.2-2 The Spectral Form Function $\Phi_x(\mu)$

2.8.3 The Autocorrelation and Spectrum of b(x)

The spectral properties of b(x) may be calculated directly from those of x, as shown in Section 2.6.3. In this example, however, we will use an approach based on equations (2.6.2-8) and (2.6.2-10), and use the differential formulae as a check.

The first step is to determine $v(x,t;bf_s)$ which, in this example means solving the equation

2.8.3-1) $$\frac{\partial v}{\partial t} = \frac{N^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x}\left[\operatorname{sgn} x\, v\right]$$

subject to the boundary condition

2.8.3-2) $$v(x,0;bf_s) = \frac{\operatorname{sgn} x}{N^2}\exp\left\{-\frac{2|x|}{N^2}\right\}$$

The solution to this problem is obtained in Appendix B (equation (B-19)). It is

2.8.3-3) $$v(x,t;bf_s) = \frac{2}{N^2\pi}e^{-\frac{|x|}{N^2}-\frac{t}{2N^2}}\int_0^\infty \frac{w e^{-\frac{N^2}{2}w^2 t}}{\left(\frac{1}{N^2}\right)^2 + w^2}\sin x w\, dw$$

Substituting this result into equation (2.6.2-8), interchanging the order of integration, and performing the x integration gives

2.8.3-4) $$r_{bb}(t) = \frac{4}{N^2\pi}\int_0^\infty \frac{w^2 e^{-\frac{N^2}{2}w^2 t - \frac{t}{2N^2}}}{\left(\frac{1}{N^4}+w^2\right)^2}\, dw \qquad t \geq 0$$

Making use of (2.6.2-9), we have

2.8.3-5) $$r_{bb}(t) = \frac{4}{N^2\pi}\int_0^\infty \frac{w^2 e^{-\frac{N^2}{2}w^2|t| - \frac{|t|}{2N^2}}}{\left(\frac{1}{N^4}+w^2\right)^2}\, dw \qquad -\infty < t < +\infty$$

This cannot be evaluated in closed form, so we proceed as before to extract certain information. Make the substitu-

tion $N^2\omega = z$.

2.8.3-6) $$\Gamma_{bb}(t) = \frac{4}{\pi}\int_0^\infty \frac{z^2 e^{-(1+z^2)\frac{|t|}{2N^2}}}{(1+z^2)^2}\,dz$$

This may be written

2.8.3-7) $$\Gamma_{bb}(t) = \rho_b\left(\frac{t}{2N^2}\right)$$

where

2.8.3-8) $$\rho_b(\xi) = \frac{4}{\pi}\int_0^\infty \frac{z^2 e^{-(1+z^2)|\xi|}}{(1+z^2)^2}\,dz$$

It is easily shown from this expression that $\rho_b(0) = \Gamma_{bb}(0) = 1$. This is as it should be, since b(x) in this example only takes on the values $\pm$ 1, so $E\{b^2(x)\}$ should be unity.

Differentiating equation (2.8.3-8),

2.8.3-9) $$\rho_b'(\xi) = -\operatorname{sgn}\xi\ \frac{4}{\pi}\int_0^\infty \frac{z^2 e^{-(1+z^2)|\xi|}}{(1+z^2)}\,dz$$

which becomes infinite at $\xi = 0$. This is to be expected from McFadden's [38] theorem. The function b(x) in this example is exactly the "infinite clipping" function which McFadden uses to study zero-crossings. The slope of the autocorrelation of the clipped signal at $t = 0^+$ is proportional to the expected number of zero-crossings per unit time (of the signal before clipping). By analogy with the Ornstein-Uhlenbeck process, we would expect the average number of zero-crossings to be infinite, and it is true in this example.

Since the integral of (2.8.3-8) cannot be evaluated in closed form, it was calculated numerically. The results are given in Table 2.8.3-1 and Figure 2.8.3-1.

TABLE 2.8.3-1

NORMALIZED AUTOCORRELATION OF b(x)

0.0	1.000
0.1	0.461
0.2	0.321
0.4	0.189
0.8	0.082
1.2	0.040

By comparison with Figure 2.8.2-1, it may be seen that the autocorrelation of b falls off much more rapidly with increasing ξ than does that of x.

To check this result by means of the differential relation, we differentiate equation (2.8.2-10) twice, with the result:

2.8.3-10) $$\Gamma''_{xx}(t) = \frac{4}{\pi}\int_0^\infty \frac{x^2 e^{-\frac{|t|}{2N^2}(1+x^2)}}{(1+x^2)^2}\,dx = \Gamma_{bb}(t)$$

The second equality follows from equation (2.8.3-7), and equation (2.6.3-16) is satisfied in this example.

The spectral density of b(x) is calculated from equation (2.6.2-10).

2.8.3-11) $$G_{bb}(\omega) = \frac{2}{\pi}\int_0^\infty dt\,\cos\omega t\,\frac{4}{\pi}\int_0^\infty \frac{z^2 e^{-(1+z^2)\frac{t}{2N^2}}}{(1+z^2)^2}\,dz$$
$$= \frac{8}{\pi^2}\int_0^\infty dz\,\frac{z^2}{(1+z^2)^2}\int_0^\infty e^{-\frac{(1+z^2)}{2N^2}t}\cos\omega t\,dt$$

The inner integral may be evaluated (Bierens de Haan [4] Table 261, 2), p 383). The result is

2.8.3-12) $$G_{bb}(\omega) = \frac{16N^2}{\pi^2}\int_0^\infty \frac{z^2\,dz}{(1+z^2)\left[(1+z^2)^2 + 4N^4\omega^2\right]}$$

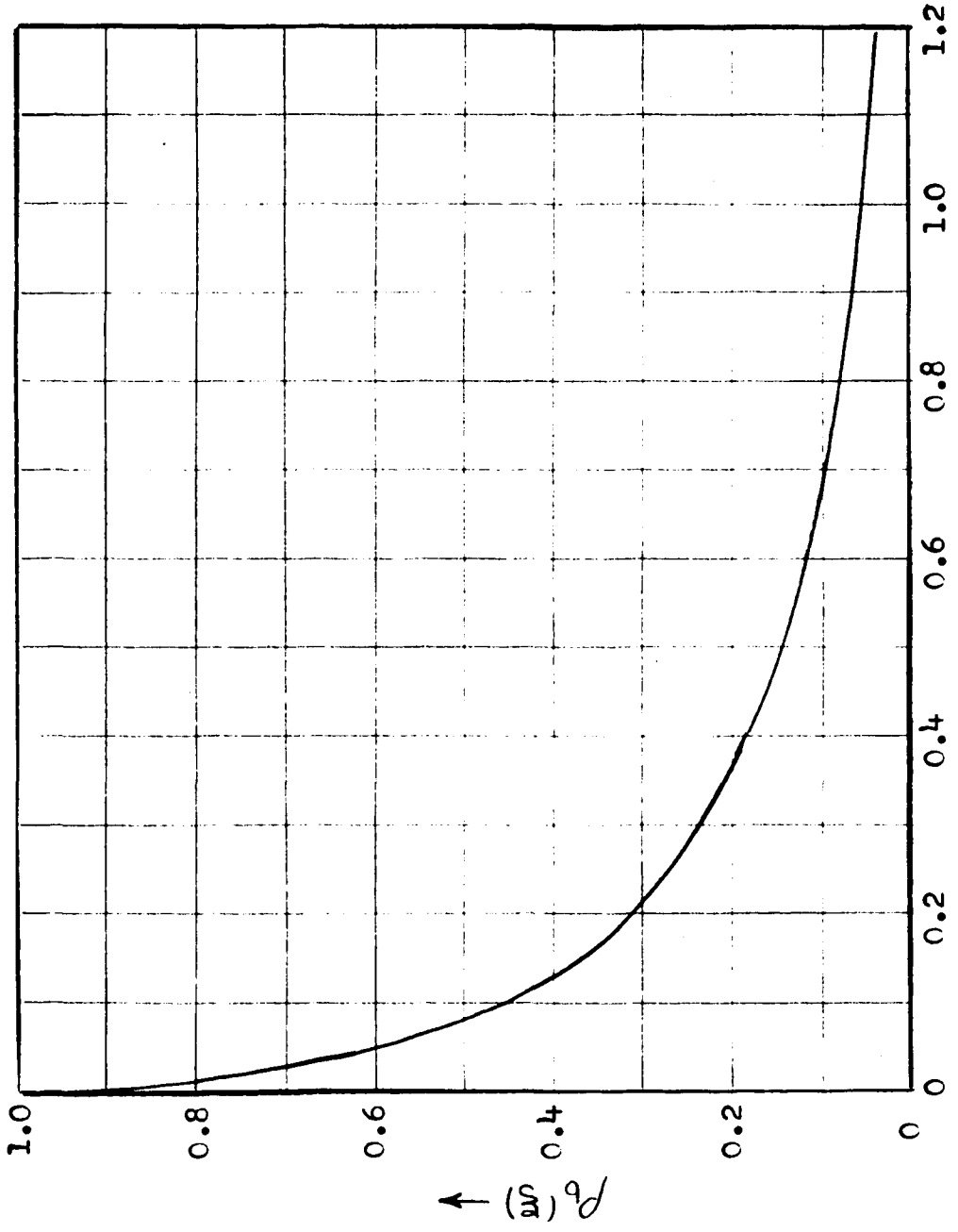


Figure 2.8.3-1 The Function $\rho_b(\xi)$

This may be written as

2.8.3-13) $$G_{bb}(\omega) = \frac{N^2}{\pi} \varphi_b(2N^2\omega)$$

where

2.8.3-14) $$\varphi_b(u) = \frac{16}{\pi} \int_0^\infty \frac{z^2 dz}{(1+z^2)\left[(1+z^2)^2 + u^2\right]}$$

Since the integrand is a rational function, it may be evaluated by contour integration, with the result

2.8.3-15) $$\varphi_b(u) = \frac{8}{u^2}\left[\sqrt{\frac{1+\sqrt{1+u^2}}{2}} - 1\right]$$

It may easily be shown that $\lim_{u\to 0} \varphi_b = 1$. For large u,

2.8.3-16) $$\varphi_b \simeq \frac{4\sqrt{2}}{u^{3/2}}$$

This is a type of asymptotic behavior which no finite lumped-constant linear system can provide. For such systems, the high-frequency asymptote is always of the form $\frac{K}{u^{2n}}$, for integer n. Equation (2.8.3-16) does not fall in this class. Nonetheless, it is still possible to define the "break frequency" as the intersection of the low- and high-frequency asymptotes. This occurs at a frequency $u = 2\sqrt[3]{4} = 3.174$. This break frequency still may serve as a rough measure of bandwidth. The function φ_b is plotted in Figure 2.8.3-2.

It is now possible to see whether equation (2.6.3-39) is satisfied. Substituting equations (2.8.2-17) and (2.8.2-19) into equation (2.6.3-39),

2.8.3-17) $$G_{bb}(\omega) = \frac{2}{\pi}\cdot\frac{N^2}{2} - \frac{1}{4\pi N^2\omega^2}\left\{4N^2\omega^2 + 8 - 8\sqrt{\frac{1+\sqrt{1+4N^4\omega^2}}{2}}\right.$$

Simplifying this and cancelling terms,

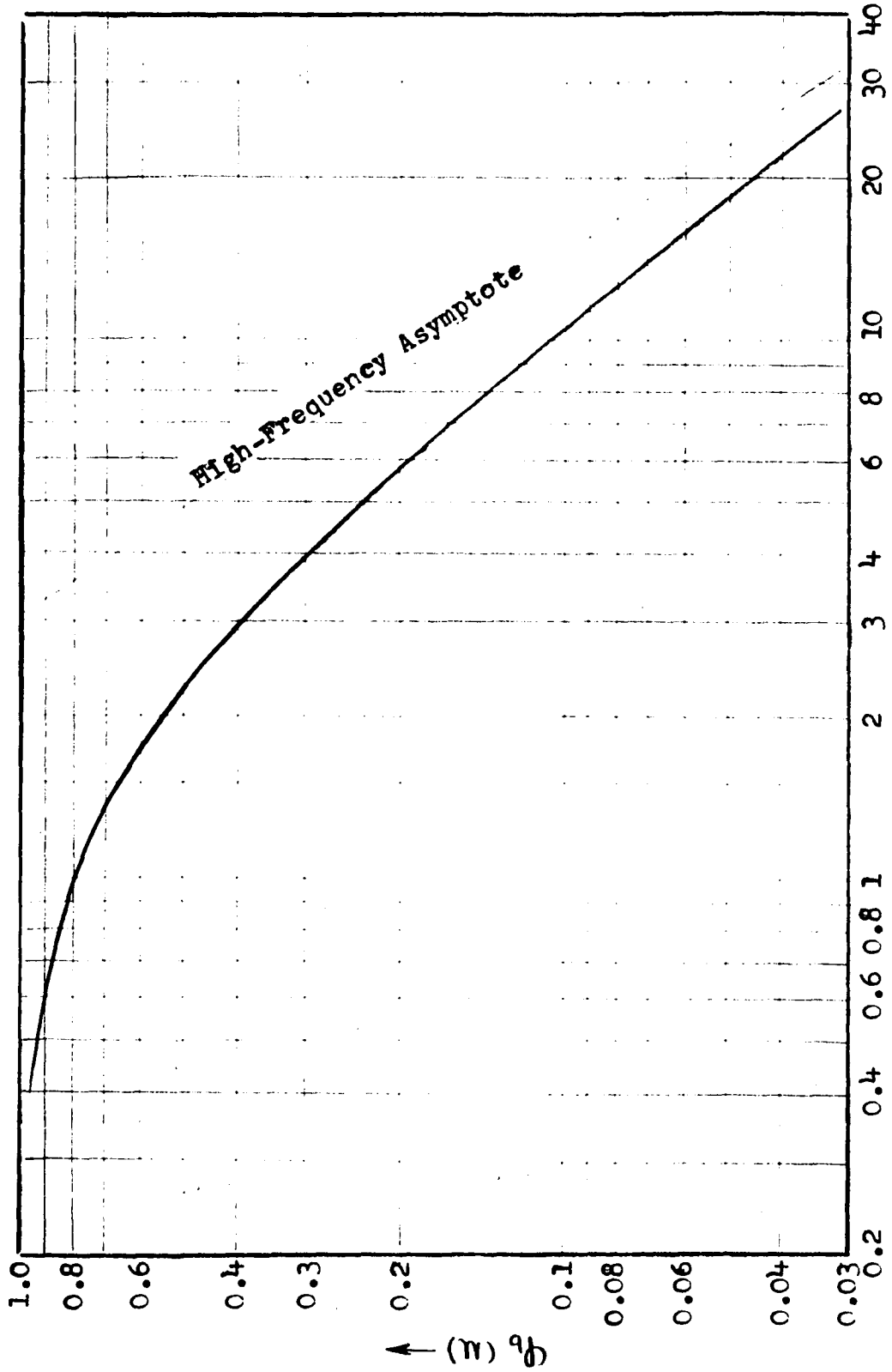


Figure 2.8.3-2 The Function $\phi_b(u)$

2.8.3-18) $$G_{bb}(\omega) = \frac{2}{\pi N^2 \omega^2}\left[\sqrt{\frac{1+\sqrt{1+4N^4\omega^2}}{2}} - 1\right]$$

which agrees with the result obtained in equations (2.8.3-12) and 2.8.3-14).

This shows that equation (2.6.3-39), which was derived from $\Gamma_{bb} = \Gamma_{xx}''$, agrees with the direct method of calculating $G_{bb}(\omega)$. This indicates that one of the conclusions of Theorem 6 is true in this case, despite the fact that b(x) does not meet the continuity requirements of the theorem. The continuity requirements are thus not necessary conditions.

In the Wiener optimum filter theory and elsewhere, use is made of the idea that an arbitrary spectrum may be approximated by a spectrum which is expressed as the ratio of two rational polynomials in ω^2. This would be awkward for the spectrum of b in this example because no such approximation can have the correct high-frequency asymptote.

2.9 Small-t Approximation of the Autocorrelation

Up to this point, formulas considered for correlations and spectra were exact. We now consider a certain type of approximation to the problem. The approximate results are far easier to compute, and, as will be seen, they retain certain features of the exact solution.

In particular, we here consider app oximations to $\Gamma(t)$ which are accurate for small t. Correspondingly, in the frequency domain the spectra $G(\omega)$ will be accurate for large ω. We begin by observing that $\Gamma_{xx}(t)$ may be expanded in a Taylor's series on one side of the origin.

2.9-1) $$\Gamma_{xx}(t) = \Gamma_{xx}(0) + \Gamma_{xx}'(0^+)t + \frac{1}{2!}\Gamma_{xx}''(0^+)t^2 + \frac{1}{3!}\Gamma_{xx}'''(0^+)t^3 + \cdots \quad t \geq 0$$

The general procedure is to approximate this series with an expression which exactly matches a certain finite number of the terms, and matches the remainder imperfectly.

Before considering these approximations themselves, we note that the coefficients in equation (2.9-1) may be evaluated by simple quadratures. By definition,

2.9-2) $$\Gamma_{xx}(0) = \int_{-\infty}^{+\infty} x^2 f_S \, dx$$

The first derivative may be obtained by the following process. Differentiate equation (2.6.2-5), and make use of the fact that v satisfies the Fokker-Planck equation.

2.9-3) $$\Gamma_{xx}'(t) = \int_{-\infty}^{+\infty} x \frac{\partial v}{\partial t}(x,t;x f_S)\, dx \qquad t \geq 0$$

$$= \int_{-\infty}^{+\infty} x \frac{\partial}{\partial x}\left[\frac{N^2}{2}\frac{\partial v}{\partial x} + bv\right] dx$$

If v and $\frac{\partial v}{\partial x}$ vanish as $|x| \to \infty$, then this may be integrated by parts to yield

2.9-4) $$\Gamma_{xx}'(t) = -\int_{-\infty}^{+\infty} b(x)\, v(x,t;xf_s)\, dx\,, \quad t \geq 0$$

Letting $t \to 0$ in this expression,

2.9-5) $$\Gamma_{xx}'(0^+) = -\int_{-\infty}^{+\infty} x\, b(x) f_s(x)\, dx = \frac{N^2}{2}\int_{-\infty}^{+\infty} x \frac{\partial f_s}{\partial x} dx = -\frac{N^2}{2}$$

The same process may be repeated. Differentiate (2.9-4) with respect to time, make use of the Fokker-Planck equation, and integrate by parts. Then set $t = 0$. The result is

2.9-6) $$\Gamma_{xx}''(0^+) = \int_{-\infty}^{+\infty} b^2(x) f_s(x)\, dx$$

Repeating the procedure once again, one obtains

2.9-7) $$\Gamma_{xx}'''(0^+) = -\frac{N^2}{2}\int_{-\infty}^{+\infty} (b')^2 f_s(x)\, dx$$

and so on to higher orders, if required. The point is that, once f_s is known, $\Gamma_{xx}^{(n)}(0^+)$ may be calculated by quadratures, provided b(x) is sufficiently differentiable.

2.9.1 The First Approximation

As a first approximation, let us consider an exponential.

2.9.1-1) $$\hat{\Gamma}_{xx}(t) = A e^{-\alpha|t|} + m^2 \qquad -\infty < t < +\infty$$

where

2.9.1-2) $$m = \int_{-\infty}^{+\infty} x f_s\, dx$$

2.9.1-3) $$A = \int_{-\infty}^{+\infty} x^2 f_s\, dx - m^2$$

2.9.1-4 $$\alpha = -\frac{\Gamma_{xx}'(0^+)}{A} = \frac{N^2}{2A}$$

Expanding this expression in a power series, $t \geq 0$,

$$\hat{\Gamma}_{xx}(t) = A\left(1 - \alpha t + \frac{\alpha^2}{2!}t^2 - \frac{\alpha^3}{3!}t^3 + \cdots\right) + m^2$$

2.9.1-5)
$$= A + m^2 - A\alpha t + \frac{A\alpha^2}{2!}t^2 - \frac{A\alpha^3}{3!}t^3 + \cdots$$

$$= \Gamma_{xx}(0) + \Gamma'_{xx}(0^+) + \frac{1}{2!}\frac{\left[\Gamma'_{xx}(0^+)\right]^2}{A} + \frac{1}{3!}\frac{\left[\Gamma'_{xx}(0^+)\right]^3}{A^2}t^3$$

This agrees exactly with (2.9-1) through the first two terms. Furthermore, $\hat{\Gamma}_{xx}(0^+) = \Gamma_{xx}(0^+)$, $\hat{\Gamma}'_{xx}(0^+) = \Gamma'_{xx}(0^+)$. The first of these statements implies that $\hat{\Gamma}_{xx}$ has the correct total power. The implication of the second will now be developed.

By a well-known theorem of Fourier transforms, (see, e.g. Titchmarsh [48], Sect. 1.8), if $\Gamma_{bb}(t)$ is $L_1(-\infty, +\infty)$, then $\lim_{\omega\to\infty} G_{bb}(\omega) = 0$. To show that these conditions are met, it is sufficient to show $\int_0^\infty |\Gamma_{bb}(t)|dt < \infty$. This can easily be done under the conditions of Theorem 6, Section 2.6.3 if, in addition, we assume that $\Gamma_{bb}(t) \geq 0$, $0 \leq t < \infty$. It appears that this is usually the case, and may be true in general, but this has not been proved. Using the results of the theorem,

2.9.1-6)
$$\int_0^\infty \Gamma_{bb}(t)dt = \int_0^\infty \Gamma''_{xx}(t)dt = \Gamma'_{xx}(t)\Big|_0^{+\infty} = -\Gamma'_{xx}(0^+) = \frac{N^2}{2}$$

The fact that $\lim_{t\to\infty} \Gamma'_{xx}(t) = 0$ is seen by substituting equation (2.6.3-30) into equation (2.6.3-24).

Since $\lim_{\omega\to\infty} G_{bb}(\omega) = 0$, taking $\lim_{\omega\to\infty}$ of both sides of equation (2.6.3-39) gives

2.9.1-7)
$$\lim_{\omega\to\infty} \omega^2 G_{xx}(\omega) = \frac{N^2}{\pi}$$

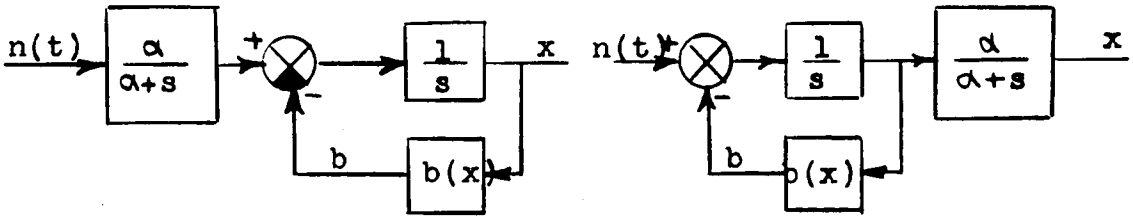
Putting it another way, for large ω,

2.9.1-8)
$$G_{xx}(\omega) \cong N^2/\pi\omega^2$$

so $\frac{N^2}{\pi\omega^2}$ is the high-frequency asymptote, independently of b. Apparently the high-frequency spectrum is determined by only the input noise and the integrator. The feedback b(x) apparently has no effect.

This is perhaps not too surprising. The integrator in the forward path of Figure 2.1-1 will attenuate high-frequencies. Thus x would have a small high-frequency component, and the same should be true for b(x). The input to the integrator is n(t)-b(t), and, for frequencies sufficiently high, one would expect the contribution from b(x) would be negligible compared to that from the white noise.

If the high-frequency behavior is in fact linear, it leads to the conjecture that, for ω sufficiently large, the two block diagrams of Figure 2.9.1-1 are approximately equivalent.



a) Prefiltering b) Post-filtering

Figure 2.9.1-1 Alternate Positions for Low-Pass Filter

This tends to support part of the conjecture of Chuang and Kazda [9] that if the input spectrum departs from constancy only at very low and very high frequencies, it may be approximated by white noise. Their assumption appears sound as to the high-frequency departure, but the effect of the low-frequency departure is still an open question.

At any rate, since $\hat{\Gamma}_{xx}$ has the correct slope at the origin, the corresponding approximate spectrum $\hat{G}_{xx}(\omega)$ will have the correct high-frequency asymptote.

Another viewpoint on the first approximation may be obtained by minimizing $E[\varepsilon^2]$ in Figure 2.9.1-2.

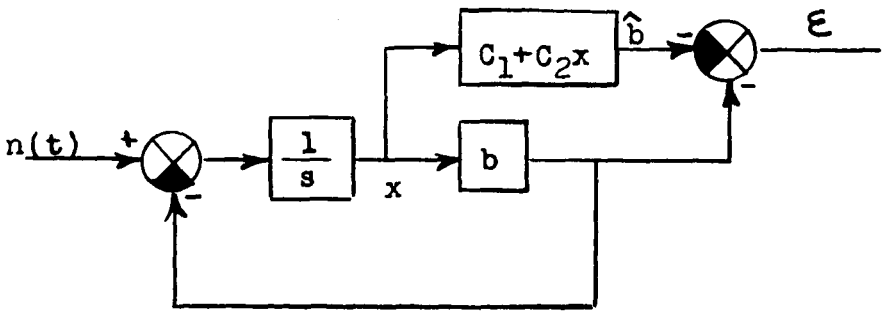


Figure 2.9.1-2 Linear Fit of b(x)

We wish to choose C_1 and C_2 is such a way that b(x) agrees with b(x) as well as possible in the mean square sense. We wish to minimize

$$2.9.1\text{-}9) \qquad E[\varepsilon^2] = \int_{-\infty}^{+\infty} [b(x) - C_1 - C_2 x]\, f_s\, dx$$

Differentiating with respect to C_1 and C_2, and setting the results equal to zero, one obtains

$$2.9.1\text{-}10) \qquad C_1 = \frac{m\,\Gamma'_{xx}(0^+)}{A} \; ; \; C_2 = -\frac{\Gamma'_{xy}(0^+)}{A}$$

Ignoring the steady-state term, we now consider the linear system

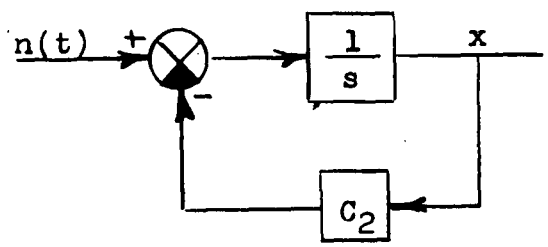


Figure 2.9.1-3 Equivalent Linear System

From equation (2.6.5-8), we see that

2.9.1-11) $$R_{xx}(t) = \frac{N^2}{2C_2} e^{-C_2|t|} = Ae^{+\frac{R'_{xx}(0^+)}{A}|t|}$$

which agrees with the first approximation, aside from the steady-state term. The parameter α chosen for the first approximation may be viewed, then, as providing a least-squares approximation to b(x).

This approximation was considered briefly in this light by Caughey and Dienes [7] and they call it the "best linear fit using exact probability". It is very much in the spirit of Booton's method [5] referred to earlier.

To review, then, the first approximation gives the correct total power, and the correct high-frequency asymptote. We are in a sense approximating the random process x(t) by an Ornstein-Uhlenbeck process.

To clarify some of these points, and show how well the first approximation works, we apply it to the infinite-clipping problem of Section 2.8. In addition, we will apply Booton's method to the same problem.

For this example, f_s is given in equation (2.8.1-3).

2.9.1-12) $$f_s(x) = \frac{1}{N^2} \exp\left\{-\frac{2|x|}{N^2}\right\}$$

From this, it follows that

2.9.1-13) $$m = 0$$

2.9.1-14) $$A = \int_{-\infty}^{+\infty} \frac{x^2}{N^2} \exp\left\{-\frac{2|x|}{N^2}\right\} dx = \frac{2}{N^2} \int_0^{\infty} x^2 e^{-\frac{2x}{N^2}} dx = \frac{N^4}{2}$$

2.9.1-15) $$\alpha = \frac{N^2}{2A} = \frac{1}{N^2}$$

Accordingly, the first approximation for the autocorrelation is

2.9.1-16) $$\hat{\Gamma}_{xx}(t) = \frac{N^4}{2} e^{-\frac{|t|}{N^2}}$$

Making the change of variable $\frac{t}{2N^2} = \xi$, and normalizing the magnitude,

2.9.1-17) $$\hat{\rho}_x(\xi) = e^{-2\xi}$$

This is the approximate normalized autocorrelation, which may be compared with the exact expression (2.8.2-10). This is done in Figure 2.9.1-4, where it may be seen that the approximate result falls off slightly more rapidly than does the exact.

The first approximate spectrum is obtained by transforming equation (2.9.1-17).

2.9.1-18) $$\hat{G}_{xx}(\omega) = \frac{2}{\pi}\int_0^\infty \frac{N^4}{2} e^{-\frac{t}{N^2}} \cos\omega t\,dt = \frac{N^6}{\pi}\frac{1}{1+N^4\omega^2}$$

Making the change of variable $2N^2\omega = \mu$, and normalizing against the correct magnitude,

2.9.1-19) $$\hat{Q}_x(\mu) = \frac{4}{5}\frac{1}{1+\frac{\mu^2}{4}}$$

This is compared with the exact result in Figure 2.9.1-5.

We have not yet touched on the problem of approximating the autocorrelation and spectrum of b(x). Two approaches suggest themselves. One continues in the spirit of Booton's approximation in assuming that b(x) is also Gaussian and

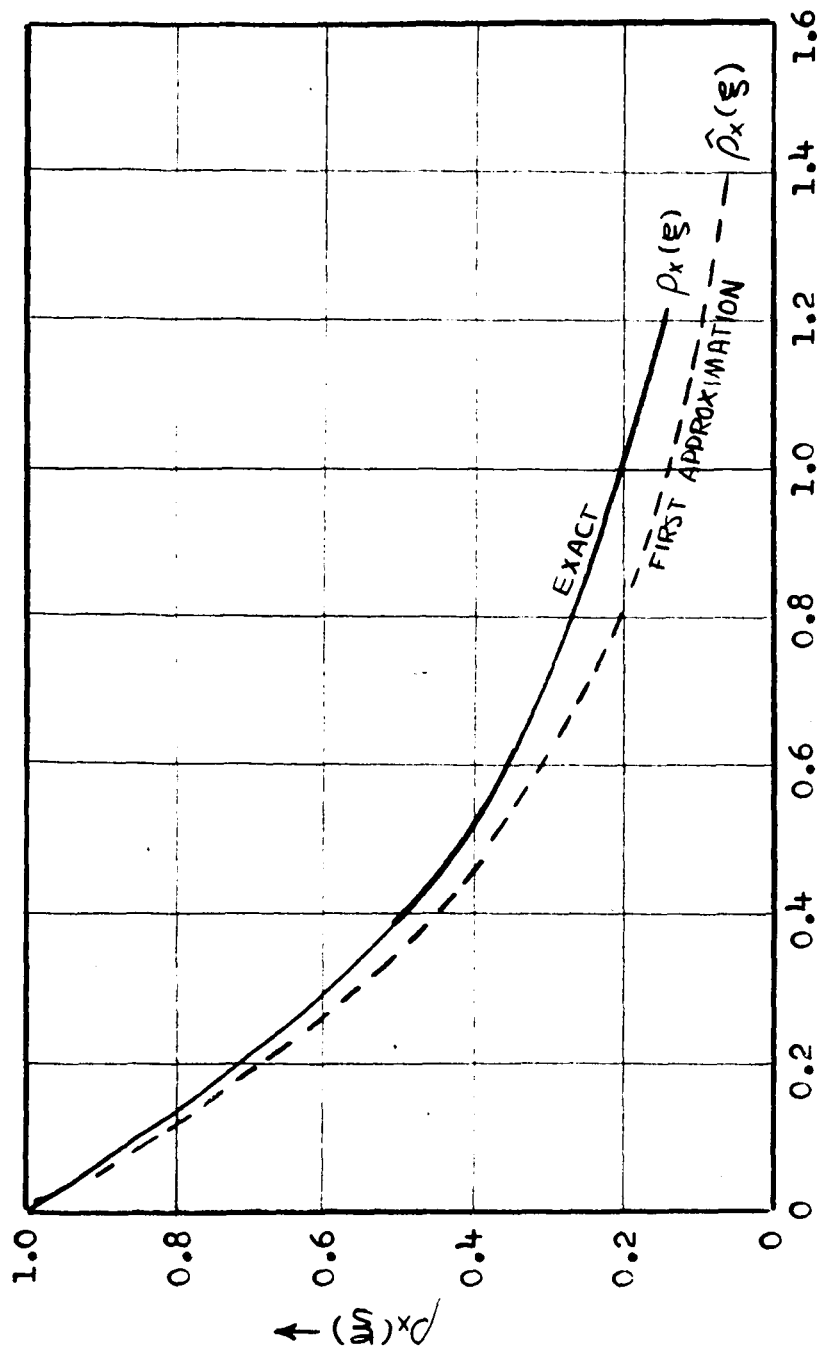


Figure 2.9.1-4 Comparison Between Exact Autocorrelation and the First Approximation

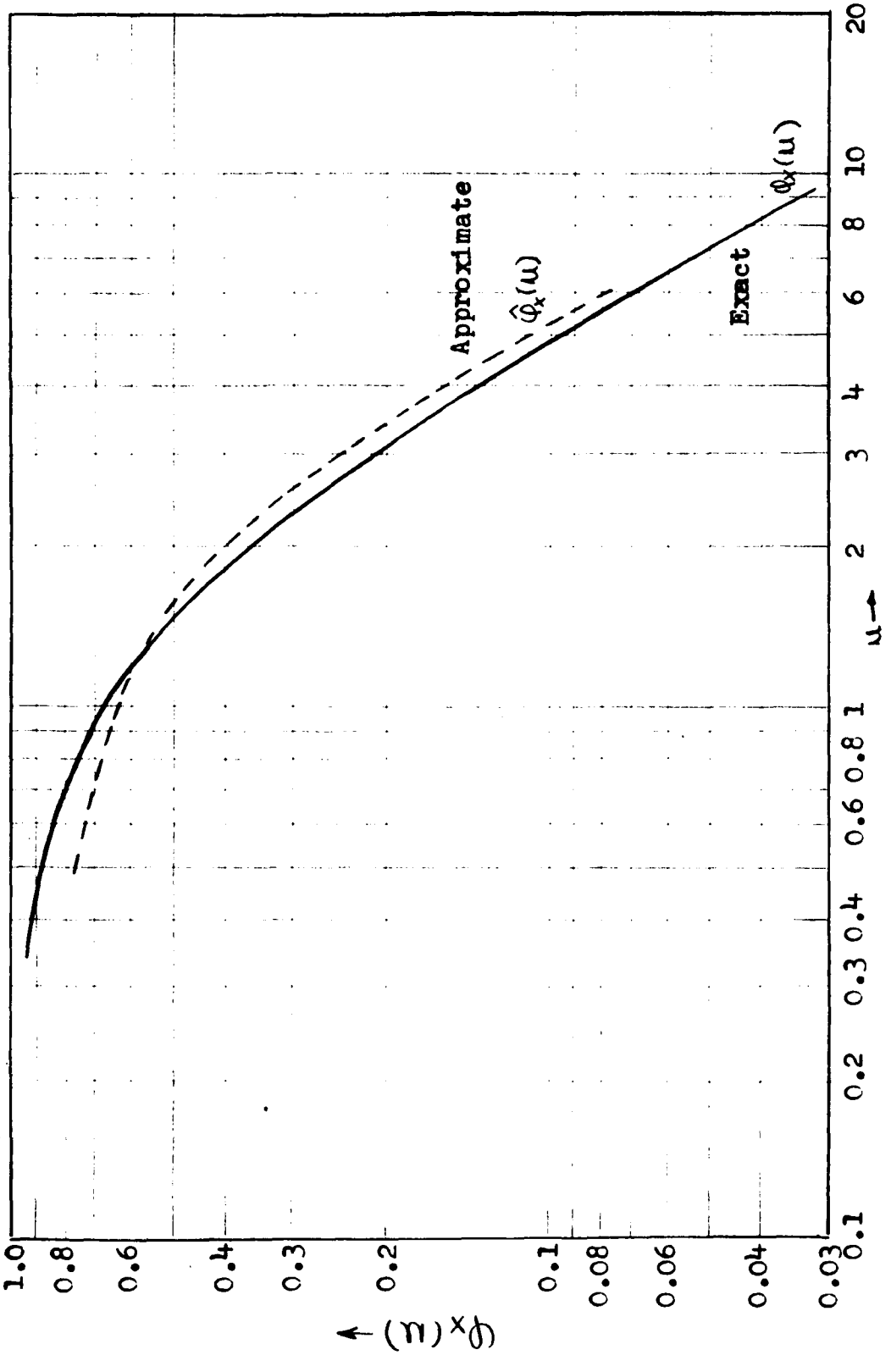


Figure 2.9.1-5 Comparison Between Exact Spectrum and First Approximation

that b(x) may be represented by $C_1 x$. In this case, the spectrum of b(x) is proportional to that of x.

A more accurate method would be to treat x as a Gaussian process, as Booton does, but put that process through the exact b(x), and determine the correlation of the result. Adopting the latter procedure in our sample problem, the calculation to be made is the following:

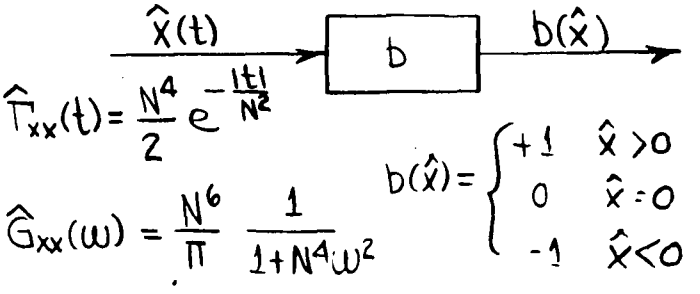


Figure 2.9.1-6 Approximate Calculation of Properties of b(x)

We need to find the autocorrelation of b(x). This is easily done using the result of Lawson and Uhlenbeck [32] p 57 ff, who analyzed this particular b(x). They show that

2.9.1-20) $$\hat{\Gamma}_{bb}(t) = \frac{2}{\pi} \sin^{-1} e^{-\frac{|t|}{N^2}}$$

Making the customary change of variable, $\xi = \frac{t}{2N^2}$,

2.9.1-21) $$\hat{\rho}_b(\xi) = \frac{2}{\pi} \sin^{-1} e^{-2|\xi|}$$

This is compared with the corresponding exact result, equation (2.8.3-8) in Figure 2.9.1-7). This approximation has the correct property that $\hat{\rho}_b'(0^+) = -\infty$.

For other b(x), of course, other methods must be used, such as those of Rice[41] and Middleton[40] .

It appears, then, that the first approximation (an

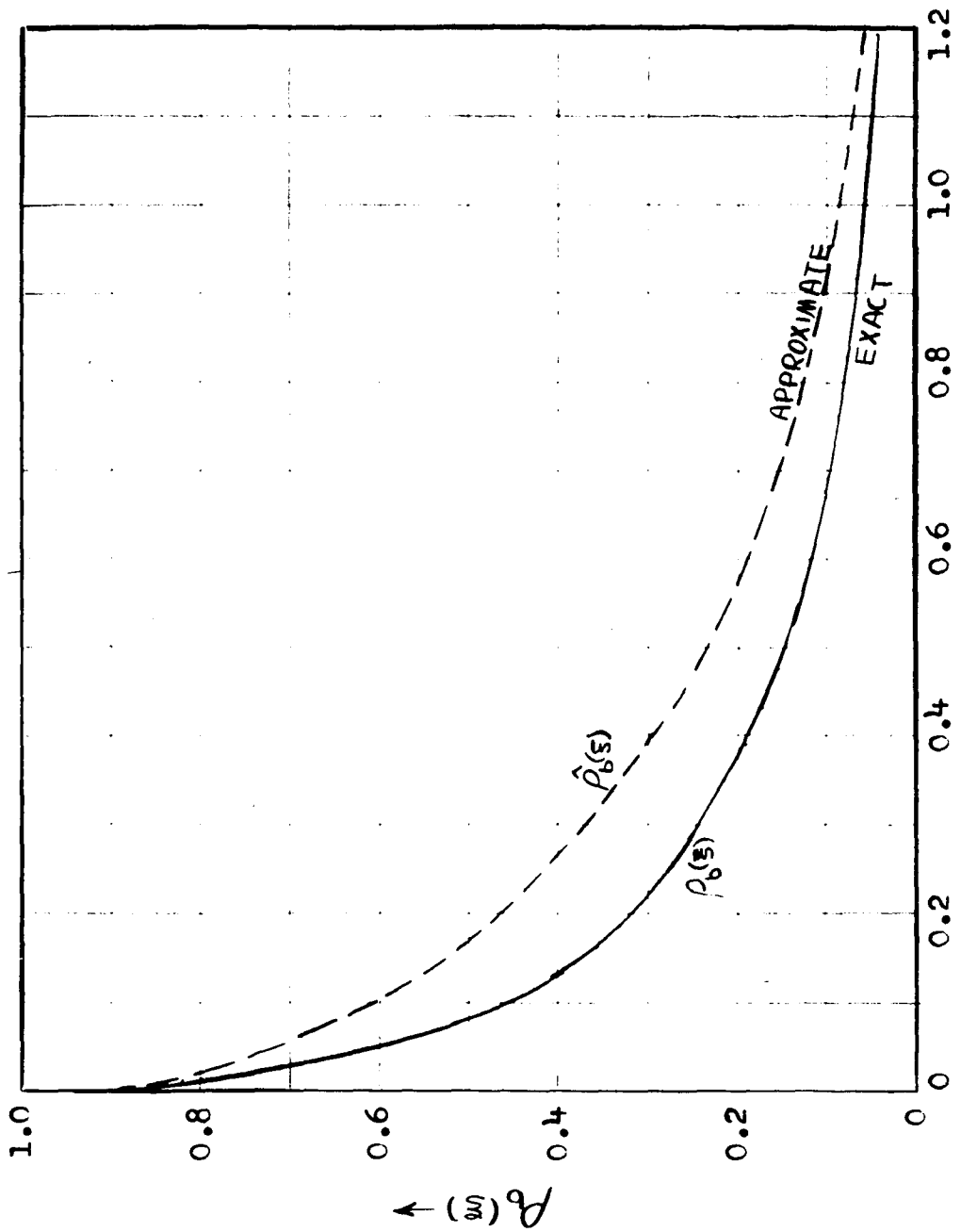


Figure 2.9.1-7 Comparison Between Exact and Approximate Autocorrelations of b(x)

exponential autocorrelation) is a reasonably accurate one. In an earlier section, we saw that if $b(x) = \beta x$, the first approximation gives an exact representation. We now ask whether there is any other $b(x)$ which gives an autocorrelation which is exactly exponential. In comparing the two series (2.9-1) and (2.9.1-5), we note that if $\Gamma_{xx}(t)$ is to admit of an exponential representation, it is necessary (for the third terms to agree) that

2.9.1-22) $$\Gamma_{xx}''(0^+) = \frac{[\Gamma_{xx}'(0^+)]^2}{A}$$

2.9.1-23) $$\int_{-\infty}^{+\infty} b^2(x) f_s(x) dx = \frac{\frac{N^4}{4}}{\int_{-\infty}^{+\infty} x^2 f_s dx - M^2}$$

In this problem, we can eliminate the mean value without loss of generality. Accordingly, for an exponential autocorrelation, it is necessary that

2.9.1-24) $$\int_{-\infty}^{+\infty} b^2 f_s dx \int_{-\infty}^{+\infty} x^2 f_s dx = \frac{N^4}{4}$$

By Schwartz' inequality,

2.9.1-25) $$\left[\int_{-\infty}^{+\infty} x^2 f_s dx \right] \left[\int_{-\infty}^{+\infty} b^2 f_s dx \right] \geq \left[\int_{-\infty}^{+\infty} x b f_s dx \right]^2$$

By equation (2.6.3-32), the integral on the right has the value $-N^2/2$, so

2.9.1-26) $$\left[\int_{-\infty}^{+\infty} x^2 f_s dx \right] \left[\int_{-\infty}^{+\infty} b^2 f_s dx \right] \geq \frac{N^4}{4}$$

Comparing this with equation (2.9.1-24) shows that in order

to have an exponential autocorrelation, the Schwartz inequality must reduce to an equality. This can happen only if b and x are proportional, i.e., $b = \beta x$, β = const. If this is the case, it was shown in Section 2.6.5 that $\Gamma_{xx}(t)$ is exponential.

However, there is no other b(x) which will give an exponential $\Gamma_{xx}(t)$.

2.9.2 A Second Approximation

There are several ways in which we could proceed to higher orders of approximation. For the second approximation we use

2.9.2-1) $$\tilde{\Gamma}_{xx}(t) = m^2 + Ae^{-\alpha|t|}\left\{1 + \frac{[\Gamma_{xx}''(0^+) - A\alpha^2](1-e^{-\alpha|t|})}{2A\alpha^2}\right\}$$

The form chosen here is somewhat arbitrary, but the motivating idea was to obtain an expression which would agree with equation (2.9-1) through the third term, while retaining the generally exponential character of Γ_{xx}. It may easily be shown, by differentiating the above expression and letting $t \to 0$, that

2.9.2-2) $$\tilde{\Gamma}_{xx}'(0^+) = \Gamma_{xx}'(0^+)$$

2.9.2-3) $$\tilde{\Gamma}_{xx}''(0^+) = \Gamma_{xx}''(0^+)$$

In the same way, one could proceed to higher orders. Of course, nothing has been said about the convergence of this process.

To find whether the second approximation is better than the first, we also apply it to the infinite-clipping prob-

lem of Section 2.8. In this case, it is easily shown that

2.9.2-4) $$\tilde{\Gamma}_{xx}(t) = \frac{N^4}{2} e^{-\frac{|t|}{N^2}} \left\{ 1 + \frac{1}{2}\left(1 - e^{-\frac{|t|}{N^2}}\right)^2 \right\}$$

Applying the same normalization and change of variable as before,

2.9.2-5) $$\tilde{\rho}_x(\xi) = e^{-2|\xi|} \left\{ 1 + \frac{1}{2}\left(1 - e^{-2|\xi|}\right)^2 \right\}$$

Figure 2.9.2-1 shows the comparison between the exact result $\Gamma_{xx}(t)$ and this second approximation. The agreement is rather good, and certainly a considerable improvement over that of the first approximation shown in Figure 2.9.1-3.

The spectral density corresponding to the second approximation is

2.9.2-6) $$\tilde{G}_{xx}(\omega) = \frac{N^6}{\pi(1+N^4\omega^2)} \left\{ 1 + \frac{6(1 - N^4\omega^2)}{(4+N^4\omega^2)(9+N^4\omega^2)} \right\}$$

Normalizing against the exact magnitude, and making the customary change of variable,

2.9.2-7) $$\tilde{\Phi}_x(u) = \frac{4}{5} \frac{1}{\left(1+\frac{u^2}{4}\right)} \left\{ 1 + \frac{24(4-u^2)}{(16+u^2)(36+u^2)} \right\}$$

This is compared with the exact result in Figure 2.9.2-2. Once again, the agreement is good, and is an improvement over the first approximation of Figure 2.9.1-4.

Apparently, then, the proposed form of the second approximation works rather well in this example. We do not know to what extent this conclusion may be extended to other problems.

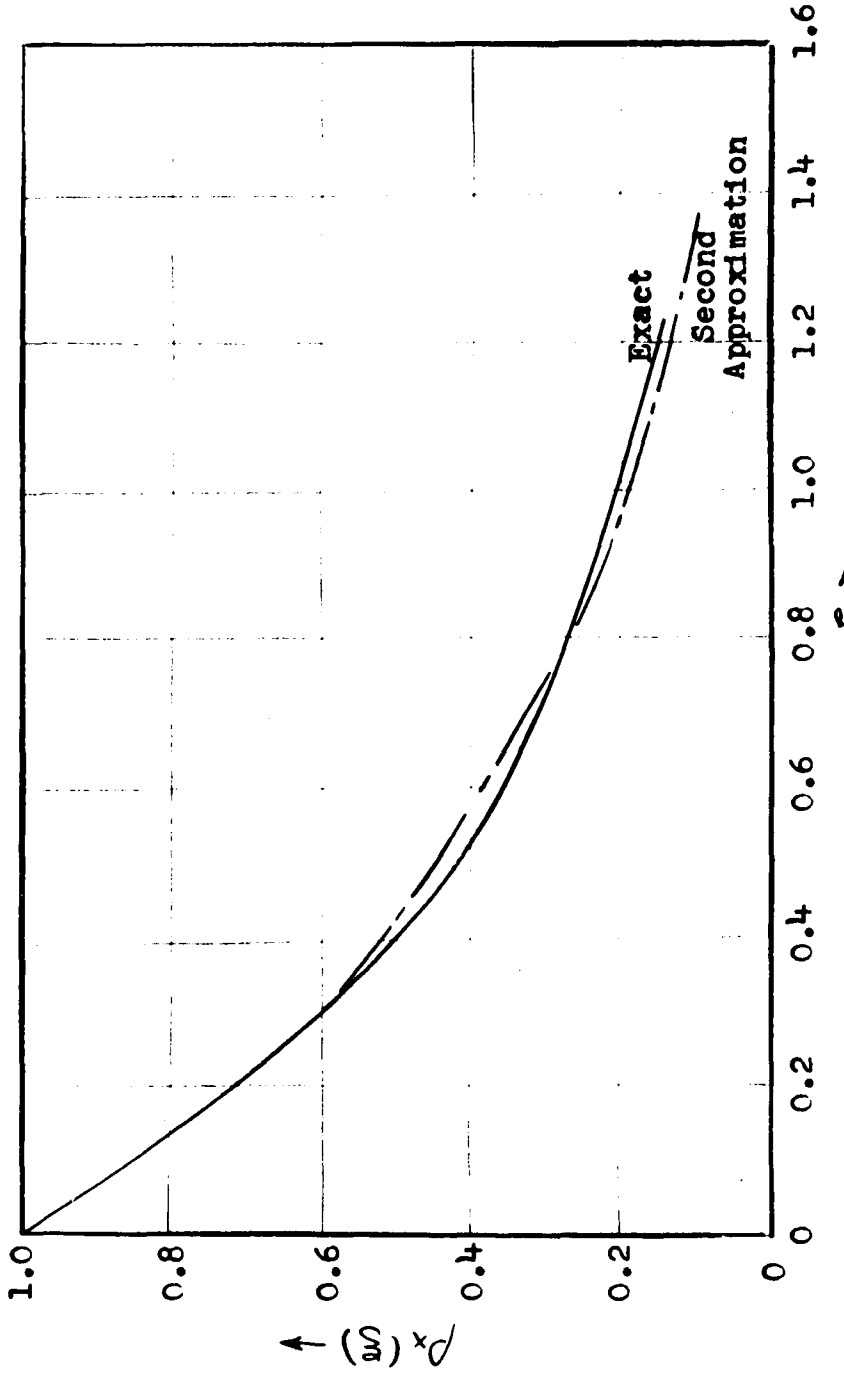


Figure 2.9.2-1 Comparison of Exact Autocorrelation with Second Approximation

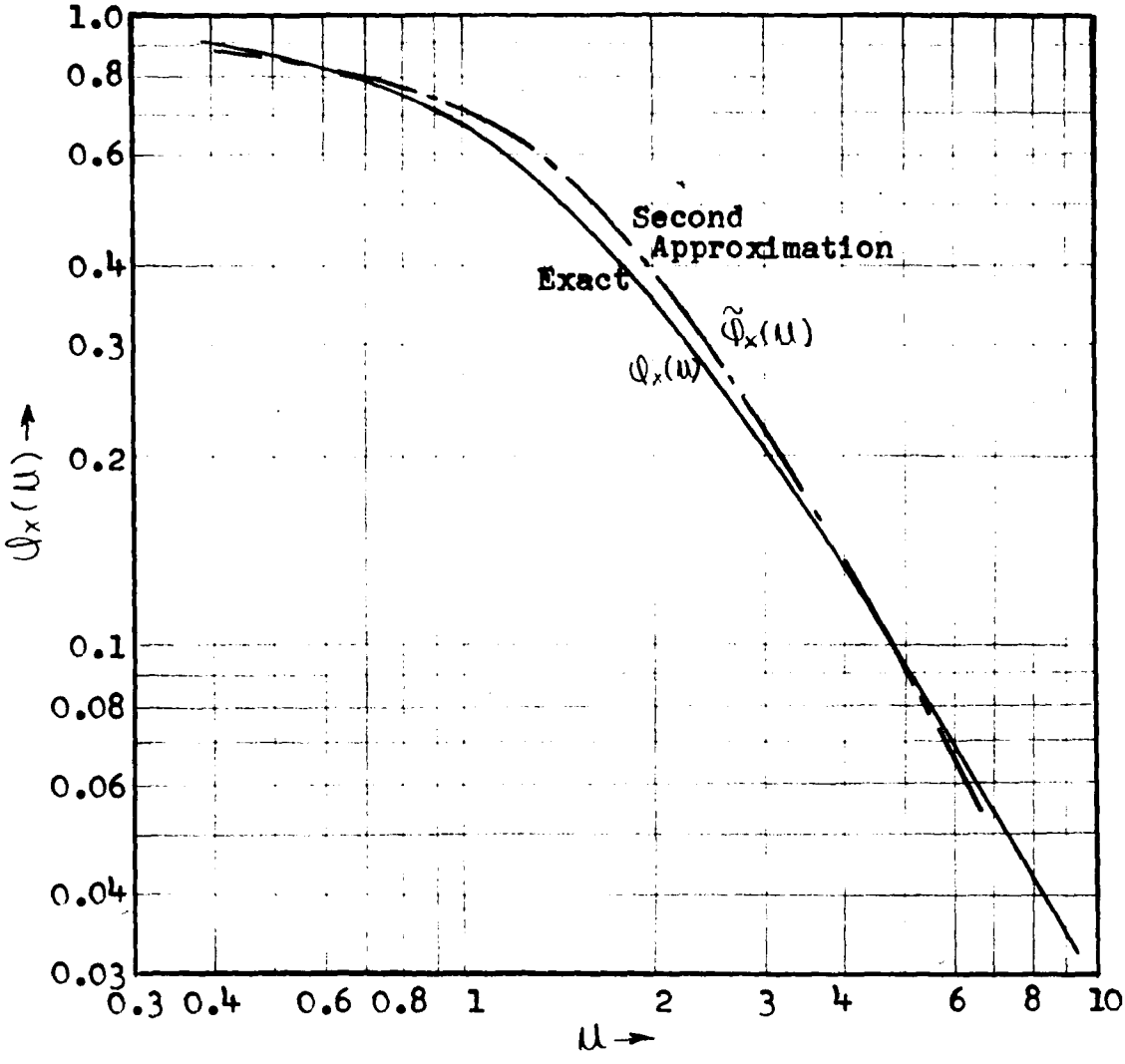


Figure 2.9.2-2 Comparison Between Exact and Second-Approximate Spectra

2.9.3 Booton's Approximation

Booton's method [5] is the only well-known one for calculating spectra of the output of nonlinear systems. Its analogy with the foregoing has already been noted. Accordingly, we apply this technique to the infinite-clipping example of Section 2.8, and see how it compares with the approximations already considered.

It should be emphasized, however, that this problem, being first-order and strongly nonlinear, may be expected to place the Booton approximation in the worst possible light. It is clear, as Booton himself states, that the loop to which the method is applied should have heavy filtering of the high frequencies, i.e. be strongly low-pass in nature. A first-order system affords the poorest possible fulfillment of this requirement. The results determined here for the Booton method are reasonably good, but they should not be taken as an indication of its accuracy when applied to higher-order systems.

To apply the Booton method to the problem at hand, the system being considered is

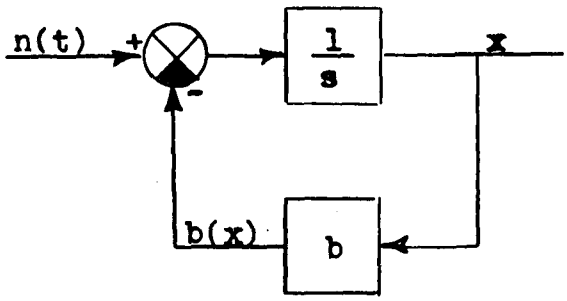


Figure 2.9.3-1 Block Diagram of First-Order System

We now assume with Booton, that the signals at all points in the loop are Gaussian. If x is Gaussian, with standard deviation σ_x , then the "equivalent gain" of the nonlinear block is

2.9.3-1) $$k_b = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_x^3} \int_{-\infty}^{+\infty} x\, b(x)\, e^{-\frac{x^2}{2\sigma_x^2}} dx$$

Introducing the pertinent b(x), which is given in equation (2.8-1), the equivalent gain becomes

2.9.3-2) $$k_b = \frac{2}{\sigma_x^3 \sqrt{2\pi}} \int_0^{\infty} x\, e^{-\frac{x^2}{2\sigma_x^2}} dx$$

This integral may be evaluated (Bierens de Haan [4] , Table 81, 5), p 122) with the result

2.9.3-3) $$k_b = \frac{1}{\sigma_x} \sqrt{\frac{2}{\pi}}$$

We now consider the "equivalent" linear system

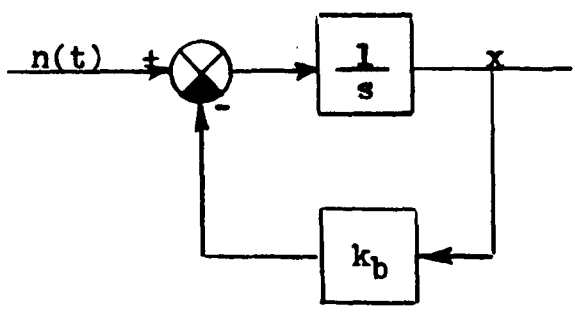


Figure 2.9.3-2 Block Diagram of "Equivalent" Linear System

The overall transfer function would be

2.9.3-4) $$\frac{X(s)}{n(s)} = \frac{1}{s + k_b}$$

Then, using a well-known result for linear systems,(see e.g. Laning and Battin [31] eq. (5.3-12), p 197) the spectrum of x is

2.9.3-5) $$\hat{\hat{G}}_{xx}(\omega) = \left|\frac{1}{j\omega + k_b}\right|^2 G_{nn}(\omega) = \frac{G_{nn}(\omega)}{\omega^2 + k_b^2}$$

where $G_{nn}(\omega)$ is the spectral density of the input noise n(t). It is now necessary to determine $G_{nn}(\omega)$ in the spectral density units being used here.

This may be done by noting that the autocorrelation of the noise has been taken to be

2.9.3-6) $$\Gamma_{nn}(t) = N^2 \delta(t)$$

If this is approximated by an exponential,

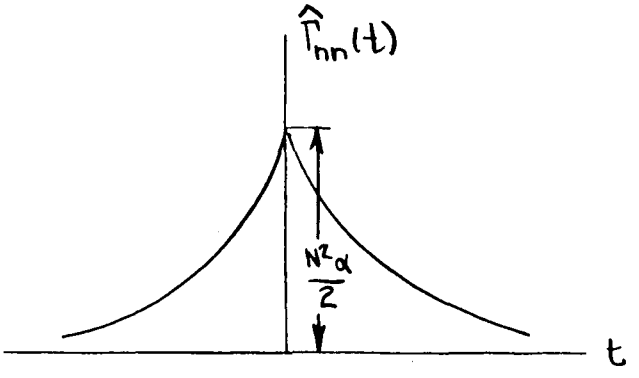


Figure 2.9.3-3 Approximate Autocorrelation for White Noise

of the form $\hat{\Gamma}_{nn}(t) = \frac{N^2\alpha}{2} e^{-\alpha|t|}$, then the approximate spectrum is

2.9.3-7) $$\hat{G}_{nn}(\omega) = \frac{\alpha N^2}{\pi} \int_0^\infty e^{-\alpha t} \cos\omega t \, dt = \frac{\alpha^2 N^2}{\pi(\alpha^2 + \omega^2)}$$

Taking the limit as $\alpha \to \infty$ gives

2.9.3-8) $$G_{nn}(\omega) = \frac{N^2}{\pi}$$

Substituting this into equation (2.9.3-5) yields

2.9.3-9) $$\hat{\hat{G}}_{xx}(\omega) = \frac{N^2}{\pi} \frac{1}{\omega^2 + k_b^2}$$

Integrating over all frequencies to get the total power,

2.9.3-10) $$\sigma_x^2 = \int_0^\infty \hat{\hat{G}}_{xx}(\omega)\, d\omega = \frac{N^2}{2k_b}$$

This equation may now be used, together with equation (2.9.3-3) to determine the "operating point". The result is

2.9.3-11) $$\sigma_x^2 = \frac{\pi N^4}{4}$$

2.9.3-12) $$k_b = \frac{2}{\pi N^2}$$

from which (2.9.3-9) gives

2.9.3-13) $$\hat{\hat{G}}_{xx}(\omega) = \frac{N^6\pi}{4} \frac{1}{1 + \frac{\pi^2 N^4}{4}\omega^2}$$

The autocorrelation may be determined from this by the relation

2.9.3-14) $$\hat{\hat{\Gamma}}_{xx}(t) = \int_0^\infty \hat{\hat{G}}_{xx}(\omega)\cos\omega t\, d\omega$$

and the result is

2.9.3-15) $$\hat{\hat{\Gamma}}_{xx}(t) = \frac{N^4\pi}{4} e^{-\frac{4}{\pi}\left(\frac{|t|}{2N^2}\right)}$$

Qualitatively this spectrum and autocorrelation behave just as do the comparable quantities for the nonlinear system. For this reason, it is possible to normalize them in much the same way. If, then, we make the substitutions

2.9.3-16) $$\frac{t}{2N^2} = \xi$$

2.9.3-17) $$2N^2\omega = \mu$$

and if we normalize the magnitudes of the two functions against the magnitude of the exact results, then

2.9.3-18) $$\hat{\hat{\rho}}_x(\xi) = \frac{\pi}{2} e^{-\frac{4}{\pi}|\xi|}$$

2.9.3-19) $$\hat{\hat{\Phi}}_x(\mu) = \frac{\frac{\pi^2}{5}}{1 + \frac{\pi^2}{16}\mu^2}$$

$\hat{\hat{\rho}}_x(\xi)$ is not a normalized autocorrelation because when $\xi = 0$, $\hat{\hat{\rho}}_x(\xi)$ does not equal unity. The reason is that the Booton approximation gives too much total power in this case. The functions above may be compared directly with the corresponding quantities derived from the exact analysis. This is done in the two figures following.

While both spectrum and autocorrelation have about the correct shape, the total power is about 50% high, so both curves lie above the correct ones. Still, the Booton approximation does predict the correct trend with input amplitude.

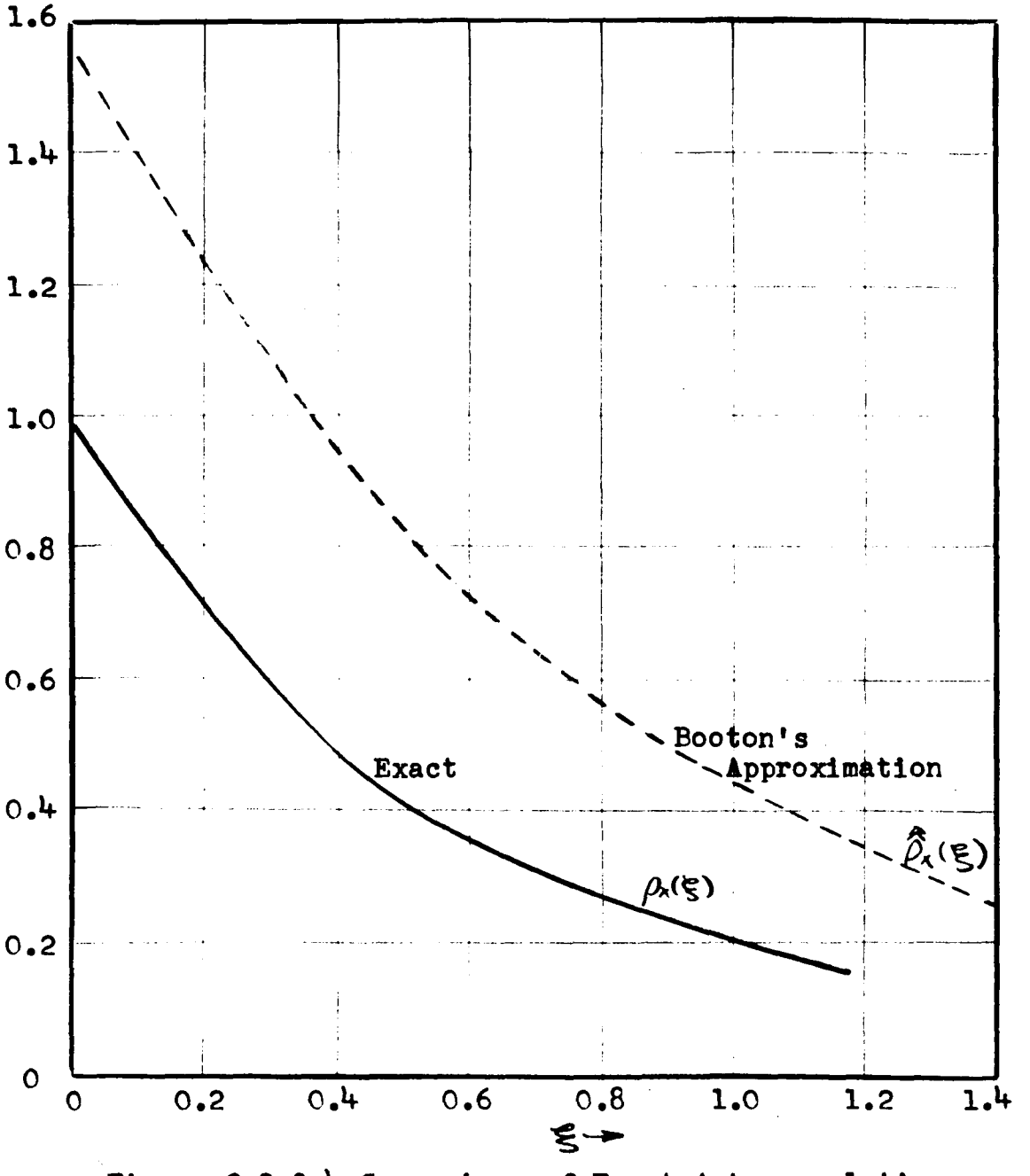


Figure 2.9.3-4 Comparison of Exact Autocorrelation With Booton's Approximation

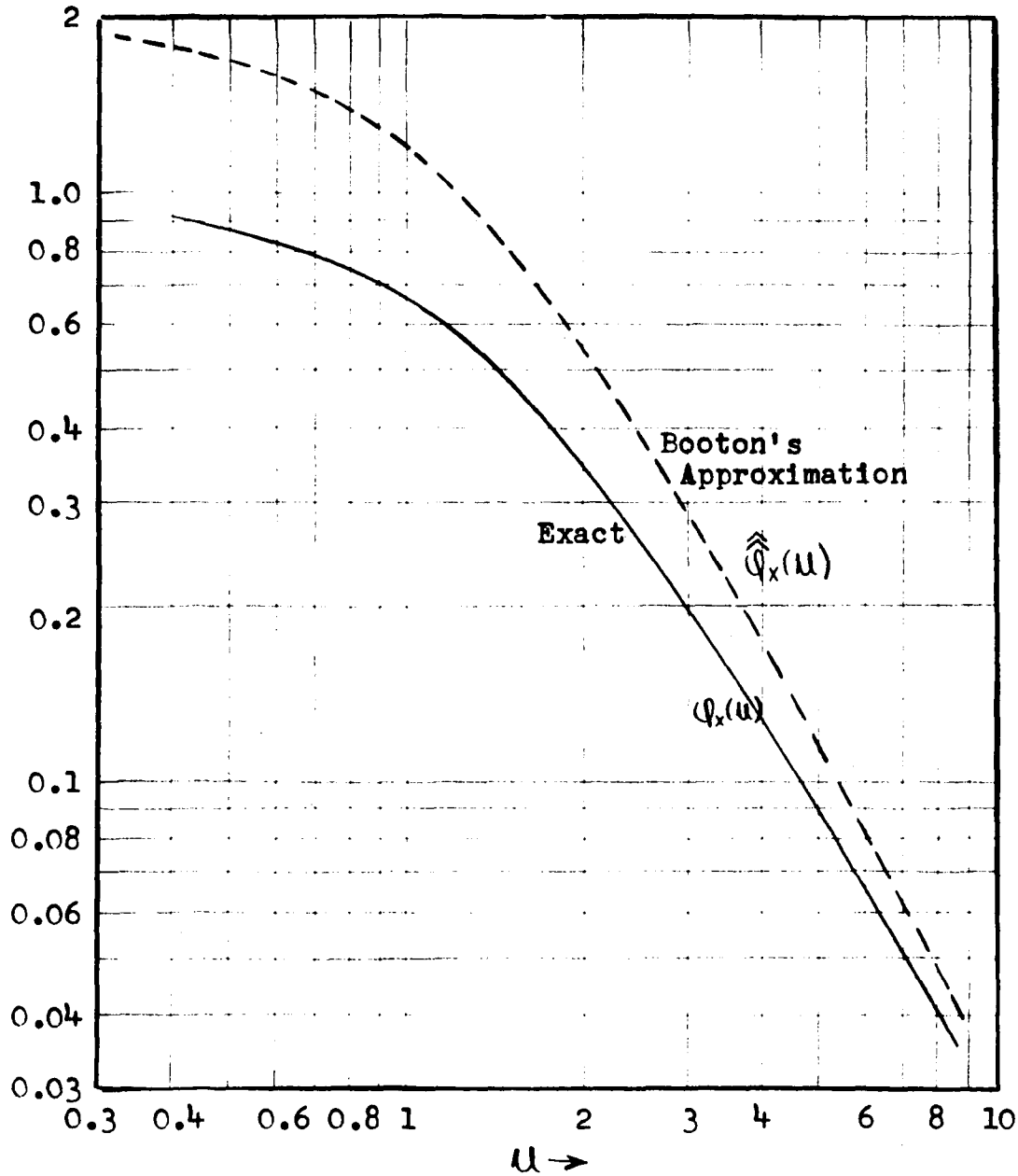


Figure 2.9.3-5 Comparison Between Exact Spectrum and Booton-Approximate Spectrum

2.10 Other Approximate Methods

To determine spectra and autocorrelations exactly, it is in principle necessary to obtain an exact analytical solution of a parabolic partial differential equation. This solution must then be integrated with respect to the space variable. It is to be expected that the number of cases in which this program can be carried through (in closed form) is small compared to the class of interesting problems.

One is naturally led, then, to consider approximations of various sorts, either analytical or numerical. An example of the former was considered in the previous section. Here we discuss, in a largely qualitative manner, certain other possibilities.

The first is that of numerical solution (approximation) of the Fokker-Planck equation. The theory and practice of numerical solution of parabolic equations is well-advanced, though possibly more attention has been given to elliptic and hyperbolic equations. At any rate, we do not discuss these matters here other than to point out one special problem.

This is the fact that the space variable has an infinite range, $-\infty < x < +\infty$. Numerical techniques cannot treat this case directly. It is necessary either to truncate the probem in some way, or to map the infinite range into some finite range.

As an example of the latter, consider a new space variable $z(x)$ where z is a unique invertible function of x. The Fokker-Planck equation may be written in terms of z.

2.10-1) $$\frac{\partial v}{\partial t} = \frac{N^2}{2}\left[z'^2 \frac{\partial^2 v}{\partial z^2} + z'' \frac{\partial v}{\partial z} \right] + z' \frac{\partial}{\partial z}\left[h(z)\, v \right]$$

where $h(z) = b\,[x(z)]$. As an example of the type of mapping we want, let

2.10-2) $$\begin{cases} z = \tan^{-1} x \\ \tan z = x \end{cases}$$

In this case, (2.10-1) becomes

2.10-3) $$\frac{\partial v}{\partial t} = \frac{N^2}{2}\cos^4 z \frac{\partial^2 v}{\partial z^2} + \left[\cos^2 z\, h(z) - N^2 \sin z \cos^3 z\right]\frac{\partial v}{\partial z} + \cos^2 z\, h'(z)\, v$$

The range $-\infty < x < +\infty$ has been mapped onto $-\frac{\pi}{2} < z < +\frac{\pi}{2}$. The transformed equation is a singular one, i.e. the coefficient of $\frac{\partial^2 v}{\partial z^2}$ vanishes on the boundary of the region. Feller [18] gave the theoretical basis for treating such boundary value problems, but they are still somewhat unusual.

Rather than make numerical approximations to the real problem, it may be possible to generate other stochastic problems which approximate the real one in some way, and solve the approximating problem. For example, Brownian motion (the Wiener process) may be viewed as the limiting case of a sequence of discrete random walk problems. It is natural, then, to expect that solving a discrete random walk problem which is sufficiently far in the sequence would permit approximation of the Brownian motion to any desired accuracy.

We do not develop this approach in detail here, but it seems clear that if both time and space are quantized, it is possible to develop a Markoff chain problem which may be

treated by well-known methods. These methods are somewhat simpler than those for continuous-parameter processes, and are well-suited for treatment by numerical means, e.g. matrix multiplications.

In any event, it appears that practical computation of spectra will require some sort of approximate approach, and such approaches should be studied, their errors and convergence investigated, etc.

CHAPTER 3

SECOND-ORDER SYSTEMS

We now turn to consideration of certain classes of second-order problems. A number of new practical difficulties arise, but conceptually, the relation between first- and second-order problems is quite close.

There are few rigorous results available for the second-order case, though some second-order problems have been considered heuristically. See, e.g. references [9], [30], [51].

3.1 Statement of the Second-Order Problem

The class of second-order problems we choose to consider is defined by the equations

$$\begin{aligned} \dot{x} &= P(x,y) + n(t) \\ \dot{y} &= Q(x,y) \end{aligned} \tag{3.1-1}$$

where n(t) is white Gaussian noise and P and Q are functions whose regularity properties will be discussed below. This is a time-invariant second-order system, and with appropriate conditions on P and Q, it is reasonable to expect x,y to be a two-dimensional Markoff process. It also seems reasonable that if P and Q satisfy Lipschitz conditions in x and y, that unique solutions of (3.1-1) will exist. It seems probable that the Lipschitz requirement is not a necessary condition.

If P and Q are once-differentiable, the Fokker-Planck equation for this case is (see, e.g. Lévy [34] p 89)

$$\frac{\partial f}{\partial t} = \frac{N^2}{2}\frac{\partial^2 f}{\partial x^2} - \frac{\partial}{\partial x}\left[P(x,y)\,f\right] - \frac{\partial}{\partial y}\left[Q(x,y)\,f\right] \tag{3.1-2}$$

The function f is now a two-dimensional conditional density function $f(x,y,t|x_o,y_o)$. It is the bivariate density at t, given that $x(0) = x_o$, $y(0) = y_o$. Accordingly (3.1-2) is to be solved subject to the boundary condition

3.1-3) $$f(x,y,0) = \delta(x-x_o)\,\delta(y-y_o)$$

This assumes P and Q are continuous and differentiable everywhere.

3.2 Cross-Correlations in the Second-Order Problem

The type of correlation problem we consider may be represented in block diagram form as shown below.

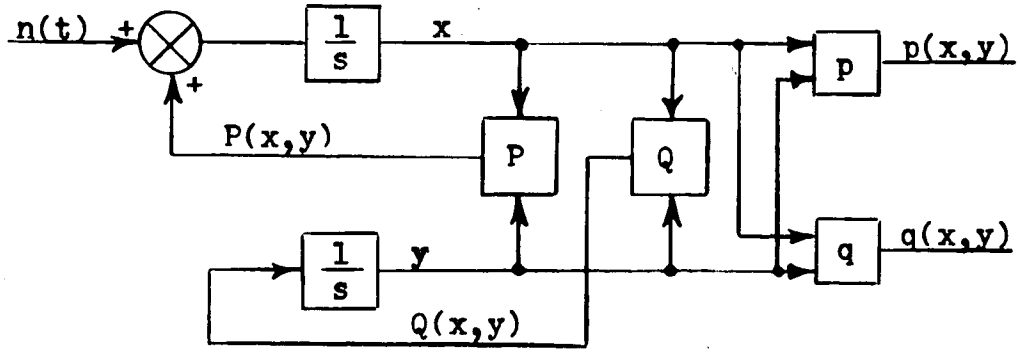


Figure 3.2-1 Block Diagram of the Second-Order Problem

The task, then, is to determine the cross-correlation between the random processes p and q.

The steady-state density $f_s(x,y)$ is now much more difficult to find. If, for example, we set $\frac{\partial f}{\partial t} = 0$ in (3.1-2), we must solve the equation

3.2-1) $$\frac{N^2}{2}\frac{\partial^2 f_s}{\partial x^2} - \frac{\partial}{\partial x}\left[P(x,y)\,f_s\right] - \frac{\partial}{\partial y}\left[Q(x,y)\,f_s\right] = 0$$

subject to the requirement that

3.2-2) $$\int_{-\infty}^{+\infty}\int f_s(x,y)\,dx\,dy = 1$$

This is a partial differential equation rather than an ordinary differential equation as in the first-order problem. It has not been solved in general, though we will consider later its solution for a certain class of P and Q.

Further, we do not know the conditions on P and Q which assure a non-dispersive steady-state. We will have to rely on physical intuition for the time being.

At any rate, assuming that $f_s(x,y)$ exists and is known, the cross-correlation between p and q will be

3.2-3) $$\Gamma_{pq}(t) = \int_{-\infty}^{+\infty}\int\int\int p(x_0,y_0)\,q(x,y)\,f_s(x_0,y_0)\,f(x,y,t\,|\,x_0,y_0)\,dx\,dy\,dx_0\,dy_0 \qquad t\geq 0$$

3.2-4) $$\Gamma_{qp}(t) = \int_{-\infty}^{+\infty}\int\int\int q(x_0,y_0)\,p(x,y)\,f_s(x_0,y_0)\,f(x,y,t\,|\,x_0,y_0)\,dx\,dy\,dx_0\,dy_0 \qquad t\geq 0$$

These two equations permit complete determination of the cross-correlation with the aid of $\Gamma_{pq}(t) = \Gamma_{qp}(-t)$.

It is possible to simplify these expressions in the same way as in the first-order case. Define

3.2-5) $$v(x,y,t\,;pf_s) = \int_{-\infty}^{+\infty}\int p(x_0,y_0)\,f_s(x_0,y_0)\,f(x,y,t\,|\,x_0,y_0)\,dx_0\,dy_0$$

It may be shown as before, by direct substitution, that this quantity satisfies the Fokker-Planck equation

3.2-6) $$\frac{\partial v}{\partial t} = \frac{N^2}{2}\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}[Pv] - \frac{\partial}{\partial y}[Qv]$$

subject to the boundary condition

3.2-7) $$v(x,y,0;pf_s) = p(x,y)f_s(x,y)$$

Using this expression, equation (3.2-3 may be written

3.2-8) $$\Gamma_{pq}(t) = \int_{-\infty}^{+\infty}\int q(x,y)\, v(x,y,t;pf_s)\,dx\,dy \quad t\geq 0$$

In an entirely analogous way,

3.2-9) $$\Gamma_{qp}(t) = \int_{-\infty}^{+\infty}\int p(x,y)\, v(x,y,t;qf_s)\,dx\,dy \quad t\geq 0$$

where $v(x,y,t;qf_s)$ is the solution of (3.2-6) subject to the initial condition $v(x,y,0;qf_s) = qf_s$.

These are the general relations from which a number of useful special cases may be derived.

Case 1: p = q In this case, the problems for Γ_{pq} and Γ_{qp} are identical, so only one need be done.

3.2-10) $$\Gamma_{pp}(t) = \Gamma_{pp}(-t) = \int_{-\infty}^{+\infty}\int p(x,y)\, v(x,y,t;pf_s)\,dx\,dy \quad t\geq 0$$

where v is the solution to the Fokker-Planck equation, subject to the initial condition $v = pf_s$.

Case 2: Autocorrelation of x - - In this case, p=q=x .

3.2-11) $$\Gamma_{xx}(t) = \Gamma_{xx}(-t) = \int_{-\infty}^{+\infty}\int x\, v(x,y,t;xf_s)\,dx\,dy \quad t\geq 0$$

Case 3: Autocorrelation of y - - Let p=q=y.

3.2-12) $$\Gamma_{yy}(t) = \Gamma_{yy}(-t) = \int_{-\infty}^{+\infty}\int y\, v(x,y,t;yf_s)\,dx\,dy \quad t\geq 0$$

Case 4: Cross-Correlation of x and y - - Let p=x and q=y.

3.2-13) $$\Gamma_{xy}(t) = \int_{-\infty}^{+\infty}\int y\, v(x,y,t;xf_s)\,dx\,dy \quad t\geq 0$$

3.2-14) $$\Gamma_{yx}(t) = \int_{-\infty}^{+\infty}\int x\, v(x,y,t;yf_s)\,dx\,dy \quad t\geq 0$$

Case 5: Autocorrelation of P - - Let p=q=P.

3.2-15) $$\Gamma_{PP}(t) = \Gamma_{PP}(-t) = \int_{-\infty}^{+\infty}\int P\, v(x,y,t;Pf_s)\,dx\,dy \quad t\geq 0$$

Case 6: Autocorrelation of Q - - Let p=q=Q.

3.2-16) $$\Gamma_{QQ}(t) = \Gamma_{QQ}(-t) = \int_{-\infty}^{+\infty}\int Q\, v(x,y,t;Qf_s)\,dx\,dy \quad t\geq 0$$

Case 7: Cross-Correlation of P and Q - - Let p=P and q=Q.

3.2-17) $$\Gamma_{PQ}(t) = \int_{-\infty}^{+\infty}\int Q\, v(x,y,t;Pf_s)\,dx\,dy \quad t\geq 0$$

3.2-18) $$\Gamma_{QP}(t) = \int_{-\infty}^{+\infty}\int P\, v(x,y,t;Qf_s)\,dx\,dy \quad t\geq 0$$

3.3 Example 4: A Linear Second-Order Problem

To illustrate the application of this method to a second-order problem, we consider a case in which the answer is known. We wish to determine the autocorrelation of y, where y is the solution of the second-order stochastic differential equation

3.3-1) $$\ddot{y} + a\dot{y} + \beta y = n(t)$$

where a, β are positive constants, n(t) is white Gaussian noise of magnitude N. If we define $\dot{y}=x$, then this equation may be written

3.3-2) $$\dot{x} = -ax - \beta y + n(t)$$

3.3-3) $$\dot{y} = x$$

which is of the form being considered here, with

3.3-4) $$\begin{cases} P = -\alpha x - \beta y \\ Q = x \end{cases}$$

To determine $f_s(x,y)$, the steady-state distribution, we use (3.2-1). We need to solve

3.3-5) $$\frac{N^2}{2}\frac{\partial^2 f_s}{\partial x^2} + \frac{\partial}{\partial x}\left[(\alpha x+\beta y) f_s\right] - \frac{\partial}{\partial y}\left[x f_s\right] = 0$$

subject to the side condition $\int_{-\infty}^{+\infty}\int f_s\,dx\,dy = 1$. It may easily be verified that the desired distribution is

3.3-6) $$f_s(x,y) = \frac{\alpha\sqrt{\beta}}{N^2\pi}\exp\left\{-\frac{\alpha}{N^2}x^2 - \frac{\beta\alpha}{N^2}y^2\right\}$$

It is now necessary to solve the equation

3.3-7) $$\frac{\partial v}{\partial t} = \frac{N^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x}\left[(\alpha x+\beta y) v\right] - \frac{\partial}{\partial y}\left[x v\right]$$

subject to the initial condition

3.3-8) $$v(x,y,0;yf_s) = y f_s(x,y)$$

For this problem, we restrict ourselves to the overdamped case: $\xi = \frac{\alpha}{2\sqrt{\beta}} > 1$. To completely solve this example would require study of two additional cases: $\xi = 1$ and $0 < \xi < 1$. To show the workability of the method, it appears sufficient to show agreement for only one of the cases.

The solution of this boundary-value problem is given in Appendix C. The result is

3.3-9) $$v(x,y,t;yf_s) = \frac{f_s(x,y)}{2\omega_n\sqrt{\xi^2-1}}\left\{(e^{\lambda_2 t} - e^{\lambda_1 t})x + (\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t})y\right\}$$

where $\omega_n = \sqrt{\beta}$, $\lambda_1 = \omega_n(-\xi + \sqrt{\xi^2-1})$, $\lambda_2 = \omega_n(-\xi - \sqrt{\xi^2-1})$.

Now to determine $T_{yy}(t)$, we substitute this result into equation (3.2-12).

$$\Gamma_{yy}(t) = \int_{-\infty}^{+\infty}\int y\, v(x,y,t;yf_s)\,dxdy$$

3.3-10) $$= \int_{-\infty}^{+\infty}\int y\left\{\frac{f_s(x,y)}{\lambda_1-\lambda_2}\left[(e^{\lambda_2 t}-e^{\lambda_1 t})x+(\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t})y\right]\right\}dxdy$$

$$= \frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_1-\lambda_2}\int_{-\infty}^{+\infty}\int xy\, f_s(x,y)\,dxdy + \frac{\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2}\int_{-\infty}^{+\infty}\int y^2 f_s\,dxdy$$

The first term vanishes because of the even symmetry of f_s. Since f_s is separable, the second integral may be separated.

3.3-11) $$\Gamma_{yy}(t) = \frac{\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2}\,\frac{a\sqrt{\beta}}{\pi N^2}\left[\int_{-\infty}^{+\infty} e^{-\frac{ax^2}{N^2}}dx\right]\left[\int_{-\infty}^{+\infty} y^2 e^{-\frac{\beta a}{N^2}y^2}dy\right]$$

Evaluating these integrals,

3.3-12) $$\Gamma_{yy}(t) = \frac{N^2}{8\omega_n^4\,\xi\sqrt{\xi^2-1}}\,(\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t})$$

which is exactly the same as the result obtained from conventional filter theory.

The corresponding spectrum may easily be shown to be

3.3-13) $$G_{yy}(\omega) = \frac{N^2}{\pi(\lambda_1^2+\omega^2)(\lambda_2^2+\omega^2)}$$

3.4 The Second-Order Problem with Discontinuous P(x,y)

It would be desirable to relax the continuity requirements on P and Q in a manner analogous to the way in which discontinuous b(x) were treated in the first-order case.

We retain the requirement that Q(x,y) be continuous and differentiable everywhere, but broaden the class of P(x,y) to include functions which are continuous and differentiable everywhere except possibly along the line $x = h(y)$, where h is a single-valued continuous function for all y. Along this line, a finite discontinuity is allowed.

The derivation here follows quite closely that of Section 2.7. We are now concerned with the bivariate density function f(x,y,t), which is assumed to be once-differentiable everywhere. We consider a class of functions $\varphi(x,y)$ which vanish outside some finite rectangle, are thrice differentiable, and whose third derivatives are uniformly bounded in both x and y. This means that there exists some $K > 0$ such that $\left|\frac{\partial^3 \varphi}{\partial x^3}\right| < K$, $\left|\frac{\partial^3 \varphi}{\partial x^2 \partial y}\right| < K$, etc. Now define

$$3.4\text{-}1) \qquad M(t) = E[\varphi(x,y)] = \iint_{-\infty}^{+\infty} \varphi(x,y) f(x,y,t)\,dx\,dy$$

We will calculate the derivative of M in two different ways.

$$3.4\text{-}2) \qquad M'(t) = \frac{d}{dt}\left\{E[\varphi(x,y)]\right\} = E\left[\frac{d}{dt}\varphi(x,y)\right]$$

The first method is easily done.

$$3.4\text{-}3) \qquad M'(t) = \iint_{-\infty}^{+\infty} \varphi(x,y) \frac{\partial}{\partial t} f(x,y,t)\,dx\,dy$$

To calculate $E\left[\frac{\partial \varphi}{\partial t}\right]$ we will actually calculate $E\left[\lim_{\Delta t \to 0} \frac{\Delta \varphi}{\Delta t}\right]$ by inverting the order of limit and expectation. We will

use

3.4-4) $$M'(t) = \lim_{\Delta t \to 0} E\left[\frac{\Delta \varphi}{\Delta t}\right]$$

Expanding φ in a bivariate Taylor's series, it may be shown that

3.4-5)
$$\frac{\Delta \varphi}{\Delta t} = \frac{\partial \varphi}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial \varphi}{\partial y}\frac{\Delta y}{\Delta t} + \frac{1}{2}\left\{\frac{\partial^2 \varphi}{\partial x^2}\frac{(\Delta x)^2}{\Delta t} + 2\frac{\partial^2 \varphi}{\partial x \partial y}\frac{\Delta x \Delta y}{\Delta t} + \frac{\partial^2 \varphi}{\partial y^2}\frac{(\Delta y)^2}{\Delta t}\right\}$$
$$+ \frac{K}{6}\left[\theta_1 \frac{(\Delta x)^3}{\Delta t} + 3\theta_2 \frac{(\Delta x)^2 \Delta y}{\Delta t} + 3\theta_3 \frac{\Delta x (\Delta y)^2}{\Delta t} + \theta_4 \frac{(\Delta y)^3}{\Delta t}\right]$$

where $|\theta_i| \leq 1$, $i = 1,2,3,4$. Now integrate equations (3.1-1) over a finite time increment from t to $t + \Delta t$.

3.4-6) $$\Delta x = \int_t^{t+\Delta t} P(x,y)\,dt + \int_t^{t+\Delta t} n(t)\,dt$$

3.4-7) $$\Delta y = \int_t^{t+\Delta t} Q(x,y)\,dt$$

The second integral of (3.4-6) is a Gaussian random variable with zero mean and variance $N^2 \Delta t$. We call this integral Δn. Then $E[\Delta n] = 0$, $E[\Delta n^2] = N^2 \Delta t$, $E[\Delta n^3] = 0$, etc. It may be shown that

3.4-8) $$\Delta x = P(x,y)\Delta t + \Delta n + O(\Delta t^{3/2})$$

3.4-9) $$\Delta y = Q(x,y)\Delta t + O(\Delta t^{3/2})$$

where P and Q are evaluated at the beginning of the interval. This may be seen by expanding the integrands in a Taylor's series about the initial value. Where P is differentiable,

3.4-10) $$\int_t^{t+\Delta t} P(x,y)\,dt = \int_t^{t+\Delta t}\left\{P(x_0,y_0) + (x-x_0)\frac{\partial P}{\partial x}\Big|_{x_1,y_1} + (y-y_0)\frac{\partial P}{\partial y}\Big|_{x_1,y_1}\right\}dt$$

3.4-10) $$= P(x_0,y_0)\,\Delta t + \left.\frac{\partial P}{\partial x}\right|_{x_1,y_1}\int_t^{t+\Delta t}(x-x_0)dt + \left.\frac{\partial P}{\partial y}\right|_{x_1,y_1}\int_t^{t+\Delta t}(y-y_0)dt$$

where x_0 and y_0 denote x(t) and y(t), while

3.4-11) $$\left.\begin{aligned} x_1 &= x_0 + c_1\Delta x \\ y_1 &= y_0 + c_1\Delta y \end{aligned}\right\} \quad 0 \le c_1 \le 1$$

The partial derivatives are finite for points not on the line of discontinuity. The quantity $(x - x_0)$ is $O(\sqrt{\Delta t})$, and integrating it over and interval Δt gives a result which is $O(\Delta t^{3/2})$. From this type of consideration, equations (3.4-8) and 3.4-9) follow.

Substituting equations (3.4-8) and (3.4 -9) into (3.4-5), and taking the expectation of the result, passing to the limit as in equation (3.4-4) gives

3.4-12) $$M'(t) = E\left\{\frac{d\varphi}{dt}\right\} = \int_{-\infty}^{+\infty}\iint\left\{P\frac{\partial\varphi}{\partial x} + Q\frac{\partial\varphi}{\partial y} + \frac{N^2}{2}\frac{\partial^2\varphi}{\partial x^2}\right\} f(x,y,t)\,dx\,dy$$

If it were not for the discontinuity of P, this expression could be integrated by parts, and equation (3.1-2) could be derived by equating the result to equation (3.4-3). Because of the discontinuity, however, we proceed differently. For example, consider the first integral of equation (3.4-12).

$$\iint_{-\infty}^{+\infty} Pf\,\frac{\partial\varphi}{\partial x}\,dx\,dy = \int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty} P\,f\,\frac{\partial\varphi}{\partial x}\,dx$$

3.4-13) $$= \int_{-\infty}^{+\infty}dy\left\{\int_{-\infty}^{h(y)} Pf\,\frac{\partial\varphi}{\partial x}\,dx + \int_{h(y)}^{+\infty} Pf\,\frac{\partial\varphi}{\partial x}\,dx\right\}$$

$$3.4\text{-}13) = \int_{-\infty}^{+\infty} dy \left\{ Pf\varphi\Big|_{h(y^-)} - Pf\varphi\Big|_{h(y^+)} - \int_{-\infty}^{h(y)} \varphi \frac{\partial}{\partial x}[Pf]\,dx - \int_{h(y)}^{+\infty} \varphi \frac{\partial}{\partial x}[Pf]\,dx \right\}$$

Proceeding in the same way with the other integrals, and equating equations (3.4-12) and (3.4-3), it may be shown that

$$\int_{-\infty}^{+\infty} \varphi \left\{ -\left[\frac{N^2}{2}\frac{\partial f}{\partial x} - Pf\right]_{h(y^-)} + \left[\frac{N^2}{2}\frac{\partial f}{\partial x} - Pf\right]_{h(y^+)} \right\} dy$$

$$3.4\text{-}14) \quad + \int_{-\infty}^{+\infty} dy \int_{-\infty}^{h(y)} \varphi \left\{ -\frac{\partial}{\partial x}[Pf] - \frac{\partial}{\partial y}[Qf] + \frac{N^2}{2}\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} \right\} dx$$

$$+ \int_{-\infty}^{+\infty} dy \int_{h(y)}^{+\infty} \varphi \left\{ -\frac{\partial}{\partial x}[Pf] - \frac{\partial}{\partial y}[Qf] + \frac{N^2}{2}\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} \right\} dx = 0$$

In order for this to be true for any φ in the class considered, it is necessary that

$$3.4\text{-}15) \quad \left[\frac{N^2}{2}\frac{\partial f}{\partial x} - Pf\right]_{h(y^-),y^-} = \left[\frac{N^2}{2}\frac{\partial f}{\partial x} - Pf\right]_{h(y^+),y^+}, \text{ all } y$$

$$3.4\text{-}16) \quad \frac{\partial f}{\partial t} = \frac{N^2}{2}\frac{\partial^2 f}{\partial x^2} - \frac{\partial}{\partial x}[Pf] - \frac{\partial}{\partial y}[Qf] \qquad \begin{matrix} x > h(y) \\ x < h(y) \end{matrix}$$

The Fokker-Planck equation holds except on the boundary, and there the continuity condition (3.4-15) holds. If, as we assume, f is continuous everywhere, $\frac{\partial f}{\partial x}$ must be discontinuous across the boundary, if $f \neq 0$. If $f = 0$, then $\frac{\partial f}{\partial x}$ must be continuous.

3.5 Relations Among the Correlations

As in the first-order case, we are able to derive relationships among the various correlations present in the system, using the Kolmogoroff equations. We consider now the bivariate transition density function $f_T(x_1y_1;x_2,y_2,t)$ which is the density associated with having values x_1,y_1 at a Certain time, and x_2,y_2 at a time t seconds later (stationarity is assumed). This density f_T satisfies the bivariate forward and backward equations of Kolmogoroff. These may be derived by standard methods (see e.g. Bharucha-Reid [3] p 160) with the result

3.5-1) $$\frac{\partial f_T}{\partial t} = \frac{N^2}{2}\frac{\partial^2 f_T}{\partial x_1^2} + P(x_1,y_1)\frac{\partial f_T}{\partial x_1} + Q(x_1,y_1)\frac{\partial f_T}{\partial y_1}$$

3.5-2) $$\frac{\partial f_T}{\partial t} = \frac{N^2}{2}\frac{\partial^2 f_T}{\partial x_2^2} - \frac{\partial}{\partial x_2}\left[P(x_2,y_2)f_T\right] - \frac{\partial}{\partial y_2}\left[Q(x_2,y_2)f_T\right]$$

(3.5-1) is the "first" or "backward" equation. (3.5-2) is the "second" or "forward" equation, which is identical with the Fokker-Planck equation which has already been derived.

If we wish to find the cross-correlation between two random processes p(x,y) and q(x,y), the formula is, by analogy with (3.2-3) and (3.2-4).

3.5-3) $$\Gamma_{pq}(t) = \iiiint_{-\infty,\ t\geq 0}^{+\infty} p(x_1,y_1)q_f(x_2,y_2)f_S(x_1,y_1)f_T(x_1,y_1;x_2,y_2,t)\,dx_1dy_1dx_2dy_2$$

3.5-4) $$\Gamma_{qp}(t) = \iiiint_{-\infty,\ t\geq 0}^{+\infty} q_f(x_1,y_1)p(x_2,y_2)f_S(x_1,y_1)f_T(x_1,y_1;x_2,y_2,t)\,dx_1dx_2dy_1dy_2$$

Now note that (3.5-3) may be written in two different ways:

$$3.5\text{-}5)\quad \Gamma_{pq}(t)=\int_{-\infty}^{+\infty}\int dx_1\,dy_1\,p(x_1,y_1)\,f_S(x_1,y_1)\int_{-\infty}^{+\infty}\int q(x_2,y_2)\,f_T(x_1,y_1;x_2,y_2,t)\,dx_2\,dy_2 \qquad t\geq 0$$

$$3.5\text{-}6)\quad \Gamma_{pq}(t)=\int_{-\infty}^{+\infty}\int q(x_2,y_2)\,dx_2\,dy_2\int_{-\infty}^{+\infty}\int p(x_1,y_1)\,f_S(x_1,y_1)\,f_T(x_1,y_1;x_2,y_2,t)\,dx_1\,dy_1 \qquad t\geq 0$$

If we now define

$$3.5\text{-}7)\quad u(x_1,y_1,t;q)=\int_{-\infty}^{+\infty}\int q(x_2,y_2)\,f_T(x_1,y_1;x_2,y_2,t)\,dx_2\,dy_2$$

$$3.5\text{-}8)\quad v(x_2,y_2,t;pf_S)=\int_{-\infty}^{+\infty}\int p(x_1,y_1)\,f_S(x_1,y_1)\,f_T(x_1,y_1;x_2,y_2,t)\,dx_1\,dy_1 \qquad t\geq 0$$

then it may be shown by direct substitution that u and v satisfy the first and second Kolmogoroff equations respectively. Further, $u(x_1,y_1,0;q) = q(x_1,y_1)$ and $v(x_2,y_2,0;pf_s) = p(x_2,y_2)f_s(x_2,y_2)$.

Using these facts, equations (3.5-5) and (3.5-6) may be written

$$3.5\text{-}9)\quad \Gamma_{pq}(t)=\int_{-\infty}^{+\infty}\int p(x_1,y_1)\,f_S(x_1,y_1)\,u(x_1,y_1,t;q)\,dx_1\,dy_1\,,\quad t\geq 0$$

$$3.5\text{-}10)\quad \Gamma_{pq}(t)=\int_{-\infty}^{+\infty}\int q(x_2,y_2)\,v(x_2,y_2,t;pf_S)\,dx_2\,dy_2\,,\quad t\geq 0$$

The second of these is identical with equation (3.2-8). The x's and y's appearing in these integrals are merely dummy variables, so the subscripts may be dropped.

We may apply (3.5-10) to calculation of the autocorrelation of y.

3.5-11) $$\Gamma_{yy}(t) = \iint_{-\infty}^{+\infty} y\, v(x,y,t;yf_s)\,dx\,dy \quad t \geq 0$$

Now differentiate, and use the "second" equation.

3.5-12) $$\Gamma'_{yy}(t) = \iint_{-\infty}^{+\infty} y\left\{\frac{N^2}{2}\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}[Pv] - \frac{\partial}{\partial y}[Qv]\right\}dx\,dy$$

Integrating by parts,

3.5-13) $$\Gamma'_{yy}(t) = \iint_{-\infty}^{+\infty} Q\, v(x,y,t;yf_s)\,dx\,dy = \Gamma_{yQ}(t) \quad t \geq 0$$

Alternatively, we could use (3.5-9) to compute $\Gamma_{yy}(t)$.

3.5-14) $$\Gamma_{yy}(t) = \iint_{-\infty}^{+\infty} y f_s\, u(x,y,t;y)\,dx\,dy \quad t \geq 0$$

Differentiate, and apply the "first" equation.

3.5-15) $$\Gamma'_{yy}(t) = \iint_{-\infty}^{+\infty} y f_s\left\{\frac{N^2}{2}\frac{\partial^2 u}{\partial x^2} + P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y}\right\}dx\,dy$$

Integrating by parts yields

3.5-16) $$\Gamma'_{yy}(t) = \iint_{-\infty}^{+\infty} u\, y\left\{\frac{N^2}{2}\frac{\partial^2 f_s}{\partial x^2} - \frac{\partial}{\partial x}[Pf_s] - \frac{\partial}{\partial y}[Qf_s]\right\}dx\,dy - \iint_{-\infty}^{+\infty} Q f_s\, u(x,y,t;y)\,dx\,dy$$

According to equation (3.2-1), the $\{\}$ expression in the first integrand is zero. Accordingly,

3.5-17) $$\Gamma'_{yy}(t) = -\iint_{-\infty}^{+\infty} Q f_s\, u(x,y,t;y)\,dx\,dy = -\Gamma_{Qy}(t) \quad t \geq 0$$

Comparison of this with equation (3.5-13) shows that

3.5-18) $$\Gamma_{yQ}(t) = -\Gamma_{Qy}(t) \quad t \geq 0$$

Since it is always true that $\Gamma_{Qy}(t) = \Gamma_{yQ}(-t)$, we have

3.5-19) $$\Gamma_{yQ}(t) = -\Gamma_{yQ}(-t)$$

and the cross-correlation between y and Q is odd-symmetric. In order for Γ_{yQ} to be continuous, then, it is necessary that $\Gamma_{yQ}(0) = 0$, from which it follows that

3.5-20) $$\Gamma'_{yy}(0^+) = 0$$

By a computation exactly similar to the derivation of equation (3.5-17), it may be shown that

3.5-21) $$\Gamma'_{yQ}(t) = -\Gamma_{QQ}(t) \quad , t \geq 0$$

Differentiating equation (3.5-13) and substituting the above result, one obtains

3.5-22) $$\Gamma''_{yy}(t) = -\Gamma_{QQ}(t) \quad , t \geq 0$$

and an analogous procedure may be applied for $t \leq 0$ with the same result. From this we see that

3.5-23) $$\Gamma''_{yy}(0^+) = -\int_{-\infty}^{+\infty}\int Q^2 f_s \, dx \, dy \leq 0$$

with the equality holding only in the trivial case $Q = 0$. In any other case, $\Gamma''_{yy}(0^+) < 0$.

Equations (3.5-17), (3.5-19), (3.5-21) and (3.5-22) can be derived from general properties of correlation functions if it is assumed that y(t) is differentiable in mean square. In this case, we can apply Loève's general formula ([36] p 471) which, in the stationary case reduces to

3.5-24) $$E\left\{ y^{(n)}(\tau)\, y^{(m)}(\tau+t) \right\} = (-1)^n \Gamma_{yy}^{(n+m)}(t)$$

Use of this formula with $y' = Q$ will give the results named above.

In these equations, derivatives are understood to be right-hand derivatives.

3.6 A Subclass of Second-Order Problems: Kramers' Equation

Rather than pursue the general case further, we now confine our attention to the subclass

3.6-1) $$\begin{cases} P(x,y) = -ax - b(y), \quad a = \text{const} > 0 \\ Q(x,y) = x \end{cases}$$

The set of equations is then

3.6-2) $$\begin{cases} \dot{x} = -ax - b(y) + n(t) \\ \dot{y} = x \end{cases}$$

which may also be written as the single second-order equation

3.6-3) $$\ddot{y} + a\dot{y} + b(y) = n(t)$$

which was considered by Kramers [30] . The Kolmogoroff equations become

3.6-4) $$\frac{\partial f_T}{\partial t} = \frac{N^2}{2}\frac{\partial^2 f_T}{\partial x_1^2} - [ax + b(y)]\frac{\partial f_T}{\partial x_1} + x_1\frac{\partial f_T}{\partial y_1}$$

3.6-5) $$\frac{\partial f_T}{\partial t} = \frac{N^2}{2}\frac{\partial^2 f_T}{\partial x_2^2} + \frac{\partial}{\partial x}\left\{[ax + b(y)]f_S\right\} - x\frac{\partial f_S}{\partial y}$$

Setting $\frac{\partial f_T}{\partial t} = 0$ in (3.6-5), the steady-state solution may be found to be

3.6-6) $$f_S(x,y) = f_0 e^{-\frac{a}{N^2}x^2 - aB(y)}$$

where

3.6-7) $$B(y) = \frac{2}{N^2}\int_0^y b(\xi)\,d\xi$$

3.6-8) $$\frac{1}{f_0} = \int_{-\infty}^{+\infty}\int e^{-\frac{a}{N^2}x^2 - aB(y)}\,dx\,dy = N\sqrt{\frac{\pi}{a}}\int_{-\infty}^{+\infty} e^{-aB(y)}\,dy$$

Since we are concerned with second-order properties, we say a function b(y) is non-dispersive if

3.6-9) $$\int_{-\infty}^{+\infty} y^2 e^{-aB(y)}\,dy < \infty, \quad \int_{-\infty}^{+\infty} b^2(y) e^{-aB(y)}\,dy < \infty$$

This is exactly the same condition as in equation (2.5-10,11) for the first-order problem. The sufficient conditions developed in Section 2.5 may accordingly be carried over to this case without change.

Applying the results of the preceeding section to this case, we have:

3.6-10) $$\Gamma''_{yy}(t) = -\Gamma_{xx}(t) \qquad -\infty < t < +\infty$$

3.6-11) $$\Gamma'_{yy}(t) = \Gamma_{yx}(t) \qquad t \geq 0$$

3.6-12) $$\Gamma_{yx}(t) = -\Gamma_{yx}(-t)$$ (right-hand derivatives)

3.6-13) $$\Gamma'_{yy}(0^+) = 0$$

Taking advantage of certain properties of Kramers' equation, we may proceed further. By equation (3.2-11),

3.6-14) $$\Gamma_{xx}(t) = \iint_{-\infty}^{+\infty} x\, v(x,y,t;xf_s)\,dx\,dy \qquad t \geq 0$$

Differentiating,

3.6-15) $$\Gamma'_{xx}(t) = \iint_{-\infty}^{+\infty} x \left\{ \frac{N^2}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x}\left[(ax+b)v\right] - x \frac{\partial v}{\partial y} \right\} dx\,dy$$

Integrating this by parts, it may be shown that

3.6-16) $$\Gamma'_{xx}(t) = -\iint_{-\infty}^{+\infty} (ax+b)\, v(x,y,t;xf_s)\,dx\,dy = \Gamma_{x2}(t)$$

3.6-17) $$\Gamma'_{xx}(t) = -a\Gamma_{xx}(t) - \Gamma_{xb}(t) \qquad t \geq 0$$

Using $v(x,y,t;bf_s)$ in the integrand of (3.6-14), and proceeding in exactly the same way, it may be shown that

3.6-18) $$\Gamma'_{bx}(t) = -a\Gamma_{bx}(t) - \Gamma_{bb}(t)$$

$\Gamma'_{xx}(t)$ may also be calculated starting from equation (3.5-9).

3.6-19) $$\Gamma_{xx}(t) = \iint_{-\infty}^{+\infty} x f_s\, u(x,y,t;x)\,dx\,dy \qquad t \geq 0$$

Differentiating,

3.6-20) $$\Gamma'_{xx}(t) = \iint_{-\infty}^{+\infty} x f_s \left\{ \frac{N^2}{2}\frac{\partial^2 u}{\partial x^2} - [ax+b]\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} \right\} dx\,dy$$

Integrating by parts yields

$$\Gamma'_{xx}(t) = \iint_{-\infty}^{+\infty} ux \left\{ \frac{N^2}{2}\frac{\partial^2 f_s}{\partial x^2} + \frac{\partial}{\partial x}[(ax+b)f_s] - x\frac{\partial f_s}{\partial y} \right\} dx\,dy$$

3.6-21) $$+ \iint_{-\infty}^{+\infty} u \left\{ 2\frac{\partial f_s}{\partial x}\frac{N^2}{2} + f_s(ax+b) \right\} dx\,dy$$

The $\{\}$ expression in the first integrand is zero from the properties of f_s. Differentiating (3.6-6),

3.6-22) $$\frac{\partial f_s}{\partial x} = -\frac{2a}{N^2} x f_s$$

Substituting this into (3.6-21), it may be shown that

3.6-23) $$\Gamma'_{xx}(t) = -a\,\Gamma_{xx}(t) + \Gamma_{bx}(t) \qquad t \geq 0$$

Comparing this with (3.6-17), $\Gamma_{bx}(t) = -\Gamma_{xb}(t)$ so $\Gamma_{bx}(t)$ is odd-symmetric and $\Gamma_{bx}(0) = 0$.

By using u(x,y,t;b) in the integrand of (3.6-19), we may show that

3.6-24) $$\Gamma'_{xb}(t) = -a\,\Gamma_{xb}(t) + \Gamma_{bb}(t)$$

Differentiating (3.6-17) gives

3.6-25) $$\Gamma''_{xx}(t) = -a\,\Gamma'_{xx}(t) - \Gamma'_{xb}(t) \qquad t \geq 0$$

Substituting (3.6-17) and (3.6-24) into this,

3.6-26) $$\Gamma''_{xx}(t) = a^2\Gamma_{xx}(t) + 2a\,\Gamma_{xb}(t) - \Gamma_{bb}(t) \qquad t \geq 0$$

By use of these formulae, it is possible to deduce most of the correlations in the system by differentiation of $\Gamma_{yy}(t)$. From the foregoing, we may show the following:

3.6-27) $\Gamma_{yx}(t) = \Gamma'_{yy}(t) \quad t \geq 0$

3.6-28) $\Gamma_{xx}(t) = -\Gamma''_{yy}(t) \quad -\infty < t < +\infty$

3.6-29) $\Gamma_{xb}(t) = -\Gamma^{(3)}_{yy}(t) + a\,\Gamma''_{yy}(t) \quad t \geq 0$

2.6-30) $\Gamma_{bb}(t) = \Gamma^{(4)}_{yy}(t) - 2a\,\Gamma^{(3)}_{yy}(t) + a^2\Gamma''_{yy}(t) \quad t \geq 0$

If b is a second-order process, this implies the existence of the first four derivatives of $\Gamma_{yy}(t)$.

Using these relations, it is possible to make statements about the high-frequency spectra of x and y. The definition of the spectral density is

3.6-31) $$G(\omega) = \frac{2}{\pi}\int_0^\infty \Gamma(t)\cos\omega t\,dt$$

If $\Gamma(t) \in L_1$, then integrating by parts,

3.6-32) $$G(\omega) = \frac{2}{\omega\pi}\left\{\Gamma(t)\sin\omega t\Big|_0^{+\infty} - \int_0^\infty \Gamma'(t)\sin\omega t\,dt\right\}$$

The first term vanishes at both limits. Integrating by parts again, and assuming $\Gamma'(\infty) = 0$,

3.6-33) $$G(\omega) = -\frac{2}{\pi}\frac{\Gamma'(0^+)}{\omega^2} - \frac{1}{\omega^2}\frac{2}{\pi}\int_0^\infty \Gamma''(t)\cos\omega t\,dt$$

If $\Gamma'(0^+)$ exists, this process may be repeated to give

3.6-34) $$G(\omega) = -\frac{2}{\pi}\frac{\Gamma'(0^+)}{\omega^2} + \frac{2}{\pi}\frac{\Gamma^{(3)}(0^+)}{\omega^4} + \frac{1}{\omega^4}\frac{2}{\pi}\int_0^\infty \Gamma^{(4)}(t)\cos\omega t\,dt$$

This may be carried on as long as the $\Gamma^{(n)}(0^+)$ which appear at each step are finite, and $\Gamma^{(n)}(\infty) = 0$.

Now we consider the high-frequency behavior of x. From

equation (3.6-33),

$$3.6\text{-}35) \quad \omega^2 G_{xx}(\omega) = -\frac{2}{\pi}\Gamma'(0^+) - \frac{2}{\pi}\int_0^\infty \Gamma_{xx}''(t)\cos\omega t\, dt$$

If all signals in the loop are second-order processes, we see from equation (3.6-26) that the integral must approach zero as $\omega \to \infty$. Accordingly the high-frequency asymptote is

$$3.6\text{-}36) \quad G_{xx}(\omega) \simeq -\frac{2}{\pi}\frac{\Gamma_{xx}'(0^+)}{\omega^2}$$

The derivative may be found by letting $t \to 0$ in equation (3.6-17).

$$3.6\text{-}37) \quad \Gamma_{xx}'(0^+) = -a\Gamma_{xx}(0) = -a\int_{-\infty}^{+\infty}\int x^2 f_s\, dx\, dy = -\frac{N^2}{2}$$

It follows that, for large ω,

$$3.6\text{-}38) \quad G_{xx}(\omega) \simeq +\frac{N^2}{\pi\omega^2}$$

Note that this asymptote is independent of b.

The high-frequency behavior of y may be deduced from equation (3.6-34).

$$3.6\text{-}39) \quad G_{yy}(\omega) = -\frac{2}{\pi}\frac{\Gamma_{yy}'(0^+)}{\omega^2} + \frac{2}{\pi}\frac{\Gamma_{yy}^{(3)}(0^+)}{\omega^4} + \frac{1}{\omega^4}\frac{2}{\pi}\int_0^\infty \Gamma_{yy}^{(4)}(t)\cos\omega t\, dt$$

By equation (3.6-13), the first term vanishes.

$$3.6\text{-}40) \quad \omega^4 G_{yy}(\omega) = \frac{2}{\pi}\Gamma_{yy}^{(3)}(0^+) + \frac{2}{\pi}\int_0^\infty \Gamma_{yy}^{(4)}(t)\cos\omega t\, dt$$

By an argument similar to that preceeding equation (3.6-36), the integral must approach zero as $\omega \to \infty$. Accordingly, for large ω,

$$3.6\text{-}41) \quad G_{yy}(\omega) \cong \frac{2}{\pi}\frac{\Gamma_{yy}^{(3)}(0^+)}{\omega^4}$$

Differentiating equation (3.6-10),

$$\Gamma_{yy}^{(3)}(t) = -\Gamma_{xx}'(t)$$

3.6-42) $$\Gamma_{yy}^{(3)}(0^+) = -\Gamma_{xx}'(0^+) = \frac{N^2}{2}$$

so equation (3.6-41) becomes

3.6-43) $$G_{yy}(\omega) = \frac{N^2}{\pi\omega^4}$$

independently of b.

The example of Section 3.3 is a special case of Kramers' equation with b(y) = β y. Note that as $\omega \to \infty$ in equation (3.3-13), the spectrum approaches the above asymptote.

CHAPTER 4

SUMMARY AND CONCLUSIONS

In the foregoing chapters, a number of questions have been considered, and a number of particular calculations made. At this point, we review these efforts briefly, to indicate which appear to be the most important, and which appear to be original with the author.

The major objective of these studies was the elucidation of methods for finding the spectra or correlations of the state variables of dynamic systems which are driven by noise. The only previous efforts along these lines are those of Tortrat [49] and those of Caughey and Dienes [7] which were discussed earlier. The present work is very much in the spirit of that of the latter-named authors, but goes beyond their efforts in a number of directions.

The fundamental equation which is used to calculate the autocorrelation was given by Caughey and Dienes for the first-order case. Extension to the second-order case is new, but quite straightforward. Recognition that inner integrals in these expressions satisfy the Kolmogoroff equations appears to be original, and is of the first importance in simplifying either analytical or numerical computation of the correlations. From the practical standpoint, this is probably the most important new idea presented here.

The consideration given to discontinuous terms in both first- and second-order equations appears to be new. While not rigorous, it does serve to suggest a solution which fits

all the known facts, and appears to be a reasonable one. Wishner [56] ,[55] obtained the result in the first-order case, but his derivation is extremely sketchy. He did not consider the second-order case at all.

The proof of convergence of the first-order distribution to the steady-state distribution is new, though the result had been obtained by Mandl [37] .

The derivative relationships among the various correlations in the problem are believed to be new in both first and second-order cases, and they are of considerable interest, both practically and theoretically.

Asymptotic results on the various spectra are new, for nonlinear problems. The small-t approximation for the first-order autocorrelation is closely allied with some of Caughey and Dienes [7] work, though a much more thorough consideration is given here.

There are several examples included in this work, but only one of these cannot be treated by older methods. This is the discontinuous b(x) of Section 2.8. This problem was treated by Caughey and Dienes [7] , though they calculated only the spectrum of x. The spectrum of b(x), with its unusual asymptotic property, is new.

The remaining examples are included for the purpose of showing that the methods developed here do at least give the correct result in those problems for which the answer is known.

There are numerous possible extensions to these studies

in addition to the obvious one to higher-order systems. First of all, some additional examples, especially of second-order problems, would be useful. The problem is one of finding cases for which the Fokker-Planck equation can be solved analytically. No thorough survey of the possibilities has been made. If this were done, it seems certain that a number of soluble cases would be found.

Departing from exact analytical treatment, there are numerous possible approximation techniques, both analytical and numerical. One which is particularly interesting is the idea of a perturbation solution for problems which are almost linear.

Also, one might try quantizing the problem in some reasonable way so that a Markoff chain problem would be generated which, in some sense, approximates the continuous problem. Practical Markoff chain computation should be much easier than the corresponding calculations for the continuous case.

It should be quite feasible to develop a small-t approximation for the second-order problem in much the same way it was done here in the first-order case.

APPENDIX A

SOLUTIONS OF THE FIRST-ORDER FOKKER-PLANCK EQUATION WITH $b(x) = \beta x$

In Sections 2.6.5 and 2.6.6, solutions to the equation were required for each of two initial conditions. In this appendix, we give the method of solution for the two cases. We define the Fourier transform of $v(x,t;g)$ by

A-1) $$V(\mu,t;g) = \int_{-\infty}^{+\infty} v(x,t;g)\, e^{-j\mu x}\, dx$$

Transforming the subject differential equation yields

A-2) $$\frac{\partial V}{\partial t} + \beta \mu \frac{\partial V}{\partial \mu} = -\frac{N^2}{2}\mu^2 V$$

This is now a first-order equation, and may readily be solved by the method of characteristics (see, e.g. Goursat [21] Vol II, Part 2, p 214 ff). The equations for the characteristic curves are

A-3) $$\frac{dt}{1} = \frac{d\mu}{\beta\mu} = \frac{dV}{-\frac{N^2}{2}\mu^2 V}$$

These equations lead to the first integrals

A-4) $$\begin{cases} \mu e^{-\beta t} = C_1 \\ V \exp\left\{\frac{N^2\mu^2}{4\beta}\right\} = C_2 \end{cases}$$

The general solution is

A-5) $$H\left[\mu e^{-\beta t},\ V \exp\left\{\frac{N^2\mu^2}{4\beta}\right\}\right] = 0$$

where H is an arbitrary function of its two arguments. An acceptable special case is

A-6) $$\Psi(\mu e^{-\beta t}) - V \exp\left\{\frac{N^2\mu^2}{4\beta}\right\} = 0$$

from which is follows that

A-7) $$V = \Psi(\mu e^{-\beta t}) \exp\left\{-\frac{N^2\mu^2}{4\beta}\right\}$$

where Ψ is an arbitrary function. It is determined by the boundary condition.

The boundary condition for the first problem is given in equation (2.6.5-3). Taking the transform of this equation gives

A-8) $$V(\mu,0;xf_s) = -\frac{i\mu N^2}{2\beta}\exp\left\{-\frac{N^2\mu^2}{4\beta}\right\}$$

Setting t = 0 in equation (A-7) and equating the result to (A-8) permits determination of the function Ψ. The complete transformed solution to the first problem is

A-9) $$V(\mu,t;xf_s) = -j\frac{N^2\mu}{2\beta}e^{-\beta t}\exp\left\{-\frac{N^2\mu^2}{4\beta}\right\}$$

It remains only to take the inverse transform

A-10) $$v(x,t;xf_s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} V(\mu,t;xf_s)e^{jx\mu}\,d\mu$$

A-11) $$v(x,t;xf_s) = \sqrt{\frac{\beta}{N^2\pi}}\,e^{-\beta t}\exp\left\{-\frac{\beta x^2}{N^2}\right\}$$

This is the solution to the problem posed by equations (2.6.5-2) and (2.6.5-3).

The second problem involves the same partial differential equation, but the boundary condition is now equation (2.6.6-2). Transforming the latter, the boundary condition becomes

A-12) $$V(\mu,0,ax^2f_s) = \frac{a}{4}\left(\frac{N^2}{\beta}\right)\left(\frac{2\beta}{N^2}-\mu^2\right)e^{-\frac{N^2\mu}{4\beta}}$$

Using this to determine the arbitrary function of (A-7),

A-13) $$V(\mu,t;ax^2f_s) = \frac{a}{4}\left(\frac{N^2}{\beta}\right)^2\left(\frac{2\beta}{N^2}-\mu^2e^{-2\beta t}\right)e^{-\frac{N^2\mu^2}{4\beta}}$$

The inverse transform yields

A-14) $$v(x,t;ax^2f_s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} V(\mu,t;ax^2f_s)e^{j\mu x}\,d\mu$$

A-15) $$v(x,t;ax^2f_s) = \frac{aN}{2\sqrt{\pi\beta}}e^{-\frac{\beta x^2}{N^2}}\left\{1-e^{-2\beta t}+\frac{2\beta}{N^2}x^2e^{-2\beta t}\right\}$$

This is the solution to the problem posed by equations (2.6.6-1) and (2.6.6-2).

APPENDIX B

SOLUTIONS OF THE FIRST-ORDER FOKKER-PLANCK EQUATION WITH b(x) = sgn x

In Section 2.8 we require the solution of

B-1) $$\frac{\partial v}{\partial t} = \frac{N^2}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x}\left[\operatorname{sgn} x \; v\right]$$

under two different initial conditions, both of which assure that $v(0,t) = 0$, $t \geq 0$. This eliminates any difficulties with the continuity conditions. Since we know this to be true, we may use $v(0,t) = 0$ as a boundary condition and solve the problem for $x \geq 0$. The solution for $x < 0$ may be determined from the fact that v is an odd-symmetric function of x. Odd symmetry of the initial condition will always make this truncation possible.

To solve the problem, we assume a solution of the form $v(x,t) = X(x)T(t)$, substitute this into the equation, and separate variables in the usual way.

B-2) $$\frac{T'}{T} = \frac{N^2}{2}\frac{X''}{X} + \frac{X'}{X} = -\lambda$$

Solving the separate equations,

B-3) $$T = C_1 e^{-\lambda t}$$

B-4) $$X = A_1 \exp\left\{-\frac{1+\sqrt{1-2N^2\lambda}}{N^2}x\right\} + A_2 \exp\left\{\frac{-1-\sqrt{1-2\lambda N^2}}{N^2}x\right\}$$

where C_1, A_1 and A_2 are arbitrary constants. If we require $A_2 = -A_1$, then the complete eigenfunction would be

B-5) $$v(x,t) = A(\lambda)\, e^{-\frac{x}{N^2} - \lambda t}\left[e^{j\frac{x}{N^2}\sqrt{2N^2\lambda - 1}} - e^{-j\frac{x}{N^2}\sqrt{2N^2\lambda - 1}}\right]$$

In order to satisfy the boundary conditions, we employ a superposition of these eigenfunctions.

B-6) $$v(x,t) = \int_{\frac{1}{2N^2}}^{+\infty} A(\lambda) e^{-\frac{x}{N^2} - \lambda t} 2j \sin \frac{x}{N^2}\sqrt{2N^2\lambda - 1}\, d\lambda$$

Now make a change of variable $w = \frac{\sqrt{2N^2\lambda - 1}}{N^2}$. Then

B-7) $$v(x,t) = e^{-\frac{x}{N^2} - \frac{t}{2N^2}} \int_0^{\infty} \bar{A}(w) e^{-\frac{N^2}{2} w^2 t} \sin xw\, dw$$

where

B-8) $$\bar{A}(w) = 2jA\left(\frac{1}{N^2} + \frac{N^2}{2} w^2\right) N^2 w$$

Now setting t = 0 gives

B-9) $$e^{+\frac{x}{N^2}} v(x,0) = \int_0^{\infty} \bar{A}(w) \sin xw\, dw$$

From this, it appears that $e^{\frac{x}{N^2}} v(x,0)$ and $\bar{A}(w)$ are a sine transform pair. The inverse transform gives (see, e.g. Titchmarsh [47] p 2, eq. 1.1.5)

B-10) $$\bar{A}(w) = \frac{2}{\pi} \int_0^{\infty} e^{\frac{x}{N^2}} v(x,0) \sin wx\, dx$$

Thus A(w) may be determined for any desired odd-symmetric boundary condition. Applying this to the boundary condition of equation (2.8.2-3), we have

B-11) $$\bar{A}(w) = \frac{2}{\pi N^2} \int_0^{\infty} e^{-\frac{x}{N^2}} x \sin wx\, dx$$

which may be integrated (Bierens de Haan [4] Table 361, 1), p 504) to give

B-12) $$\bar{A}(w) = \frac{4N^4 w}{\pi (1 + N^4 w^2)^2}$$

Substituting this result into equation (B-7) gives

B-13) $$v(x,t) = \frac{4N^4}{\pi} e^{-\frac{x}{N^2} - \frac{t}{2N^2}} \int_0^\infty \frac{w e^{-\frac{N^2 t}{2} w^2}}{(1+N^4 w^2)^2} \sin x w \, dw \qquad t \geq 0 \quad x \geq 0$$

By an entirely analogous procedure, we can show that, for $x<0$,

B-14) $$v(x,t) = -\frac{4N^4}{\pi} e^{\frac{x}{N^2} - \frac{t}{2N^2}} \int_0^\infty \frac{w e^{-\frac{N^2 t}{2} w^2}}{(1+N^4 w^2)^2} \sin x w \, dw \qquad x \leq 0$$

The two results may be combined in a single formula.

B-15) $$v(x,t) = \operatorname{sgn} \frac{4N^4}{\pi} e^{-\frac{|x|}{N^2} - \frac{t}{2N^2}} \int_0^\infty \frac{w e^{-\frac{N^2 t}{2} w^2}}{(1+N^4 w^2)^2} \sin x w \, dw$$

which is the solution to the first problem.

In the second problem (section 2.8.3), we have the same differential equation, but a different boundary condition. The new boundary condition, equation (2.8.3-2) also has odd symmetry, so the procedure will be exactly the same down through equation (B-10). Using the boundary condition of equation (2.8.3-2),

B-16) $$\bar{A}(w) = \frac{2}{N^2 \pi} \int_0^\infty e^{-\frac{x}{N^2}} \sin wx \, dx$$

This may be readily integrated (Bierens de Haan [4] Table 261, 1) p 383) to give

B-17) $$\bar{A}(w) = \frac{2}{\pi} \frac{N^2 w}{1+N^4 w^2}$$

Substituting this into equation (B-7),

B-18) $$v(x,t) = \frac{2}{N^2 \pi} e^{-\frac{x}{N^2} - \frac{t}{2N^2}} \int_0^\infty \frac{w e^{-\frac{N^2}{2} w^2 t}}{(\frac{1}{N^2})^2 + w^2} \sin x w \qquad x \geq 0$$

Combining this with the analogous expression for $x < 0$,

B-19) $$v(x,t) = \frac{2}{N^2\pi} e^{-\frac{|x|}{N^2} - \frac{t}{2N^2}} \int_0^\infty \frac{w e^{-\frac{N^2}{2}w^2 t}}{\left(\frac{1}{N^2}\right)^2 + w^2} \sin x w \, dw \qquad -\infty < x < \infty$$

which is the solution to the second problem.

The method employed here was suggested by Bateman pp 218-219.

APPENDIX C

SOLUTION OF THE FOKKER-PLANCK EQUATION FOR THE SECOND-ORDER LINEAR PROBLEM

The objective is to solve equation (3.3-7), subject to the boundary condition $v(x,y,0;yf_s) = yf_s$. As the first step, we simplify notation by defining $\beta = \omega_n^2$, $a = 2\xi\omega_n$, so that the equation may be written

$$\text{C-1)} \qquad \frac{\partial v}{\partial t} = \frac{N^2}{2}\frac{\partial^2 v}{\partial x^2} + 2\xi\omega_n x\frac{\partial v}{\partial x} + \omega_n^2 y\frac{\partial v}{\partial x} - x\frac{\partial v}{\partial y} + 2\xi\omega_n v$$

and the boundary condition becomes

$$\text{C-2)} \qquad v(x,y,0;yf_s) = \frac{2\xi\omega_n^2}{\pi N^2} y\, e^{-\frac{2\xi\omega_n}{N^2}x^2 - \frac{2\xi\omega_n^3}{N^2}y^2}$$

Now we apply the procedure suggested by Wang and Uhlenbeck [51] . A linear transformation of variables from x,y to new variables x_1, y_1 is made. In particular, it is the same linear transformation which reduces the set of equations

$$\text{C-3)} \qquad \begin{aligned} \dot{x} &= -\alpha x - \beta y \\ \dot{y} &= x \end{aligned}$$

to normal form. This is a well-known procedure (see, e.g. Cunningham [11] p 86 ff). The appropriate transformation is

$$\text{C-4)} \qquad x_1 = x + \omega_n\left(\xi - \sqrt{\xi^2-1}\right) y$$

$$\text{C-5)} \qquad y_1 = x + \omega_n\left(\xi + \sqrt{\xi^2-1}\right) y$$

The inverse is

$$\text{C-6)} \qquad x = \frac{\xi+\sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}}\, x_1 - \frac{\xi-\sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}}\, y_1$$

$$\text{C-7)} \qquad y = -\frac{1}{2\omega_n\sqrt{\xi^2-1}}\, x_1 - \frac{1}{2\omega_n\sqrt{\xi^2-1}}\, y_1$$

Applying this transformation to the Fokker-Planck equation

(C-1) gives

$$\text{C-8)} \quad \frac{\partial v}{\partial t} = \frac{N^2}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right)^2 v - \lambda_2 \frac{\partial}{\partial x_1}[x_1 v] - \lambda_1 \frac{\partial}{\partial x_2}[x_2 v]$$

where

$$\text{C-9)} \quad \lambda_1 = \omega_n\left(-\xi + \sqrt{\xi^2-1}\right)$$
$$\lambda_2 = \omega_n\left(-\xi - \sqrt{\xi^2-1}\right)$$

and transforming the boundary condition,

$$\text{C-10)} \quad v(x_1, y_1, 0) = \frac{\xi\omega_n(-x_1+y_1)}{\pi N^2\sqrt{\xi^2-1}} \exp\frac{\omega_n^2\xi^2}{N^2(\xi^2-1)}\left[\frac{x_1^2}{\lambda_1} + \frac{2x_1y_1}{\xi\omega_n} + \frac{y_1^2}{\lambda_2}\right]$$

To solve this problem, we make use of a bivariate Fourier transformation.

$$\text{C-11)} \quad V(\theta_1, \theta_2, t) = \int_{-\infty}^{+\infty}\int v(x_1, y_1, t)\, e^{-j\theta_1 x_1 - j\theta_2 y_1}\, dx_1 dy_1$$

Applying this transformation to equation (C-7),

$$\text{C-12)} \quad \lambda_2\theta_1\frac{\partial V}{\partial \theta_1} + \lambda_1\theta_2\frac{\partial V}{\partial \theta_2} - \frac{\partial V}{\partial t} = \frac{N^2}{2}(\theta_1+\theta_2)^2 V$$

while the boundary condition transforms to

$$\text{C-13)} \quad V(\theta_1, \theta_2, 0) = \frac{jN^2}{4\xi\omega_n^3}(\lambda_1\theta_1 + \lambda_2\theta_2)\, 2\omega_n\sqrt{\xi^2-1}\cdot \exp\left\{-\frac{1}{2}\left[\sigma_1^2\theta_1^2 + 2\sigma_1\sigma_2\left(\frac{1}{\xi}\right)\theta_1\theta_2 + \sigma_2^2\theta_2^2\right]\right\}$$

where

$$\text{C-14)} \quad \sigma_1^2 = -\frac{\lambda_1 N^2}{2\omega_n^2} \quad ; \quad \sigma_2^2 = -\frac{\lambda_2 N^2}{2\omega_n^2}$$

Equation (C-12) may be solved by the method of characteristics. The subsidiary equations are

$$\text{C-15)} \quad \frac{d\theta_1}{\lambda_2\theta_1} = \frac{d\theta_2}{\lambda_1\theta_2} = \frac{dt}{-1} = \frac{dV}{\frac{N^2}{2}(\theta_1+\theta_2)^2 V}$$

The equation

C-16) $$\frac{d\theta_1}{\lambda_2\theta_1} = -dt$$

yields the elementary solution

C-17) $$\theta_1 e^{\lambda_2 t} = C_1$$

while the equation

C-18) $$\frac{d\theta_2}{\lambda_1\theta_2} = -dt$$

gives another elementary solution:

C-19) $$\theta_2 e^{\lambda_1 t} = C_2$$

Finally, the equations

C-20) $$\theta_2 \frac{dV}{V} = \frac{N^2}{2}(\theta_1+\theta_2)^2 \frac{d\theta_2}{\lambda_1}$$

C-21) $$\theta_1 \frac{dV}{V} = \frac{N^2}{2}(\theta_1+\theta_2)^2 \frac{d\theta_2}{\lambda_1}$$

may be added, and the result integrated to give a third elementary solution.

C-22) $$V \exp\left\{-\frac{N^2}{2}\left[\frac{\theta_1^2}{2\lambda_2} + \frac{2\theta_1\theta_2}{\lambda_1+\lambda_2} + \frac{\theta_2^2}{2\lambda_1}\right]\right\} = C_3$$

These three solutions lead to a general solution of the form

C-23) $$V = \Psi\left[\theta_1 e^{\lambda_2 t}, \theta_2 e^{\lambda_1 t}\right] \exp\frac{N^2}{2}\left[\frac{\theta_1^2}{2\lambda_2} + \frac{2\theta_1\theta_2}{\lambda_1+\lambda_2} + \frac{\theta_2^2}{2\lambda_1}\right]$$

where Ψ is an arbitrary differentiable function of the two arguments. To determine this function, the boundary condition must be applied. If this is done, the solution is

C-24) $$V(\theta_1,\theta_2,t) = \frac{jN^2\sqrt{\xi^2-1}}{2\xi\omega_n^2}\left(\lambda_1\theta_1 e^{\lambda_2 t} + \lambda_2\theta_2 e^{\lambda_1 t}\right) \cdot \exp\left\{-\frac{1}{2}\left[\sigma_1^2\theta_1^2 + 2\sigma_1\sigma_2\left(\frac{1}{\xi}\right)\theta_1\theta_2 + \sigma_2^2\theta_2^2\right]\right\}$$

Taking the inverse Fourier transform of this, and applying the inverse linear transformation,

$$\text{C-25)}\quad v(x,y,t;yf_s) = \frac{f_s(x,y)}{2\omega_n\sqrt{\zeta^2-1}}\left\{(e^{\lambda_2 t} - e^{\lambda_1 t})x + (\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t})y\right\}$$

which is the desired solution. It may be verified directly that this satisfied the equation and the boundary condition.

APPENDIX D

AN H-THEOREM FOR APPROACH TO THE STEADY-STATE

Here we consider an alternative approach to demonstration of convergence to the steady-state distribution. While this approach is not rigorous as it stands, it places the problem in a different light, and shows the analogy between our problem and certain problems in statistical mechanics.

Consider the quantity

D-1) $$H(t) = \int_{-\infty}^{+\infty} v(x,t;g) \log\left[\frac{v(x,t;g)}{f_s(x)} \right] dx$$

where v(x,t;g is the semi-group associated with the Fokker-Planck equation. If $\int_{-\infty}^{+\infty} g(x)dx = 1$ and $g(x) \geq 0$, $-\infty < x < +\infty$, then v(x,t;g) may be interpreted as the probability density at $t \geq 0$, given the initial density g(x).

The analogy between this function and Boltzmann's H-function is immediately apparent. H(t) looks very much like a negentropy, and it behaves like one.

Since -log y is a convex function of y, it follows from Jensen's inequality (see, e.g. Hardy, Littlewood and Pólya [23] Theorem 204, p 150) that

D-2) $$H(t) \geq -\log\left\{ \int_{-\infty}^{+\infty} v(x,t;g) \frac{f_s(x)}{v(x,t;g)} dx \right\} = -\log\left\{ \int_{-\infty}^{+\infty} f_s(x)dx \right\} = 0$$

provided $H(t) < \infty$. The equality holds if and only if $v(x,t;g) = f_s(x)$. Multiplying equation (D-1) by -1, and applying Jensen's inequality, it may further be shown that

D-3) $$H(t) \leq \log\left\{ \int_{-\infty}^{+\infty} \frac{v^2(x,t;g)}{f_s(x)} dx \right\} \qquad t \geq 0$$

with equality if and only if $v(x,t;g) = f_s(x)$. A sufficient condition for $H(0) < \infty$ is $\frac{g^2(x)}{f_s(x)} \in L_1(-\infty, +\infty)$. Inequalities (D-2) and (D-3) would conflict if $\int_{-\infty}^{+\infty} \frac{v^2}{f_s} dx < 1$. This cannot happen, however. We see this by an application of the Schwartz inequality.

$$\text{D-4)} \quad \int_{-\infty}^{+\infty} \frac{v^2}{f_s} dx = \int_{-\infty}^{+\infty} f_s dx \int_{-\infty}^{+\infty} \frac{v^2}{f_s} dx \geq \left[\int_{-\infty}^{+\infty} \sqrt{f_s} \frac{v}{\sqrt{f_s}} dx \right]^2 = 1$$

By simple inequality arguments, we have shown that H(t) is bounded from below if it is finite.

Now differentiate H(t) with respect to time. If the derivative can be moved inside the integral sign of equation (D-1), then,

$$\text{D-5)} \quad H'(t) = \int_{-\infty}^{+\infty} \frac{\partial v}{\partial t} \left\{ 1 + \log \left[\frac{v(x,t;g)}{f_s} \right] \right\} dx$$

The first integral vanishes if $\int_{-\infty}^{+\infty} v dx = 1$ can be differentiated under the integral sign. Integrating the remaining term by parts,

$$\text{D-6)} \quad H'(t) = \left[\frac{N^2}{2} \frac{\partial v}{\partial x} + bv \right] \log \left[\frac{v(x,t;g)}{f_s(x)} \right] \Bigg|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \left[-\frac{f_s'}{f_s} + \frac{1}{v} \frac{\partial v}{\partial x} \right] \left[\frac{N^2}{2} \frac{\partial v}{\partial x} + bv \right] dx$$

If v and $\frac{\partial v}{\partial x}$ go to zero sufficiently fast as $|x| \to \infty$, then the first term will vanish. From equation (2.5-8), $f_s' = -2bf_s/N^2$. Using this, the equation becomes

$$\text{D-7)} \quad H'(t) = -\frac{2}{N^2} \int_{-\infty}^{+\infty} \frac{1}{v} \left\{ \frac{N^2}{2} \frac{\partial v}{\partial x} + bv \right\}^2 dx$$

From this, we see that $H'(t) \leq 0$, with equality only if $v = f_s$. If these steps are justified,

$$H(t) > 0 \quad , \quad H'(t) < 0 \qquad v \neq f_s$$

D-8)
$$H(t) = 0 \quad , \quad H'(t) = 0 \qquad v = f_s$$

The analogy between this procedure and the proof of Boltzmann's H-theorem is apparent. The conclusions are frequently drawn, from relations (D-8) that $\lim_{t\to\infty} H(t) = 0$, and $v(x,t;g)$ converges pointwise to $f_s(x)$. These conclusions may be true, but they do not follow from the above relations.

If $H(t) > 0$ implies $H'(t) < 0$, and $H(t) \geq 0$, then H is a bounded monotone function. By a standard theorem of analysis, it must approach a limit. That limit, however, need not be zero. This may be shown by an example. Suppose a function H obeys the differential equation

D-9)
$$H'(t) = \begin{cases} -H(t) & H \leq a \\ -H(t)+a & H > a \end{cases}$$

In this case, $H(t) > 0$ implies $H'(t) < 0$, but if $H(0) > a$, then $\lim_{t\to\infty} H(t) = a$. If $H(0) < a$, $\lim_{t\to\infty} H(t) = 0$.

In order to show that $H(t) \to 0$, it would be sufficient to establish also that $H'(t)$ does not become infinitesmally small unless $H(t)$ is infinitesmally small. For example, if

D-10)
$$H'(t) = -\varphi(H)$$

where $\varphi \geq 0$, $\varphi(0) = 0$, $\varphi'(x) > 0$, $0 < x < a$, then $\lim_{t\to\infty} H(t) = 0$. This question is closely related to asymptotic stability in the Lyapunov sense. (See, e.g. Hahn [22], Theorems 4.1 and 4.2) H is analogous to the Lyapunov function. Here, H depends on the function v, which is a point in a function space. In Lyapunov theory, H customarily depends on a

vector in a finite-dimensional Euclidean phase space. Nonetheless, proof that $H(t)\to 0$, given (D-10) would follow closely the proof of Hahn's Theorem 4.2, cited above.

Even if all this is established, there remains a question as to just what it implies as to convergence of v. In order to prove that $\lim_{t\to\infty} v(x,t;g) = f_s(x)$, $\int_{-\infty}^{+\infty} g(x)dx = 1$, (where the limit is in L_1 norm) it appears necessary to prove a slightly stronger result than the H-Theorem mentioned above. Consider the function

D-11) $$J(t) = \int_{-\infty}^{+\infty} \frac{v^2(x,t;g)}{f_s}\,dx - 1$$

It was shown in equation (D-4) that the integral is greater than or equal to unity, with equality only if $v = f_s$. Therefore,

D-12) $$J(t) \geq 0$$

with equality only if $v = f_s$. If differentiating under the integral is valid,

D-13) $$J'(t) = 2\int_{-\infty}^{+\infty} \frac{v}{f_s}\frac{\partial}{\partial x}\left\{\frac{N^2}{2}\frac{\partial v}{\partial x} + bv\right\}dx$$

Integrating by parts,

D-14) $$J'(t) = \frac{2v}{f_s}\left[\frac{N^2}{2}\frac{\partial v}{\partial x} + bv\right]\Bigg|_{-\infty}^{+\infty} - 2\int_{-\infty}^{+\infty}\left[\frac{N^2}{2}\frac{\partial v}{\partial x} + bv\right]\left[\frac{f_s\frac{\partial v}{\partial x} - v\frac{\partial f_s}{\partial x}}{f_s^2}\right]dx$$

If $v\to 0$ sufficiently fast as $|x|\to\infty$, then the first term will vanish. In the integral, substitute for $\frac{\partial f_s}{\partial x}$ using equation (2.5-8).

D-15) $$J'(t) = -\frac{4}{N^2}\int_{-\infty}^{+\infty}\frac{1}{f_s}\left[\frac{N^2}{2}\frac{\partial v}{\partial x} + bv\right]^2 dx$$

Clearly, $J'(t) \leq 0$, with equality only if $v(x,t;g) = f_s(x)$. The unique condition for $J(t) = 0$ is also $v(x.t;g) = f_s(x)$. The situation here is exactly the same as with $H(t)$. If $J(t) > 0$, then $J'(t) < 0$, and we expect that $\lim_{t\to\infty} J(t) = 0$, but the proof is lacking. If this can be shown, however, then L_1 convergence of v to f_s will follow. From the Schwartz inequality, we observe that

$$\text{D-16)} \quad \int_{-\infty}^{+\infty} f_s(x)dx \int_{-\infty}^{+\infty} \frac{[v(x,t;g) - f_s(x)]^2}{f_s(x)} dx \geq \left[\int_{-\infty}^{+\infty} |v(x,t;g) - f_s(x)| dx \right]^2$$

The first integral on the left is unity. Expanding the second integral, it follows that

$$\text{D-17)} \quad \int_{-\infty}^{+\infty} \frac{v^2}{f_s} dx - 1 = J(t) \geq \left[\int_{-\infty}^{+\infty} |v(x,t;g) - f_s(x)| dx \right]^2 \equiv \left[\|v - f_s\| \right]^2$$

If $J(t) \to 0$, then $v \to f_s$ in L_1 norm. Also, $H(t) \to 0$. This may be seen from (D-3).

$$\text{D-18)} \quad H(t) \leq \log \left\{ \int_{-\infty}^{+\infty} \frac{v^2}{f_s} dx \right\} \leq \int_{-\infty}^{+\infty} \frac{v^2}{f_s} dx - 1 = J(t)$$

Thus $J(t) \geq H(t)$, and $\lim_{t\to\infty} H(t) = 0$ would follow from $\lim_{t\to\infty} J(t) = 0$. If the latter proposition can be proved rigorously, all the desired results will follow.

The argument as it stands has a number of gaps, and is included here only to indicate analogies of our class of nonlinear noise problems with Boltzmann's H-theorem, and with the Lyapunov stability theory.

LIST OF REFERENCES

1. A.A. Andronov, L.S. Pontriagin and A.A. Witt, On the Statistical Investigation of Dynamical Systems, J. Exper. Theor. Phys. vol 3 (1933) p 165.
2. H. Bateman, Partial Differential Equations of Mathematical Physics, Cambridge University Press, 1944.
3. A.T. Bharucha-Reid, Elements of the Theory of Markov Processes and Their Applications, New York, McGraw-Hill Book Co., 1960.
4. David Bierens de Haan, Nouvelles Tables d'Intégrales Définies, New York, Hafner Pub. Co., 1957 (ed. of 1867).
5. R.C. Booton, Jr., The Analysis of Nonlinear Control Systems with Random Inputs, IRE Trans. Circuit Th. volCT-1 (Dec. 1954) pp 32-34.
6. T.K. Caughey, Response of Van der Pol's Oscillator to Random Exitation, ASME Trans. vol. 81E (J. Appl. Mech., Sept. 1959) pp 345-348.
7. T.K. Caughey and J.K. Dienes, Analysis of a Nonlinear System with a White Noise Input, J. Appl. Phys. vol 32 (1961) pp 2476-79.
8. S. Chandrasekhar, Stochastic Problems in Physics and Astronomy, Rev. Mod. Phys. vol. 15 (1943) pp 1-89.
9. K. Chuang and L.F. Kazda, Study of Nonlinear Systems with Random Inputs, AIEE Trans. vol. 78 Part II (Applications and Industry no. 42, May 1959) pp 100-105.
10. R. Courant and D. Hilbert, Methods of Mathematical Physics Vol I, New York, Interscience Publishers, Inc., 1953.

11. W.J. Cunningham, Introduction to Nonlinear Analysis, New York, McGraw-Hill Book Co., 1958.

12. D.A. Darling and A.J.F. Siegert, On the Distribution of Certain Functionals of Markoff Processes, Santa Monica, The Rand Corp. P-429, April 20, 1954.

13. W.B. Davenport, Jr. and William L. Root, Introduction to the Theory of Random Signals and Noise, New York, McGraw-Hill Book Co., 1958.

14. J.L. Doob, The Brownian Movement and Stochastic Equations Ann. Math. vol 43 (1942) pp 351-369.

15. J.L. Doob, Stochastic Processes, New York, John Wiley & Sons, Inc., 1953.

16. O.I. Elgerd, Study of Asynchronously Exited Oscillations in Nonlinear Control Systems, IRE Intl. Conv. Rec. vol 8 pt. 4 (1960) pp 23-36.

17. W. Feller, Zur Theorie der Stochastischen Prozesse (Existenz- und Eindeutigkeitsätze), Math. Annalen vol. 113 (1936) pp 113-160.

18. W. Feller, The Parabolic Differential Equations and the Associated Semi-Groups of Transformations, Ann. Math. vol. 55 (1952) pp 468-519.

19. W. Feller, Diffusion Processes in One Dimension, Trans. Amer. Math. Soc. vol. 77 (1954) pp 1-31.

20. M.A. Garstens, Noise in Nonlinear Oscillators, J. Appl. Phys. vol. 28 (1957) pp 353-365.

21. Edouard Goursat, A Course in Mathematical Analysis, trans. E.R. Hedrick and Otto Dunkel, New York, Dover Publications, Inc., 1959 (ed. of 1917)

22. Wolfgang Hahn, Theorie und Anwendung der Direkten Methode von Ljapunov, Berlin, Springer-Verlag, 1959.

23. G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities 2nd ed., Cambridge, University Press, 1952.

24. Einar Hille, On the Integration Problem for Fokker-Planck's Equation in the Theory of Stochastic Processes, Proc. 11th Scandanavian Math. Congr., Strondheim, 1949, Oslo, Johhan Grundt Tanums Forlag, 1952.

25. Einar Hille, Les Probabilités Continues en Chaine, C.R. Acad. Sci. Paris, vol. 230 (1950) pp 34-35.

26. Einar Hille and R.S. Phillips, Functional Analysis and Semi-Groups, Vol 31 of American Mathematical Society Colloqium Publications, Providence, American Math. Soc. 1957.

27. Kiyosi Ito, On a Stochastic Integral Equation, Proc. Jap. Acad. nos. 1-4, (1946) pp 32-35.

28. Kiyosi Ito, On Stochastic Differential Equations, Mem. Amer. Math. Soc., no. 4 (1951).

29. A. Kolmogoroff, Über die Analytischen Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. vol. 104 (1931) pp 415-458.

30. H.A. Kramers, Brownian Motion in a Field of Force, and the Diffusion Model of Chemical Reactions, Physica, vol. 7 (1940) pp 284-304.

31. J.H. Laning and R.H. Battin, Random Processes in Automatic Control, New York, McGraw-Hill Book Co., 1956.

32. James L. Lawson and G.E. Uhlenbeck, Threshold Signals, New York, McGraw-Hill Book Co., 1950.

33. H.R. Leland, Input-Output Cross-Correlation Functions for some Memory-Type Nonlinear Systems, AIEE Trans. vol. 79, part II (Appl. and Ind. no. 49, July 1960) pp 219-223.

34. Paul Lévy, Processus Stochastiques et Mouvement Brownien Paris, Gauthier-Villars, 1948.

35. Michel Loève, See Lévy [34].

36. Michel Loève, Probability Theory, 2nd ed. Princeton, D. van Nostrand, 1960.

37. P. Mandl, Les Proprietes Limites des Repartitions de Probabilite des Processus de Markov Bornes, Trans. 2nd Prague Conf. on Info. Theory, June 1-6, 1959. Prague, Publ. House, Czech. Acad. Sci., 1960.

38. J.A. McFadden, The Axis-Crossing Intervals of Random Functions, IRE Trans. Info. Th. Vol IT-2 (1956) pp 146-150.

39. J.A. McFadden, The Probability Density of the Output of a Filter when the Input is a Random Telegraphic Signal: Differential-Equation Method, IRE Trans. Circuit Th. vol. CT-6 (1959) pp 228-233.

40. David Middleton, An Introduction to Statistical Communication Theory, New York, McGraw-Hill Book Co., 1960.

41. S.O. Rice, Mathematical Analysis of Random Noise, Bell Syst. Tech. J. vol. 23 (1944) pp 282-322 and vol. 24 (1945) pp 46-156.

42. F. Sales Valles, On the Continuity of the Covariance of Random Functions of Second Order, Collect. Math. vol. 11 (1959) pp 69-75.

43. Y. Sawaragi and Y. Sunahara, The Statistical Studies on the Response of Automatic Control Systems with a Nonlinear Element of Zero-Memory Type, Part IV, On the Statistical Evaluation of the Closed-Loop Response of Nonlinear Control Systems Subjected to a Random Command Signal Contaminated by a Random Noise, Tech. Rept. Res. Inst. Kyoto Univ. vol 9 (1959) pp 209-222.

44. A.J.F. Siegert, On the Roots of Markoffian Random Functions, Santa Monica, The Rand Corp. RM-447, Sept. 5,1950.

45. A.J.F. Siegert, Passage of Stationary Processes Through Linear and Nonlinear Devices, Santa Monica, The Rand Corp. P-419, Oct. 29, 1953.

46. T.M. Stout, Block Diagram Transformations for Systems with One Nonlinear Element, AIEE Trans. vol 75, part II (Applications and Industry no. 25, July 1956) pp 130-140.

47. C.L. Tang, The Behavior of Nonlinear Oscillating Systems in the Presence of Noise, Proc. IRE vol 48 (1960) p 1493.

48. E.C. Titchmarsh, Introduction to the Theory of Fourier Integrals, 2nd ed. London, Oxford Univ. Press, 1948.

49. A. Tortrat, Les Processus Strictement Stationnaires de Markoff et leurs Corrélations, J. Math. Pur. et Appl. vol. 32 (1953) pp 281-333.

50. G.E. Uhlenbeck and L.S. Ornstein, On the Theory of Brownian Motion, Phys. Rev. vol 36 (1930) pp 823-841.

51. Ming Chen Wang and G.E. Uhlenbeck, On the Theory of Brownian Motion II, Rev. Mod. Phys. vol. 17 (1945) pp 323-342.

52. Norbert Wiener, Differential Space, J. Math. Phys. vol. 2 (1923) pp 131-174.

53. Norbert Wiener, Generalized Harmonic Analysis, Acta Math. vol. 55 (1930) pp 117-258.

54. J.C. West and P.N. Nikiforuk, The Response of Remote-Position-Control Systems with Hard-Spring and Non-Linear Characteristics to Step-Function and Random Inputs, Proc. IEE vol 102, part II (1955) pp 575-595.

55. R.P. Wishner, On Markov Processes in Control Systems, Report R-116, Coordinated Science Laboratory, Univ. of Illinois, Urbana, Ill., June 1960.

56. R.P. Wishner and M.E. van Valkenburg, On Markov Processes in Control Systems with Quantizers, Proc. Natl. Electr. Conf., 1960, pp 72-78.

57. K. Yosida, An Operator Theoretical Treatment of Temporally Homogeneous Markoff Processes, J. Math. Soc. Japan, vol 1 (1949) pp 244-253.