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ACTIONS OF THE GROUPS SO(3) AND Sp(1) ON THE SPHERES S^4 AND S^5

by
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INTRODUCTION

Montgomery and Zippin have shown (see [18], Theorem 6.7.1, p. 260) that if G is a compact connected group which acts effectively on euclidean 3-space, E^3, then G is either S^1, the circle group, or SO(3), the group of proper orthogonal transformations of E^3, and that the action of G is (topologically) equivalent respectively to the group of all rotations about an axis or the group of all proper orthogonal transformations.

In the case of groups acting on the 3-dimensional sphere, S^3, the situation is somewhat more complicated, since S^1 may act on S^3 without any fixed points. However, all actions of S^1 on S^3 have been classified to within equivalence ([11]).

In a much more general context it has been shown by Montgomery, Samelson, and Yang ([15]) that if G is a compact connected Lie group acting differentiably on E^n such that there exists an (n - 2)-dimensional orbit, then the action of G is linear in a suitable coordinate system. In this proof strong use is made of the differentiability.

We shall be concerned with actions of SO(3) and Sp(1), the group of unit quaternions, on the 4-dimensional sphere, S^4, and the 5-dimensional sphere, S^5. We are interested in classifying all such actions to within equivalence. No differentiability assumptions are made.

In the case of actions on S^4 the problem is completely solved. To within equivalence there are only the two linear actions of SO(3) on S^4 (see Definition 1.8). There is only one effective action of Sp(1)

on S^4 to within equivalence, again a linear action. The results for S^5 are less complete. We need to restrict ourselves to cases in which there exists a 3-dimensional orbit. There is then only one effective action of Sp(1) on S^5 and it is equivalent to a linear action. There are either two or three classes of actions of SO(3) on S^5 with a 3-dimensional orbit. Two of them contain linear actions. The ambiguity in the third case will be explained later.

Our method is roughly as follows. In Chapter II we prove two theorems which characterize the orbit space.

THEOREM 6.1. Let G be a compact connected Lie group acting on the n-sphere $S^n = S$ with an (n - 1)-dimensional orbit. Let X be the set of points lying on (n - 1)-dimensional orbits and Y = S - X. Then S^*, the orbit space, is a closed arc and Y^* consists of exactly two points, the end points of S^*. All orbits of X are equivalent.

THEOREM 7.1. Let G be a compact, connected, semi-simple Lie group acting on the n-sphere S with the highest dimension of any orbit (n - 2). Let X be the set of points on (n - 2)-dimensional orbits and let Y = S - X. Then S^* is a closed 2-cell with Y^* as simple closed curve boundary. Furthermore, all orbits of X are equivalent.

Assume now that we are given an action of SO(3) or Sp(1) on either S^4 or S^5. In the case of actions on S^5 we assume there is a 3-dimensional orbit. Let X and Y be as in Theorems 6.1 and 7.1 and let us denote our group by G. Then Y is invariant under the action of G and Y^* is either a simple closed curve or consists of exactly two points. Furthermore, all orbits of Y are either fixed points, or are homeomorphic to S^2, the 2-sphere, or P^2, the real projective plane.

It is a consequence of Theorems 6.1 and 7.1 that X is the product of an open cell and G(x), where G(x) is a typical orbit of X. The subgroups of SO(3) and Sp(1) are completely classified. This enables us to describe all possible orbits G(x). By Alexander duality we have $H^{n-i-1}(Y) = H_i(X)$ $(n = 4 \text{ or } 5)$.

From this point on it is a process of elimination. We try to "patch together" the spaces X and Y so as to get a sphere. By using the above restrictions on X and Y plus certain general theorems on transformation groups we are able to show that there are at most altogether seven equivalence classes of actions of SO(3) and Sp(1) on S^4 and S^5.

All known actions of SO(3) and Sp(1) on S^4 and S^5 are linear. There are exactly six inequivalent linear actions of SO(3) and Sp(1) on S^4 and S^5.

The possible non-linear action (see Section 19) occurs in the case of SO(3) acting on S^5. We are able to show that if such an action exists there is a cross section for S^*. Hence, by a method given in Section 2, a model space for the action can be constructed. The question of whether or not such an action exists is equivalent to the question of whether or not the model space is homeomorphic to S^5. We have not been able to answer this question.

In an appendix we briefly discuss actions of SO(3) on S^5 such that the highest dimension of any orbit is two. Using a result of Bing ([1], [2]) we construct an example which indicates the difficulty of classifying such actions.

CHAPTER I

PRELIMINARIES

Section 1. Definitions

Let G be a topological group and M a topological space.

DEFINITION 1.1. G acts as a transformation group on M if there is given a continuous map $f : G \times M \to M$ satisfying the following conditions (we denote $f(g, x)$ by $g \cdot x$):

1) $e \cdot x = x$ for every $x \in M$;

2) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for every $g_1, g_2 \in G$ and every $x \in M$.

The maps $h_g : M \to M$, defined by $h_g(x) = g \cdot x$, are homeomorphisms of M onto itself.

Let G act as a transformation group on the space M.

DEFINITION 1.2. The orbit of any $x \in M$ is the subset of M defined by $G(x) = \{g \cdot x : g \in G\}$.

DEFINITION 1.3. The stability group, or isotropy group, of any x in M is the closed subgroup of G defined by $G_x = \{g \in G : g \cdot x = x\}$.

Throughout this paper, even if we are concerned with the action of a particular group, such as SO(3), we shall denote the stability group of a point x by G_x and the orbit of x by G(x).

DEFINITION 1.4. The orbits G(x) and G(y) are equivalent if G_x and G_y are conjugate subgroups of G.

DEFINITION 1.5. Put $G_0 = \bigcap_{x \in M} G_x$. The action of G is effective if $G_0 = \{e\}$. G_0 is a closed normal subgroup of G. The action of G on M induces an effective action of the factor group G/G_0 on M.

DEFINITION 1.6. Let G act as a transformation group on the spaces M and N by $(g, m) \to g \cdot m$, resp. $(g, n) \to g * n$. A continuous map φ of M into N is <u>equivariant</u> with respect to the actions of G on M and N if $\varphi(g \cdot m) = g * \varphi(m)$ for every $g \in G$, $m \in M$.

DEFINITION 1.7. Let G act as a transformation group on M in two ways. The two actions of G are <u>equivalent</u> if there exists a homeomorphism of X onto itself which is equivariant with respect to the given actions.

In this paper we shall be concerned with classifying certain actions to within equivalence.

Let H be a closed subgroup of G. We define an action of G on G/H, the space of left cosets, by the rule $g \cdot (g_1 H) = g g_1 H$.

For each $x \in M$ we define the mapping $\varphi'_x : G \to M$ by $\varphi'_x(g) = g \cdot x$. φ'_x induces a 1-1 continuous equivariant mapping φ_x of G/G_x onto the orbit $G(x)$. If G is compact, φ_x is a homeomorphism. If G is a compact Lie group G/G_x is a manifold and hence so is $G(x)$.

The set of all orbits of M forms a partition of M into disjoint sets. Let M^* denote the orbit space of M, that is, the space whose points are the orbits $G(x)$ of M, with the standard decomposition topology. Let $T : M \to M^*$ denote the natural map; it is an open map. If $x \in M$, we shall occasionally denote $T(x)$ by x^*; we also write $M^* = M/G$. If G is compact M^* is Hausdorff. If M is compact, resp. connected, resp. locally connected, so is M^*.

DEFINITION 1.8. A <u>linear action</u> of G on S^{n-1} is the action obtained from a (continuous) representation $D : G \to GL(n, R)$ by

$$g \cdot v = \frac{1}{|D(g)(v)|} D(g)(v), \text{ for } g \in G, v \in S^{n-1}.$$

Here GL(n, R) is the group of non-singular $n \times n$ real matrices and $|v|$ denotes the norm of v.

All homology groups used are singular; all cohomology groups are Čech. $H_n(X ; G)$ (resp. $H_n(X, A ; G)$) denotes the n-th homology group of X (resp. n-th relative homology group of (X, A)) with coefficients in the (discrete) group G. $H^n(X ; G)$ and $H^n(X, A ; G)$ have similar meanings for cohomology. If integral coefficients are used, reference to G will be omitted. $H_*(X)$ (resp. $H^*(X)$) denotes the (weak) direct sum of the $H_n(X)$ (resp. $H^n(X)$).

Z denotes the group of integers and Z_n the integers reduced modulo n. When used as a coefficient group Q denotes the rationals.

Section 2. Cross sections and model spaces

Let G act on M, with projection $T : M \to M^*$.

DEFINITION 2.1. A cross section of M^* is a map $s : M^* \to M$ such that $T \circ s = 1_{M^*}$.

Here 1_{M^*} denotes the identity map of M^*.

DEFINITION 2.2. A local cross section at $x^* \in M^*$ is a map of a neighborhood U^* of x^* into M such that $T \circ s = 1_{U^*}$.

The most important result concerning the existence of cross sections is the following theorem, due to Gleason ([9]).

2.3. Let a compact Lie group G act on a Hausdorff space M. Let $p \in M$ have a neighborhood U such that if $x \in U$, then $G_x = \{e\}$. Then there exists a local cross section at p^*.

This generalizes to the following:

2.4. Let a compact Lie group G act on a locally compact Hausdorff space M. Let $p \in M$ have a neighborhood U such that if

$x, y \in U$, then G_x and G_y are conjugate. Then there is a local cross section at p^*; it may be chosen so that G_x is constant along the cross section. The last statement means: if s is defined on V^*, then $G_{s(x^*)} = G_{s(y^*)}$ for any $x^*, y^* \in V^*$.

PROOF. We may assume that U is compact. Let $E = \{x \in G(U) : G_x = G_p\}$. Then E is compact. Let $N(G_p)$ denote the normalizer of G_p. Then $N(G_p)/G_p$ acts on E such that the stability group of every point is trivial. Let E^{**} denote the orbit space of E under this action. Then there are natural maps $T_1 : E \to E^{**}$ and $T_2 : E^{**} \to T(U)$ such that $T_2 \circ T_1 = T|E$ (here $T|E$ denotes the restriction of T to E). T_2 is 1-1, continuous, and onto, hence a homeomorphism since E^{**} is compact. 2.4 now follows from 2.3.

2.5. Let G be a compact Lie group which acts on a locally compact Hausdorff space M in such a way that all orbits of M are equivalent. Assume that M^* is a contractible space. Then there is a cross section for M^* on which G_x is constant. Let $p \in M$. Then M^* is homeomorphic to $M^* \times G(p)$.

PROOF. Let $E = \{x \in M : G_x = G_p\}$. Then $N(G_p)/G_p$ acts principally on E. E is a principal fiber bundle with base space M^* and group $N(G_p)/G_p$ (see [6], Exposé VI, for definitions). From 2.3 we see that E is locally trivial. The result follows from ([6], Exposé VIII, Corollary 1 of Theorem 2).

A space M_1, together with an action of G on M_1, is said to be a <u>model space</u> for the action of G on M if there is an equivariant homeomorphism of M_1 onto M. In practice it often happens that by means of a model space we are able to get a concrete realization of an action which is given rather arbitrarily.

Suppose G is compact, M compact Hausdorff, and there is a cross section s for M^*. There is then a standard method for constructing a model space for the action of G on M.

Let $\varphi: M^* \times G \to M$ be defined by $\varphi(x^*, g) = g \cdot s(x^*)$. φ is onto. Define an action of G on $M^* \times G$ by $g_1 \cdot (x^*, g) = (x^*, g_1 g)$. It follows immediately that φ is equivariant. Let M_1 be the space whose points are the sets $\varphi^{-1}(x)$, $x \in M$. M_1 is given the usual decomposition space topology. M_1 is Hausdorff since M is Hausdorff. φ induces a homeomorphism φ_1 of M_1 onto M. M_1 can alternatively be described as the decomposition space of $M^* \times G$ whose points are the sets $(x^*) \times (gG_{s(x^*)})$. There is an induced action of G on M_1 and φ_1 is equivariant.

The following proposition gives a sufficient condition for the equivalence of two actions.

PROPOSITION 2.6. Let a compact group G act on a compact Hausdorff space M in two ways. Let M^* and $M^{*'}$ be the corresponding orbit spaces. Assume that there exist cross sections s and s' for M^* and $M^{*'}$. Assume further that there exists a homeomorphism f of M^* onto $M^{*'}$ such that $G_{s(x^*)} = G_{s' \circ f(x^*)}$ for every $x^* \in M^*$. Then the two actions of G are equivalent.

PROOF. Let M_1 be the model space for the first action of G on M and M_2 the model space for the second action. Let $\varphi_1 : M_1 \to M$ and $\varphi_2 : M_2 \to M$ be the corresponding maps. Let $\psi_1 : M^* \times G \to M_1$ and $\psi_2 : M^* \times G \to M_2$ be the natural maps of a space onto a decomposition space. Define $\psi : M^* \times G \to M^{*'} \times G$ by $\psi(x^*, g) = (f(x^*), g)$. The map $\psi_2 \circ \psi$ maps $M^* \times G$ onto M_2. The

condition $G_{s(x^*)} = G_{s'\circ f(x^*)}$ implies that $\psi_2 \circ \psi$ is constant on sets $(x^*)\times(gG_{s(x^*)})$. Hence, we get an induced map ψ' of M_1 onto M_2 such that the following diagram is commutative.

$$\begin{array}{ccc} M^* \times G & \xrightarrow{\psi} & M^{*\prime} \times G \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ M_1 & \xrightarrow{\psi'} & M_2 \end{array}$$

The condition $G_{s(x^*)} = G_{s'\circ f(x^*)}$ implies that ψ' is 1-1. Hence, ψ' is a homeomorphism. ψ and ψ' are equivariant. The mapping $\varphi_2 \circ \psi' \circ \varphi_1^{-1}$ of M onto M is an equivariant homeomorphism and the proposition is proved.

Section 3. SO(3) and its subgroups

SO(3) is the group of all 3×3 real orthogonal matrices of determinant +1. Euclidean 3-space E^3 is given with coordinates x_1, x_2, x_3 and basis vectors e_1, e_2, e_3 ($e_1 = (1, 0, 0)$, etc.). SO(3) acts in the usual way on E^3 and on the 2-sphere

$$S^2 = \left\{(x_1, x_2, x_3) \in E^3 : \sum_{i=1}^{3} x_i^2 = 1\right\}.$$

The real projective plane P^2 is the set of all unordered pairs $[p, -p]$, $p \in S^2$. P^2 is given the usual decomposition topology. SO(3) acts on P^2 by $g([p, -p]) = [g(p), g(-p)]$.

SO(3) has the following conjugacy classes of non-trivial subgroups (see ([23], pp. 16-18) for a discussion of the finite subgroups of SO(3)):

1) The groups S_p of all rotations of E^3 about the axis determined by a point p of S^2 are defined by $S_p = \{g \in SO(3) : g(p) = p\}$. S_p is isomorphic to the multiplicative group of all complex numbers of norm 1. Since $S_p = S_{-p}$ there is a natural 1-1 correspondence between the set of groups $\{S_p\}$ and the points of P^2. The map $g \longrightarrow g(p)$ of $SO(3)$ onto S^2 induces an equivariant homeomorphism of $SO(3)/S_p$ onto S^2.

2) For $p \in S^2$, define $N_p = \{g \in SO(3) : g(p) = \pm p\}$. N_p is the normalizer of S_p in $SO(3)$. S_p is a subgroup of N_p of index two. N_p is isomorphic to the group of all rotations, proper or improper, of the plane. Let $q \in S^2$, $q \neq \pm p$. Then if q is orthogonal to p, $N_p \cap N_q \approx Z_2 \oplus Z_2$, otherwise $N_p \cap N_q = \{e\}$.

   N_p is the stability group of $[p, -p] \in P^2$; it has only $[p, -p]$ as fixed point and $N_p \longleftrightarrow [p, -p]$ is a 1-1 correspondence between the system $\{N_p\}$ and P^2. The map $g \longrightarrow g([p, -p])$ induces an equivariant homeomorphism of $SO(3)/N_p$ onto P^2.

3) Finite subgroups of S_p. For any integer n, let $S_{p,n}$ be the group of rotations about the axis determined by p through an angle which is an integral multiple of $2\pi/n$. $S_{p,n}$ is a cyclic subgroup of S_p of order n.

4) Finite subgroups of N_p. Let $g \in SO(3)$ be such that $g(p) = -p$. We define $D_{p,n}$ to be the subgroup of $SO(3)$

generated by $S_{p,n}$ and g. $D_{p,n}$ depends on g and is determined only to within conjugacy in N_p. The groups $D_{p,n}$ are called dihedral groups. $S_{p,n}$ is a subgroup of $D_{p,n}$ of index two.

5) The tetrahedron group. Let H_T be the group of all symmetries of the regular tetrahedron. H_T is determined only to within conjugacy in SO(3). H_T is isomorphic to $\mathcal{A}_4$, the group of even permutations of four letters.

6) The group of the cube. Let H_C denote the group of all symmetries of the cube. H_C is determined only to within conjugacy in SO(3). H_C is isomorphic to $\mathcal{S}_4$, the group of all permutations of four letters.

7) The group of the icosahedron. Let H_I denote the group of all symmetries of the icosahedron. H_I is isomorphic to $\mathcal{A}_5$, the group of even permutations of five letters.

This completes the enumeration of the subgroups of SO(3). Note that SO(3) has no normal subgroups. Thus, every non-trivial action of SO(3) is effective.

For later reference we note the following fact:

3.1. If $n > 2$ then N_p is the only subgroup of the type 2) which contains $D_{p,n}$. $D_{p,2}$ is contained in exactly three subgroups of this type.

As an example, let $D_{e_1,2}$ be the group consisting of the identity and rotations through angle π about the axes e_1, e_2, and e_3. Then $D_{e_1,2}$ is included in N_{e_1}, N_{e_2}, and N_{e_3}.

Section 4. Sp(1) and its subgroups

Let Q denote the algebra of quaternions over the reals. We write $q = x_0 + x_1 i + x_2 j + x_3 k$ with the usual quaternion multiplication. The quaternions of norm 1 form, under multiplication, a group, Sp(1), the symplectic group of order 1. Sp(1) is topologically a 3-sphere. Let Q_0 be the set of quaternions with $x_0 = 0$; we identify Q_0 with E^3. Every $q \in Sp(1)$ can be written as $\cos\theta + (\sin\theta)q'$ with $q' \in Q_0' = Sp(1) \cap Q_0$. For a fixed $q' \in Q_0'$ the elements $\cos\theta + (\sin\theta)q'$ form a 1-parameter subgroup, denoted by $S_{q'}'$, isomorphic to the complex numbers of norm 1. All 1-parameter subgroups of Sp(1) are of this form. The element -1 is the only element of Sp(1) of order two.

The inner automorphism of Q by $q_1 \in Sp(1)$ leaves $Q_0 = E^3$ invariant and induces a rotation in it. This amounts to a mapping $\tau : Sp(1) \longrightarrow SO(3)$. τ is a homomorphism and is known to be a covering of order two. The element $\tau(\cos\theta + (\sin\theta)q)$ is a rotation of angle 2θ about the axis q.

Sp(1) has the following conjugacy classes of non-trivial subgroups:

1) 1-parameter subgroups S_q'. Note that $\tau^{-1}(S_q) = S_q'$.
2) Normalizers of 1-parameter subgroups. Let N_q' denote the normalizer of S_q'. As an explicit example consider $S_k' = \{\cos\theta + (\sin\theta)k : 0 \leq \theta \leq 2\}$. Then $N_k' = S_k' \cup iS_k'$. Again $\tau^{-1}(N_q) = N_q'$.
3) Finite subgroups of 1-parameter subgroups. For every integer n, S_q' has a cyclic subgroup of order n. Let us denote this subgroup by $S_{q,n}'$. Then $\tau^{-1}(S_{q,n}) = S_{q,2n}'$.

4) Finite subgroups of N_q' not contained in S_q'. Note that $\tau(N_q') = N_q$ and $N_q - S_q$ contains only elements of order two. If $g \in SO(3)$ is of order two, then $\tau^{-1}(g) \subset Q_0'$. Hence, any such subgroup must contain an element of Q_0'. But if $q \in Q_0'$, $q^2 = -1$. Hence, every such subgroup must contain -1. The order of such a subgroup must be a multiple of four. For every integer n we have a conjugacy class of subgroups of N_q'. Each subgroup in the conjugacy class contains $S_{q,n}'$ as a subgroup of index two. We denote such a subgroup by $D_{q,n}'$. Then $D_{q,n}' = \tau^{-1}(D_{q,n})$ for a suitable choice of $D_{q,n}$.

5) $\tau^{-1}(H_T)$

6) $\tau^{-1}(H_C)$

7) $\tau^{-1}(H_I)$

To see that this list is complete note that if H' is a subgroup of Sp(1), then $\tau(H') = H$ is a subgroup of SO(3). If $-1 \in H'$, then $\tau^{-1}(H) = H'$. If $-1 \notin H'$, then τ maps H' isomorphically onto H. A check of the above list shows that all groups of the form $\tau^{-1}(H)$, for H a subgroup of SO(3), are included. On the other hand, if τ maps H' isomorphically onto H, then H cannot have any elements of order two since -1, which is in the kernel of τ, is the only element of Sp(1) of order two. A check of the list of subgroups of SO(3) shows that all of these contain elements of order two except the groups $S_{q,n}$ for odd n. But the only subgroup of Sp(1) which is mapped isomorphically by τ onto $S_{q,n}$ is the group $S_{q,n}'$, which is already included in our list.

4.1. Let Sp(1) act effectively as a transformation group on a space M. Then there must exist $x \in M$ such that $G_x = \{e\}$ or $G_x = S'_{q,n}$ with n odd. In particular, there must be a 3-dimensional orbit.

PROOF. There must be an $x \in M$ with $-1 \notin G_x$. But the only subgroups of Sp(1) with this property are those mentioned in 4.1, as one checks from the above list.

The representations of SO(3) and Sp(1) have been completely classified to within equivalence (see ([8]) for a discussion of these representations). The representations of SO(3) correspond to representations of Sp(1) for which -1 is in the kernel. For each half-integer ℓ there is an irreducible representation D_ℓ of Sp(1) by $(2\ell + 1) \times (2\ell + 1)$ unitary matrices of determinant +1. If ℓ is an integer then D_ℓ gives a representation of SO(3) and the representation is equivalent to a representation by $(2\ell + 1) \times (2\ell + 1)$ real orthogonal matrices. For our purposes if ℓ is an integer we shall consider D_ℓ as being a representation of Sp(1) (or SO(3)) in $SO(2\ell + 1)$. If ℓ is not an integer we consider D_ℓ as being a representation of Sp(1) in $SO(4\ell + 2)$ (instead of in $SU(2\ell + 1)$).

Any representation of Sp(1) is equivalent to a direct sum of irreducible representations.

Section 5. Coset spaces of SO(3) and Sp(1)

We compute the homology groups of all possible coset spaces $SO(3)/H$ and $Sp(1)/H'$. These spaces are the possible orbits under SO(3) or Sp(1). Since $SO(3)/H$ is homeomorphic to $Sp(1)/\tau^{-1}(H)$ it suffices to consider coset spaces of Sp(1).

As for the 1-dimensional subgroups, $Sp(1)/S'_q$ is homeomorphic to S^2 and $H_0(S^2) \approx Z$, $H_1(S^2) = 0$, and $H_2(S^2) \approx Z$. Similarly, $Sp(1)/N'_q$ is homeomorphic to P^2 and $H_0(P^2) \approx Z$, $H_1(P^2) \approx Z_2$, and $H_2(P^2) = 0$.

If V is a finite subgroup of Sp(1), then Sp(1) is the universal covering space of Sp(1)/V. The group of covering translations will be given by the action of V on Sp(1) from the right. Hence, all covering translations are orientation preserving and Sp(1)/V is orientable. Thus, $H_0(Sp(1)/V) \approx H_3(Sp(1)/V) \approx Z$. Since Sp(1)/V is orientable $H_2(Sp(1)/V)$ has no torsion. From covering space theory, $\pi_1(Sp(1)/V) \approx V$. Thus, $H_1(Sp(1)/V) \approx V/[V, V]$ where $[V, V]$ denotes the commutator subgroup of V. Hence, $H_1(Sp(1)/V)$ is finite and by Poincaré duality $H_2(Sp(1)/V) = 0$. We now need to compute $V/[V, V]$ for the finite subgroups of Sp(1).

1) $[S'_{p,n}, S'_{p,n}] = \{e\}$ since $S'_{p,n}$ is abelian. Thus, $S'_{p,n}/[S'_{p,n}, S'_{p,n}] \approx Z_n$.

2) Let g be a generator of $S'_{p,2n} \subset D'_{p,n}$. Let V be the subgroup generated by g^2. If $q_1, q_2 \in (D'_{p,n} - S'_{p,2n})$, then $q_1^{-1} = \overline{q_1} = -q_1$, $q_2^{-1} = \overline{q_2} = -q_2$. Thus, $q_1q_2q_1^{-1}q_2^{-1} = q_1q_2(-q_1)(-q_2) = (q_1q_2)^2 \in V$. If $q \in S'_{p,2n}$, $q_1 \in D'_{p,n}$ then $q_1q\overline{q_1} = \overline{q}$. Thus, $q_1q\overline{q_1}\,\overline{q} = \overline{q}^2 \in V$. It follows that $[D'_{p,n}, D'_{p,n}] = V$. Let $q \in (D'_{p,n} - S'_{p,2n})$ Then, $q^2 = -1$. For n even, $-1 \in K$; for n odd, $-1 \notin K$. Hence, for n even, $D'_{p,n}/[D'_{p,n}, D'_{p,n}] = D'_{p,n}/V \approx Z_2 \oplus Z_2$. For n odd, $D'_{p,n}/[D'_{p,n}, D'_{p,n}] = D'_{p,n}/V \approx Z_4$.

3) Let $H_T' = \tau^{-1}(H_T)$. H_T is isomorphic to $\mathcal{A}_4$ and thus $[H_T, H_T] \approx Z_2 \oplus Z_2$. Hence, $H_T/[H_T, H_T] \approx Z_3$. Since τ maps $[H_T', H_T']$ onto $[H_T, H_T]$ and $[H_T, H_T]$ contains elements of order two, it follows that $[H_T', H_T'] = \tau^{-1}([H_T, H_T])$ and hence $H_T'/[H_T', H_T'] \approx H_T/[H_T, H_T] \approx Z_3$.

4) Let $H_C' = \tau^{-1}(H_C)$. Then, H_C is isomorphic to $\mathcal{S}_4$, thus $[H_C, H_C] \approx \mathcal{A}_4$ and $H_C/[H_C, H_C] \approx \mathcal{S}_4/\mathcal{A}_4 \approx Z_2$. As in 3) above we obtain $H_C'/[H_C', H_C'] \approx H_C/[H_C, H_C] \approx Z_2$.

5) Let $H_I' = \tau^{-1}(H_I)$. H_I is isomorphic to $\mathcal{A}_5$ and thus $H_I/[H_I, H_I] = 0$. As in 3) we find $H_I'/[H_I', H_I'] = 0$.

For convenience we list the homology groups of the coset spaces of Sp(1) and SO(3).

Coset Spaces of Sp(1)	The Corresponding Coset Spaces of SO(3)	Homology Groups H_0	H_1	H_2	H_3
$Sp(1)/\{1\}$		Z	0	0	Z
$Sp(1)/\{1, -1\}$	$SO(3)/\{e\}$	Z	Z_2	0	Z
$Sp(1)/S_{p,n}'$	$SO(3)/S_{p,n/2}$ (for n even)	Z	Z_n	0	Z
$Sp(1)/D_{p,2n}'$	$SO(3)/D_{p,2n}$	Z	$Z_2 \oplus Z_2$	0	Z
$Sp(1)/D_{p,2n+1}'$	$SO(3)/D_{p,2n+1}$	Z	Z_4	0	Z
$Sp(1)/H_T'$	$SO(3)/H_T$	Z	Z_3	0	Z
$Sp(1)/H_C'$	$SO(3)/H_C$	Z	Z_2	0	Z
$Sp(1)/H_I'$	$SO(3)/H_I$	Z	0	0	Z
$Sp(1)/S_p'$	$SO(3)/S_p$	Z	0	Z	0
$Sp(1)/N_p'$	$SO(3)/N_p$	Z	Z_2	0	0

Table 5.1. The homology groups of the coset spaces of SO(3) and Sp(1).

Note that $SO(3)/H_I$ and $Sp(1)/H_I'$ have the same homology groups as the 3-sphere. This will cause difficulty later on.

CHAPTER II

CHARACTERIZATION OF THE ORBIT SPACE

In this chapter we prove theorems which characterize the orbit space if a compact connected Lie group acts on the n-sphere with (n - 1)- or (n - 2)-dimensional orbits.

Section 6. Groups on S^n with (n - 1)-dimensional orbits

THEOREM 6.1. Let G be a compact connected Lie group acting on the n-sphere $S^n = S$ with an (n - 1)- dimensional orbit. Let X be the set of points on (n - 1)-dimensional orbits and Y = S - X. Then S^* is a closed arc and Y^* consists of exactly two points, the end points of S^*. All orbits of X are equivalent.

PROOF. By a theorem of Mostert ([19]) S^* is either: 1) an open arc; 2) a half-open arc; 3) a closed arc; or 4) a simple closed curve. Since S^* is compact, 1) and 2) are ruled out. By ([17], Corollary 2 of Theorem 2) S^* must be simply connected. Hence, S^* is a closed arc. The fact that Y^* consists of exactly two points follows from Alexander duality. In ([19]) it is also proved that all orbits of X are equivalent.

Section 7. Groups on S^n with (n - 2)-dimensional orbits

THEOREM 7.1. Let G be a compact connected semi-simple Lie group acting on the n-sphere $S^n = S$ with the highest dimension of any orbit (n - 2). Let X be the set of points on (n - 2)-dimensional orbits and let Y = S - X. Then S^* is a closed 2-cell with Y^* as simple closed curve boundary. Furthermore, all orbits of X are equivalent.

This theorem in a sense generalizes Lemma 2 of ([15]) . There is also a certain amount of overlap with ([4]). The hypothesis that G be semi-simple is not superfluous, as is shown by well known actions of the circle group on S^3. However, this hypothesis may be replaced by the existence of a fixed point. In this case we get Lemma 2 of ([15]).

The rest of this chapter will be devoted to the proof of Theorem 7.1.

In ([18] , p. 241) it is shown that X is an open subset of S. By ([16]), dim $Y \leq (n - 2)$. Hence, X is an open connected subset of S. Let $x \in X$ and let G_x^* be the identity component of G_x. By ([18] , p. 216) there is a neighborhood N of x such that $y \in N$ implies that G_y is conjugate to a subgroup of G_x. Since G_x and G_y are Lie groups of the same dimension, this implies that G_x^* and G_y^* are conjugate. Since X is connected it follows that if $y \in X$, then G_y^* is conjugate to G_x^*.

For $x \in X$, let $m(x)$ be the number of components of G_x. Since $m(x)$ is constant on each orbit, we get an induced function $m(x^*)$ defined on X^*. Let $r = \inf.\{m(x) : x \in X\}$. Let $C = \{x \in X : m(x) = r\}$. Then by ([18] , p. 216) C is an open subset of X. In ([14]) it is shown that C is connected. This gives

7.2. All orbits of C are equivalent.

Section 8. The slice

Let G be a compact Lie group acting on a space M.

DEFINITION 8.1. A slice at $p \in M$ is a closed set K containing p which satisfies the following conditions:

1) $G_p(K) = K$

2) If $g \in (G - G_p)$, then $g(K) \cap K = \emptyset$

3) There exists a compact cell R, containing e, in G which is a local cross section of the cosets of G_p in G at e such that the map $\varphi : R \times K \longrightarrow R(K)$ given by $\varphi(r, k) = r \cdot k$ is a homeomorphism onto a neighborhood of p in M.

Note that 2) implies that if $q \in K$, then $G_q \subset G_p$.

In ([17]) it is proved that if M is metric, complete, separable, and finite dimensional, then there exists a slice at every point of M.

Let M be a manifold and K be a slice at $p \in M$. Let T be the projection of M onto M^*. The following properties are immediate:

8.2. Let U^* be a closed neighborhood of p^* in M^* such that $U^* \subset T(K)$. Then, $T^{-1}(U^*) \cap K$ is a slice at p.

8.3. Let K^{**} be the orbit space of K under the action of G_p. Then $T : K \longrightarrow T(K)$ induces a map $T_1 : K^{**} \longrightarrow T(K)$. T_1 is a homeomorphism.

8.4. There is a subset K_1 of K which includes p and is invariant under G_p, and there is an open cell $R_1 \subset R$ containing e such that $\varphi | R_1 \times K_1 : R_1 \times K_1 \longrightarrow R_1(K_1)$ is a homeomorphism of $R_1 \times K_1$ onto an open neighborhood of p in M.

8.5. Let G(p) be of maximum dimension. Then G_p^* acts trivially on K.

We identify K^{**} and T(K) by means of the homeomorphism T_1 throughout the paper.

LEMMA 8.6. Let M be a manifold of dimension n and let the highest dimension of any orbit be (n - 2). Let $p \in M$ and dimension

$G(p) = (n - 2)$. Then we may choose as slice at p a closed 2-cell K_p.

PROOF. Let R_1 and K_1 be as in 8.4. By a theorem of Young ([22]) K_1 is a 2-manifold. G_p^* acts trivially on K_1 so that we get an induced action of the finite group $H_p = G_p/G_p^*$ on K_1. Let W be a closed 2-cell neighborhood of p in K_1. Let V be a closed 2-cell neighborhood of p such that $H_p(V) \subset W$. Let K_p be the component of $H_p(V)$ containing p. By a theorem of Whyburn ([20], p. 108, Theorem 2.6) K_p is a closed 2-cell. It is easy to see that K_p is invariant under the action of G_p and the other properties of the slice are easily checked.

By ([5] and [12]) the action of H_p on the 2-cell K_p is linear in a suitable coordinate system.

Section 9. Orientability of orbits

Let G be a compact connected Lie group acting on an orientable n-manifold M. We wish to examine the relation between the orientability of an orbit and the action of the stability group on the slice. First it is necessary to study the orientability of a coset space.

Let H be a closed subgroup of G. For $g \in G$ define $\ell_g' : G/H \longrightarrow G/H$ by $\ell_g'(g_1H) = gg_1H$. In particular, if $h \in H$, then ℓ_h' maps eH into eH.

LEMMA 9.1. G/H is orientable if and only if, for every $h \in H$, ℓ_h' preserves orientation locally at eH.

PROOF. Let H^* be the identity component of H. G/H^* is an orientable manifold since H^* is connected. The natural map $\pi : G/H^* \longrightarrow G/H$ is a covering map. The group of covering translations

is H/H^*, acting on G/H in the usual way (induced from the action r_h of H by $r_h(gH^*) = gh^{-1}H^*$). G/H is orientable if and only if all the covering transformations are orientation-preserving, i.e., if the r_h are orientation-preserving. This in turn is equivalent to $\ell_h \circ r_h$ preserving orientation at the fixed point eH^*, where the left translation ℓ_h is defined by $\ell_h(gH^*) = hgH^*$ (ℓ_h is orientation-preserving since G is connected). The map π sets up a local homeomorphism of G/H and G/H^* and transforms $\ell_h \circ r_h$ into ℓ'_h. This proves Lemma 9.1.

Since G is connected and M is orientable the maps h_g defined by $h_g(x) = g \cdot x$ will be orientation-preserving. Let p and K_p be as in Lemma 8.6. Then, in a suitable coordinate system for K_p, $H_p = G_p/G_p^*$ acts as a finite group of linear transformations of K_p.

LEMMA 9.2. $G(p)$ is non-orientable if and only if there is an element of H_p which reverses orientation in K_p.

PROOF. Choose the $(n - 2)$-cell R as in Definition 8.1, transversal to G_p at e. We may choose R so that it is invariant under inner automorphisms I_g by elements g of G_p. We define an action of G_p on $R \times K_p$ by $\alpha_g(g_1, x) = (gg_1g^{-1}, g \cdot x)$. The map $\varphi: R \times K_p \longrightarrow M$ is equivariant, that is, $\varphi \circ \alpha_g = h_g \circ \varphi$ for every $g \in G_p$. Set $\varphi(R \times K_p) = L$. Since R is an $(n - 2)$-cell, K_p a 2-cell, and φ a homeomorphism, L is an n-cell in M. Since φ is equivariant, L is invariant under the action of G_p. Let $i : (L, L - p) \subset (M, M - p)$ denote the injection map. Then $i_* : H_n(L, L - p) \longrightarrow H_n(M, M - p)$ is an isomorphism onto by excision. Consider the following diagram:

$$\begin{array}{ccccc} H_n(R \times K_p, R \times K_p - (e, p)) & \xrightarrow{\varphi_*} & H_n(L, L - p) & \xrightarrow{i_*} & H_n(M, M - p) \\ \downarrow (\alpha_g)_* & & \downarrow (h_g)_* & & \downarrow (h_g)_* \\ H_n(R \times K_p, R \times K_p - (e, p)) & \xrightarrow{\varphi_*} & H_n(L, L - p) & \xrightarrow{i_*} & H_n(M, M - p) \end{array}$$

Since φ is equivariant the diagram is commutative. All maps are isomorphisms onto. Since h_g is orientation-preserving $(h_g)_*$ is the identity isomorphism on $H_n(M, M - p)$. This implies that $(\alpha_g)_*$ is the identity isomorphism. This means that h_g reverses the orientation of K_p if and only if I_g reverses the orientation of R. R can be identified with a neighborhood of eG_p in G/G_p; Lemma 9.2 follows now from Lemma 9.1.

We recall (8.3) that $K_p/G_p = K_p^{**}$ is homeomorphic with $T(K) = G(K)/G$. If K_p is a closed 2-cell and G_p acts on K_p as a finite group of rotations, then K_p^{**} is again a closed 2-cell with p^{**} (the image of p) as an interior point. If G_p acts on K_p as a group of rotations and reflections, then the orbit space is homeomorphic to a closed 2-cell with p^{**} as a boundary point.

Suppose now that M is the n-sphere $S^n = S$ and let X, Y, and C be as in Section 7. Let $X - C = D \cup E$ where D is the union of the orientable orbits in $X - C$ and E is the union of the non-orientable orbits in $X - C$. As a consequence of Lemma 8.5, Lemma 9.2, and the remarks above we obtain:

9.3. X^* is a connected 2-manifold with boundary E^*.

9.4. $C^* \cup D^*$ is a connected 2-manifold (without boundary) and D^* is a discrete subset of $C^* \cup D^*$.

9.5. All orbits of C are orientable.

9.6. $C^* \cup E^*$ is an open subset of S^*.

9.7. If $x^* \in D^*$, then $m(x^*) = kr$ for some integer $k > 1$.

Suppose $G(p)$ is non-orientable. Since H_p contains reflections of K_p, H_p has even order $2m$, and $E \cap K_p$ consists of m straight lines in K_p through p. From the properties of the slice it follows that p has a neighborhood N such that $N \cap E$ is homeomorphic to the product of an $(n-2)$-cell and $E \cap K_p$. This gives:

9.8. Let D^2 denote the unit disk in E^2 and let A be the subset of D^2 given in polar coordinates by:

$$A = \left\{(r, \theta) \in D^2 : \theta = \frac{k}{2m} \cdot 2\pi, \quad k = 1, 2, \ldots, 2m\right\}.$$

Then p has a neighborhood N such that $N \cap E$ is homeomorphic to the product of A with a closed $(n-2)$-cell.

Section 10. Proof that $D = \emptyset$

We assume henceforth that G is a compact, connected, semi-simple Lie group. Let H be a closed subgroup of G. From the exact homotopy sequence for the fiber space $(G, H, G/H)$ we see immediately that $\pi_1(G/H)$ is finite. Thus, $H^1(G/H ; Q) = 0$.

LEMMA 10.1. $E \cup Y \neq \emptyset$.

PROOF. If $E \cup Y = \emptyset$ then, by 9.4, S^* is a compact 2-manifold. By ([17], Corollary 2 to Theorem 2) S^* is simply connected. Thus, S^* is the 2-sphere S^2. Since G is semi-simple, $H^1(T^{-1}(x^*) ; Q) \approx H^1(G/G_x ; Q) = 0$ for every $x^* \in S^*$. Hence, by the Begle-Vietoris mapping theorem as formulated in ([3]), $T^* : H^2(S^* ; Q) \longrightarrow H^2(S ; Q)$ is an injection. This is a contradiction since $H^2(S ; Q) = 0$ (it is implicit in our assumptions that $n > 2$). This proves the lemma.

By an arc we mean a space homeomorphic to the closed unit interval I. An open arc is a space homeomorphic to the open unit interval. We shall often denote an arc with end points a and b by ab. Similarly, the notation abcd denotes an arc with end points a and d which is such that the points a, b, c, d occur in that order along the arc. If a group H acts on a space M and if J^* is an arc in $M^* = M/H$, an arc J in M is said to cross section M if T maps J homeomorphically onto J^*. It is proved in ([17]) that if a compact connected Lie group acts on a Hausdorff space M, then every arc in M^* admits a cross sectioning arc in M.

Let A be a subspace of a topological space B. A point $a \in A$ is accessible from $B - A$ if there exists an arc $ab \subset B$ such that $(ab - a) \subset (B - A)$. Every point of a closed $(n - 2)$-dimensional subset A of an n-manifold M is accessible from $M - A$.

From 2.5 and 7.2 we obtain:

10.2. Let a^*b^* be an arc in C^*. Then there exists a cross sectioning arc ab in C on which G_x is constant.

This can be strengthened as follows:

LEMMA 10.3. Let a^*b^* be an arc in X^* such that $b^* \in D^*$, $(a^*b^* - b^*) \subset C^*$. Then there is a cross sectioning arc ab such that G_x is constant on $(ab - b)$.

PROOF. Let $b \in T^{-1}(b^*)$ and K_b be a closed 2-cell slice at b. Then G_b acts on K_b so that the base space is $T(K_b)$. Let c^*b^* be a sub-arc of a^*b^* such that $c^*b^* \subset T(K_b)$. Let cb be a cross sectioning arc in K_b for c^*b^*. By 10.2 there exists a cross sectioning arc a'c' for a^*c^* on which G_x is constant. We may translate this by an element

of G to get a cross sectioning arc ac. Then the arc $(ac \cup cb] = ab$ has the desired properties.

LEMMA 10.4. $D = \phi$.

PROOF. S^* has a countable basis for open sets since S has this property. Since D^* is a discrete subset of S^* it must be countable. Each point of D^* corresponds to a compact $(n - 2)$-dimensional orbit. Since Y is closed, it follows by the sum theorem from dimension theory that dimension $(D \cup Y) \leq (n - 2)$. Since, furthermore, $D \cup Y$ is closed by 9.6, it follows that every point of $D \cup Y$ is accessible from $C \cup E$.

Assume $D \neq \phi$. Let $a^* \in D^*$. Then by accessibility arguments there is an arc $J^* = a^*b^*$ such that $\left(a^*b^* - (a^* \cup b^*)\right) \subset C^*$ and $b^* \in \left(E^* \cup Y^*\right)$. It is an easy consequence of Lemma 10.3 that there is a half-open arc J' with end point a which cross sections $J^* - b^*$ and is such that G_x is constant on $J' - a$. Let $x \in (J' - a)$. Then $T^{-1}(J^* - b^*)$ is homeomorphic to $[0, 1) \times G/G_x$ with $0 \times G/G_x$ "collapsed" to G/G_a. Let $m(a^*) = kr$. Consider the exact cohomology sequence of $(T^{-1}(J^*), T^{-1}(a^*) \cup T^{-1}(b^*))$.

$$H^{n-2}(T^{-1}(a^*) \cup T^{-1}(b^*)) \xrightarrow{\delta} H^{n-1}(T^{-1}(J^*), T^{-1}(a^*) \cup T^{-1}(b^*))$$

$$\longrightarrow H^{n-1}(T^{-1}(J^*)) \longrightarrow H^{n-1}(T^{-1}(a^*) \cup T^{-1}(b^*)).$$

$H^{n-1}(T^{-1}(a^*) \cup T^{-1}(b^*)) = 0$, $H^{n-2}(T^{-1}(a^*)) \approx Z$, and $H^{n-2}(T^{-1}(b^*)) \approx Z_2$ or 0 depending on whether $b^* \in E^*$ or $b^* \in Y^*$. $(T^{-1}(J^*), T^{-1}(a^*) \cup T^{-1}(b^*))$ is an orientable relative $(n - 1)$-manifold and hence $H^{n-1}(T^{-1}(J^*), T^{-1}(a^*) \cup T^{-1}(b^*)) \approx Z$. Let β be a generator of $H^{n-2}(T^{-1}(a^*))$ and β' a generator of

$H^{n-1}(T^{-1}(J^*), T^{-1}(a^*) \cup T^{-1}(b^*))$. Then it follows that $\delta(\beta) = \pm k\beta'$. But this implies that $H^{n-1}(T^{-1}(J^*)) \approx Z_k$. By Alexander duality this implies that $H_0(S - T^{-1}(J^*)) \approx Z_k$, which is impossible. Hence $D = \emptyset$.

Section 11. Completion of the proof of Theorem 7.1

In this section we complete the proof of Theorem 7.1. We prove that S^* is a closed 2-cell with $E^* \cup Y^*$ as simple closed curve boundary. Then we prove that $E^* = \emptyset$ which, combined with Lemma 10.4, shows that all orbits of X are equivalent.

LEMMA 11.1. C^* is an open 2-cell.

PROOF. By 9.4 C^* is a connected 2-manifold. It suffices to prove that every simple closed curve in C^* separates it into two components, exactly one of which is compact (see [24], p. 340).

Let J^* be a simple closed curve in C^*. By Gleason's theorem $T^{-1}(J^*)$ is a closed $(n - 1)$- manifold. By Alexander duality $T^{-1}(J^*)$ separates S into two components, U_1 and U_2. The Alexander duality theorem implies further that every point of $T^{-1}(J^*)$ is accessible from both U_1 and U_2. It will suffice to show that either $(E \cup Y) \cap U_1 = \emptyset$ or $(E \cup Y) \cap U_2 = \emptyset$. If both these sets are non-empty then, using the accessibility of $E \cup Y$ from C and the accessibility of $T^{-1}(J^*)$ from U_1 and U_2, we find an arc $x^*y^*z^*$ in S^* such that $x^* \in (E^* \cup Y^*) \cap U_1^*$, $y^* \in J^*$, $z^* \in (E^* \cup Y^*) \cap U_2^*$, and $(x^*y^* - (x^* \cup y^*)) \subset (U_1^* \cap C^*)$, $(y^*z^* - (y^* \cup z^*)) \subset (U_2^* \cap C^*)$. By arguments like those in the proof of Lemma 10.4 it follows that $H^{n-1}(T^{-1}(x^*y^*z^*)) \approx Z$. By the Alexander duality theorem $T^{-1}(x^*y^*z^*)$ separates S into two components, V_1 and V_2. $T^{-1}(J^* - y^*)$ is a connected set not intersecting $T^{-1}(x^*y^*z^*)$

and thus must be contained in either V_1 or V_2, let us say V_1. Then V_2 is a connected set which does not intersect $T^{-1}(J^*)$ and hence must lie in either U_1 or U_2. But this is impossible since $T^{-1}(x^*) \subset U_1$, $T^{-1}(x^*) \cap \overline{V}_2 \neq \phi$, $T^{-1}(z^*) \subset U_2$, $T^{-1}(z^*) \cap \overline{V}_2 \neq \phi$. This completes the proof of the lemma.

LEMMA 11.2. $E^* \cup Y^*$ is a simple closed curve.

PROOF. By ([21], p. 32) it is sufficient to prove that $E^* \cup Y^*$ is connected, non-degenerate (contains more than one point), has no cut point, but is disconnected by the omission of any two distinct points. By 2.5 and Lemma 11.1, C is the product of an open 2-cell and a typical orbit $G(x)$. Thus, $H_{n-1}(C) = 0$ and, by Alexander duality, $H^0(E \cup Y) = 0$. Hence, $E \cup Y$ is connected. We use the semi-simplicity of G to prove that $E^* \cup Y^*$ is non-degenerate. By Alexander duality $Q \approx H_{n-2}(C ; Q) \approx H^1(E \cup Y ; Q)$. Each inverse image $T^{-1}(x^*)$ is connected, and, as mentioned in Section 10, the semi-simplicity of G implies that $H^1(T^{-1}(x^*) ; Q) = 0$. Thus, by the Begle-Vietoris mapping theorem, $T^* : H^1(E^* \cup Y^* ; Q) \longrightarrow H^1(E \cup Y ; Q)$ is an isomorphism onto. Thus, $H^1(E^* \cup Y^* ; Q) \approx Q$ and $E^* \cup Y^*$ is non-degenerate.

Let $z^* \in (E^* \cup Y^*)$. We wish to show that $(E^* \cup Y^*) - z^*$ is connected. Suppose not. Then $E^* \cup Y^* = U^* \cup V^*$ with U^* and V^* closed non-degenerate sets intersecting in z^*. Let $K = S - T^{-1}(U^*)$ and $L = S - T^{-1}(V^*)$. Then $K \cap L = C$ and $K \cup L = S - T^{-1}(z^*)$. Let $u^* \in U^*$ and $v^* \in V^*$ be points distinct from z^* and choose $x^* \in C^*$. Let u^*x^* and v^*x^* be arcs with $(u^*x^* - u^*) \subset C^*$, $(v^*x^* - v^*) \subset C^*$. Such arcs exist by accessibility. Considering the inverse images

$T^{-1}(u^*x^*)$ and $T^{-1}(v^*x^*)$ we see that $T^{-1}(x^*)$ carries an $(n-2)$-cycle which does not bound in $K \cap L$ but which bounds in K and in L. From the Mayer-Vietoris sequence for $(K \cup L ; K, L)$ we see that $H_{n-1}(K \cup L) \neq 0$. But this contradicts, by Alexander duality, the fact that $H^0(T^{-1}(Z^*)) \approx Z$. Thus, $E^* \cup Y^*$ has no cut points.

Let x^* and y^* be distinct points of $E^* \cup Y^*$. By accessibility there is an arc x^*y^* such that $(x^*y^* - (x^* \cup y^*)) \subset C^*$. $H^{n-1}(T^{-1}(x^*y^*) ; Q)$ is isomorphic to Q and thus $T^{-1}(x^*y^*)$ separates S into two components, U and V. By the Begle-Vietoris mapping theorem

$$H^1(T^{-1}(x^*y^*) ; Q) \approx H^1(x^*y^* ; Q) = 0.$$

Hence, by Alexander duality, $H_{n-2}(U ; Q) \oplus H_{n-2}(V ; Q) \approx H_{n-2}(U \cup V ; Q) = H_{n-2}(S - T^{-1}(x^*y^*) ; Q) \approx H^1(T^{-1}(x^*y^* ; Q) = 0$. Thus, $H_{n-2}(U ; Q) \approx H_{n-2}(V ; Q) = 0$. Neither $U \cap (E \cup Y)$ nor $V \cap (E \cup Y)$ can be empty. Suppose, for example, that $U \cap (E \cup Y) = \phi$. Then U is an invariant subset of C and hence is the product of an orbit and a subset of C^*. But such a subset must carry a rational $(n-2)$-cycle. Thus, $((E \cup Y) - (T^{-1}(x^*) \cup T^{-1}(y^*)))$ is not connected. This finishes the proof of the lemma.

LEMMA 11.3. S^* is a closed 2-cell with $E^* \cup Y^*$ as simple closed curve boundary.

PROOF. An arc x^*y^* in S^* is said to span $E^* \cup Y^*$ if $(E^* \cup Y^*) \cap x^*y^* = (x^* \cup y^*)$. By ([21], p. 92) it is sufficient to prove that every arc of S^* that spans $E^* \cup Y^*$ separates S^* and that no closed proper subset of such an arc separates S^*. Let x^*y^* be an arc which spans $E^* \cup Y^*$. Then $H^{n-1}(T^{-1}(x^*y^*)) \approx Z$ and hence $T^{-1}(x^*y^*)$

separates S. Thus, x^*y^* separates S^*. If γ is a closed proper subset of x^*y^*, then $T^{-1}(\gamma)$ does not carry an $(n-1)$-cycle and hence does not separate S. Thus γ does not separate S^*.

LEMMA 11.4. $E^* = \emptyset$.

PROOF. By ([18], p. 241) E is an open subset of $E \cup Y$. Assume $E \neq \emptyset$. Let E_1 be a component of E and let $\overline{E_1}$ denote the closure of E_1. Then E_1^* is an open arc or a 1-sphere and $\overline{E_1^*}$ is a closed arc or a 1-sphere. By 9.8 each point of E_1 has a neighborhood (in E_1) which is either homeomorphic to an $(n-1)$-cell or is homeomorphic to a finite number of intersecting $(n-1)$-cells. Also, dimension $(\overline{E_1} - E_1) \leq (n-3)$. By the method used in ([14], pp. 139-140) it can be shown that $H^{n-1}(\overline{E_1}\ ; Z_2) \neq 0$. But $S - \overline{E_1}$ is connected, which contradicts Alexander duality. Thus, $E = \emptyset$.

This lemma completes the proof of Theorem 7.1.

CHAPTER III

ACTIONS OF SO(3) AND Sp(1) ON S^4

In this chapter we consider actions of SO(3) and Sp(1) on the 4-dimensional sphere S^4. We show that, to within equivalence, there are exactly two actions of SO(3) on S^4 and exactly one effective action of Sp(1) on S^4. All of these actions are linear actions.

Section 12. The representation D_2

The representation D_2 of SO(3), which defines an action of SO(3) on S^4, can be described as follows: The underlying vector space is the space $\mathcal{M}$ of 3×3 real symmetric matrices of trace zero, and SO(3) acts on $\mathcal{M}$ by $g(m) = gmg^{-1}$ (conjugation of the matrix m by the matrix g). Using $tr(m_1 m_2)$ (tr = trace) as inner product in $\mathcal{M}$ we identify S^4 with the set of all m of norm 1. This inner product is invariant under SO(3).

We shall construct a cross section for this action. Let S^1 be the set of all diagonal matrices of S^4. Since every symmetric matrix can be orthogonally diagonalized, it follows that every orbit of SO(3) on S^4 intersects S^1. S^1 is a great circle on S^4.

Let V_4 be the group of diagonal matrices of SO(3), consisting of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $m \in S^1$ has distinct eigenvalues, then its stability group G_m is V_4; this follows immediately from the fact that any matrix commuting with m must be a diagonal matrix. If the first two eigenvalues of m are equal, then $g \in SO(3)$ commutes with m exactly if $g(e_3) = \pm e_3$, that is, if $g \in N_{e_3}$.

One verifies that there are six points of S^1 with two equal eigenvalues. Writing only the diagonal coordinates, we write them as

$$a = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right),$$

$$b = \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),$$

$$c = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),$$

$$a' = -a,$$

$$b' = -b, \text{ and}$$

$$c' = -c.$$

Let H be the subgroup of SO(3) which leaves S^1 invariant. Then S^1/H is homeomorphic to $S^4/SO(3)$. H acts on S^1 as a group of order six generated by reflections of S^1 through the axes aa', bb' and cc'. Thus, there are exactly six arcs on S^1 which serve as cross sections for $S^4/SO(3)$. For example, one such arc is ab'. This gives:

12.1. Let $i, j \in \{1, 2, 3\}$, $i \neq j$. Then there exists a cross sectioning arc cd for S^* such that $G_c = N_{e_i}$, $G_d = N_{e_j}$, and $G_x = V_4$ on $(cd - (c \cup d))$.

Section 13. Actions of SO(3) on S^4 with 3-dimensional orbits

Let SO(3) act on the 4-sphere $S^4 = S$ with at least one 3-dimensional orbit. Let X be the set of points on 3-dimensional orbits and $Y = S - X$. By Theorem 6.1, S^* is an arc, X^* is an open arc, and Y^* consists of exactly two points. Furthermore, all orbits of X are equivalent. From 2.5 it follows that X is homeomorphic to $G(x) \times (0, 1)$, where G(x) is a typical orbit of X. The only orbits of dimension less than three under an action of SO(3) are homeomorphic to P^2, S^2, and $\{p\}$ (a point). Thus, Y is homeomorphic to one of the following spaces:

1) $S^2 \cup S^2$,

2) $S^2 \cup P^2$,

3) $S^2 \cup \{p\}$,

4) $P^2 \cup P^2$,

5) $P^2 \cup \{p\}$,

6) $\{p\} \cup \{p\}$.

LEMMA 13.1. Y is homeomorphic to $P^2 \cup P^2$.

PROOF. If Y has a fixed point p, then letting $p = p_\infty$ we get an induced action of SO(3) on E^4. But then, by ([18], p. 252), the other exceptional orbit must be a fixed point. Hence, cases 3) and 5) are ruled out.

G(x), the typical orbit of X, is homeomorphic to a coset space SO(3)/V, where V is a finite subgroup of SO(3). Since X is homeomorphic to $G(x) \times (0, 1)$, G(x) is a deformation retract of X. Hence, $H_*(G(x)) \approx H_*(X)$.

By the Alexander duality theorem

$$H^2(Y) \approx H_1(X) \approx H_1(G(x)).$$

We recall from Section 5 that $H_1(G(x))$ is finite. Hence, $H^2(Y)$ is finite. Since $H^2(S^2) \approx Z$, Y cannot have an orbit homeomorphic to S^2. Thus, cases 1), 2), and 3) are ruled out.

Suppose Y is homeomorphic to $\{p\} \cup \{p\}$. Then, X is homeomorphic to $E^4 - \{p\}$. But $E^4 - \{p\}$ is simply connected, and thus G(x) must be simply connected. But SO(3) has no simply connected 3-dimensional coset spaces. Thus, case 6) is ruled out. Hence, Y is homeomorphic to $P^2 \cup P^2$.

Let $x \in X$ and let $G_x = V$. Let τ: Sp(1) $\longrightarrow$ SO(3) be the covering map (see Section 5) and set $V' = \tau^{-1}(V)$. We have

$$H^2(Y) \approx H^2(P^2) \oplus H^2(P^2) \approx Z_2 \oplus Z_2.$$

Hence, $H_1(X) \approx H_1(SO(3)/V) \approx V'/[V', V'] \approx Z_2 \oplus Z_2$.

Let $y \in Y$. Then $G_y = N_p$ for some $p \in S^2$. We may choose y so that $G_y = N_{e_1}$. By ([18] , p. 216) there exists a neighborhood N of y such that if $z \in N$, then G_z is conjugate to a subgroup of N_{e_1}. Since all orbits of X are equivalent we may assume $V \subset N_{e_1}$. From Table 5.1 we see that $V = D_{e_1,n}$ for some even n.

Let M be a space and let J = ab be an arc. Let f be a map of J - (a $\cup$ b) into M. We say that a point $x \in M$ belongs to the <u>cluster set</u> of f at a (resp. b) if there exists a sequence $\{t_i\} \in (J - (a \cup b))$ approaching a (resp. b) such that $\{f(t_i)\}$ approaches x. Let D be a closed 2-cell, D^0 its interior, and let f be a map of D^0 into M. We define the cluster set of f at a point

$d \in (D^0 - D)$ in a similar manner. A necessary and sufficient condition that f can be extended to J (resp. D) is that each cluster set contain exactly one point.

The following result is immediate:

13.2. Let M be a compact metric space, J an arc with interior J^0, and D a closed 2-cell with interior D^0. Let f be a map of J^0 (resp. D^0) into M and let $d \in (J - J^0)$ (resp. $D - D^0$). Then the cluster set of f at d is connected.

LEMMA 13.3. There exists a cross sectioning arc ab for S^* such that G_x is constant on $ab - (a \cup b)$.

PROOF. By 2.5 there is a cross section s for X^* on which G_x is constant. We may assume that $G_x = V = D_{e_1,n}$ on this cross section.

In the standard action of SO(3) on P^2 there is a 1-1 correspondence between stability groups G_x and points x of P^2 (see Section 3). By 3.1, if $n > 2$ there is only one subgroup of the form N_p which contains $D_{e_1,n}$. There are exactly three subgroups of this form which contain $D_{e_1,2}$. For any point x in the cluster set of s at either end point we must have $G_x \subset V$ and G_x must be of the form N_p. Thus, the cluster set of s at either end point is finite and connected, hence a point. Thus, s can be extended to a cross section for S^*. This cross section will be an arc which we denote by ab. It follows that G_x is constant on $ab - (a \cup b)$.

LEMMA 13.4. $G_a \neq G_b$.

PROOF. Let $s : S^* \longrightarrow S$ denote our cross section and let f be a homeomorphism of I onto S^* such that $s \circ f(0) = a$ and $s \circ f(1) = b$.

Let $\varphi : I \times SO(3) \to S$ be given by $\varphi(t, g) = g \cdot (s \circ f(t))$. Let M be the space $I \times (SO(3)/V)$ with $0 \times (SO(3)/V)$ "collapsed" to $SO(3)/G_a$ and $1 \times (SO(3)/V)$ "collapsed" to $SO(3)/G_b$. Then, as in Section 2, φ induces a homeomorphism of M onto S. We show that if $G_a = G_b$, then $H_2(M) \neq 0$ and hence M is not homeomorphic to S.

Let π denote the natural projection of M onto I. Let $K = \pi^{-1}([0, 1/2])$, $L = \pi^{-1}([1/2, 1])$. Then $SO(3)/G_a$ is a deformation retract of K, and $SO(3)/G_b$ is a deformation retract of L. $SO(3)/G_a$ and $SO(3)/G_b$ are homeomorphic to P^2. $K \cap L$ is homeomorphic to $SO(3)/V$. Consider the Mayer-Vietoris sequence for K and L:

$$13.5. \quad H_2(M) \xrightarrow{d} H_1(K \cap L) \xrightarrow{i_{1*} \oplus i_{2*}} H_1(K) \oplus H_1(L).$$

Here i_1 is the injection $(K \cap L) \subset K$ and i_2 is the injection $(K \cap L) \subset L$. Now $H_1(K \cap L) \approx H_1(SO(3)/D_{e_1,n}) \approx Z_2 \oplus Z_2$, and $H_1(K) \approx H_1(P^2) \approx H_1(L) \approx Z_2$. Thus, i_{1*} and i_{2*} must have non-trivial kernels. If $G_a = G_b$, then clearly kernel i_{1*} = kernel i_{2*}. Thus, $i_{1*} \oplus i_{2*}$ has a non-trivial kernel which implies by 13.5 that $H_2(M) \neq 0$. Since M is homeomorphic to S, this is a contradiction. Thus, $G_a \neq G_b$.

THEOREM 13.6. Let $SO(3)$ act as a transformation group on the 4-dimensional sphere S^4 such that there exists a 3-dimensional orbit. Then the action is equivalent to the action of $SO(3)$ on S^4 induced by the representation D_2.

PROOF. We have already shown that there is an arc ab in S which cross sections S^* and that $G_x = D_{e_1,n}$, for some even n, on $ab - (a \cup b)$. Furthermore, G(a) and G(b) are homeomorphic to P^2 and $G_a \neq G_b$. Since $G_a \subset D_{e_1,n}$, $G_b \subset D_{e_1,n}$, it follows from 3.1 that

$n = 2$, hence that $G_x = D_{e_1,2}$ on $ab - (a \cup b)$. By translation of our cross sectioning arc we may assume that, on $ab - (a \cup b)$, G_x is the group V_4 defined in Section 12. (Remember that $D_{e_1,2}$ is determined only to within conjugacy in N_{e_1}.) Hence G_a and G_b must each be one of the three groups N_{e_1}, N_{e_2}, and N_{e_3}. Recalling 12.1 we now apply 2.6 to prove Theorem 13.6.

Section 14. Actions of SO(3) on S^4 with 2-dimensional orbits

Let SO(3) act as a transformation group on the 4-sphere $S^4 = S$ with the highest dimension of any orbit two. Let X be the set of points which lie on 2-dimensional orbits and $Y = S - X$. Then, by Theorem 7.1, S^* is a closed 2-cell with Y^* as simple closed curve boundary and all orbits of X are equivalent. It follows from 9.5 that all orbits of X must be homeomorphic to S^2. All orbits of Y must be fixed points since SO(3) has no 1-dimensional coset spaces. Hence Y is a simple closed curve.

LEMMA 14.1. There exists a cross section s for S^* such that G_x is constant on $s(X^*)$.

PROOF. By 2.5 there exists a cross section for X^* on which G_x is constant. Since every orbit of Y is a fixed point it follows immediately that this can be extended to a cross section for S^*.

THEOREM 14.2. Let SO(3) act as a transformation group on the 4-dimensional sphere S^4 with the highest dimension of any orbit two. Then the given action is equivalent to the action of SO(3) on S^4 induced by the representation $D_1 + D_0 + D_0$.

PROOF. Let s be the given cross section. Since G_x is constant on $s(X^*)$ we may assume that $G_x = S_{e_1}$ on $s(X^*)$.

To obtain $D_1 + D_0 + D_0$ we write $E^5 = E^3 \times E^2$, let $SO(3)$ operate in standard fashion on E^3 and by identity on E^2, and restrict the operation to S^4, the unit sphere in E^5. A cross section of the action is given by the set

$$\{(x_1, 0, 0, x_4, x_5) : 0 \leq x_1 \leq 1,\ x_1^2 + x_4^2 + x_5^2 = 1\};$$

this is a 2-cell. For points on the interior ($x_1 > 0$) the stability group is S_{e_1}, for points on the boundary ($x_1 = 0$) it is $SO(3)$. It is now immediate by 2.6 that the two actions are equivalent.

Section 15. Actions of $Sp(1)$ on S^4

Let $Sp(1)$ act effectively as a transformation group on the 4-dimensional sphere $S^4 = S$. By 4.1 there must be a 3-dimensional orbit. Let X be the space of points which lie on 3-dimensional orbits and $Y = S - X$. By Theorem 6.1, S^* is a closed arc, Y^* consists of two points, the end points of S^*, and all orbits of X are equivalent.

LEMMA 15.1. Y consists of two fixed points.

PROOF. By 2.5, X is homeomorphic to $(0, 1) \times Sp(1)/V$, where V is a finite subgroup of $Sp(1)$. By 4.1, $V = S'_{q,n}$ with n odd or $V = \{e\}$. Hence, by Table 5.1, $H_1(X) \approx H_1(Sp(1)/V) \approx 0$ or Z_n with n odd. Thus, by Alexander duality, $H^2(Y)$ is isomorphic to either Z_n with n odd or to 0. But each orbit of Y is homeomorphic to S^2, P^2, or $\{p\}$. Hence each orbit of Y is a fixed point.

THEOREM 15.2. Let $Sp(1)$ act effectively as a transformation group on the 4-dimensional sphere. Then the action is equivalent to the action of $Sp(1)$ on S^4 induced by the representation $D_{1/2} + D_0$.

PROOF. X is homeomorphic to $(0, 1) \times (Sp(1)/V)$, where V is equal to either $S'_{q,n}$ or $\{e\}$. By Alexander duality

$$0 = H^2(Y) \approx H_1(X) \approx H_1(Sp(1)/V).$$

By Table 5.1 $V = \{e\}$. By 2.5 there is a cross section for X^* and it is immediate that this cross section can be extended to give a cross section for S^*.

$D_{1/2} + D_0$ can be obtained by writing $E^5 = E^4 \times E^1$, identifying E^4 with the quaternions Q, and operating with $Sp(1)$ by $g\cdot(q, x_5) = (gq, x_5)$. A cross section on $S^4 \subset E^5$ consists of all $(x_1, 0, 0, 0, x_5)$ with $0 \leq x_1 \leq 1$, $x_1^2 + x_5^2 = 1$; this is an arc. The stability groups are clearly such that 2.6 applies to prove 15.2.

CHAPTER IV

ACTIONS OF SO(3) AND Sp(1) ON S^5 WITH 3-DIMENSIONAL ORBITS

In this chapter we will be concerned with actions of SO(3) and Sp(1) on the 5-dimensional sphere S^5 such that there exists a 3-dimensional orbit. We will show that, to within equivalence, there is only one such action of Sp(1) on S^5 which is effective. This action is linear. Similarly, to within equivalence, there are at most three such actions of SO(3) on S^5. Two of them are linear. In the remaining case the situation is more complicated: Suppose we are given an action of SO(3) on S^5 with a 3-dimensional orbit such that all lower dimensional orbits are fixed points. If such an action exists then we show that there is a cross section for S^*. Using this cross section we construct a model space for the action. However, we cannot tell whether or not our model space is homeomorphic to S^5. Thus, we are not sure whether or not there exists an action of SO(3) on S^5 with the prescribed properties.

Section 16. Homology relations

Let SO(3) or Sp(1) act on $S^5 = S$ such that there exists a 3-dimensional orbit. Let X be the set of points on 3-dimensional orbits and $Y = S - X$. By Theorem 7.1, S^* is a closed 2-cell with Y^* as simple closed curve boundary. Furthermore, all orbits of X are equivalent. Since X^* is an open 2-cell it follows by 2.5 that X is homeomorphic to $X^* \times G(x)$, where G(x) is a typical orbit of X. Thus, $H_*(X) \approx H_*(G(x))$. G(x) is homeomorphic to $Sp(1)/V'$ where V' is a

finite subgroup of Sp(1). From Table 5.1 we see that

$$H_0(Sp(1)/V') \approx H_3(Sp(1)/V') \approx Z,$$

$$H_2(Sp(1)/V') = 0,$$

$$H_1(Sp(1)/V') \approx V'/[V', V'] .$$

By Alexander duality $H^{5-i-1}(Y) \approx H_i(X)$. This gives:

16.1. 1) $H^1(Y) \approx H_3(X) \approx Z$

2) $H^2(Y) \approx H_2(X) \approx 0$

3) $H^3(Y) \approx H_1(X) \approx V'/[V', V']$.

In particular, $H^3(Y)$ is finite.

We shall show that the space Y is completely determined by knowledge of its cohomology groups. Once we know Y we can determine the subgroup V'. Then we show that we can find a cross section for S^* with prescribed stability groups along the cross section. By Theorem 2.6 this completely characterizes the action.

We know that all orbits of Y have dimension ≤ 2. Thus, the orbits of Y are either fixed points or are homeomorphic to S^2 or P^2.

Section 17. Case 1. Y has an orbit homeomorphic to S^2

This section will be devoted to the proof of the following theorem:

THEOREM 17.1. Let SO(3) act on S^5 such that there is a 3-dimensional orbit and an orbit homeomorphic to S^2. Then the action is equivalent to the action of SO(3) on S^5 induced by the representation $D_1 + D_1$.

The proof of Theorem 17.1 will be given in a series of lemmas. We use the notation introduced earlier.

LEMMA 17.2. If Y has an orbit homeomorphic to S^2, then all orbits of Y are homeomorphic to S^2.

PROOF. Let B be the subset of Y of points whose orbits are homeomorphic to S^2. By ([18], p. 241), B is an open subset of Y. Assume $B \neq Y$. Let B_0 be a component of B and let $\overline{B_0}$ denote the closure of B_0. Then B_0^* is an open arc and $\overline{B_0^*}$ is a closed arc or a 1-sphere. By 2.5, B_0 is homeomorphic to $(0, 1) \times S^2$. Thus, $(\overline{B_0}, \overline{B_0} - B_0)$ is an orientable relative 3-manifold. By ([7], pp. 314-315) $H^3(\overline{B_0}, \overline{B_0} - B_0) \approx Z$. Now $\overline{B_0} - B_0$ will be homeomorphic to one of the following spaces: a) $P^2 \cup P^2$; b) P^2; c) $\{p\}$; d) $\{p\} \cup P^2$; or e) $\{p\} \cup \{p\}$. In any case $H^3(\overline{B_0} - B_0) \approx H^2(\overline{B_0} - B_0) \approx 0$ or Z_2. Thus, from the exact cohomology sequence for $(\overline{B_0}, \overline{B_0} - B_0)$, it follows that $H^3(\overline{B_0}) \approx Z$. Let $\overline{B_1} = Y - B_0$. Then $\overline{B_0} \cap \overline{B_1}$ is again homeomorphic to one of the spaces listed above and $H^2(\overline{B_0} \cap \overline{B_1}) \approx H^3(\overline{B_0} \cap \overline{B_1}) \approx 0$ or Z_2. From the Mayer-Vietoris sequence for $(Y, \overline{B_0}, \overline{B_1}, \overline{B_0} \cap \overline{B_1})$ it follows that $H^3(Y)$ contains Z as a subgroup. This contradicts 16.1.

LEMMA 17.3. If $x \in X$, then $G_x = \{e\}$.

PROOF. Since all orbits of Y are equivalent, it follows from Gleason's theorem that Y is a connected 3-manifold. Since $H^3(Y)$ is finite, Y is non-orientable and hence $H^3(Y) \approx Z_2$. Now X is the product of a 2-cell and $SO(3)/G_{x_0}$ for some $x_0 \in X$. Thus, $Z_2 \approx H^3(Y) \approx H_1(X) \approx H_1(SO(3)/G_{x_0})$. Take $y \in Y$ such that $G_y = S_{e_1}$. Then, if x is near y, G_x is conjugate to a subgroup of S_{e_1}. We may choose $x_0 \in X$ such that $G_{x_0} \subset S_{e_1}$. From Table 5.1 we see that either

$G_{x_0} = \{e\}$ or $G_{x_0} = H_C$. But $H \not\subset S_{e_1}$. Hence $G_{x_0} = \{e\}$. Since all orbits of X are equivalent, this says that $G_x = \{e\}$ for every $x \in X$.

LEMMA 17.4. Let $J = \{y \in Y : G_y = S_{e_1}\}$. Then J is a simple closed surve and $T|J : J \longrightarrow Y^*$ is a 2-1 covering map.

PROOF. Each orbit of Y intersects J in exactly two points. Hence $T|J$ is 2-1. It follows from Gleason's theorem that $T|J$ is a covering map. Thus, J is either a simple closed curve or a disjoint union of two simple closed curves. Assume $J = J_1 \cup J_2$, where J_1 and J_2 are disjoint simple closed curves. Then J_1 and J_2 each define a cross section for Y^*. By considering the corresponding model spaces we see that Y is homeomorphic to $S^1 \times S^2$, which is an orientable manifold. This contradicts 16.1. Hence J is a simple closed curve.

Let M be the space $I \times S^2$ with $0 \times S^2$ and $1 \times S^2$ identified by the antipodal map. Let SO(3) act on $I \times S^2$ by acting on the second factor. This induces an action of SO(3) on M. It is an easy consequence of Lemma 17.4 that M is a model space for the action of SO(3) on Y.

The natural projection $\pi_1 : I \times S^2 \longrightarrow I$ induces the map $T : M \longrightarrow M^*$. M^* is naturally homeomorphic to the space I with its end points identified. A cross section of M is given by a map $f : I \longrightarrow S^2$ such that f(0) and f(1) are antipodal points. Any cross section of M^* is of this form.

We now construct a cross section for S^*. For purposes of description we identify S^* with the closed unit disk in E^2. The construction takes place in several stages. We first define local cross sections at points of Y^*. These cross sections are then

"patched together" to give a cross section of an annular neighborhood of Y^*. The extendability of this cross section to a cross section of S^* reduces to a simple homotopy question. If the cross section is not extendable we may modify it so that it becomes extendable.

Choose $y_0 \in Y$ such that $G_{y_0} = S_{e_1}$. Let K be a slice at y_0. By cutting down the size of K (see 8.2) we may assume that T(K) is defined in polar coordinates by

$$T(K) = \left\{(r, \theta) \in S^* : r_0 \leq r \leq 1,\ \theta_1 \leq \theta \leq \theta_2\right\}.$$

S_{e_1} acts on K so that the orbit space is T(K). S_{e_1} acts on $K \cap X$ without fixed points and the orbit space is $T(K) \cap X^*$ which is homeomorphic to a half-plane. Thus, by 2.5, there exists a cross section for $T(K) \cap X^*$ in $K \cap X$. Each orbit of Y intersects K in at most one point. Since K is compact it follows from 13.2 that the cross section can be extended to give a cross section s_0 for T(K) in K.

Let N be the space $I \times I \times SO(3)$ with $(0, v) \times SO(3)$ "collapsed" to $SO(3)/S_{e_1}$ for each v, $0 \leq v \leq 1$. Let SO(3) act on $I \times I \times SO(3)$ by $g_1 \cdot (w, v, g) = (w, v, g_1 g)$. This induces an action of SO(3) on N. The orbit space $N^* = N/SO(3)$ can be identified with $I \times I$ and the projection $T_0 : N \longrightarrow N^*$ is induced by the projection $\pi : (I \times I) \times SO(3) \longrightarrow I \times I$.

Let M_0 be the model space for $T^{-1}(T(K)) = G(K)$ constructed by the method of Section 2, using the cross section s_0. Since $I \times I$ is homeomorphic to T(K), it is immediate that there is an equivariant

homeomorphism of N onto M_0. Thus, N serves as a model space for G(K).

The following lemma will allow us to "patch together" our local cross sections.

LEMMA 17.5. 1) Let $s' : (I \times 1) \cup (0 \times I) \longrightarrow N$ be a cross section such that $G_{s'(0, v)} = S_{e_1}$, $0 \leq v \leq 1$. Then s' can be extended to a cross section of $I \times I$; 2) Let $s' : (I \times 1) \cup (0 \times I) \cup (I \times 0) \longrightarrow N$ be a cross section such that $G_{s'(0,v)} = S_{e_1}$, $0 \leq v \leq 1$. Then s' can be extended to a cross section of $I \times I$.

PROOF. We prove only 2). Let $N_1 = I \times I \times (SO(3)/S_{e_1})$. Then there are natural maps $T_1 : N \longrightarrow N_1$ and $T_2 : N_1 \longrightarrow I \times I$ such that $T_2 \circ T_1 = T_0$. Let $f' = T_{1 \circ s'}$. Then it is immediate that f' can be extended to a map $f: I \times I \longrightarrow N_1$ such that $T_{2 \circ f} = 1_{(I \times I)}$. Note that f is a homeomorphism of $I \times I$ into N_1.

We now want to define a map $f_1 : f(I \times I) \longrightarrow N$ such that $f_1 \circ f$ is a cross section which extends s'. The set $T_1^{-1}(f(I \times I)) = L$ is compact. S_{e_1} acts on L. S_{e_1} acts on $T_1^{-1}(f((0, 1] \times I))$ without fixed points and the orbit space is $f((0, 1] \times I)$. Every point of $T_1^{-1}(f(0 \times I))$ is fixed under S_{e_1}. Using 2.5 and 13.2 we construct a cross section for L/S_{e_1}. The space $(I \times I) \times S_{e_1}$ with $(0, v) \times S_{e_1}$ "collapsed" to a point for each v, $0 \leq v \leq 1$, serves as a model space for L; this space is a solid cylinder.

The problem of extending s' now reduces to the following problem. Let φ_1 and φ_2 be maps of $(0, 1]$ into S^1, the unit circle in E^2; we wish to define a homotopy of φ_1 and φ_2. This can be done by considering the covering map $\psi : E^1 \longrightarrow S^1$, lifting the maps

φ_1 and φ_2 to maps φ_1' and φ_2' of $(0, 1] \to E^1$, and homotopically deforming φ_1' into φ_2'. This completes the proof of 17.5.

LEMMA 17.6. There exists a cross section for S^*.

PROOF. We shoose points $p_1, \ldots, p_n \in Y$ and slices $K_1, \ldots, K_n$ at (resp.) $p_1, \ldots, p_n$ such that the $T(K_i)$ are defined in polar coordinates by

$$T(K_1) = \{(r, \theta) : r_0 \leq r \leq 1, 0 \leq \theta \leq \theta_1\}$$

$$T(K_2) = \{(r, \theta) : r_0 \leq r \leq 1, \theta_1 \leq \theta \leq \theta_2\}$$

$$\ldots$$

$$T(K_n) = \{(r, \theta) : r_0 \leq r \leq 1, \theta_{n-1} \leq \theta \leq 2\pi\}.$$

We may assume that $G_{p_i} = S_{e_1}$, $i = 1, \ldots, n$. By a change of coordinates in S^* we may assume that $\theta_1 = \pi$. Let the annulus A be given by $A = \{(r, \theta) : r_0 \leq r \leq 1\}$. We define a cross section for A.

Let s_1' be a cross section for $T(K_1)$ such that $G_{s_1'}(1, \theta) = S_{e_1}$, $0 \leq \theta \leq \pi$; we have seen that such cross sections exist. We now modify this cross section. Let $g_\theta \in SO(3)$ be the rotation about the axis e_3 through an angle θ with orientation chosen so that $g_{\pi/2}(e_1) = e_2$. Let $p_\theta \in S^2$ be defined by: $p_\theta = (\cos \theta)e_1 + (\sin \theta)e_2$ for $0 \leq \theta \leq \pi$, $p_\theta = -e_1$ for $\pi \leq \theta \leq 2\pi$. We define a new cross section s_1 for $T(K_1)$ by

$$s_1(r, \theta) = g_\theta \cdot s_1'(r, \theta), \quad 0 \leq \theta \leq \pi.$$

For this cross section we have $G_{s_1}(1, \theta) = S_{p_\theta}$. In particular, $G_{s_1}(1, 0) = G_{s_1}(1, \pi) = S_{e_1}$ (remember that $S_{e_1} = S_{-e_1}$).

Let $Y_1^* = \{(r, \theta) : r = 1, \pi \leq \theta \leq 2\pi\}$. Let $E_1 = \{x \in T^{-1}(Y_1^*) : G_x = S_{e_1}\}$. Then E_1 consists of two disjoint arcs, each of which is mapped homeomorphically by T onto Y_1^*. From our discussion of the cross sections of the model space M for Y, it can be seen that $s_1(1, \pi)$ and $s_1(1, 0)$ must lie on the same arc of E_1. Thus, s_1 can be extended to a cross section, still denoted by s_1, of $T(K_1) \cup Y_1^*$.

By successive applications of 17.5 1) we extend s_1 to $Y_1^* \cup (\bigcup_{i=1}^{n-1} T(K_i))$. By an application of 17.5 2) s_1 is extended to $A = \bigcup_{i=1}^{n} T(K_i)$. We note that $G_{s_1(1,\theta)} = S_{p_\theta}$.

Let $B = \{(r, \theta) : 0 \leq r \leq r_0\}$ and $C = \{(r, \theta) : r = r_0\}$. Since all orbits of $T^{-1}(B)$ are equivalent, it follows from 2.5 that $T^{-1}(B)$ is homeomorphic to $B \times SO(3)$ and $T^{-1}(C)$ is homeomorphic to $C \times SO(3)$. A cross section of C is thus given by a map $\psi: C \to SO(3)$. Such a cross section is extendable to B if and only if the map ψ is homotopic to the constant map. Since $\pi_1(SO(3)) \approx Z_2$ there is precisely one homotopy class of cross sections of C that cannot be extended to B.

Now $s_1 | C$ gives a cross section for C. If this cross section is extendable, we extend it to a cross section s_1 of S^*.

Let $h_\theta \in SO(3)$ be the rotation about p_θ through angle π. By considering the covering map $\tau: Sp(1) \to SO(3)$, we see that $\theta \to h_\theta$ represents the non-trivial element of $\pi_1(SO(3))$. If s_1 is not extendable we modify it by $s_2(r, \theta) = h_\theta \cdot s_1(r, \theta)$. Note that $G_{s_2(r,\theta)} = G_{h_\theta \cdot s_1(r,\theta)} = h_\theta G_{s_1(r,\theta)} h_\theta^{-1} = h_\theta S_{p_\theta} h_\theta^{-1} = S_{h_\theta(p_\theta)} = S_{p_\theta}$. The two maps $\theta \to s_1(r_0, \theta)$ and $\theta \to s_2(r_0, \theta)$ define different

elements of $\pi_1(T^{-1}(B)) \approx \pi_1(B \times SO(3))$. Thus, s_2 can be extended to a cross section of B. This proves 17.6.

We have shown that if SO(3) acts on S^5 with a 3-dimensional orbit and an orbit homeomorphic to S^2, then we can find a cross section s for S^* such that $G_{s(1,\theta)} = S_{p_\theta}$ and $G_{s(r,\theta)} = \{e\}$ for $r < 1$. It follows from 2.6 that all such actions are equivalent.

THEOREM 17.1. Let SO(3) act on S^5 such that there is a 3-dimensional orbit and an orbit homeomorphic to S^2. Then the action is equivalent to the action of SO(3) on S^5 induced by the representation $D_1 + D_1$.

PROOF. To realize $D_1 + D_1$ we write E^6 as $E^3 \times E^3$, let SO(3) operate by $g\cdot(x, y) = (g(x), g(y))$, and restrict the operation to S^5, the unit sphere in E^6. It is sufficient to show that this action has a 3-dimensional orbit and a 2-sphere orbit. The point $\frac{1}{\sqrt{2}}(e_1, e_2) \in S^5$ has stability group $S_{e_1} \cap S_{e_2} = \{e\}$ and thus $G\left(\frac{1}{\sqrt{2}}(e_1, e_2)\right)$ is 3-dimensional. The point $\frac{1}{\sqrt{2}}(e_1, e_1)$ has stability group S_{e_1}; thus, $G\left(\frac{1}{\sqrt{2}}(e_1, e_1)\right)$ is homeomorphic to S^2.

It is possible to construct an explicit cross section for this action, but as the details are complicated, the construction is omitted.

Section 18. Case 2. Y has an orbit homeomorphic to P^2

In this section we prove the following theorem:

THEOREM 18.1. Let $SO(3)$ act on S^5 with a 3-dimensional orbit and an orbit homeomorphic to P^2. Then the action is equivalent to the action of $SO(3)$ on S^5 induced by the representation $D_2 + D_0$.

First note that there cannot be any orbits homeomorphic to S^2 by Lemma 17.2. Thus, all orbits of Y are either fixed points or are homeomorphic to P^2. Let B be the subset of Y of points which lie on orbits homeomorphic to P^2 and let F be the set of fixed points of Y. By ([18], p. 241), B is an open subset of Y. Since Y^* is a simple closed curve, B^* will be a simple closed curve if $F = \emptyset$, otherwise each component of B^* will be an open arc. Similarly each component of F^* will be an arc or a point.

By Theorem 7.1 all orbits of X are equivalent. Choose $b \in B$ such that $G_b = N_{e_1}$. If x is sufficiently close to b, then G_x is conjugate to a subgroup of G_b. Every neighborhood of b will intersect X. Hence we may find $x_0 \in X$ such that $G_{x_0} \subset N_{e_1}$. Set $G_{x_0} = V$.

We shall need the following result which is proved in ([19]):

18.2. Let X be an open n-cell, N a finite polyhedron, and $I' = [0, 1)$, the half open interval. Let D be the space obtained by identifying $0 \times N$ to a point in $I' \times N$ and let $M = D \times X$. Then, if M is an $(n + m + 1)$-manifold, N has the integral homology groups of an n-sphere.

LEMMA 18.2. F^* has no components which are arcs.

PROOF. Assume that F^* has a component which is an arc. Let us identify S^* with the closed unit disk in E^2. Let R be defined in polar coordinates by

$$R = \{(r, \theta) : 1/2 < r \leq 1, \ \theta_1 < \theta < \theta_2\}$$

where θ_1 and θ_2 are so chosen that $R \cap Y^* \subset F^*$. Then $T^{-1}(R)$ is an open connected subset of S, thus a manifold. All orbits of $T^{-1}(R) \cap X$ are equivalent and the orbit space is $R \cap X^*$, which is an open 2-cell. Hence by 2.5 there is a cross section for $R \cap X^*$ on which the stability group is constant. We may choose the cross section so that $G_x = V$ on this cross section. Since $T^{-1}(R) \cap Y$ consists of fixed points this cross section may be extended to give a cross section of R. Using this cross section we may construct a model space by the method of Section 2. From the model space we see that $T^{-1}(R)$ is a space of the type described in 18.1 with X an open 1-cell, $N = SO(3)/V$, and $M = T^{-1}(R)$. But 18.1 implies that N has the homology of a 3-sphere. This contradicts Table 5.1, since V is a finite subgroup of N_{e_1}.

LEMMA 18.3. There are exactly two fixed points. V is a dihedral group $D_{e_1,n}$ with n even.

PROOF. We first show that $F \neq \phi$. Assume $F = \phi$. Then all orbits of Y are homeomorphic to P^2. Let $E = \{x \in Y : G_x = N_{e_1}\}$. Then E is compact and every orbit of Y meets E exactly once. Thus, $T : E \longrightarrow Y^*$ is a homeomorphism. It follows easily that Y is homeomorphic to $E \times P^2$. Using the Künneth formula we obtain $H_1(Y) = Z \oplus Z_2$, $H_2(Y) = Z_2$. By the universal coefficient theorem $H^2(Y) = Z_2$ which contradicts 16.1. Thus, $F \neq \phi$.

Assume there is exactly one fixed point. Then B^* is an open arc and B is homeomorphic to $(0, 1) \times P^2$. Hence (Y, F) is a non-orientable relative 3-manifold and $H^3(Y, F) \approx Z_2$. Since F is

a point $H^3(Y) \approx Z_2$. Thus, $H_1(SO(3)/V) \approx Z_2$. Since $V \subset N_{e_1}$, it follows from Table 5.1 that $V = \{e\}$.

Let us identify S^* with the unit disk in E^2 by a homeomorphism. Choose $y \in B$ such that $G_y = N_{e_1}$. Let K be a slice at y. We may choose K so that T(K) is given in polar coordinates by

$$T(K) = \{(r,\theta) : 1/2 \leq r \leq 1, \ \theta_1 \leq \theta \leq \theta_2\}.$$

Define R by

$$R = \{(r, \theta) : 1/2 < r \leq 1, \ \theta_1 < \theta < \theta_2\}.$$

N_{e_1} acts on K so that the orbit space is T(K). N_{e_1} acts on $K \cap X$ and since $G_x = \{e\}$ for $x \in X$, all orbits of $K \cap X$ under the action of N_{e_1} are equivalent. The orbit space for $K \cap X$ under this action is $T(K) \cap X^*$. Thus, by 2.5, there exists a cross section in $K \cap X$ for $T(K) \cap X^*$. Every point of $K \cap Y$ is fixed under the action of N_{e_1}. Thus, the cross section may be extended to a cross section of T(K) in K. This cross section also serves as a cross section for T(K) under the action of SO(3). Let J be the space obtained by "collapsing" $0 \times SO(3)$ to $SO(3)/N_{e_1}$ in $[0, 1) \times SO(3)$. Then, using the cross section, it can be shown that $T^{-1}(R)$ is homeomorphic to $(0,1) \times J$. Since $T^{-1}(R)$ is a 5-manifold it follows from 18.1 that N_{e_1} must have the homology groups of a 1-sphere, which is a contradiction. Thus, we see that there is not exactly one fixed point.

Assume that there are two or more fixed points. Let B_1 be a component of B. Then B_1^* is an open arc and B_1 is homeomorphic to $(0, 1) \times P^2$. Thus, $(\overline{B_1}, \overline{B_1} - B_1)$ is a non-orientable relative 3-manifold and $H^3(\overline{B_1}, \overline{B_1} - B_1) \approx Z_2$. Since $\overline{B_1} - B_1$ consists of two points

$H^3(\overline{B_1}) \approx Z_2$. If we consider the Mayer-Vietoris sequence for $(\overline{B_1}, Y - B_1)$ we see that $H^3(Y) \approx H^3(\overline{B_1}) \oplus H^3(Y - B_1)$. Repeated application of the Mayer-Vietoris sequence shows that $H^3(Y)$ is a direct sum of Z_2 factors, with one factor for each component of B. But $H^3(Y) \approx H_1(SO(3)/V)$. We have assumed that B has at least two components. From an inspection of Table 5.1 we see that V must be a dihedral group $D_{e_1,n}$, with n even, and hence that B must have exactly two components. Thus, there are exactly two fixed points.

LEMMA 18.4. There exists a cross section for S^* in S.

PROOF. X^* is an open 2-cell and all orbits of X are equivalent. Thus, by 2.5, there is a cross section for X^* on which the stability group is constant. We may choose the cross section such that $G_x = V$ on the cross section. By 3.1, V is contained in at most three groups of the form N_p. Let $s: X^* \longrightarrow X$ be our cross section. Then the cluster set of s at each point of Y^* consists of at most three points. Since this cluster set must be connected by 13.2, it reduces to one point. Hence we may extend s to a cross section of S^*. This proves the lemma.

As we have already seen B has exactly two components. Let B_1 and B_2 denote these components. Then B_1^* and B_2^* are both open arcs.

LEMMA 18.5. G_x is constant on $s(B_i^*)$, $i = 1, 2$.

PROOF. Let $x_0 \in s(B_i^*)$. Then x_0 is a limit point of points with stability group V and hence $G_{x_0} \subset V$. By 3.1, there are only three possibilities for G_x, namely $G_x = N_{e_1}$, N_p, and N_q, where p and q are perpendicular unit vectors in the plane generated by e_2 and e_3. There is a neighborhood N of x_0 such that $x \in N$ implies that $G_x \subset g^{-1}G_{x_0}g$

for some $g \in SO(3)$. By restricting N we may require g to lie in an arbitrarily small neighborhood of e. But N_{e_1}, N_p, and N_q are not conjugate to one another by elements in an arbitrarily small neighborhood of e. Thus, the set $A = \{x \in s(B_i^*) : G_x = G_{x_0}\}$ is open. The same argument shows that A is closed. By the connectivity of $s(B_i^*)$, $A = s(B_i^*)$.

LEMMA 18.6. 1) If $x_1 \in s(B_1^*)$, $x_2 \in s(B_2^*)$, then $G_{x_1} \neq G_{x_2}$; 2) $V = D_{e_1,2}$.

PROOF. Let us identify S^*, and also $s(S^*)$, with the closed unit disk in E^2 by a homeomorphism. Let γ be a straight line segment in $s(S^*)$ from a point $x_1 \in s(B_1^*)$ to a point $x_2 \in s(B_2^*)$. Let $\gamma^* = T(\gamma)$. Then $T^{-1}(\gamma^*)$ is homeomorphic to $I \times SO(3)/V$ with $0 \times SO(3)/V$ "collapsed" to $SO(3)/G_{x_1}$ and $1 \times SO(3)/V$ "collapsed" to $SO(3)/G_{x_2}$. Let $S' = S - F$. Since F consists of exactly two points, S' is homeomorphic to $E^5 - \{0\}$. Thus, $H_*(S') \approx H_*(S^4)$. Using our cross section and remembering that G_x is constant on each $s(B_i^*)$ we see that S' is homeomorphic to $T^{-1}(\gamma^*) \times (0, 1)$. Thus, $H_*(S') \approx H_*(T^{-1}(\gamma^*))$. But $T^{-1}(\gamma^*)$ is a space of the type considered in Section 13 (see in particular Lemma 13.4). In this section it was shown that if $H_*(T^{-1}(\gamma^*)) \approx H_*(S^4)$ then $G_{x_1} \neq G_{x_2}$. From this it follows that $V = D_{e_1,2}$.

We are now ready to prove the main theorem of this section.

THEOREM 18.1. Let SO(3) act on S^5 with a 3-dimensional orbit and with an orbit homeomorphic to P^2. Then the action is equivalent to the action of SO(3) on S^5 induced by the representation $D_2 + D_0$.

PROOF. Let V_4 be the group of diagonal matrices of $SO(3)$. We may assume that $V = V_4$. We may also assume without loss of generality that $G_x = N_{e_1}$ on $s(B_1^*)$ and $G_x = N_{e_2}$ on $s(B_2^*)$.

One description of the action of $SO(3)$ on S^5 induced by the representation $D_2 + D_0$ is as follows: We take the "suspension" of the action of $SO(3)$ on S^4 induced by D_2. More precisely we have the following: S^5 is homeomorphic to $I \times S^4$ with $0 \times S^4$ and $1 \times S^4$ each identified to a point. The action of $SO(3)$ on S^4 given by D_2 induces an action of $SO(3)$ on $I \times S^4$ by the rule $g\cdot(t, x) = (t, g\cdot x)$. This in turn induces an action of $SO(3)$ on S^5 in the obvious manner. Consider the cross section for the action of $SO(3)$ on S^4 given by D_2 which is represented by the arc cb' described in Section 12. Note that $G_c = N_{e_1}$, $G_{b'} = N_{e_2}$ and $G_x = V_4$ on the interior of this cross section. If we suspend this cross section we get a cross section s' for the action induced by $D_2 + D_0$. The stability groups are clearly such that Proposition 2.6 applies to prove Theorem 18.1.

Section 19. Case 3. All points of Y are fixed points

In this section we consider actions of $SO(3)$ on S^5 such that all orbits are either 3-dimensional or are fixed points. We show that, to within equivalence, there is at most one such action. However, we are unable to determine whether any such actions exist.

We use the terminology of Section 16. Since all orbits of Y are fixed points and Y^* is a simple closed curve, Y is a simple closed curve. Thus, $H^3(Y) = 0$. Hence, by Alexander duality, $H_1(X) = 0$. But $H_1(X) \approx H_1(G(x)) \approx H_1(SO(3)/V)$ where $G(x)$ is a

typical orbit of X and V is a finite subgroup of SO(3). From Table 5.1 we see that V must be an icosahedral group H_I.

By 2.5 there is a cross section for X^* such that $G_x = H_I$ on this cross section. Since all points of Y are fixed points, this cross section can be extended to a cross section s for S^*.

Let D be the closed unit disk in E^2 and S^1 the unit circle in E^2. Let M be the space $D \times (SO(3)/H_I)$ with $p \times (SO(3)/H_I)$ "collapsed" to a point for each $p \in S^1$. Let SO(3) act on $D \times (SO(3)/H_I)$ by the rule $g_1 \cdot (x, gH_I) = (x, g_1 gH_I)$. This action induces an action of SO(3) on M.

Since we have found a cross section with prescribed stability groups it follows from 2.6 that there is at most one action of SO(3) on S^5 such that all points of Y are fixed points. If there is such an action then it follows from the construction of Section 2 that the space M is a model space for the action. The question of whether or not there is such an action reduces to the question of whether or not M is homeomorphic to S^5.

M can alternatively be described in the following two ways: 1) M is the join of S^1 and $SO(3)/H_I$; 2) M is the double suspension of $SO(3)/H_I$.

LEMMA 19.1. M is simply connected and has the same integral homology groups as S^5. Thus, M has the same homotopy type as S^5. M has the local homology groups of a 5-manifold.

PROOF. The suspension of a connected polyhedron is always simply connected. Hence M is simply connected. Again since $SO(3)/H_I$ has the integral homology groups of the 3-sphere, it follows immediately

that the double suspension of $SO(3)/H_I$ has the integral homology groups of the 5-sphere. By ([6], Exposé III, p. 10) it follows that M and S^5 are of the same homotopy type. The statement concerning the local homology groups follows from the fact that $SO(3)/H_I$ has the integral homology of the 3-sphere.

It can also be shown that M has the same local homotopy groups as S^5 in the sense of ([10]). In view of these facts it becomes quite difficult to prove that M is not homeomorphic to S^5.

It should be noted that M is triangulable. If M is homeomorphic to S^5, in fact if M is locally euclidean, then it provides a counter-example to the "Sphere-Conjecture" (see ([13]) for a discussion of this conjecture). This conjecture states that if X is a triangulated n-manifold, then the closed star of every vertex is an n-cell. However, M has vertices whose closed star is homeomorphic to the cone over the (single) suspension of $SO(3)/H_I$. The suspension of $SO(3)/H_I$ is not homeomorphic to S^4, since we may remove two points and it is no longer simply connected.

We have not been able to determine whether or not M is homeomorphic to S^5.

The results of this section are summarized in the following somewhat indecisive theorem.

THEOREM 19.2. There is at most one action of $SO(3)$ on S^5 such that all orbits are either 3-dimensional or fixed points. Such an action exists if and only if S^5 is homeomorphic to the double suspension of $SO(3)/H_I$.

Section 20. Actions of Sp(1) on S^5

In this section we consider effective actions of Sp(1) on $S^5 = S$. We show that, to within equivalence, there is exactly one such action. This action is the action induced by the representation $D_0 + D_0 + D_{1/2}$.

Let Sp(1) act effectively on S^5. By 4.1 there must exist a 3-dimensional orbit. We shall use the notation of Section 16. By 4.1 there must exist $x \in X$ such that $G_x = \{e\}$ or $G_x = S'_{q',n}$ for odd n. Hence by 16.1 and by Table 5.1, $H^3(Y)$ is isomorphic to either 0 or Z_n for odd n. But the analysis of Sections 17 and 18 shows that $H^3(Y)$ cannot possibly be isomorphic to Z_n for odd n. Thus, $H^3(Y) = 0$. Thus, $G_x = \{e\}$ for every $x \in X$. It also follows from Sections 17 and 18 that all orbits of Y are fixed points.

By 2.5 there exists a cross section for X^*. Since all orbits of Y are fixed points this cross section can be extended to give a cross section for S^*.

THEOREM 20.1. Let Sp(1) act effectively on S^5. Then the action is equivalent to the action of Sp(1) on S^5 induced by the representation $D_{1/2} + D_0 + D_0$.

PROOF. The proof is similar to that of Theorem 15.2. We omit the details.

APPENDIX

ACTIONS OF SO(3) ON S^5 WITH 2-DIMENSIONAL ORBITS

There is only one linear action of SO(3) on S^5 such that the highest dimension of any orbit is two. This is the action induced by the representation $D_1 + 3D_0$. The base space is homeomorphic to B^3, the closed unit ball in E^3. The interior points of B^3 correspond to 2-sphere orbits and the boundary points of B^3 correspond to fixed points.

Let $\dot{B}^3$ ($= S^2$) denote the boundary of B^3. Then S^5 is homeomorphic to $B^3 \times S^2$ with $b \times S^2$ "collapsed" to a point for each $b \in \dot{B}^3$.

In ([1]) Bing defines a decomposition of E^3 into points and tame arcs such that the decomposition space, B, is not homeomorphic to E^3. In ([2]) it is shown that $B \times E^1$ is homeomorphic to E^4.

Let us identify $(B^3 - \dot{B}^3)$, the interior of B^3, with E^3. We decompose B^3 into points and tame arcs of E^3 (as in ([1])), and points of $\dot{B}^3$. Let B' be the resulting decomposition space. Then B' is not homeomorphic to B^3. Let M be the space $B' \times S^2$ with $b \times S^2$ "collapsed" to a point for each $b \in \dot{B}^3$ (we consider $\dot{B}^3$ as a subset of B'). The action of SO(3) on $B' \times S^2$ given by $g \cdot (b, p) = (b, g(p))$ induces an action of SO(3) on M. The orbit space M^* is homeomorphic to B'.

It follows from ([2]) that M is a manifold. By a straightforward calculation one shows that M is simply connected and that $H_*(M) \approx H_*(S^5)$.

If M is not homeomorphic to S^5 then it provides a contradiction to the Poincaré conjecture as generalized by Hurewicz. If M is homeomorphic to S^5 then the action of SO(3) on M defines a non-linear action of SO(3) on S^5.

We suspect that M is homeomorphic to S^5 but we have not been able to prove this. If M is homeomorphic to S^5 it should be possible by modifying the construction of B to construct an infinite number of non-equivalent actions of SO(3) on S^5.

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