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Lyapunov Stability Analysis of an RMRAC-based Adaptive Sigmoid Super-Twisting Sliding Mode under Matched and Unmatched Uncertainties

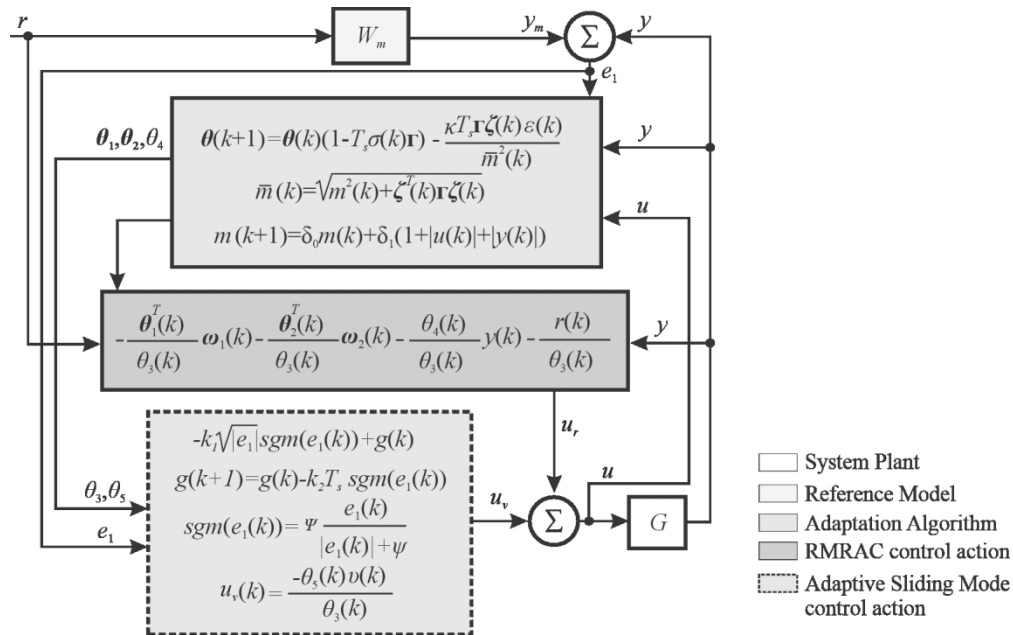
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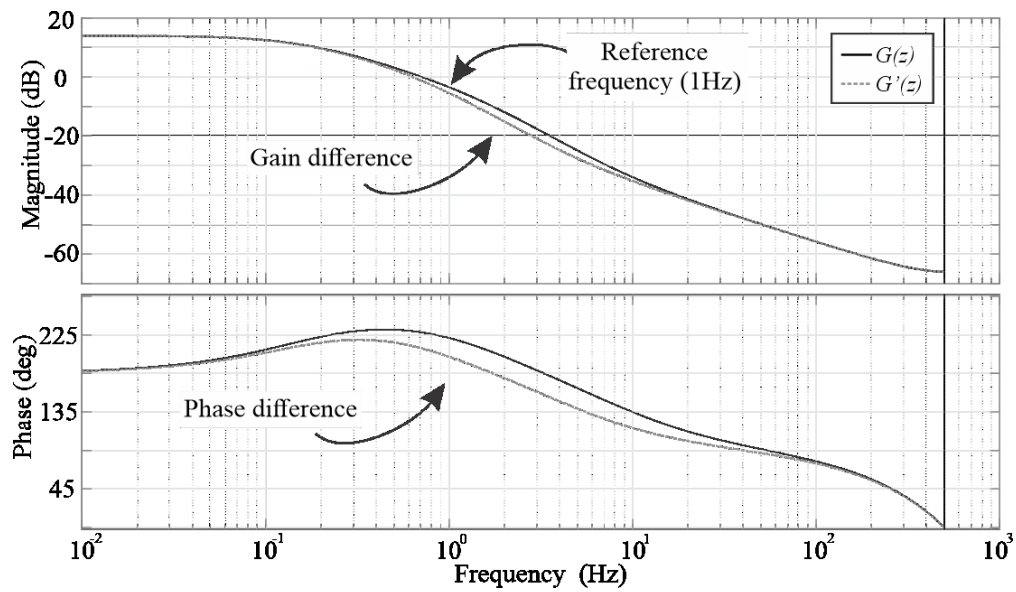
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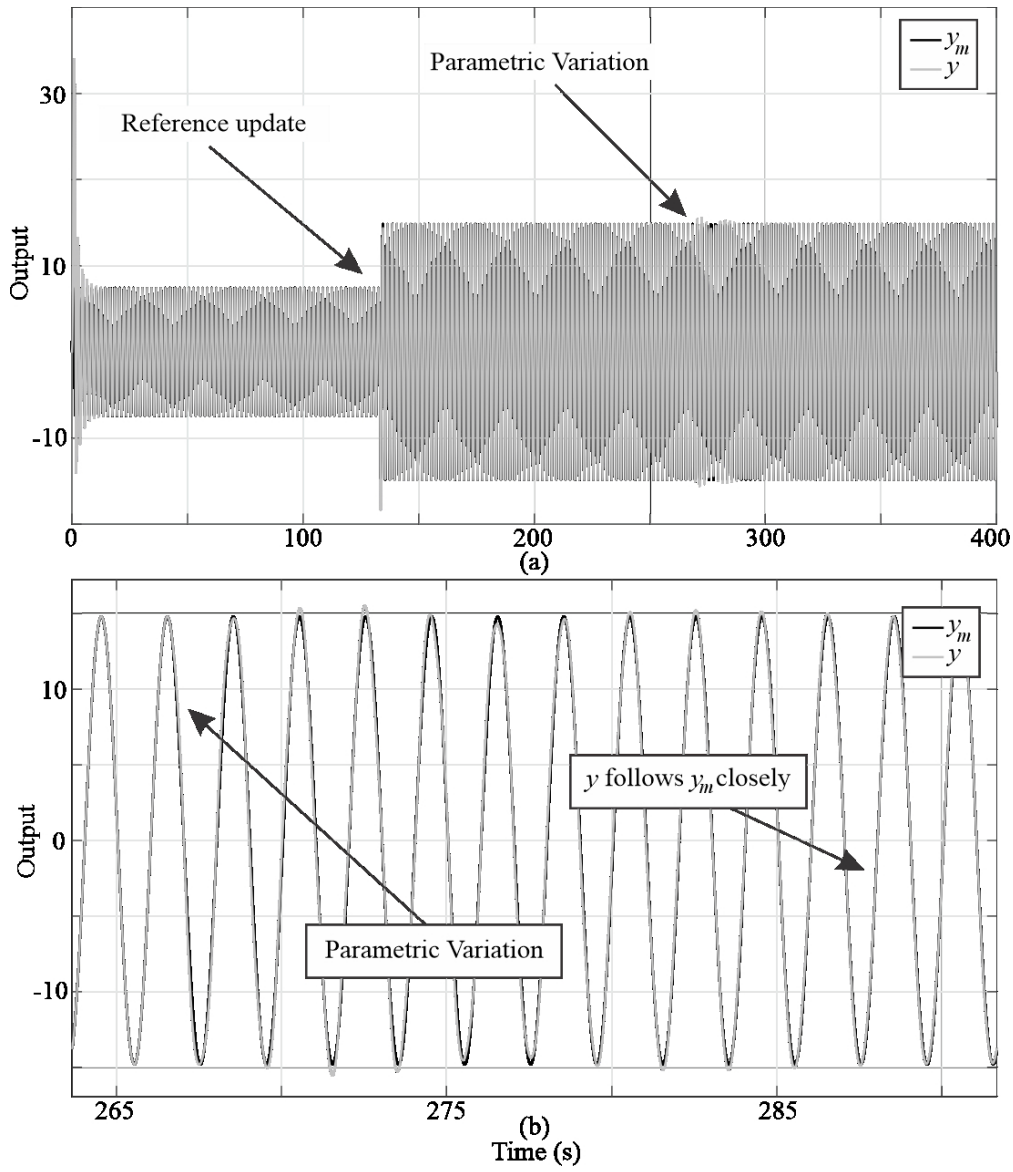
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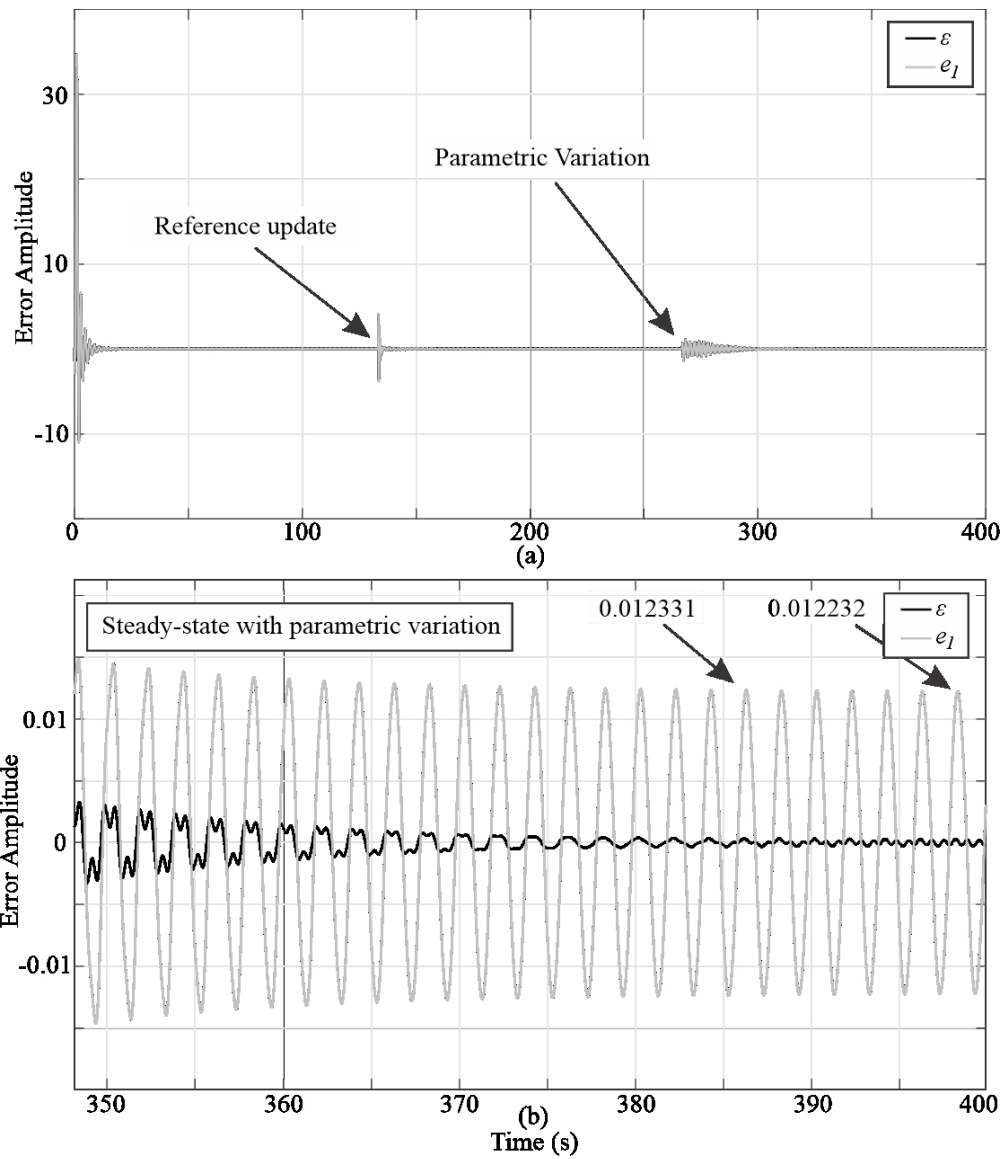
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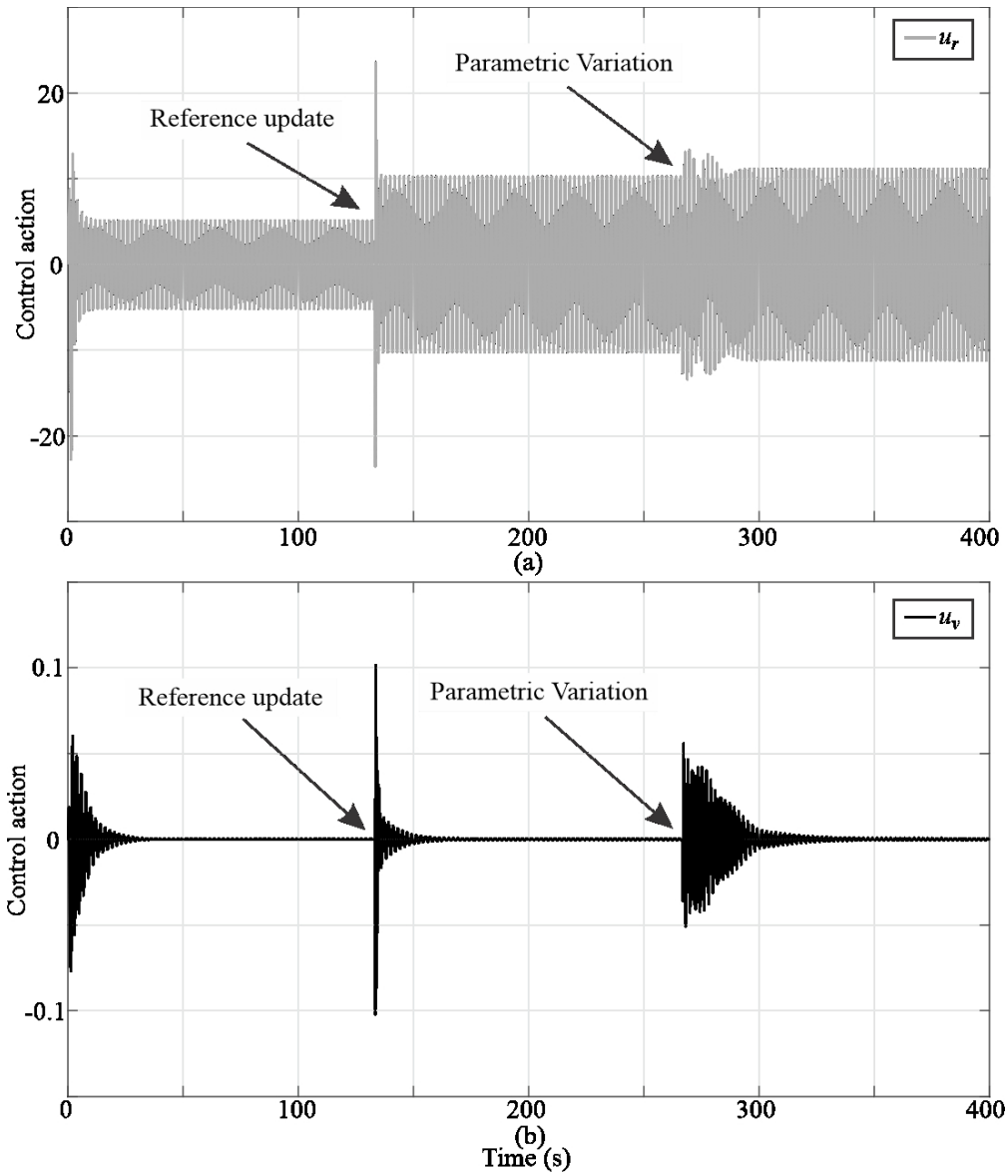
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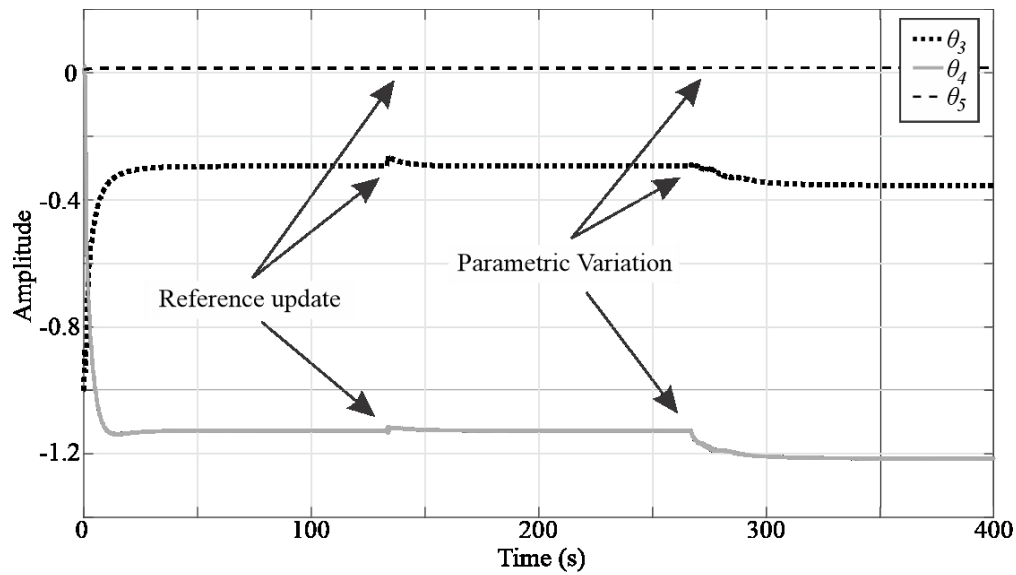
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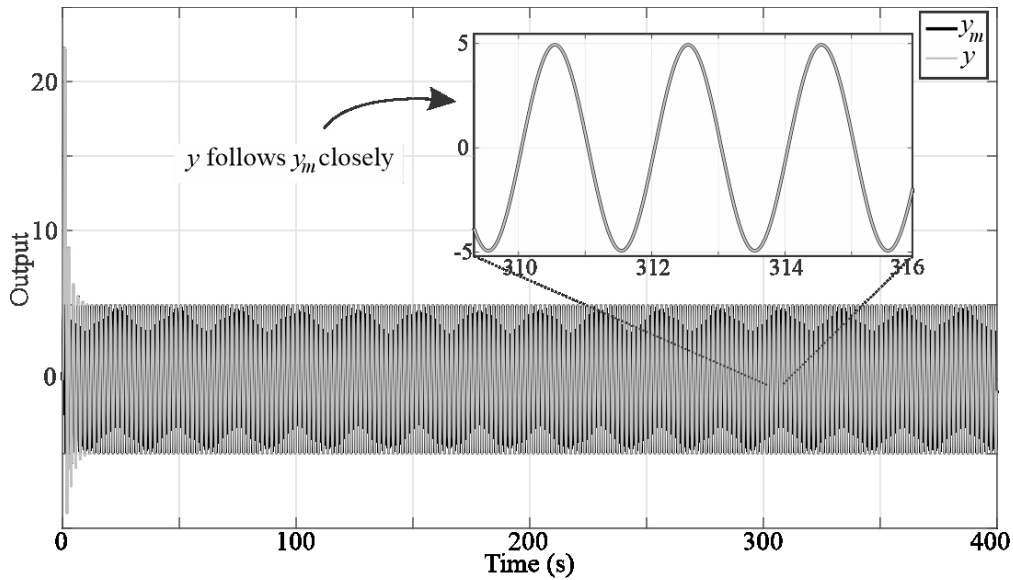
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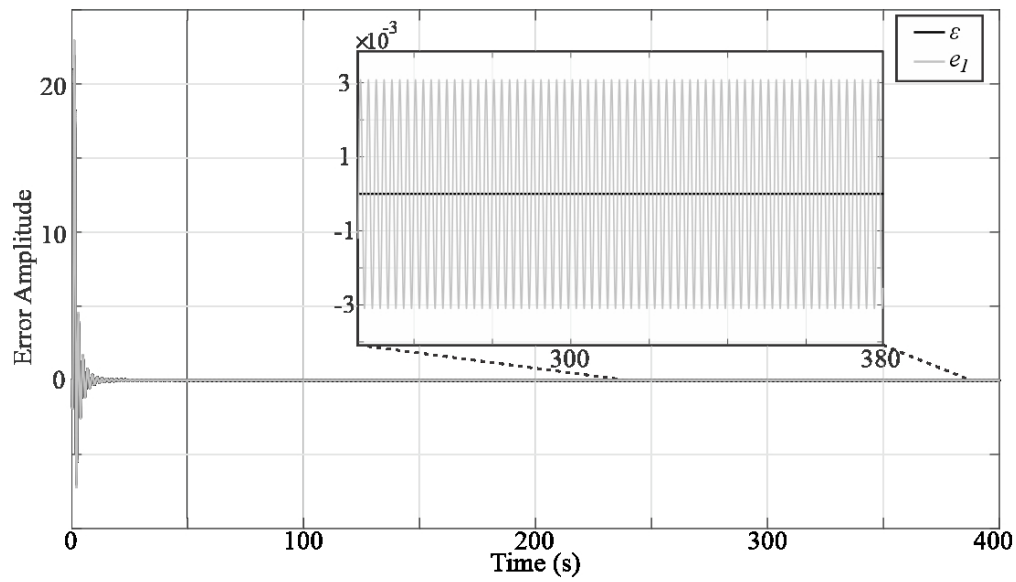
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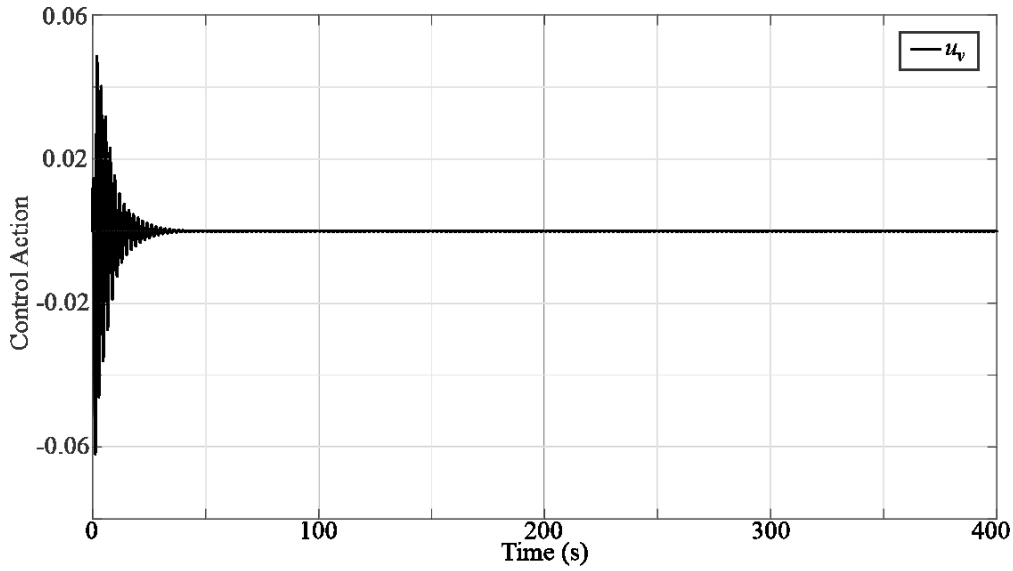
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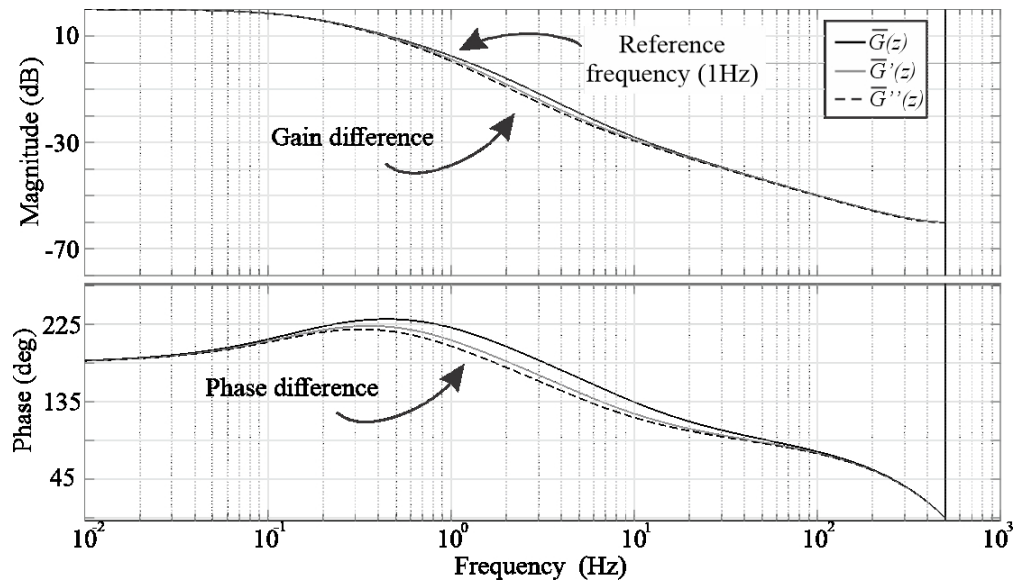
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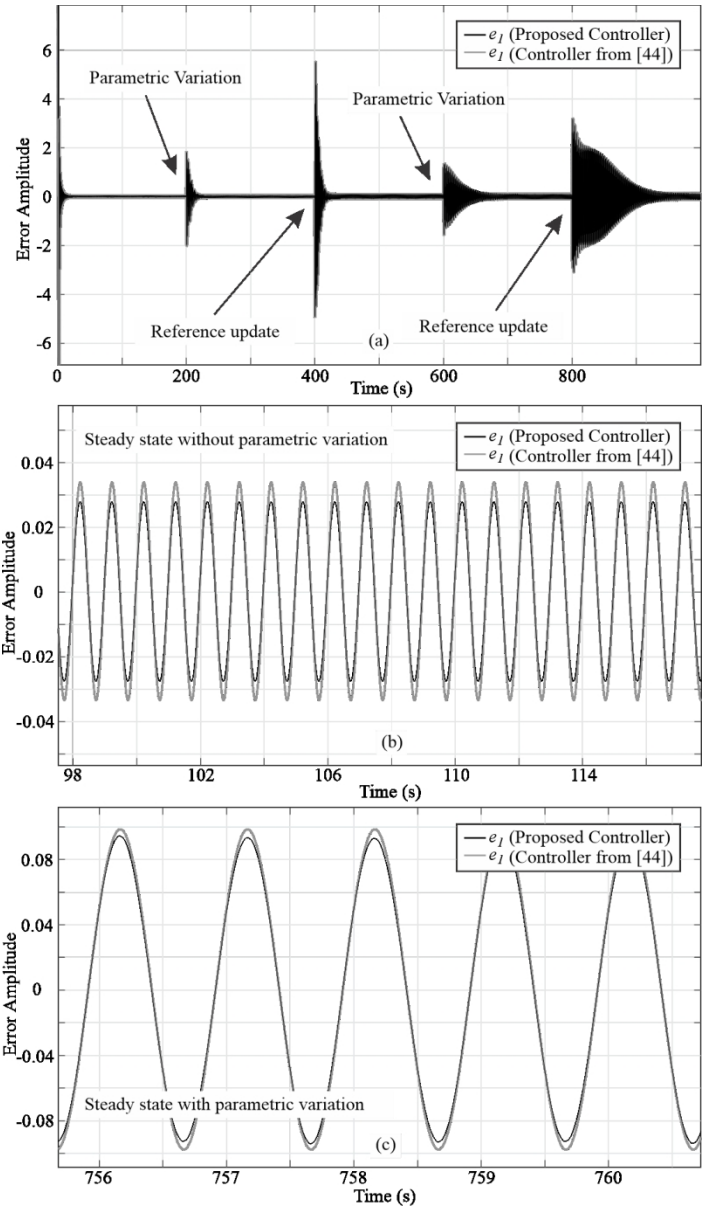
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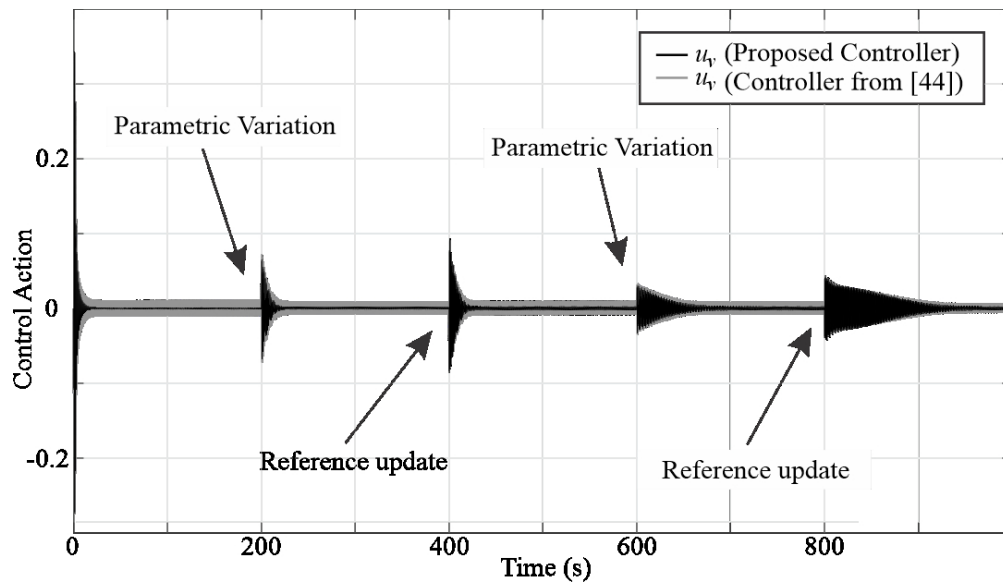
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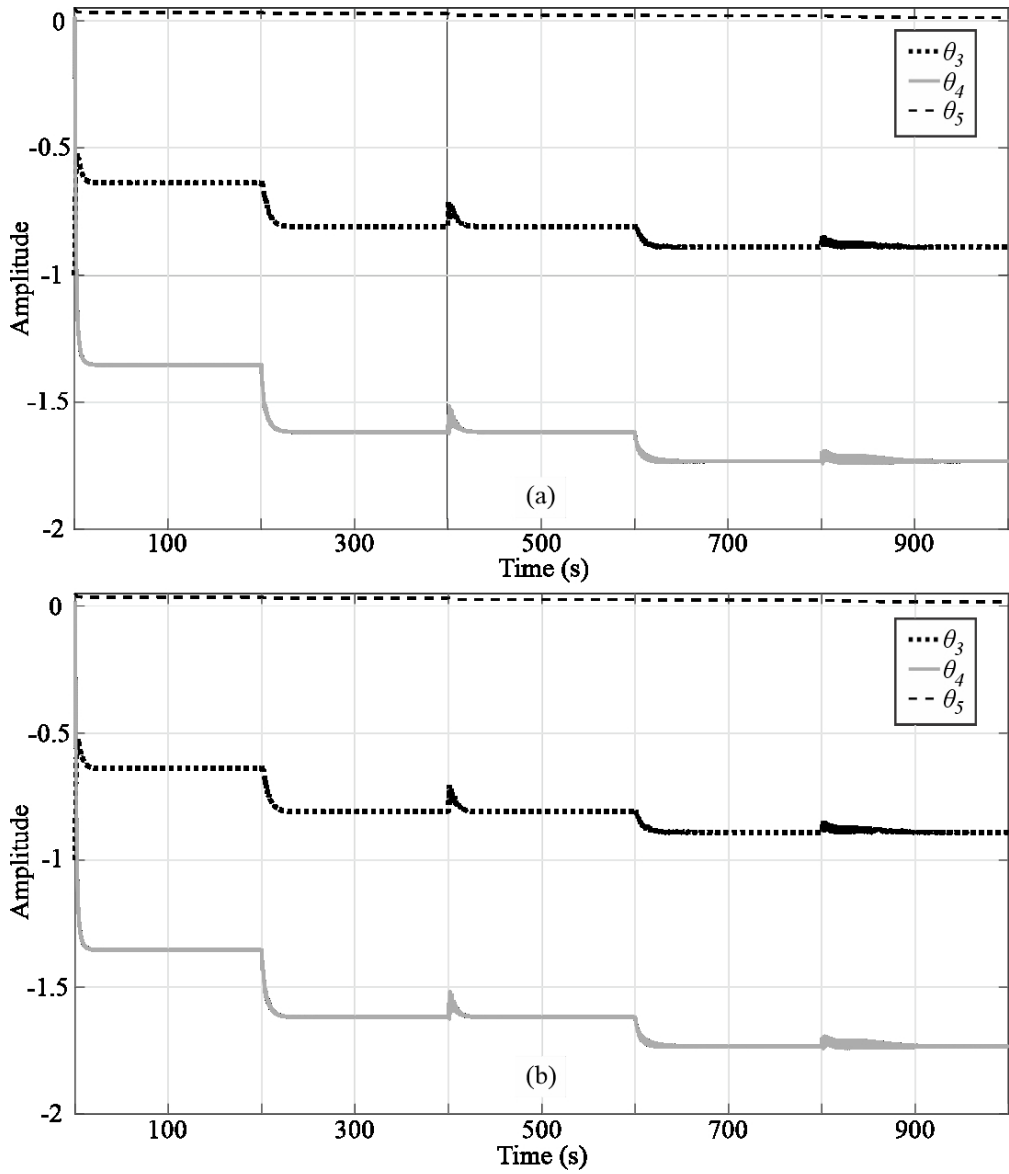
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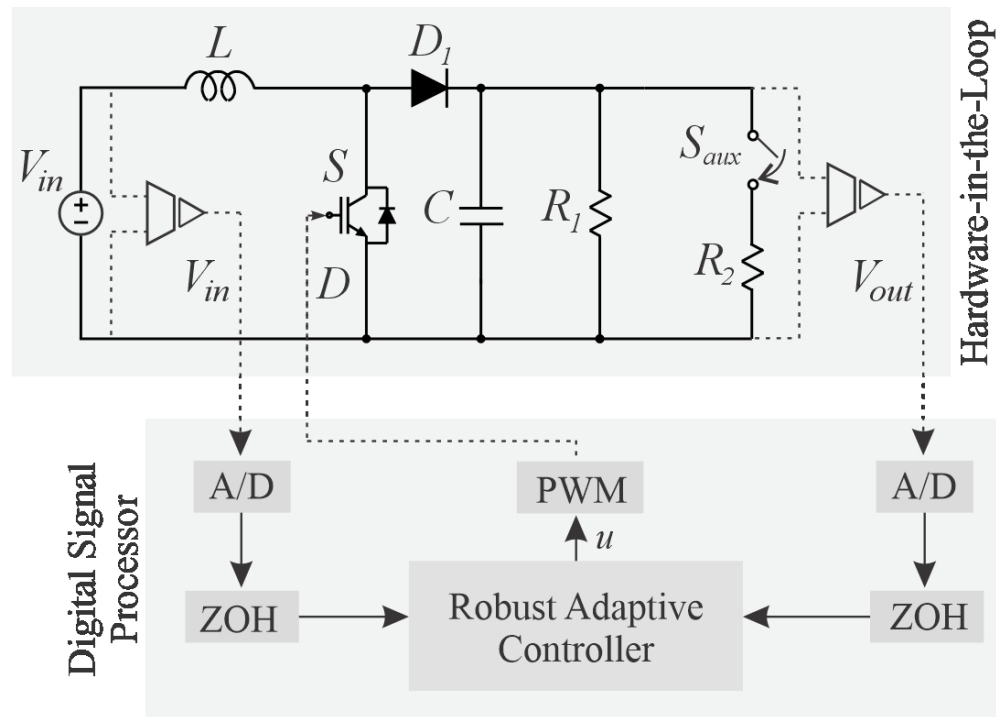
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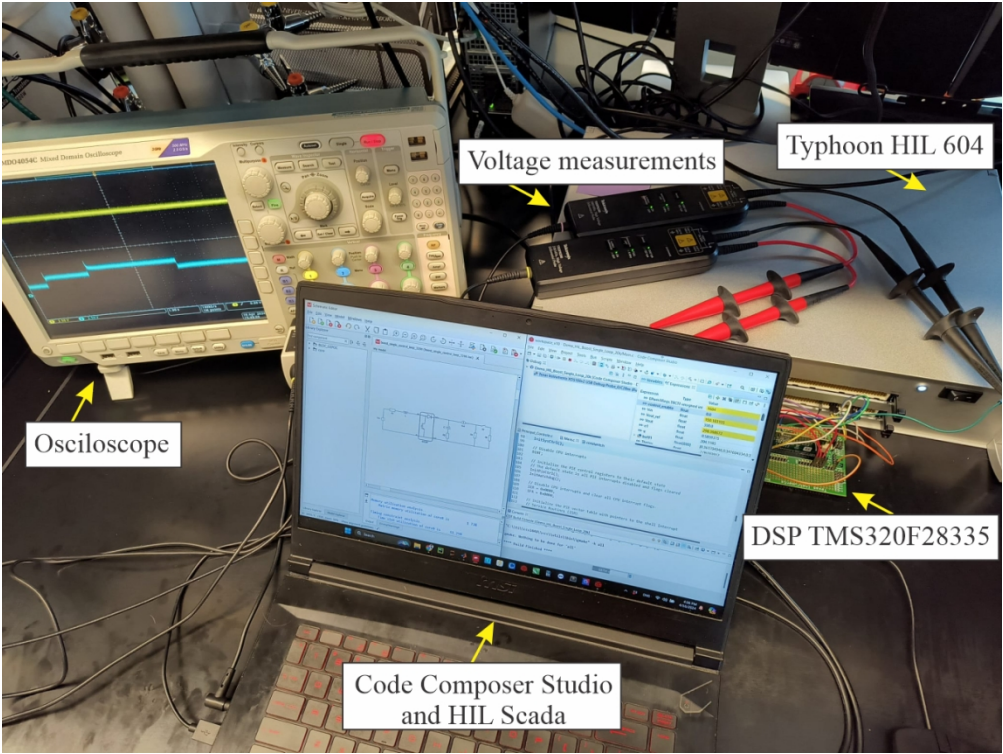
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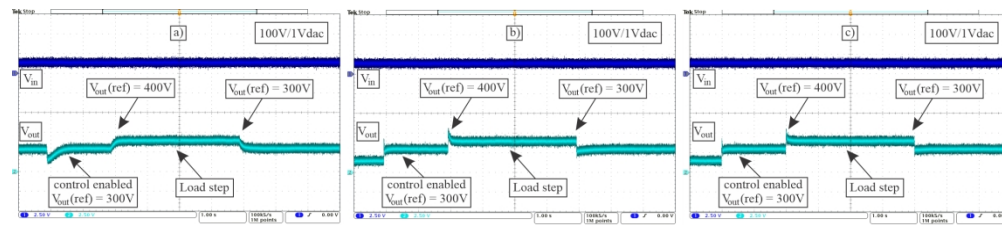
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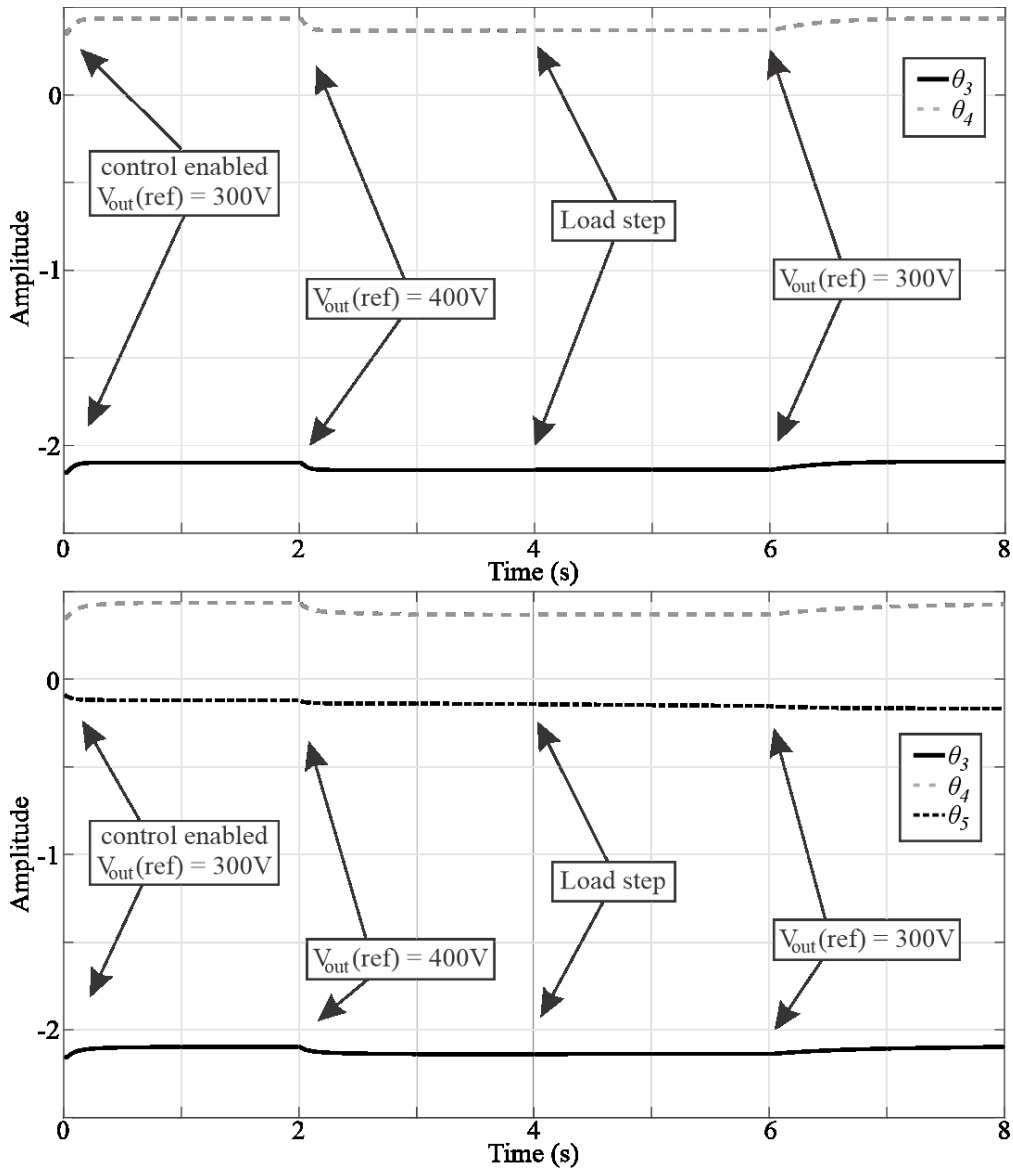
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- Explain why the paper is relevant to the journal as described in the aims and scope?

The International Journal of Robust and Nonlinear Control focuses on research that advances the understanding and application of robust and nonlinear control methodologies. Our paper, titled "Lyapunov Stability Analysis of an RMRAC-based Adaptive Sigmoid Super-Twisting Sliding Mode under Matched and Unmatched Uncertainties," is highly aligned with the aims and scope of the journal for the following reasons:

Robust Control Framework: Our work introduces a discrete-time Robust Model Reference Adaptive Controller (RMRAC) combined with an Adaptive Sigmoid Super-Twisting Sliding Mode (ASSTSM). This robust control framework is designed to handle both matched and unmatched uncertainties in a system, contributing significantly to the field of robust control.

Nonlinear Control Strategies: The incorporation of a sigmoid function in the super-twisting sliding mode action represents a novel approach to tackle the chattering phenomenon in steady-state. This aligns with the journal's interest in nonlinear control strategies, providing a unique solution for enhancing the performance of the control system.

Hybrid Adaptive High-Order SMC: Our paper introduces a hybrid adaptive high-order Sliding Mode Control (SMC) that addresses issues related to robustness, performance, and chattering suppression. This hybrid approach, grounded in RMRAC theory, showcases the intersection of robust and nonlinear control, making it highly relevant to the journal's scope.

In summary, our paper contributes to the journal by presenting a comprehensive study on a novel hybrid control strategy that combines robust and nonlinear control methodologies, aligning closely with the themes of robustness, nonlinearity, and control system performance emphasized in the aims and scope.

- Explain why the theoretical contributions in the paper are novel?

Our paper makes novel theoretical contributions that distinguish it from existing literature as follows:

Hybrid Adaptive High-Order SMC: Our control strategy represents a unique hybrid approach by combining the principles of robust model reference adaptive control (RMRAC) and high-order sliding mode control. This synthesis allows for adaptability to uncertainties while leveraging the benefits of sliding mode control. The resulting controller, named RMRAC-ASSTSM, provides enhanced robustness and performance compared to existing models in the literature.

Lyapunov Stability Analysis: The convergence of the tracking error for a residual set in steady state is mathematically guaranteed, even if the plant is subject to matched and unmatched dynamics. Moreover, conditions for tracking error to go to zero on steady-state are also provided; The proposed controller can be applied in unstable and nonminimum-phase plants because it is designed considering a partial model of the plant, which allows using reduced order reference model

Differentiation from Variable Structure Model Reference Adaptive Control (VS-MRAC): While related to model reference-based techniques, our work significantly deviates from Variable Structure Model Reference Adaptive Control (VS-MRAC) structures commonly found in the literature. The control development, control action and adaptation algorithm modeling is different, as well as the stability analysis. The adaptation of a sigmoid function in the reaching function of the RMRAC-based structure and the overall hybrid nature of our proposed controller differentiate it from existing approaches, offering a new perspective on the integration of model reference and sliding mode control.

- Provide a list of keywords that must correspond with the topics covered by the journal and dealt with in the paper

Adaptive control, Direct adaptive controllers, Super-twisting, RMRAC, SMC, Chattering suppression.

RESEARCH ARTICLE

Lyapunov Stability Analysis of an RMRAC-based Adaptive Sigmoid Super-Twisting Sliding Mode under Matched and Unmatched Uncertainties

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Summary

This paper presents a discrete-time robust model reference adaptive controller and adaptive sigmoid super-twisting sliding mode (RMRAC-ASSTSM) and its stability analysis using Lyapunov stability theory. This control structure is robust to matched and unmatched dynamics. In addition, the chattering phenomenon tends to be suppressed in the steady state due to adaptive super-twisting sliding mode action using a sigmoid function. The performance of the proposed control structure is corroborated with simulation results, considering a second-order non-minimum phase unstable plant, where it can be seen that RMRAC-ASSTSM regulation errors converge to a finite residual set when the plant presents unmodeled dynamics. A comparison is also presented with a similar RMRAC-based control structure with an adaptive super-twisting sliding mode implemented with a sign function.

KEYWORDS:

Adaptive control, direct adaptive controllers, super-twisting, RMRAC, SMC, chattering suppression, hardware-in-the-loop.

1 | INTRODUCTION

Model-free controllers, such as ^{1,2,3}, are advanced control strategies that have provided excellent performance for complex systems. Among these, sliding mode controllers have stood out due to their robustness. However, they can have their performance reduced due to chattering⁴, which can be increased because of system uncertainties. Conversely, adaptive controllers stand out due to their high performance in controlling complex systems subject to uncertainties, such as robot manipulators^{5,6}, aviation⁷, cancer treatment⁸, autonomous underwater vehicles⁹, satellite systems¹⁰, nuclear power plants¹¹, power electronics¹², electric machines^{13,14,15}, renewable energy systems¹⁶, and so on. These controllers can be classified as direct¹⁷, indirect¹⁸, or learning-based controllers¹⁹. Direct adaptive controllers require the design of a reference model to impose a desired behavior on the closed-loop system²⁰. On the other hand, indirect adaptive controllers estimate the system parameters while controlling them. In this kind of controller, there is no need for a reference model, but it has a greater computational burden in comparison with direct adaptive control strategies. Alternatively, there are adaptive controllers based on learning approaches. These controllers use fuzzy theory, neural networks, and similar techniques to provide robustness and high performance for adaptive controllers. Recently, adaptive strategies have been used to modify parameters of sliding mode controllers (SMCs) in response to system perturbations, combining adaptability from auto-tuning algorithms with robustness from SMC strategies^{21,22,23,24,25,26}. However,

even high-order SMCs are affected by the chattering phenomenon and can present regulation problems due to severe unmodeled dynamics in the system^{27,28}.

Chattering, which can damage hardware in some situations, is a relevant problem for practical applications. Therefore, many researchers have investigated solutions to mitigate this effect. Classical techniques such as constant reaching law (CRL), constant plus proportional reaching law (CPPL), and power rate reaching law (PRRL) can be used to reduce the convergence time and moderately attenuate the chattering effects^{29,30}. However, they can decrease the robustness of the control performance close to the surface³¹. On the other hand, these techniques have stimulated the development of other laws aiming to eliminate chattering. An enhanced PRRL was presented in³², where the reaching law has a perturbation estimation. A modified exponential reaching law (ERL) was proposed in³³, where it employs a sinusoidal term to reduce the chattering. A generalized Gao's reaching law for higher-relative-degree sliding variables was introduced in³⁴, where a time-varying sliding surface was used in the initial transient regime. A dead-zone sliding mode reaching law with disturbance compensation was developed in³⁵, where disturbances are compensated for by the high-order difference function. A fractional PRRL was presented in³⁶, where SMC reduced its activity near the sliding surface, dynamically accommodating variations in switching function. A composite reaching law, formed by exponential, power, and difference function was proposed in³⁷. As can be seen, chattering is a persistent problem for SMC application, and many researchers aim to design a technique to eliminate it or at least make it irrelevant in practical systems (see interesting discussions in^{38,39,40}).

The literature contains several interesting control approaches. However, there is still a lack of adaptive solutions that act online to alleviate chattering phenomena. In general, the adaptability of the SMC-based controllers is given by learning techniques (see^{41,42,43} and other works), which are designed to run offline. In this circumstance, if the system is subject to several sources of uncertainties, there is a high probability of chattering increases during the control process. Different from what is found in the literature, this work proposes a novel hybrid adaptive high-order SMC regarding the issues of robustness, performance, and chattering suppression, which is incorporated into the control system for acting in real time. The developed control strategy is grounded in the robust model reference adaptive control (RMRAC) theory used to make a high-order SMC adaptive. For the sliding mode control action, a super-twisting algorithm with a sigmoid function on both SMC parcels is considered. The resulting controller is named the robust model reference adaptive controller and adaptive sigmoid super-twisting sliding mode (RMRAC-ASSTSM). This controller emerges as an enhancement from⁴⁴, where a sign function was implemented in the adaptive STSM control action as well as for the integral of the sliding variable. However, for the proposed controller, the reaching function is based in the sigmoid function but is adaptive. The resulting reaching function tends to mitigate the chattering more effectively when compared with a sign function. While the proposed control structure also combines a model reference-based technique with sliding mode control, it is substantially different from variable structure model reference adaptive control (VS-MRAC) structures found in the literature. For example, the works^{45,46} presented different equations for the control action and the adaptation algorithm. In addition, it is not uncommon that several algorithms for different order systems need to be presented because the control algorithm is not easily expandable to high-order systems. Similarly, the papers^{47,48} also followed the methodology from⁴⁵, where the sign function was presented along with all adaptive gains in the control algorithm.

In this improved controller, the sliding mode term pertains to the internal gains vector, and it appears in only one adaptive gain. Additionally, two different control laws are implemented in the proposed structure, one being for the RMRAC and the other for the adaptive high-order SMC term. Since the high-order SMC term tends to actuate expressively during transient periods, it is expected that the RMRAC control law presents dominant behavior across all implementation scenarios. Furthermore, it will be shown in the stability analysis that the high-order SMC control action amplitude is linked to chattering. The robustness enhancement of the controller using RMRAC philosophy provides relevant chattering suppression, as the SMC control action tends to act on system transients, leading to a residual set in the steady state.

The main contributions of this work are:

- A new robust model reference and adaptive sigmoid super-twisting sliding mode controller is proposed, where the stability analysis is developed using Lyapunov stability theory and proved in discrete time, supplying the controller constraints and ensuring robustness under matched and unmatched dynamics. In addition, proof of boundedness of all closed-loop signals is also presented.
- The convergence of the tracking error for a residual set in steady state is mathematically guaranteed, even if the plant is subject to matched and unmatched dynamics. Moreover, conditions for tracking error to go to zero in steady state are also provided.

- The applicability limitation of a super twisting sliding mode controller in plants with unitary relative degree ($n^* = 1$) is eliminated, because RMRAC action tends to increase its contribution to keeping the system globally stable, while the contribution of adaptive STSM action is not altered for plants with $n^* > 1$; The only information required to design the controller is the relative degree of the nominal part of the plant and its gain signal.
- The proposed controller can be applied in unstable and nonminimum-phase plants because it is designed considering a partial model of the plant, which allows using a reduced-order reference model.

The remainder of this work is organized as follows: Section II introduces the RMRAC-ASSTSM structure. Section III provides the stability analysis of the proposed controller. Section IV presents the simulation results, applying the controller in an unstable non-minimum phase plant under severe parametric variation, where the feasibility of the controller is corroborated. Additionally, a comparative analysis is provided, contrasting the proposed controller with a similar RMRAC-based control structure featuring an adaptive super-twisting sliding mode implemented using a sign function. Furthermore, Section IV includes a case study illustrating the application of the controller in a step-up DC-DC power converter within a controller hardware-in-the-loop (C-HIL) setup. In this scenario, the control algorithm operates on a digital signal processor (DSP), showcasing its practical implementation and comparing it with other control structures. Section V gives the final conclusions of this work.

2 | CONTROLLER MATHEMATICAL DEVELOPMENT

This section presents the plant and reference model assumptions, as well as the control action, the adaptation algorithm, and the steps for implementing the discrete-time control algorithm.

2.1 | Plant and model reference assumptions

Consider a general single-input single-output (SISO) plant, which is given by

$$G(z) = [G_0(z)(1 + \mu\Delta_m(z)) + \mu\Delta_a(z)] = \frac{y}{u}, \quad (1)$$

where y and u are plant output and input, respectively. Furthermore, $G(z)$ is the complete plant transfer function, while $G_0(z)$ is the modeled part. In addition, $\mu\Delta_a(z)$ and $\mu\Delta_m(z)$ are additive and multiplicative unmodeled dynamics, respectively. Additionally, μ is a scalar, which with no loss of generality can be considered as the same value for both additive and multiplicative dynamics.

In addition, (1) can be rewritten as

$$y = G_0(z)u + \mu\eta(z), \quad (2)$$

where $\mu \in [0, \bar{\mu})$ and $\bar{\mu} \in \mathbb{R}^+$. Moreover, $\eta(z)$ is

$$\eta(z) = \Delta G(z)u, \quad (3)$$

and $\Delta G(z)$ is

$$\Delta G(z) = G_0(z)\Delta_m(z) + \Delta_a(z), \quad (4)$$

where $\mu\Delta G(z)$ represents the uncertainties of the considered plant, related by scalar μ . In addition, η and $\Delta G(z)$ are presented in detail in Lemma 1 in the Appendix section.

The modeled part of $G(z)$, described by $G_0(z)$, can be expressed as

$$G_0(z) = k_p \frac{N(z)}{D(z)}, \quad (5)$$

which needs to satisfy assumptions A_1 - A_4 ,

- A_1 : $N(z)$ is an $m - th$ order stable Schur monic polynomial.
- A_2 : $D(z)$ is an $n - th$ order Schur monic polynomial, where $n_p > m_p$.
- A_3 : k_p is an unknown gain, but its signal is known.
- A_4 : $N(z)$ and $D(z)$ coefficients are unknown.

Remark 1:: The sign of k_p is essential (as noted in A_3) because it impacts the control action⁴⁹. Consequently, since the control action involves division by $\theta_3(0)$, knowing the sign of k_p enables the selection of $\theta_3(0)$ with the correct polarity (matching that of k_p). This prevents division by zero in the calculation of the control action during its adaptation process.

The assumptions A_1 - A_4 are similar to different direct-type adaptive controllers^{50,51}, since they used to have severe restrictions in considering the system plant. However, the proposed structure lies in a combination of RMRAC and adaptive high-order SMC, which generates two separate control actions. Despite the STSM algorithm being suitable for systems with a relative degree of 1, the plant assumptions mainly align with RMRAC theory. Therefore, this algorithm is not limited to systems with a relative degree of 1. As the order of the system increases, the complexity of RMRAC also increases, while the contribution of STSM remains constant.

As the controller is direct, it necessitates a reference model, a selection left to the designer. Hence, the controller's aim is to compel the output of the closed-loop plant to follow the output of the reference model. Consequently,

$$W_m(z) = \frac{y_m}{r}, \quad (6)$$

where $W_m(z)$ is the reference model transfer function, y_m is the reference model output, and r is the reference signal. Additionally, W_m can be rewritten in a general form as

$$W_m(z) = k_m \frac{1}{D_m(z)}. \quad (7)$$

The reference model is subject to the following assumptions:

A_5 : $D_m(z)$ is a stable $(n_p - m_p)$ -order monic polynomial.

A_6 : k_m is a constant high-frequency gain, with the same sign as k_p .

Similar assumptions are also found in other RMRAC-based controllers.

2.2 | Control law and control actions

The control law is written as

$$\bar{\theta}^T(k)\bar{\omega}(k) + r(k) = 0, \quad (8)$$

where $\bar{\theta}^T(k) = [\theta_1^T(k) \theta_2^T(k) \theta_3(k) \theta_4(k)]$ is the gain vector and $\bar{\omega}(k) = [\omega_1^T(k) \omega_2^T(k) u(k) y(k)]$ is an auxiliary vector. Also, $\theta_1^T(k)$ and $\theta_2^T(k)$ have dimension $1 \times (n_p - 1)$. Furthermore, ω_1^T and ω_2^T ⁴⁹ are

$$\omega_1 = (zI - F)^{-1}qu, \quad (9)$$

and

$$\omega_2 = (zI - F)^{-1}qy, \quad (10)$$

where the auxiliary filters ω_1 and ω_2 are vectors of order $(n-1)$. In addition, F is a stable matrix and the pair (F, q) is controllable.

From (8) follows

$$\theta_1^T(k)\omega_1(k) + \theta_2^T(k)\omega_2(k) + \theta_3(k)u(k) + \theta_4(k)y(k) + r(k) = 0. \quad (11)$$

Moreover, from (11),

$$u(k) = \frac{-\theta_1^T(k)\omega_1(k) - \theta_2^T(k)\omega_2(k) - \theta_4(k)y(k) - r(k)}{\theta_3(k)}. \quad (12)$$

Thus, (12) is the control action regarding RMRAC. For inclusion of an adaptive super-twisting sliding mode control action into (8) comes

$$\theta^T(k)\omega(k) + r(k) = 0, \quad (13)$$

where $\theta(k)^T = [\theta_1^T(k) \theta_2^T(k) \theta_3(k) \theta_4(k) \theta_5(k)]$, $\omega(k) = [\omega_1^T(k) \omega_2^T(k) u(k) y(k) v(k)]$. In addition,

$$v(k) = -k_1 \sqrt{|e_1(k)|} \text{sgm}(e_1(k)) + g(k), \quad (14)$$

where $g(k)$ is

$$g(k+1) = g(k) - k_2 T_s \text{sgm}(e_1(k)), \quad (15)$$

k_1 and k_2 being bounded positive constants, and T_s is the sampling period. Furthermore, the sigmoid function is

$$\text{sgm}(e_1(k)) = \Psi \frac{e_1(k)}{|e_1(k)| + \psi}. \quad (16)$$

Here, Ψ and ψ represent positive constants defined by the designer. It is important to highlight that this is a substantial difference from⁴⁴. The tracking error is regarded as the sliding variable and is formulated as $e_1(k) = y(k) - y_m(k)$.

Additionally, (13) can be expanded as

$$u(k) = \frac{-\theta_1^T(k)\omega_1(k) - \theta_2^T(k)\omega_2(k) - \theta_4(k)y(k)}{\theta_3(k)} - \frac{\theta_5(k)v(k) - r(k)}{\theta_3(k)}. \quad (17)$$

Next, (17) can be divided in two parts,

$$u(k) = u_r(k) + u_v(k), \quad (18)$$

where

$$u_r(k) = \frac{-\theta_1^T(k)\omega_1(k) - \theta_2^T(k)\omega_2(k) - \theta_4(k)y(k) - r(k)}{\theta_3(k)}, \quad (19)$$

and

$$u_v(k) = \frac{-\theta_5(k)v(k)}{\theta_3(k)}. \quad (20)$$

2.3 | Adaptation algorithm

The adaptation algorithm is based on a gradient algorithm. The continuous adaptation law is

$$\dot{\theta} = -\sigma\Gamma\theta - \frac{\Gamma\zeta\epsilon}{\bar{m}^2}, \quad (21)$$

where Γ is a square symmetric gain matrix that determines the convergence rate of the gains, ζ is a regressor vector, and ϵ is the augmented error. The augmented error description and proof are presented in the Appendix section. Furthermore, σ is the σ -modification function⁵², implemented to avoid parameters drifting during system transients and to improve system robustness⁵³. This algorithm uses a fixed Γ , whose values are non-null only in the main diagonal of this matrix. Therefore, the adopted gradient algorithm presents reduced computational burden when compared with other online gain adjustment strategies, such as recursive least squares.

The difference equation of (21) is

$$\theta(k+1) = (\mathbf{I} - T_s\sigma(k)\Gamma)\theta(k) - \kappa \frac{T_s\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)}, \quad (22)$$

where κ is an auxiliary gain to accelerate parameters convergence, because T_s is often small and should impose a slow adaptation dynamic.

The majorant signal, $\bar{m}$, which is used to ensure boundedness of all closed-loop signals, can be represented as

$$\bar{m}(k) = \sqrt{m^2(k) + \zeta^T(k)\Gamma\zeta(k)}. \quad (23)$$

and

$$\begin{aligned} m(k+1) &= \delta_0 m(k) + \delta_1 (1 + |u(k)| + |y(k)|), \\ m_0 &\geq \frac{\delta_1}{1 - \delta_0}, \end{aligned} \quad (24)$$

so that δ_0 , δ_1 , and δ_2 are chosen by the designer. However, they must respect the following constraints:

$$\begin{aligned} 0 < \delta_2 < \delta_0 < 1, \quad \max_i |\lambda_i| < \delta_0 - \delta_2; \\ i &= 1, 2, \dots, n^* + n - 1, \end{aligned} \quad (25)$$

where λ_i are the i -th poles of $W_m(z)$.

As previously explained, a σ -modification function⁵² is incorporated into the algorithm. This function is given by

$$\sigma(k) = \begin{cases} 0 & \text{if } \|\theta(k)\| < M_0 \\ \sigma_0 \left(\frac{\|\theta(k)\|}{M_0} - 1 \right) & \text{if } M_0 \leq \|\theta(k)\| < 2M_0, \\ \sigma_0 & \text{if } \|\theta(k)\| \geq 2M_0 \end{cases} \quad (26)$$

where σ_0 and M_0 must be designed to respect the following condition,

$$0 \leq \sigma(k) \leq \sigma_0, \quad M_0 \geq 2\|\theta^*\|. \quad (27)$$

The selection of these majorant and switching σ -modification functions is motivated by their ability to enhance the robustness of the algorithm, as demonstrated in⁵⁰. The stability analysis establishes that the proposed algorithm ensures boundedness and minimal residual tracking errors for any bounded initial conditions. In the absence of plant uncertainties, the residual tracking

error diminishes to zero. The only a priori information needed for implementing the algorithm is an upper bound for the norm of the desired controller parameter vector.

Figure 1 presents a block diagram showing the general workflow of the proposed controller.

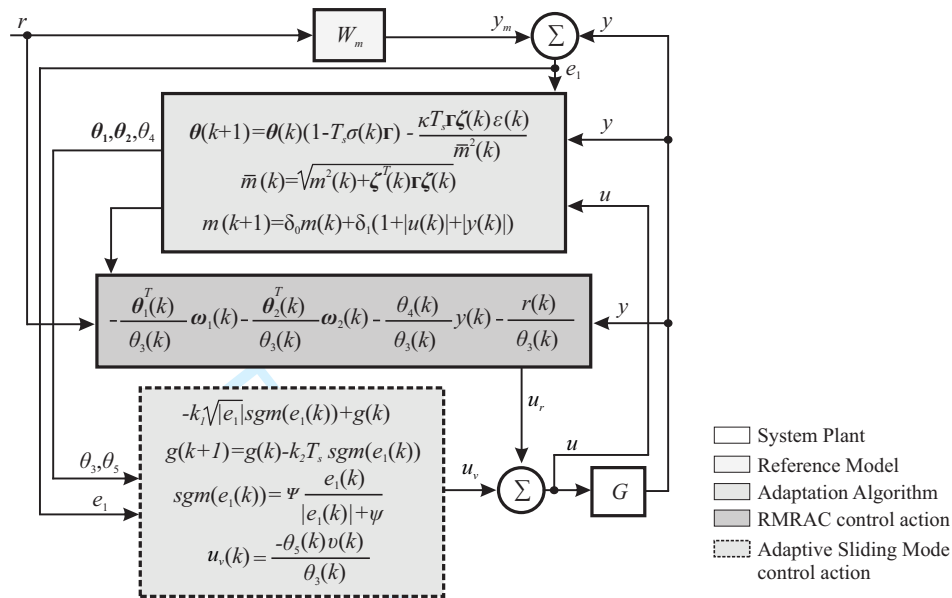


FIGURE 1 General block diagram of robust model reference adaptive controller and adaptive sigmoid super-twisting sliding mode controller.

The steps for implementing the discrete-time control algorithm are given as follows:

1. Update of reference: $r(k)$
2. Update of reference model output: $y_m(k)$
3. Update of tracking error: $e_1(k)$
4. Update of regressor vector: $\zeta(k)$
5. Update of augmented error: $\epsilon(k)$
6. Update of σ -modification function: $\sigma(k)$
7. Update of gains norm: $\|\theta(k)\|$
8. Update of majorant signal: $\bar{m}(k+1)$
9. Update of adaptive gains: $\theta(k+1)$
10. Update of complete control action: $u(k)$

The order of equations is not unique because it depends on the used discretization technique, mainly on the gradient algorithm, which directly affects the samples required for calculating the gains (e.g.: forward and backward Euler). In this work, we adopted the forward Euler technique, as you can see that the update of θ has the index $(k+1)$. Therefore, the same algorithm can be implemented in many forms, as long as the correct time samples and variables are always respected.

Remark 2: As presented in^{54,38,39}, the super-twisting algorithm was designed for plants with a unitary relative degree. However, this limitation does not apply to the control structure proposed in this work. Therefore, if the relative degree of the plant

increases, the contribution of the RMRAC control action tends to be intensified because the adaptation gains are straightforwardly related to the relative degree of the reference model. Hence, the contribution of the adaptive super-twisting control action does not increase (in terms of adaptive parameters), regardless of the relative degree of the system.

3 | STABILITY ANALYSIS

This section presents a stability analysis using Lyapunov stability theory. As a complement to the stability analysis, the proof of boundedness of closed-loop signals is shown in the Appendix section.

3.1 | Lyapunov stability analysis

Theorem 1: Let the system be defined by (1)–(5), subject to assumptions A1 to A4. The reference model is represented by (6)–(7) and is subject to assumption A5 and A6, based on the control law shown in (17) and the gradient-based parameter adaptation algorithm presented in (22)–(25). Then, the closed-loop system is robust and the tracking error $e_1(k) \rightarrow 0$ when $k \rightarrow \infty$, or at least $e_1(k)$ is small on average and belongs to a residual set.

The proof for **Theorem 1** will be presented below, divided into two subsections: the first considers $\mu = 0$, while the second addresses $\mu \neq 0$. This division is intended to enhance the readability of the proof. However, both subsections should be considered as one proof for the attending **Theorem 1**.

3.1.1 | Particular case of the stability analysis ($\mu = 0$)

Proof:

Consider the following Lyapunov candidate in a discrete-time domain:

$$V(k) = V_1(k) + V_2(k), \quad (28)$$

where

$$V_1(k) = \phi^T(k) \Gamma^{-1} \phi(k), \quad (29)$$

and

$$V_2(k) = e_1^2(k). \quad (30)$$

The finite variation of the Lyapunov candidate (28) is

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k) = -\phi^T(k+1) \Gamma^{-1} \phi(k+1) - \phi^T(k) \Gamma^{-1} \phi(k) + e_1^2(k+1) - e_1^2(k), \quad (31)$$

where $\Delta V_1(k)$ can be expressed as

$$\Delta V_1(k) = \phi^T(k+1) \Gamma^{-1} \phi(k+1) - \phi^T(k) \Gamma^{-1} \phi(k), \quad (32)$$

and $\Delta V_2(k)$ is

$$\Delta V_2(k) = e_1^2(k+1) - e_1^2(k). \quad (33)$$

Taking into consideration (29), comes

$$\phi^T(k) \Gamma^{-1} \phi(k) = [\phi(k+1) - \Delta \phi(k)]^T \Gamma^{-1} [\phi(k+1) - \Delta \phi(k)], \quad (34)$$

where

$$\Delta \phi(k) = \phi(k+1) - \phi(k). \quad (35)$$

Moreover, (35) is equivalent to

$$\phi(k)^T \Gamma^{-1} \phi(k) = \phi^T(k+1) \Gamma^{-1} \phi(k+1) - \phi^T(k+1) \Gamma^{-1} \Delta \phi(k) - \Delta \phi^T(k) \Gamma^{-1} \phi(k+1) + \Delta \phi^T(k) \Gamma^{-1} \Delta \phi(k). \quad (36)$$

In addition, it is true that

$$\phi^T(k) \Gamma^{-1} \phi(k) = \phi^T(k+1) \Gamma^{-1} \phi(k+1) - 2\phi^T(k+1) \Gamma^{-1} \Delta \phi(k) + \Delta \phi^T(k) \Gamma^{-1} \Delta \phi(k). \quad (37)$$

Considering $\Delta V_1(k)$, replacing (37) in (32), it follows that

$$\Delta V_1(k) = \phi^T(k+1)\Gamma^{-1}\phi(k+1) - \phi^T(k+1)\Gamma^{-1}\phi(k+1) + 2\phi^T(k+1)\Gamma^{-1}\Delta\phi(k) - \Delta\phi^T(k)\Gamma^{-1}\Delta\phi(k). \quad (38)$$

Thus,

$$\Delta V_1(k) = 2\phi^T(k+1)\Gamma^{-1}\Delta\phi(k) - \Delta\phi^T(k)\Gamma^{-1}\Delta\phi(k). \quad (39)$$

Furthermore, (39) can be rewritten as

$$\Delta V_1(k) \leq 2\phi^T(k+1)\Gamma^{-1}\Delta\phi(k). \quad (40)$$

Taking into consideration (22), subtracting $\theta^*(k)$ from both sides of the equation, it can be obtained:

$$\phi(k+1) = \phi(k) - T_s\sigma(k)\Gamma\theta(k) - \kappa \frac{T_s\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)}, \quad (41)$$

and

$$\Delta\phi(k) = \phi(k+1) - \phi(k) = -T_s\sigma(k)\Gamma\theta(k) - \kappa \frac{T_s\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)}. \quad (42)$$

Replacing (41) and (42) in (40), comes

$$\Delta V_1(k) \leq 2 \left(\phi(k) - \sigma(k)T_s\Gamma\theta(k) - \kappa \frac{T_s\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)} \right)^T \Gamma^{-1} \left(-\sigma(k)T_s\Gamma\theta(k) - \kappa \frac{T_s\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)} \right), \quad (43)$$

or

$$\begin{aligned} \Delta V_1(k) \leq & -2\sigma(k)T_s\phi^T(k)\theta(k) + 2\sigma^2(k)T_s^2\theta^T(k)\Gamma\theta(k) + \\ & \frac{2\sigma(k)T_s^2\kappa\zeta^T(k)\Gamma\theta(k)\epsilon(k)}{\bar{m}^2(k)} - \frac{2T_s\kappa\phi^T(k)\zeta(k)\epsilon(k)}{\bar{m}^2(k)} + \frac{2\sigma(k)T_s^2\kappa\theta^T(k)\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)} + \frac{2T_s^2\kappa^2\Gamma\zeta^T(k)\zeta(k)\epsilon^2(k)}{\bar{m}^2(k)\bar{m}^2(k)}. \end{aligned} \quad (44)$$

From (44), it is true that

$$\begin{aligned} \Delta V_1(k) \leq & -2\sigma(k)T_s\phi^T(k)\theta(k) + 2\sigma^2(k)T_s^2\theta^T(k)\Gamma\theta(k) - \\ & \frac{2T_s\kappa\phi^T(k)\zeta(k)\epsilon(k)}{\bar{m}^2(k)} + \frac{4\sigma(k)T_s^2\kappa\theta^T(k)\zeta(k)\epsilon(k)}{\bar{m}^2(k)} + \frac{2T_s^2\kappa^2\Gamma\zeta^T(k)\zeta(k)\epsilon^2(k)}{\bar{m}^2(k)\bar{m}^2(k)}. \end{aligned} \quad (45)$$

Replacing (163) into (45), it can be obtained that

$$\begin{aligned} \Delta V_1(k) \leq & -2T_s\sigma(k)\phi^T(k)\theta(k) + 2\sigma^2(k)T_s^2\theta^T(k)\Gamma\theta(k) - \\ & \frac{2T_s\kappa\phi^T(k)\zeta(k)(\phi^T(k)\zeta(k) + \mu\eta(k))}{\bar{m}^2(k)} + \frac{4\sigma(k)T_s^2\kappa\theta^T(k)\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)} + \frac{2T_s^2\kappa^2\Gamma\zeta^T(k)\zeta(k)\epsilon^2(k)}{\bar{m}^2(k)\bar{m}^2(k)}, \end{aligned} \quad (46)$$

or

$$\begin{aligned} \Delta V_1(k) \leq & -2T_s\sigma(k)\phi^T(k)\theta(k) + 2\sigma^2(k)T_s^2\theta^T(k)\Gamma\theta(k) - \\ & \frac{2T_s\kappa(\phi^T(k)\zeta(k))^2}{\bar{m}^2(k)} + \frac{2T_s\kappa\phi^T(k)\zeta(k)\mu\eta(k)}{\bar{m}^2(k)} + \frac{4\sigma(k)T_s^2\kappa\theta^T(k)\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)} + \frac{2T_s^2\kappa^2\Gamma\zeta^T(k)\zeta(k)\epsilon^2(k)}{\bar{m}^2(k)\bar{m}^2(k)}. \end{aligned} \quad (47)$$

Therefore,

$$\begin{aligned} \Delta V_1(k) \leq & -2T_s\sigma(k)\phi^T(k)\theta(k) + 2\sigma^2(k)T_s^2\theta^T(k)\Gamma\theta(k) - \frac{T_s\kappa(\phi^T(k)\zeta(k))^2}{\bar{m}^2(k)} - \frac{T_s\kappa(\phi^T(k)\zeta(k))^2}{\bar{m}^2(k)} - \\ & \frac{2T_s\kappa\phi^T(k)\zeta(k)\mu\eta(k)}{\bar{m}^2(k)} + \frac{4\sigma(k)T_s^2\kappa\theta^T(k)\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)} + \frac{2T_s^2\kappa^2\Gamma\zeta^T(k)\zeta(k)\epsilon^2(k)}{\bar{m}^2(k)\bar{m}^2(k)}, \end{aligned} \quad (48)$$

and (48) can be rewritten as

$$\begin{aligned} \Delta V_1(k) \leq & -2T_s\sigma(k)\phi^T(k)\theta(k) + 2\sigma^2(k)T_s^2\theta^T(k)\Gamma\theta(k) - \frac{T_s\kappa(\phi^T(k)\zeta(k))^2}{\bar{m}^2(k)} - \frac{T_s\kappa(\phi^T(k)\zeta(k) + \mu\eta(k))^2}{\bar{m}^2(k)} - \\ & \frac{T_s\kappa(\mu\eta(k))^2}{\bar{m}^2(k)} + \frac{4\sigma(k)T_s^2\kappa\theta^T(k)\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)} + \frac{2T_s^2\kappa^2\Gamma\zeta^T(k)\zeta(k)\epsilon^2(k)}{\bar{m}^2(k)\bar{m}^2(k)}. \end{aligned} \quad (49)$$

In view of

$$\begin{aligned} & 2\sigma^2(k)T_s^2\theta^T(k)\Gamma\theta(k) + \frac{4\sigma(k)T_s^2\kappa\theta^T(k)\Gamma\zeta(k)\epsilon(k)}{\bar{m}^2(k)} + \\ & \frac{2T_s^2\kappa^2\Gamma\zeta^T(k)\zeta(k)\epsilon^2(k)}{\bar{m}^2(k)\bar{m}^2(k)} = 2T_s^2 \left(\sigma(k)\theta(k) + \frac{\kappa\zeta(k)\epsilon(k)}{\bar{m}^2(k)} \right)^T \Gamma \left(\sigma(k)\theta(k) + \frac{\kappa\zeta(k)\epsilon(k)}{\bar{m}^2(k)} \right), \end{aligned} \quad (50)$$

(49) can be rewritten according to

$$\Delta V_1(k) \leq -2T_s \sigma(k) \phi^T(k) \theta(k) - \frac{T_s \kappa (\phi^T(k) \xi(k))^2}{\bar{m}^2} - \frac{T_s \kappa (\phi^T(k) \xi(k) + \mu \eta(k))^2}{\bar{m}^2(k)} + \frac{T_s \kappa (\mu \eta(k))^2}{\bar{m}^2(k)} + 2T_s^2 \|\Gamma\| \cdot \left\| \sigma(k) \theta(k) + \frac{\kappa \xi(k) \epsilon(k)}{\bar{m}^2(k)} \right\|^2, \quad (51)$$

which can be also expressed as

$$\Delta V_1(k) \leq -2T_s \sigma(k) \phi^T(k) \theta(k) - \frac{T_s \kappa (\phi^T(k) \xi(k))^2}{\bar{m}^2} - \frac{T_s \kappa (\phi^T(k) \xi(k) + \mu \eta(k))^2}{\bar{m}^2(k)} + \frac{T_s \kappa (\mu \eta(k))^2}{\bar{m}^2(k)} + 2T_s^2 \|\Gamma\| \left(\|\sigma(k) \theta(k)\|^2 + \left\| \frac{\kappa \xi(k) \epsilon(k)}{\bar{m}^2(k)} \right\|^2 \right). \quad (52)$$

In addition, it is true that

$$\Delta V_1(k) \leq -2T_s \sigma(k) \phi^T(k) \theta(k) - \frac{T_s \kappa (\phi^T(k) \xi(k))^2}{\bar{m}^2} - \frac{T_s \kappa (\phi^T(k) \xi(k) + \mu \eta(k))^2}{\bar{m}^2(k)} + \frac{T_s \kappa (\mu \eta(k))^2}{\bar{m}^2(k)} + 2T_s^2 \sigma^2(k) \|\Gamma\| \cdot \|\theta(k)\|^2 + 2T_s^2 \|\Gamma\| \cdot |\kappa|^2 \left\| \frac{\xi(k)}{\bar{m}(k)} \right\|^2 \left\| \frac{\epsilon(k)}{\bar{m}(k)} \right\|^2. \quad (53)$$

Rewriting (53), comes

$$\Delta V_1(k) \leq -2T_s \sigma(k) \phi^T(k) \theta(k) - \frac{T_s \kappa (\phi^T(k) \xi(k))^2}{\bar{m}^2(k)} + \frac{T_s \kappa (\mu \eta(k))^2}{\bar{m}^2(k)} + 2T_s^2 \sigma^2(k) \|\Gamma\| \cdot \|\theta(k)\|^2. \quad (54)$$

Considering the equality

$$(\phi - \theta)^T (\phi - \theta) = \|\theta^*\|^2 = \|\phi\|^2 - 2\phi^T \theta + \|\theta\|^2, \quad (55)$$

therefore,

$$-2\phi^T \theta \leq \|\theta^*\|^2 - \|\theta\|^2, \quad (56)$$

and

$$-2T_s \sigma(k) \phi^T(k) \theta(k) \leq T_s \sigma(k) \|\theta^*\|^2 - T_s \sigma(k) \|\theta(k)\|^2. \quad (57)$$

Replacing (57) in (54) and reorganizing the terms, it follows that

$$\Delta V_1(k) \leq -T_s \sigma(k) \|\theta(k)\|^2 + T_s \sigma(k) \|\theta^*\|^2 + 2T_s^2 \sigma^2(k) \|\Gamma\| \cdot \|\theta(k)\|^2 - T_s \kappa \frac{\epsilon^2(k)}{\bar{m}^2(k)} \left(1 - 2T_s^2 \|\Gamma\| \cdot |\kappa|^2 \frac{\|\xi(k)\|^2}{\bar{m}^2(k)} \right) - \frac{T_s \kappa (\phi^T(k) \xi(k))^2}{\bar{m}^2(k)} + \frac{T_s \kappa (\mu \eta(k))^2}{\bar{m}^2(k)}. \quad (58)$$

Therefore, in view of (58):

I) Based on (26), it can be concluded from (54), and consequently (58), that

$$-(T_s \sigma(k) \|\theta(k)\|^2 + T_s \sigma(k) \|\theta^*\|^2 + 2T_s^2 \sigma^2(k) \|\Gamma\| \cdot \|\theta(k)\|^2) \leq 0, \quad (59)$$

or yet,

$$T_s \sigma(k) (-\|\theta(k)\|^2 + \|\theta^*\|^2 + 2T_s \sigma(k) \|\Gamma\| \cdot \|\theta(k)\|^2) \leq 0. \quad (60)$$

Considering (26) and (60), when $\|\theta(k)\| \geq 2M_0$, it is possible to find values for T_s , Γ , and $\sigma(k)$ such that $8T_s \sigma(k) \|\Gamma\| M_0^2 < 1$. For values of $0 \leq \|\theta(k)\| \leq M_0$, $\sigma(k) = 0$ and (60) is null. Additionally, for range $M_0 < \|\theta(k)\| < 2M_0$, $\sigma(k) \rightarrow 0$ according to $\|\theta(k)\| \rightarrow M_0$ and (60) is negative and tends to zero when $\|\theta(k)\| \rightarrow M_0$. Moreover, $\|\theta^*(k)\| \leq M_0$.

II) $\frac{\|\xi(k)\|^2}{\bar{m}^2(k)} < 1$ because $\bar{m}^2(k) > \|\xi(k)\|^2$.

For this proof, see also⁵². From this work, it can be inferred that as T_s is defined by the designer and often is considerably small,

$$2T_s^2 \|\Gamma\| \cdot |\kappa|^2 \frac{\|\xi(k)\|^2}{\bar{m}^2(k)} < 1, \quad (61)$$

and as T_s , κ , and Γ are values defined by the designer, it is possible to choose values such that (61) is less than 1. Therefore, $\left(1 - 2T_s^2 \|\Gamma\| \cdot |\kappa|^2 \frac{\|\xi(k)\|^2}{\bar{m}^2(k)} \right)$ is positive, and $-T_s \kappa \frac{\epsilon^2(k)}{\bar{m}^2(k)} \left(1 - 2T_s \|\Gamma\| \cdot |\kappa|^2 \frac{\|\xi(k)\|^2}{\bar{m}^2(k)} \right)$ is a negative term.

III) As T_s and κ are positive constants that are chosen by the designer, the term $-\frac{T_s \kappa (\phi^T(k) \xi(k))^2}{\bar{m}^2(k)}$ can then be negative. In this way, the condition $\Delta V_1(k) \leq 0$ is satisfied for this term.

Considering I–III and disregarding the terms that meet the stability criteria, (58) can be rewritten as

$$\Delta V_1(k) \leq \frac{T_s \kappa (\mu \eta(k))^2}{\bar{m}^2(k)}. \quad (62)$$

Furthermore, $\Delta V_2(k)$ is written as

$$\Delta V_2(k) \leq e_1^2(k+1) - e_1^2(k). \quad (63)$$

As $\Delta V(k) = \Delta V_1(k) + \Delta V_2(k)$, as previously defined, and considering (62)–(63), $\Delta V(k)$ can be rewritten as

$$\Delta V(k) \leq \kappa \frac{T_s (\mu \eta(k))^2}{\bar{m}^2(k)} + e_1^2(k+1) - e_1^2(k). \quad (64)$$

In view of (64), when the unmodeled dynamics is null ($\mu = 0$), there is a $\theta = \theta^*$ so that $\phi = [0]$ due to the controllability of $G_0(z) = G(z)$. Therefore, the augmented error ϵ can be expressed as

$$\epsilon = \phi^T \xi + \mu \eta = 0, \quad (65)$$

according to Lemma 1, presented in the Appendix.

Hence, it implies $e_1(k) \rightarrow 0$ when $k \rightarrow \infty$ and $\Delta V(k)$ from (64) is smaller than or equal to zero.

3.1.2 | Generalization of the stability analysis for systems subjected to uncertainties ($\mu \neq 0$)

The previous analysis considers that $\mu = 0$. Now, the generalization of the stability analysis for systems subjected to uncertainties is provided ($\mu \neq 0$).

In this regard, due to the controllability of $G_0(z)$, it can be assumed

$$e_1(k) \propto \mu \eta(k), \quad (66)$$

where $\eta(k) = \Delta(z)u$ and $\Delta(z)$ is a stable transfer function. Furthermore, the system's tracking error can be expressed as

$$e_1(k) = \alpha_0 \mu \eta(k), \quad (67)$$

where $\alpha_0 > 1$.

Considering

$$\eta(k) = \frac{\alpha_1}{(z - \alpha_2)} u(k) + \xi(k), \quad (68)$$

and

$$\eta(k+1) = \alpha_2 \eta(k) + \alpha_1 u(k) + \xi_1(k), \quad (69)$$

where α_1 is a positive scalar and α_2 is chosen such that it is contained within a unit radius circle.

Replacing (18) into (69), comes

$$\eta(k+1) = \alpha_2 \eta(k) + \alpha_1 u_r(k) + \alpha_1 u_v(k) + \xi_1(k). \quad (70)$$

Moreover, multiplying both sides of (70) by $\alpha_0 \mu$,

$$\alpha_0 \mu \eta(k+1) = \alpha_0 \mu \alpha_2 \eta(k) + \alpha_0 \mu \alpha_1 u_r(k) + \alpha_0 \mu \alpha_1 u_v(k) + \alpha_0 \mu \xi_1(k), \quad (71)$$

and substituting (20) in (71), it follows that

$$\alpha_0 \mu \eta(k+1) = \alpha_0 \mu \alpha_2 \eta(k) + \alpha_0 \mu \alpha_1 u_r(k) + \theta_{SSTS M}(k) v(k) + \alpha_0 \mu \xi_1(k), \quad (72)$$

being

$$\theta_{SSTS M}(k) = -\frac{\alpha_0 \mu \alpha_1 \theta_5(k)}{\theta_3(k)}, \quad (73)$$

$$v(k) = -k_1 \sqrt{|e_1(k)|} \text{sgm}(e_1(k)) + g(k), \quad (74)$$

$$g(k+1) = g(k) - k_2 T_s \text{sgm}(e_1(k)). \quad (75)$$

Next, replacing (67) in (72), it follows that

$$e_1(k+1) = \alpha_2 e_1(k) + \alpha_0 \mu \alpha_1 u_r(k) + \theta_{SSTS M}(k) v(k) + \alpha_0 \mu \xi_1(k), \quad (76)$$

or yet,

$$e_1(k+1) = \alpha_2 e_1(k) + \bar{u}_{SSTS M}(k) + \alpha_0 \mu \xi_2(k), \quad (77)$$

where

$$\bar{u}_{SSTS M}(k) = \theta_{SSTS M}(k) v(k), \quad (78)$$

and

$$\xi_2(k) = \alpha_1 u_r(k) + \xi_1(k). \quad (79)$$

Subtracting $e_1(k)$ from both sides of (77), it is possible to obtain

$$\Delta e_1(k) = (\alpha_2 - 1)e_1(k) + \bar{u}_{SSTS M}(k) + \alpha_0 \mu \xi_2(k). \quad (80)$$

Considering the equality

$$e_1^2(k+1) - e_1^2(k) = e_1^2(k+1) - (e_1(k+1) - \Delta e_1(k))^2 = 2e_1(k+1)\Delta e_1(k) - \Delta e_1^2(k), \quad (81)$$

and replacing (81) into (64), it can be written as

$$\Delta V(k) \leq \kappa \frac{T_s(\mu\eta(k))^2}{\bar{m}^2(k)} + 2e_1(k+1)\Delta e_1(k) - \Delta e_1^2(k). \quad (82)$$

Moreover, using (76) and (80), (81) can be expressed as

$$2e_1(k+1)\Delta e_1(k) = 2[\alpha_2 e_1(k) + \bar{u}_{SSTS M}(k) + \alpha_0 \mu \xi_2(k)](\alpha_2 - 1)e_1(k) + 2\bar{u}_{SSTS M}(k)(\alpha_2 - 1)e_1(k) + 2\alpha_0 \mu \xi_2(k)(\alpha_2 - 1)e_1(k), \quad (83)$$

and also,

$$2e_1(k+1)\Delta e_1(k) = 2\alpha_2(\alpha_2 - 1)e_1^2(k) + 2\alpha_2\bar{u}_{SSTS M}(k)e_1(k) + 2\alpha_2\alpha_0\mu e_1(k)\xi_2(k) + 2\bar{u}_{SSTS M}(k)(\alpha_2 - 1)e_1(k) + 2\bar{u}_{SSTS M}^2(k) + 4\alpha_0\mu\bar{u}_{SSTS M}(k)\xi_2(k) + 2\alpha_0\mu(\alpha_2 - 1)e_1(k)\xi_2(k) + 2(\alpha_0\mu)^2\xi_2^2(k). \quad (84)$$

Considering (80),

$$-\Delta e_1^2(k) = -[(\alpha_2 - 1)e_1(k) + \bar{u}_{SSTS M}(k) + \alpha_0 \mu \xi_2(k)][(\alpha_2 - 1)e_1(k) + \bar{u}_{SSTS M}(k) + \alpha_0 \mu \xi_2(k)]. \quad (85)$$

Thus,

$$-\Delta e_1^2(k) = -(\alpha_2 - 1)^2 e_1^2(k) - \bar{u}_{SSTS M}^2(k) - (\alpha_0 \mu)^2 \xi_2^2(k) - 2(\alpha_2 - 1)e_1(k)\bar{u}_{SSTS M}(k) - 2\alpha_0 \mu(\alpha_2 - 1)e_1(k)\xi_2(k) - 2\alpha_0 \mu \bar{u}_{SSTS M}(k)\xi_2(k). \quad (86)$$

Replacing (84) and (86) into (81), it follows that

$$2e_1(k+1)\Delta e_1(k) - \Delta e_1^2(k) = 2\alpha_2(\alpha_2 - 1)e_1^2(k) + 2\alpha_2\bar{u}_{SSTS M}(k)e_1(k) + 2\mu\alpha_0\alpha_2 e_1(k)\xi_2(k) + \bar{u}_{SSTS M}^2(k) + 2\mu\alpha_0\bar{u}_{SSTS M}(k)\xi_2(k) + (\alpha_0\mu)^2\xi_2^2(k) - (\alpha_2 - 1)^2 e_1^2(k), \quad (87)$$

and it also follows that

$$2e_1(k+1)\Delta e_1(k) - \Delta e_1^2(k) = (\alpha_2^2 - 1)e_1^2(k) + 2\alpha_2\bar{u}_{SSTS M}(k)e_1(k) + 2\mu\alpha_0\alpha_2 e_1(k)\xi_2(k) + (\bar{u}_{SSTS M}(k) + \alpha_0\mu\xi_2(k))^2. \quad (88)$$

Replacing (88) in (82), it is true that

$$\Delta V(k) \leq \kappa \frac{T_s(\mu\eta(k))^2}{\bar{m}^2(k)} + (\alpha_2^2 - 1)e_1^2(k) + 2\alpha_2\bar{u}_{SSTS M}(k)e_1(k) + 2\mu\alpha_0\alpha_2 e_1(k)\xi_2(k) + (\bar{u}_{SSTS M}(k) + \alpha_0\mu\xi_2(k))^2. \quad (89)$$

Furthermore, (89) can be written as

$$\Delta V(k) \leq a + b \leq 0, \quad (90)$$

where $a = \kappa \frac{T_s(\mu\eta(k))^2}{\bar{m}^2(k)} + (\alpha_2^2 - 1)e_1^2(k)$ and $b = 2\alpha_2\bar{u}_{SSTS M}(k)e_1(k) + 2\mu\alpha_0\alpha_2 e_1(k)\xi_2(k) + (\bar{u}_{SSTS M}(k) + \alpha_0\mu\xi_2(k))^2$.

Next, replacing (67) in a , it follows that

$$a = \kappa \frac{T_s(\mu\eta(k))^2}{\bar{m}^2(k)} + (\alpha_2^2 - 1)\alpha_0^2(\mu\eta(k))^2 \leq 0. \quad (91)$$

Given that $\alpha_2^2 < 1$, $\alpha_0 > 1$, and $\frac{\kappa T_s}{\bar{m}^2(k)}$ are considerably small, it becomes feasible to select values for κ , T_s , and α_0 such that inequality (91) is less than or equal to 0. Consequently, the stability analysis for the a term concludes, leaving the analysis of the b term pending.

Considering (90), for b , the necessary condition follows

$$\bar{u}_{SSTS M}(k) = -\mu\alpha_0\xi_2(k), \quad (92)$$

and, also,

$$2\alpha_2\bar{u}_{SSTS M}(k)e_1(k) + 2\mu\alpha_0\alpha_2e_1(k)\xi_2(k) + (\bar{u}_{SSTS M}(k) + \alpha_0\mu\xi_2(k))^2 = 0, \quad (93)$$

and consequently $\Delta V(k) \leq 0$, being in this condition a particular case.

However, for b in (90), it is necessary in a general form that

$$2\alpha_2\bar{u}_{SSTS M}(k)e_1(k) + 2\mu\alpha_0\alpha_2e_1(k)\xi_2(k) + (\bar{u}_{SSTS M}(k) + \alpha_0\mu\xi_2(k))^2 \leq 0, \quad (94)$$

which can be rewritten as,

$$(\bar{u}_{SSTS M}(k) + \mu\alpha_0\xi_2(k))^2 + 2\alpha_2e_1(k)(\bar{u}_{SSTS M}(k) + \mu\alpha_0\xi_2(k)) \leq 0. \quad (95)$$

From (95), it is true that

$$2\alpha_2e_1(k)(\bar{u}_{SSTS M}(k) + \mu\alpha_0\xi_2(k)) \leq -(\bar{u}_{SSTS M}(k) + \mu\alpha_0\xi_2(k))^2, \quad (96)$$

where

$$2\alpha_2e_1(k) \leq -\bar{u}_{SSTS M}(k) - \mu\alpha_0\xi_2(k). \quad (97)$$

In this configuration, $\bar{u}_{SSTS M}(k)$ has to satisfy the following inequality:

$$\bar{u}_{SSTS M}(k) \leq -2\alpha_2e_1(k) - \mu\alpha_0\xi_2(k). \quad (98)$$

Remark 3: Referring to equations (73) and (78), and considering adaptation of θ_5 , it becomes feasible to derive (98) to account for some unmodeled dynamics. By substituting (98) into (89), and disregarding the term a , which has previously been demonstrated, there is

$$\Delta V(k) \leq 2\alpha_2e_1(k)[-2\alpha_2e_1(k) - \mu\alpha_0\xi_2(k)] + 2\mu\alpha_0\alpha_2e_1(k)\xi_2(k) + (-2\alpha_2e_1(k))^2. \quad (99)$$

Expanding the terms of (99), it is true that

$$\Delta V(k) \leq -4\alpha_2^2e_1^2(k) - 2\mu\alpha_0\alpha_2\xi_2(k)e_1(k) + 2\mu\alpha_0\alpha_2\xi_2(k)e_1(k) + 4\alpha_2^2e_1^2(k) \leq 0, \quad (100)$$

and, at last,

$$\Delta V(k) \leq 0. \quad (101)$$

Remark 4: Should either the specific or the general case not be met, with reference to equations (73) and (78), $\bar{u}_{SSTS M}(k)$ can alternatively be expressed as

$$-\mu\alpha_0\alpha_1\frac{\theta_5(k)}{\theta_3(k)}v(k) \leq -2\alpha_2e_1(k) - \mu\alpha_0\xi_2(k), \quad (102)$$

and, consequently,

$$\frac{\theta_5(k)}{\theta_3(k)} \leq \frac{2\alpha_2e_1(k)}{\mu\alpha_0\alpha_1v(k)} + \frac{\xi_2(k)}{\alpha_1v(k)}. \quad (103)$$

However, if conditions (92) or (98) are not met, the control signal $\bar{u}_{SSTS M}(k)$ becomes persistently exciting. Consequently, the adaptive algorithm continuously adjusts, ensuring the validity of (92) or (98). Even if the algorithm fails to find a solution set leading to zero tracking error, a residual error arises due to unmodeled dynamics. Nevertheless, the tracking error remains small on average, surpassing an upper limit, as indicated by (58)–(62). A similar approach for proving the convergence of the residual set for $e_1(k)$ is detailed in^{55,50}.

It is worth mentioning that the incorporation of the adaptive high-order SMC term in the controller brings attention to an interesting aspect related to chattering. In cases where chattering occurs at low levels, it tends to be advantageous for the parameter identifier adaptive algorithm. The persistent excitement generated by chattering aids in meeting the adaptability requirements of a reference “rich in frequency”⁵⁶, which, in turn, assists θ in converging to θ^* . In summary, it can be affirmed that all terms

of $V(k)$ tend to zero or negligible values as $k \rightarrow \infty$. Consequently, $e_1(k) \rightarrow 0$ as $k \rightarrow \infty$ or, at the very least, $e_1(k)$ will be small on average and confined to a residual set. ■

4 | SIMULATION RESULTS

This section presents simulation results, where the developed controller is applied for regulation of a non-minimum-phase and unstable system under structured uncertainties, with a reduced-order reference model. Additionally, a comparison with the controller presented in⁴⁴ is provided. Simulation is carried out using Matlab®.

4.1 | Unstable and non-minimum phase plant

This subsection discusses the results of applying the developed controller in an unstable and non-minimum phase plant.

4.1.1 | System description

The plant transfer function is

$$G(s) = \frac{5 - \alpha s}{(s - 1)(1 + \alpha s)}, \quad (104)$$

where $\alpha = 0.1$, which, discretized with the zero-order holder (ZOH) method with a sampling rate (T_s) equal to 1 kHz, is

$$G(z) = \frac{-0.0009706z + 0.00102}{z^2 - 1.991z + 0.991}, \quad (105)$$

which is an unstable and non-minimum-phase plant. Additionally, a relevant parametric variation was imposed on (105) in order to evaluate the robustness of the control algorithm, increasing α and constituting a structured uncertainty. When this occurs, the complete transfer function can be rewritten as

$$G'(s) = \frac{5 - \alpha' s}{(s - 1)(1 + \alpha' s)}, \quad (106)$$

where $\alpha' = 0.18$. The discretized system ($T_s = 1$ kHz) with parametric variation is

$$G'(z) = \frac{-0.0009839z + 0.001012}{z^2 - 1.995z + 0.9955}. \quad (107)$$

Figure 2 presents the complete plant without parametric variation (G_z) and with parametric variation (G'_z). When there is parametric variation (α'), the plant changes considerably, especially close to the signal of interest (1 Hz), imposing a significantly difficult-to-control structure.

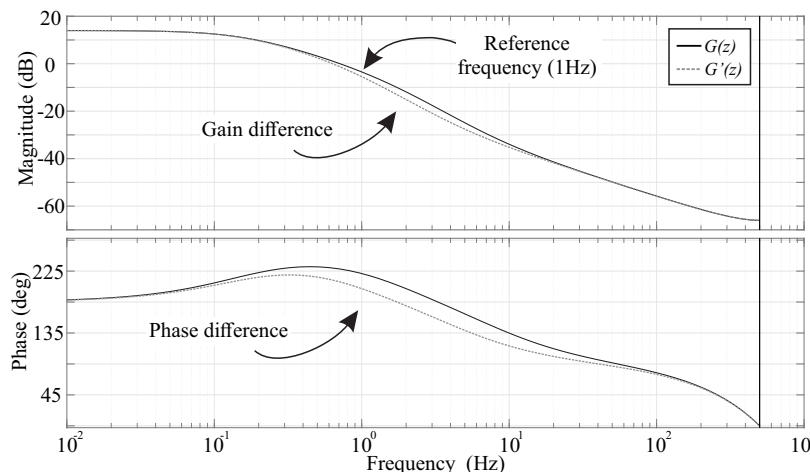


FIGURE 2 Bode diagram of $G(z)$ and $G'(z)$.

4.1.2 | Controller design

The nominal part of the plant (minimum phase) considered in designing the reference model is

$$G_0(s) = \frac{5}{s-1}, \quad (108)$$

which, discretized ($T_s = 1$ kHz) using ZOH, results in

$$G_0(z) = \frac{0.005003}{z-1.001}, \quad (109)$$

and the chosen reference model is expressed by

$$W_m(s) = \frac{20}{s+20}, \quad (110)$$

which, discretized with the same T_s , is

$$W_m(z) = \frac{0.0198}{z-0.9802}. \quad (111)$$

The complete plant $G(z)$ is of second order, but only the nominal part of the plant $G_0(z)$ is considered for controller design. As the relative degree of $G_0(z)$ is equal to one, $W_m(z)$ can be designed as a first-order transfer function. The reference model is designed to be similar to the plant near the frequency of interest (in this case, 1 Hz). This is a common practice in adaptive controllers (see^{57,49,58}). Furthermore, both the nominal plant $G_0(z)$ and the reference model $W_m(z)$ agree with Assumptions A_1 – A_6 , presented in Section II.

The designed controller parameters are: $\Gamma = 5 \mathbf{I}_{3 \times 3}$, where $\mathbf{I}$ is the identity matrix, $\kappa = 1$, $\sigma_0 = 0.1$, $M_0 = 2.5$, $M = 4$, $\delta_0 = 0.7$, and $\delta_1 = 1$. In addition, k_1 and k_2 are set to 1 and 0.001, respectively. The constant k_2 is chosen to be a small value because it is related to the tracking error integral to adaptive STSM controllers. Additionally, the initial values of adaptive gains are defined randomly as $\theta(0) = [-1 \ 0 \ 0.1]$, corresponding to $\theta_3(0)$, $\theta_4(0)$ and $\theta_{STSM}(0)$, respectively. As the adopted reference model is first order, θ_1 and θ_2 are not implemented because there are no internal filters ω_1 and ω_2 .

4.1.3 | Results for a system with parametric variation

Reference r is a sinusoidal signal with period of 1 s. The reference signal amplitude is 7.5 until instant $t = 133.33$ s. After this period, the amplitude is increased to 15. A parametric variation is imposed at $t = 266.667$ s. When $t = 400$ s, the simulation is finished.

Figure 3(a) shows the regulated signal (y) and the reference model output (y_m). As can be seen, the plant output follows the reference model closely, even in a scenario of considerable parametric variation: a few cycles after the imposed parametric variation (α') at $t = 266.667$, the system is properly regulated again, as observed in Fig. 3(b).

Figure 4(a) presents the tracking error (e_1) and augmented error (ϵ), where initial overshoot and reference tracking at the amplitude change instant is better analyzed. As can be seen, the system is severely disturbed at start-up as well as at $t = 133.33$, when a 100% variation in the reference is imposed. At $t = 266.667$, when there is a parametric variation in the plant, gains are accommodated in a solution set close to θ^* ; consequently, this transient regime is reduced. In addition, both e_1 and ϵ tend to a residual set at steady state, as demonstrated in the proofs and corroborated in Fig. 4(b).

Figures 5(a) and (b) show u_r (from the RMRAC) and u_v (from the adaptive sigmoid STSM), respectively. Both control actions are disturbed during the transients, but they tend to accommodate at steady state. Moreover, the adaptive STSM control action actively contributes during transients, stimulating system convergence. However, its value tends to an irrelevant value in steady state, mitigating the chattering effect. Although there is a residual value in steady state, its waveform presents smooth and non-square transitions, which are obtained with the adaptive sigmoid function.

Figure 6 shows the adaptive gains θ . As can be observed, the adaptation of gains is intense at startup because the gains were chosen randomly. This initial gain modification explains the overshoot. However, the controller acts quickly to reduce the tracking error. If $\theta(0)$ is initialized with values close to their converged values, the overshoot tends to decrease considerably. Furthermore, when the reference amplitude changes, the gains are smoothly adapted. However, when the parametric variation is imposed, they alter relevantly because the plant parameters changed, and consequently they need to converge to another solution set to ensure global stability.

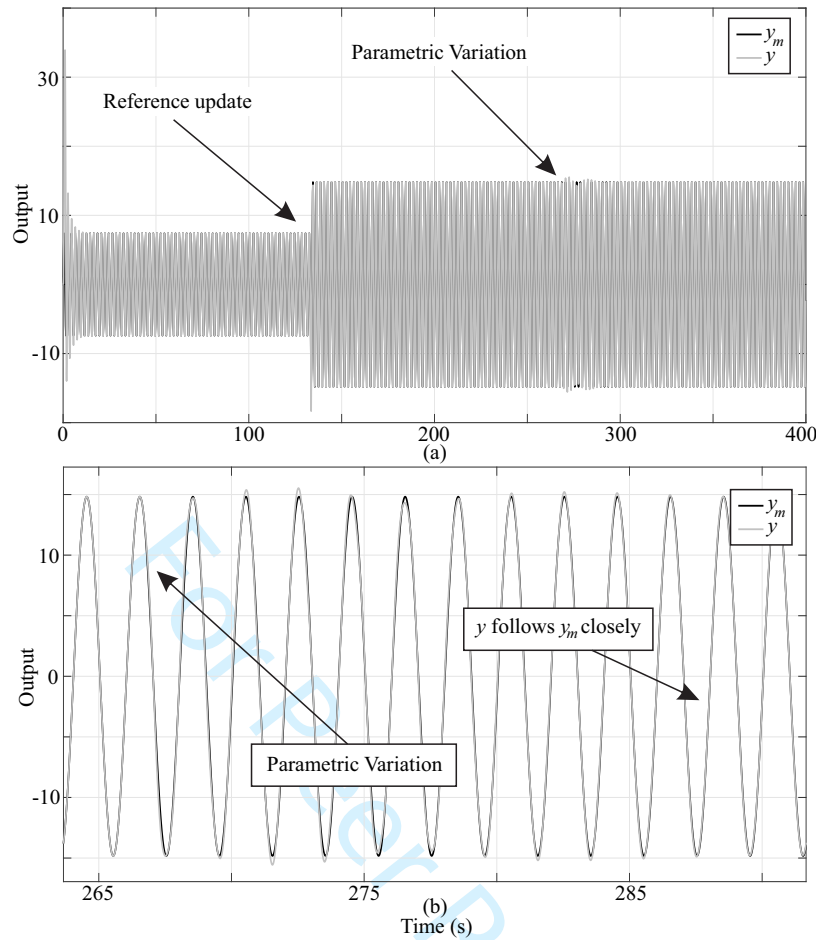


FIGURE 3 Plant output (y) and reference model output (y_m). (a) general overview; (b) detail of steady-state with parametric variation (α').

4.1.4 | Results for a system without parametric variation

As presented in the stability analysis of Section III, the control action u_v and the errors must tend to zero in a scenario where the plant does not have parametric variation. In order to verify this behavior, the following results are presented, where r is a sinusoidal reference signal with period of 1 s and amplitude of 5. When $t = 400$ s, the simulation is finished.

Figure 7 shows the reference model y_m and regulated signal y . Note that the plant output signal can perfectly track the reference model. Figure 8 presents e_1 and e . It is observed that in steady state the tracking and augmented errors tend to zero. Furthermore, it turns out that e , minimized by the adaptation algorithm, has amplitudes around 10^{-5} , while the tracking error e_1 is close to 10^{-3} (corresponding to an error of 0.06%). This difference occurs because even if the controlled plant (unstable and non-minimum phase) does not present imposed parametric variation, there are unmodeled dynamics, as the reference model used was chosen based on a reduced model of the plant (see (104)–(111)).

Figure 9 shows the adaptive sigmoid STSM control action u_v in a scenario without parametric variation. It can be seen that u_v presents high dynamics at start-up, during the transient regime, but tends to zero in steady state. Thus, after convergence, the RMRAC action should dominate the complete control action response. However, if there are new disturbances, u_v must be reactivated, as could be seen in the previous subsection. This behavior is also in agreement with that obtained through the controller mathematical stability analysis by means of Lyapunov theory.

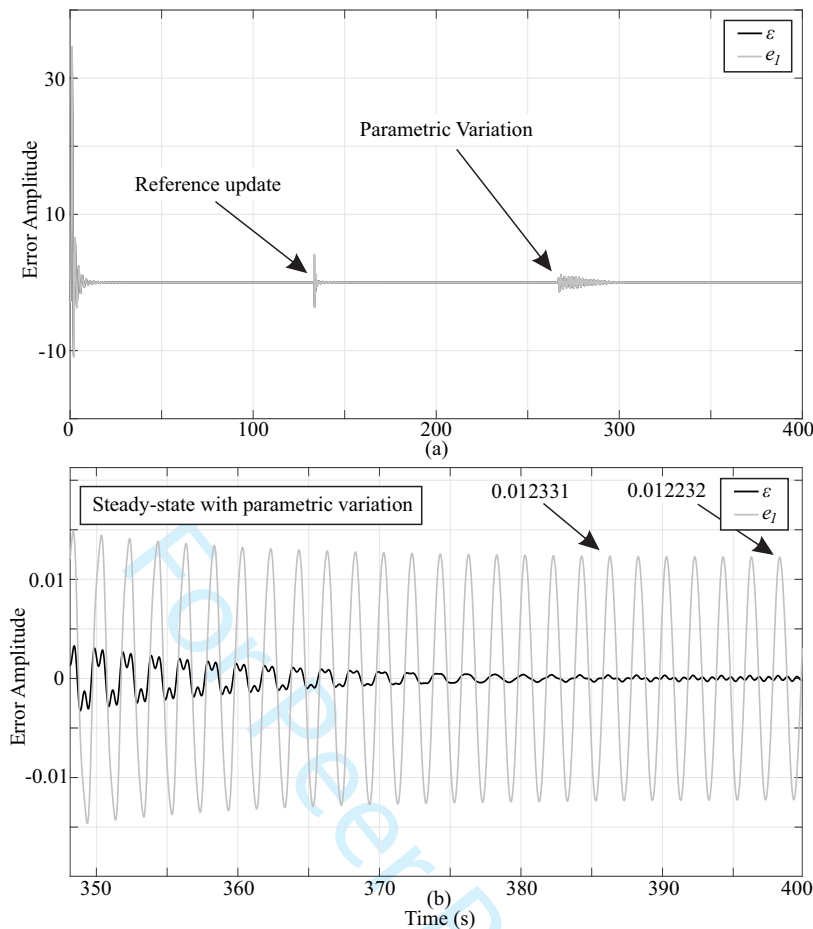


FIGURE 4 Tracking and augmented error. (a) general overview; (b) detail of steady state with parametric variation (α').

4.2 | Comparison with previous adaptive super-twisting sliding mode controller

Since the proposed controller emerges as an enhancement from⁴⁴, this section presents the comparison of both control structures dealing with a non-minimum phase and unstable second order plant.

4.2.1 | System description

The system plant can be described as

$$\bar{G}(s) = 2 \frac{5 - \beta s}{(s - 1)(1 + \beta s)}, \quad (112)$$

where $\beta = 0.1$, which, when discretized with the ZOH method with a $T_s = 1$ kHz, is

$$\bar{G}(z) = \frac{0.001941z + 0.002041}{z^2 - 1.991z + 0.991}, \quad (113)$$

A parametric variation was imposed on (113), increasing β . When this occurs, the new system transfer function is

$$\bar{G}'(s) = 2 \frac{5 - \beta' s}{(s - 1)(1 + \beta' s)}, \quad (114)$$

where $\beta' = 0.15$. The discretized system ($T_s = 1$ kHz) with parametric variation can be written as

$$\bar{G}'(z) = \frac{-0.001961z + 0.002028}{z^2 - 1.994z + 0.9943}. \quad (115)$$

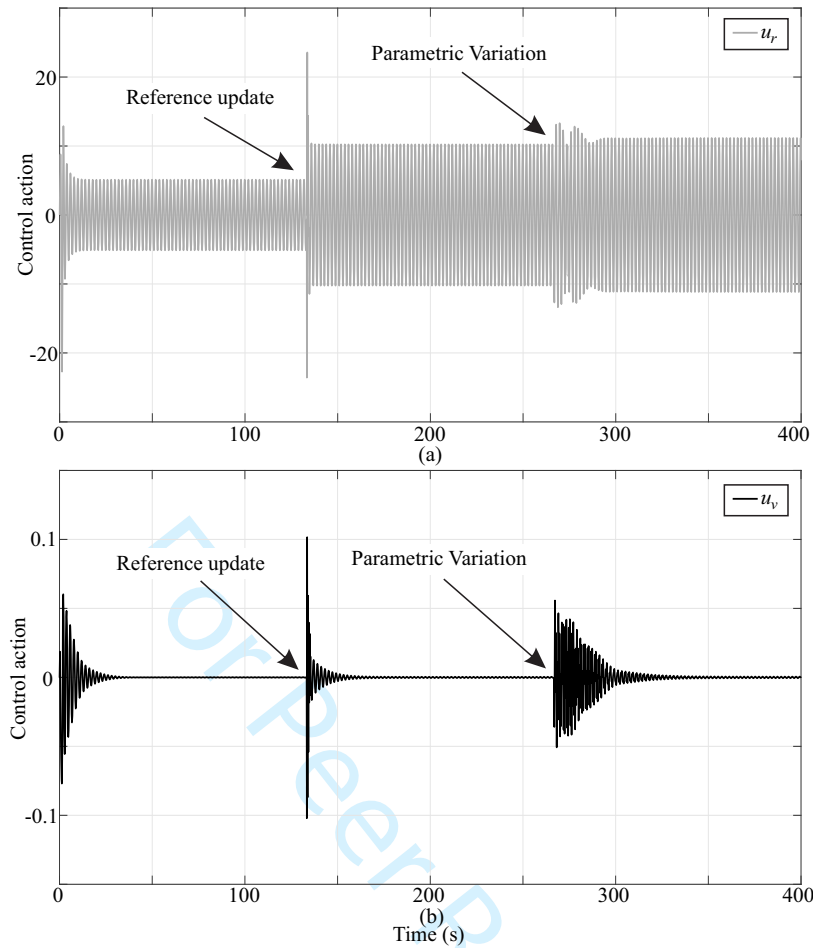


FIGURE 5 Control actions. (a) u_r ; (b) u_v .

In order to stress the system even more, a second parametric variation is also considered. The system in continuous time can be expressed as

$$\bar{G}''(s) = 2 \frac{5 - \beta'' s}{(s - 1)(1 + \beta''' s)}, \quad (116)$$

where $\beta'' = 0.175$ and $\beta''' = 0.18$. The discretized system ($T_s = 1$ kHz) with parametric variation is as follows

$$\bar{G}''(z) = \frac{-0.001912z + 0.001968}{z^2 - 1.995z + 0.9955}. \quad (117)$$

Figure 10 presents the complete plant without parametric variation (G_z) and the plant with first ($\bar{G}_z'$) and second parametric variations ($\bar{G}_z''$). The signal y to be controlled is at 1 Hz. It is possible to observe in Fig. 10 the mismatch between plants near the frequency of interest, as well as for high frequencies, requesting a significant robustness from the control system to keep the system stable.

4.2.2 | Controller design

Regarding the reference model, since the considered plant (113) is similar to the one presented in the last section (105) and the regulated signal is also at low frequency, the same reference model design from (108)–(111) was considered. It is important to highlight that the complete plant $G(z)$ is of second order, but only the nominal part of the plant $G_0(z)$ is considered for controller design, as previously discussed. The reference model is designed to be similar to the plant near the frequency of interest (in this case, 1 Hz). Since the controller was proven robust to matched and unmatched dynamics, the modeling simplification is possible.

The designed controller parameters for both structures (proposed controller and⁴⁴) are: $\Gamma = 60 \mathbf{I}_{3 \times 3}$, $\kappa = 1.25$, $\sigma_0 = 0.1$, $M_0 = 2.5$, $M = 4$, $\delta_0 = 0.7$, and $\delta_1 = 1$. In addition, k_1 and k_2 are set to 1 and 0.01, respectively. As previously discussed, the

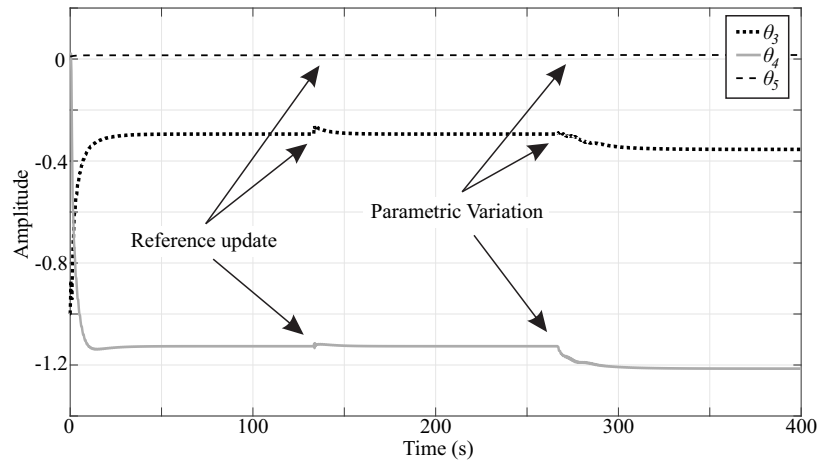


FIGURE 6 Adaptive gains θ .

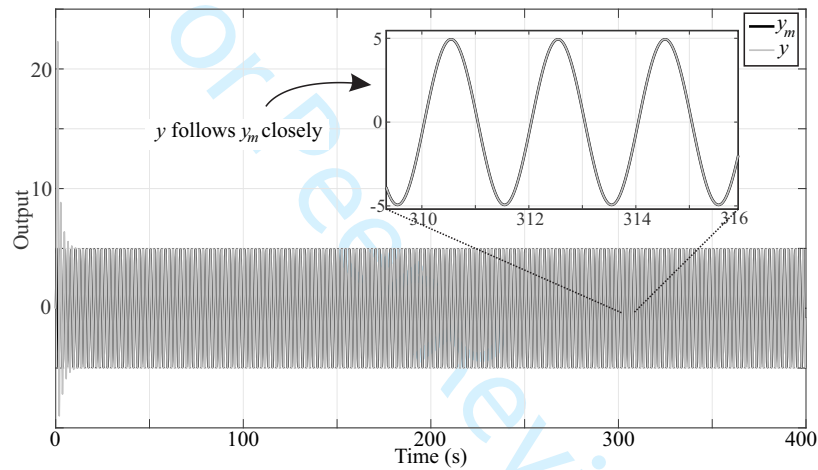


FIGURE 7 Plant output (y) and reference model output (y_m) (without parametric variation).

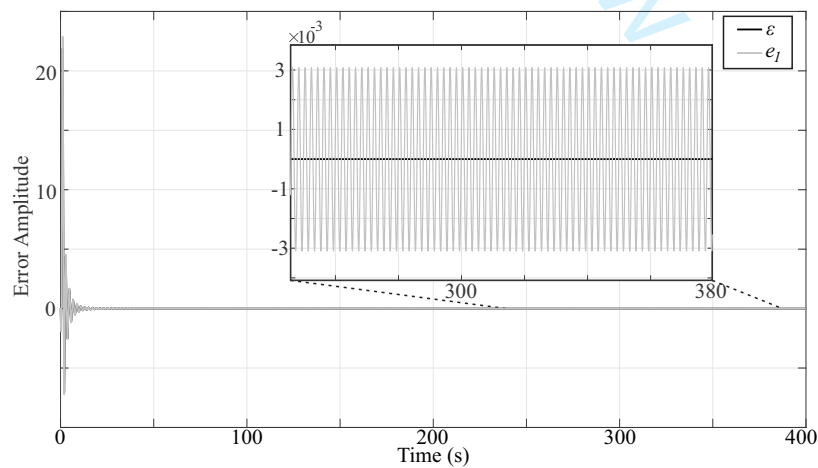


FIGURE 8 Tracking and augmented error (without parametric variation).

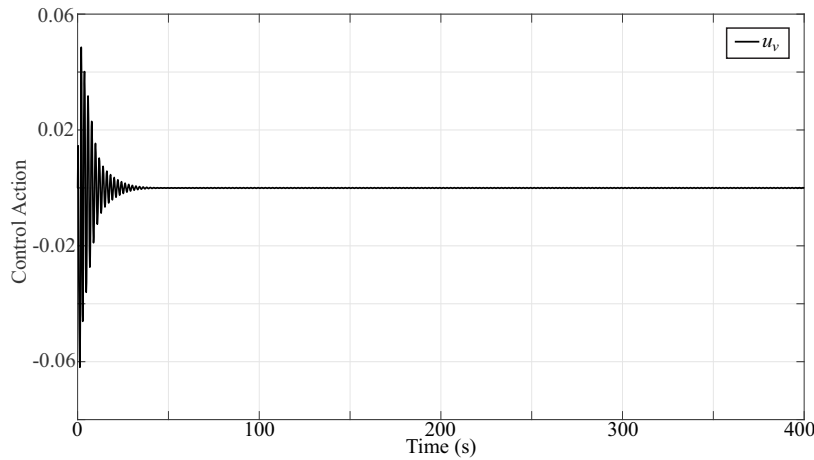


FIGURE 9 u_v control action (without parametric variation).

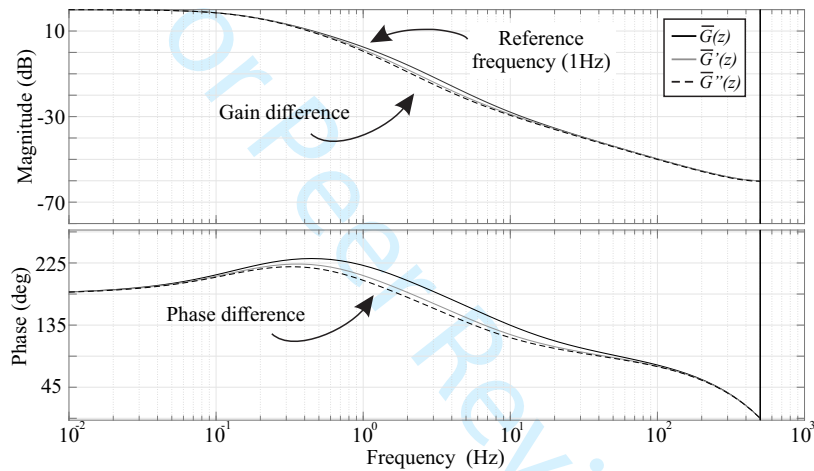


FIGURE 10 Bode diagram of $\bar{G}(z)$, $\bar{G}'(z)$ and $\bar{G}''(z)$.

constant k_2 is chosen to be a small value because it is related to the tracking error integral to adaptive STSM controllers. For the proposed structure, $\Psi = 1$ and $\psi = 0.25$. Furthermore, the initial values of the adaptive gains vector were defined randomly as $\theta(0) = [-1 \ 0 \ 0.05]$, corresponding to $\theta_3(0)$, $\theta_4(0)$, and $\theta_{STSM}(0)$, respectively. Note that with the exception of ψ and Ψ , which are exclusive to the proposed controller, all other parameters are the same for both controllers. This is developed to keep a fair comparison regarding robustness and performance between controllers. As the adopted reference model is first order, θ_1 and θ_2 are not implemented because there are no internal filters ω_1 and ω_2 .

4.2.3 | Comparison results

The simulation routine is the following: the reference r is a sinusoidal signal with period of 1 s. The reference signal amplitude is 10 until instant $t = 400$ s. After this period, the amplitude is increased to 20. Then, at $t = 800$ s the reference amplitude is updated to 25. A parametric variation (check (115)) is imposed at $t = 200$ s, and $\bar{G}(z)$ is updated to $\bar{G}'(z)$. When $t = 600$ s a new parametric variation (check (117)) is considered. In this instant, $\bar{G}'(z)$ is updated to $\bar{G}''(z)$. When $t = 1000$ s, the simulation is finished.

Figure 11 presents a comparison of the tracking error e_1 between both control structures. In Figure 11(a), the tracking error behavior throughout the entire experiment is depicted. It is evident that both control structures effectively regulate the plant, causing the tracking error to converge to a residual value in steady state. Additionally, it is noteworthy that both parametric variations and reference updates induce a significant transient response in the system. Upon closer inspection of Fig. 11(b)

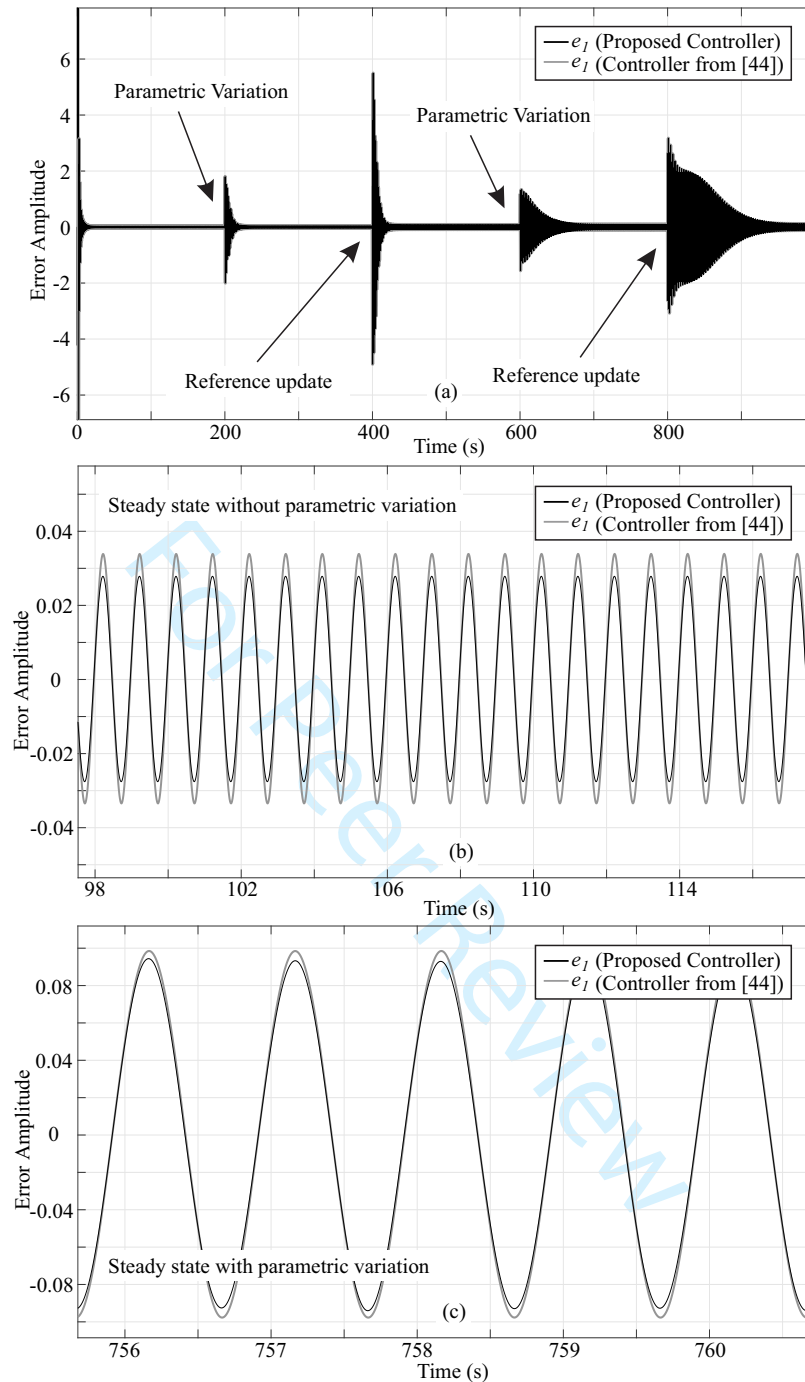


FIGURE 11 Tracking error (e_1) comparison. (a) general overview; (b) detail with the system at steady state without parametric variation; (c) detail with the system at steady state with parametric variation.

and 11(c), which represent details of the tracking in steady state without the presence of unmodeled dynamics and with the presence of unmodeled dynamics, respectively, the superiority of the proposed controller becomes apparent. During the steady state without parametric variation, from $t = 100$ to $t = 200$ s, the error amplitude for the previous controller was 0.033, while the proposed controller presented 0.027. For the period after the parametric variation and reference updates, from $t = 750$ to $t = 800$ s, the e_1 for the previous controller was 0.089, while the proposed controller presented 0.086. The tracking error for both scenarios is minimized when employing the proposed control structure. This superior performance is attributed to the presence of the adaptive super-twisting sliding mode with sigmoid, which will be elaborated on in the following paragraph.

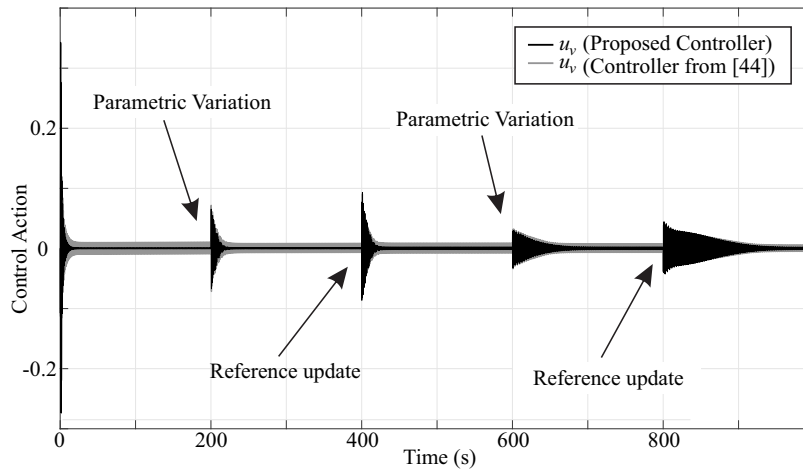


FIGURE 12 u_v control action comparison.

Figure 12 depicts the adaptive high-order sliding mode control (SMC) action for both controllers. It is evident that in terms of transient dynamics, both structures exhibit considerable similarity. The control action u_v tends to increase its amplitude in response to the system encountering unmodeled dynamics and disturbances, thereby altering its dynamics. This adjustment aids the adaptive gains in finding a new solution set to maintain system stability. However, when considering steady-state performance, a notable difference emerges: the proposed controller exhibits significantly less amplitude compared with the controller from⁴⁴. For the system without parametric variation, from $t = 100$ to $t = 200$ s, the previous control structure presented a u_v amplitude of 0.01, while the proposed controller presented $1 \cdot 10^{-3}$. Even for the system with severe parametric variation, from $t = 980$ to $t = 1000$ s, the previous control structure presented a u_v amplitude of $6 \cdot 10^{-3}$, while the proposed controller presented $2 \cdot 10^{-3}$. This distinction is crucial, as it is expected that the modified adaptive super-twisting sliding mode control action approaches zero as closely as possible in steady state. This reduction is aimed at minimizing chattering and enhancing regulation performance, representing a significant benefit derived from the modified adaptive sigmoid term in the proposed controller.

Figure 13 displays the adaptive gains vector θ for both control structures. Concerning the adaptive gains, it is evident that both controllers exhibit similar dynamics in both transient and steady-state conditions. The corresponding adaptive gains from both controllers converge to the same final value, indicating a consistent behavior for both systems, which is expected. The proposed structure, incorporating the modified adaptive sigmoid, maintains the same fast actuation behavior as the controller in⁴⁴. However, it enhances system regulation in steady state, as the control action u_v can assume minimal values, reducing chattering and improving overall system tracking performance, as presented in Figures 11 and 12 and discussed previously in this section.

4.3 | Practical example: a case study

This section aims to present a practical example to demonstrate the feasibility of the controller in regulating real-world control problems. For this purpose, the case study involves a step-up DC-DC power converter implemented in a controller hardware-in-the-loop (C-HIL) environment, with the control algorithm running on a digital signal processor (DSP).

4.3.1 | System description

In modern power systems, the penetration of distributed energy resources (DERs) has significantly impacted grid dynamics and control, introducing complexities and necessitating new control mechanisms for dealing with the power converters^{59,60}. Power electronics play an important role in managing these resources efficiently, allowing seamless integration and utilization within the grid⁶¹. Among DERs, photovoltaic (PV) systems often require step-up (DC-DC) voltage conversion to match their generated voltage with grid or load requirements before the DC-AC step^{62,63}. In this situation, a boost converter is a type of DC-DC converter that steps up the input voltage to a higher level by employing switching elements, such as controlled switches and diodes, along with inductive and capacitive elements. Boost converters efficiently convert energy, making them a suitable

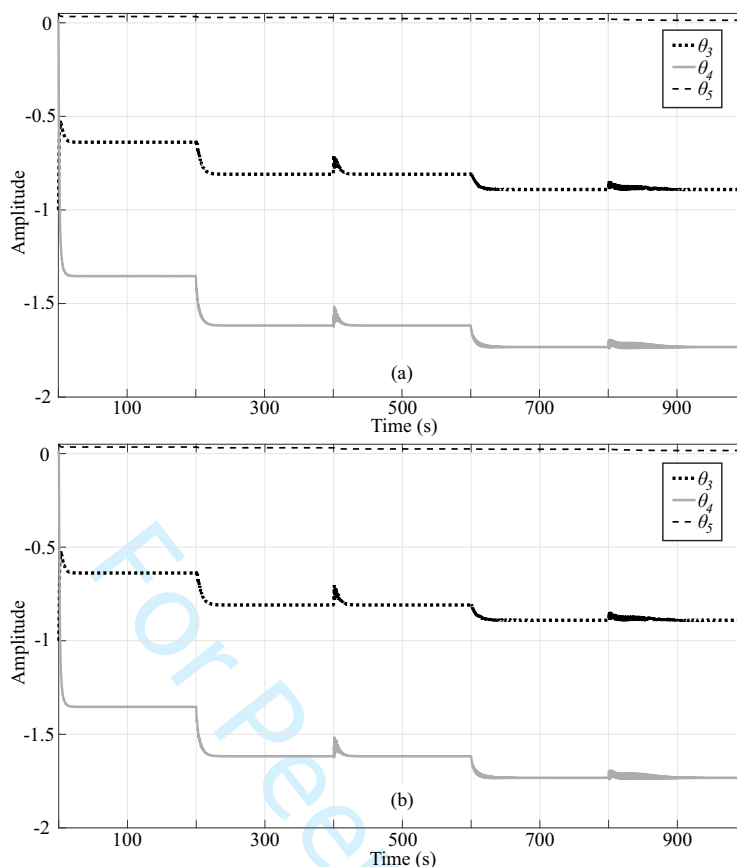


FIGURE 13 Adaptive gains θ comparison. (a) Proposed controller; (b) Controller from⁴⁴.

alternative in modern PV systems for providing grid integration capabilities. Fig. 14 presents the electrical model of a boost converter.

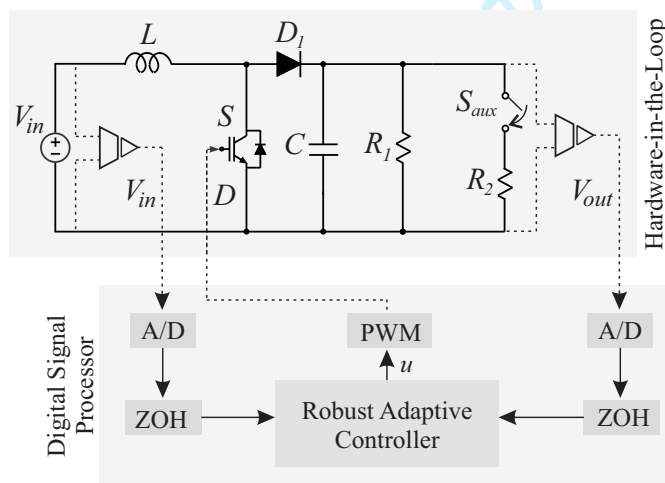


FIGURE 14 Boost converter electrical diagram.

Considering Fig. 14, V_{in} represents the input voltage, L denotes the inductor, C signifies the capacitor, D_1 indicates the diode, S denotes the controlled switch, S_{aux} signifies an auxiliary switch for load steps, and R_1 and R_2 represent the load. Additionally,

V_{out} is the regulated output voltage and D is the duty cycle, corresponding to the pulse width modulation (PWM) signal generated by the control routine. The modeling and design of boost converters have been extensively explored in the literature. In this context, they will be briefly described here. The state space matrix can be obtained using Kirchoff's current law (KCL) and Kirchoff's voltage law (KVL) to model the power converter. For the first step of operation (switch S ON), it follows:

$$\begin{bmatrix} \dot{i}_L(t) \\ \dot{v}_C(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{R_o C} \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_{in}(t)], \quad (118)$$

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [v_{in}(t)]. \quad (119)$$

For the second step of operation (switch S OFF), the following equations are obtained:

$$\begin{bmatrix} \dot{i}_L(t) \\ \dot{v}_C(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{R_o C} \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_{in}(t)], \quad (120)$$

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [v_{in}(t)]. \quad (121)$$

The considered power converter model contains two states, i_L and v_c , the current in the inductor and the voltage across the capacitor, respectively. Also, it contains one input, v_{in} , and one output, v_{out} , corresponding to the voltage to be controlled (equal to v_c). Observe that R_o corresponds to the equivalent load (only R_1 or a parallel combination of both R_1 and R_2). The boost converter was designed according with⁶⁴. The obtained values for the power converter were $D = 0.702$, $V_{in} = 150$ V, $V_{out} = 400$ V, $L = 1.8$ mH, $C = 470$ μ F, and $R_o = 49$ Ω (worst case scenario). The maximum power of the power converter is $P = 3200$ W.

After performing the small ripple approximation and the equilibrium system, the following transfer function relating the output voltage (v_{out}) and the duty cycle (D) can be obtained, as

$$\frac{v_{out}}{D}(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}, \quad (122)$$

where $b_2 = 4.859 \times 10^{40}$, $b_1 = 5.41 \times 10^{42}$, $b_0 = 1.875 \times 10^{46}$, $a_2 = 1.036 \times 10^{38}$, $a_1 = 4.5 \times 10^{39}$, and $a_0 = 1.088 \times 10^{43}$.

Discretizing (122), considering $T_s = 0.00005$ s (1/20 kHz) as sampling time and the ZOH method, the discrete-time transfer function can be written as

$$\frac{v_{out}}{D}(z) = \frac{468.9z^2 - 934.9z + 466.4}{z^2 - 1.998z + 0.9978}. \quad (123)$$

For more information regarding the boost converter modeling and design, refer to^{64,65}.

4.3.2 | Results comparison

The outcomes were achieved within a C-HIL setup. A Typhoon HIL 604 facilitated the execution of the DC-DC power converter dynamics to obtain the results, while the control algorithms were implemented into a TMS320F28335 DSP. Figure 15 illustrates the C-HIL setup configuration.

The implemented DSP control algorithms have A/D readings, signal conditioning, control law calculations, and PWM generation for the power converter switch. Thus, the same code that is implemented in the C-HIL environment can be used for a real prototype. The sampling and switching frequency is 20 kHz. The V_{in} is set to 150 V. The steps of the realized experiment are as follows: (i) When the controller is enabled (there is $t = 0$ s), the output voltage reference is set to 300 V, and it remains with this amplitude until $t = 2$ s. (ii) At $t = 2$ s, the output voltage reference is updated to 400 V. (iii) When $t = 4$ s, the switch S_{aux} closes (see Fig. 14) and a load step is performed (the equivalent load changes from 124 Ω to 49 Ω). The converter keeps this load value until the end of the experiment. (iv) At $t = 6$ s, the output voltage reference is set back to 300 V. (v) When $t = 8$ s, the experiment is finished. For comparison with the proposed structure, an RMRAC and a PI were also implemented and will be described next.

Given the robustness of the proposed adaptive controller to both matched and unmatched dynamics, as discussed in Sections 2 and 3, a reduced-order plant G_0 can be selected for reference model design. This approach is further elaborated and detailed

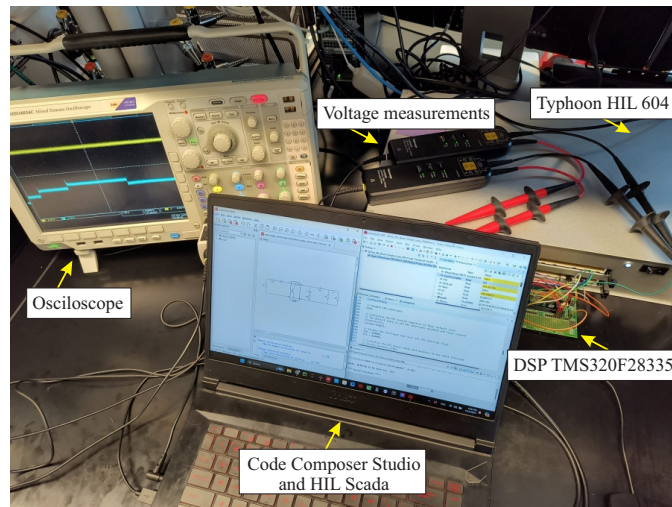


FIGURE 15 C-HIL setup.

in Section 4.1. Consequently, for simplification purposes, a first-order plant is chosen, and the corresponding reference model is designed accordingly.

$$W_m(z) = \frac{0.259181}{z - 0.740818}, \quad (124)$$

obtained with a ZOH method and a $T_s = 0.00005$ s.

The controller parameters designed for the proposed structure and for the RMRAC are: $\Gamma = 1$, $\kappa = 20$, $\sigma_0 = 0.18$, $M_0 = 10$, $M = 4$, $\delta_0 = 0.7$, and $\delta_1 = 1$. In addition, k_1 and k_2 are set to 5 and 0.001, respectively. For the proposed structure, $\Psi = 5$ and $\psi = 0.15$. Furthermore, the initial values of the adaptive gains vector are defined as $\theta(0) = [-2.15 \ 0.37 \ -0.07]$, corresponding to $\theta_3(0)$, $\theta_4(0)$, and $\theta_{STSM}(0)$, respectively. For obtaining the $\theta(0)$, the controllers were initialized with random gains and the values observed near convergence. This is a common practice to improve the adaptive controller performance in practical applications. Observe that with the exception of $\theta_{STSM}(0)$, k_1 , k_2 , ψ , and Ψ , which are exclusive to the proposed controller, the other parameters are the same for both adaptive structures, keeping a fair comparison regarding robustness and performance between the controllers. As the adopted reference model is first order, θ_1 and θ_2 are not implemented because there are no internal filters ω_1 and ω_2 , as also explained in Section 4.2.2.

The PI controller was tuned considering the transfer function presented in (123), and the open loop system has a phase margin of 89.7 degrees and a gain margin of 8.86 dB. The following equation represents the tuned PI controller:

$$C_{PI}(z) = \frac{-7.821389 \cdot 10^{-7}z + 1.41458 \cdot 10^{-6}}{z - 1}. \quad (125)$$

Fig. 16 presents the regulated voltage V_{out} and the input voltage V_{in} . Observe that the readings in the oscilloscope are in 100V/1 Vdac scale. This happens because Typhoon HIL input and output signals are limited to -10 to 10 V. As observed in Figs. 16(a),(b),(c), the input voltage V_{in} remains at 150 V for the whole experiment. Regarding the controlled output voltage, all evaluated control structures were able to regulate it properly. However, some dynamics differences are clearly observable between the controllers. First, the PI controller keeps the output voltage well regulated, but it is seen to have significant transient periods at startup, as well as when the reference is updated. The same behavior is not observable in the adaptive structures, which tend to regulate the signal much faster. The proposed structure shown in Fig. 16(c) also outperforms the RMRAC in 16(b). Note that the transients are reduced with the proposed controller due to the adaptive second-order sliding mode action. The same phenomena could be observed in the previous Sections 4.1 and 4.2. Also, it is important to highlight that both adaptive structures kept the same adaptive gains and tuning parameters so as to maintain a fair comparison. Regarding the tracking accuracy, the root mean square error of the voltage tracking error (e_1) was calculated. For the period comprising $t = 1$ s to $t = 2$ s (when the reference was 300 V), the root mean square (RMS) value was 0.2092 V for the PI, 0.1976 V for the RMRAC, and 0.1971 V for the proposed controller. From $t = 3$ s to $t = 5$ s (when the reference was 400 V), the RMS value was 0.1971 V for the PI, 0.1952 V for the RMRAC, and 0.1947 V for the proposed structure. Regarding the startup transient, the PI controller requests approximately 0.5 s for the signal to be in steady state, the RMRAC requests 0.1 s, and the proposed controller, 0.05 s. It is also observable that the load step performed at $t = 4$ s barely created a transient, due to the robustness of the evaluated controllers.

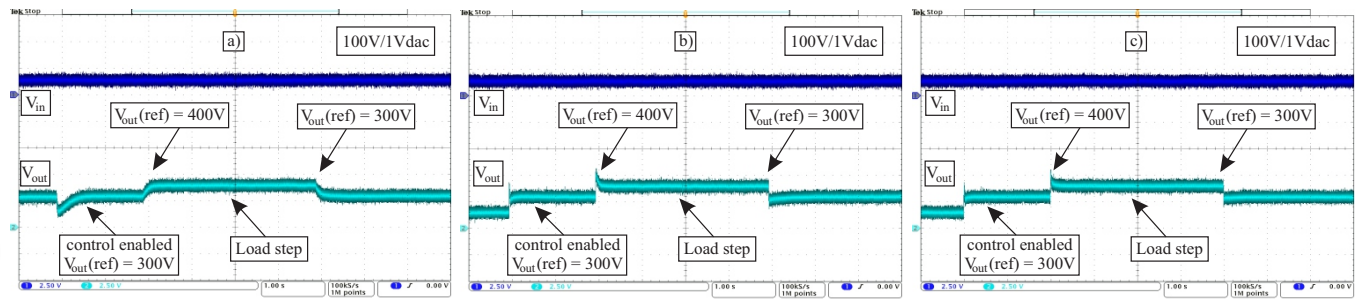


FIGURE 16 Regulated voltages. (a) PI controller; (b) RMRAC; (c) proposed controller.

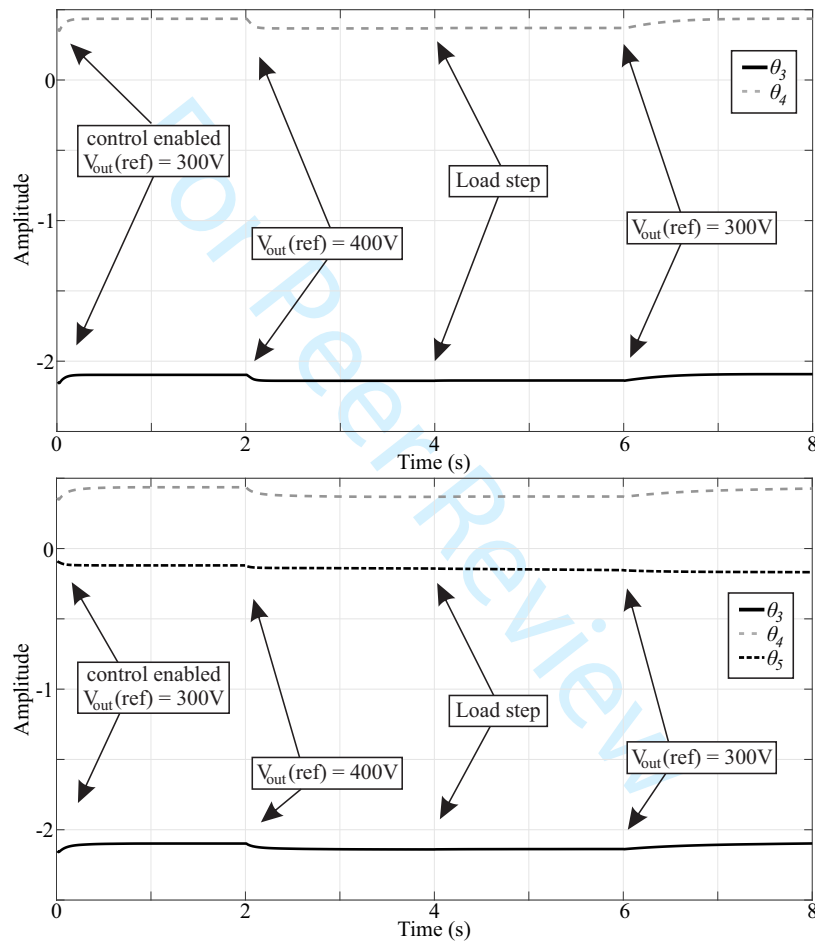


FIGURE 17 Adaptive gains θ . (a) RMRAC; (b) proposed controller.

Fig. 17 shows the adaptive gains θ for both adaptive controllers. The behavior of the adaptive gains is similar in both controllers. The main difference lies in the presence of the θ_5 , coming from the adaptive second order sliding mode term, present in the proposed structure. However, it is possible to observe that both adaptive controllers show transients in the startup and also when the reference is updated, but their gains converge at steady state. It is also important to highlight that the performance of the proposed adaptive controller can be significantly enhanced by taking into account the complete order of the system for reference model design, rather than relying on reduced-order models. Furthermore, optimizing the controller tuning parameters and initial values can further improve its performance. By fine-tuning these parameters, the controller can achieve faster convergence, better tracking accuracy, and increased robustness to disturbances.

5 | CONCLUSION

This paper presented a discrete-time Lyapunov stability analysis of an RMRAC-based adaptive sigmoid super-twisting sliding mode controller. The tendency of the tracking error to a residual set in steady state was proved. Furthermore, chattering is related to the unmodeled dynamics of the plant, but by using the proposed controller, there are some conditions in which it is possible to force the convergence of the tracking error to zero, thanks to the adaptive STSM control action. In addition, stability proofs demonstrated that the developed controller is globally stable and robust, even in the presence of matched and unmatched dynamics. **Simulation results using a reduced-order reference model showed the feasibility of the control structure, which was applied for the regulation of an unstable and non-minimum phase plant under a severe parametric variation, as well as for regulation of a DC-DC power converter, where the controller presented good performance and robustness with very limited chattering, demonstrating its feasibility for practical applications.**

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6 | APPENDIX

6.1 | Development of Error Equations

Lemma I: Tracking error between plant output $y(k)$ and reference model output $y_m(k)$ is expressed as

$$e_1 = y - y_m = W_m(z)\phi^T \omega + \mu\eta, \quad (126)$$

and the augmented error ϵ is

$$\epsilon = \phi^T \zeta + \mu\eta. \quad (127)$$

These proofs can be found in⁵⁰ but are also reproduced here in a more detailed way.

Proof: Based on control law $\theta^T \omega + r = 0$ and subtracting $\theta^{*T} \omega$ from the control law, it follows

$$\theta^T \omega - \theta^{*T} \omega + r = -\theta^{*T} \omega. \quad (128)$$

As ϕ is defined as $\phi \triangleq \theta - \theta^*$, (128) can be rewritten, according to

$$\phi^T + \theta^{*T} \omega + r = 0. \quad (129)$$

Expanding (129), it follows

$$\phi^T \omega + [\theta_1^{*T} \omega_1 + \theta_2^{*T} \omega_2 + \theta_3^* u + \theta_4^* y + \theta_5^* v] + r = 0. \quad (130)$$

As $\omega_1 = (zI - F)^{-1} qu$ and $\omega_2 = (zI - F)^{-1} qy$, (130) can be expressed as

$$\phi^T \omega + [\theta_1^{*T} (zI - F)^{-1} qu + \theta_2^{*T} (zI - F)^{-1} qy + \theta_3^* u + \theta_4^* G(z)u + \theta_5^* v] + r = 0. \quad (131)$$

In addition, as $y = G(z)u$, it follows

$$\phi^T \omega + [\theta_1^{*T} (zI - F)^{-1} qu + \theta_2^{*T} (zI - F)^{-1} qG(z)u + \theta_3^* u + \theta_4^* G(z)u + \theta_5^* v] + r = 0. \quad (132)$$

Therefore, $f_1(z) \triangleq (zI - F)^{-1} q$. Thus,

$$\phi^T \omega + [\theta_1^{*T} f_1(z)u + \theta_2^{*T} f_1(z)G(z)u + \theta_3^* u + \theta_4^* G(z)u + \theta_5^* v] + r = 0. \quad (133)$$

Considering the controllability of G_0 , which should respect Assumptions A_5 – A_6 , there is a $\theta = \theta^*$ such that $\phi = [0]$. Given this, (133) can be rewritten as

$$[\theta_1^{*T} f_1(z)u + \theta_2^{*T} f_1(z)G_0(z)u + \theta_3^* u + \theta_4^* G_0(z)u + \theta_5^* v] + r = 0. \quad (134)$$

It is highlighted that $G_0(z)$ is now considered in the Lemma's proof and not $G(z)$ anymore, because $G_0(z)$ controllability is assumed for $\phi = [0]$. When $y = y_m$, it follows

$$y = W_m(z)r = G_0(z)u, \quad (135)$$

or

$$r = W_m(z)^{-1} G_0(z)u. \quad (136)$$

Replacing (136) in (134), it also follows

$$[\theta_1^{*T} f_1(z)u + \theta_2^{*T} f_1(z)G_0(z)u + \theta_3^* u + \theta_4^* G_0(z)u + \theta_5^* v] + W_m(z)^{-1} G_0(z)u. \quad (137)$$

Rewriting (137), comes

$$G_0(z)u = -W_m(z)[\theta_1^{*T} f_1(z)u + \theta_2^{*T} f_1(z)G_0(z)u + \theta_3^* u + \theta_4^* G_0(z)u + \theta_5^* v]. \quad (138)$$

Plant output y can be rewritten as

$$y = G(z)u = G_0(z) [1 + \mu \Delta_m(z)] u + \mu \Delta_a(z)u. \quad (139)$$

Replacing (138) in (139) and rearranging the terms, it follows

$$y = -W_m(z)[\theta_1^{*T} f_1(z)u + (\theta_2^{*T} f_1(z) + \theta_4^*)G_0(z)u + \theta_3^* u + \theta_5^* v] [1 + \mu \Delta_m(z)] + \mu \Delta_a(z)u. \quad (140)$$

Moreover, $f_2(z) \triangleq \theta_2^{*T} f_1(z) + \theta_4^*$. Therefore, (140) can be expressed as

$$y = -W_m(z) [\theta_1^{*T} f_1(z)u + f_2(z)G_0(z)u + \theta_3^* u + \theta_5^* v] [1 + \mu \Delta_m(z)] + \mu \Delta_a(z)u. \quad (141)$$

Adding and subtracting $W_m(z)f_2(z)G(z) [1 + \mu \Delta_m(z)] u$ results in

$$y = -W_m(z)[\theta_1^{*T} f_1(z)u + f_2(z)G_0(z)u + \theta_3^* u + \theta_5^* v] [1 + \mu \Delta_m(z)] + W_m(z)f_2(z)G(z)[1 + \mu \Delta_m(z)]u - W_m(z)f_2(z)G_0(z) [1 + \mu \Delta_m(z)] u + \mu \Delta_a(z)u. \quad (142)$$

Replacing (133) in (142), it is obtained

$$y = W_m(z) [\phi^T \omega + r] - W_m(z)[\theta_1^{*T} f_1(z)u + f_2(z)G(z)u + \theta_3^* u + \theta_5^* v]\mu \Delta_m(z) + W_m(z)f_2(z)G(z)[1 + \mu \Delta_m(z)]u - W_m(z)f_2(z)G_0(z) [1 + \mu \Delta_m(z)] u + \mu \Delta_a(z)u. \quad (143)$$

Next, it is true that

$$y = W_m(z) [\phi^T \omega + r] - \mu \Delta_m(z)W_m(z)[\theta_1^{*T} f_1(z)u + \theta_3^* u + \theta_5^* v] + W_m(z)f_2(z)G(z)u - W_m(z)f_2(z)G_0(z) [1 + \mu \Delta_m(z)] u + \mu \Delta_a(z)u. \quad (144)$$

Replacing (1) into (144), it follows

$$y = W_m(z) [\phi^T \omega + r] - \mu \Delta_m(z)W_m(z)[\theta_1^{*T} f_1(z)u + \theta_3^* u + \theta_5^* v] + W_m(z)f_2(z)[G_0(z)[1 + \mu \Delta_m(z)] + \mu \Delta_a(z)]u - W_m(z)f_2(z)G_0(z)[1 + \mu \Delta_m(z)]u + \mu \Delta_a(z)u. \quad (145)$$

Besides,

$$y = W_m(z) [\phi^T \omega + r] - \mu \Delta_m(z)W_m(z)[\theta_1^{*T} f_1(z)u + \theta_3^* u + \theta_5^* v] + W_m(z)f_2(z)\mu \Delta_a(z)u + \mu \Delta_a(z)u. \quad (146)$$

Rearranging (146), it follows

$$y = W_m(z) [\phi^T \omega + r] + \mu \Delta_1(z)u + \mu \Delta_2(z)v, \quad (147)$$

where

$$\Delta_1(z) = -\Delta_m(z)W_m(z)\theta_1^{*T}f_1(z) - \Delta_m(z)W_m(z)\theta_3^* + W_m(z)f_2(z)\Delta_a(z) + \Delta_a(z), \quad (148)$$

and

$$\Delta_2(z) = -\Delta_m(z)W_m(z)\theta_5^*. \quad (149)$$

As the reference model output is $y_m = W_m(z)r$, subtracting it from both sides of (147) results in

$$e_1 = y - y_m = W_m(z)\phi^T \omega + \mu \eta_1 + \mu \eta_2, \quad (150)$$

where η_1 and η_2 are, respectively,

$$\eta_1 = \Delta_1(z)u, \quad (151)$$

$$\eta_2 = \Delta_2(z)v. \quad (152)$$

Furthermore, (150) can be written as

$$e_1 = W_m(z)\phi^T \omega + \mu \eta, \quad (153)$$

and

$$\eta = \eta_1 + \eta_2. \quad (154)$$

Note that (126) was obtained in (150) and, consequently, in (153).

It is also true that

$$\theta^T W_m(z)\omega - W_m(z)\theta^T \omega = \phi^T W_m(z)\omega - W_m(z)\phi^T \omega. \quad (155)$$

Adding e_1 in both sides of (155), it follows

$$e_1 + \theta^T W_m(z)\omega - W_m(z)\theta^T \omega = \phi^T W_m(z)\omega - W_m(z)\phi^T \omega + e_1. \quad (156)$$

Considering (153) and rewriting (156), it also follows

$$e_1 + \theta^T W_m(z)\omega - W_m(z)\theta^T \omega = \phi^T W_m(z)\omega - W_m(z)\phi^T \omega + W_m(z)\phi^T \omega + \mu \eta, \quad (157)$$

or yet,

$$e_1 + \theta^T W_m(z)\omega - W_m(z)\theta^T \omega = \phi^T W_m(z)\omega + \mu \eta. \quad (158)$$

In view of the control law, (158) can be rewritten as

$$e_1 + \theta^T \zeta + W_m(z)r = \phi^T \zeta + \mu \eta, \quad (159)$$

and ζ is a regressor vector, according to

$$\zeta = W_m(z)\omega. \quad (160)$$

Moreover, (159) can be also expressed as

$$\epsilon = e_1 + \theta^T \zeta + W_m(z)r = \phi^T \zeta + \mu \eta, \quad (161)$$

or as

$$\epsilon = e_1 + \theta^T \zeta + y_m = \phi^T \zeta + \mu \eta. \quad (162)$$

Hence, the augmented error ϵ can be written as

$$\epsilon = \phi^T \zeta + \mu \eta. \quad (163)$$

■

Remark 5: $\mu \eta$ is not measurable or observable. Hence, (163) cannot be used to implement the adaptive control structure, but it can be considered for mathematical purposes. However, observe that (162) is analogous to

$$\epsilon = y + \theta^T \zeta, \quad (164)$$

which is measurable and is used in the adaptive control algorithm for implementation.

6.2 | Boundedness of closed-loop signals

One of the adaptation algorithm terms presented in (22), which has paramount importance for controller robustness, is the majorant signal, $m(k)$, as this term limits the perturbation of $\eta(k)$, ensuring that all signals in the closed loop are bounded.

Lemma II: The following closed-loop signals are bounded by a constant, or else it can be said that they belong to the space L_∞ .

$$(i) \frac{|y(k)|}{m(k)} \in L_\infty$$

$$(ii) \frac{|u(k)|}{m(k)} \in L_\infty$$

$$(iii) \frac{|\eta(k)|}{m(k)} \in L_\infty$$

$$(iv) \frac{||\omega(k)||}{m(k)} \in L_\infty$$

$$(v) \frac{||\zeta(k)||}{m(k)} \in L_\infty$$

Proof: For (i) and (ii), it follows that

$$\bar{m}^2(k) = m^2(k) + \zeta^T(k)\Gamma\zeta(k), \quad (165)$$

and

$$m(k+1) = \delta_0 m(k) + \delta_1(1 + |u(k)| + |y(k)|). \quad (166)$$

Thus, the sign limitation proof is given as

$$\frac{|y(k)|}{m(k)} = \frac{|y(k)|}{\delta_0 m(k-1) + \delta_1(1 + |u(k-1)| + |y(k-1)|)}, \quad (167)$$

or,

$$\frac{|y(k)|}{m(k)} \leq c_2 + \epsilon(k), \quad (168)$$

where c_2 is a positive constant and $\epsilon(k)$ is a term that decays exponentially to zero when $k \rightarrow \infty$. That is, $\frac{|y(k)|}{m(k)}$ is limited. Or

yet, $\frac{|y(k)|}{m(k)} \in L_\infty$.

In a similar way,

$$\frac{|u(k)|}{m(k)} = \frac{|u(k)|}{\delta_0 m(k-1) + \delta_1(1 + |u(k-1)| + |y(k-1)|)}, \quad (169)$$

or,

$$\frac{|u(k)|}{m(k)} \leq c_3 + \epsilon(k), \quad (170)$$

where c_3 is a positive constant and $\epsilon(k)$ is a term that decays exponentially to zero when $k \rightarrow \infty$. Then, $\frac{|u(k)|}{m(k)}$ is limited. Or,

$\frac{|u(k)|}{m(k)} \in L_\infty$.

Regarding (iii), $\eta = \Delta(z)u$ is a stable and strictly proper transfer function, as presented in Section II. Therefore, as $u(k)$ is limited, $\eta(k)$ is also limited. Besides,

$$\frac{|\eta(k)|}{m(k)} = \Delta(z) \frac{|u(k)|}{\delta_0 m(k-1) + \delta_1(1 + |u(k-1)| + |y(k-1)|)}, \quad (171)$$

or yet,

$$\frac{|\eta(k)|}{m(k)} \leq c_4 + \epsilon(k), \quad (172)$$

being c_4 is a positive constant. In this way, $\frac{|\eta(k)|}{m(k)} \in L_\infty$.

Considering (iv), as $\omega(k)^T = [\omega_1(k)^T \ \omega_2(k)^T \ y(k) \ u(k)]^T$ where ω_1^T e ω_2^T are auxiliary vectors of order $(n-1)$, $u(k)$ and $y(k)$ are limited, as presented in (i) and (ii), respectively. Therefore, the proof follows with

$$\frac{||\omega(k)||}{m(k)} = \frac{||\omega(k)||}{\delta_0 m(k-1) + \delta_1(1 + |u(k-1)| + |y(k-1)|)}, \quad (173)$$

or yet,

$$\frac{||\omega(k)||}{m(k)} \leq c_5 + \epsilon(k), \quad (174)$$

where c_5 is a positive constant. Then, $\frac{||\omega(k)||}{m(k)} \in L_\infty$.

The proof of (v) is given knowing $\zeta = W_m(z)\omega$, where $W_m(z)$ is a stable transfer function and $\omega(k)$ is limited and proved earlier in (iv). Therefore,

$$\frac{||\zeta(k)||}{m(k)} = \frac{||\omega(k)||}{\delta_0 m(k-1) + \delta_1(1 + |u(k-1)| + |y(k-1)|)}, \quad (175)$$

$$\frac{||\zeta(k)||}{m(k)} \leq c_6 + \epsilon(k), \quad (176)$$

where c_6 is a positive constant. Then, $\frac{||\zeta(k)||}{m(k)} \in L_\infty$. ■

Author Contributions

Guilherme Vieira Hollweg: Conceptualization, Formal analysis, Investigation, Writing—Original draft;
Paulo Jefferson Dias de Oliveira Evald: Methodology, Formal analysis, Writing—review & editing;
Hilton Abílio Grundling: Formal analysis, Mathematical review, Writing—review;
Wencong Su: Funding acquisition, Project administration.

Conflicts of interest/Competing interests

There are no conflicts of interest.

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Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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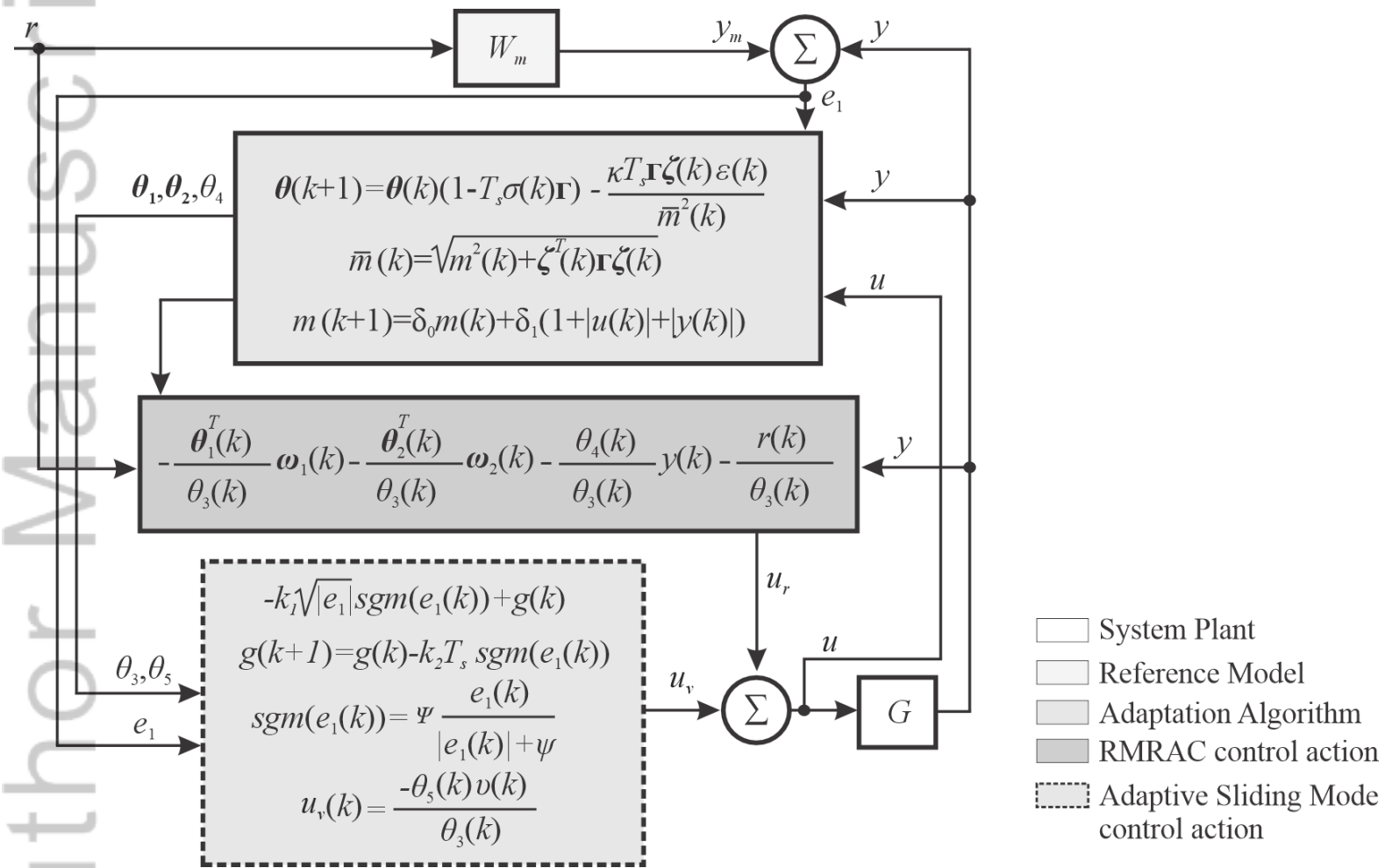


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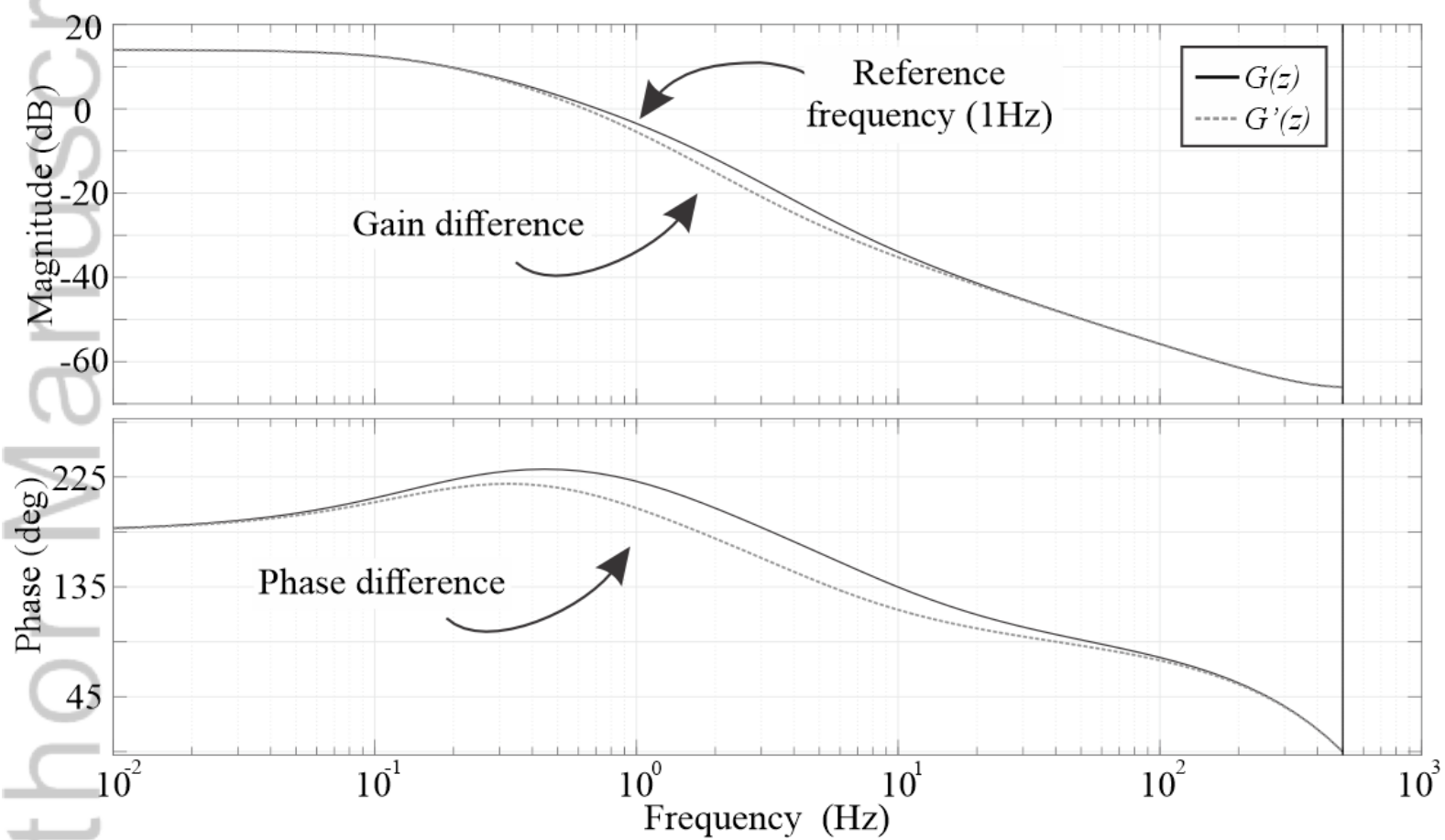


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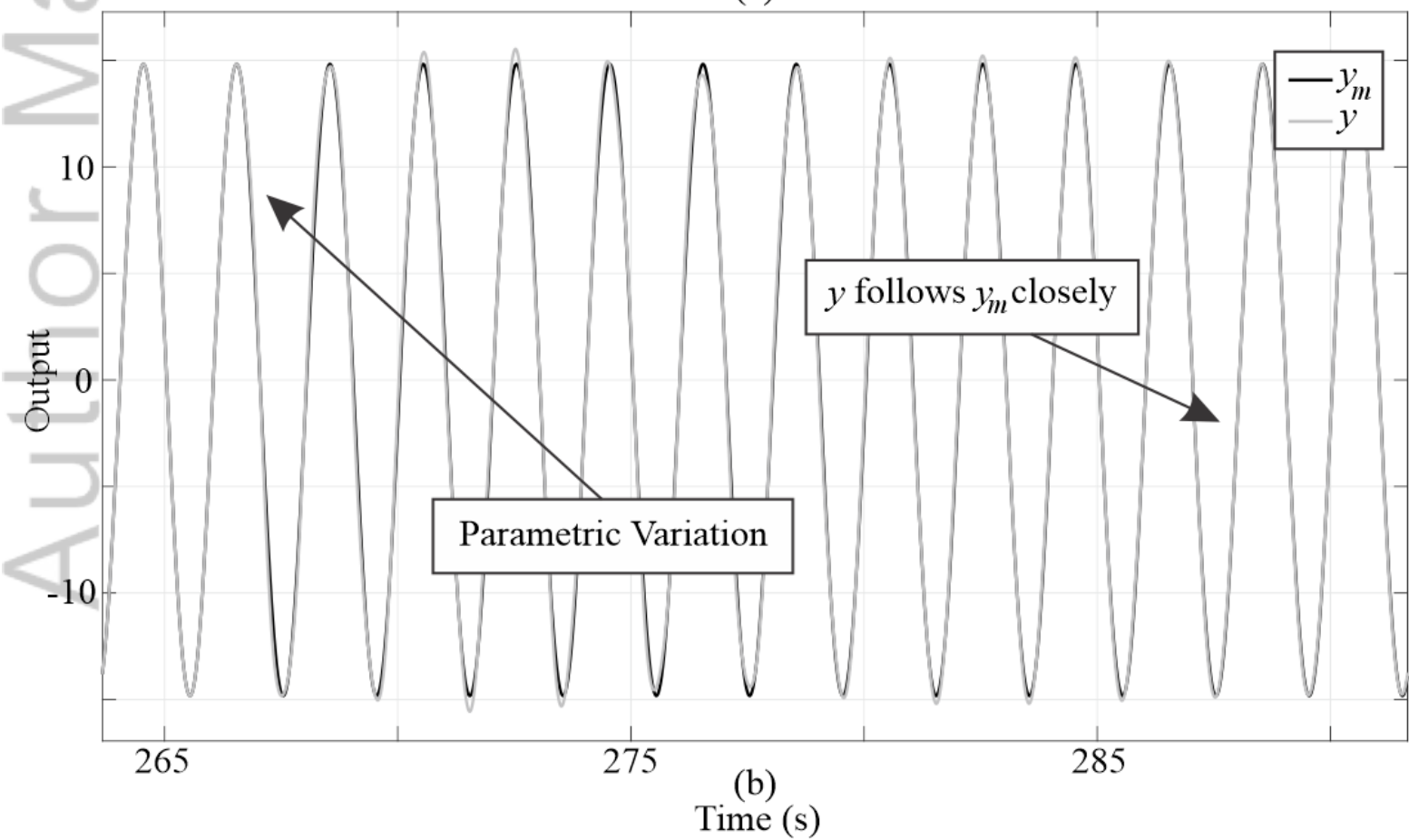
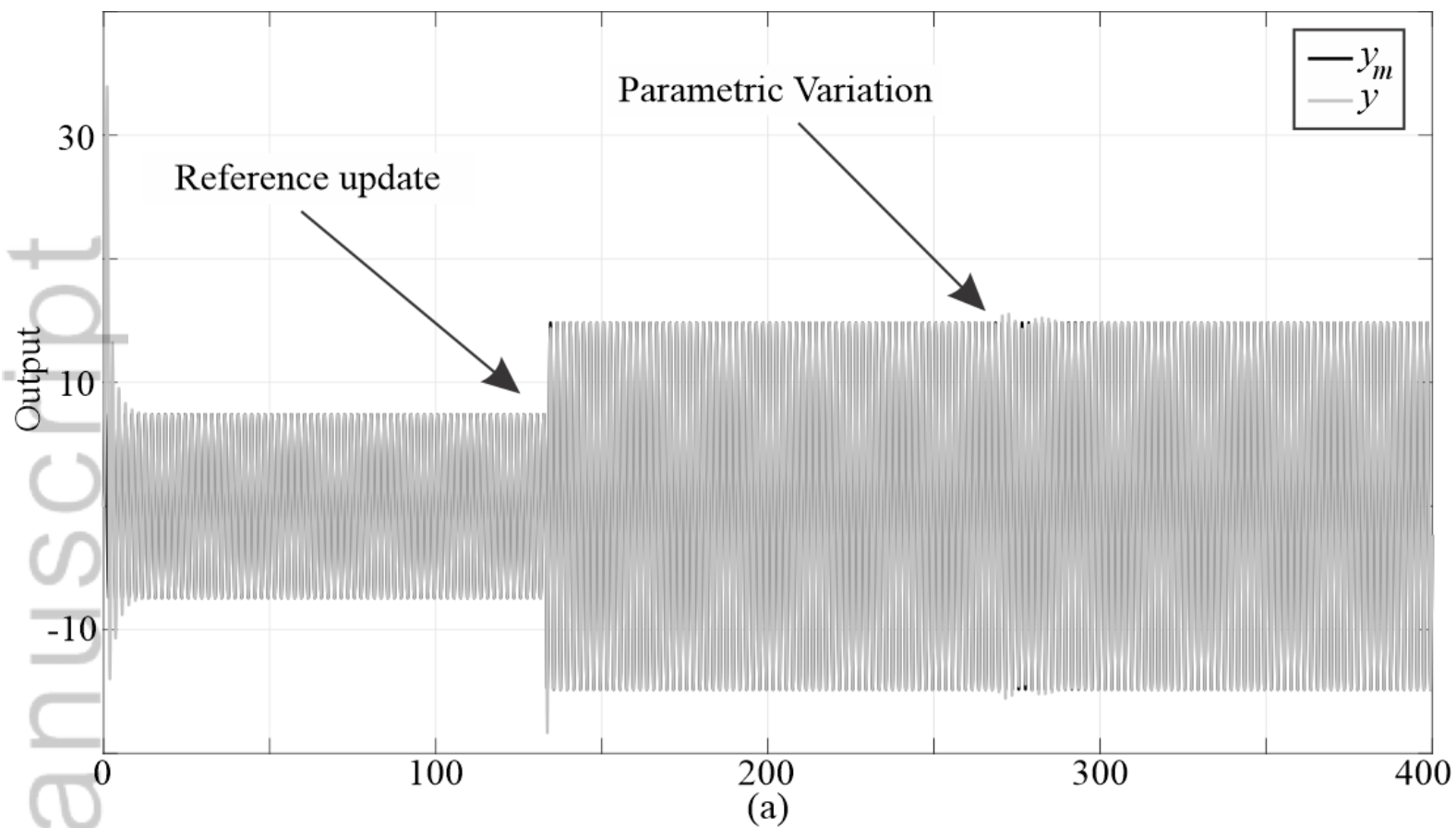


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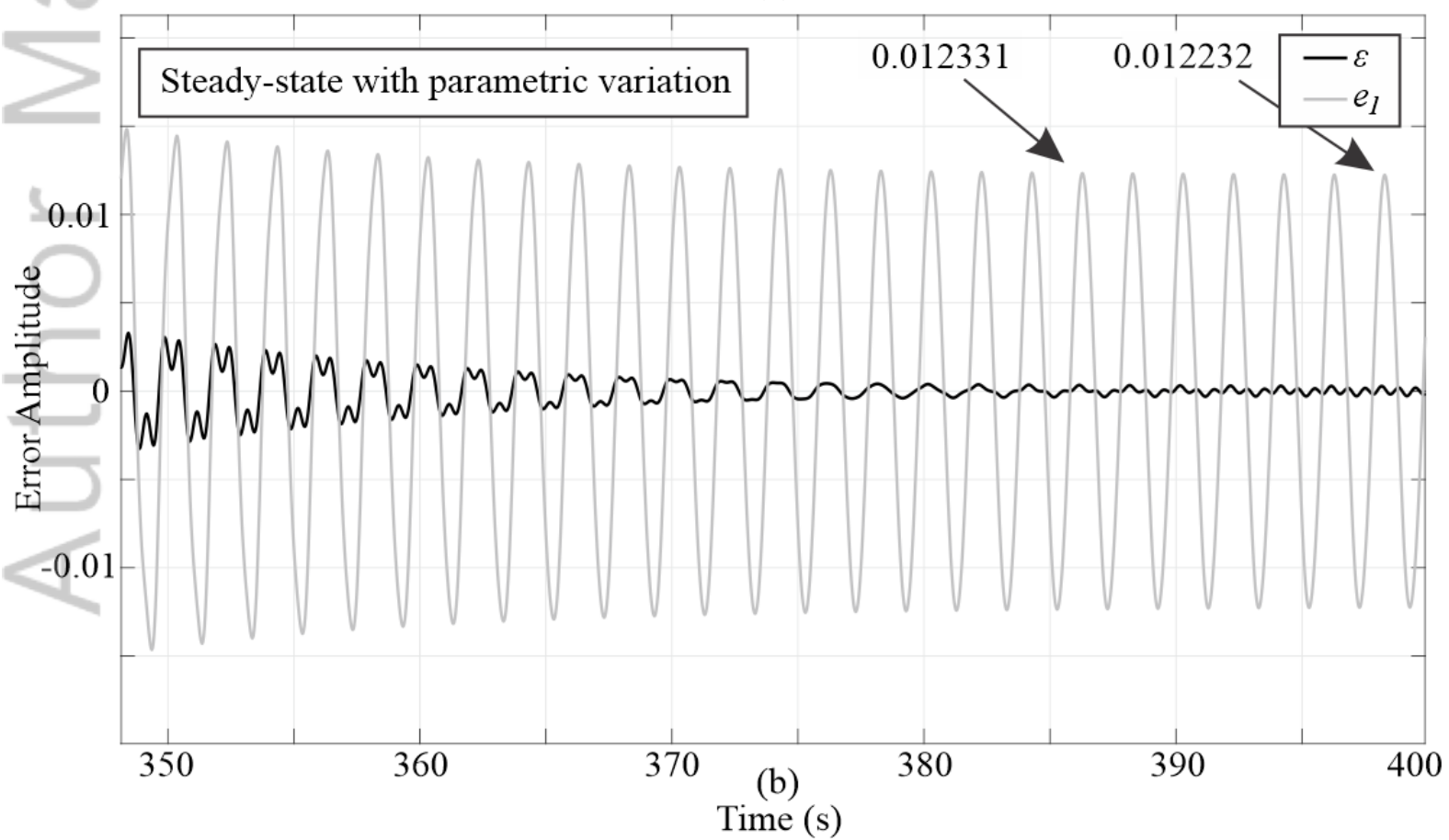
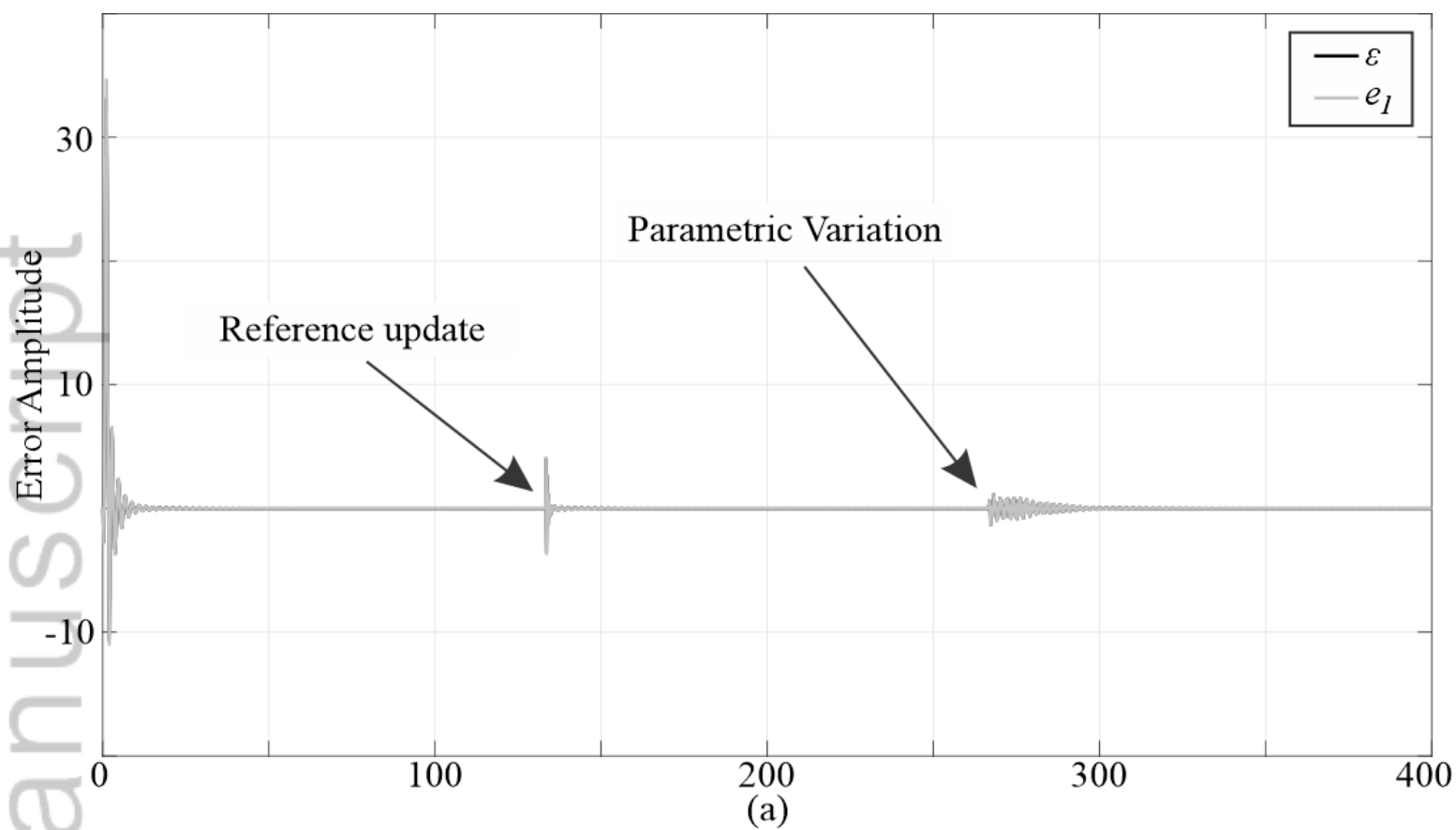


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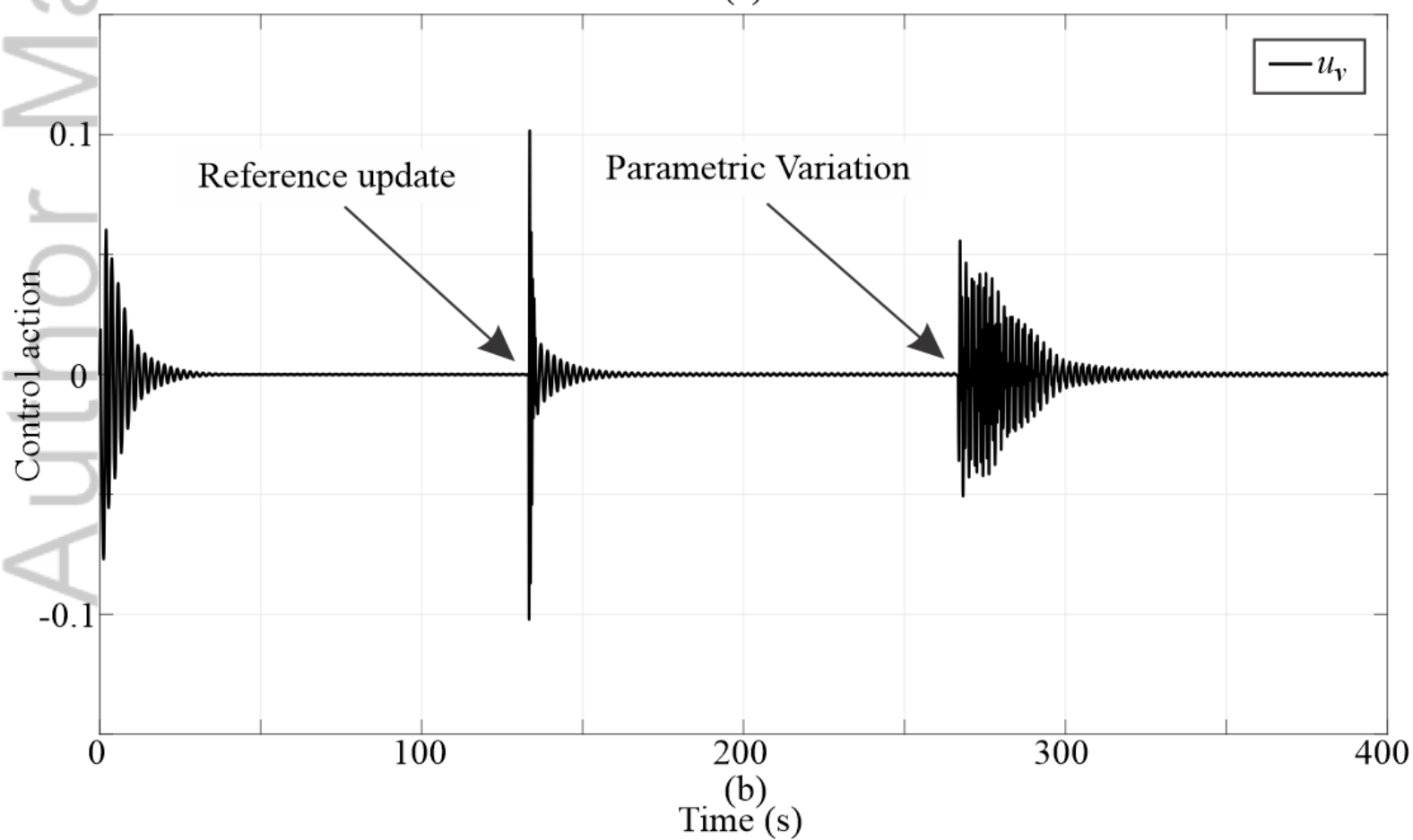
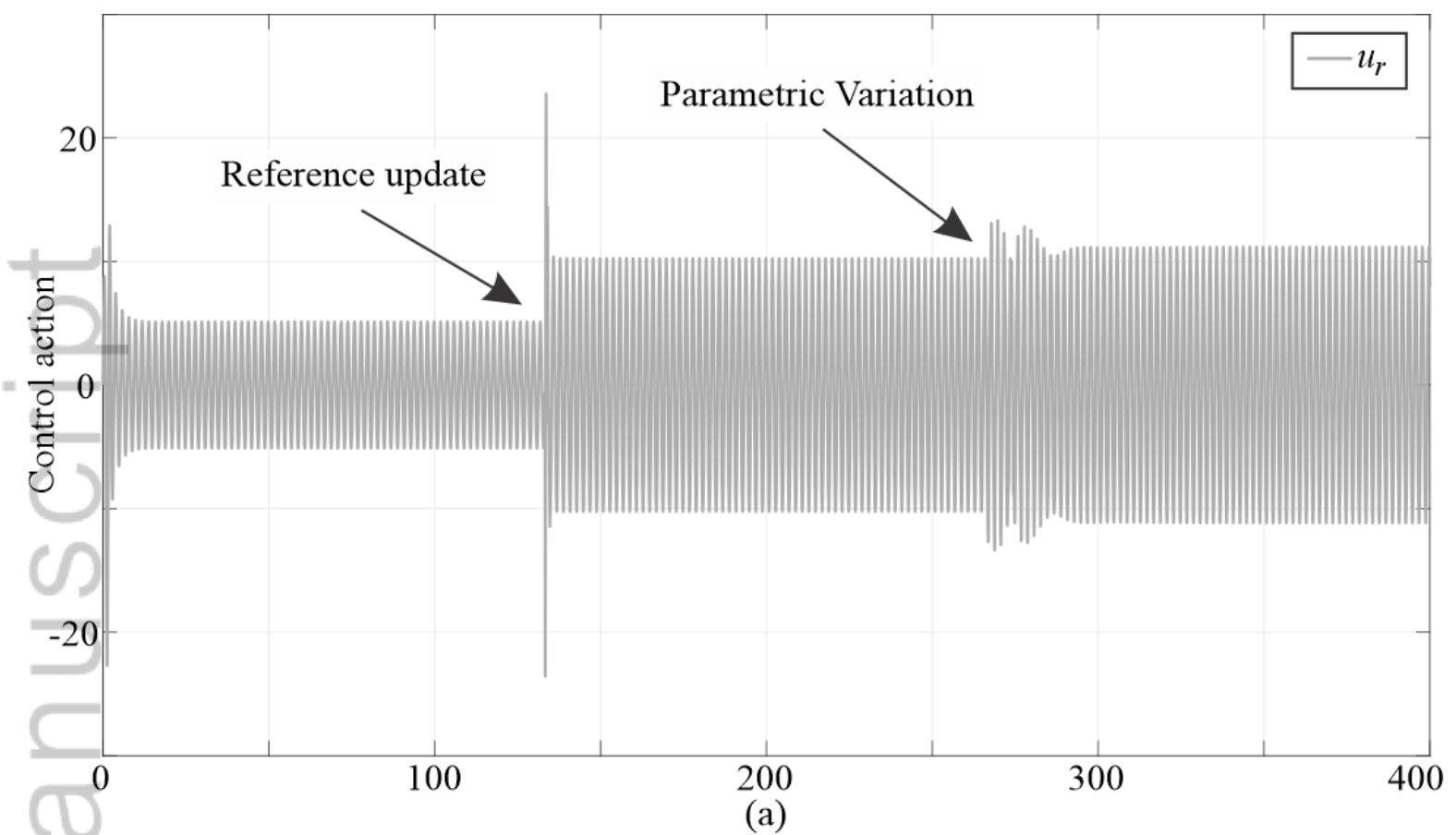


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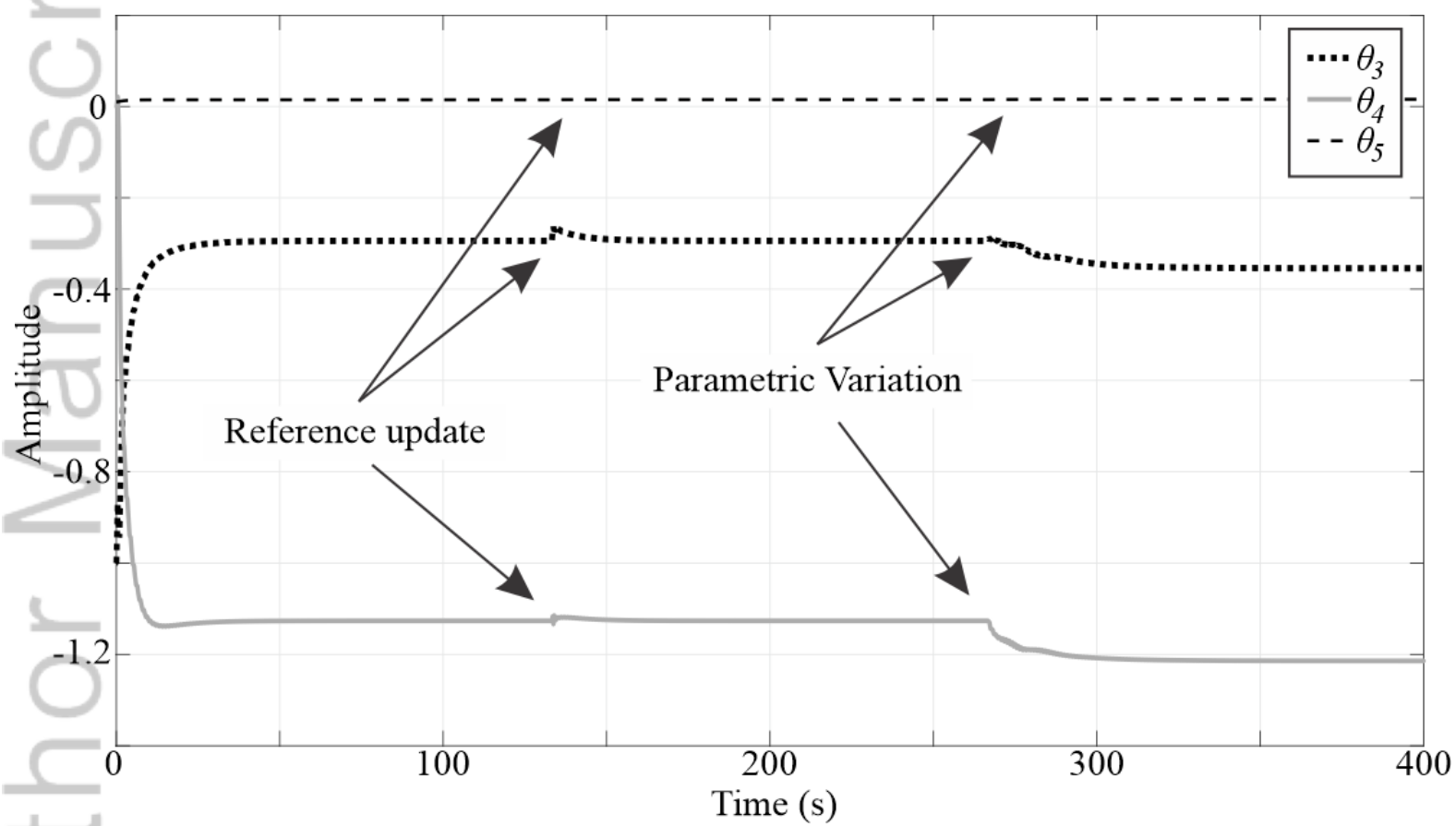


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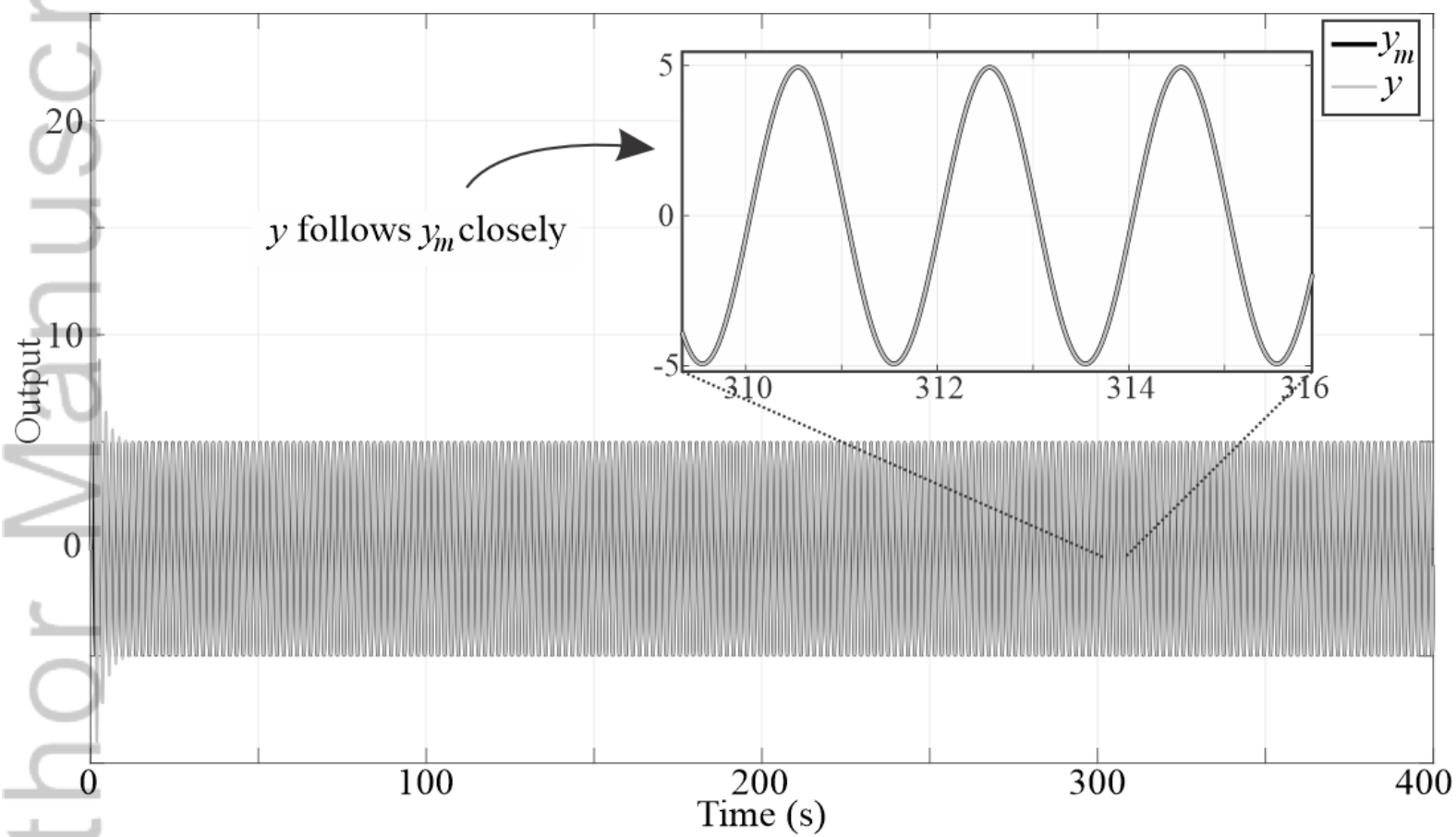


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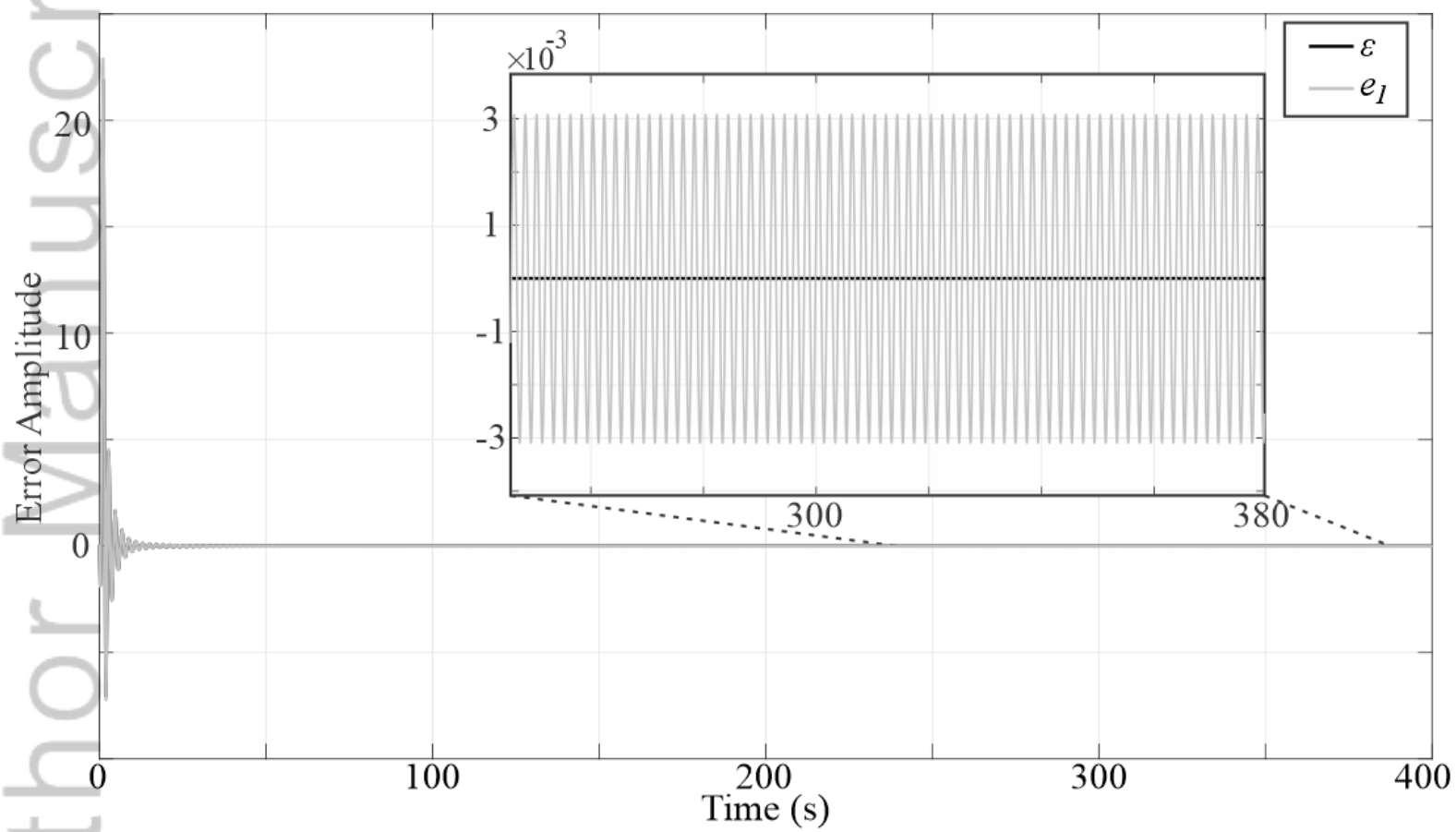


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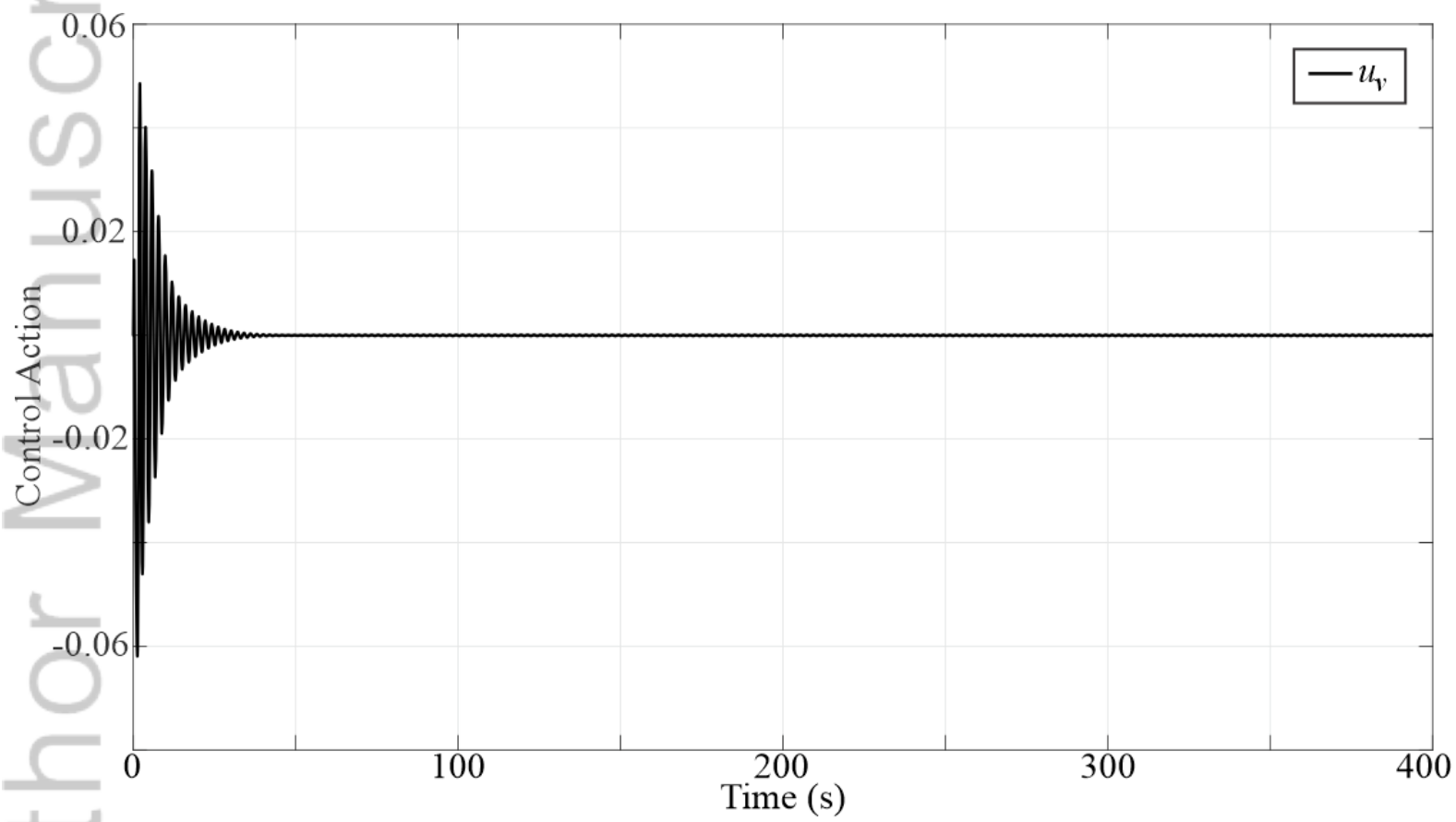


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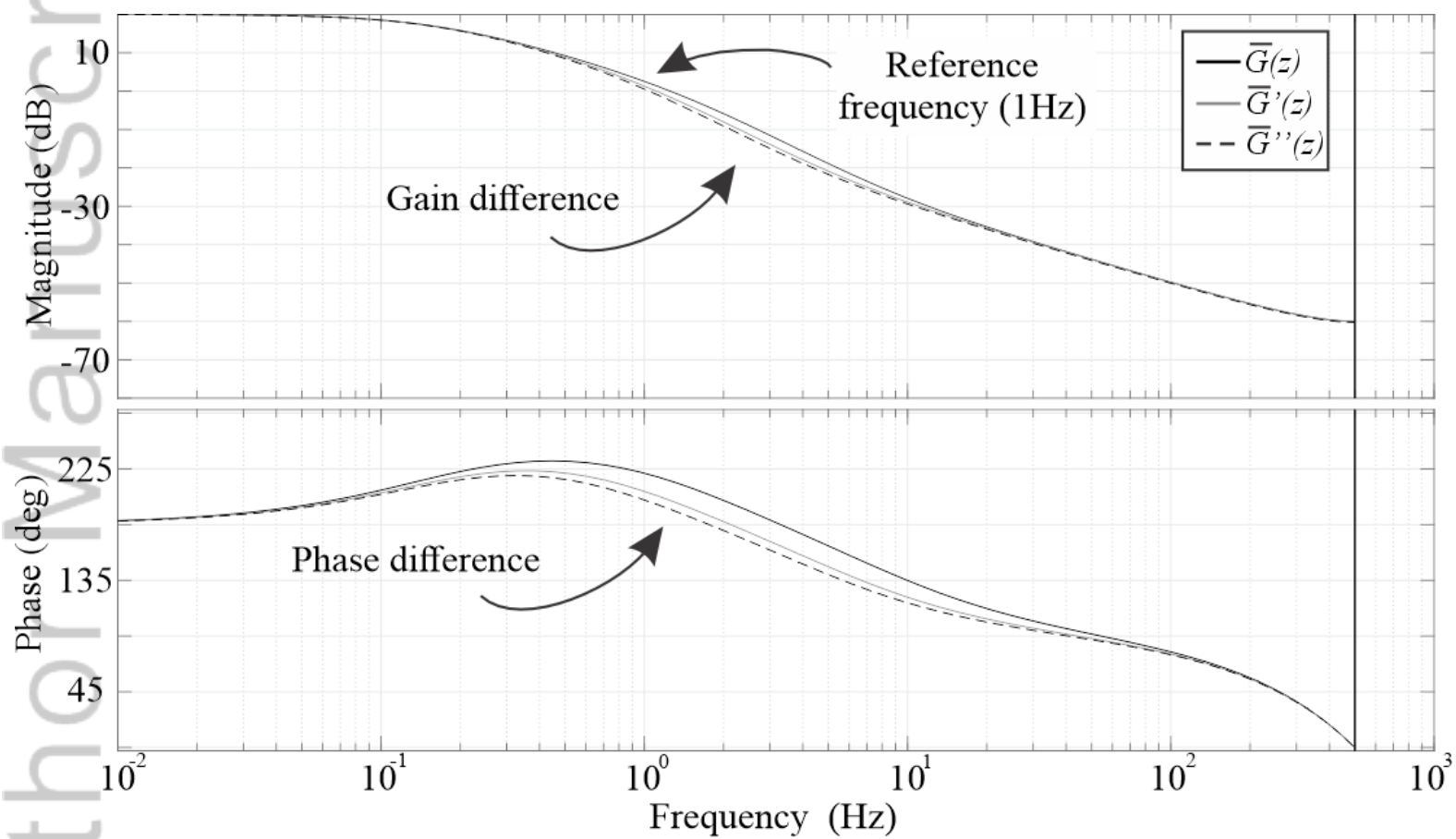


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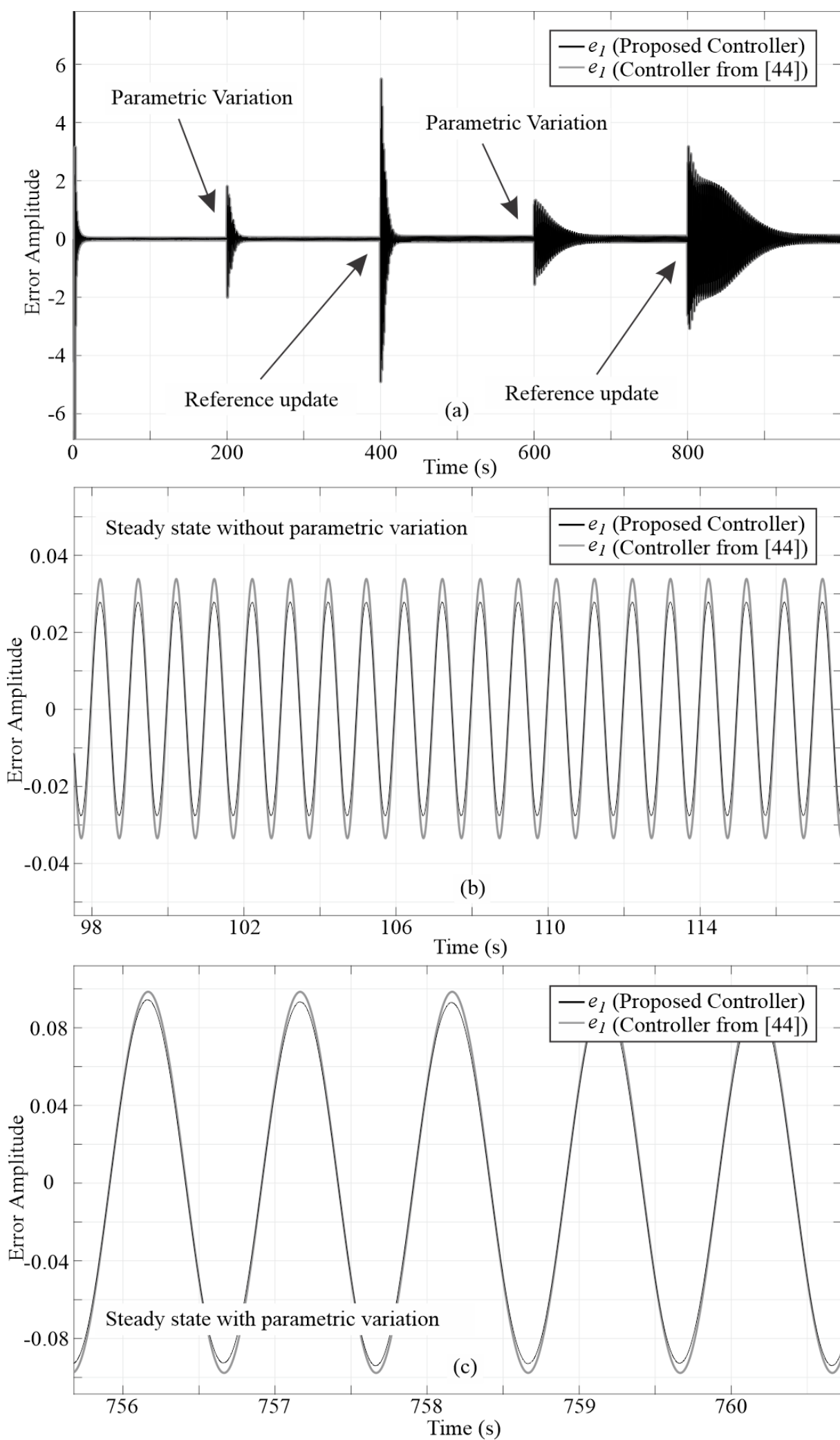


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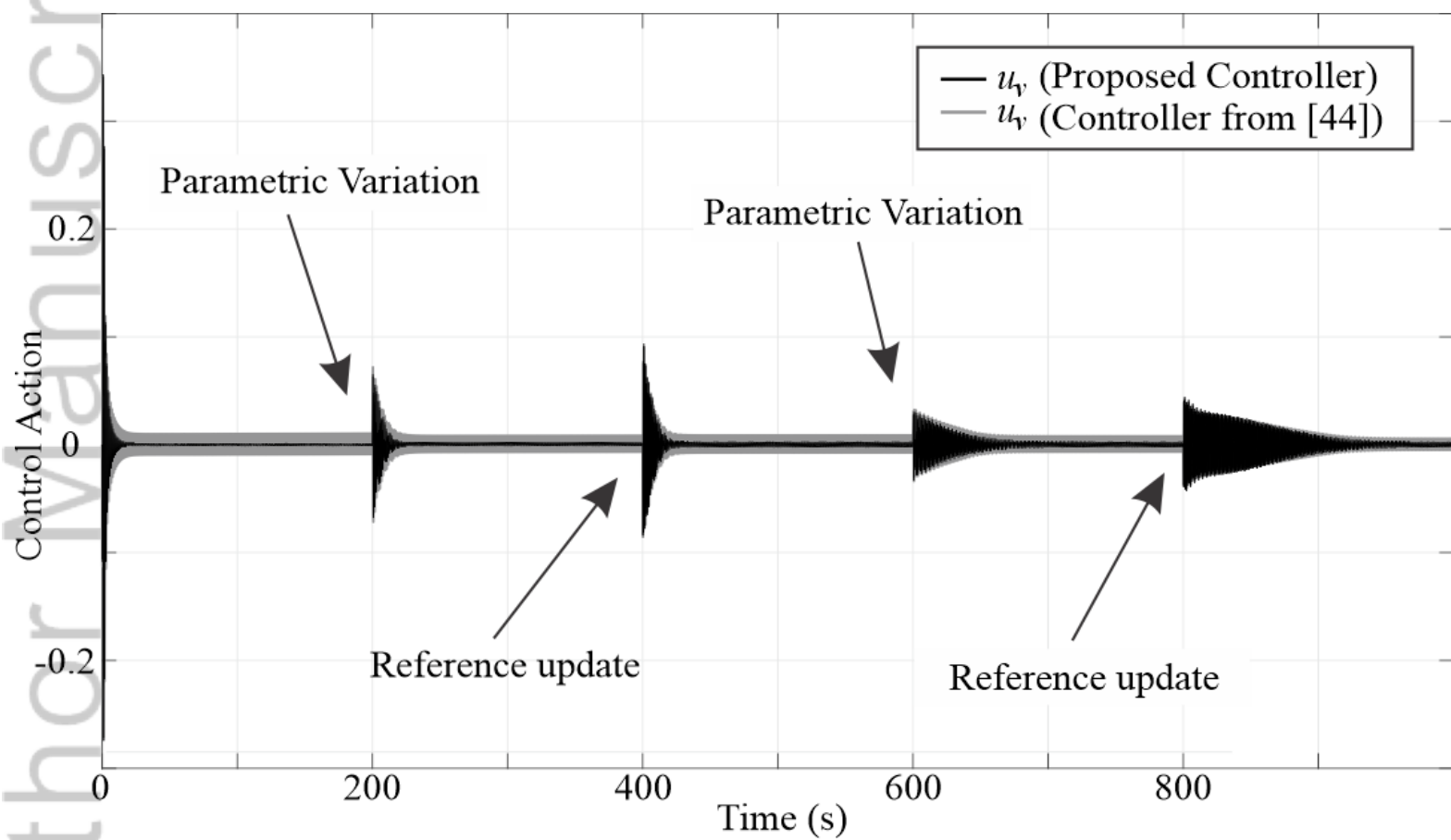


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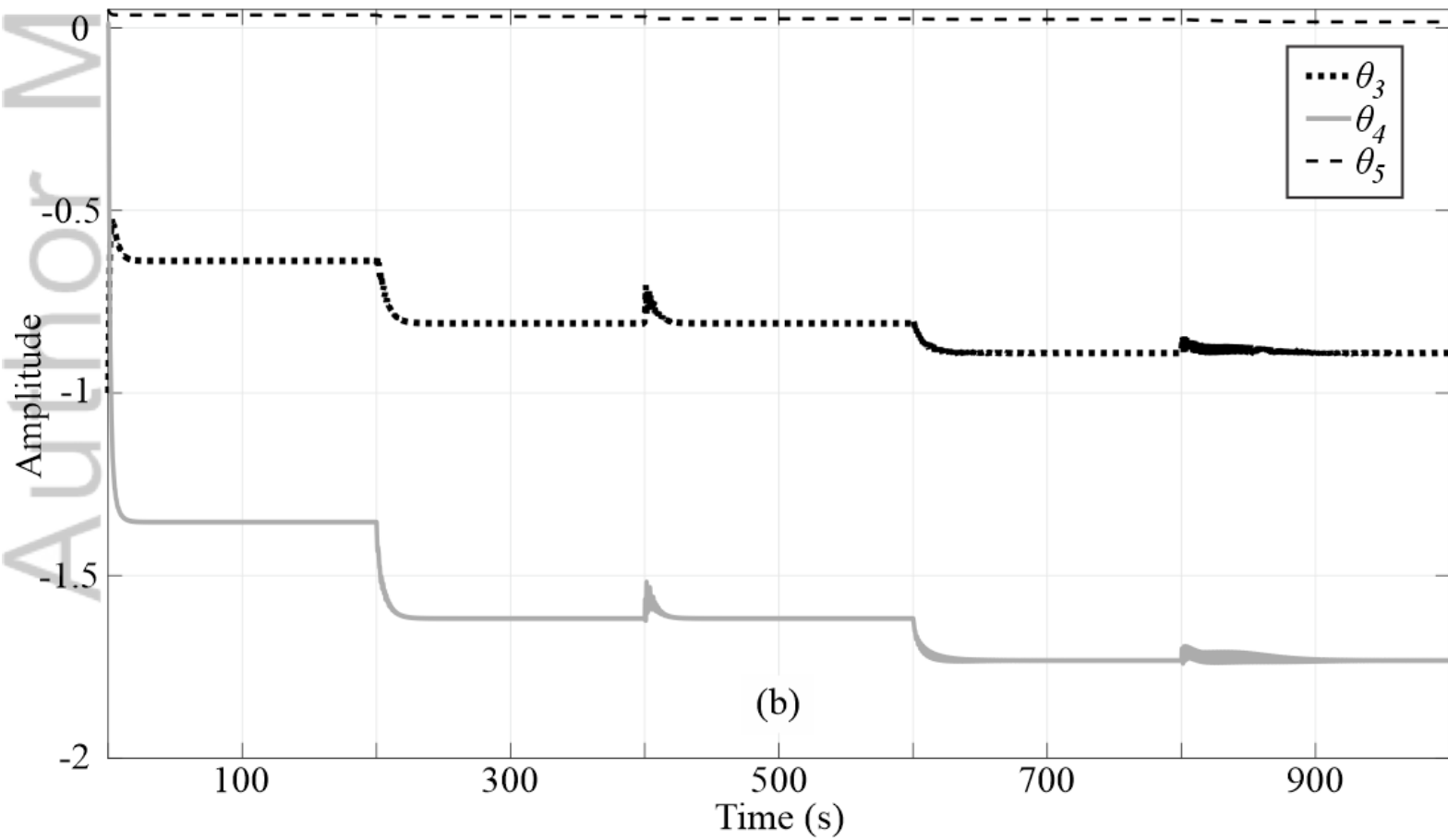
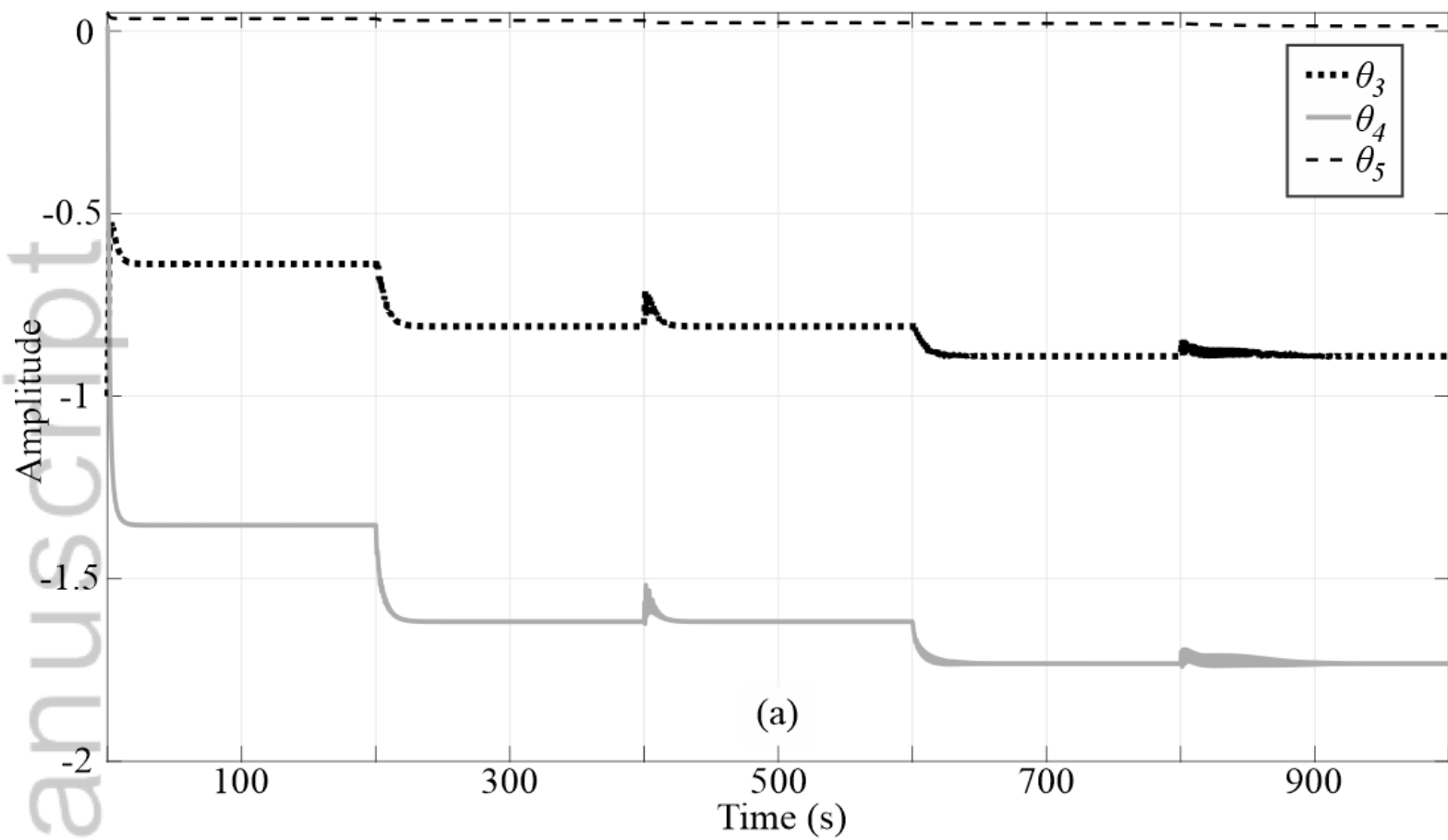
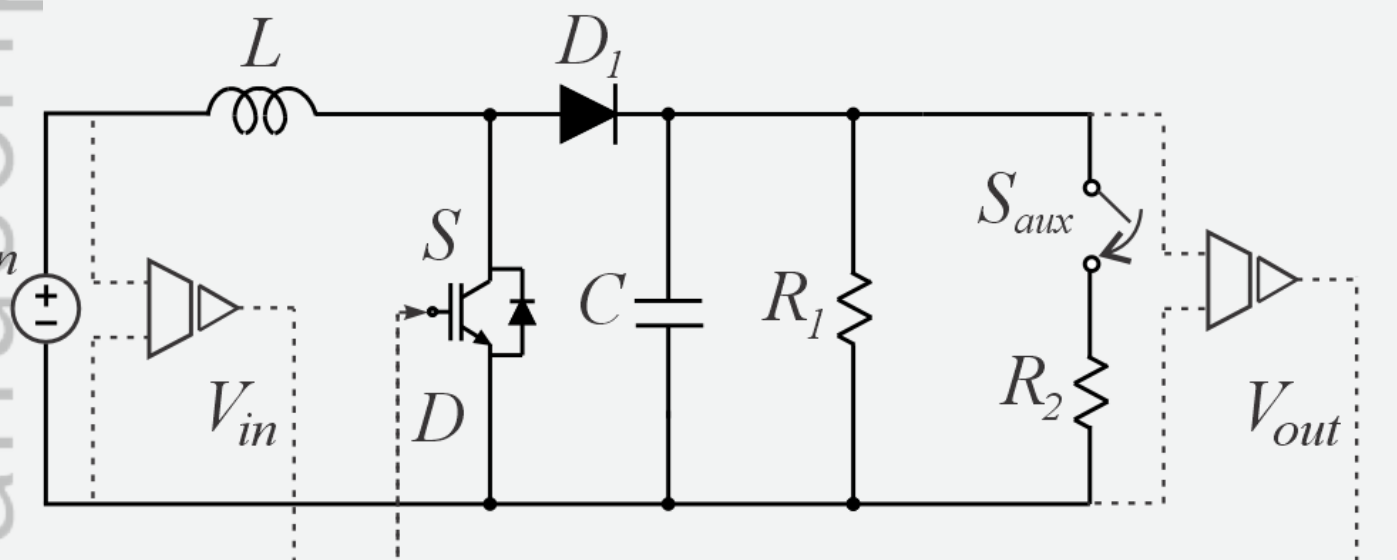


Fig_13.png

Digital Signal
Processor



Hardware-in-the-Loop

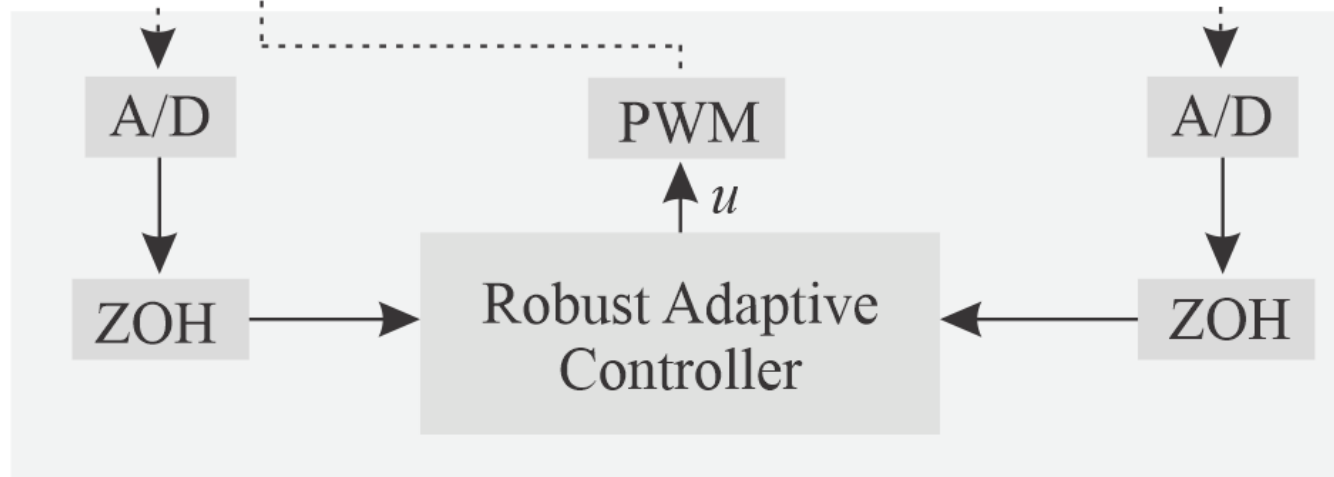


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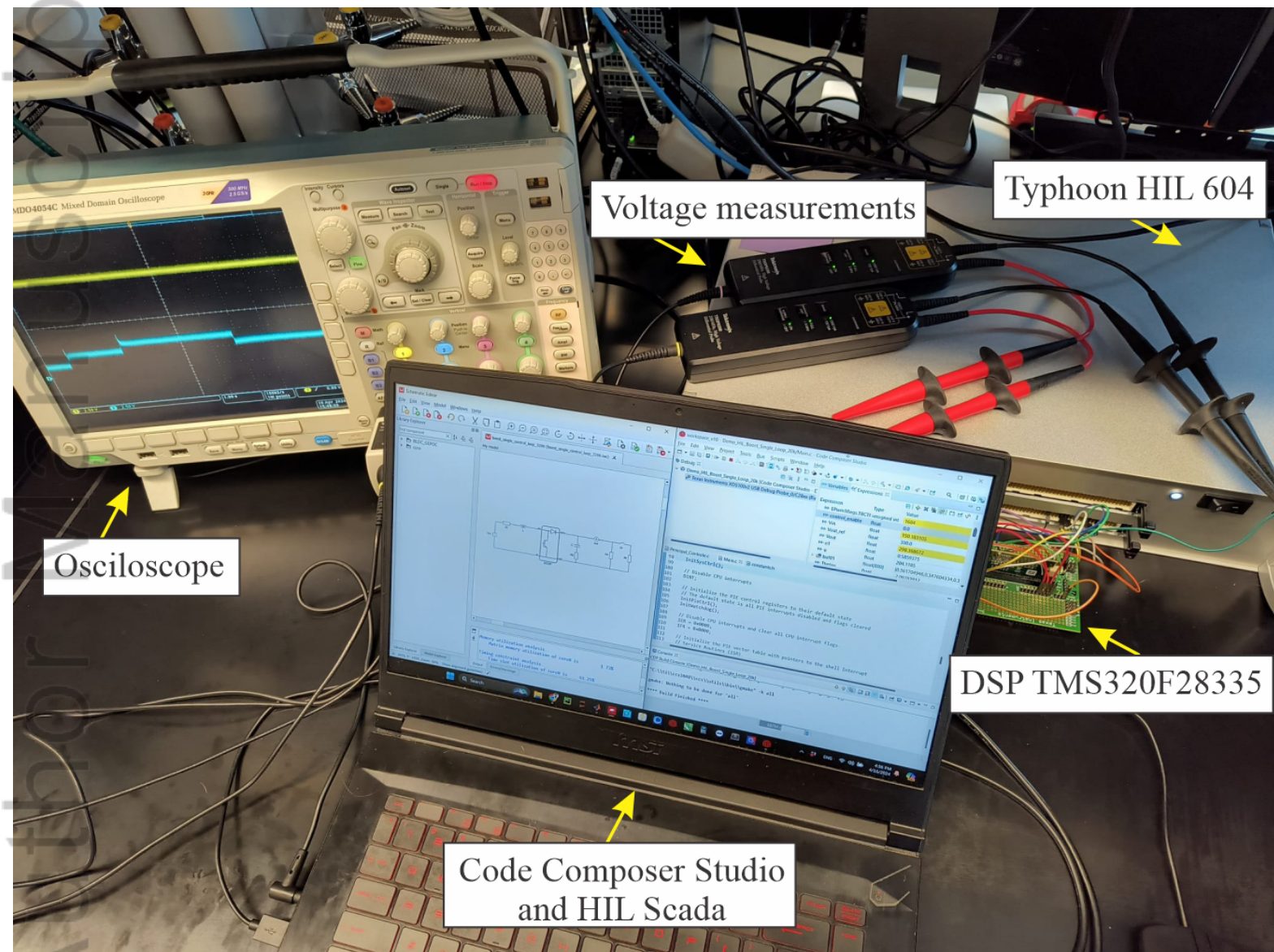
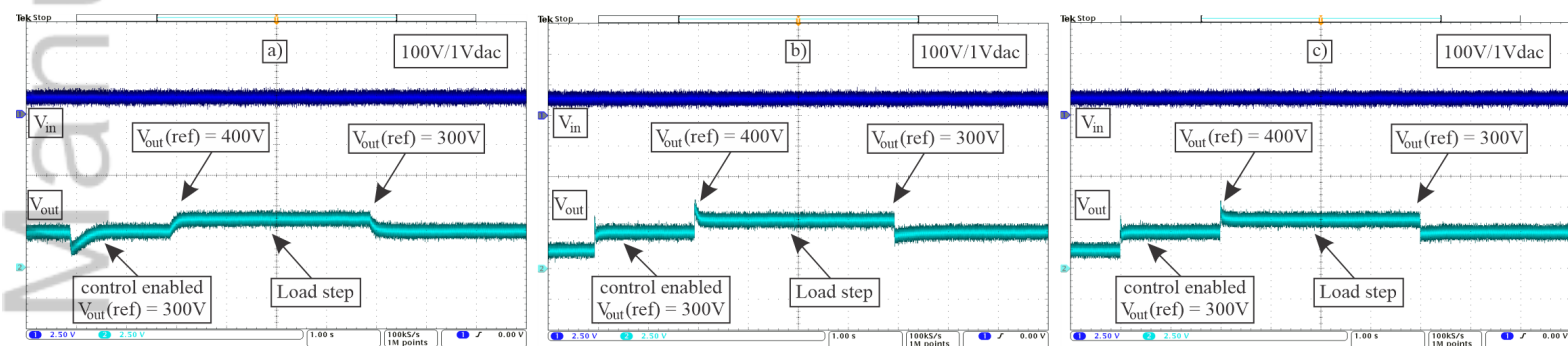


Fig_15.png



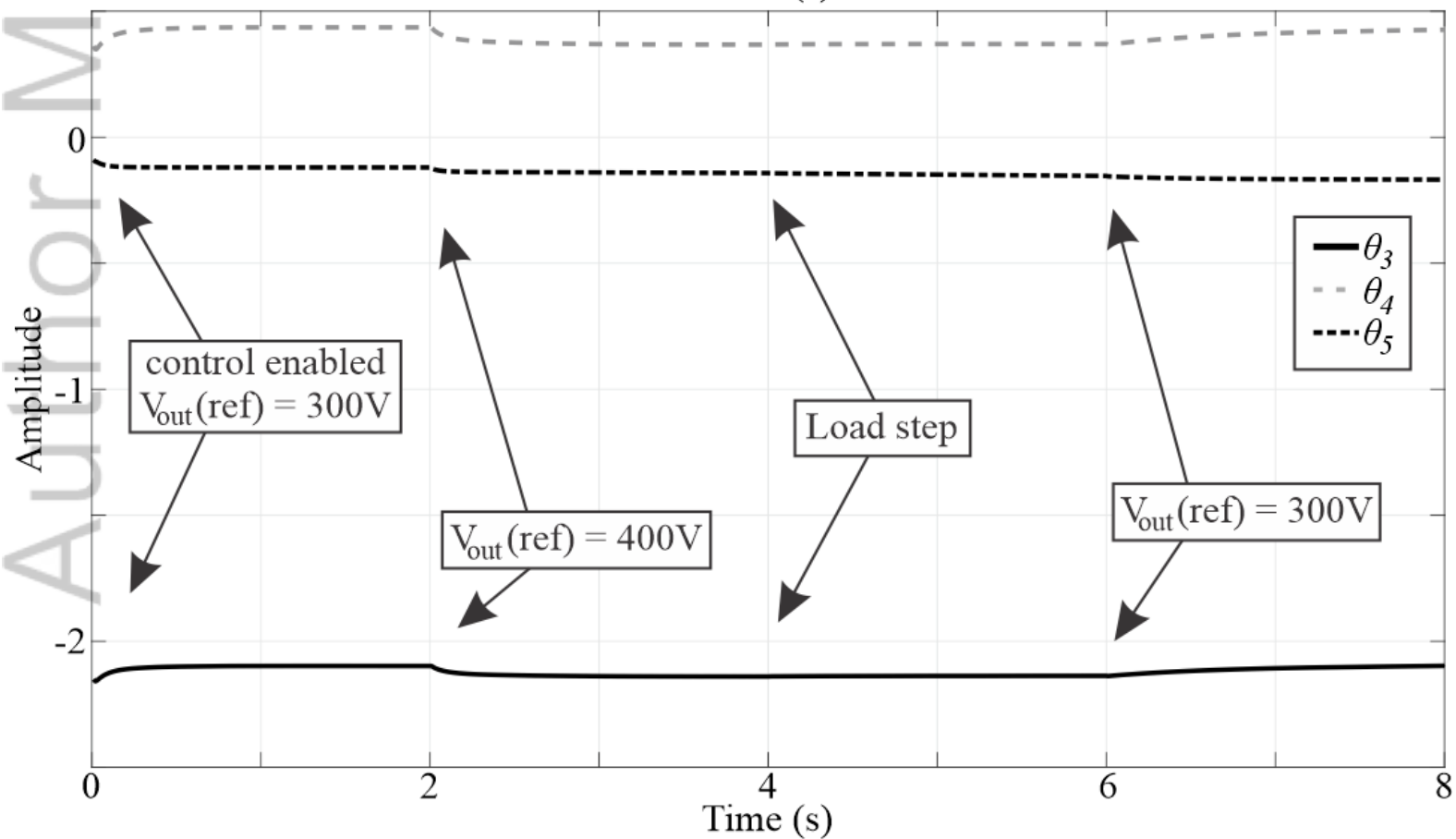
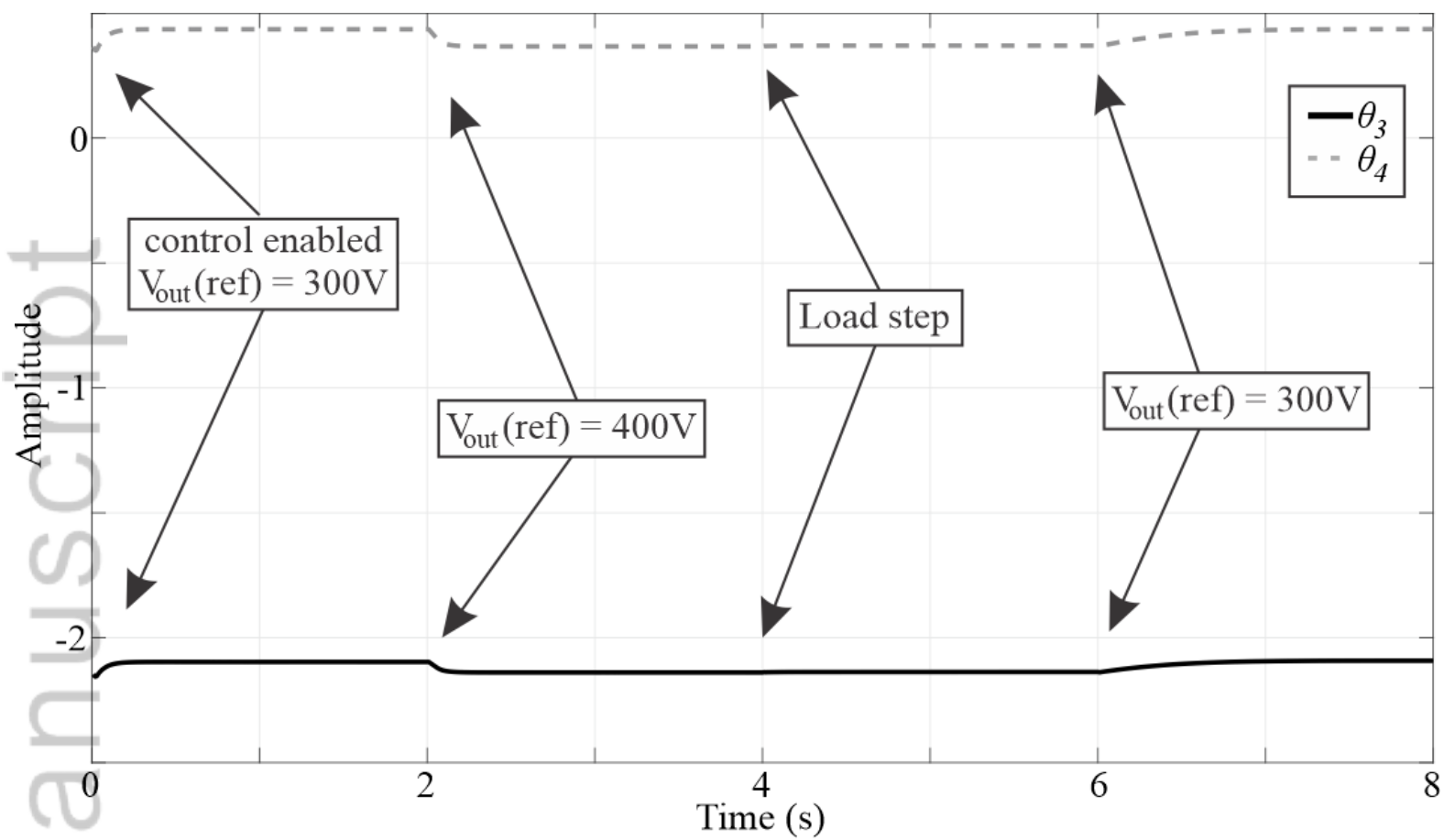


Fig_17.png