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STABILITY OF THE CONDUCTION REGIME

OF NATURAL CONVECTION

1. FLOW IN AN INCLINED SLOT

2. FLOW IN A NARROW ROTATING ANNULUS

by

Seppo Antero Korpela

**A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mechanical Engineering)
in The University of Michigan
1972**

Doctoral Committee:

**Professor Vedat S. Arpaci, Chairman
Associate Professor George S. Springer
Assistant Professor Charles M. Vest
Professor James O. Wilkes**

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NOMENCLATURE

a	thermal diffusivity
c	scaled wave speed of disturbance
D	operator, $\partial/\partial x$
$\bar{f}$	base flow distribution for vertical slot
Fr	Froude number, $\Omega^2 R/g$
g	gravity
Gr	Grashof number, $g \beta \Delta T L^3 / \nu^2$
h	aspect ratio
H	height of the slot/annulus
k	magnitude of the wave vector $k = \sqrt{\alpha^2 + \beta^2}$
L	width of the slot/annulus
N	number of terms in the truncated series, eq.(33, 45, 65)
m	azimuthal wave number
p	pressure
P	pressure above the hydrostatic value
Pr	Prandtl number, ν/a
R	Reynolds number, mean radius of the annulus
Ra	Rayleigh number, GrPr
S	parameter defined in eq.(68)
t	time
T	temperature
T_h	temperature of the hot wall
T_c	temperature of the cold wall
T_0	reference temperature $(T_h + T_c)/2$
ΔT	temperature difference $T_h - T_c$

Ta	Taylor number $2\Omega L^2/\nu$
u	velocity component in x-direction
U	reference velocity, $g\delta\Delta TL^2/\nu$
v	velocity component in y-direction
$\hat{V}$	velocity vector
w	velocity component in z-direction
X	coefficient vector, eq.(49)
Y	coefficient vector, $\left\{ \frac{dX}{dt}, X \right\}^T$

Greek Letters

α	wave number in z-direction
β	wave number in y-direction
χ	coefficient of thermal expansion
δ	angle of inclination from the vertical
ξ	x-component of vorticity
η	ratio R_I/R_0 of the radii of the annulus
θ	temperature above T_0
κ	parameter defined in eq.(36)
λ	parameter defined in eq.(35)
μ	parameter defined in eq.(35)
ν	kinematic viscosity
ρ	density, parameter defined in eq.(36)
σ_i	non-dimensional wave speed
Φ	stream function
χ	phase angle of disturbance, $\tan^{-1} \frac{\beta}{\alpha}$
Ψ	stream function
Ω	rotation rate

INTRODUCTION

In this thesis an investigation on the stability of natural convection 1) in an inclined slot, and 2) in a vertical annulus rotating about its axis is reported. Considering the basic state of the inclined slot to be a vertical one and defining the aspect ratio h , for both cases 1 and 2, to mean the height to width ratio of the space occupied by the fluid, the analysis is limited to those cases in which h is large. The space filled with the fluid is considered closed by the inclusion of a top and a bottom so that across any section the net mass flux at steady state is zero. In this type of a system and for large h the basic fluid motions which develop, if a small temperature difference is imposed between the vertical side walls, are characterized by antisymmetry with fluid rising adjacent to the hot wall and sinking next to the cold one. Away from the ends the flow is parallel and corresponds to what is known as the conduction regime according to the way the heat is transferred across the gap. This regime, according to the experiments of Elder (1965), occurs for $Ra/h < 300$ (Ra is the Rayleigh number of the flow).

By further considering the gap width of the annulus to be small compared to the mean radius, the two problems become closely related. In fact, the ratio of the centrifugal force to the gravity force acting on the fluid rotating in the annulus can be identified with the tangent of the angle of inclination in the analysis of the tilted slot.

PART I
STABILITY OF NATURAL CONVECTION
IN AN INCLINED SLOT

CHAPTER I

LITERATURE SURVEY

1.1 Vertical Slot

The stability of the conduction regime in a vertical slot was first analyzed by Gershuni (1953). By using polynomials of smallest possible order as approximating functions, he constructed the Galerkin equations, which, as an algebraic eigenvalue problem, lead to a solution representable in terms of physically meaningful parameters. The problem was reconsidered later by his colleagues Birikh (1967) and Rudakov (1967a, b) who, due to the introduction of high speed digital computers, found it feasible to consider higher order approximating functions.

Birikh treated the zero Prandtl number case, which, due to the vanishing of the temperature perturbations, leads to a reduction in the order of the differential equations. The calculations of the eigenvalues showed the critical one to be real so that the instability was concluded to set in as stationary convection.

Rudakov confirmed Birikh's results in the zero Prandtl number limit and extended them to include the thermal effects. In his first paper (1967a), he solved the problem by considering a regular perturbation scheme treating the Grashof number wave number product as a small parameter. By this method he obtained the eigenvalue spectra when the Prandtl number was fairly small. Cases up to $Pr = 1.26$ were considered. In the second paper he solved the problem by Galerkin method and was able to extend his results up to $Pr = 10$ beyond which he indicated poor convergence. Also, in this range he found the onset of instability to be stationary.

Similar results at about the same time were obtained by Vest and Arpaci (1969), who also provided additional insight into the phenomenon through their experimental investigation. From their photographs the flow is seen to be two-dimensional and cellular with the observed wave number in good agreement with the calculated one.

An asymptotic analysis of this problem has been given by Gotoh and Satoh (1966). It was limited to small Prandtl and wave numbers and led to a critical Grashof number of 77,000, with the instability setting in as traveling waves. Compared to a critical Grashof number of 7932 given by Vest and Arpaci for stationary stability the traveling wave mode can be ruled out as a physically significant case. Very recently two new papers appeared along the same line by Gotoh and Ikeda (1971a, b). The first dealt with the inviscid equations for which neutrally stable solutions of zero phase speed were found in the limit of zero Prandtl number. The second treated the stability of unsteady natural convection due to a sudden application of a temperature difference between the plates in a quasi-steady manner and the relationship of this to the convection in a finite slot was explored. At infinite time the solution of the base flow corresponds to the conduction regime of the steady convection and their results, confirming the earlier results of Gotoh and Satoh, remain therefore in contradiction to the stationary stability curve. They do point out, however, a possibility of a new branch of the critical curve which might give a lower Grashof number. An indication was made by them that the analysis of this would follow in a subsequent paper.

In the same volume appeared also the experimental work of Oshima (1971). He has made detailed measurements of the velocity and temperature distributions both in the horizontal and vertical directions with water

as the test fluid. No comparison from his results can be made in regard to stability of the conduction regime, however, because of the low aspect ratio of his slot. At the aspect ratio of 20 and $Ra = 20,000$ the flow does not appear to deviate significantly from the solution of the base flow of the conduction regime but by increasing the temperature so that $Ra = 347,000$ sufficient distortion has taken place in the velocity distribution so that the flow remains stable. The critical Rayleigh number corresponding to conduction regime is approximately 60,000. As these results definitely correspond to the finite slot there seems then to be no discrepancy with the results of Rudakov and it appears safe to conclude that the instability for $Pr < 10$ sets in as cellular convection. This is not the case, however, when Prandtl number is large.

Gill and Kirkham (1970) were first to consider the large Prandtl number case. The relation

$$Gr_c = 9.4 \times 10^3 Pr^{-1/2}, \quad 1$$

giving an asymptotic formula for the critical Grashof number as the Prandtl number tends to infinity summarizes their results. They found the critical wave number to be 2.5 and the phase speed equal to 8.5×10^{-3} . The fact that the phase speed is greater than the maximum base flow velocity ($= 8.03 \times 10^{-3}$) is particularly important since this results in a shift of the turning point of the equations to the outside of the flow domain, so that if the problem were to be solved by an asymptotic method proper account of this would have to be taken.

Although, as has been discussed, the behavior at both limits of small and large Prandtl numbers are known, there exists a region from $Pr = 10$ to say $Pr = 1000$ which remains to be investigated. In addition to the

results presented in this dissertation for this range, Baxi (1971) in conjunction with a problem of natural convection with a periodic base flow and Gözüm (1972) in connection with a problem on the effect of viscoelasticity on natural convection have obtained results in this range of parameters.

1.2 Inclined Slot

Gershuni (1955) was also first to study the stability of the conduction regime in an inclined slot. This early work suffered in accuracy for the same reason as his accompanying paper on the convection in the vertical slot. A more thorough study was undertaken later by Birikh et al (1968) and by Gershuni and Zhukhovitskii (1969). In the first of these papers, transverse disturbances, i.e. disturbances which lead to cells with an axis perpendicular to the direction of the base flow, are considered. The calculation of the critical Grashof number was performed for Prandtl numbers of 0.2, 1.0 and 5.0 which fall, as the calculations show, into the range in which the instability sets in as stationary convection.

In the paper by Gershuni and Zhukhovitskii the analysis is extended to include three-dimensional disturbances. In a manner of Koppel (1964) and Gage and Reid (1968) a transformation is introduced which enables the generation of the stability boundary from the knowledge of the behavior of the corresponding two-dimensional problem. The transformations are such that the two-dimensional disturbances whether transverse or longitudinal, turn out to always be the more critical ones when the angle of inclination is varied from -90° to $+90^\circ$, with the angle zero corresponding to vertical orientation. Since the instability sets in through stationary states for the Prandtl numbers considered, the resulting flow

patterns are cells. These change their orientation from transverse to longitudinal abruptly at a certain angle which varies with the Prandtl number. For nearly horizontal plates heated from below the neutral curves for all disturbances tend to the same limit giving the well known degeneracy of the Benard problem. It is worth noting, however, that any small inclination from the horizontal leads to the occurrence of longitudinal cells first. This was also shown by Liang and Acrivos (1970) for the slightly inclined Benard problem with free boundaries. Using the angle of inclination from the horizontal as a small parameter they reached the above conclusion by solving the first order perturbation equations.

The first experiments dealing with heat transfer between inclined plates were performed by DeGraaf and Van der Held (1953). Although their primary interest was the determination of heat transfer across the gap, they also made visual observations of the flow patterns which developed. They reported the occurrence of longitudinal cells for $\delta = 70^\circ - 80^\circ$, but observed that the flow became turbulent for angles less than 60° .

Recently Kurzweg (1970) published a note on the inclined slot problem. Using silicone oil of $Pr = 480$ he observed that the flow became unstable as longitudinal cells for angles $\delta > 15^\circ$. No results were given for small angles of inclination.

Additional experimental results have been provided by Hart (1971) who investigated primarily the effect of aspect ratio on the stability of this flow. His experimental results for water, $Pr = 6.7$ and air, $Pr = 0.71$ show a good agreement with his theoretical results as well as with the calculations of Gershuni and Zhukhovitskii (1969).

To date, the stability of the conduction regime for the inclined slot has not been published when $Pr > 6.7$. In this dissertation the results for the case of positive angles are given in this regime.

CHAPTER II

THEORETICAL ANALYSIS

2.1 Basic Equations

Consider a slot as shown in Figure 1, inclined from the vertical by an angle δ , with the dimensions of height H and width L to be such that the aspect ratio $H/L \gg 1$. A temperature difference of ΔT is imposed between the sidewalls, considered isothermal, in such a manner that $\delta > 0$ corresponds to heating from below. A top and a bottom are included for the slot so that a parallel flow, driven by buoyancy forces, develops which has an antisymmetric velocity and temperature profile with respect to the middle plane of the slot.

The slot is assumed to contain a Newtonian fluid with constant properties except for density when it occurs in the gravity term. With respect to this term the equation of state of the fluid is approximated by the linear relation which also defines the coefficient of thermal expansion

$$\gamma = - \left(\frac{\partial \rho}{\partial T} \right)_p = - \frac{\rho - \rho_o}{T - T_o} . \quad 2$$

In this relation the subscript "o" denotes the reference state which is taken to be the hydrostatic condition. It is further assumed that viscous dissipation and the work of compression are negligible. Finally the temperature level is assumed to be low enough so that the radiant heat flux is negligible. This description of the fluid constitutes the so called Boussinesq approximation and leads to the following set of equations governing the fluid motions

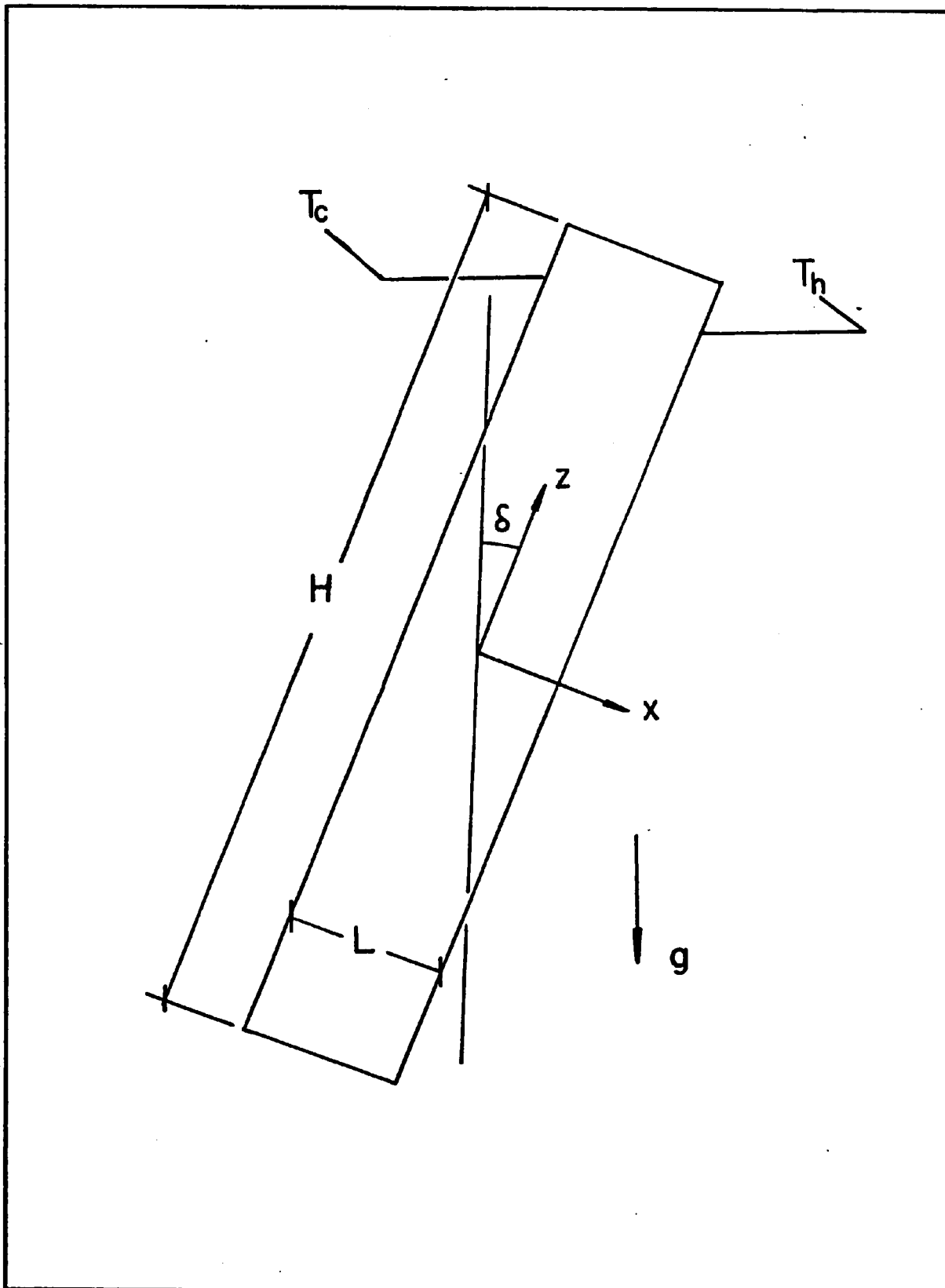


Figure 1. Inclined Slot.

$$\vec{\nabla} \cdot \vec{V} = 0,$$

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho_0} \vec{\nabla} P + [1 - \gamma(T - T_0)] g(\hat{i} \sin \delta - \hat{k} \cos \delta) + \nu \nabla^2 \vec{V}, \quad 3$$

$$\frac{DT}{Dt} = \alpha \nabla^2 T.$$

Noting that the hydrostatic problem is governed by the equation -

$$0 = -\frac{1}{\rho_0} \vec{\nabla} P_0 + g(\hat{i} \sin \delta - \hat{k} \cos \delta), \quad 4$$

the equations 3 can be brought into a somewhat more convenient form

$$\vec{\nabla} \cdot \vec{V} = 0,$$

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho_0} \vec{\nabla} (P - P_0) - \gamma(T - T_0) g(\hat{i} \sin \delta - \hat{k} \cos \delta) + \nu \nabla^2 \vec{V}, \quad 5$$

$$\frac{DT}{Dt} = \alpha \nabla^2 T.$$

2.2 Base Flow

That solution of the governing equations, which is experimentally observed to exist for small temperature differences is called the base flow. The stability analysis then addresses itself to the stability of this solution as a function of the parameters involved in the description of the equations of the motion.

Denoting the base flow quantities by a bar over them the following dependence on the coordinates is taken

$$\begin{aligned} \bar{u} &= 0, & \bar{v} &= 0, & \bar{w} &= \bar{w}(x), \\ \bar{\theta} &= \bar{\theta}(x), & \text{and} & & \bar{P} &= \bar{P}(x), \end{aligned} \quad 6$$

where the definitions

$$\bar{\Theta}(x) = \bar{T}(x) - T_0, \quad \bar{P}(x) = \bar{p}(x, z) - p_0(x, z) \quad 7$$

have been made. The pressure which is measured above the hydrostatic pressure is not a function of the z-coordinate even if the hydrostatic pressure is.

Substituting the definitions in equations 6 and 7 into equation 5 gives the system governing the base flow, namely

$$\begin{aligned} 0 &= -\frac{1}{\rho_0} \frac{d\bar{P}}{dx} - g\gamma \sin \delta \bar{\Theta}, \\ 0 &= g\gamma \cos \delta \bar{\Theta} + \nu \frac{d^2 \bar{w}}{dx^2}, \\ 0 &= \frac{d^2 \bar{\Theta}}{dx^2}. \end{aligned} \quad 8$$

The boundary conditions to be satisfied at the rigid walls are

$$\bar{w}(\pm L/2) = 0, \quad \bar{\Theta}(+L/2) = +\frac{\Delta T}{2}, \quad \bar{\Theta}(-L/2) = -\frac{\Delta T}{2} \quad 9$$

and the inclusion of the top and the bottom for the slot requires that

$$\rho_0 \int_{-L/2}^{+L/2} \bar{w}(x) dx = 0 \quad 10$$

The last equation gives a balance for the mass flux across a section of the slot and is written in a manner consistent with the Boussinesq approximation. Solving this system gives

$$\begin{aligned} \bar{\Theta}(x) &= \Delta T \left(\frac{x}{L} \right), \\ \bar{w}(x) &= \frac{g\gamma \Delta T L^2}{\nu} \frac{1}{6} \left[\frac{1}{4} \left(\frac{x}{L} \right) - \left(\frac{x}{L} \right)^3 \right] \cos \delta. \end{aligned} \quad 11$$

By replacing the variables $\bar{\Theta}$, $\bar{w}$ and x by the scaled quantities $\bar{\Theta} \Delta T$,

$\bar{w}$ $\bar{v}$, and x/L where,

$$\bar{v} = \frac{g \Delta T L^2}{\nu}, \quad 12$$

the solutions take the nondimensional forms

$$\begin{aligned} \bar{\theta}(x) &= x, \\ \bar{w}(x) &= \bar{f}(x) \cos \delta = \frac{1}{6} \left(\frac{1}{4} x - x^3 \right) \cos \delta \end{aligned} \quad 13$$

For the subsequent equations as in equation 13 no special symbols are used to distinguish the dimensional variables from the dimensionless ones since the meaning is always made clear by the presence or absence of dimensionless parameters in the equations.

2.3 Stability Formulation

The equations to be developed in this section are those which govern the spatial and temporal behavior of infinitesimal disturbances introduced into the base flow. It is convenient to write the equations in their non-dimensional forms which are obtained by scaling the spatial coordinates, velocity components, and temperature by L , U , and ΔT , respectively, as in the last section and by replacing the time, t and pressure, p by their scaled counterparts tL^2/ν and $P/\rho_0 U^2$.

The non-dimensional equations governing the fluid motions written in the component forms are then

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\ \frac{\partial u}{\partial t} + Gr \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) &= -Gr \frac{\partial P}{\partial x} - \sin \delta \theta + \nabla^2 u, \end{aligned}$$

$$\frac{\partial v}{\partial t} + Gr \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -Gr \frac{\partial p}{\partial y} + \nabla^2 v, \quad 14$$

$$\frac{\partial w}{\partial t} + Gr \left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -Gr \frac{\partial p}{\partial z} + \cos \delta \theta + \nabla^2 w,$$

$$\frac{\partial \theta}{\partial t} + Gr \left(u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} \right) = \frac{1}{Pr} \nabla^2 \theta,$$

where the dimensionless parameter

$$Gr = \frac{\sigma L}{\nu} \text{ is the Grashof number, } Pr = \frac{\nu}{\alpha} \text{ is the Prandtl number,} \quad 15$$

To study the initial evolution of infinitesimal disturbances the following linearization

$$\begin{aligned} u &= \epsilon u', & v &= \epsilon v', & w &= \bar{w} + \epsilon w', \\ \theta &= \bar{\theta} + \epsilon \theta' & \text{and} & & p &= \bar{p} + \epsilon p' \end{aligned} \quad 16$$

is carried out. By collecting terms which are of equal order in ϵ , the terms of order one give the equations for the base flow. From the terms which are $O(\epsilon)$ the following equations are obtained

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0,$$

$$\frac{\partial u'}{\partial t} + Gr \bar{w} \frac{\partial u'}{\partial z} = -Gr \frac{\partial p'}{\partial x} - \sin \delta \theta' + \nabla^2 u', \quad 17$$

$$\frac{\partial v'}{\partial t} + Gr \bar{w} \frac{\partial v'}{\partial z} = -Gr \frac{\partial p'}{\partial y} + \nabla^2 v',$$

$$\frac{\partial \psi'}{\partial t} + Gr \left(u' \frac{d\bar{\psi}}{dx} + \bar{\psi} \frac{\partial \psi'}{\partial z} \right) = -Gr \frac{\partial p'}{\partial z} + \cos \delta \Theta' + \nabla^2 \psi',$$

$$\frac{\partial \Theta'}{\partial t} + Gr \left(u' \frac{d\bar{\Theta}}{dx} + \bar{\Theta} \frac{\partial \Theta'}{\partial z} \right) = \frac{1}{Pr} \nabla^2 \Theta',$$

which with the boundary conditions

$$u' = \psi' = \bar{\psi}' = \Theta' = 0 \quad \text{at} \quad x = \pm 1/2$$

pose an eigenvalue problem. The equations 17 in turn are seen to accept solutions of the form

$$q'(x, y, z, t) = q(x, t) \exp(i\alpha z + i\beta y), \quad 18$$

where the wave numbers α and β are taken as real so that the solutions remain bounded as z and y tend to infinity. With the aid of the relation 18 and defining $D \equiv \partial/\partial x$ the equations 17 become

$$Du + i(\beta \bar{\psi} + \alpha \bar{\psi}) = 0,$$

$$\frac{\partial u}{\partial t} + i\alpha Gr \bar{f} \cos \delta u = -Gr Dp - \sin \delta \Theta + [D^2 - (\alpha^2 + \beta^2)]u,$$

$$\frac{\partial \bar{\psi}}{\partial t} + i\alpha Gr \bar{f} \cos \delta \bar{\psi} = -i\beta Gr p + [D^2 - (\alpha^2 + \beta^2)]\bar{\psi},$$

19

$$\begin{aligned} \frac{\partial \bar{\psi}}{\partial t} + i\alpha Gr \bar{f} \cos \delta \bar{\psi} + Gr D \bar{f} \cos \delta u = & -i\alpha Gr p \\ & + \cos \delta \Theta + [D^2 - (\alpha^2 + \beta^2)]\bar{\psi}, \end{aligned}$$

$$\frac{\partial \Theta}{\partial t} + i\alpha Gr \bar{f} \cos \delta \Theta + Gr D \bar{\Theta} u = \frac{1}{Pr} [D^2 - (\alpha^2 + \beta^2)]\Theta,$$

with the boundary conditions

$$U = v = w = \theta = 0 \quad \text{at} \quad x = \pm 1/2$$

The eigenvalues of this system are now found to determine the stability of the flow under consideration. Since, however, there exists some experimental evidence as to the type of disturbances which render the system unstable, this knowledge is used to advantage in the following development.

2.3.1 Equations Governing the Longitudinal Disturbances

Since it has been shown experimentally that longitudinal cells are a possible outcome for some range of the parameters and since these cells are characterized by $\alpha = 0$ the equations 19 in this case are reduced to

$$DU + i\beta v = 0,$$

$$\frac{\partial U}{\partial t} = -Gr D_P - \sin \delta \theta + (D^2 - \beta^2)U,$$

$$\frac{\partial v}{\partial t} = -i\beta Gr P + (D^2 - \beta^2)v, \quad 20$$

$$\frac{\partial w}{\partial t} + Gr D \bar{f} \cos \delta U = \cos \delta \theta + (D^2 - \beta^2)w,$$

$$\frac{\partial \theta}{\partial t} + Gr D \bar{\theta} U = \frac{1}{Pr} (D^2 - \beta^2) \theta.$$

In this set the z-momentum equation is seen to decouple from the remaining set, leading to certain simplifications. In particular, since the stability problem can now be solved independently of the z-momentum, the effect of the shape of the base flow velocity profile does not influence the analysis.

Introducing now the stream function by the definitions

$$u = -i\beta\psi \quad \text{and} \quad w = D\psi \quad 21$$

and eliminating the pressure, equations 20 are brought into the form

$$\begin{aligned} \frac{\partial}{\partial t}(D^2 - \beta^2)\psi &= i\beta \sin\delta \Theta + (D^2 - \beta^2)^2\psi, \\ \frac{\partial \Theta}{\partial t} &= i\beta Gr D\bar{\Theta}\psi + \frac{1}{Pr}(D^2 - \beta^2)\Theta, \\ \Theta = \psi = D\psi &= 0 \quad \text{at} \quad x = \pm 1/2. \end{aligned} \quad 22$$

These equations are closely related to those governing the classical Benard convection. Delaying further discussion to Section 2.4.1, the case when $\alpha \neq 0$ is considered next.

2.3.2 Application of Squire's Transformation to the Three-Dimensional Case

Gershuni and Zhukhovitskii (1969) noted that a transformation similar to the type first given by Squire (1933) can be applied to equations 19. Writing accordingly

$$\begin{aligned} \tilde{\alpha}\tilde{w} &= \alpha w + \beta v, & \tilde{\alpha}^2 &= \alpha^2 + \beta^2, \\ \tilde{u} &= u, & \tilde{Pr} &= Pr, & \tilde{p}\tilde{Gr} &= pGr, \\ \sin\tilde{\delta}\tilde{\Theta} &= \sin\delta\Theta, & \sin\tilde{\delta}\tilde{\Theta}r &= \sin\delta\Theta r, \\ \text{and} & & \tilde{\alpha}\tilde{Gr}\cos\tilde{\delta} &= \alpha Gr\cos\delta, \end{aligned} \quad 23$$

the transformed equations are

$$D\tilde{U} + i\tilde{\alpha}\tilde{w} = 0,$$

$$\frac{\partial \tilde{U}}{\partial t} + i\tilde{\alpha}\tilde{G}_r\bar{f}\cos\tilde{\delta}\tilde{U} = -\tilde{G}_r D\tilde{P} - \sin\tilde{\delta}\tilde{\Theta} + (D^2 - \tilde{\alpha}^2)\tilde{U},$$

$$\begin{aligned} \frac{\partial \tilde{w}}{\partial t} + i\tilde{\alpha}\tilde{G}_r\bar{f}\cos\tilde{\delta}\tilde{w} + \tilde{G}_r D\bar{f}\cos\tilde{\delta}\tilde{U} = & -i\tilde{\alpha}\tilde{G}_r\tilde{P} \\ & + \cos\tilde{\delta}\tilde{\Theta} + (D^2 - \tilde{\alpha}^2)\tilde{w} \end{aligned} \quad 24$$

$$\frac{\partial \tilde{\Theta}}{\partial t} + i\tilde{\alpha}\tilde{G}_r\bar{f}\cos\tilde{\delta}\tilde{\Theta} + \tilde{G}_r D\tilde{\Theta}\tilde{U} = \frac{1}{Pr}(D^2 - \tilde{\alpha}^2)\tilde{\Theta},$$

$$\tilde{U} = \tilde{w} = \tilde{\Theta} = 0 \quad \text{at} \quad x = \pm 1/2.$$

Introducing now the stream function as before by the relations

$$\tilde{U} = -i\tilde{\alpha}\tilde{\Phi} \quad \text{and} \quad \tilde{w} = D\tilde{\Phi} \quad 25$$

and eliminating the pressure, the equations 24 become

$$\begin{aligned} \frac{\partial}{\partial t}(D^2 - \tilde{\alpha}^2)\tilde{\Phi} + i\tilde{\alpha}\tilde{G}_r\cos\tilde{\delta}[\bar{f}(D^2 - \tilde{\alpha}^2)\tilde{\Phi} - D\bar{f}\tilde{\Phi}] \\ = \cos\tilde{\delta}D\tilde{\Theta} + i\tilde{\alpha}\sin\tilde{\delta}\tilde{\Theta} + (D^2 - \tilde{\alpha}^2)^2\tilde{\Phi}, \end{aligned} \quad 26$$

$$\frac{\partial \tilde{\Theta}}{\partial t} + i\tilde{\alpha}\tilde{G}_r\cos\tilde{\delta}\bar{f}\tilde{\Theta} - i\tilde{\alpha}\tilde{G}_r D\tilde{\Theta}\tilde{\Phi} = \frac{1}{Pr}(D^2 - \tilde{\alpha}^2)\tilde{\Theta}$$

$$\tilde{\Phi} = D\tilde{\Phi} = \tilde{\Theta} = 0 \quad \text{at} \quad x = \pm 1/2$$

These equations are identical in structure to those obtained by setting $\beta = 0$ at the outset in equations 19 and proceeding in a similar manner as before in the introduction of the stream function and in the elimina-

tion of the pressure. The $\beta = 0$ case follows, of course, also trivially from equations 23.

To obtain the results for the three-dimensional problem the inverse transformations are introduced. From a given solution of the transformed equations the characteristic wave numbers of the three-dimensional problem, for a given angle of inclination, are obtained from the relations

$$\alpha = \tilde{\alpha} \frac{\tan \delta}{\tan \tilde{\delta}} \quad \text{and} \quad \beta = \sqrt{\tilde{\alpha}^2 - \alpha^2} \quad 27$$

Since β must be real $\tan \delta \leq \tan \tilde{\delta}$ i.e. $\delta \leq \tilde{\delta}$. This gives a limitation on the applicability of the procedure, namely the value of the angle of inclination of the three-dimensional problem must be smaller than the corresponding two-dimensional angle. The Grashof number associated with the calculated wave numbers is then obtained from

$$Gr = \tilde{Gr} \sqrt{\sin^2 \tilde{\delta} + \left(\frac{\tilde{\alpha}}{\alpha}\right)^2 \cos^2 \tilde{\delta}} \quad 28$$

This completes the formulation of the stability equations, the solution of which is now considered.

2.4 Solution of the Eigenvalue Problem

For a number of stability problems the orthogonal expansion method is known to converge rather slowly when simple orthogonal functions are used as generating functions. This was the case for the classical problems of plane Poiseuille flow and for the plane Couette flow treated by Dolph and Lewis (1958) and Gallegher and Mercer (1962, 1965), respectively. They have shown the convergence to deteriorate for large αR (α being the wave number of the disturbance and R the Reynolds number of the base flow). It is suggestive to draw from the WKBJ type of analysis

(see Lin (1966), p. 116) and attribute this to the rapidly oscillating nature of the solutions when αR is large and to the steep variation of the solutions near the turning points. In terms of an orthogonal expansion then a large number of terms would generally have to be included to account for all the harmonics of the rapidly oscillating solutions. Similarly Rudakov (1967) indicated slow convergence for the vertical slot for $Pr > 10$ casting doubts on the ability to treat the present problem with a reasonable amount of computations, by an orthogonal expansion technique.

On the other hand, an extrapolation of the results of Gill and Kirkham (1970) shows that a transition to the region in which the instability is likely to occur for traveling waves could take place shortly beyond $Pr = 1$. Since this would also occur at substantially lower values of Grashof number, difficulty of the above type might not exist. This provided sufficient reason to reconsider the problem with the hope of obtaining reasonably fast convergence in the overstable range. In addition to the Galerkin method, the method of Chandrasekhar (1961), actually going back in one form to Stodola and Vianello (see e.g. Hildebrand (1949), p. 318), was employed as it seemed promising in at least reducing the order of the matrices involved in the analysis of the stationary neutral states. This is so, since, as it is applied here, only the temperature perturbations are expressed in terms of an orthogonal set, the perturbations for the stream functions being obtained as a partial solution of the governing equations. Further discussion on the relative merits as applied to the present problem are deferred to Chapter III.

2.4.1 Solution for Longitudinal Waves

Since the proof for the exchange of stabilities has been given for

the Bénard problem with a similar differential equation, Chandrasekhar (1961), the time derivatives can be dropped from equations 22 which become

$$\begin{aligned} (D^2 - \beta^2)^2 \psi &= -i\beta \sin \delta \Theta, \\ (D^2 - \beta^2) \Theta &= -i\beta GrPr D\bar{\Theta} \psi, \end{aligned} \quad 29$$

$$\Theta = \psi = D\psi = 0 \quad \text{at} \quad x = \pm 1/2$$

and by eliminating the Θ

$$(D^2 - \beta^2)^3 \psi = -\beta^2 Ra D\bar{\Theta} \sin \delta \psi, \quad 30$$

$$\psi = D\psi = (D^2 - \beta^2)^2 \psi = 0 \quad \text{at} \quad x = \pm 1/2$$

For a linear base temperature distribution the solution has been given by Jeffries (1928) and others, the onset of instability occurring at

$$Ra \sin \delta = 1708 \quad \text{with} \quad \beta = 3.12 \quad 31$$

where $Ra = GrPr$ is the Rayleigh number.

2.4.2 Solution for Transverse Waves

2.4.2.1 Galerkin Method

Because the Galerkin method has a long history in terms of solutions of hydrodynamic stability problems, Di Prima (1953), Dolph and Lewis (1958), Finlayson and Scriven (1965), Gallagher and Mercer (1962, 1965), Rudakov (1967), Vest and Arpaci (1970) and others, its application to the present problem will be only briefly discussed. An authoritative treatment of the basic method can be found in Kantorovitch and Krylov (1964).

In accordance with this, the unknown functions in equations 26, which are expressed in short as

$$\begin{aligned} L_1(\tilde{\Phi}, \tilde{\Theta}) &= 0, \\ L_2(\tilde{\Theta}, \tilde{\Phi}) &= 0, \\ \tilde{\Theta} = \tilde{\Phi} = D\tilde{\Phi} &= 0 \text{ at } x = \pm 1/2 \end{aligned} \quad 32$$

are written in terms of the orthogonal expansions

$$\begin{aligned} \tilde{\Phi}(x, t) &= \sum_{n=1}^{\infty} a_n(t) C(\lambda_n x) + i b_n(t) S(\mu_n x) \text{ and} \\ \tilde{\Theta}(x, t) &= \sum_{n=1}^{\infty} d_n(t) \sin x_n x + i e_n(t) \cos p_n x. \end{aligned} \quad 33$$

The functions $C(\lambda_n x)$ and $S(\mu_n x)$, written in their normalized forms as

$$\begin{aligned} C(\lambda_n x) &= \frac{\cosh \lambda_n x}{\cosh \lambda_n / 2} - \frac{\cos \lambda_n x}{\cos \lambda_n / 2}, \\ S(\mu_n x) &= \frac{\sinh \mu_n x}{\sinh \mu_n / 2} - \frac{\sin \mu_n x}{\sin \mu_n / 2}, \end{aligned} \quad 34$$

satisfy the boundary conditions imposed on $\tilde{\Phi}$ if the parameters λ_n and μ_n are solutions of the transcendental equations

$$\begin{aligned} \tanh \lambda / 2 + \tan \lambda / 2 &= 0 \quad \text{and} \\ \coth \mu / 2 - \cot \mu / 2 &= 0. \end{aligned} \quad 35$$

Since only two boundary conditions need to be satisfied by the temperature, $\sin x_n x$ and $\cos p_n x$ are chosen as the approximating functions with

$$k_n = 2n\pi \quad \text{and} \quad p_n = (2n-1)\pi \quad n=1, 2, \dots \quad 36$$

By orthogonalizing now the first of equations 32 by $C(\lambda_n x)$ and $S(\mu_n x)$ in turn and similarly the second by $\sin k_n x$ and $\cos p_n x$ the following set is obtained

$$\begin{aligned} \langle L_1 \left(\sum_{n=1}^{\infty} a_n(t) C(\lambda_n x) + i b_n(t) S(\mu_n x) \right), C(\lambda_m x) \rangle &= 0, \\ \langle L_1 \left(\sum_{n=1}^{\infty} a_n(t) C(\lambda_n x) + i b_n(t) S(\mu_n x) \right), S(\mu_m x) \rangle &= 0, \\ \langle L_2 \left(\sum_{n=1}^{\infty} d_n(t) \sin k_n x + i e_n(t) \cos p_n x \right), \sin k_m x \rangle &= 0, \\ \langle L_2 \left(\sum_{n=1}^{\infty} d_n(t) \sin k_n x + i e_n(t) \cos p_n x \right), \cos p_m x \rangle &= 0, \end{aligned} \quad 37$$

where the real inner products are defined by

$$\langle u, v \rangle = \int_{-1/2}^{+1/2} u v dx. \quad 38$$

The inner products needed in the present problem have been tabulated by Vest (1967). This set of equations can now be written in the matrix form

$$B \frac{dX}{dt} + C X = 0, \quad 39$$

where the unknown coefficients of the expansions are the elements of the vector

$$X = \{ a_1, \dots, a_n, \dots, b_1, \dots, b_n, \dots, d_1, \dots, d_n, \dots, e_1, \dots, e_n, \dots \}^T. \quad 40$$

The elements of the matrices B and C are real due to the symmetric nature of the operators L_1 and L_2 . Finally truncating each of the series to N terms gives a system of 4N equations the stability of which is determined by the eigenvalues of the matrix $-(B^{-1}C)$.

2.4.2.2 Chandrasekhar's Method

For the purpose of applying the Chandrasekhar's method the equations 26 are written in the form

$$L^2 \tilde{\Phi} = \left[\frac{\partial}{\partial t} L + i\tilde{\alpha} \tilde{G} r N \right] \tilde{\Phi} + H \tilde{\Theta}, \quad 41$$

$$L \tilde{\Theta} = \left[P r \frac{\partial}{\partial t} + i\tilde{\alpha} \tilde{R}_a \bar{\omega} \right] \tilde{\Theta} - i\tilde{\alpha} \tilde{R}_a D \tilde{\Theta} \tilde{\Phi}, \quad 42$$

where

$$L = D^2 - \tilde{\alpha}^2, \quad N = \bar{\omega} L - D^2 \bar{\omega}, \quad H = -\cos \tilde{\delta} D - i\tilde{\alpha} \sin \tilde{\delta}. \quad 43$$

Solving now for the function $\tilde{\Phi}$ on the right hand side of equation 42 in terms of $\tilde{\Theta}$ and substituting this into the right hand side of equation 41 gives

$$L^2 \tilde{\Phi} = \frac{1}{i\tilde{\alpha} \tilde{R}_a D \tilde{\Theta}} \left[\frac{\partial}{\partial t} L + i\tilde{\alpha} \tilde{G} r N \right] \left[P r \frac{\partial}{\partial t} + i\tilde{\alpha} \tilde{R}_a \bar{\omega} - L \right] \tilde{\Theta} + H \tilde{\Theta} \quad 44$$

In this equation $\tilde{\Theta}$ can be written in terms of the orthogonal expansion

$$\tilde{\Theta} = \sum_{n=1}^{\infty} d_n(t) \sin k_n x + i e_n(t) \cos e_n x, \quad 45$$

where

$$k_n = 2n\pi \quad \text{and} \quad e_n = (2n-1)\pi \quad n=1,2,\dots \quad 46$$

giving an equation of the form

$$L^2 \tilde{\Phi} = f_e + i f_o. \quad 47$$

The nonhomogeneous terms on the right, in which the subscripts "e" and "o" denote evenness and oddness, besides being functions of x , depend on

time through the coefficients d_n and e_n and their derivatives. By substituting the solution of equation 47 which satisfies the boundary conditions back into equation 42 and orthogonalizing it with respect to $\sin k_m x$ and $\cos \ell_m x$, the resulting system written in matrix form is

$$A \frac{d^2 X}{dt^2} + B \frac{dX}{dt} + CX = 0, \quad 48$$

where

$$X = \{d_1, \dots, d_n, \dots, e_1, \dots, e_n, \dots\}^T \quad 49$$

The matrices are infinite in order and consist of elements which are again real. At this stage no approximations have been made and the only one which needs to be introduced is the truncation of the series for the temperature to N terms.

Rewriting now the truncated system of order $2N$ in the form

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \left\{ \frac{dY}{dt} \right\} + \begin{bmatrix} 0 & -I \\ \bar{A}^T B & \bar{A}^T C \end{bmatrix} \left\{ Y \right\} = 0, \quad 50$$

where $Y = \left\{ \frac{dX}{dt}, X \right\}^T$, the stability characteristics of the problem are determined by finding the eigenvalues of the matrix

$$\begin{bmatrix} 0 & I \\ -\bar{A}^T B & -\bar{A}^T C \end{bmatrix}. \quad 51$$

Since this matrix is of order $4N$ the Galerkin and Chandrasekhar methods result in solving an algebraic eigenvalue problem of the same order and the convergence can be compared strictly in terms of N .

In general the eigenvalues (c) will be complex. Besides being functions of the parameters Gr , Pr , and δ which are easily varied in an experiment they are also functions of the wave numbers α and β . For the flow to be unstable it is required that for a given Gr , Pr , and δ the real part of c be negative for all α and β . The neutral stability for a given wave is defined by the condition of vanishing of the real part of one of the eigenvalues while the other modes are damped, i.e. have negative real parts of c . If the imaginary part of the eigenvalue vanishes when the real part does the onset of instability is said to set in through stationary states, otherwise the term overstability or instability as traveling waves is used.

An alternative to finding all eigenvalues exists if the instability sets in through stationary states, for then the stability boundary can be found by investigating the sign change of the determinant of the matrix C . In this case the determinant to be evaluated in the Chandrasekhar method has half the size of that generated by the Galerkin equations.

2.4.2.3 Numerical Solution of the Eigenvalue Matrix

The eigenvalues were found in both of the methods by converting the eigenvalue matrix to an upper almost triangular form (Hessenberg form) by similarity transformations and subsequently using the double QR iteration due to Francis (1961) on the transformed matrix. These procedures exist as subroutines in the I.B.M. "Scientific Subroutine Package". The investigation of the sign change of the determinant can be effectively carried out by performing the LU factorization on the matrix C . The general discussion of this decomposition can be found in Forsythe and Moler (1967).

CHAPTER III

RESULTS AND CONCLUSIONS

The two methods were first checked against the known results for small Prandtl numbers. After this their convergence was studied to determine which one ought to be used for the bulk of the calculations. On general grounds it was expected that the Chandrasekhar method would show slower convergence in those cases in which the temperature perturbations are both unimportant and require a large number of terms in their representation. The first condition is satisfied for small Prandtl numbers although it is not clear why in this region the temperature perturbations would require a large number of terms. The extreme case of $Pr = 0$ leads to complete failure of the method since in this case the temperature perturbations vanish identically.

The Galerkin method for $Pr = 0$, on the other hand, as has also been shown by others, gives a convergence to within 0.1% for $N = 4$ (N is the number of terms in each expansion). These results are given in Table I. Because a variational principle in terms of an adjoint system can be shown to exist, the Grashof number can be sought as a problem of finding an extremum. As is shown in Appendix I, if the adjoint system has the same boundary conditions as the original one, the same base functions can be used in the expansion of both the stream function and its adjoint. Finding the extremum with the coefficients treated as parameters the Galerkin equations are obtained. As a result of this, the solution for the Grashof number must get progressively better as more base functions are included. This is verified by the results in Table I.

For non-zero Prandtl numbers the comparison of Galerkin and Chandrasekhar methods was carried out by fixing an α close to the critical one,

and calculating the critical Grashof number by the different approximations. The results of these computations are shown in Table II. In this table it should be noted that as $Pr \leq 10$ corresponds to stationary stability and the investigation of the sign change of the determinant could be performed with great efficiency as compared to the calculation of the whole eigenvalue spectrum, the results for $Pr = 10$ are accordingly more accurate than the more global results for $Pr = 100$ and 1000 . If the comparison in the stationary range is made for equal N , it is seen that the Chandrasekhar method lags slightly in convergence the more pronounced effect being for lower Prandtl numbers. On the other hand, a comparison in terms of the order of the matrix involved reverses the situation with Galerkin method lagging now slightly. In the overstable range the comparison is made strictly in terms of N as the complete eigenvalue spectrum is needed. Both methods show quite similar convergence behavior. In particular for both $Pr = 100$ and 1000 the large improvement in the value of the critical Grashof number occurs on the second and fourth approximation for both methods.

Since the convergence study did not reveal very striking differences between the two methods it seemed immaterial which one ought to have been employed in the subsequent calculations. For the stationary states the Chandrasekhar method was used with an eight order approximation providing the critical Grashof number accurate to 0.05%. These are given in Table III for various angles of inclination. The values for the vertical slot compare favorably with those published by Rudakov (1967), Vest and Arpaci (1969), and Baxi (1971) as well as the unpublished results of Gözüüm (1972). For the inclined slot they are in as close agreement with the results of Gershuni and Zhukhovitskii (1969) and of Hart (1971), as can be read from their graphs.

α	N			
	4	5	6	7
2.50	7988.6	7993.2	7994.0	-
2.55	7958.9	7963.7	7964.6	-
2.60	7938.0	7943.1	7944.0	-
2.65	7926.1	7931.4	7932.4	7932.6
2.70	7923.2	7928.9	7929.9	7930.1
2.75	7929.8	7935.8	7936.9	-
2.80	-	7952.6	7953.7	-

Table I. $Gr_{crit} \cos \delta$ for $Pr = 0$. Effect of wave number and convergence.

Chandrasekhar									
Pr	N	1	2	3	4	5	6	7	8
0.001		9200	6530	8395	7960	7944	7930	7925	7920
0.01		8490	6425	8235	7845	7833	7820	7815	7810
0.1		6125	6160	7480	7350	7360	7355		
1		--	--	--	8035	7995	7990		
10		--	--	--	--	8815	7950	7910	7900
100		245	520	555	770	755	750		
1000		77	160	160	325	245	250		

Galerkin									
Pr	N	1	2	3	4	5	6	7	8
0.001		6445	7650	7900	7915	7920			
0.01		6385	7550	7795	7805	7810			
0.1		6085	7105	7335	7350	7355			
1		7870	7885	8000	7985	7990			
10		--	--	--	10685	8385	7950	7905	7900
100		360	580	560	790	770	755		
1000		110	170	165	240	230	250		

Table II. Comparison of the Chandrasekhar and Galerkin method as a function of the Prandtl Number.

The investigation of the higher Prandtl numbers, which has been largely left untreated, shows the existence of an oscillatory mode which becomes critical when $Pr = 12.7$. Through the transition as is shown in Figures 2 and 3, the wave number drops to a value which is about one third of that associated with stationary stability. The corresponding wave speed is first below the maximum base flow velocity but asymptotes to a value 4% above it as the Prandtl number is increased. Ten terms were needed in the Galerkin equations in this transition regime ($12.7 < Pr < 70$) to achieve an accuracy of 0.5%.

To give an indication of the eigenvalue spectra in this range Figures 4 and 5 for $Pr = 5$ and $Pr = 50$, respectively, are included. Following Rudakov the eigenvalues were labeled as isothermal or non-isothermal in accordance whether they originate from the momentum or the energy equation, respectively. This identification could be made conveniently since for $Gr = 0$ the two equations are only loosely coupled. By holding α and Prandtl number fixed and varying the Grashof number the spectra were obtained. At low values of Grashof number all eigenvalues were found to be real; the first four for $Pr = 5$ and the first thirteen for $Pr = 50$ originating from the energy equation. An increase in Grashof number resulted in the real levels combining into complex conjugate pairs. The wave speeds corresponding to the complex conjugate pairs are shown in the lower portion of the Figures 4 and 5 and are scaled by a factor of $(\alpha Gr)^{-1}$ to facilitate comparison with the base flow velocity.

Rudakov's (1967a, 1967b) observations of the eigenvalue spectra show that stationary stability is associated with the first isothermal level. In fact, for $Pr < 0.32$ this level stays real throughout the variation of the Grashof number past the point of instability. For $0.4 < Pr < 10$,

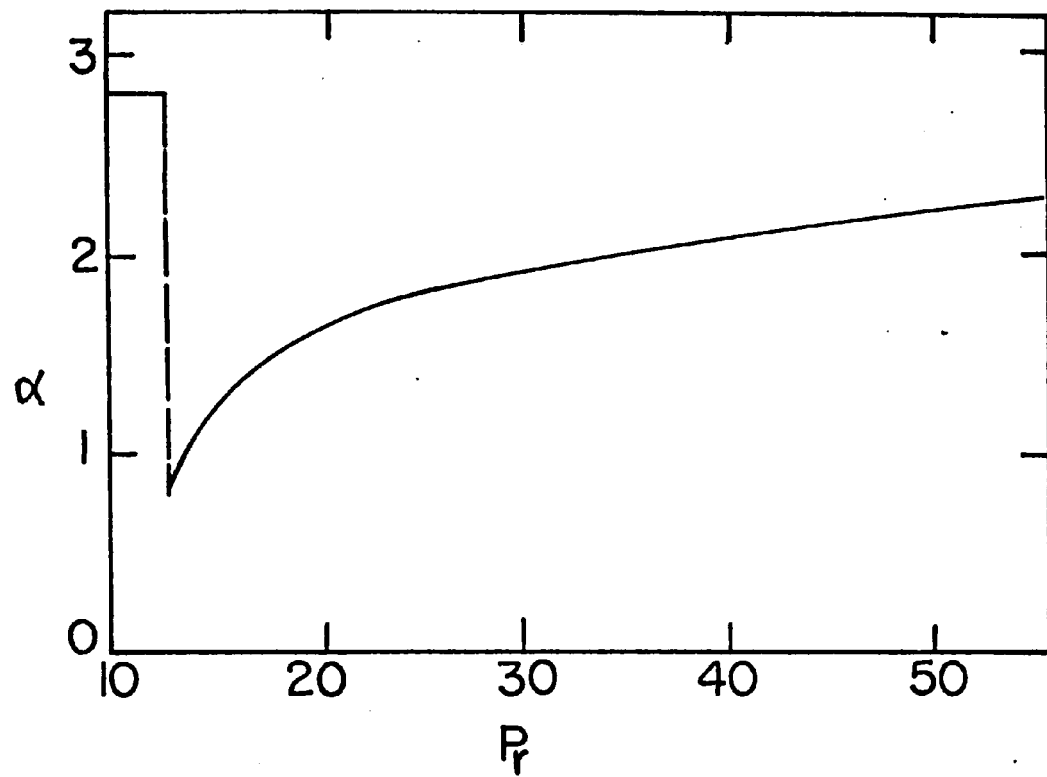


Figure 2. The Variation of the Critical Wave Number (α) as a Function of the Prandtl Number (Pr). The Dimensional Wave Number is $L^{-1}\alpha$.

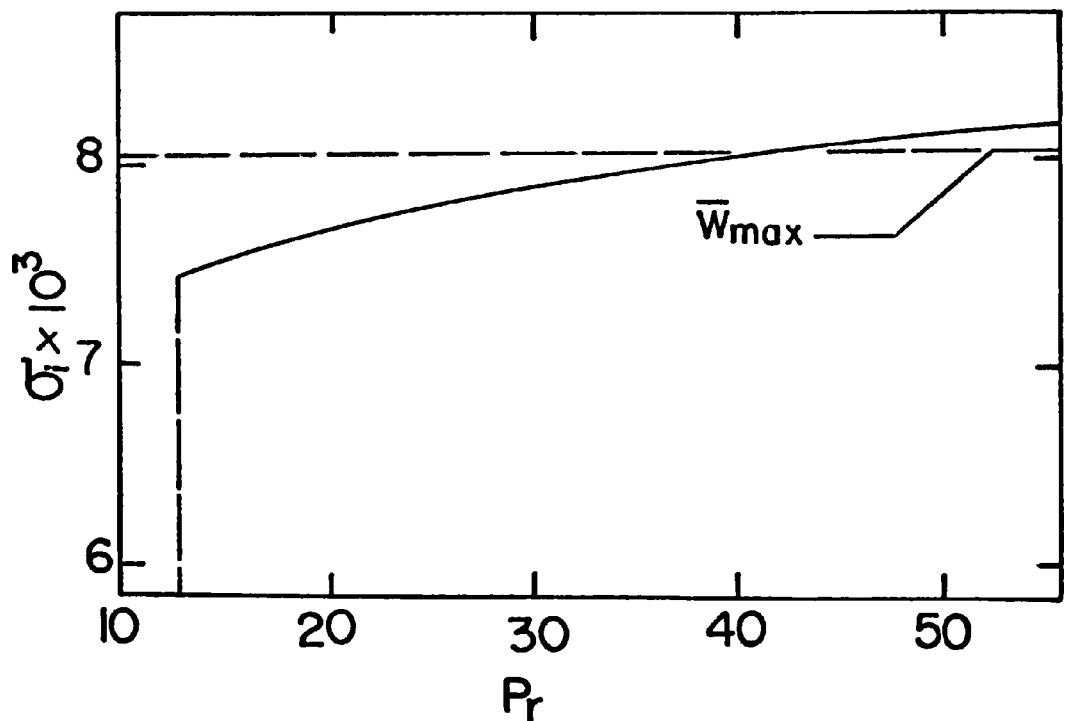


Figure 3. The Wave Speed (σ_i) at Neutral Stability as a Function of the Prandtl Number (Pr). The Dimensional Wave Speed is $g\delta\Delta T L^2 \nu^{-1} c_i$.

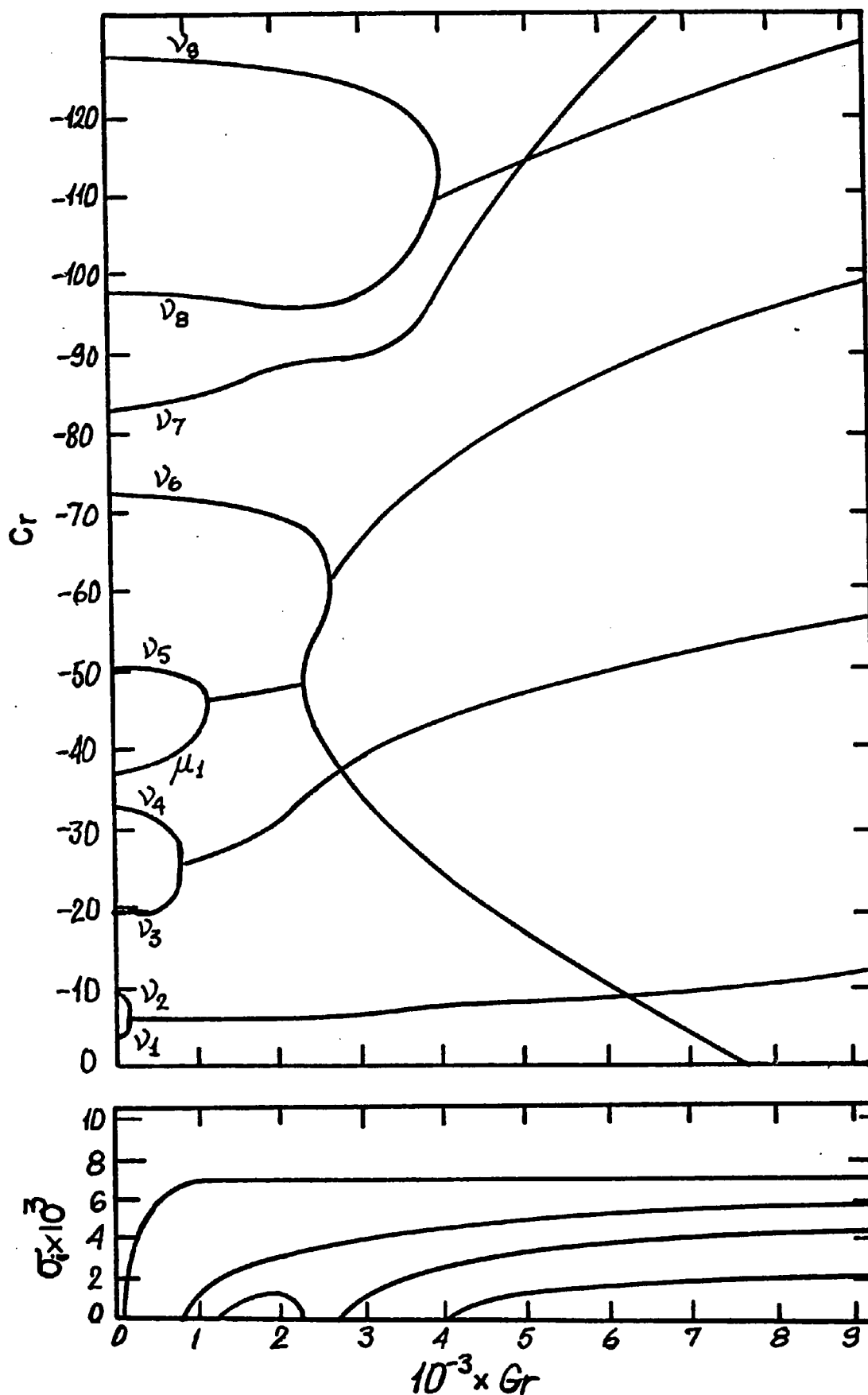


Figure 4. Non-dimensional Damping Factors and Wave Speed for $\alpha=2.65$, $Pr=5$. The Definition $\alpha Gr \sigma_i = c_i$ Relates the Ordinates.

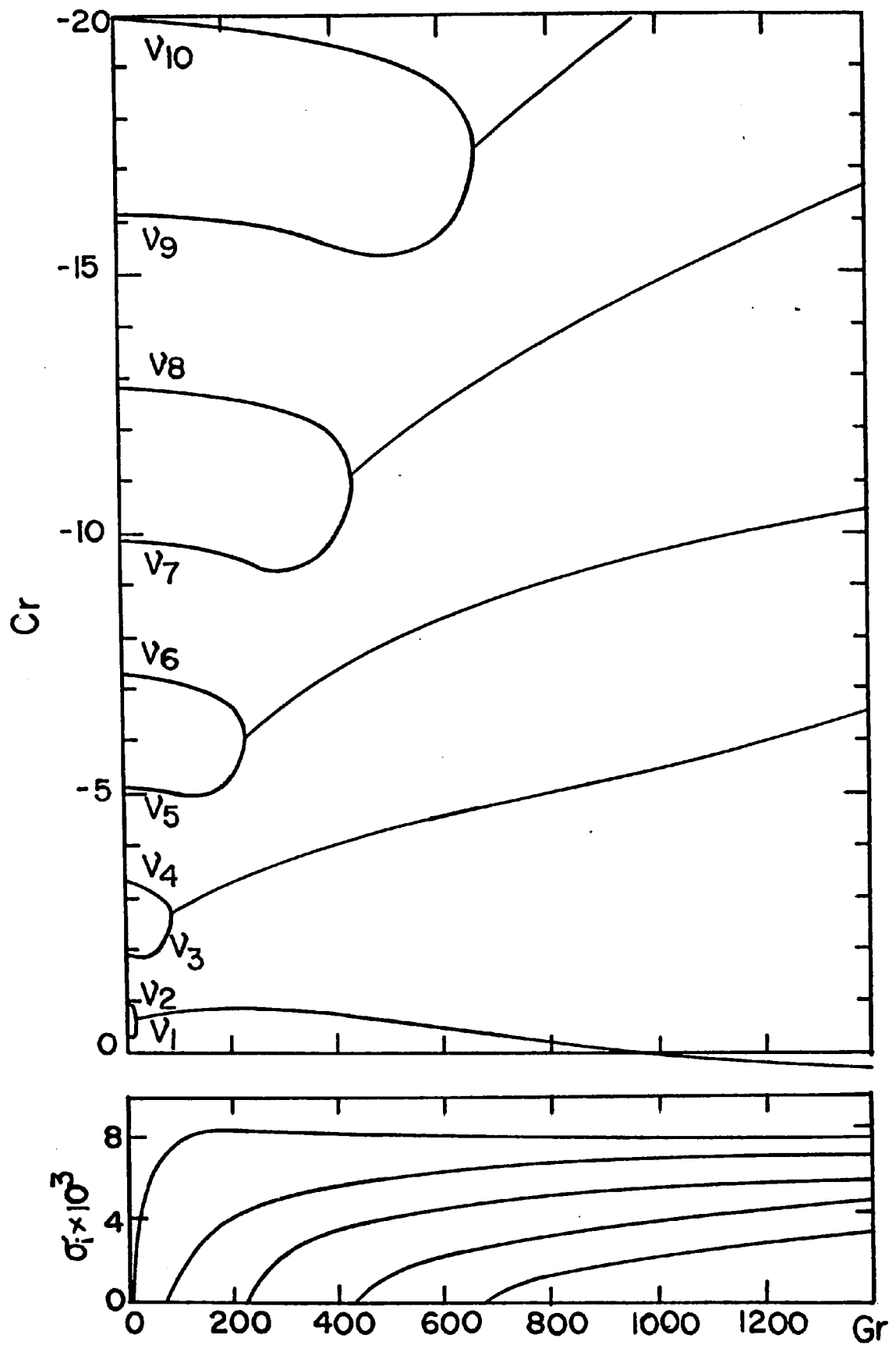


Figure 5. Non-dimensional Damping Factor and Wave Speed for $\alpha=2.65$, $Pr=50$.

however, the lowest isothermal level combines first with some non-isothermal ones to result in a complex value but bifurcates upon a further increase in Grashof number into two real levels again. The lower branch of this bifurcation as shown in Figure 4, determines the point of instability. Although for $Pr = 0.4$ the interaction of the lowest isothermal level is with the first non-isothermal one, the increase in Prandtl number leads to an increasing population of non-isothermal levels in the lower portion of the spectrum. Thus, for $Pr = 5$ the interaction is with the fifth non-isothermal level and for $Pr = 50$ with the fourteenth one. As the non-isothermal levels are brought down with an increase in Prandtl number there comes a point at which the lowest non-isothermal level becomes the critical one. This occurs at $Pr = 12.7$. At this point there are two waves of different wavelengths which are equally likely to become unstable; one associated with stationary stability leading to stationary convection, the other, corresponding to overstability, presumably leads to turbulence. Since a random disturbance consists of waves of all wavelengths both of these waves are always present in a physical system subjected to arbitrary disturbances.

The interaction of the lowest isothermal level with the various other levels during the course of varying the Prandtl number is manifested in the behavior of the neutral stability curve for $Pr < 12.7$. Although no abrupt changes of the slope of neutral curve of the type when the transition to overstability occurs were detected, it would seem that this type of behavior should not be ruled out a priori. As the Prandtl number range was sampled only at a finite number of points, although some attempts were made in trying to discover discontinuous dependence in the wave number, these were not found. Since the overstability is

always associated with the same eigenvalue, the neutral curve also shows smoother dependence.

The values of critical Grashof number are given for the transition region in Table IV and the associated curves of the critical states in Figure 6. Since the oscillatory states vary as $GrPr^{1/2} = \text{constant}$ and the longitudinal modes as $GrPr = \text{constant}$, these curves intersect in a manner such that the instability is governed by transverse disturbances for angles close to zero and longitudinal ones when the angle of inclination is increased beyond a certain definite value. This value, being a function of Prandtl number, is $1/2$ degree for $Pr = 850$, 6 degrees for $Pr = 2$, and 13 degrees for $Pr = 1$, the last set of figures being identical to those given by Gershuni and Zhukhovitskii. From the form of Figure 6 the effect of angle of inclination cannot be deduced with certainty as the angle occurs both as a parameter and as a part of forming the ordinate. The tendency would be to say that the increase in the angle of inclination is associated with destabilizing the flow. This is true only when $Pr > 0.4$. For values less than this the flow is first destabilized and then stabilized as the angle is varied from 0° to 90° . Gershuni showed that the minimum condition for $Pr = 0.2$ occurs at an angle of 25° . The figures as obtained here for $Pr = 0.1$, 0.01 , and 0.001 are 10° , 1.7° , and 0.84° , respectively.

As a final comment, due to the shape of the $\delta = 0^\circ$ curve there is still a possibility that for certain angles in the range of 1 to 2 degrees the mode of instability can change from transverse stationary to longitudinal stationary followed by a transition to transverse oscillatory and then back to longitudinal stationary as the Prandtl number is increased. Since there is little practical motivation for the calcula-

tion of the detailed structure in this region of the stability diagram, this was not done.

tan δ	Prandtl number					
	0	0.001	0.01	0.1	1	10
0.00	7932	7920	7812	7350	7940	7872
0.01	7932	7919	7808	7332	7913	7845
0.1	7972	7956	7827	7217	7715	1717
1	11218	11167	10741	8289	2415	241.5
10	79716	77344	60480	16241	1717	171.7
∞	∞	1707762	170776	17078	1708	170.8

Table III. The critical Grashof number as a function of the angle of inclination for various Prandtl numbers.

Pr	Gr	α	σ_i
13	6200	0.82	7.50×10^{-3}
14	4710	1.02	7.53×10^{-3}
15	3910	1.25	7.55×10^{-3}
20	2390	1.65	7.69×10^{-3}
25	1890	1.85	7.81×10^{-3}
30	1600	1.95	7.86×10^{-3}
35	1410	2.05	7.98×10^{-3}
100	746	2.35	8.16×10^{-3}
1000	253	2.85	8.37×10^{-3}

Table IV. Critical states in the transition region.

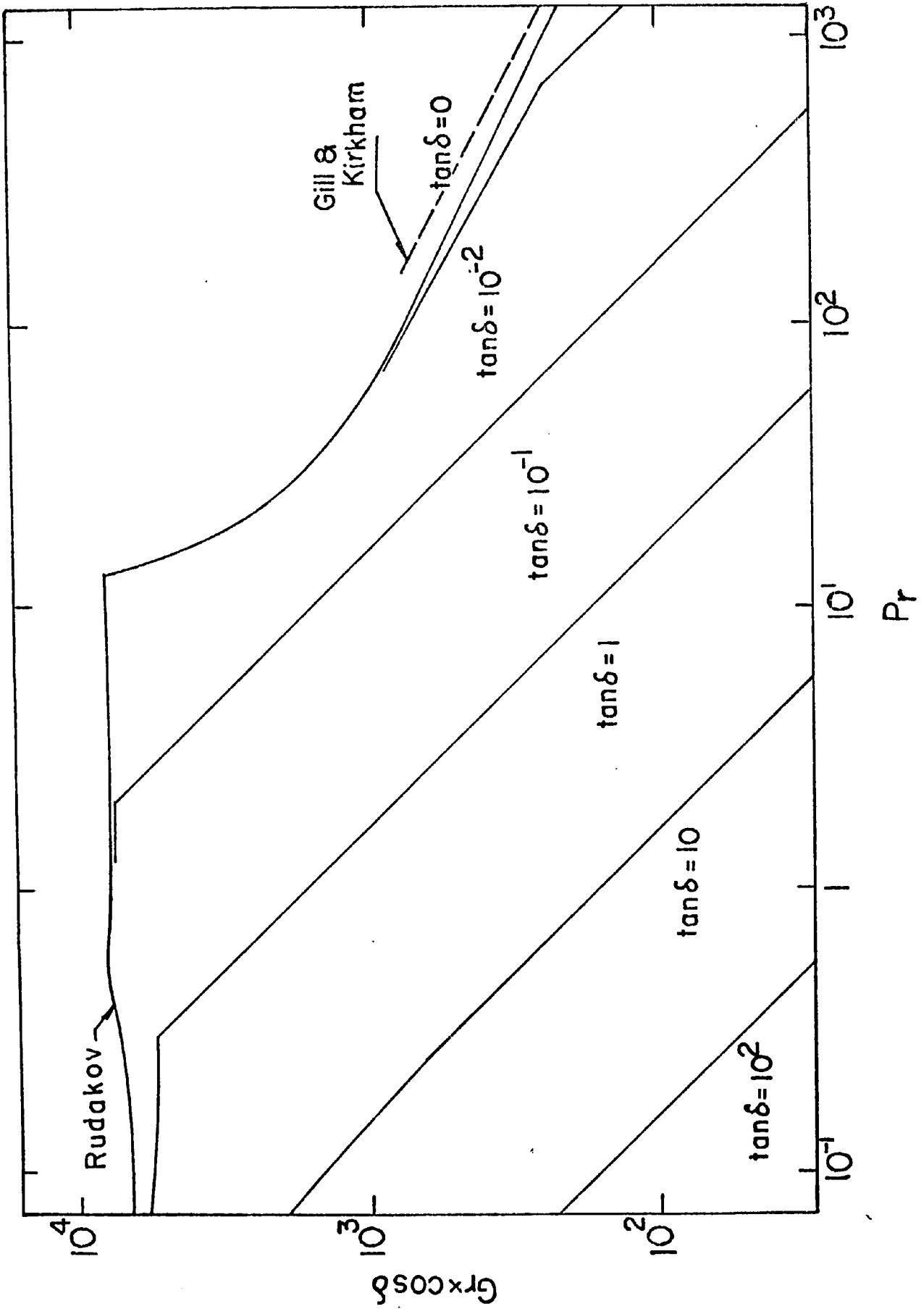


Figure 6. The States of Neutral Stability as a Function of Prandtl Number (Pr) for Positive Angles of Inclination (δ).

CHAPTER IV

EXPERIMENTAL INVESTIGATION

The apparatus used in the experimental investigation was that constructed by Vest (1967). Because its details have been discussed by him in length, this aspect will be considered only briefly here. The test section consisted of two aluminum plates of dimensions 25x7x0.25 inches between which a Plexiglass frame could be inserted. Each plate had been designed to be a part of a heat exchanger, the function of which is to keep the plate at a constant, uniform temperature. The cavity formed by the aluminum plates and the Plexiglass frame constitute the slot which is occupied by the fluid. It could be illuminated by shining a light through the top, or a side section, to study the transverse and longitudinal cells, respectively.

The temperatures of the side walls were recorded by Copper-Constantan thermocouples. To check the uniformity of the temperature nine thermocouples were used on the hot side, the respective number on the cold side being three.

Since the conduction regime is given by the condition $GrPr/h < 300$, an experimental apparatus of reasonable dimensions is only feasible for small Prandtl numbers. For air which was used in these experiments the minimum aspect ratio required is 19. In the actual experiment this ratio was 33 so that the conduction regime was adequately approximated.

Following the procedure by Vest, the tests were performed by bringing the system into a quasi-steady state and allowing the temperature of the hot wall to increase at a nominal rate of 0.3 deg F/min. Insertion of cigarette smoke allowed visual observations to be made of the developing flow patterns and so also determination of the state of stability of

the flow. From the recorded temperatures the critical Grashof numbers, shown in Figure 7 for various angles, were calculated. With this method of observing the flow the point of instability could be bracketed within a certain range of Grashof numbers. The results show the theoretical curve to fall close to, although not entirely within, the bracketed region.

The photographs of the typical cell structure are included in Figures 8 and 9. The Grashof number for the transverse cells at $\delta = 0^\circ$ in Figure 8 is about 9000. It was found that by letting the Grashof number exceed the critical one by a reasonable amount, the cell pattern developed rapidly at a time much less than the characteristic time for diffusion of the smoke, thus providing better contrast in the photograph. In Figure 8 the flow is up on the right side of the figure. The boundaries of the slot are identified with arrows to help distinguish the flow from its reflection on the aluminum walls. The wave length of the cells is approximated to be about 2.5 ± 0.2 times the gap width, this corresponding to a wave number ($\alpha = 2\pi/\lambda$) also 2.5 ± 0.2 ; in good agreement with the theory.

The longitudinal rolls of Figure 9 correspond to $\delta = 30^\circ$ and $Gr = 5500$. In this photograph the reflections of the flow from the aluminum walls are more pronounced. The cross section of the rolls is approximately square. The wave length of this pattern is measured across two cells since the flow in two adjacent rolls is in a direction opposite to one another. The wave number for square cells is equal to $\pi = 3.14$ which compares favourably with 3.12 from the theoretical calculation.

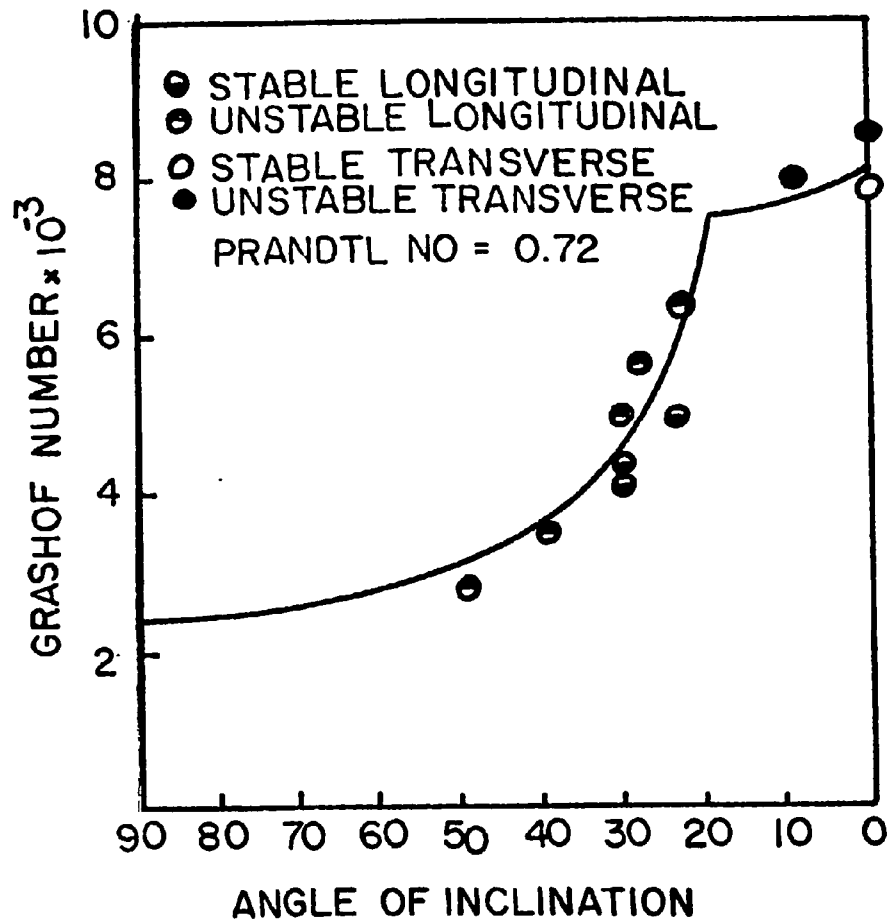


Figure 7. Comparison of the Experimental Data for Air ($Pr=0.72$) with the Theory.

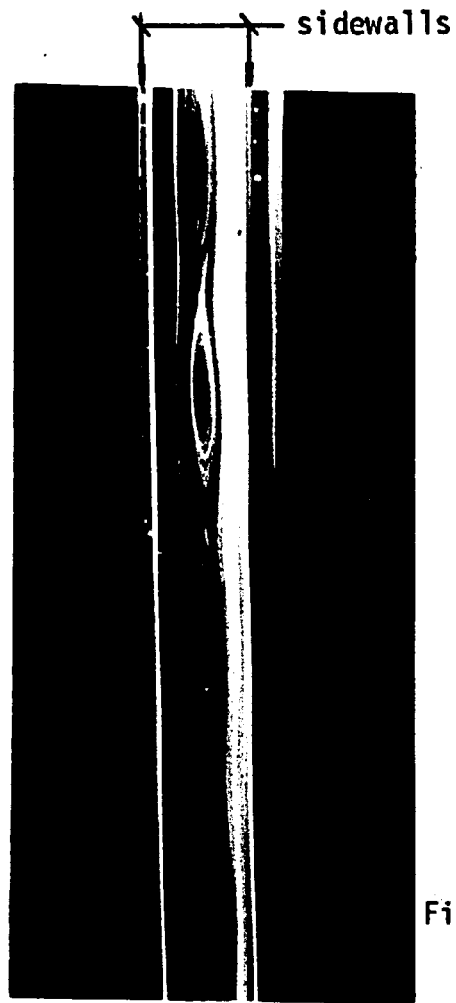


Figure 8. Photograph of the Convection Cells of Air for the Vertical Slot at $Gr = 9000$.

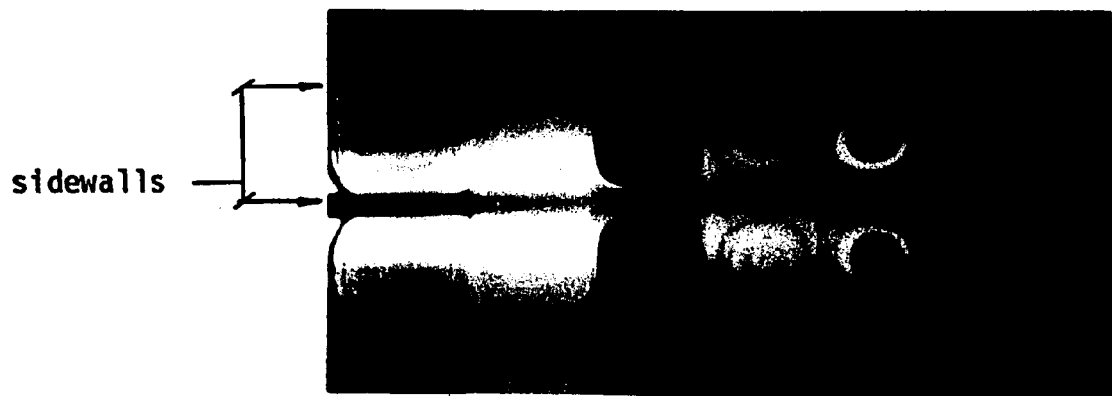


Figure 9. Photograph of the Longitudinal Rolls of Air at an Angle of Inclination $\delta = 30^\circ$ and $Gr = 5500$.

PART II
STABILITY OF NATURAL CONVECTION
IN A NARROW ROTATING ANNULUS

CHAPTER V

LITERATURE SURVEY

This part of the thesis reports on the effect of rotation on the stability of the conduction regime of natural convection in a cylindrical annulus. Because the annulus is considered infinite, the vertical component of velocity is the only one present, simplifying the study of the problem considerably. The pattern of flow is in sharp contrast to observations made in experiments in which the vertical and horizontal dimensions of the annulus are of comparable dimensions.

The important feature in the finite case is the development of a temperature gradient in the vertical direction leading to a motion of the fluid which is primarily zonal in its direction and varying linearly in the vertical coordinate. The instability of this flow, which has been termed baroclinic, has important meteorological implications and as a result has received considerable attention in recent years. Experimentally it has been studied by Fultz (1951, 1953), Hide (1953, 1958), and Fowles and Hide (1965) among others, and theoretically by Davies (1956, 1959), Kuo (1954, 1956, 1957), Brindley (1960), Lorentz (1962), and Barcilon (1964). A more complete bibliography can be found in the monograph of Greenspan (1968), which contains a survey of the problem.

When the height of the annulus is increased the vertical temperature gradient diminishes and with it the zonal flow. The effect of the unstable stratification of the fluid in the radial direction now becomes important in an analysis of the instability. This feature was studied by Chandrasekhar (1954), who by neglecting the gravity term considered a fluid quiescent in a rotating frame and heated radially from the outer cylinder.

The present investigation includes the effect of the gravity and with it a base flow. In addition to the unstable stratification in the radial direction the vorticity distribution of the base flow can be a cause of instability as has been shown in Part I.

CHAPTER VI

THEORETICAL ANALYSIS

6.1 Base Flow

Limiting the discussion again to the conduction regime an annulus as in Figure 10, rotating about its axis with an angular velocity Ω and being heated from the outside, is considered. The realization of the conduction regime necessitates taking $H/L \gg 1$ where H and L are the height and the gap width of the annulus, respectively. Also the ratio R/L is taken large, R being the average radius, to isolate the effects of curvature from the problem.

With this description the Boussinesq equations governing the base flow are

$$\begin{aligned} 0 &= -\frac{1}{\rho_0} \frac{d\bar{P}}{dx} - \Omega^2 R x \bar{\Theta}, \\ 0 &= g x \bar{\Theta} + \nu \frac{d^2 \bar{w}}{dx^2}, \\ 0 &= \frac{d^2 \bar{\Theta}}{dx^2}, \end{aligned} \tag{52}$$

with

$$\bar{w}(\pm L/2) = 0, \quad \bar{\Theta}(+L/2) = \frac{\Delta T}{2}, \quad \bar{\Theta}(-L/2) = -\frac{\Delta T}{2} \tag{53}$$

and

$$\rho_0 \int_{-L/2}^{+L/2} \bar{w}(x) dx = 0 \tag{54}$$

The solution of this set gives for the base quantities

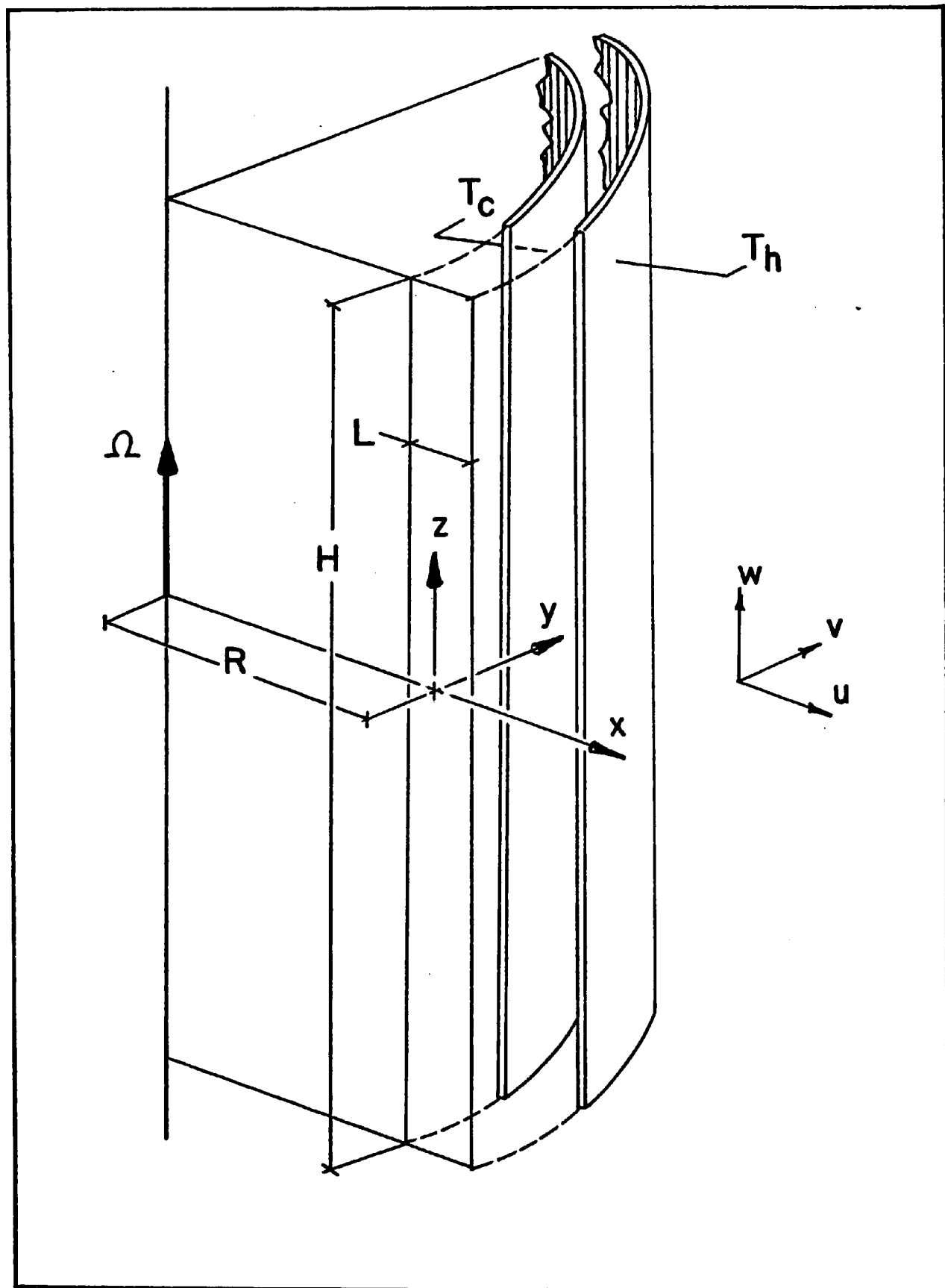


Figure 10. Rotating Annulus

$$\bar{\Theta}(x) = \Delta T \left(\frac{x}{L} \right) \quad \text{and}$$

$$\bar{w}(x) = \frac{28 \Delta T L^2}{\nu} \left[\frac{1}{6} \left[\frac{1}{4} \left(\frac{x}{L} \right) - \left(\frac{x}{L} \right)^3 \right] \right],$$

55

which differ from the equations 11, governing the inclined slot, only in the velocity term by the factor $\cos \delta$.

6.2 Stability Formulation

Following the development in Section 2.3 and non-dimensionalizing as before leads, after linearization, to the set

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0,$$

$$\frac{\partial u'}{\partial t} + Gr \bar{w} \frac{\partial u'}{\partial z} - Ta v' = -Gr \frac{\partial p'}{\partial x} - Fr \Theta' + \nabla^2 u',$$

$$\frac{\partial v'}{\partial t} + Gr \bar{w} \frac{\partial v'}{\partial z} + Ta u' = -Gr \frac{\partial p'}{\partial y} + \nabla^2 v', \quad 56$$

$$\frac{\partial w'}{\partial t} + Gr \bar{w} \frac{\partial w'}{\partial z} + Gr \frac{d\bar{w}}{dx} u' = -Gr \frac{\partial p'}{\partial z} + \Theta' + \nabla^2 w',$$

$$\frac{\partial \Theta'}{\partial t} + Gr \bar{w} \frac{\partial \Theta'}{\partial z} + Gr \frac{d\bar{\Theta}}{dx} u' = \frac{1}{Pr} \nabla^2 \Theta',$$

$$u' = v' = w' = \Theta = 0 \quad \text{at} \quad x = \pm 1/2.$$

The two new parameters occurring in these equations are

$$Fr = \frac{Q^2 R}{g}, \quad \text{the Froude number and}$$

$$Ta = \frac{2 Q L^2}{\nu}, \quad \text{the Taylor number.}$$

57

Eliminating the pressure and defining the x component of vorticity as $\zeta = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$ the resulting equations can be written as

$$\begin{aligned} \frac{\partial}{\partial t} \nabla^2 u' + Gr \bar{w} \frac{\partial}{\partial z} \nabla^2 u' - Gr \frac{d^2 \bar{w}}{dx^2} \frac{\partial u'}{\partial z} + Ta \frac{\partial \zeta'}{\partial z} \\ = - Fr \left(\frac{\partial \theta'}{\partial y^2} + \frac{\partial^2 \theta'}{\partial z^2} \right) - \frac{\partial^2 \theta'}{\partial x \partial z} + \nabla^4 u', \end{aligned}$$

$$\frac{\partial \zeta'}{\partial t} + Gr \bar{w} \frac{\partial \zeta'}{\partial z} + Gr \frac{d \bar{w}}{dx} \frac{\partial u'}{\partial y} - Ta \frac{\partial u'}{\partial z} = \frac{\partial \theta'}{\partial y} + \nabla^2 \zeta', \quad 58$$

$$\frac{\partial \theta'}{\partial t} + Gr \bar{w} \frac{\partial \theta'}{\partial z} + Gr D \bar{\theta} u' = \frac{1}{Pr} \nabla^2 \theta', \quad \text{and}$$

$$u' = \frac{\partial u'}{\partial x} = \zeta' = \theta' = 0 \quad \text{at} \quad x = \pm 1/2.$$

Assuming now that the disturbances have the spatial dependence in the form

$$q'(x, y, z, t) = q(x, t) \exp(i k \cos x z + i k \sin x y) \quad 59$$

and denoting $D \equiv \frac{\partial}{\partial x}$ the following set is obtained,

$$\frac{\partial}{\partial t} (D^2 - k^2) u + i k \cos x Gr [\bar{w} (D^2 - k^2) - D^2 \bar{w}] u$$

$$+ i k \cos x Ta \zeta = Fr k^2 \theta - i k \cos x D \bar{\theta} + (D^2 - k^2)^2 u,$$

$$\frac{\partial \zeta}{\partial t} + i k \cos x Gr \bar{w} \zeta + i k \sin x Gr D \bar{w} u \quad 60$$

$$- i k \cos x Ta u = i k \sin x \theta + (D^2 - k^2) \zeta,$$

$$\frac{\partial \theta}{\partial t} + i k \cos x Gr \bar{w} \theta + Gr D \bar{\theta} u = \frac{1}{Pr} (D^2 - k^2) \theta,$$

$$u = Du = \zeta = \theta = 0 \quad \text{at} \quad x = \pm 1/2$$

The stability equations in this form are the most convenient for the study of three-dimensional disturbances. Although it is useful at this point to look for a simplification in the form of the limiting cases of the two-dimensional problems, it is clear that in the absence of transformations to reduce the problem to an equivalent two-dimensional one, the complete solution must include the treatment of the three-dimensional disturbances. In addition, on physical grounds there is no reason to rule these out, mainly due to the presence of the Coriolis term.

In terms of the two limiting two-dimensional problems it is noted that no substantial simplifications occur in the case when the orientation of the waves is horizontal ($\chi = 0$). On the other hand when the disturbances are vertical ($\chi = \pi/2$) the equations simplify sufficiently to warrant a separate discussion. This will be done below after which the solution of the general problem is presented.

6.2.1. Stability Equations of the Longitudinal Mode

Letting $\cos \chi = 0$ and $\sin \chi = 1$ in the equations 60, the following set is obtained

$$\frac{\partial}{\partial t} (D^2 - k^2) U = k^2 Fr \Theta + (D^2 - k^2)^2 U$$

$$\frac{\partial \zeta}{\partial t} + i k \cos \chi Gr D \bar{\omega} U = i k \omega \chi \Theta + (D^2 - k^2) \zeta$$

$$\frac{\partial \Theta}{\partial t} + Gr D \bar{\Theta} U = \frac{1}{Pr} (D^2 - k^2) \Theta \quad 61$$

$$U = D U = \zeta = \Theta = 0 \quad \text{at} \quad x = \pm 1/2$$

The great simplification in these equations is due to the decoupling of the vorticity equation. This, together with the absence of Taylor number from the remaining set, render the equations similar in structure to those governing the Benard problem. They are

$$\frac{\partial}{\partial t} (D^2 - k^2) U = k^2 Fr \Theta + (D^2 - k^2)^2 U$$

$$\frac{\partial \Theta}{\partial t} + Gr D \bar{\Theta} U = \frac{1}{Pr} (D^2 - k^2) \Theta \quad 62$$

$$U = DU = \Theta = 0 \quad \text{at} \quad x = \pm 1/2$$

6.3 Solution of the Eigenvalue Problem

Due to its simplicity, the longitudinal mode can be resolved forthwith. For stationary disturbances as is the case here the time derivatives are dropped which, with the elimination of the temperature give the equation

$$(D^2 - k^2)^3 U = -k^2 Fr Ra D \bar{\Theta} U$$

63

$$U = DU = (D^2 - k^2)^2 U \quad \text{at} \quad x = \pm 1/2$$

For a conduction regime for which $D \bar{\Theta} = 1$ the classical solution of the Benard problem is obtained with the onset of instability at

$$Ra Fr = 1708 \quad \text{and} \quad k = 3.12 \quad 64$$

Applying now the Galerkin procedure to equations 60 a matrix eigenvalue problem is generated. The expansions

$$\psi(x,t) = \sum_{n=1}^{\infty} a_n(t) C(\lambda_n x) + i b_n(t) S(\mu_n x)$$

$$\xi(x,t) = \sum_{n=1}^{\infty} d_n(t) \sin k_n x + i e_n(t) \cos e_n x \quad 65$$

$$\theta(x,t) = \sum_{n=1}^{\infty} f_n(t) \cos e_n x + i g_n(t) \sin k_n x$$

which have been more fully defined in Section 2.4.2.1 result after orthogonalization in the matrix equation of the form

$$B \frac{d\bar{x}}{dt} + C \bar{x} = \Delta \quad 66$$

with the matrices B and C containing only real elements. The order of the matrices, after truncation of each of the series to N terms is 6N.

During this phase of the study a new subroutine appeared tailored to solve an eigenvalue problem of the form of equation 66 without inverting the matrix B. The procedure, developed by Moler and Stewart (1971), called the QZ algorithm, is a generalization of the QR method (Wilkinson (1965)) and reduces to it when $B = I$. It has an advantage over the I.B.M. subroutine in using only orthogonal transformations which stabilize the calculation. This procedure was used in the subsequent calculations when the full spectrum of eigenvalues was calculated.

CHAPTER VII

RESULTS AND CONCLUSIONS

7.1 Results for Centrifugal Instability

When the centrifugal forces are large enough the instability leads to the formation of vertical cells. As was shown above, the instability occurs at

$$RaFr = 1708 \quad \text{and} \quad K = 3.12 \quad 67$$

To compare this with the results of the wide gap problem treated by Chandrasekhar (1954), the parameters defined by him are first transformed to those used here. In particular, if the Froude number is defined in terms of the average radius, the parameter S given by him as

$$S' = \frac{\Omega^2 \bar{R}^4 \Delta T}{\alpha \nu \ln R_0/R_i} \quad 68$$

can be written as

$$S = RaFr \frac{2}{(1+\eta)(1-\eta)^3 \ln' 1/\eta} \quad 69$$

where $\eta = R_i/R_0$ and the subscripts on R relate to the inner and outer cylinders, respectively. By the use of these relations the results in Table 4 in Chandrasekhar's paper can be given in terms of $RaFr$. The actual numerical values are shown in Table V. Only values up to $\eta = 0.5$ were calculated by him so that a comparison at values close to $\eta = 1.0$ which should correspond very closely to the limiting case given here cannot be made. The tendency, in the values depicted, to the limiting value, is however present.

The number of waves can also be related to the limiting solution. By measuring the wavelength at the average radius, and equating this to

the solution of the narrow gap problem an approximate relation for the number of waves is obtained. This is given by

$$m = \frac{k(1+\eta)}{2(1-\eta)} \quad 70$$

in which $k = 3.12$. A comparison with the entries in Table VI shows that the correct number of waves is predicted in each case; even at $\eta = 0.2$.

7.2 Results for $Pr = 0$.

The results for the onset of instability due to the vorticity distribution, and in particular the case when the problem is completely hydrodynamic are now presented. This condition is obtained by setting the Prandtl number to zero which in turn leads to the vanishing of the temperature perturbations and consequently, the problem becoming also independent of the Froude number. The parametric set left consists of Gr , Ta , k , and χ .

In the presence of rotation it might be expected that the motions as a result of instability are three-dimensional. This follows from the fact that the Coriolis forces lie in the plane perpendicular to the rotation vector, whereas the motions due to instability in the absence of rotation are in vertical planes containing the axis of rotation. Three-dimensional motions are then a likely outcome when both of these effects are important. The Taylor number is a measure of their relative importance. Since the zero Taylor number case has been discussed in Part I, the large Taylor number limit is of interest.

If it is accepted that when the Taylor number is large $\cos \chi \ll \sin \chi$ an assumption to be justified a posteriori, the equations 60 for $Pr = 0$ and instability setting in through stationary states reduce to

m	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	1st appr. $\eta=0.5$	2nd appr. $\eta=0.5$
1	3697			12054	12327
2	2032	2130		3673	3777
3	2479	1929	1879	2209	2278
4	3832	2423	1800	1819	1880
5			2244	1801	1864
6				2002	2061

Table V. Critical states for the wide gap problem

η	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Eq. 70	30	14	9	6	5	4	3	2	2
Table V.	-	-	-	-	5	4	3	2	-

Table VI. Critical wave number for the wide gap problem

$$ik \cos x T_a \zeta = (D^2 - k^2)^2 U,$$

$$ik \sin x Gr D\bar{\omega} U - ik \cos x T_a U = (D^2 - k^2) \zeta, \quad 71$$

$$U = DU = \zeta = 0 \quad \text{at} \quad x = \pm 1/2.$$

Elimination of ζ in these equations leads to

$$(D^2 - k^2)^3 U = -k^2 [Gr T_a \sin x \cos x D\bar{\omega} - T_a^2 \cos^2 x] U, \quad 72$$

$$U = DU = (D^2 - k^2)^2 U = 0 \quad \text{at} \quad x = \pm 1/2.$$

Although the equation 71 can be solved easily enough for example, by the methods of Chandrasekhar (1961), at present only the behavior of the parameters in this limit are of interest. Multiplying the equation 72 with U and integrating gives the condition

$$\Lambda = -\frac{1}{k^2} \frac{\langle U, (D^2 - k^2)^3 U \rangle}{\|U\|^2} \quad 73$$

The parameter Λ is defined by

$$\Lambda = Gr T_a \sin x \cos x D\bar{\omega}(\xi) - T_a^2 \cos^2 x \quad 74$$

in which use has been made of the integral mean value theorem

$$\int_{-1/2}^{+1/2} D\bar{\omega}(x) U^2(x) dx = D\bar{\omega}(\xi) \langle U, U \rangle = D\bar{\omega}(\xi) \|U\|^2. \quad 75$$

From the relation 74 the value of minimum Gr as function of T_a for a fixed x is easily obtained. It occurs when

$$Ta_c = \Lambda^{1/2} / \cos \chi. \quad 76$$

Alternatively with Ta held constant

$$\chi_c = \tan^{-1} \sqrt{1 + \frac{Ta^2}{\Lambda}}. \quad 77$$

From the last expression χ is seen to tend to $\pi/2$ as Ta approaches infinity. The critical Grashof number follows the relation

$$Gr_c = \frac{2\Lambda^{1/2}}{D\bar{w}(3)} \left[1 + \frac{1}{2} \left(\frac{\Lambda}{Ta^2} \right) \right] + O(Ta^{-4}) \quad 78$$

The usefulness of the relations 76-78, even though they give an indication of the trend to be expected, is limited since numerical values for the quantities Λ and $D\bar{w}(3)$ are not known. The procedure which was adopted was to first determine these quantities from the complete problem for very large values of Taylor number and then extrapolate the results to lower values and so obtain an approximation useful as an initial guess during a search of the critical state. The value of Λ obtained in this manner for $Ta = 20000$ was 1725, and that for $D\bar{w}(3) = 0.029$. With this preliminary effort the neutral stability curves were calculated for the full problem.

In Figure 11 the variation of the critical Grashof number is shown as a function of the magnitude of the wave vector k , for three Taylor numbers. The phase of the wave vector χ was kept constant on these curves at a value close to the critical one. The dependence of the minimum value of k was found to depend only weakly on the Taylor number, the values falling between the limits 2.69 - 3.12, the lower one corresponding to the case of no rotation, the latter was approached as the Taylor number was increased. In Figure 12 the neutral stability curves

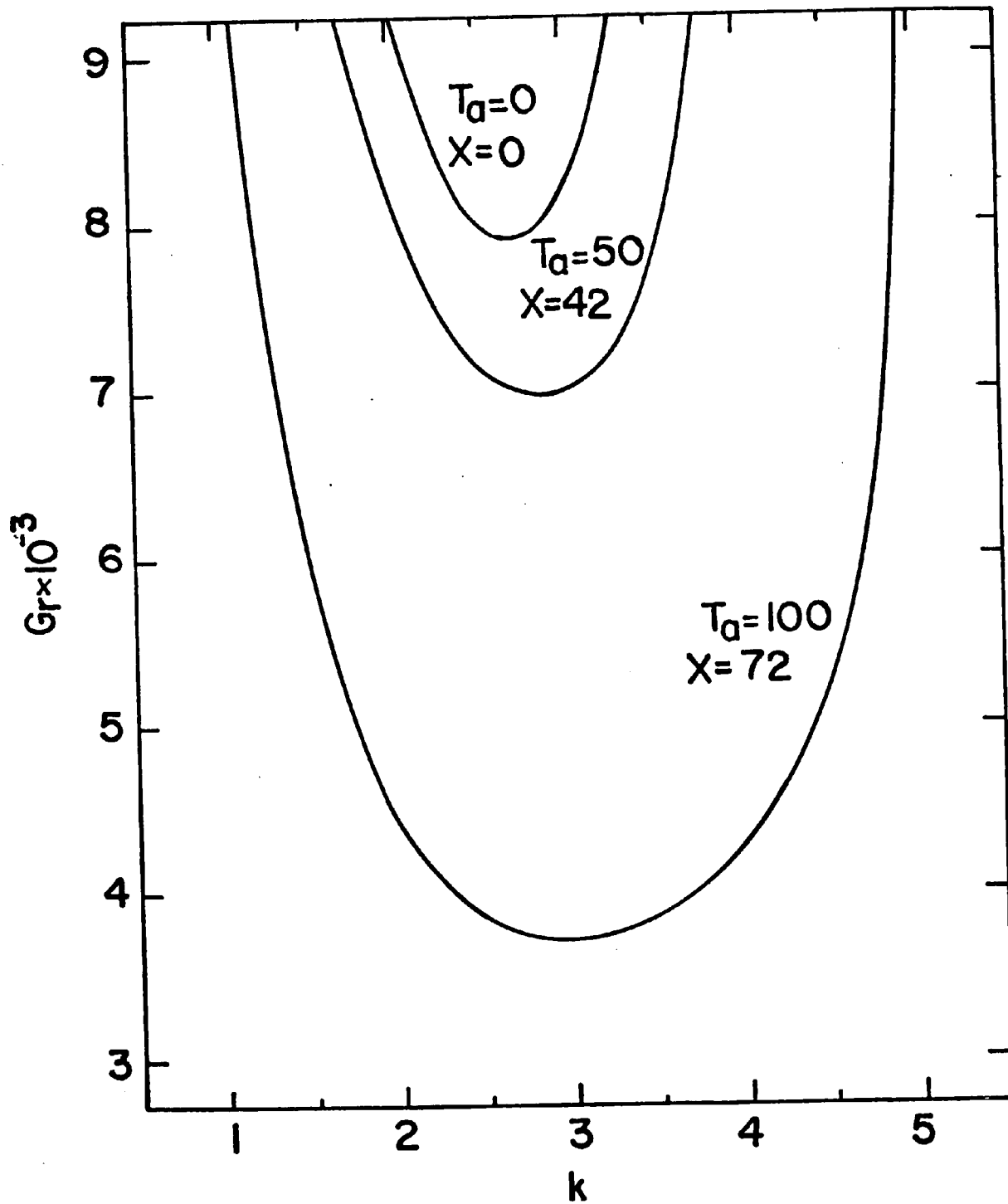


Figure 11. The Variation of the Critical Grashof Number as a Function of the Magnitude of the Wave Vector (k). The Minimum of Each Curve Defines a Critical Point.

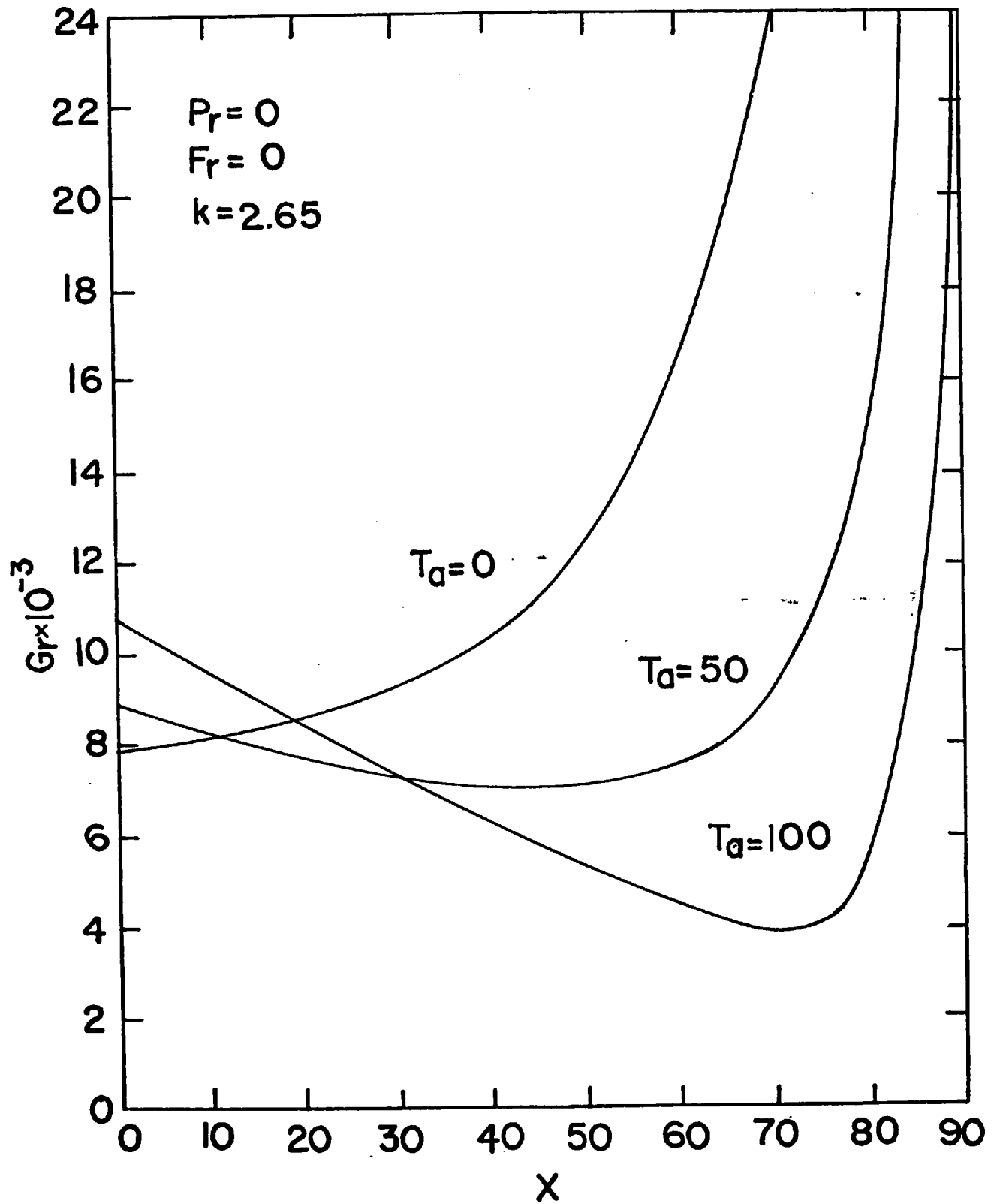


Figure 12. The Variation of the Critical Grashof Number as a Function of the Phase of the Wave Vector (χ).

for the same Taylor numbers are shown as a function of the phase angle χ . A constant value of $k = 2.65$ is associated with each of the curves. The three-dimensional nature of the problem is well exhibited in this figure as the critical points corresponding to the minimum of each curve, shifts to the right tending toward the value of 90° . This gives validity to the approximations undertaken in deriving equations 71. The physical interpretation of this shift is associated with the formation of rolls, as a result of instability, which are oriented horizontally in the absence of rotation; tilting to approximately 45° when $Ta = 50$ and assuming, on further increase of Taylor number, a structure which is predominantly vertical.

The collection of the critical states is summarized in Figure 13. The critical Grashof number starting from a value of 7932 in the absence of rotation diminishes to 2867 which is given as an asymptote in Figure 13 for very large values of Taylor number.

7.3 Results for $Pr \neq 0$.

For non-zero Prandtl numbers the effect of rotation is manifested in both the Froude and the Taylor number. To fix ideas in this situation a sketch of a three-dimensional stability diagram is given in Figure 14. With the axes of Froude number and the square of the Taylor number an increase in the rotation rate, with the other parameters held fixed, corresponds to following a ray from the origin, oriented in a direction compatible with the scales of the axes. The lower left hand portion in Figure 14 corresponds to centrifugal instability, termed here as Class I instability. The top portion is associated primarily with the vorticity distribution of the base flow and is basically of the

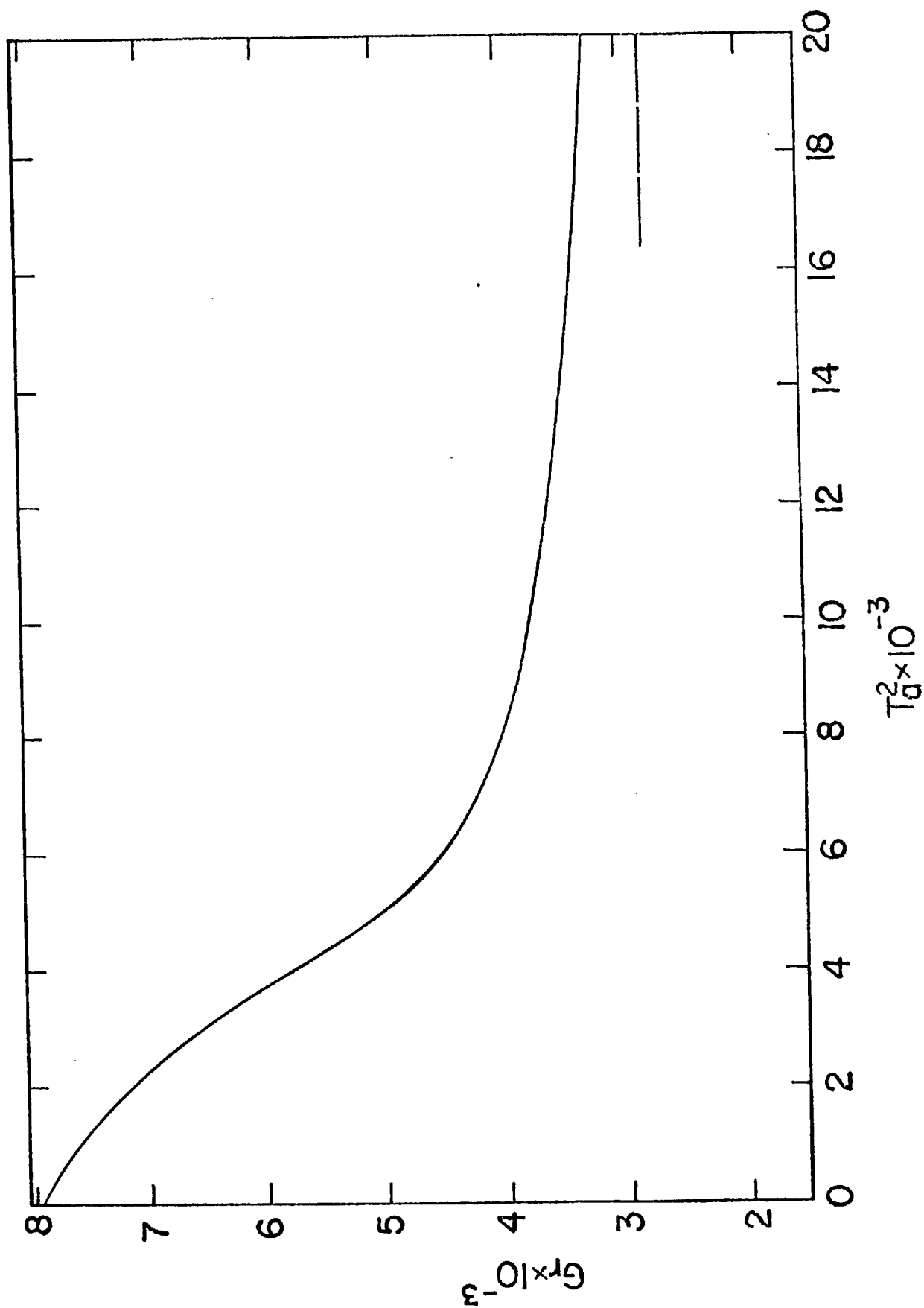


Figure 13. The States of Neutral Stability for $Pr=0$.

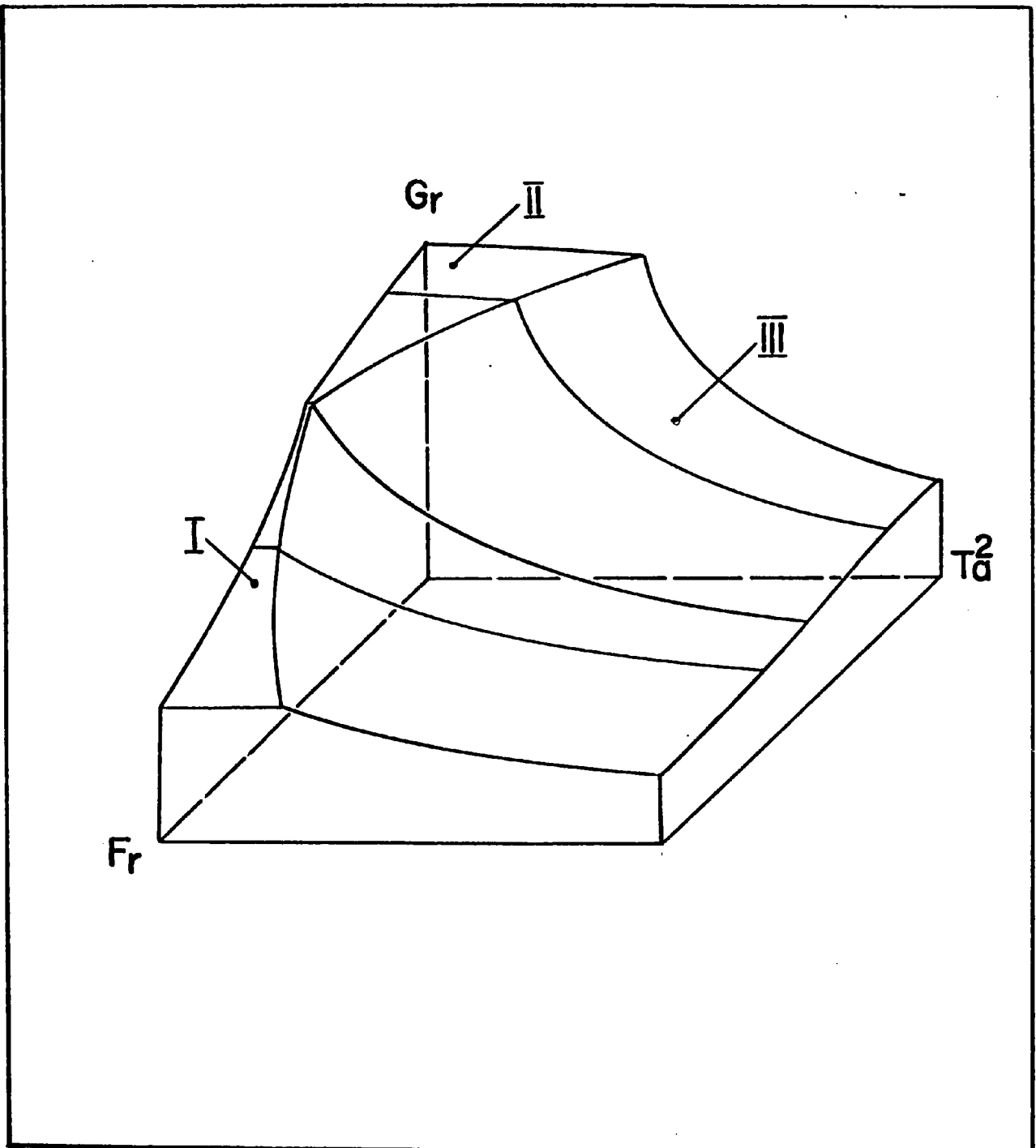


Figure 14. A Sketch of the Neutral Stability Surface for $Pr = 0.72$ Indicating the Regions of Class I, II, and III Type of Instability.

type of instability which occurs in the absence of rotation for low Prandtl numbers. This is called Class II type. Finally, on the right side of the figure the Coriolis forces play an important part. This instability is called Class III type. For $Pr = 0.72$ (air) the surfaces of constant Froude number are shown in Figure 15. The point of division of the instability classes I and II occurs at $Fr = 0.34$, for $Ta = 0$, the lower values of Fr corresponding to Class II type. The transition points between Classes II and III is seen to occur at about $Ta = 33$ for $Fr = 0$, this point moving to lower values of Taylor number as the Froude number is increased. An idea of how this transition occurs for $Fr = 0$ as a function of the phase angle is displayed in Figure 16. The Taylor number is the parameter in these curves. The value of $k = 2.2$ which gives the minimum Grashof number at the minima of the curves of Figure 16 is held constant for the entire set. Considering the curves for $Ta = 35$ and 40 it is noted that basically two values of minima occur, the first corresponding to Class II type of instability, and quite insensitive to Taylor number variation, takes place near $\chi = 25^\circ$. The other minimum associated with Class III instability depends strongly on Taylor number and occurs in the neighborhood of $\chi = 70^\circ$. Between these two instabilities there is a stable region which extends its peak to very large values of Grashof number. On increasing the Taylor number the peak is brought down, indeed for $Ta = 60$ it has been brought down sufficiently so that one can speak only of one minimum for the curve.

A spectrum of eigenvalues is shown in Figure 17 for a typical case near the nose of the Class III instabilities. The damping rates are plotted as a function of Grashof number for values of $\chi = 71.5^\circ$, $k = 2.2$ and $Ta = 35$. The lowest eigenvalue stays real throughout the variation

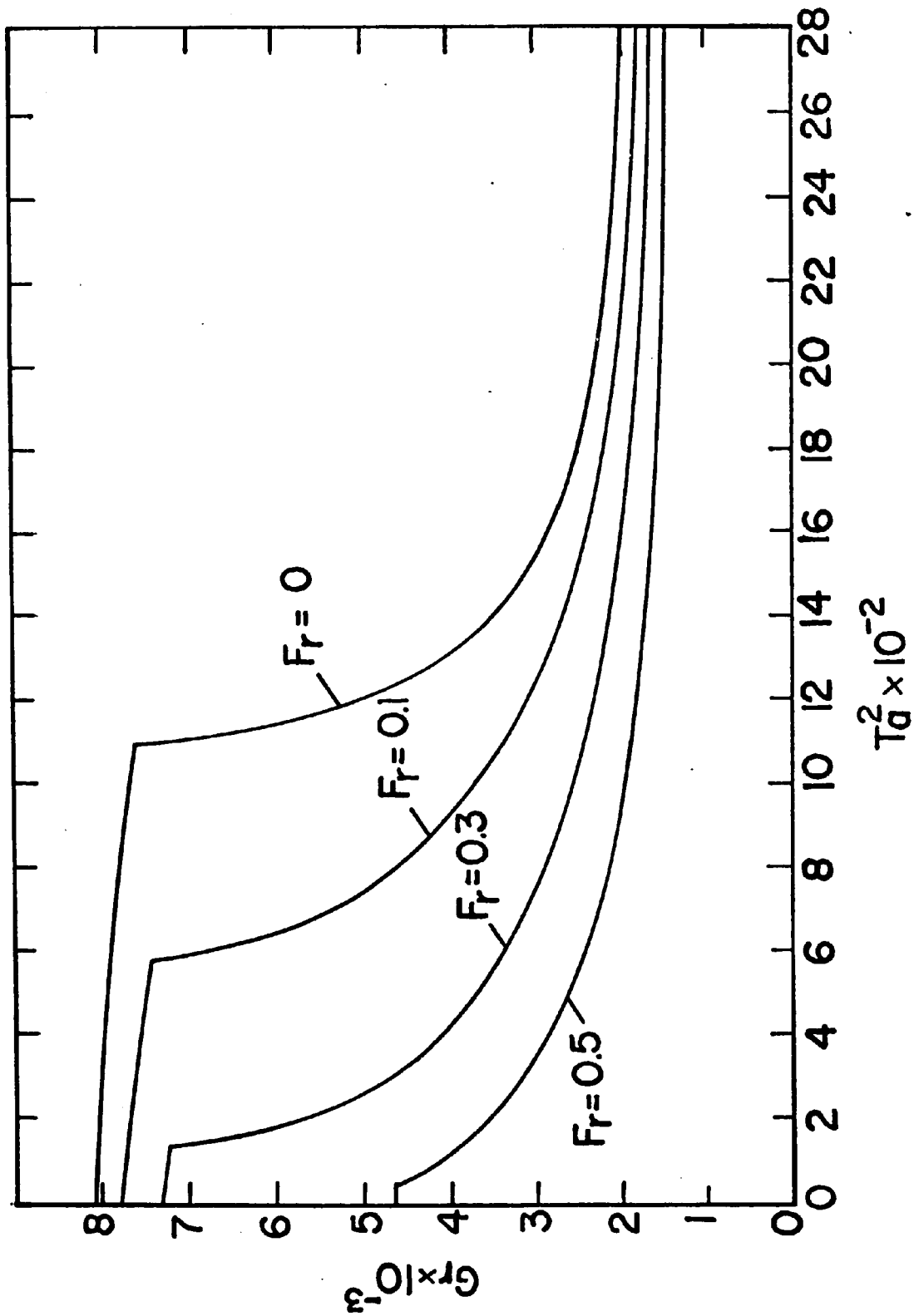


Figure 15. The States of Neutral Stability for Air ($Pr = 0.72$).

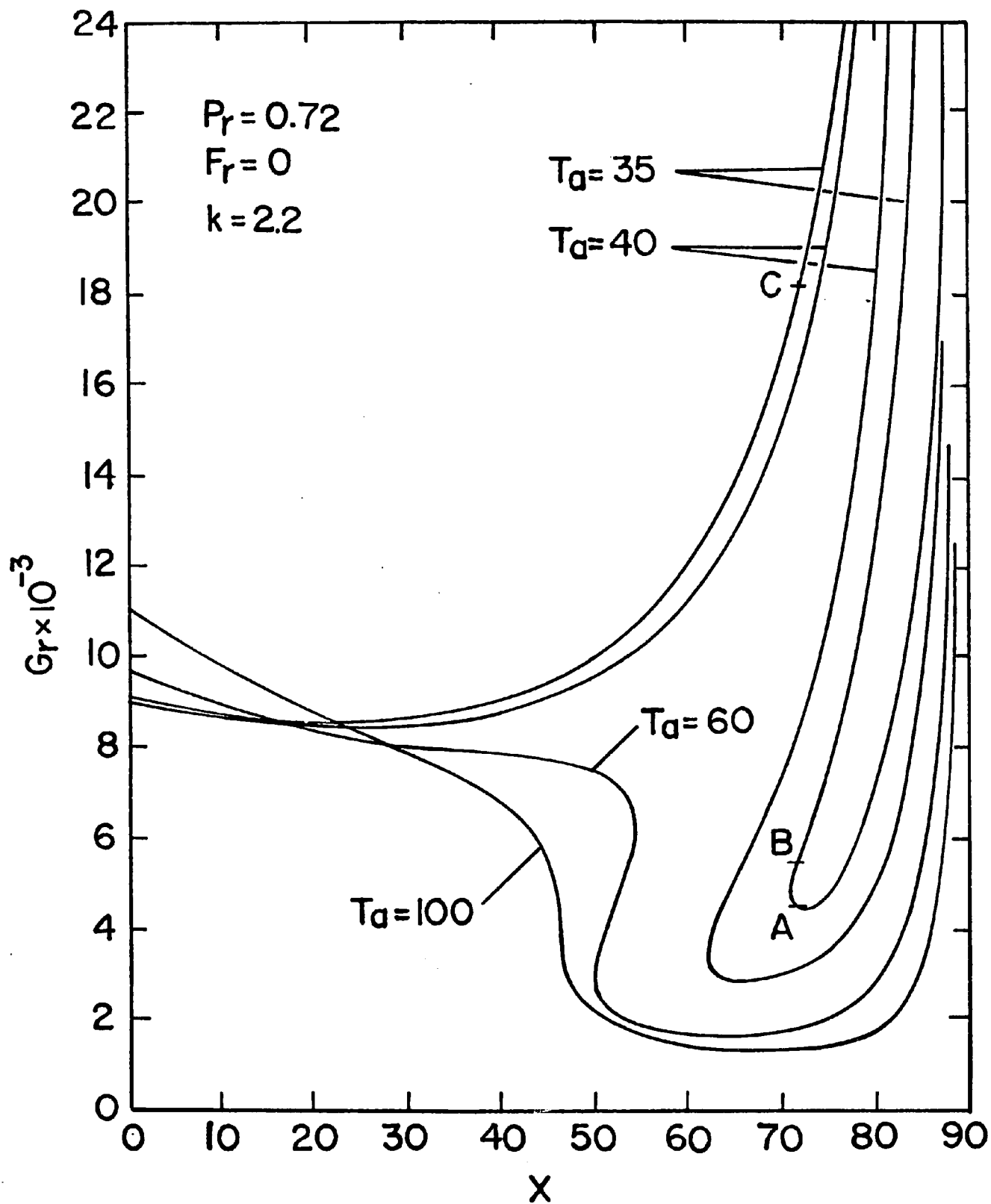


Figure 16. The Dependence of the Critical Grashof Number on the Phase Angle (X) in the Transition Region for Class II and III type of Instabilities for Air.

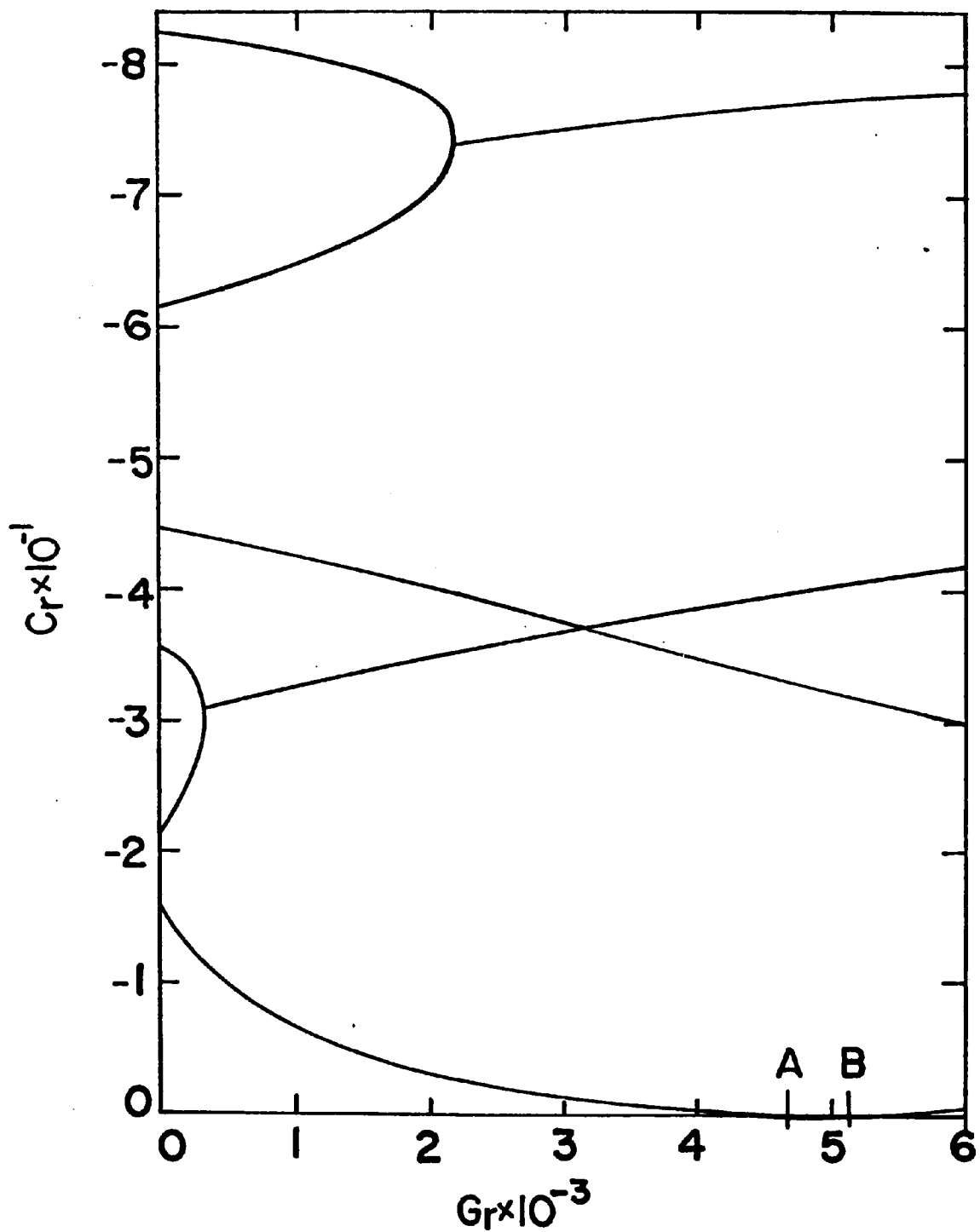


Figure 17. The Damping Rates for Air at $Ta = 35$, $\chi = 71.5$ as a Function of the Grashof Number.

of the Grashof number becoming unstable at the point A and after traversing the unstable region, stable again at the point B. The second and third eigenvalues combine into complex conjugate pairs at $Gr = 400$, the real part of which stays in the damped region for all Grashof numbers considered. The fourth eigenvalue is associated with Class II type of instability and becomes critical at the point C in Figure 16.

On decreasing the Taylor number there is an increase in the minimum of the instability of the Class III type. This tendency causes finally a shift in the instability to the Class II type, the transition point for $Fr = 0$ being at $Ta = 33$, as has been mentioned. With this transition there is a discontinuity in the slope of the critical curve, as well as in the phase angle. The critical k also changes abruptly from 2.2 for Class III instability to 2.9 for the Class II type. What actually happens in a physical situation depends on the relative amplification rates of the corresponding disturbances and after the initial growth on the non-linear terms of the equations which we neglected in the study of the onset of instability.

For water ($Pr = 6.7$) the situation is similar to that for air. In Figure 18 the critical Grashof number in the transition region between Class II and III type of instabilities are given as a function of χ . The Class III instability is now concentrated to very narrow regions for $Ta = 32$ and the slope of the neutral curve as shown in Figure 19 is very high. The Class I type of instability becomes active for small values of Froude number which is evident on Figure 19 in which it is the dominant instability for $Fr = 0.1$ for low Taylor numbers.

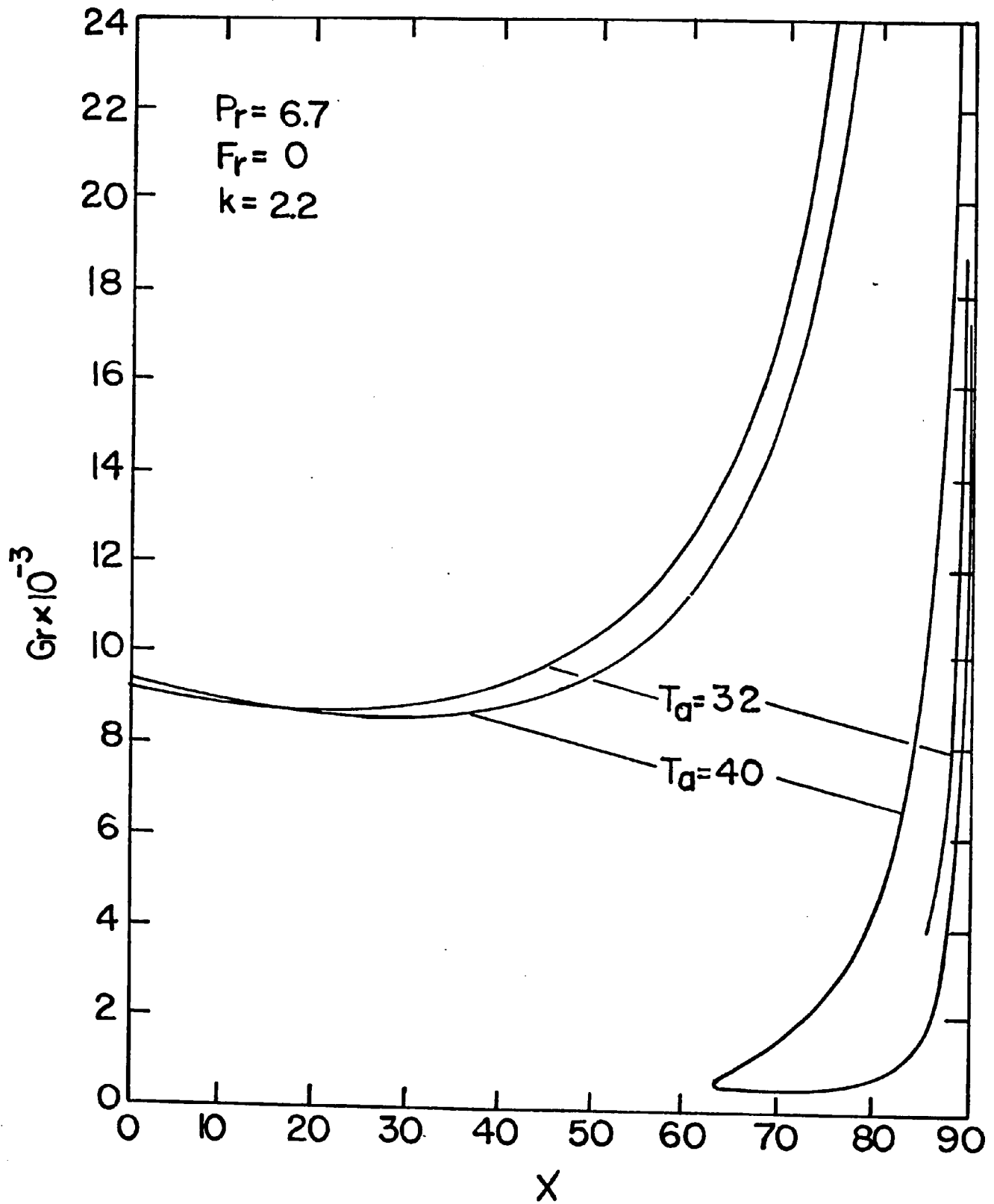


Figure 18. The Dependence of the Critical Grashof Number on the Phase Angle (X) for Water ($Pr = 6.7$).

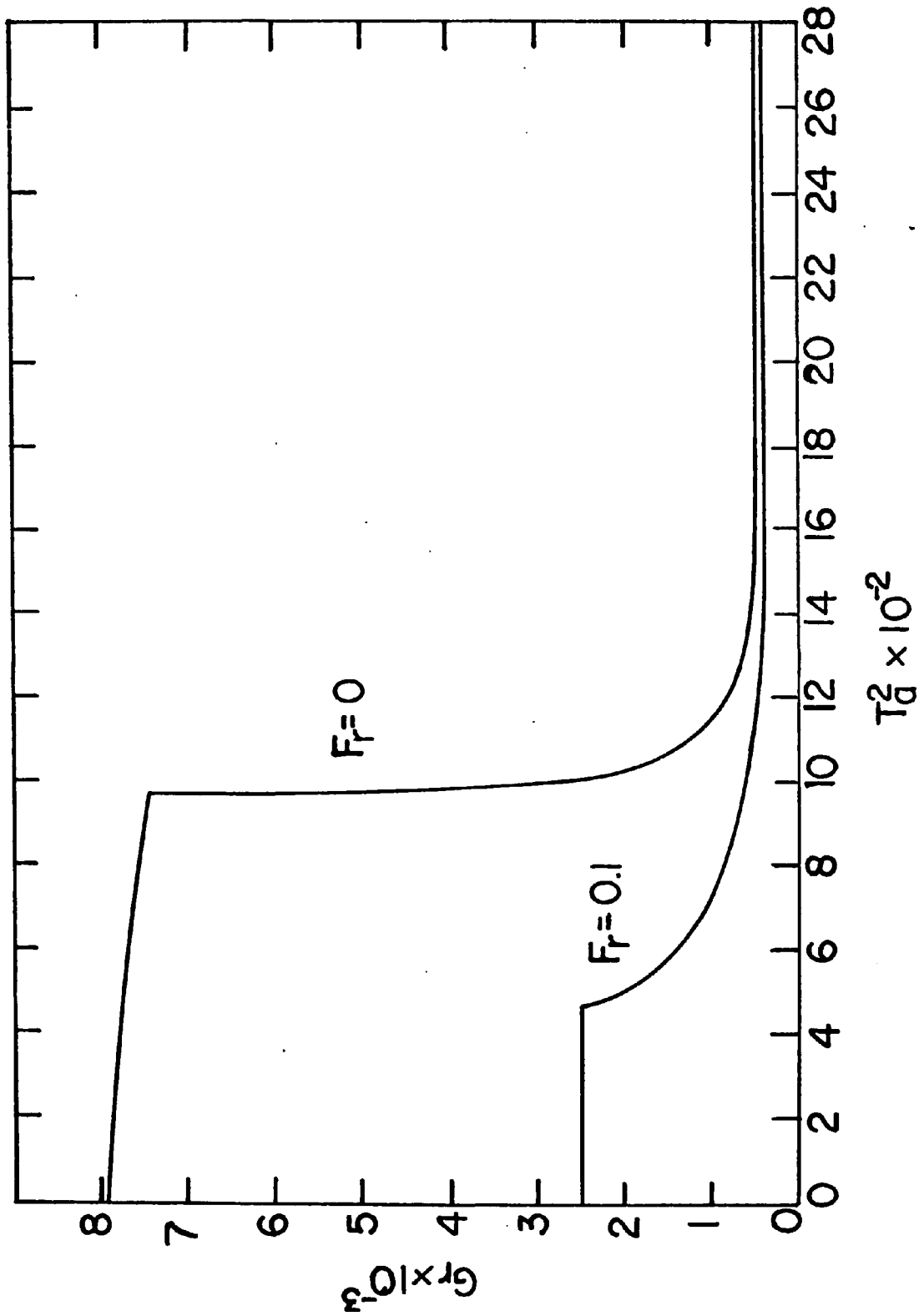


Figure 19. The States of Neutral Stability for Water ($Pr = 6.7$).

APPENDIX
A VARIATIONAL PRINCIPLE IN
TERMS OF AN ADJOINT EQUATION

Roberts (1960) has shown the existence of a variational principle in terms of an adjoint system for a set of equations which are nonself-adjoint. His method is applied here for the present problem.

Considering first the simplest case when T_a , F_r , and P_r are all zero, the equations 58 reduce to

$$LU = i\alpha Gr MU, \quad 79$$

$$U = DU = 0 \quad \text{at} \quad x = \pm 1/2,$$

where

$$L = (D^2 - \alpha^2 - c)(D^2 - \alpha^2), \quad 80$$

$$M = \bar{w}(D^2 - \alpha^2) - \bar{w}''.$$

The adjoint problem takes the form

$$LU^* = i\alpha Gr^* M^* U^*, \quad 81$$

$$U^* = DU^* = 0 \quad \text{at} \quad x = \pm 1/2,$$

where

$$M^* = \bar{w}(D^2 - \alpha^2) + 2\bar{w}' \quad 82$$

Taking now a particular characteristic value and its associated function, for both the original and its adjoint problem, and orthogonal-

izing the equation 81 in terms of the characteristic function u_i , the following integrals are seen to be equal

$$\begin{aligned}
 i\alpha Gr_j^* \int_{-1/2}^{+1/2} u_i M^* u_j^* dx &= \int_{-1/2}^{+1/2} u_i L u_j^* dx \\
 &= \int_{-1/2}^{+1/2} u_j^* L u_i dx = i\alpha Gr_i \int_{-1/2}^{+1/2} u_j^* M u_j dx \\
 &= i\alpha Gr_i \int_{-1/2}^{+1/2} u_i M^* u_j^* dx
 \end{aligned} \tag{83}$$

From these it is seen that

$$i\alpha (Gr_i - Gr_j^*) \int_{-1/2}^{+1/2} u_i M^* u_j^* dx = 0 \tag{84}$$

Now if $Gr_i \neq Gr_j^*$ then

$$\int_{-1/2}^{+1/2} u_i M^* u_j^* dx = 0 \tag{85}$$

Also if it is assumed that the set $\{u_i\}$ forms a complete set of functions, then for $i=j$ the relation 84 implies that the function $M^* u^*_j$ is orthogonal to all functions of the set $\{u_i\}$ which in turn requires that $M^* u^*_j = 0$. From equation 81 it follows then that $u^*_j = 0$. This is not true, however, since u^*_j is the eigensolution of the set 81 leading to a contradiction so that the relation

$$(Gr_j - Gr_j^*) \int_{-1/2}^{+1/2} u_j M^* u_j^* dx = 0 \tag{86}$$

must imply that $Gr_j = Gr_j^*$. This proves that the equation 81 together with its boundary conditions is the adjoint problem for the set 79.

Furthermore, the sets $\{u_j\}$ and $\{M^* u_j^*\}$ are bi-orthogonal and from
83 the eigenvalue can be represented as the ratio

$$i\alpha Gr_i = \frac{\int_{-1/2}^{+1/2} u_i^* L u_i dx}{\int_{-1/2}^{+1/2} u_i^* M u_i dx} = \frac{I_1}{I_2} \quad 87$$

Now considering the effect of a small variation in u_i and u_i^* on Gr_i , it follows that

$$\begin{aligned} i\alpha \delta Gr_i &= \frac{1}{I_2} (\delta I_1 - \frac{I_1}{I_2} \delta I_2) \\ &= \left[\int_{-1/2}^{+1/2} (\delta u_i^* L u_i + u_i^* L \delta u_i) dx - i\alpha Gr_i \int_{-1/2}^{+1/2} (\delta u_i^* M u_i + u_i^* M \delta u_i) dx \right] / \int_{-1/2}^{+1/2} u_i^* M u_i dx \\ &= \left[\int_{-1/2}^{+1/2} \delta u_i^* (L u_i - i\alpha Gr_i M u_i) dx + \int_{-1/2}^{+1/2} \delta u_i (L u_i^* - i\alpha Gr_i M u_i^*) dx \right] / \int_{-1/2}^{+1/2} u_i^* M u_i dx \\ &= 0 \end{aligned} \quad 88$$

proving the variational principle.

Finally, from equation 87 if u_i and u_i^* are expanded in terms of the same set of orthogonal functions the Galerkin equations follow from the problem of finding an extremum for Gr_i as a function of the coefficients a_n , b_n and d_n , e_n of the expansions representing the functions u_i and u_i^* , respectively. Since the approximating functions chosen form a complete orthogonal set, the adding of more terms into the series does not change the value of the previously calculated coefficients. Since in any truncated set the coefficients are also chosen to solve the extremum problem, each additional term added to the series provides a better

approximation for the eigenvalue, leading for it, a sequence of approximations which converge to the actual eigenvalue.

In constructing a sequence of this type by increasing the number of terms in the series the other parameters of the problem are kept fixed. Although this does not represent any practical difficulty when the instability sets in as stationary convection as the parameters are provided as input data, this is not the case when the flow becomes overstable, since the wave speed is not known a priori. In solving the problem as an eigenvalue problem in the wave speed, the neutral stability would in general be associated with different values of both Gr and $Im(c)$ from one approximation to the next.

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