Gravitational Lensing by a Naked Singularity in Vaidya Spacetime

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In this paper we explored how the telescope image of circular astronomical objects are distorted by the gravitational lensing of a black hole in Vaidya spacetime, which is a dynamic spacetime for a spherically symmetric collapsing star. When the accumulation of mass is slow enough, a naked singularity can be formed. We used numerical ray tracing technique to get the telescope image of a distant ring across the black hole and explored qualitative changes of the image before and after the formation of a naked singularity. A derivation for the criterion of naked singularity formation and physical constraints of an effective photon sphere are provided for Vaidya spacetime. To investigate the problem, we invented a simple iterative algorithm and projection method to find telescope images that can be useful for related research in the future.

I. INTRODUCTION

Previous work has provided us tools to discover naked singularity in Janis-Newman-Winicour (JNW) spacetime, which is a static spacetime of a naked singularity with photon sphere, using time difference in gravitational lensing [1]. We wish to provide an alternative tool for dynamical spacetime, to discover Vaidya naked singularities.

A method we found to determine whether a black hole in Vaidya spacetime is a naked singularity is by looking at the eccentricity of telescope images formed when observing a distant ring across the black hole from various angle. This method is useful when looking at a black hole closed by, which might not be practical.

A globally naked singularity in Vaidya spacetime can be developed under the constraint: $\sup_{v>0} m'(v) \leq 1/16$, and it can not be formed if $\exists v_0 > 0$, s.t. $m(v_0)/v_0 \geq 1/4$ [2]. These arguments are weaker versions of Wheeler's results but more illuminating from a physics point of view. For the specific kind of linearly increasing mass function we are interested in $m'(v) = \mu > 0$, the neccesary and sufficient boundary between naked singularity formation and non-naked singularity formation is strictly $1/16$.

In order to set up the ray tracing correctly, one need to avoid tracing light from below the photon sphere of a black hole. Since Vaidya spacetime is dynamic, there exists no rigorous definition of a photon sphere, but light can still get trapped below a particular surface. Previous formulation of an extended photon surface is explored by Ellis et al [3]. Instead we will provide an approximation to the photon sphere in Vaidya spacetime that captures similar meaning where light can orbit in a close circular pattern certain radius away from the singularity. It happens when the accumulation of mass is slow for a massive black hole. The effective photon sphere of a Vaidya black hole has a $9m'(v)m(v)$ linear correction term from the direct Schwarzschild photon sphere approximation $3m(v)$ with a 17% increase when $m'(v) = 1/16$, the threshold for naked singularity formation. We won't be using this result specifically when performing numerical ray tracing, but it is something to keep in mind of.

To investigate the problem, we invented a simple iterative algorithm called the step algorithm and a planar projection method to find telescope images that can be useful for related research in the future. Detailed introduction of these algorithms are given in the appendix.

II. FORMATION OF GLOBALLY NAKED SINGULARITY IN VAIDYA SPACETIME

The Vaidya metric is well suited to describe the spacetime of a black hole, accrediting or radiating null energy over time. It can be obtained by replacing the $m = GM/c^2$ term in the Eddington–Finkelstein coordinate to $m(v) = GM(v)/c^2$, where $M(v)$ is a function for the mass of the black hole over the advanced time coordinate v.

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{1}$$

Outgoing null geodesic in Vaidya spacetime is written as

$$r'(v) = \frac{1}{2} - \frac{m(v)}{r(v)}, \tag{2}$$

To find the criterion for a globally naked singularity to form in this spacetime, we want to find a family of null trajectories such that

$$\lim_{v\to 0} r(v) = 0 \qquad \lim_{v\to\infty} r(v) = \infty, \tag{3}$$

Locally naked singularity satisfies boundary condition (3a), but not (3b). In this paper we are interested in globally naked singularity with non-decreasing mass function $m(v)$ that starts from zero and stops accumulating at some time $v_f > 0$.

The apparent horizon of a Vaidya black hole is at $2m(v)$, which is strictly smaller than its event horizon. Therefore whenever $r(v) < 2m(v)$, light is trapped, and it will stay inside the apparent horizon forever. We will show this result by defining the quantity $\Delta(v) = r(v) - 2m(v)$. For $v > 0$, since $m'(v), r(v) \geq 0$, the fol-

lowing inequality holds

$$\Delta'(v) = r'(v) - 2m'(v) = \frac{1}{2} - \frac{m(v)}{r(v)} - 2m'(v),$$
$$= \frac{\Delta(v)}{2r(v)} - 2m'(v) \leq \frac{\Delta(v)}{2r(v)}, \tag{4}$$

Therefore whenever $\Delta(v) \leq 0$, $\Delta'(v) \leq 0$. So, we can conclude that once $\Delta(v_0) \leq 0$, $\Delta(v) \leq 0$ for all $v > v_0$. And if $\Delta(v_0) \geq 0$, $\Delta(v) \geq 0$ for all $v \in (0, v_0)$.

A constant numerical threshold of the accumulation rate that sufficiently allows the formation of a naked singularity in Vaidya spacetime can be obtained by considering a more general family of curves with $\Delta_k(v) = r(v) - km(v)$ for $k > 0$ according to Wheeler [2]. $\Delta'_k(v)$ can be expressed as

$$\Delta'_k(v) = r'(v) - km'(v),$$
$$\Delta'_k(v) = \frac{1}{2} - \frac{m(v)}{r(v)} - km'(v),$$
$$\Delta'_k(v) = \left(\frac{1}{2} - \frac{1}{k} - km'(v)\right) + \frac{1}{k} - \frac{m(v)}{r(v)},$$
$$\Delta'_k(v) = k\left(\frac{1}{2k} - \frac{1}{k^2} - m'(v)\right) + \frac{\Delta_k(v)}{kr(v)}, \tag{5}$$

Define

$$f(k) = \frac{1}{2k} - \frac{1}{k^2} \tag{6}$$

The maximum of $f(k)$ is at $1/16$, see figure (1), which will be useful later. Choose $C \in (0, 1/16)$, the line $y = C$ intersects with $f(k)$ at two values k_- and k_+ where $2 < k_- < 4 < k_+$.

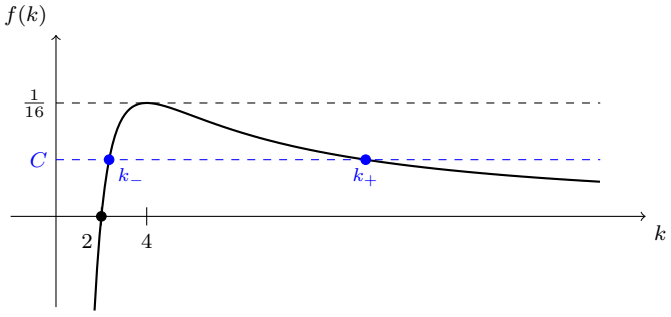


FIG. 1. Visualization of k_-, k_+ and $f(k)$

Assuming $\sup_{0\leq v\leq v_0} m'(v) < C$, we can find a solution $r(v)$ of the Vaidya geodesic such that $k_-m(v_0) \leq r(v_0) \leq k_+m(v_0)$, which means $\Delta_{k_+}(v_0) \leq 0$.

Note that all terms in equation (5) are non-negative. If $f(k) - m'(v) > 0$, then $\Delta'_k(v) > \Delta_k(v)/kr(v)$, which is the case when $m'(v) < C < 1/16$ on the interval $(0, v_0)$. Therefore once $\Delta_k(v_0) > 0$, $\Delta_k(v) > 0$ for all $v > v_0$, which implies once $\Delta_k(v_0) < 0$, $\Delta_k(v) < 0$, for all $v \in (0, v_0)$.

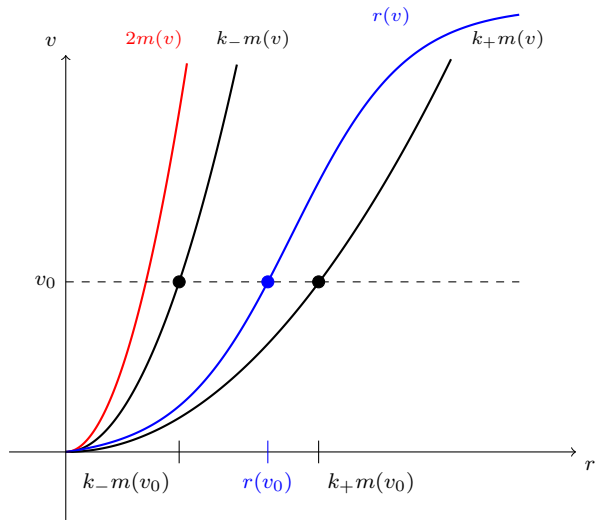


FIG. 2. Example of a null curve extending from 0 to ∞ bounded by $k_-m(v)$ when $v > v_0$ and $k_+m(v)$ when $v < v_0$.

Since $\Delta_{k_+}(v_0) \leq 0$, we know $r(v) < \Delta_{k_+}(v)$ for all $v \in (0, v_0)$. In polar coordinate $r < 0$ is undefined. Since $m(v)$ starts from zero, the bound $k_+m(v)$ therefore squeezes $r(v)$ to zero as v goes to zero.

Because $\Delta_{k_-}(v_0) \geq 0$, we know $\Delta_{k_-}(v) \geq 0$ for all $v > v_0$. Eventually, if not already passed, v_f will be reached, where the black hole is reduced to Schwarzschild. $r(v_f) > 2m(v_f)$ ensures the photon to escape to infinity. An illustration of the proof is given in figure (2).

The family of null curves described above are able to escape to infinity from the origin, giving us a sufficient but not necessary condition to the formation of naked singularity

$$\sup_{v>0} m'(v) \leq \frac{1}{16} \tag{7}$$

This criterion is not directly mentioned in the original paper. A stronger constraint $\sup_{v>0} m(v)/v < 1/16$ is developed by Wheeler in [2].

We can also find a lower bound for the accumulation rate to prevent naked singularity formation. Since $m(v), r(v) > 0$ for all $v > 0$, and $r'(v) < \frac{1}{2}$ with equality in the Minkowskian limit, in order for a photon to start from zero and escape to infinity, we must have $2m(v) < r(v) < \frac{v}{2}$. Note that we integrate $r'(v)$ from $\lim_{v\to 0} r(v) = 0$. Therefore a neccesary but not sufficient condition for the creation of a naked singularity can be obtained

$$\sup_{v>0} \frac{m(v)}{v} \geq \frac{1}{4} \tag{8}$$

If we restric the mass function to be a linearly increasing function starting from zero, where $m'(v) = \mu > 0$, then $1/16$ becomes a strict boundary between naked singularity formation and non-naked singularity formation. This is because when v is small, we can ignore higher order

terms in the Taylor expansion of $r(v) = a_0 + a_1 v + O(v^2)$, the solution of the geodesic equation

$$r'(v) = \frac{1}{2} - \frac{\mu v}{r(v)}, \tag{9}$$

Since $r(0) = 0$, we know that $a_0 = 0$. Assume linear term of $r(v)$ we obtained

$$0 = 2a_1^2 - a_1 + 2\mu \tag{10}$$

This gives us

$$a_1 = \frac{1 \pm \sqrt{1 - 16\mu}}{4} \tag{11}$$

if $\mu > 1/16$, then a_1 does not have a real solution. Therefore if $m'(v) > 1/16$, we can not have the trajectory $r(v) \to 0$ as $v \to 0$, making the singularity no longer locally and globally naked. A rigorous proof is given by Kuroda in 1984 [4].

III. EFFECTIVE PHOTON SPHERE

Since Vaidya spacetime is dynamic, a photon sphere where a photon traveling tangent to this surface will stay tangent to this surface does not exist. Instead we provide an alternative notion of a photon sphere in Vaidya spacetime that captures the meaning of a light trap surface where light remains in close circular orbit by frozing the spacetime to be static at a particular v, assuming the dynamics of spacetime do not change much comparing to the time scale for a photon to travel around the black hole as v progresses. We perform this froze of dynamical spacetime by treating $m'(v)$ and $m(v)$ in the null geodesic equations as constants. This approximation should be valid when $m'(v) < 1/16$ where a massive naked singularity is present.

The radial geodesic equation in Vaidya spacetime is parametrized by τ as follow

$$\begin{aligned}\ddot{r} = &-\left(\frac{m'(v)}{r} + \frac{m(v)(r-2m(v))}{r^3}\right)\dot{v}^2 \\ &+ (r-2m(v))\sin^2(\theta)\dot{\phi}^2 + \frac{2m(v)}{r^2}\dot{v}\dot{r} \\ &+ (r-2m(v))\dot{\theta}^2,\end{aligned} \tag{12}$$

Notice that the only difference between the radial geodesic in Vaidya metric and Schwarzschild metric is the additional $m'(v)\dot{v}^2/r$ term. Null constraint is given by

$$\left(1 - \frac{2m(v)}{r}\right)\dot{v}^2 = 2\dot{v}\dot{r} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2, \tag{13}$$

By inserting the null constraint into the radial geodesic, we obtained

$$\begin{aligned}\ddot{r} = &-\frac{m'(v)}{r-2m(v)}\left(2\dot{v}\dot{r} + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \\ &+ (r-3m(v))\left(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2\right),\end{aligned} \tag{14}$$

We are interested in circular orbit where $\dot{r} = 0$ and $\dot{\phi} = 0$. By setting $\ddot{r} = 0$, not allowing radial acceleration, we get

$$\begin{aligned}0 &= -\frac{m'(v)}{r-2m(v)}r^2 + r - 3m(v), \\ 0 &= (1-m'(v))r^2 - 5m(v)r + 6m^2(v)\end{aligned} \tag{15}$$

Solving for r we get

$$r_{eph} = \frac{\left(5 \pm \sqrt{1+24m'(v)}\right)m(v)}{2(1-m'(v))} \tag{16}$$

Since by setting $m'(v) = 0$, the positive solution is reduced to the Schwarzschild approximation $r_{ph} = 3m(v)$, we will focus on the positive solution. For $m'(v) < 1/16$, $\sqrt{1+24m'(v)} \approx 1 + 12m'(v)$. Effective photon sphere is obtained

$$r_{eph} \approx \left(\frac{1+2m'(v)}{1-m'(v)}\right)3m(v) \tag{17}$$

Using $(1+2x)/(1-x) = 1 + \sum_{n=0}^{\infty} 3x^n$, we can express alternatively

$$r_{eph} \approx \left(1 + \sum_{n=0}^{\infty} 3m'(v)^n\right)3m(v) \approx (1+3m'(v))\,3m(v), \tag{18}$$

Due to accumulation, effective photon sphere in Vaidya spacetime has a $9m'(v)m(v)$ first order correction term to the direct Schwzrzschild approximation. For $m'(v) < 1/16, m(v) < 1$, the error between equation (16) and (18) is within 1.5%. At $m'(v) = 1/16$, effective photon sphere has a radius of $r_{eph} = 3.51m(v)$, which is about 17% larger than the Schwarzschild approximation $r_{ph} = 3m(v)$.

IV. NUMERICAL RAY TRACING

Now we begin the computational aspect of this project. We will first start from the problem setup of the circle lensing method to discover naked singularity in Vaidya spacetime, get a feel for the general lensing feature by looking at various telescope images at different angle and accumulation rate, then analyze how eccentricity of telescope images change with respect to different looking angles and accumulation rates. The algorithms used for image generation are described in appendix A. The link to the github repository is provided in appendix B.

A. Circle Lensing Problem Setup

The problem statement is as follow: Based on an initial shooting angle, we wish to find the set of initial conditions that gets us to a ring fromed under the end of its initial trajectory across the black hole. After finding the set

of inital conditions, we then plot their trajectories and project them onto a plane close to the observer to perform smooth interpolation on the resulting points set with size about 100 to get the telescope image. Figure 3 is an illustration of the problem set up.

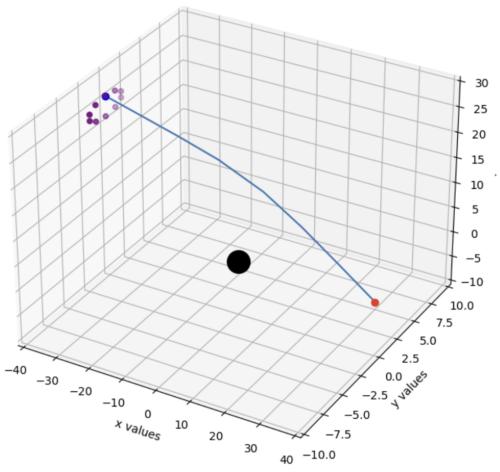


FIG. 3. An illustration of the circle lensing method set up.

We have 4 parameters of interest, the initial mass of the black hole for Vaidya spacetime, $m(v)$, (if the spacetime is Schwarzschild, m will stay constant over time). The radius of the circle, r, which we want to get the preimage of, the distance the light travels from the origin to the target, $2x$, the number of points on the circle, and the angle of the initial projection θ away from the x-axis. The initial ϕ angle is measured counter clockwise from the x axis. ϕ angle is needed to specify initial condition of a point on the ring, but it is not needed to specify the height of the ring due to spherical symmetry. The highest point of the ring is determined by θ and x alone.

B. Telescope Images

Now we will analyze general lensing features of Vaidya spacetime in three cases: Schwarzschild case, where $m'(v) = 0$, Vaidya spacetime with naked singularity case, where $m'(v) = 1/16$, and Vaidya spacetime without naked singularity case, where $m'(v) = 1/4$. We always fix observer's time coordinate v_0 to be after the final time of accumulation with sufficient enough time for lensing. The radius of the circle is fixed at $r = 2$ and initial and final masses are fixed at $m_0 = 0$ and $m_f = 1$.

To get a feel for the kind of lensing images available, we ploted telescope images formed by a ring 80 units across the black hole at eight looking angles that partition $\theta = \pi/10$ to $\theta = \pi/6$ evenly. We fixed observer's time coordinate at $v_0 = 85$ to ensure sufficient lensing features at the same time being able to distinguish the three cases.

We organized them into three columns for side by side comparison in figure (4). Most of the telescope images are elipses. A trend appeares where the larger the looking angle, the closer the projection is to a circle, or in another word, eccentricity increases with increasing looking angle. We also see that the steeper the slope of increase the closer the image is to Schwarzschild, and the lensing effect becomes more severe.

When the looking angle is small enough while it still allows light to pass over the effective photon sphere, a bean shaped telescope image appears.

How does a bean shaped image form? You might ask. By looking at figure (5), we can gain some qualitative insights. The left and right image in figure (5) are the 3D and 2D cross sections of ten trajectories that maps a circle to a bean. Now, zoom in onto the lowest and the second lowest blue dots, trace back their trajectories you will find out that the trajectory that gets from red dot to the lowest blue dot has a higher maximum than the trajectory that gets from red dot to the second lowest blue dot, but the slope during inclination and declination are steeper for the trajectory to the lowest blue dot.

If we cut off the plane of projection at around $x = 5$, since all the trajectories are climbing with different rates proportional to their initial angle, the projection onto that plane will be ordered in the same way. But since we are cutting off at $x = -40$, the declination rate also needs to be taken into account.

An example can be two projectiles with different initial angles. Their trajectories intersect at one point after the maximum distance of the object with higher launching angle. If we place a wall before the point of intersection, the ball with higher launching angle will land above the ball with lower launching angle. If we place a wall after the point of intersection, the ball with higher launching angle will land below the ball with lower launching angle. The argument holds for all motions with symmetric trajectories. In gravitational lensing, trajectories are symmetric across $x = 0$ in the Schwarzschild case and are almost symmetric in the Vaidya case (in Vaidya case there is more lensing in the second half of the trajectory than in the first half, since the black hole is still accumulating mass).

The specific angle that makes the telescope image transitions from an elipse to a bean shaped image given r and x is called the critical angle θ_c. We define θ_c formally as the angle where the lowest points of the telescope image is no longer the preimage of the lowest point on the circle. After computing θ_c at $m'(v) = 1/16$, $m'(v) = 1/4$ and $m'(v) = 0$ respectively, as well as three additional accumulation rates at $m'(v) = 1/20$, $m'(v) = 1/2$ and $m'(v) = 1/3$ from reasonable distance above the critical angle, we can plot the eccentricity versus looking angle graph. In this case, we are interested in lensing images obtained from observer starts observing slightly further from the accumulation time, at $v_0 = 60$, and distance across black hole is fixed at $x = 40$ as usual.

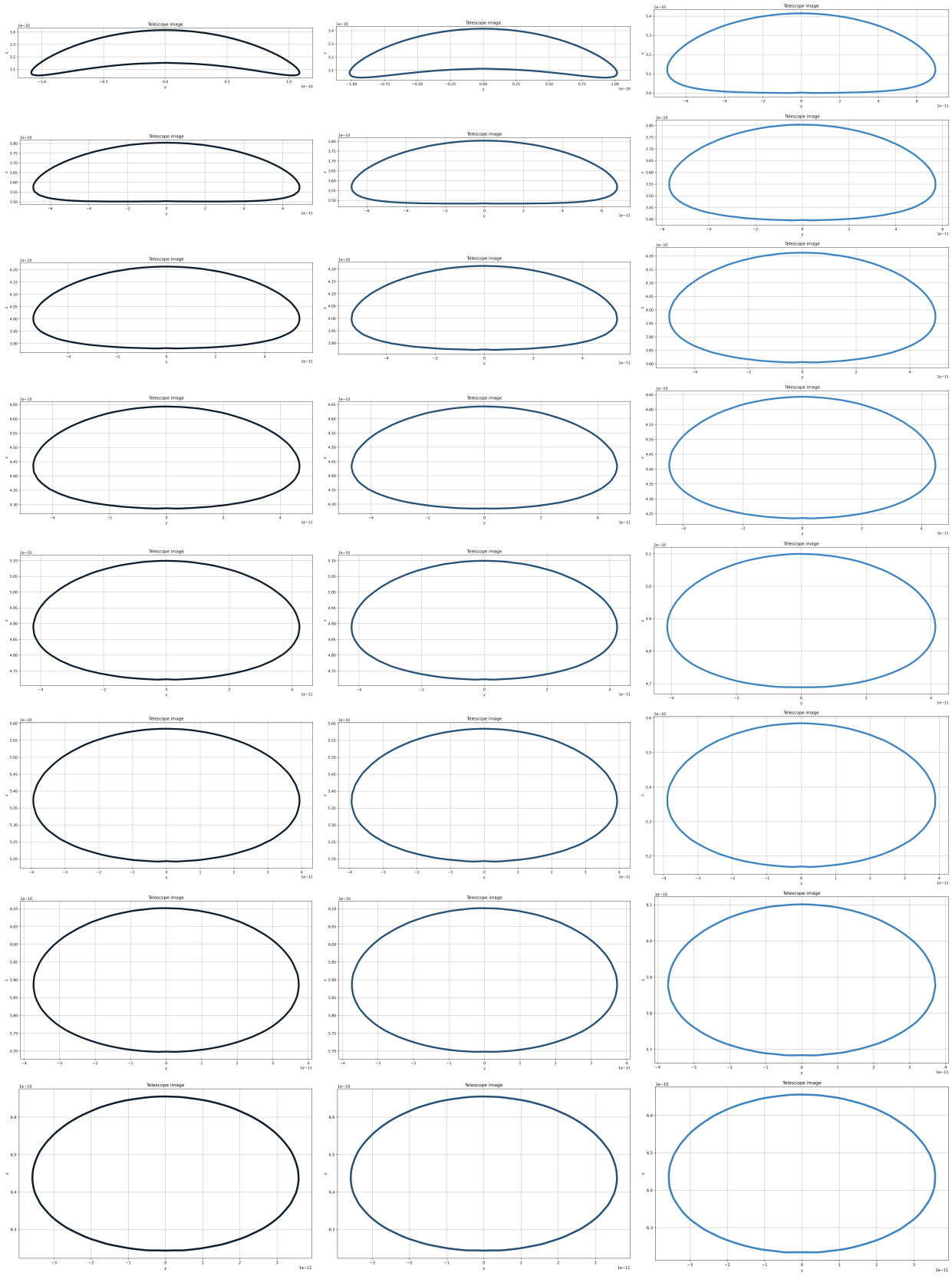


FIG. 4. Telescope images of lensed ring in $m'(v) = 0$, $m'(v) = 1/4$ and $m'(v) = 1/16$ cases respectively. From left column to the right. The observer time coordinate is at $v_0 = 85$. We see eccentricity increases with increasing looking angle, decreases with increasing accumulation rate.

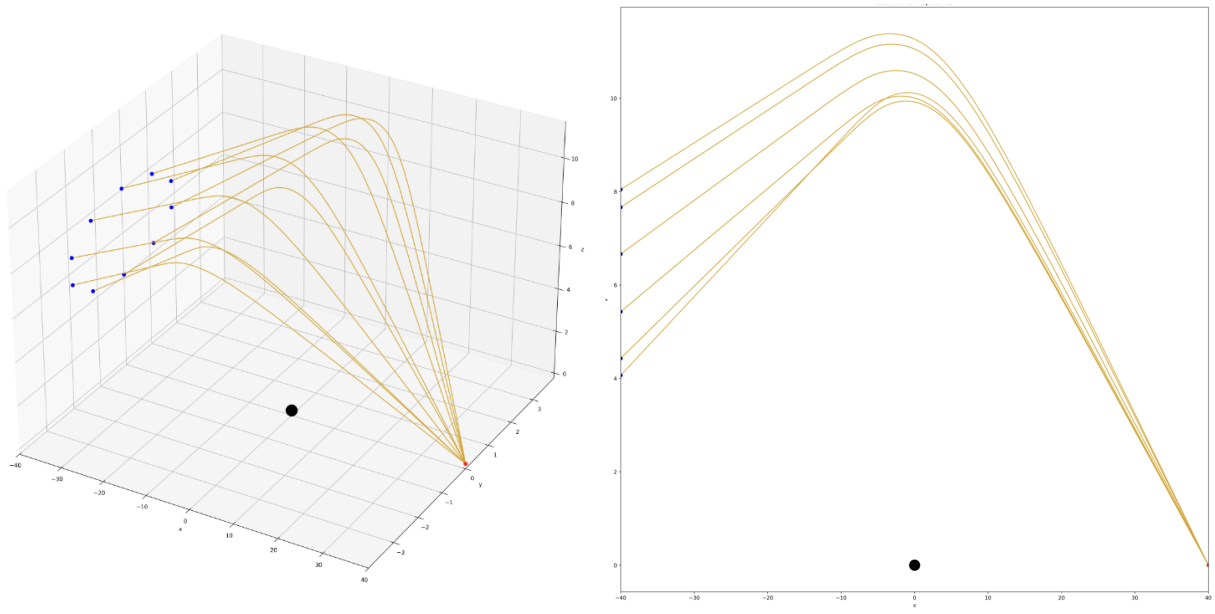


FIG. 5. Example image of the 3D light trajectories for a Vaidya black hole with a bean shaped telescope image (left) and its side view cross section (right). The looking angle for this particular example is $\theta = \pi/10$ with an accumulation rate of $m'(v) = 1/4$ from $m_0 = 0$ to $m_f = 1$, when the observer looks back from $v_0 = 85$ (which is the second picture in figure (4)). There are 10 evenly distributed trajectories in this example, which is 3 times less than what we actually used in the lensing image before smooth interpolation.

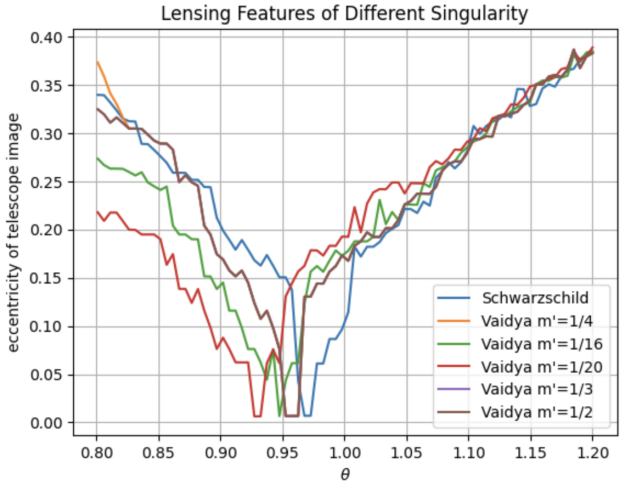


FIG. 6. Eccentricity versus lensing angle for accumulation rates $m'(v) = 0$, $m'(v) = 1/2$, $m'(v) = 1/3$, $m'(v) = 1/4$, $m'(v) = 1/16$, $m'(v) = 1/20$ respectively at observer time coordinate $v_0 = 60$ and distance $x = 40$ across the black hole.

As we can see in figure 6, the eccentricity of lensing images decreases as shooting angle increases initially until 0, near 0.95, which indicates a circle, then the eccentricity starts to increase again but with major and semi-major axis switched. The curve before and after the point indicate circular image formation is about linear, but with the decreasing segment starting at different point and the increasing segment combined into one path. The instability in the graph is due to error of image projection in the step algorithm. This is introduced in appendix B1.

The slopes of the graph in the linear segment at the beginning declination phase are about the same, but starting at different initial positions at $\theta = 0.8$. Based on the lensing feature graph in figure 6, decreasing the accumulation rate can shift the looking angle that corresponds to a circle lensing image to the left, as well as decreasing the starting eccentricity during declination. We can find the curve corresponding to accumulation rate $m'(v) = 1/16$, then all curves in the eccentricity versus θ graph on the left of which indicate naked singularity, and all the curves on the right of which indicate non-naked singularity.

Note that this is only a specific case of analysis for $v_0 = 65$ and $x = 40$. The method to determine threshold of naked singularity formation using slope is the same for all cases with different v_0 and x. We chose $v_0 = 65$ with the default value $x = 40$ so that the differences between curves of various accumulation rates are more obvious.

V. CONCLUSION

Based on previous theorem on the constraint of accumulation rate for naked singularity formation in Vaidya spacetime, we found telescope images of distant circular objects across a black hole and an effective photon sphere $r_{eph} \approx (1 + 3m'(v))3m(v)$ with 17% greater than the Schwarzschild approximation. A black hole in Vaidya spacetime, given that the observer starts observing slightly later than the black hole accumulation time, have the eccentricity of lensing images decreases linearly

with increasing looking angle initially at different height. A neccesary and sufficient threshold for naked singularity formation in Vaidya spacetime under constant accumulation rate is $m'(v) = 1/16$.

VI. ACKNOWLEDGMENT

We wish to thank the University of Michigan's department of Mathematics for providing us this opportunity to conduct research at the lab of geometry and Dr. James Wheeler at the department of mathematics and Dr. James Liu at the department of physics to supervise this project.

Appendix A: Numerical Method

In section 4a through 4c, we introduced the circle lensing method to find the naked singularity in Vaidya spacetime. The algorithm used to find preimage of each point and the way we form image on the plane are shown in the following section.

1. Step Algorithm

It is not an easy task to find the preimage of a point, since the geodesic equations are nonlinear. Instead we perform an iterative method to find the preimage of a set of points by perturbing the initial condition of the highest point whose initial condition is already known. We call this method the step algorithm.

Intuitively, if we shoot a photon backwards in time from the observer's location to a point at a distance with a larger initial θ angle, the point would have a higher z coordinate. If we shoot a photon with a large ϕ angle, the point would deviates to the right, and vice versa. It turned out to be pretty accurate for small circle that is far enough distance from the observer and not too close to the black hole.

To find the preimage of the point $p_1 = (y_1, z_1)$ given preimage of $p_0 = (y_0, z_0)$, we first compare the values between y_1 and y_0, z_1 and z_0. If $z_0 > z_1$, we increase θ by a certain amount; If $y_0 > y_1$, we increase ϕ by a certain amount, and vice versa, then record and keep updating the final landing point. Once we are at a certain distance within the target, we decrease the stepsize, which is the amount we want to change θ and ϕ, by one order of magnitude.

After trials and error, we set up a configuration that optimized both runtime and accuracy for an error tolerance of 0.01. When the landing point is above 1 distance away from the target point, we allow the step size to be 10^{-4}. When the landing point is above 0.1 distance away from the target point, we allow the step size to be 10^{-5}, and the step size is reduced to 10^{-6} below 0.1. We always start perturbing from the highest point, see figure 7 for an illustration.

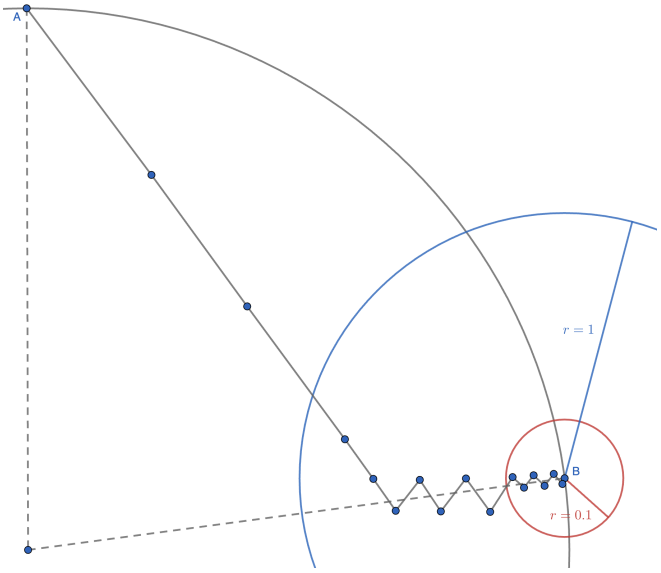


FIG. 7. An illustration of the step algorithm, where the projection starts from the top point A, then iteratively getting closer and closer to the target point B.

2. Image Formation

To obtain the telescope image of the distorted ring, we make a slanted plane of cross section very close to the observer's location about 10^{-9} unit, one tenth of the integration step size, with its unit normal vector pointing θ above the x-axis, then plot the intersection of the light trajectory with the plane. We do so by extracting the (x, y) values of the image on a vertical plane of projection, then perform another projection onto the slanted plane.

$$x' = x\left(\frac{y_{\max}^2 + d^2}{d^2 + y y_{\max}}\right) \tag{A1}$$

$$y' = \frac{y_{\max}}{\cos\theta} - \frac{\sqrt{y_{\max}^2 + d^2}(y_{\max} - y)d}{d^2 + y_{\max} y} \tag{A2}$$

where (x', y') is the new coordinate of each preimages. $y_{\max}$ is the highest point of the original image, and $d = 10^{-9}$ is the distance between the plane and the observer location.

This is not a realistic image, because telescope lense in real life has a parabolic shape with image formed after its focal point, but it is close to real telescope when the distortion is not too severe. Even when the distortion is severe, plane of cross section method still captures the general qualitative lensing feature. For each image, we chose 100 points evenly space around the circle then perform smooth interpolation of their preimages.

A detailed derivation of the coordinate transformation formula is as follow:

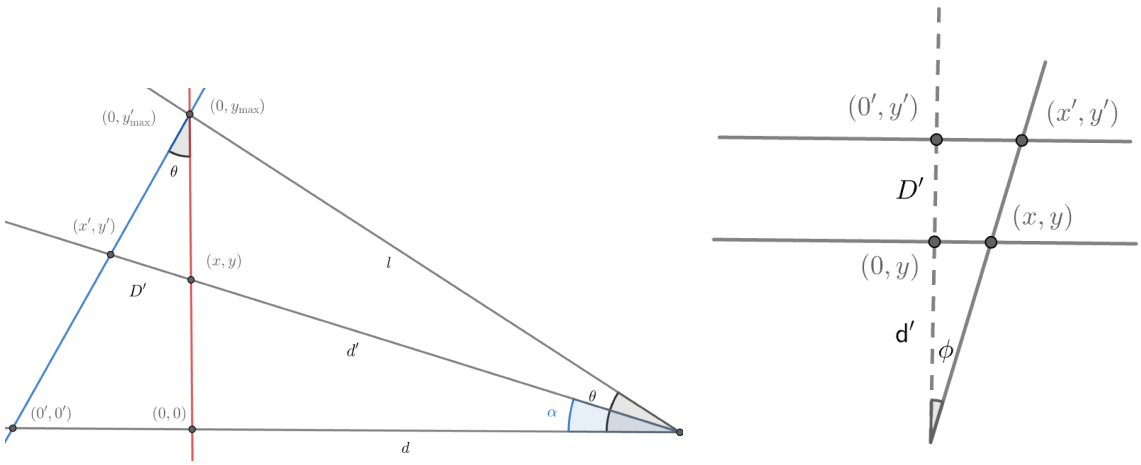


FIG. 8. A ray is passing through two planes at once, intersecting at (x, y) and (x', y'). The original plane is vertical, a distance d away from the observer, the plane of projection is slanted with angle θ with respect to the vertical plane. The top ray is orthogonal to the slanted plane. It passes through the highest point in the telescope image, with height denoted by $y_{\max}$. Figure on the left is the side view of the situation cut by the x-z plane, and figure on the right is the top down view of the situation restricted to the plane of intersection at a specifc inclination angle α. We are using the left figure to derive y' as a function of y, and using the right figure to derive x' as a function of x.

Starting with x-coordinate, by observing the geometry shown in figure 10, we can find the following relations

$$\frac{x}{d'} = \frac{x'}{d' + D'} = \tan\phi \tag{A3}$$

$$d = d' \cos\alpha \tag{A4}$$

$$\frac{l}{d' + D'} = \cos(\theta - \alpha) \tag{A5}$$

$$l^2 = y_{\max}^2 + d^2 \tag{A6}$$

$$\frac{y}{d} = \tan\alpha \tag{A7}$$

Rearanging (A1), (A4), plug (A6) into (A5), we have

$$x' = x\left(\frac{d' + D'}{d'}\right) \tag{A8}$$

$$d' = \frac{d}{\cos\alpha} \tag{A9}$$

$$\frac{\sqrt{y_{\max}^2 + d^2}}{d' + D'} = \cos(\theta - \alpha) \tag{A10}$$

$$\frac{y}{d} = \tan\alpha \tag{A11}$$

Plug (A10) and (A9) into (A8), and apply trignometric identity on (A11) we obtain

$$x' = x\left(\frac{\sqrt{y_{\max}^2 + d^2}/\cos(\theta - \alpha)}{d/\cos\alpha}\right) \tag{A12}$$

$$\cos\alpha = \frac{d}{\sqrt{y^2 + d^2}} \tag{A13}$$

Plug (A13) into (A12), notice by observing the geometry, $\cos\theta = d/l$, $\sin\theta = y_{\max}/l$, and by applying trig-identity $\sin\alpha = y/d'$ we have

$$x = x'\frac{\sqrt{y_{\max}^2 + d^2}\cos\alpha}{d\cos(\theta - \alpha)} \tag{A14}$$

$$\cos\alpha = \frac{d}{\sqrt{y^2 + d^2}} \tag{A15}$$

and

$$\begin{aligned}\cos(\theta - \alpha) &= \cos\theta\cos\alpha + \sin\theta\sin\alpha \\ &= \frac{d}{\sqrt{y_{\max}^2 + d^2}}\frac{d}{\sqrt{y^2 + d^2}} + \frac{y_{\max}}{l}\frac{y}{d'} \\ &= \frac{d^2}{\sqrt{(y_{\max}^2 + d^2)(y^2 + d^2)}} + \frac{y_{\max}}{\sqrt{y_{\max}^2 + d^2}}\frac{y\cos\alpha}{d} \\ &= \frac{d^2 + y_{\max}y}{\sqrt{(y_{\max}^2 + d^2)(y^2 + d^2)}}\end{aligned} \tag{A16}$$

Plug (A16), (A15) into (A14) we obtained (A1)

$$x' = x\left(\frac{y_{\max}^2 + d^2}{d^2 + yy_{\max}}\right) \tag{A17}$$

Now lets calculate the y-coordinate transformation. Note that all the points passed by the same inclination angle α have the same height, y value. Therefore y' is independent of x. We can take $x = 0 = x'$ to simplify our analysis. Observing the geometry, we have $y'_{\max} = l\tan\theta$, and by trig-identity,

$$\begin{aligned}y' &= y'_{\max} - (y'_{\max} - y) = y'_{\max} - l\tan(\theta - \alpha) \\ &= \sqrt{y_{\max}^2 + d^2}\left(\tan\theta - \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}\right)\end{aligned} \tag{A18}$$

Observing the geometry, $\tan\theta = y_{\max}/d, \tan\alpha = y/d$, so

$$y' = \sqrt{y_{\max}^2 + d^2}\left(\frac{y_{\max}}{d} - \frac{y_{\max}/d - y/d}{1 + (y_{\max}/d)(y/d)}\right) = \sqrt{y_{\max}^2 + d^2}\left(\frac{y_{\max}}{d} - \frac{d(y_{\max} - y)}{d^2 + y_{\max}y}\right) \tag{A19}$$

Since $\cos\theta = d/l$, we have

$$y' = \frac{y_{\max}}{\cos\theta} - \frac{\sqrt{y_{\max}^2 + d^2}(y_{\max} - y)d}{d^2 + y_{\max}y} \tag{A20}$$

which is (A2)

Appendix B: GitHub Repository

You can find all the code, except the eccentricity analysis code, for this project in the following GitHub repository link:

https://github.com/jeffyujianfu/vaidya.git

The code for eccentricity analysis can be found in the following google colab project:

https://colab.research.google.com/drive/1UUoeOpwVOa8k19UEiETUG52udaXrx5xp

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