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Technical Note

SOME CANONICAL FORMS FOR ARBITRARY SWITCHING FUNCTIONS USING A SINGLE THRESHOLD DEVICE

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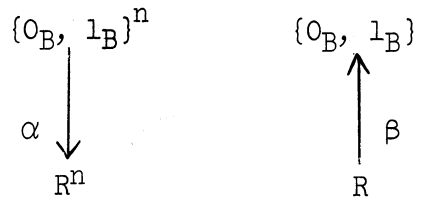
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I. INTRODUCTION

This report gives constructive existence proofs for the realizability of an arbitrary switching function as (1) a threshold function of positive product functions and (2) a threshold function of "parity" functions. The second result is essentially a restatement of a result of Coleman,[1] and is repeated here because of its close connection with the first and, what we believe to be, a much more simple derivation. Both of the above are then shown to imply similar results for positive sums and "equals" functions. Finally, we mention the problem of minimizing the derived canonical forms.

II. POSITIVE PRODUCT FORM

Let F_B: $\{0_B, 1_B\}^n \rightarrow \{0_B, 1_B\}$ be the set of switching functions of n variables. Then we associate the domain and range of F with <u>real</u> vector spaces R^n and R by maps α and β.



$$\alpha(x_1 \ldots x_n) = (y_1 \ldots y_n)$$

$$y_i = 1_R \longleftrightarrow x_i = 1_B$$

$$y_i = 0_R \longleftrightarrow x_i = 0_B$$

$\beta(z) = w$

$$w = 1_B \longleftrightarrow z \underset{z \in R}{>} 0$$

$$w = 0_B \longleftrightarrow z \leq 0$$

We have "injected" $\{0, 1\}^n$ into R^n by means of the "natural" map which identifies the Boolean 1 and 0 with the real 1 and 0. The return map β is simply a "threshold" map with threshold 0.

Any map g whatsoever from R^n to R induces a corresponding map g' from $\{0, 1\}^n$ to $\{0, 1\}$ by requiring commutativity of the diagram:

$$\begin{array}{ccc} \{0_B, 1_B\}^n & \xrightarrow[g']{} & \{0_B, 1_B\} \\ \alpha \downarrow & & \uparrow \beta \\ R^n & \xrightarrow{g} & \{R\} \end{array}$$

i.e., $g' = \beta g \alpha$.

Since α is 1-1 and β is onto, it is clear that for any $h' : \{0, 1\}^n \to \{0, 1\}$, there will be maps h such that $h' = \beta h \alpha$. More specifically, we associate a particular <u>polynomial</u> function P with h' such that

$$P(y) = + \epsilon \longleftrightarrow h'(\alpha^{-1}(y)) = 1$$

$$P(y) = - \epsilon \longleftrightarrow h'(\alpha^{-1}(y)) = 0$$

$$y \in \alpha\{0, 1\}^n$$

Since the set $\alpha\{0, 1\}^n$ is finite, we are assured that a polynomial exists having prescribed values that are set by the La Grange interpolation formula. (See Van der Waerdon.[3])

Because we are only concerned with the behavior of the polynomial function on the particular set $\alpha\{0_B, 1_B\}^n$ there is an equivalence relation induced on polynomials by the relation

$$x_i^n = x_i$$

which holds just because x_i takes only the values 0_R and 1_R. Therefore, it will always be possible to pick a polynomial having no powers greater than one.

Example: Let $f_B \epsilon F_B$ be defined by the Boolean form $xy + \bar{y}\bar{z}$, and pick $\epsilon = 1$. The appropriate polynomial is of the form

$$a_0 + a_1x + a_2y + a_3z + a_4xy + a_5xy + a_6yz + a_7xyz.$$

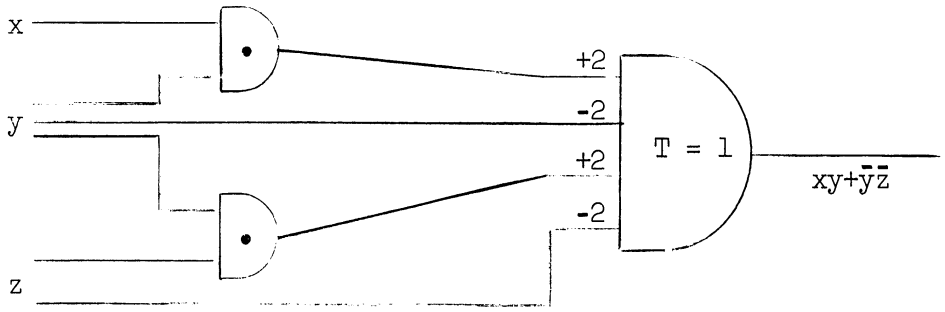
By specializing the arguments and requiring the polynomial to take the appropriate value, we obtain the following triangular system of linear equations:

$$\begin{aligned}
a_0 &= 1 \\
a_0 + a_1 &= 1 \\
a_0 + a_2 &= -1 \\
a_0 + a_3 &= -1 \\
a_0 + a_1 + a_2 + a_4 &= 1 \\
a_0 + a_1 + a_3 + a_5 &= -1 \\
a_0 + a_2 + a_3 + a_6 &= -1 \\
a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 &= 1
\end{aligned}$$

The resulting polynomial is easily computed to be

$$P_f(x,yz) = 1 - 2y - 2z + 2xy + 2yz .$$

The computation of the entire function $\beta P_f \alpha = f$ may now be done with "and" gates and a single threshold function. This is again possible because of the identity of Boolean multiplication and real multiplication on the domain of interest. The resulting realization of the above example in this form is:



Conjunctive Form

III. PARITY FUNCTION FORM

The result for parity or "exclusive or" functions is obtained by picking a different injection of $\{0_B, 1_B\}^n$ into R^n. Specifically, we redefine

$$\alpha(x_1 \ldots x_n) = (y_1 \ldots y_n)$$

$$y_i = 1_R \longleftrightarrow x_i = 0_B$$

$$y_i = -1_R \longleftrightarrow x_i = 1_B$$

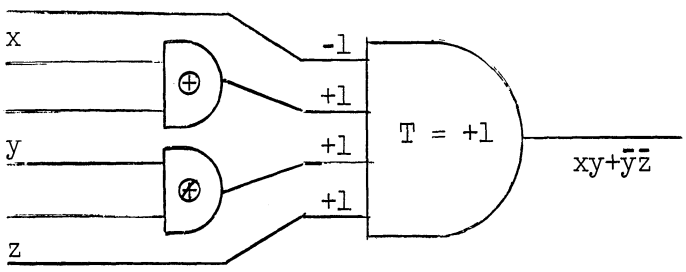
Previously we had a homomorphism between $\left\{\{0_B, 1_B\}, \cdot_B\right\}$ and $\left\{\{0_R, 1_R\}, \cdot_R\right\}$. Now the homomorphism is between $\left\{\{0_B, 1_B\}, \oplus_B\right\}$ and $\left\{\{1_R, -1_R\}, \cdot_R\right\}$. Again we can apply the La Grange interpolation theorem to obtain the existence of a polynomial realizing an arbitrary switching function. Again we can be assured of obtaining a polynomial with no powers greater than one. The justification

of this last step depends on a different relation however. The relation is $x^2 = 1$, and it may be used to reduce an arbitrary power modulo 2.

Before computing the particular polynomial, it is convenient to transform the values taken by the component product functions to 0 and 1. This may be done by $\{t^1 = 1/2 - 1/2\ t\}$. A dual transformation on the polynomial will result in a new polynomial function with the same values on the transformed product terms as the old polynomial had on the untransformed terms. The new form of the product terms now more closely resembles the "ring sum" operation when the "hardware" identification is made of $\{0_B, 1_B\}$ with $\{0_R, 1_R\}$. The resulting polynomial will then have the form:

$$a_0 + a_1(x) + a_2(y) + a_3(z) + a_4(x \oplus y) + a_5(x \oplus z) + a_6(y \oplus z) + a_7(x \oplus y \oplus z)$$

and a linear system may be solved for the a_i's. For this particular linear system which will not be triangular, a more efficient computation is given in Coleman's paper.[1] Both of these utilize the fact that the set of $\oplus$ functions is orthogonal with respect to an appropriate inner product, and their coefficients may therefore be computed independently of one another. For the example given in Section II, the circuit realization is:



"Exclusive or" Form

IV. DUAL FORMS

The result for "or's" and "equal's" needs certain simple properties of dual functions. The dual of a function $f(x)$ is defined $f^D = \bar{f}(\bar{x})$, where x is a vector and f may be a vector valued function. Then we have

1. $[f(g(x))]^D = \bar{f}(g(\bar{x}))$

$= \bar{f}(\bar{\bar{g}}(\bar{x}))$

$= \bar{f}(\overline{\bar{g}^D}(x))$

$= f^D(g^D(x))$

2. If $f(x)$ is linearly separable, so is $f^D(x)$.

   a. If $f(x)$ is linearly separable, $f(\bar{x})$ is also since for any injection of $\{0_B, 1_B\}$ into the real numbers, there is a linear affine transformation which interchanges the images of 0_B and 1_B. The adjoint transformation will carry a set of weights realizing $f(x)$ into one realizing $f(\bar{x})$.

   b. Multiplying a set of weights realizing $f(x)$ by -1 yields a set of weights realizing $\bar{f}(x)$.

The existence of a realization of an arbitrary function $f(x)$ as a linearly separable function of positive sums now follows from the fact that its dual, $f^D(x)$, is realizable as a linearly separable function of positive products. $f^D(x)$ may be expressed as:

$f^D(x) = \ell(P(x))$

ℓ linearly separable
P a vector of positive product functions

$f(x) = [\ell(P(x))]^D$

$= \ell^D(P^D(x))$

and ℓ^D is linearly separable while P^D is in the form of positive sums.

A similar argument from the " $\oplus$ " form gives the " = " form.

V. MINIMIZATION

In the above example, certain terms dropped out. In general, the fewer the terms appearing in the polynomial, the cheaper the corresponding realization. The problem of determining that polynomial having the fewest terms is made non-trivial by the fact that for threshold devices, the values taken by the polynomial need only satisfy inequalities. This replacement of a single polynomial with a convex set of polynomials results in a "tree search" of the type familiar to integer programmers. The difficulties of finding an efficient algorithm to minimize a particular form may be seen from the following observation. It may happen that the space spanned by a subset of terms contains a "feasible solution" while not containing the minimal solution. Thus the linear process of eliminating terms one at a time must be replaced by a branching search. It is not known if other relations exist which would shorten this search. Finally, the main factor in determining the length of any minimization algorithm appears to depend primarily on the length of an algorithm for the determination of linear separability for a given function. (See, for example, Muroga.[2])

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3. Van der Waerdon, B. L., Modern Algebra, Revised English edition, Ungar Publishing Co., 1953.

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