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AUTOMORPHISMS OF THE PROJECTIVE UNITARY GROUPS

By
John H. Walter

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ABSTRACT

AUTOMORPHISMS OF THE PROJECTIVE
UNITARY GROUPS

by

John H. Walter

Let E be a right vector space of dimension n over a sfield K which possesses an involutive anti-automorphism J. Let f be a non-degenerate hermitian sesquilinear form defined on E relative to the anti-automorphism J. The unitary group $U_n(K,f)$ is the group of linear transformations of E which leave f invariant. Dencte by P(E) the projective space consisting of the 1-dimensional subspaces of E. Designate by C_n the normal subgroup of $U_n(K,f)$ consisting of those elements which leave invariant the 1-dimensional subspaces of E. Then the group $PU_n(K,f) = U_n(K,f)/C_n$ is the projective unitary group; it is the factor group of $U_n(K,f)$ which acts faithfully on P(E). The object of this dissertation is to determine the automorphisms of $PU_n(K,f)$.

We require that n be greater than 4 and not 6. For the sake of convenience, we assume that n is finite although our methods directly and easily extend to the infinite dimensional case. Then we show that every automorphism φ of $PU_n(K,f)$ is of the form $\varphi(\bar{t}) = \bar{r}^{-1}\bar{t}\bar{r}$ where $\bar{r}$ is a collineation of P(E) induced by a semilinear transformation r of E relative to an automorphism σ of K such that for all x and y in E

$$f(r(x),r(y)) = e_r f(x,y)^{\sigma}$$

where e_r is an element of K independent of x and y.

The method of proof is as follows. The elements $\bar{t}$ of $PU_n(K,f)$ are

considered to be cosets in $U_n(K,f)$. If $\bar{u}$ is an involution of $PU_n(K,f)$ and u is in $\bar{u}$, then $u^2(x) = x\gamma$ for all x in E where $\gamma^J\gamma = 1$ and γ is in Z, the center of K. We classify the involutions of $PU_n(K,f)$ according to relations satisfied by γ. In particular, if $\gamma = \lambda^2$ with λ in Z and $\lambda^J\lambda = 1$, $\bar{u}$ is said to be an involutions of the first kind. Such involutions, considered as cosets of $U_n(K,f)$ contain involutions of $U_n(K,f)$. If they contain involutions with 1-dimensional eigenspaces, they are said to be extremal involutions. It has been shown by Dieudonné and Rickart that the characterization of such involutions strictly in terms of the multiplicative structure of $PU_n(K,f)$ leads to the characterization of the automorphisms of $PU_n(K,f)$. Our principal theorem yields this characterization for $PU_n(K,f)$.

If $\bar{\mathfrak{s}}$ is a set of involutions of $PU_n(K,f)$, denote by $c(\bar{\mathfrak{s}})$ those involutions of $PU_n(K,f)$ which commute with the elements of $\bar{\mathfrak{s}}$. In particular, if $\bar{u}$ and $\bar{v}$ are distinct commuting involutions, denote by $\nu(\bar{u},\bar{v})$ the order of $c(c(\bar{u},\bar{v}))$ and by $\nu_{\bar{u}}$ the maximum of $\nu(\bar{u},\bar{v})$ for all involutions $\bar{v}$ commuting with $\bar{u}$. Let $\bar{\mathfrak{D}}$ be the set of involutions $\bar{u}$ for which $\nu_{\bar{u}} = 4$. For the general linear and unitary groups, Rickart showed that $\bar{\mathfrak{D}}$ consists only of extremal involutions. In the case of the projective unitary groups, we show that $\bar{\mathfrak{D}}$ contains the extremal involutions along with two other classes of involutions. To distinguish extremal involutions from these other involutions group-theoretically, we denote by $\omega(\bar{u},\bar{v},\bar{w})$ the order of $c(c(\bar{u},\bar{v},\bar{w}))$ where $\bar{u}$, $\bar{v}$, and $\bar{w}$ are distinct mutually commuting involutions in $\bar{\mathfrak{D}}$ and by $\omega_{\bar{u}}$ the maximum of $\omega(\bar{u},\bar{v},\bar{w})$ for all such $\bar{v}$ and $\bar{w}$. (It is important to note that it is necessary to require that $\bar{u}$, $\bar{v}$, and $\bar{w}$ be in $\bar{\mathfrak{D}}$. For this reason the first part of this dissertation is devoted to the characterization of $\bar{\mathfrak{D}}$.)

Let $\bar{\mathfrak{g}}$ denote the set of involutions for which $\omega_{\bar{u}} = 8$. Let $\kappa_{\bar{u}}$ denote the maximum number of involutions of $\bar{\mathfrak{g}}$ contained in any maximal set

of commuting involutions containing $\bar{u}$. Then our principal theorem states that an extremal involution $\bar{u}$ is characterized by the following group-theoretical conditions: $\bar{u}$ is in $\bar{\mathfrak{g}}$, $\kappa_{\bar{u}} = n$ and any product of two or three involutions of $\bar{\mathfrak{g}}$ commuting with $\bar{u}$ is not in $\bar{\mathfrak{k}}$. If $n \neq 8$ and $n \neq 12$, it is sufficient to require merely that $\bar{u}$ be in $\bar{\mathfrak{g}}$. From this point on, the arguments of Rickart and Dieudonné yield the final result.

PREFACE

I would like to acknowledge my indebtedness to Professor Jean Dieudonné, at whose suggestion I investigated this problem. His encouragement and criticism have proved invaluable and have made the writing of this dissertation a valuable educational experience.

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INTRODUCTION

This dissertation is devoted to the determination of the automorphisms of the projective unitary groups. As such it is a contribution to the work on the general problem of determining the automorphisms of the classical groups. In the case of real or complex vector spaces, this problem has been solved for some time by infinitesimal methods. To O. Schreier and B. van der Waerden [8] goes the credit for the first algebraic attack on this problem.[1] Following them, J. Dieudonné greatly extended their results (e.g., see [5]) to many more classes of groups and to cases where the vector space was in some cases over an arbitrary sfield (division ring or skew-field). As a general rule, the more complicated the group, the more restrictions Dieudonné placed on the sfield until, when dealing with the projective unitary groups, he required that the sfield be a finite field. Dieudonné's methods turn on obtaining a group-theoretical characterization of a certain type of involution and then applying the fundamental theorem of projective geometry. Dieudonné relies heavily on the structure of the groups to obtain his results.

C. Rickart ([9] and [10]) broke away from this by using a device which had its origin in a paper by G. Mackey [7] and was able to obtain a more uniform method of attack on the problem. For the most part, Rickart dealt with the cases that Dieudonné had already considered. His method generalized to infinite dimensional spaces over sfields and did not need Dieudonné's requirement that the index of the form be greater than zero in the case of unitary and orthogonal groups.

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[1]Numbers in brackets refer to the bibliography at the end of this dissertation.

By refining the method of Rickart, we have been able to solve the problem for the projective unitary groups. We have also made use of Dieudonné's characterization of the centralizer of an involution. The problem is solved providing that the dimension of the underlying vector space is greater than 4 and not 6 and that the sfield is not of charcteristic 2 and not the finite field of three elements. For low dimensions, the methods of Dieudonné, Rickart, and myself all fail. In this connection, see Hua [6].

Along with Dieudonné and Rickart, we have approached the problem by studying the way the automorphisms transform the involutions of a projective unitary group. We have distinguished a certain class of involutions, which are called "extremal involutions". They are defined geometrically in terms of the way they act on the underlying vector space. Their importance comes from the fact that they are associated with the 1-dimensional subspaces of the underlying vector space. Our principal result is the characterization of these involutions in terms of the multiplicative structure of the projective unitary group (Theorem 5.5). Once this is done, the arguments of Rickart and Dieudonné apply. A summary of them is given in the appendix.

CHAPTER I

PRELIMINARIES[2]

1.1 Definitions of Certain Classical Groups

Let E be a right vector space of dimension n over a sfield K. The group of semilinear transformations[3] of E onto itself is denoted by $\Gamma L_n(K)$ and is called the group of collineations. The set of transformations h_λ, such that $h_\lambda(x) = x\lambda$ with λ in K for all x in E, forms the subgroup $H_n(K)$ of homothetic mappings; h_λ is semilinear with respect to the inner automorphism $\bar{\lambda} : \phi \to \phi^{\bar{\lambda}} = \lambda^{-1}\phi\,\lambda$. Sometimes the transformation h_λ will be denoted by λ and $h_\lambda(x)$ by $\lambda(x)$. Denote by P(E) the (n-1)-dimensional projective space formed by taking the 1-dimensional subspaces of E as points. The quotient group $P\Gamma L_n = \Gamma L_n(K)/H_n$ can be identified as the group of projective collineations acting on P(E). It is the factor group of ΓL_n which acts faithfully on P(E).

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[2]This chapter takes its point of departure from Dieudonne's treatise, [2], Chapter I. Many statements are given here without proof. For these proofs and for more details concerning the various groups of transformations and the characterization of the various kinds of involutions which we shall introduce, the reader is referred to the aforementioned treatise.

[3]A function t of E into itself is semilinear with respect to an automorphism τ of K if t(x) is additive in x and $t(x\phi) = t(x)\phi^{\tau}$ for ϕ in K.

Suppose, furthermore, that K possesses an involutive[4] anti-automorphism J. Let f be a nondegenerate hermitian sesquilinear form[5] defined on E with respect to J. The subgroup $\Gamma U_n(K,f)$ of $\Gamma L_n(K)$ consisting of elements t such that

$$f(t(x),t(y)) = e_t f(x,y)^\tau ,$$

where τ is the automorphism of K associated with t and e_t is an element of K, the "multiplier of t", depending only on t, is called the group of semisimilitudes. The fact that f is hermitian implies that $e_t^J = e_t$. The normal subgroup $U_n(K,f)$ of $\Gamma U_n(K,f)$ consisting of linear transformations with multiplier 1 is the unitary group. For t in $U_n(K,f)$, we have the familiar relation

$$f(t(x),t(y)) = f(x,y) .$$

The subgroups $P\Gamma U_n(K,f)$ and $PU_n(K,f)$ corresponding to $\Gamma U_n(K,f)$ and $U_n(K,f)$ under the natural homomorphism of $\Gamma L_n(K)$ onto $P\Gamma L_n(K)$ are called, respectively, the group of projective semisimilitudes and the projective unitary group. They can be identified as the factor groups $\Gamma U_n(K,f)/H_n(K)$ and $U_n(K,f)/H_n(K)\cap U_n(K,f)$. With one exception, the group $H_n(K)\cap U_n(K,f)$ is the center C_n of $U_n(K,f)$, as is shown, for example, in [4]. This exception, which occurs when K is the galois field GF(3) and n = 2, is excluded from the groups we shall consider. The center C_n consists of those homothetic transformation h_λ for which λ is in the center Z of K and $\lambda^J\lambda = 1$. The corresponding set of such elements in Z forms

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[4]In the case that K is commutative, J is naturally an automorphism, which may be the identity.

[5]A function f: E x E → K is said to be sesquilinear with respect to an anti-automorphism J, if f(x,y) is additive in x and y and $f(x\alpha,y\beta) = \alpha^J f(x,y)\beta$ for α and β in K. A sesquilinear form f is said to be hermitian if $f(y,x) = f(x,y)^J$ and is said to be nondegenerate if f(x,y) = 0 for x in E and all y in E implies that x = 0.

a multiplicative subgroup, which we shall denote by C. The centralizer of $U_n(K,f)$ in $\Gamma L_n(K,f)$ is $H_n(K)$. This will be shown in Section 1.10.

With certain restrictions on the dimension n of E and on the sfield K, we shall determine the automorphisms of $PU_n(K,f)$ by characterizing certain involutions of $PU_n(K,f)$ called extremal involutions, which are associated with 1-dimensional subspaces of E. These are defined in Section 1.5. Once this is done, known methods of Dieudonné [5] and Rickart [10] will yield our result (cf. Appendix). As Rickart points out, we can obtain even more information. Indeed, we can conclude that if $PU_m(L,g)$ is another projective unitary group acting on a right L-space F of dimension m, then n = m, K and L are isomorphic, and f and g are related by a given isomorphism between E and F (cf. Rickart [10], Theorems 4.1 and 4.3 or Theorem A.2 of this dissertation).

Throughout this dissertation we restrict ourselves to the case where K is not of characteristic 2 and is not the finite field GF(3). We assume that the dimension n of E is finite as a matter of convenience. Our result extends easily and directly to the infinite dimensional case. It is to be noted that we include in our considerations the projective orthogonal groups $PO_n(K,f)$, which are the specializations of the projective unitary groups to the case where K is commutative and J is the identity.

A sesquilinear form f is said to be reflexive if f(y,x) = 0 implies that f(x,y) = 0. With the exception of the case where f is a skew-symmetric bilinear form, which can occur only when K is commutative, Birkhoff and von Neumann [1] have shown that there exists an element d in K such that the sesquilinear form g = df is hermitian. It then follows that $U_n(K,f) = U_n(K,g)$. So our result extends to this case also.

In passing, we also note that if f is hermitian and $d^J = d$, then $g = df$ is again hermitian. But if $d^J = -d$, then $g = df$ is skew-hermitian; that is, $g(y,x) = -g(x,y)^J$.

1.2 Classification of the Involutions of $PU_n(K,f)$

We shall use the methods of linear algebra and for that reason we look upon an element $\bar{t}$ in $PU_n(K,f)$ as a coset in $U_n(K,f)$ and deal with the elements t of $\bar{t}$. In particular, if $\bar{u}$ is an involution of $PU_n(K,f)$, then an element u of $U_n(K,f)$ in $\bar{u}$ is called a projective involution. If $\bar{u}$ is an involution in $P\Gamma U_n(K,f)$, then $\bar{u}$ will be called a semilinear involution and u of $\Gamma U_n(K,f)$ in the coset $\bar{u}$ will be called a projective semilinear involution. The relation $u^2 = \gamma$ with γ in C_n characterizes projective involutions.

We shall now distinguish various types of projective involutions u where $u^2 = \gamma$ in C_n:

(i) u is of the first kind if $\gamma = \lambda^2$ where λ is in Z and $\lambda^J\lambda = 1$.

(ii) u is of the second kind if $\gamma = \lambda^2$ where λ is in Z and $\lambda^J\lambda = -1$.

(iii) u is of the third kind if $\gamma = \lambda^2$ where λ is not in Z.

(iv) u is of the fourth kind if γ is not a square in K.

Note that if $\gamma = \lambda^2$ and λ is in Z, then $\gamma^J\gamma = 1$ implies that $(\lambda^J\lambda)^2 = 1$ and thus that $\lambda^J\lambda = \pm 1$. So in this case u is of either the first or second kind. If the projective involution u is of the k^{th} kind, then so is every projective involution τu with τ in C_n. So we can say that the involution $\bar{u}$ of $PU_n(K,f)$ is of the k^{th} kind. An extremal involution is a particular type of involution of the first kind.. Because the identity $\bar{1}$ can obviously be distinguished from other involutions of the first kind, we shall exclude it from the set of involutions of the first kind and the elements of C_n from the set of projective involutions of the first kind.

1.3 Definition of the Invariants $\nu_{\bar{u}}$ and $\omega_{\bar{u}}$

We shall characterize the extremal involutions of $PU_n(K,f)$ by using group theoretical invariants similar to those introduced by Mackey [7]. Let $\mathcal{I}$ be a set of involutions of $PU_n(K,f)$ and define $c(\mathcal{I})$ to be those involutions of $PU_n(K,f)$ which commute with every element of $\mathcal{I}$. In particular if, $\bar{u}$ and $\bar{v}$ are distinct commuting involutions in $PU_n(K,f)$, denoted by

$\nu(\bar{u},\bar{v})$ the number of elements in $c(c(\bar{u},\bar{v}))$

$\nu_{\bar{u}}$ the maximum of $\nu(\bar{u},\bar{v})$ for all involutions $\bar{v} \neq \bar{u}$ such that $\bar{u}\bar{v} = \bar{v}\bar{u}$, $\bar{u} \neq 1$, and $\bar{v} \neq 1$.

In the case of the unitary and general linear groups, Rickart ([9] and [10]) characterized extremal involutions by what corresponds to our $\nu_{\bar{u}}$. It is easily seen that $c(c(\bar{u},\bar{v}))$ always contains the elements $\bar{1}$, $\bar{u}$, $\bar{v}$, and $\overline{uv}$. So the minimum of $\nu_{\bar{u}}$ is at least 4. Rickart showed that an involution is extremal if and only if $\nu_{\bar{u}} = 4$ in the cases he considered. In the case of the projective unitary groups, this is no longer true. So we carry Rickart's method one step further.

Let $\bar{\mathcal{L}}$ denote the set of involutions $\bar{u}$ of $PU_n(K,f)$ for which $\nu_{\bar{u}} = 4$. We shall show that $\bar{\mathcal{L}}$ contains the extremal involutions along with the involutions of the third and fourth kinds. If $\bar{u}$, $\bar{v}$, and $\bar{w}$ are distinct mutually commuting involutions in $\bar{\mathcal{L}}$, denote by

$\omega(\bar{u},\bar{v},\bar{w})$ the number of elements in $c(c(\bar{u},\bar{v},\bar{w}))$

$\omega_{\bar{u}}$ the maximum of $\omega(\bar{u},\bar{v},\bar{w})$ for all $\bar{v}$ and $\bar{w}$ subject to the condition that $\bar{u}$, $\bar{v}$, and $\bar{w}$ are distinct mutually commuting involutions in $\bar{\mathcal{L}}$.

Now the minimal value of $\omega_{\bar{u}}$ is at least 8. We shall show that $\omega_{\bar{u}} = 8$ if and only if $\bar{u}$ is extremal with two possible exceptions; viz., when $n = 8$ or 12, cases which we shall handle by other means. It is important to

note that it is necessary to require that $\bar{v}$ and $\bar{w}$ be in $\bar{\mathfrak{L}}$ in order to obtain our result. For this reason, we will devote the first part of this dissertation to the characterization of $\bar{\mathfrak{L}}$.

If $\bar{u}$ and $\bar{v}$ are commuting involutions of $PU_n(K,f)$ and u is in $\bar{u}$ and v is in $\bar{v}$ such that $u^2 = \gamma$ in C_n and $v^2 = \beta$ in C_n, we have $uv = \alpha vu$ with α in C_n. Multiplying on the left and right by u and v, respectively, we obtain $\beta\gamma = \alpha uvuv = \alpha^2\beta\gamma$. So $\alpha^2 = 1$ and $\alpha = \pm 1$. This means that u and v commute or anticommute. Replacing u by ηu or v by ηv with η in C_n does not change the value of α. So the same result holds for any involution u in $\bar{u}$ and any involution v in $\bar{v}$.

It is convenient to denote the set of elements of $U_n(K,f)$ determined by a set $\bar{\mathfrak{S}}$ in $PU_n(K,f)$ by $\mathfrak{S}$. Conversely, if $\mathfrak{S}$ is a set of elements in $U_n(K,f)$, we write $\bar{\mathfrak{S}}$ for the set of elements in $PU_n(K,f)$ which has representatives in $\mathfrak{S}$. Likewise, when $\bar{\mathfrak{S}}$ is a set of projective involutions $c(\mathfrak{S})$ and $c(c(\mathfrak{S}))$, denote those projective involutions determined respectively by $c(\bar{\mathfrak{S}})$ and $c(c(\bar{\mathfrak{S}}))$. Then $c(\mathfrak{S})$ can be characterized as the set of projective involutions which commute or anticommute with the elements of $\mathfrak{S}$.

1.4 Characterizations of Subspaces

In the discussion of the different types of projective involutions we use some fundamental concepts and properties of subspaces connected with a hermitian sesquilinear form f. A vector x is said to be isotropic if $f(x,x) = 0$ and is said to be nonisotropic otherwise. A vector y is orthogonal to a vector x if $f(x,y) = 0$. If S is a subspace, the orthogonal subspace $S^{\perp}$ is the subspace consisting of all the vectors orthogonal to every vector of S. A nonisotropic subspace S is a subspace

such that $S \wedge S^{\perp}$ contains only the zero vector, and otherwise S is said to be an isotropic subspace. If $S \subseteq S^{\perp}$, then S is a totally isotropic subspace. In particular, the space E is nonisotropic since f is nondegenerate.

If every vector of a basis for S is nonisotropic, then the basis is said to be nonisotropic. A nonisotropic subspace possesses an orthogonal nonisotropic basis and any subset of orthogonal nonisotropic vectors can be extended to such a basis. We denote the dimension of a subspace S by dim S or, if there should be any confusion, by $\dim_K S$. Then $\dim S + \dim S^{\perp} = n$.

The index ν of the form f is the maximum of the dimensions of the totally isotropic subspaces of E. Every totally isotropic subspace is contained in a maximal totally isotropic subspace of dimension ν. If $[r]$ denotes the greatest integer less than or equal to a rational number r, $\nu \leq [n/2]$. In particular, when n is even and $\nu = m = n/2$, $E = S \oplus T$ (direct sum) where S and T are totally isotropic subspaces. A 2-dimensional subspace or plane, as it is sometimes called, generated by two nonisotropic nonorthogonal isotropic vectors is nonisotropic, for its orthogonal subspace is complementary. Now let S and T be two maximal totally isotropic subspaces such that $E = S \oplus T$. If x is in S, then $S \subseteq (xK)^{\perp}$. So $T \nsubseteq (xK)^{\perp}$ and there exists a vector y in T such that $f(x,y) \neq 0$. Thus the plane generated by x and y will be nonisotropic, as its orthogonal subspace is complementary. By replacing y by $y\phi$ with ϕ in K, we can take $f(x,y)$ to be any nonzero element in K.

1.5 Involutions of the First Kind

Let $\bar{u}$ be a projective involution of the first kind and choose a representative u in $\bar{u}$. Then $u^2 = \gamma$ and $\gamma^2 = \lambda^2$ where λ is in C. The projective involution $\lambda^{-1}u$ is also in $\bar{u}$ and is easily seen to be an involution; so we can assume that u is an involution; that is, $\gamma = 1$. Then E is the direct sum $E = U^+ \oplus U^-$ of two subspaces such that $u(x) = \pm x$ according as x is in $U^{\pm}$, respectively. Since, for x in U^+ and y in U^-, $f(x,y) = f(u(x),u(y)) = -f(x,y) = 0$, we have U^+ and U^- orthogonal. Because they are complementary, they are nonisotropic. If $\dim U^+ = p$, u is said to be a p-involution. As u and -u are both in $\bar{u}$, we can always take $p \leq [n/2]$. Then $\bar{u}$ will also be called a p-involution. Since we do not consider $\bar{1}$ as an involution of the first kind, $p > 0$. If $p = 1$ and u a projective involution of the first kind, then we say that u is a minimal involution and that $\bar{u}$ is an extremal involution. Our aim is to characterize the extremal involutions group-theoretically.

It is readily verified that an element t of $U_n(K,f)$ or even of $\Gamma L_n(K)$ commutes with u if and only if $t(U^+) = U^+$ and $t(U^-) = U^-$. Similarly t anticommutes with u if and only if $t(U^+) = U^-$ and $t(U^-) = U^+$. In this latter case we have $\dim U^+ = \dim U^- = n/2 = m$. That is no element t of $\Gamma L_n(K,f)$ can anticommute with u unless u is an m-involution.

1.6 Involutions of the Second Kind

Let $\bar{u}$ be an involution of the second kind in $PU_n(K,f)$ and choose a representative u in $\bar{u}$. Then $u^2 = \gamma$ where $\gamma = \lambda^2$ with λ in Z and $\lambda^J\lambda = -1$. Again form $u' = \lambda^{-1}u$. Then u' is an involution in $\Gamma U_n(K,f)$ with multiplier -1 and $E = U^+ \oplus U^-$ where U^+ and U^- are the positive and negative eigenspaces of u'. Also $u(x) = \pm x\lambda$ according

as x is in $U^{\pm}$, respectively. Let x and y be both in U^+ or both in U^-. Then $f(x,y) = f(u'(x),u'(y)) = -f(x,y) = 0$; so U^+ and U^- are totally isotropic. As $E = U^+ \oplus U^-$, $\dim U^+ = \dim U^- = m = n/2$ and this case can occur only when n is even and the index ν of f is m.

The conditions for an element t of $\Gamma L_n(K)$ to commute or anti-commute with u are the same as for projective involutions of the first kind.

Let now u and v be two commuting projective involutions of the first or second kind. Then v leaves U^+ and U^- invariant. So if x is in U^+ or U^-, $x \pm v(x)$ is in U^+ or U^-, respectively. It is easily seen that $x \pm v(x)$ is in V^+ or V^-, respectively. As $x = \frac{x+v(x)}{2} + \frac{x-v(x)}{2}$, we have $U^+ = U^+ \cap V^+ \oplus U^+ \cap V^-$ and $U^- = U^- \cap V^+ \oplus U^- \cap V^-$; so write $E = P_1 \oplus P_2 \oplus P_3 \oplus P_4$ where $P_1 = U^+ \cap V^+$, $P_2 = U^+ \cap V^-$, $P_3 = U^- \cap V^+$, and $P_4 = U^- \cap V^-$.

Now suppose that u and v are of the first kind. Then if x is in P_1 and y is in P_2, P_3, or P_4, then $u(y) = -y$ or $v(y) = -y$. Hence, as $f(x,y) = f(u(x),u(y)) = f(v(x),v(y))$, $f(x,y) = -f(x,y) = 0$. This means that P_1 is orthogonal to $P_2 \oplus P_3 \oplus P_4$. Hence P_1 is nonisotropic. Similarly P_2, P_3, and P_4 are nonisotropic and all the P_i's are orthogonal.

1.7 Involutions of the Third Kind

Let $\bar{u}$ be an involution of the third kind and choose u in $\bar{u}$. Then $u^2 = \gamma$ in C_n where $\gamma = \lambda^2$ and λ is not in Z. We treat the following generalization, which arises in subsequent developments. Let σ be an involutive automorphism of K and let u be a semilinear projective involution of $\Gamma U_n(K,f)$ relative to the automorphism σ with multiplier e, such that $u^2 = \gamma$ in C_n with $\gamma = \lambda\lambda^{\sigma}$. In case $\lambda^{\sigma} = \lambda$, we furthermore assume that λ is not in Z.

Then $u' = \lambda^{-1}u$ is a semilinear involution in $\Gamma U_n(K,f)$ with respect to the automorphism $\sigma' = \overline{\lambda^{-1}}\sigma = \sigma\bar{\lambda}$ and with multiplier $e' = (\lambda\lambda^J)^{-1}e$. It is readily verified that $\sigma'^2 = 1$ and $\sigma' \neq 1$. Let K_1 be the subsfield of K consisting of elements left fixed by σ'. Then if ϕ is in K, $\phi = \frac{\phi + \phi^{\sigma'}}{2} + \frac{\phi - \phi^{\sigma'}}{2}$ where $\phi + \phi^{\sigma'}$ is in K_1 and $\phi - \phi^{\sigma'}$ is in the left and right K_1-subspace K_1' consisting of elements η of K such that $\eta^{\sigma'} = -\eta$. If Θ is an arbitrary element of K_1', then $K_1' = K_1\Theta = \Theta K_1$ and $K = K_1 \oplus K_1\Theta$. Thus E can be considered as a right vector space E^* over K_1 and $\dim_{K_1} E^* = 2n$. As a transformation of E^*, u' is a linear involution. So $E^* = U^+ \oplus U^-$ where U^+ and U^- are the positive and negative eigenspaces of u', respectively. If x is in U^+, $x\Theta$ is in U^- and vice versa. So $U^- = U^+\Theta$ and $E^* = U^+ \oplus U^+\Theta$; that is, a K_1-basis for U^+ is a K-basis for E.

An element t of $\Gamma U_n(K,f)$ relative to an automorphism τ where $\lambda^\tau = \lambda$ and with multiplier e_t, which commutes with u, commutes also with h_λ^{-1} and thus with u'. Considering the effect of u't and tu' on a vector $x\phi$ with ϕ in K, we can conclude that $\sigma'\tau = \tau\sigma'$ and so $K_1^\tau = K_1$. Thus t can be restricted to a semilinear transformation t_1 acting on U^+. Hence, we can reduce the analysis of such transformations t to that of transformations acting on U^+. But in order to do this effectively, we have to consider t_1 as an element of some group $\Gamma U_n(K_1,f_1)$.

Unfortunately the form f cannot in general be restricted to a sesquilinear form acting on U^+. The principal hindrance is that K_1 is not necessarily left invariant by J. Indeed, it can be shown that $\phi^{J\sigma'} = e^{-1}\phi^J e$ for ϕ in K_1 (Dieudonné [4], Section 14). Dieudonné shows that e can always be taken to be 1 by multiplying the form f by a scalar d^{-1} such that $e = d^{1-\sigma'}$ and $d^J = \pm d$. In fact, if $e \neq -1$, it suffices to take $d = 1 + e$. If $e = -1$ and there exists an element ψ of K_1' such that

$\psi^J \neq -\psi$, take $d = \psi+\psi^J$. In both of these cases $d^J = d$, and so the form $g = d^{-1}f$ will be a hermitian sesquilinear form relative to the anti-automorphism $T = J\bar{}: \phi \to \phi^T = d^{-1}\phi^J d$.

However, if $\psi^J = -\psi$ for all elements ψ of K_1' and if ϕ is any element of K_1, we have $(\phi\psi)^J = -\phi\psi = \psi^J\phi^J = -\psi\phi^J$. So $\phi^J = \psi^{-1}\phi\psi$. This means that the restriction of J is an automorphism of K_1; hence K_1 is commutative. Now let d be any element of K_1'. Then $d^{\sigma'} = -d$; so $e = -1 = d^{1-\sigma'}$. But now $g = d^{-1}f$ is a skew-hermitian sesquilinear form relative to the anti-automorphism $T = J\bar{}$. However, $\phi^T = d^{-1}\phi^J d = d^{-2}\phi d^2$ for ϕ in K_1 as d is in K_1'. But d^2 is in K_1, which is commutative; so $\phi^T = \phi$. Thus in this case the restriction of the anti-automorphism T to K_1 is the identity. In the second paragraph below, we shall show that this situation does not occur.

In any case, g is a nondegenerate hermitian or skew-hermitian sesquilinear form defined on E and u' is a semilinear involution in $\Gamma U_n(K,g)$ with multiplier 1. It is easily seen that $U_n(K,g) = U_n(K,f)$ and $\Gamma U_n(K,f) = \Gamma U_n(K,g)$. Furthermore now $K_1^T = K_1$; and, if x and y are in U^+, $g(x,y) = g(u'(x),u'(y)) = g(x,y)^{\sigma'}$ and so is in K_1. Thus the restriction f_1 of g to U^+ is a hermitian or skew-hermitian sesquilinear form such that $f_1: U^+ \times U^+ \to K_1$. As a K_1-basis for U^+ is a K-basis for E, f_1 is nondegenerate. So we can now form the group $U_n(K_1,f_1)$ acting on the K_1-space U^+.

In the case that f_1 is skew hermitian, we have seen that the restriction of T is the identity and K_1 is commutative. This means that f_1 is a skew symmetric bilinear form and $U_n(K_1,f_1)$ is actually the symplectic group $Sp_n(K_1,f_1)$. However, then every vector x of $U^+$is such that $f_1(x,x) = 0$. But $f_1(x,x) = d^{-1}f(x,x)$ and since a K_1-basis for U^+

is a K-basis for E, U^+ must contain a vector x such that $f(x,x) \neq 0$. Hence we have a contradiction; so we can always suppose that there exists an element ψ in K_1' such that $\psi^J \neq -\psi$ and thus f_1 will always be a hermitian sesquilinear form.

Now let us return to the consideration of the elements t which commute with u and which were previously described. Since g is a scalar multiple of f, $\Gamma U_n(K,f) = \Gamma U_n(K,g)$ and $U_n(K,f) = U_n(K,g)$. Therefore t can be considered to be in $\Gamma U_n(K,g)$ and, if t is linear, to be in $U_n(K,g)$. As an element of $\Gamma U_n(K,g)$, t has multiplier $e_t' = d^{-1}e_t d$. Now

$$\begin{aligned} g(u'(t(x),u'(t(y))) &= e_t'^{\sigma'}\, g(x,y)^{\tau\sigma'} \\ &= g(t(u'(x)),\, t(u'(y))) \\ &= e_t' g(x,y)^{\sigma'\tau} . \end{aligned}$$

Since $\tau\sigma' = \sigma'\tau$, it follows that $e_t'^{\sigma'} = e_t'$ and hence e_t' is in K_1. Then clearly the restriction t_1 of t to $U^{+\prime}$ is an element of $\Gamma U_n(K_1,f_1)$. If t is in $U_n(K,f)$, t_1 is in $U_n(K_1,f_1)$. The element t_1 can be extended to the original transformation t acting on E be setting $t(x\Theta) = t_1(x)\Theta^\tau$ for x in U^+. This defines t on U^-, which is what is needed, as $E = U^+ \oplus U^-$. Not every element of $\Gamma U_n(K_1,f_1)$ can be extended to a transformation of $\Gamma U_n(K,f)$ in general.[6] However, if t_1 is in $U_n(K_1,f_1)$, the above process will always lead to a transformation t in $U_n(K,f)$ of which t_1 is the restriction. If t_1 is a projective involution of the k^{th} kind in $U_n(K_1,f_1)$, t will be a projective involution of the k^{th} kind in $U_n(K,f)$.

To summarize, we have the following result.

Let u be a projective semilinear involution in $\Gamma U_n(K,f)$ relative to an automorphism σ of K such that $u^2 = \gamma$ in C_n where $\gamma = \lambda\lambda^\sigma$ with λ in K.

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[6] Cf. Dieudonné [2], Section 14.

Assume furthermore that λ is not in Z if $\lambda^\sigma = \lambda$. Set $\sigma' = \overline{\lambda^{-1}}\sigma$ and let K_1 be the fixed subsfield of K by σ' and E_1 be the vector space E considered as a right K_1-space. Then $E = U^+ \oplus U^-$ where $u(x) = \pm x\lambda$ for x in $U^\pm$, respectively. An element t of $\Gamma U_n(K,f)$ relative to an automorphism τ of K such that $\lambda^\tau = \lambda$ can be restricted to a transformation t_1 in $\Gamma U_n(K_1,f_1)$. If t is in $U_n(K,f)$, t_1 is in $U_n(K_1,f_1)$. An element t_1 of $U_n(K_1,f_1)$ can be extended to an element of t in $U_n(K,f)$. If t_1 is a projective involution of the first kind in $U_n(K_1,f_1)$, t is of the first kind in $U_n(K,f)$, and conversely.

In certain cases we must discuss projective semilinear involutions which anticommute with u. In order to do this, we must first further investigate the structure of the sfield K_1. Let Z_1 be the center of K_1 and let $\bar{\Theta}$ be the automorphism of K given as $\bar{\Theta}$: $\phi \to \phi^{\bar{\Theta}} = \Theta^{-1}\phi\Theta$. Since $(\Theta^2)^{\sigma'} = \Theta^2$, Θ^2 is in K_1. Furthermore, for ϕ in K_1, $\phi^{\bar{\Theta}\sigma'} = \phi^{\bar{\Theta}}$; so $\bar{\Theta}$ leaves K_1 and Z_1 invariant. Hence, the restriction of $\bar{\Theta}$ to Z_1, which we again denote by $\bar{\Theta}$, is an involutive automorphism of Z_1. Thus either every element of Z_1 commutes with Θ (in the case that the restriction of $\bar{\Theta}$ to Z_1 is the identity) or, by galois theory, $Z_1 = Z_0 \oplus Z_0\lambda'$ where every element of Z_0 commutes with Θ and $\lambda'^{\bar{\Theta}} = -\lambda'$.

Let Z' be the subfield of Z_1 which is left fixed by $\bar{\Theta}$ (this is Z_1 in the first case and Z_0 in the second). Then every element of Z' commutes with every element of K_1 and with Θ. Since $K = K_1 \oplus K_1\Theta$, this means that $Z' \subseteq Z$; hence $Z' \subseteq Z \cap K_1$. On the other hand $K_1 \cap Z \subseteq Z'$. Therefore, $Z' = Z \cap K_1$ and we have the following result.

If $\bar{\Theta}$ is the identity on Z_1, $Z_1 = K_1 \cap Z$. If $\bar{\Theta}$ is an involution on Z_1 with fixed subfield Z_0, then $Z_0 = K_1 \cap Z$.

In the important case where u is a projective involution of the third kind in $U_n(K,f)$, $\sigma' = \overline{\lambda^{-1}}$ and so λ is in Z_1. As K_1 is the fixed subsfield of K by $\sigma' = \overline{\lambda^{-1}}$, $Z \subseteq K_1$. So $\bar{\theta}$ is an involution on $Z_0 = K_1 \cap Z = Z$ and we can take $\lambda' = \lambda$. Therefore $Z_1 = Z \oplus Z\lambda$; that is, $Z_1 = Z(\lambda)$.

In general, when σ' leaves Z_1 fixed and $\bar{\theta}$ does not, we have $Z_1 = Z \oplus Z\lambda'$. In this case K_1 is an interior galois subsfield of K and, according to a result of Dieudonné ([3], p.182), $K = K_1 \oplus K_1\psi$ where ψ is an element of K whose square $\psi^2 = \alpha$ is in Z. However, this implies that $\psi^{\sigma'} = -\psi$. Indeed, $\psi^{\sigma'} = a + b\psi$ with a and b in K_1. Therefore, $\psi = \psi^{\sigma'^2} = a(b+1) + b^2\psi$. Thus either $a = 0$ or $b = -1$. In the first case, $b = \pm 1$. But $b \neq 1$ as ψ is not in K_1; so $b = -1$ and $\psi^{\sigma'} = -\psi$. In the second case $\psi^{\sigma'} = a - \psi$. But $(\psi^{\sigma'})^2 = \psi^{2\sigma'} = \alpha^{\sigma'} = \alpha$. On the other hand, $(\psi^{\sigma'})^2 = (a - \psi)^2 = a^2 + \psi^2 - (\psi a + a\psi)$. Then $\alpha = a^2 + \alpha$. So $a = 0$ and $\psi^{\sigma'} = -\psi$. Since θ was chosen arbitrarily in K_1', we can henceforth assume $\theta = \psi$, so that $\theta^2 = \alpha$ is in Z and $\bar{\theta}^2 = 1$.

Now we consider the case that t in $\Gamma U_n(K,f)$ anticommutes with u where u is a projective involution of the third kind in $\Gamma U_n(K,f)$. Let τ be the automorphism of K corresponding to t and e_t be the multiplier of t. Assume that $\lambda^\tau = \lambda$. Then t commutes with $h_{\lambda-1}$ and thus anticommutes with u'. As $\theta^{\sigma'} = -\theta$, h_θ anticommutes with u'. So $h_\theta t$ is a projective semilinear involution of $\Gamma U_n(K,f)$ relative to the automorphism $\tau\bar{\theta}$ which commutes with u'. The multiplier of $h_\theta t$ in $\Gamma U_n(K,g)$ is $d^{-1}\theta^J e_t \theta d$ which again must be in K_1 by the argument given previously in this section. Furthermore $K_1^{\tau\bar{\theta}} = K_1$; so $h_\theta t$ can be restricted to a transformation t_1 of $\Gamma U_n(K_1,f_1)$ acting on U^+. We say in this case that t_1 <u>represents</u> t. To summarize we have the following result.

Let u be a projective involution of the third kind in $U_n(K,f)$ and t a semilinear transformation of $\Gamma U_n(K,f)$ with respect to the automorphism τ of K which anticommutes with u. Let $u^2 = \gamma$ where $\gamma = \lambda^2$ and assume furthermore that $\lambda^\tau = \lambda$. Then t is represented by a transformation t_1 in $\Gamma U_n(K_1,f_1)$, which is semilinear relative to the automorphism $\tau\bar{\theta}$.

1.8 Involutions of the Fourth Kind

Let $\bar{u}$ be an involution of the fourth kind in $PU_n(K,f)$ and choose u in $\bar{u}$. Then $u^2 = \gamma$ in C_n where now γ is not a square in K. Again we consider a generalization, which arises in subsequent developments. Let σ be a given involutive automorphism of K and take u to be a projective semilinear involution in $\Gamma U_n(K,f)$ relative to the automorphism σ and with multiplier e such that $u^2 = \gamma$ where γ is in C and is not of the form $\lambda\lambda^\sigma$ for λ in K.

Dieudonné ([2], Section 13) has shown that there exists a quadratic extension L of K obtained by adjoining a square root ρ of γ such that $\phi\rho = \rho\phi^\sigma$ for ϕ in K. Obviously L is not commutative when $\sigma \neq 1$, even though K may be. So the consideration of sfields is necessary at this point. Now E can be considered as a vector space F over L by setting $x\rho = u(x)$. The L-dimension of F is $m = n/2$. Consequently this type of projective involution exists only when n is even. The anti-automorphism J can be extended to L by setting $\rho^J = \rho e^\sigma \gamma^{-1} = e\rho^{-1}$. Corresponding to f, we define a new nondegenerate hermitian sesquilinear form g by

$$g(x,y) = f(x,y) + \rho f(x,u(y))^\sigma \gamma^{-1} .$$

In $\Gamma U_m(L,g)$ u is a homothetic transformation h_ρ with multiplier e.

It is easily seen that if t is an element of $\Gamma U_n(K,f)$ which is relative to an automorphism t of K and which commutes with u, t can be considered to be in $\Gamma U_m(L,g)$ relative to the extension of the automorphism τ to L obtained by setting $\rho^\tau = \rho$. Conversely, if t is an element of $\Gamma U_m(L,g)$ which is relative to an automorphism τ of L leaving K invariant, t can be considered as an element of $\Gamma U_n(K,f)$ relative to the restriction of τ to K which commutes with u.

Let σ be the automorphism of L defined by the relation $\sigma: \rho \to -\rho$. Clearly σ commutes with any automorphism τ which leaves invariant the fixed field K of σ. Therefore, if t is an element of $\Gamma U_n(K,f)$ which is relative to an automorphism τ of K and which anticommutes with u, t can be considered to be in $\Gamma U_m(L,g)$ relative to the automorphism $\sigma\tau$ of L. Conversely, any element t of $\Gamma U_m(L,g)$ which is relative to an automorphism of the form $\sigma\tau$ where $\rho^\tau = \rho$ and $K^\tau = K$ can be considered as an element of $\Gamma U_n(K,f)$ which is relative to the automorphism τ of K and which anticommutes with u.

In the later chapters, it is necessary to identify the center Z_L of L when u is a projective involution of the fourth kind in $U_n(K,f)$, that is, when $e = 1$ and $\sigma = 1$. Then clearly ρ is in Z_L; so $Z(\rho) \subseteq Z_L$. On the other hand, if $\emptyset = \xi + \rho\eta$ is in Z_L with ξ and η in K, ξ and η must commute with every element of K. This implies that ξ and η are in Z; so $Z_L \subseteq Z(\rho)$. Consequently, the center Z_L of L is the quadratic extension $Z(\rho)$ of Z when u is a projective involution of the fourth kind in $U_n(K,f)$.

In Section 4.2, we make reference to the case where $U_n(K,f)$ is the orthogonal group $O_n(K,f)$ and u is a projective involution of the fourth kind in $O_n(K,f)$. Then K is commutative and J is the identity by

definition of $O_n(K,f)$. Also $u^2 = \gamma$ is in C_n. But now $\gamma^J\gamma = \gamma^2 = 1$; so $\gamma = \pm 1$. As γ is not a square in K, $\gamma = -1$, $\rho = i$, and $L = K(i)$ where $i = \sqrt{-1}$. In this case, the extension of the identity automorphism J of K to L is obtained by setting $i^J = i^{-1} = -i$.

1.9 A Theorem on Cyclic Kummer Fields

The following well-known theorem on cyclic Kummer fields is useful in the discussion of projective involutions.

Let Z be a field containing the primitive n^{th} roots of unity and Z' the splitting field of an irreducible equation $x^n-\alpha = 0$. Then if β is an element of Z' such that β^n is in Z, either β is in Z or $\beta = \lambda\rho$ where λ is a root of $x^n-\alpha$ and ρ is in Z.

We shall apply the theorem when n = 2, in which case the proof is trivial. Since the characteristic of the fields with which we deal is always different from 2, our hypothesis will be always satisfied.

1.10 Some Working Lemmas

LEMMA 1.1 If K is not of characteristic 2, then for every pair of orthogonal nonisotropic vectors x and y in E and for any α in K, at least one of the vectors $x + y(\alpha-1)$, $x + y\alpha$, and $x + y(\alpha+1)$ is nonisotropic.

Compute first

$$f(x+y\beta,x+y\beta) = f(x,x) + \beta^J f(y,y)\beta \ . \tag{1.1}$$

The lemma is trivial if $\alpha = 0$ or $\alpha = \pm 1$. So suppose that $\alpha \neq 0$ and $\alpha \neq \pm 1$ and that $x+y\alpha$ and $x+y(\alpha\pm1)$ are all isotropic. Then (1.1) vanishes for $\beta = \alpha$ and $\beta = \alpha \pm 1$. Setting $f(y,y) = \xi \neq 0$, we have

$$-f(x,x) = \alpha^J\xi\alpha = (\alpha\pm1)^J\xi(\alpha\pm1) \ .$$

From this it follows that

$$\xi = \pm(\alpha^J \xi + \xi\alpha) .$$

Since $\xi \neq 0$, this is impossible and proves the lemma.

LEMMA 1.2 Let r be a projective semilinear involution in $\Gamma U_n(K,f)$ relative to an automorphism σ of K such that $r^2 = \epsilon$ with ϵ in Z and $\sigma^2 = 1$. Let K_0 be the subsfield of K left fixed by σ. Assume that K_0 is not of characteristic 2 and is not the finite field GF(3). Let S be a nonisotropic subspace of dimension greater than 1. If for every nonisotropic vector x in S, $r(x) = x\delta_x$ with δ_x in K, then $r(x) = x\delta$ for every nonisotropic vector x in S where σ is in K and is independent of x and σ is the inner automorphism $\bar{\delta}$: $\phi \to \phi^{\bar{\delta}} = \delta^{-1}\phi\delta$.

It is an immediate consesquence of thes lemma that if $\sigma = 1$, δ is in Z.

Proof: Let x_0 be an arbitrary nonisotropic vector in S. Then $r(x_0) = x_0\delta_{x_0}$. Set $\delta = \delta_{x_0}$. Let $\overline{\delta^{-1}}$ be the inner automorphism $\overline{\delta^{-1}}$: $\phi \to \phi^{\overline{\delta^{-1}}} = \delta\phi\,\delta^{-1}$ and let K_1 be the subsfield of K left fixed by $\pi = \overline{\delta^{-1}}\sigma$. Now $r^2 = \epsilon$ implies $\delta\delta^{\sigma} = \epsilon$, which is in Z. From this it follows that $\pi^2 = 1$. By the argument of Section 1.7, $[K:K_1] = 1$ or 2 according as $\pi = 1$ or not. Now K_1 is not the field GF(3); for, if K is infinite, this is impossible as $[K:K_1] \leq 2$. If K is finite, it is commutative; so $\bar{\delta} = 1$, $\pi = \sigma$, and $K = K_0$, which is not GF(3) by assumption.

Let x_0' be a nonisotropic vector of S orthogonal to x_0. One always exists as S is nonisotropic of dimension greater than 1. Then $r(x_0') = x_0'\delta_{x_0'}$. Again set $\delta_{x_0'} = \delta'$. By Lemma 1.1 there exists an element α in K_1 such that $x_0 + x_0'\alpha$ is also nonisotropic. Because $K_1 \neq$ GF(3), we can always choose $\alpha \neq 0$ by choosing α in Lemma 1.1 not to be

0 or $\pm$ 1. Then $r(x_o+x_o'\alpha) = (x_o+x_o'\alpha)\delta_{x_o+x_o'\alpha}$. On the other hand, $r(x_o+x_o'\alpha) = r(x_o) + r(x_o')\alpha^\sigma$. From these two equations we conclude that

$$x_o(\delta_{x_o+x_o'\alpha} - \delta) = x_o'(\delta'\alpha^\sigma - \alpha\delta_{x_o+x_o'\alpha}) .$$

Since x_o and x_o' are linearly independent with respect to the sfield K, it follows that $\delta = \delta_{x_o+x_o'\alpha}$ and then that $\delta'\alpha^\sigma = \alpha\delta$. But $\alpha^\pi = \alpha$ as α is in K_1. So $\alpha^\sigma = \alpha^{\bar{\delta}} = \delta^{-1}\alpha\delta$ and $\delta\alpha^\sigma = \alpha\delta$. Consequently, $\delta'\alpha^\sigma = \delta\alpha^\sigma$ and $\delta' = \delta$.

However, x_o' was chosen to be an arbitrary nonisotropic vector in S orthogonal to x_o. Hence $\delta = \delta_{x_o'\phi}$ where ϕ is in K. Thus $r(x_o'\phi) = x_o'\phi\delta = r(x_o')\phi^\sigma = x_o'\delta\phi^\sigma$. So $\phi\delta = \delta\phi^\sigma$ and $\phi^\sigma = \delta^{-1}\phi\delta = \phi^{\bar{\delta}}$. Consequently, $r(x_o\phi) = r(x_o)\phi^\sigma = x_o\delta\phi^\sigma = x_o\phi\delta$. As S in nonisotropic, there exists an orthogonal nonisotropic basis $x_o, x_1, \ldots, x_{s-1}$ for S where dim S = s. Then if x is any vector in S, $x = x_o\alpha_o + x_1\alpha_1 + \ldots + x_{s-1}\alpha_{s-1}$, $r(x) = r(x_o\alpha_o) + r(x_1\alpha_1) + \ldots + r(x_{s-1}\alpha_{s-1}) = x\delta$, which completes the proof of the lemma.

LEMMA 1.3 Let S be a subspace of E of dimension greater than 1 and let t be an element of $\Gamma U_n(K,f)$ relative to an automorphism σ of K. Then if $t(x) = x\delta_x$ with δ_x in K for every vector x of S, $t(x) = x\delta$ with δ in K where δ is independent of x in S and σ is the inner automorphism: $\bar{\delta}$: $\phi \to \phi^{\bar{\delta}} = \delta^{-1}\phi\delta$.

Let x be an arbitrary vector in S. Then $r(x) = x\delta_x$. For convenience, set $\delta = \delta_x$. Let y be a vector in S which is linearly independent of x. Then we have $r(y) = y\delta_y$ and $r(x+y) = (x+y)\delta_{x+y}$. But $r(x+y) = x\delta_x + y\delta_y$. So $\delta_x = \delta_{x+y} = \delta_y$. Similarly $\delta = \delta_x = \delta_y\phi$ for ϕ in K. So as argued in the proof of Lemma 1.2, $\phi^\sigma = \delta^{-1}\phi\delta$ and $r(x) = x\delta$ for all x in S where, in this case, it is only necessary to choose

a linearly independent basis for S and not an orthogonal nonisotropic bases. This proves the lemma.

In particular, if an element r in $\Gamma U_n(K,f)$ relative to an automorphism σ of K commutes with every element of $U_n(K,f)$, it commutes with every 1-involution. Hence r leaves all 1-dimensional nonisotropic subspaces invariant. Then by Lemma 1.2, $r(x) = x\delta$ for all x in E where δ is in K. Furthermore, σ is the inner automorphism $\bar{\delta}$ of K; so r is in H_n. It is easily seen that every element of H_n is in the centralizer of $U_n(K,f)$ in $\Gamma U_n(K,f)$. Thus the centralizer of $U_n(K,f)$ in $\Gamma U_n(K,f)$ is H_n and the center of $U_n(K,f)$ is $H_n \cap U_n(K,f) = C_n$.

CHAPTER II

INVOLUTIONS OF THE FIRST KIND

In this chapter $\bar{u}$ will represent an involution of $PU_n(K,f)$ of the first kind. We shall assume first that $n > 3$. Later on we shall take $n > 4$ and exclude the case $n = 6$. The usual restrictions on the sfield K shall hold (cf Section 1.1). In accordance with Section 1.5, we choose the representative u of $\bar{u}$ which is a p-involution of $U_n(K,f)$ with $p \leq [n/2]$. Then $E = U^+ \oplus U^-$ where U^+ and U^- are the positive and negative subspaces of E, respectively. We shall show that $\nu_{\bar{u}} = 4$ if $\bar{u}$ is a 1-involution, that is, extremal, while $\nu_{\bar{u}} > 4$ if $\bar{u}$ is a p-involution with $p > 1$.

Let $\bar{v}$ be an involution of $PU_n(K,f)$ commuting with $\bar{u}$. If $\bar{u}$ is extremal, we have

LEMMA 2.1 If $\bar{u}$ is an extremal involution and $\bar{v}$ an involution commuting with $\bar{u}$, then a projective involution v in $\bar{v}$ must commute with any projective involution u in $\bar{u}$. Furthermore $\bar{v}$ is either of the first or third kind.

As we remarked in Section 1.5, v cannot anticommute with u in $\bar{u}$ as dim $U^+ = 1$ while dim $E = n > 2$. Consequently $v(U^+) = U^+$; so if x is in U^+, $v(x) = x\mu$ with μ in K. But then $v^2(x) = x\mu^2 = x\beta$; so $\beta = \mu^2$ and v is not of the fourth kind. If v is of the first or second kind, we have $E = V^+ \oplus V^-$ and, as in Section 1.6, $E = P_1 \oplus P_2 \oplus P_3 \oplus P_4$ and $U^+ = P_1 \oplus P_2 = U^+ \cap V^+ \oplus V^+ \cap V^-$. But, as U^+ is 1-dimensional, U^+ is contained in V^+ or V^-; that is, V^+ or V^- contains a nonisotropic vector and consequently cannot be totally isotropic. Thus $\bar{v}$ is not of the second kind.

2.1 Extremal Involution Commuting with an Involution of the First Kind

Let $\bar{u}$ be an extremal involution and u a 1-involution. Let $\bar{v}$ be an involution of the first kind distinct from $\bar{u}$ and commuting with $\bar{u}$ and v a p-involution in $\bar{v}$ with $p \leq [n/2]$. Assume $n > 3$. We then have $E = P_1 \oplus P_2 \oplus P_3 \oplus P_4$ where the subspaces P_i are nonisotropic, or 0-dimensional. Indeed, as dim $U^+ = 1$, one of $P_1 = U^+ \cap V^+$ or $P_2 = U^+ \cap V^-$ must contain only the 0 vector. Since $\bar{u} \neq \bar{v}$, $U^- \neq V^-$; and $U^- = P_3 \oplus P_4$ where dim $P_3 > 0$ and dim $P_4 > 0$. Thus E is decomposed into the direct sum of three orthogonal nonzero nonisotropic subspaces.

As dim $E > 3$, at least one of the P_i is of dimension greater than 1. Let x be a nonisotropic vector in P_j where dim $P_j > 1$. Let w be the 1-involution with subspaces $W^+ = xK$ and $W^- = (xK)^\perp$; w is in $U_n(K,f)$ be Section 1.6. As W^+ is in P_j, it is left invariant by u and v. Then $W^- = (W^+)^\perp$ is left invariant by u and v, and, consequently, u and v commute with w. That is, w is in c(u,v); so any r in c(c(u,v)) must commute with w, as w is a 1-involution. Then $r(x) = x\delta_x$ and this must hold for every nonisotropic vector x in P_j. Consequently by Lemma 1.2, $r(x) = x\delta_j$ with δ_j in Z.

Furthermore r must commute with u and commute or anticommute with v, as u and v are in c(u,v). However, r cannot anticommute with v, for then would $r(P_1) = P_2$ whereas dim $P_1 \neq$ dim P_2. So r must leave all P_i invariant. Consequently if dim $P_k = 1$ and x is in P_k, $r(x) = x\delta_x$. But $r^2(x) = x\epsilon$ for all x in E with ϵ in C. Therefore $\delta_x^2 = \delta_j^2$. But if an element ϵ of a sfield has a square root δ in the center, it has but two square roots $\pm\delta$ inasmuch as the polynomial $x^2 - \epsilon$ then factors. So $\delta_x = \pm\delta_j$ and δ_x is in Z. Consequently, for all

vectors $y = x\phi$ with ϕ in K, $r(y) = y\delta_x$. Set $\delta_k = \delta_x$ and $\delta = \delta_3$. Then for x in P_i where $\dim P_i > 0$, $r(x) = \pm x\delta$. This gives two choices in sign for $r(x)$ for x in P_4 and in P_1 or P_2, whichever is not zero. So there are but four possibilities for r, each of which is a representative of $\bar{1}$, $\bar{u}$, $\bar{v}$, or $\overline{uv}$. Therefore $\nu(\bar{u},\bar{v}) = 4$.

In subsequent cases, situations similar to that above arise; and when parts of the proofs are identical with what we have just given, they will not be rendered in such detail.

2.2 Extremal Involution Commuting with an Involution of the Third Kind

Let $\bar{u}$ be an extremal involution and $\bar{v}$ be an involution of the third kind. Again take u in $\bar{u}$ to be a 1-involution and v in $\bar{v}$ such that $v^2 = \beta$ in C_n with $\beta = \mu^2$ where μ is not in Z. As $\dim E > 2$, v must commute with u. With regard to v, form the group $U_n(K_1,f_1)$ acting on the space V^+ where K_1 is the centralizer of μ in K (cf. Section 1.7). The center Z_1 of K_1 is then $Z(\mu)$. The restriction u_1 of u to V^+ is a 1-involution in $U_n(K_1,f_1)$ with eigenspaces $U_1^+ = U^+ \cap V^+$ and $U_1^- = U^- \cap V^+$. For the remainder of this section, we assume merely that u_1 is the restriction of a projective involution u commuting with v such that u_1 is of the first kind in $U_n(K_1,f_1)$. This generalization will be needed in Chapter III.

The K_1-subspaces U_1^+ and U_1^- are orthogonal and nonisotropic (with respect to f_1), at least one of which, say U_1^+, is of K_1-dimension greater than 1 as $n > 2$. Let x be a nonisotropic vector in U_1^+ and w_1 be the 1-involution of $U_n(K_1,f_1)$ with subspaces $W_1^+ = xK_1$ and $W_1^- = (xK_1)^\perp$. Then w commutes with u_1 and its extension w to E is a 1-involution of $U_n(K,f)$ commuting with u and v. Consequently w is in $c(u,v)$ and r in $c(c(u,v))$ must commute with it.

Now r or $h_\Theta r = r'$ commutes with $v' = \mu^{-1}v$. Since $h_\Theta w = wh_\Theta$, r' as well as r commutes with w. Then the restriction r_1 of r or r' to E_1 commutes with w_1. Therefore $r_1(W_1^+) = W_1^+$ and $r(x) = x\delta_x$. If r commutes with v', then clearly r_1 is a projective involution of $U_n(K_1,f)$ such that $r^2 = \epsilon$. If r anticommutes with v', then r_1 is a projective semilinear involution relative to the automorphism $\bar{\Theta}$ of K_1 and $r_1^2 = \epsilon\Theta^2$ where $\epsilon = r^2$ and ϵ in Z. But Θ was chosen in Section 1.7 so that $\Theta^2 = \alpha$ is in Z. So $r_1^2 = \epsilon\alpha$ is in Z and $\bar{\Theta} = 1$. In both cases, r_1 satisfies the hypothesis of Lemma 1.2. Consequently $r(x) = x\delta$ with δ in K_1 for all x in U_1^+. Furthermore, since $\bar{\Theta}$ is not an inner automorphism of K_1, r_1 cannot be semilinear with respect to $\bar{\Theta}$. Consequently r_1 is a projective involution in $U_n(K_1,f_1)$ and δ is in $Z(\mu)$. As $r_1^2 = \epsilon$, $\delta^2 = \epsilon$ where ϵ is in Z. But then the theorem on cyclic fields (cf. Section 1.9) yields that δ is in Z or $\delta = \delta_o\mu$ with δ_o in Z.

If dim $U_1^- > 1$, the same argument shows that $r(x) = x\delta^-$ with δ^- in $Z(\mu)$ for x in U_1^-. Again $(\delta^-)^2 = \delta^2$; so $\delta^- = \pm\delta$. If dim $U_1^- = 1$, then as in Section 2.1, $r_1(U_1^-) = U_1^-$ and $r(x) = x\delta_x$ for x in U_1^-. But $\delta_x^2 = \delta^2$; so $\delta_x = \pm\delta$ and is in $Z(\mu)$. This implies that δ_x is independent of x in U_1^-; so we set $\delta_x = \delta^-$.

Since u_1 is of the first kind in $U_n(K_1,f_1)$, $u_1^2 = \gamma$ where $\gamma = \lambda^2$ and λ is in $Z(\mu)$. On the other hand $u^2 = \gamma$ with γ in Z. Then by the theorem on cyclic fields, λ is in Z or $\lambda = \lambda_o\mu$ with λ_o in Z. If the former is the case, $\bar{u}$ is an involution of the first kind and we can always pick u in $\bar{u}$ to be an involution in $U_n(K,f)$. If the latter is the case, we can consider the $\lambda = \mu$ by replacing the given u by $\lambda_o^{-1}u$, which is also in $\bar{u}$. Now let l_1, u_1, v_1, and u_1v_1 be

the restrictions of 1, u, v, and uv respectively to V^+. The identity is 1_1 and v_1 is the homothetic mapping h_μ. If u is an involution in $U_n(K,f)$, u_1 is an involution with subspaces U_1^+ and U_1^- and u_1v_1 is a projective involution such that $u_1v_1(x) = \pm x\mu$ for x in $U_1^{\pm}$, respectively. If u is a projective involution of the third kind in $U_n(K,f)$, then $u_1(x) = \pm x\mu$ according as x is in U_1^+ or U_1^- and $u_1v_1(x) = \pm x\beta$ with β in C for x in $U_1^{\pm}$ respectively.

As r_1 is in $U_n(K_1,f_1)$ and U_1^+ and U_1^- are nonisotropic, $\delta^T\delta = 1$. If δ is in Z, the $\delta^T = \delta^J$ and h_δ is in C_n. If $\delta = \delta_o\mu$, by looking at $v_1^{-1}r_1$, we can similarly conclude that $\delta_o{}^J\delta_o = 1$ and h_{δ_o} is in C_n. So in any case, there exists an element h_η in C_n such that $h_\eta r_1$ coincides with one of 1_1, u_1, v_1 or u_1v_1. Then r will always be in one of the cosets $\overline{1}$, $\overline{u}$, $\overline{v}$, or $\overline{uv}$. Consequently, we have $\nu(\overline{u},\overline{v}) = 4$. In the case with which we began, this yields

THEOREM 2.1 If the dimension of E is greater than 3, and $\overline{u}$ is an extremal involution in $PU_n(K,f)$, $\nu_{\overline{u}} = 4$.

2.3 Nonextremal Involutions of the First Kind

Let $\overline{u}$ be a p-involution with $p > 1$. We now restrict n to be greater than 4 and $n \neq 6$. We shall exhibit an involution $\overline{v}$ commuting with $\overline{u}$ such that $\nu(\overline{u},\overline{v}) > 4$. Pick u in $\overline{u}$ to be a p-involution in $U_n(K,f)$ with subspaces U^+ and U^-. Let P_1 be a 1-dimensional nonisotropic subspace of U^+ and P_3 a 2-dimensional nonisotropic subspace of U^-. Set $V^+ = P_1 \oplus P_3$, $V^- = (V^+)^\perp$, $P_2 = U^+ \cap V^-$, and $P_4 = U^- \cap V^-$. Then dim P_2 = p-1 > 0 and dim P_4 = n-p-2. Since $p \leq [n/2]$ and $n \geq 5$, we have dim $P_4 > 0$. So P_1, P_2, P_3, and P_4 are orthogonal nonisotropic subspaces of dimension greater than zero. Since $U^- = P_2 \oplus P_4$, $E = P_1 \oplus P_2 \oplus P_3 \oplus P_4$.

Let v be the 3-involution with subspaces V^+ and V^-; then clearly v commutes with u.

Let r_i (i = 1,2,3,4) be the involution of $U_n(K,f)$ with positive eigenspaces $R_i^+ = P_i$ and negative eigenspace $R_i^- = \bigoplus_{j\neq i} R_j$. Each of u, v, and uv has a positive eigenspace which is a sum of two P_i; so $\bar{r}_i$ is not one of $\bar{1}$, $\bar{u}$, $\bar{v}$, or $\overline{uv}$. We shall show that r_i is in c(c(u,v)). First if t in C(u,v) commutes with both u and v, then t leaves invariant U^+, U^-, V^+, and V^-; consequently t leaves P_1, P_2, P_3, and P_4 invariant. So t leaves R_i^+ and R_i^- invariant and thus must commute with r_i. On the other hand t cannot anticommute with v because v is a 3-involution and $n \neq 6$. If t commutes with v and anticommutes with u, then $t(P_1) = P_3$; but this is impossible as dim $P_1 \neq$ dim P_3. So therefore r_i is in c(c(u,v)). This proves

THEOREM 2.2 If the dimension of E is greater than 4 and not 6 and if $\bar{u}$ is a nonextremal involution of the first kind, $\nu_{\bar{u}} > 4$.

In other words, nonextremal involutions of the first kind are not in $\bar{\mathcal{S}}$.

CHAPTER III

INVOLUTIONS OF THE THIRD KIND[7]

In this chapter $\bar{u}$ will represent an involution of the third kind. We pick a representative u in $\bar{u}$. Then $u^2 = \gamma$ in C_n and $\gamma = \lambda^2$ where λ is not a square in Z. Form the group $\Gamma U_n(K_1,f_1)$ acting on the space $E_1 = U^+$ as in Section 1.7. Then if v is a projective involution in $U_n(K,f)$ commuting with u, its restriction v_1 is a projective involution in $U_n(K_1,f_1)$ such that $v_1^2 = \beta$ with β in C. If v is a projective involution in $U_n(K,f)$ anticommuting with u, then v is represented by a projective semilinear involution in $\Gamma U_n(K_1,f_1)$ relative to the automorphism $\bar{\theta}$ of K_1. Furthermore, since $\theta^2 = \alpha$ is in Z, $\bar{\theta}^2 = 1$ and $v_1^2 = \beta\alpha$ where $\beta\alpha$ is in Z. We shall consider first the case that v commutes with u. We distinguish four cases in accordance as to whether v_1 is of the first, second, third, or fourth kind in $U_n(K_1,f_1)$. We have already dealt with the case the v_1 is of the first kind in $U_n(K_1,f_1)$ in Section 2.2. There we proved the following result.

PROPOSITION 3.1. *Let* v *be a projective involution in* $U_n(K,f)$ *commuting with* u *such that its restriction* v_1 *to* E_1 *is a projective involution of the first kind in* $U_n(K_1,f_1)$. *If* $n > 3$, *then* $\nu(\bar{u},\bar{v}) = 4$.

We shall show that this result holds in the three other cases although with different restrictions on n.

[7] It is not necessary to determine whether involutions of the second kind are in $\bar{\mathfrak{I}}$. The argument in Chapter V allows us to pass over this case.

3.1 Commuting Projective Involution whose Restriction is of the Second Kind

Let v_1 be of the second kind in $U_n(K_1,f_1)$. Assume $n > 4$. Then $v_1^2 = \beta$ in C_n, $\beta = \mu^2$ with μ in $Z(\lambda)$, and $\mu^T\mu = -1$. Also $E_1 = V_1^+ \oplus V_1^-$ where V_1^+ and V_1^- are totally isotropic subspaces of E_1 such that $v_1(x) = \pm x\mu$ for x in $V_1^{\pm}$, respectively. Furthermore, dim V_1^+ = dim V_1^- = $n/2 > 2$. Let x be a vector in V_1^+; then there exists a vector y in V_1^- such that $f_1(x,y) \neq 0$. Let W^+ be the subspace of E_1 generated by x and y. Since x and y are nonorthogonal isotropic vectors, W^+ is nonisotropic (cf. Section 1.4). Set $W_1^- = (W_1^+)^\perp$ and let w_1 be the 2-involution in $U_n(K_1,f_1)$ with subspaces W_1^+ and W_1^-. As $W_2^+ = xK_1 \oplus yK_1$, v_1 leaves W_1^+ invariant. As v_1 is in $U_n(K_1,f_1)$, v_1 leaves $W_1^- = (W_1^+)^\perp$ invariant. Consequently w_1 commutes with v_1 and the extension w of w_1 to E is a 2-involution commuting with u and v. The choice of the vector y in V_1^- is not unique. Indeed, let y'' be a vector in $V_1^- \cap W_1^-$ and set $y' = y+y''$. Then $f_1(x,y') = f_1(x,y) \neq 0$ and y' is linearly independent of y. Set $W_1'^+ = xK_1 \oplus y'K_1$ and $W_1'^- = (W_1'^+)^\perp$. Then if w_1' is the 2-involution of $U_n(K_1,f_1)$ with W_1 and $W_1'^-$ as its subspaces, w_1' commutes with v_1, and its extension w' to E commutes with u and v.

Not let r be in $c(c(u,v))$. As $n > 4$, r commutes with w and w'. If r anticommutes with u, we form $r' = h_\Theta r$; r' also commutes with w and w'. The restriction r_1 of r, in case r commutes with u, or of r', in case r anticommutes with u, then commutes with w_1 and w_1'. This means that $r_1(W_1^+ \cap W_1'^+) = W_1^+ \cap W_1'^+$; that is, $r_1(x) = x\delta_x$. Furthermore, since r_1 is linear or semilinear with respect to $\bar{\Theta}$ and $r_1^2 = \epsilon$ with ϵ in Z, we have by Lemma 1.3 that $r_1(x) = x\delta$ with δ in K_1 for all x in

V_1^+. Furthermore, the possibility that r_1 is a semilinear transformation of E_1 is eliminated since $\bar{\theta}$ is not an inner automorphism of K_1. So r_1 must be linear and δ is in $Z(\lambda)$. Similarly, for x in V_1^-, $r_1(x) = x\delta^-$ with δ^- in $Z(\lambda)$.

Now $r_1^2 = \epsilon$ in Z. This implies that $\delta^2 = (\delta^-)^2 = \epsilon$. Hence, $\delta^- = \pm\delta$; and by the theorem on cyclic fields (cf. Section 1.9), δ is in Z or $\delta = \delta_0\lambda$ with δ_0 in Z. If δ is in Z, take x in V_1^+ and y in V_1^- such that $f_1(x,y) \neq 0$. Then $f_1(x,y) = f_1(r_1(x),r_1(y)) = \pm\delta^T\delta f_1(x,y)$. So $\delta^T\delta = \pm 1$ according as $\delta^- = \pm\delta$. As δ is in Z, $\delta^T = \delta^J$. So we have that δ is in C in case $\delta^- = \delta$. If $\delta^- = -\delta$, then the above argument applied to v_1r_1 shows that $v_1r_1(x) = x\delta\mu$ for all x in E_1 where $\delta\mu$ is in C. If $\delta = \delta_0\lambda$ with δ_0 in Z, replace r_1 by u_1r_1 is the above arguments. We then obtain that $u_1r_1(x) = x\delta_0\gamma$ with $\delta_0\gamma$ in C or $v_1u_1r_1(x) = x\delta_0\mu\gamma$ with $\delta_0\mu\gamma$ in C. In any case, there exists a homothetic transformation h_η in C_n such that $h_\eta r_1$ coincides with the restrictions of 1, u, v, or uv to E_1. Hence r is in $\bar{1}$, $\bar{u}$, $\bar{v}$, or $\overline{uv}$. This yields the following result.

PROPOSITION 3.2 *Let v be a projective involution in $U_n(K,f)$ commuting with u such that its restriction v_1 to E_1 is a projective involution of the second kind in $U_n(K_1,f_1)$. If $n > 4$, $\nu(\bar{u},\bar{v}) = 4$.*

3.2 Commuting Projective Involution whose Restriction is of the Third Kind

Let v_1 be of the third kind in $U_n(K_1,f_1)$. Assume $n > 2$. Then $v_1^2 = \beta$ with β in C and $\beta = \mu^2$ where μ is not in $Z(\lambda)$. Clearly, then v is the restriction of an involution of the third kind in $U_n(K_1,f_1)$. Then apply the analysis of Section 1.7 to v_1. Let K_2 be

the subsfield of K_1 left elementwise invariant by $\bar{\mu}$. Then $K_1 = K_2 \oplus K_2\psi$ where $\psi^{\bar{\mu}} = -\psi$ and ψ^2 is in $Z(\lambda)$. The center of K_2 is $Z(\lambda,\mu)$. Also E_1 as a right vector space E_1^* over K_2 is the direct sum of the eigenspaces of v_1: $E_1^* = V_1^+ \oplus V_1^-$. Set $E_2 = V_1^+$ and form the group $\Gamma U_n(K_2,f_2)$ acting on E_2 as in Section 1.7. Then an element t of $U_n(K,f)$ which commutes with both u and v can be restricted to an element t_2 in $U_n(K_2,f_2)$. Conversely each such element t_2 can be extended first to an element t_1 in $U_n(K_1,f_1)$ commuting with v_1 and then to an element t of $U_n(K,f)$ commuting with u and v.

A projective involution t of $U_n(K,f)$ which anticommutes with u is represented by a projective semilinear involution in $U_n(K_1,f_1)$ relative to the automorphism $\bar{\theta}$ of K_1. In order to restrict or represent t_1 in $\Gamma U_n(K_2,f_2)$ as in Section 1.7, the relation $\mu^{\bar{\theta}} = \mu$ must be satisfied. In general, this will not be the case; but we shall show that there exists an element χ in K such that $\chi^{\bar{\lambda}} = -\chi$, $\mu^{\bar{\chi}} = \mu$, and χ^2 is in $Z(\mu)$. This will enable us to carry out the analysis of projective involutions which anticommute with u or v.

First note that $\lambda^{\bar{\psi}} = \lambda$ as ψ is in K_1 and λ is in $Z_1 = Z(\lambda)$. We now interchange the roles of u and v to obtain the same relation between χ and μ. Clearly, we have that μ is in K but not in Z. So let K_1' be the fixed subsfield of K by $\overline{\mu^{-1}} = \bar{\mu}$. Then $K = K_1' \oplus K_1'\psi$ where ψ is the element chosen above. Since μ is in K_1, $\mu\lambda = \lambda\mu$; so λ is in K_1'. Clearly, λ is not in the center $Z(\mu)$ of K_1'; for, otherwise μ would be in $Z(\lambda)$. So let K_2' be the fixed subsfield of K_1' by $\bar{\lambda}$. Then $K_1' = K_2' \oplus K_2'\chi$ where now χ is chosen as in Section 1.7 so that $\chi^{\bar{\lambda}} = -\chi$ and χ^2 is in $Z(\mu)$. We cannot say that χ^2 is in Z as is the

case with Θ, but this will not hinder us. To summarize, we have the following relations:

i) $K = K_2 \oplus K_2\psi \oplus K_2\chi \oplus K_2\psi\chi$;

ii) $\lambda^{\bar{\varkappa}} = -\lambda$; $\mu^{\bar{\varkappa}} = \mu$; $\lambda^{\bar{\psi}} = \lambda$; $\mu^{\bar{\psi}} = -\mu$;

iii) χ^2 is in $Z(\mu)$; ψ^2 is in $Z(\lambda)$.

We now replace Θ by χ. This does not alter the situation with respect to the projective involutions commuting with u or v. Let t be a projective involution in $U_n(K,f)$ which anticommutes with u. Then t is represented by a projective involution t_1 in $\Gamma U_n(K_1,f_1)$ which is semilinear with respect to the involutive automorphism $\bar{\varkappa}$ of K_1. Furthermore, if t commutes or anticommutes with v, t_1 commutes or anticommutes with v_1. Then by Section 1.7, t_1 can be restricted to a semilinear projective involution in $\Gamma U_n(K_2,f_2)$ relative to the automorphism $\bar{\varkappa}$ of K_2 in case t_1 commutes with v_1 or relative to the automorphism $\overline{\chi\psi}$ of K_2 in case t_1 anticommutes with v_1. If t commutes with u and anticommutes with v, then the restriction t_1 of t to E_1 anticommutes with v_1 and hence is represented by a projective semilinear involution t_2 relative to the automorphism $\bar{\psi}$ of K_2.

The extension w of a 1-involution w_2 of $U_n(K_2,f_2)$ to E is a 1-involution in c(u,v). Consequently, a projective involution r of c(c(u,v)) commutes with w as $n > 2$. Therefore, if r_2 represents r in $\Gamma U_n(K_2,f_2)$, r_2 commutes with w_2. This means that r_2 leaves every 1-dimensional nonisotropic space of E_2 invariant. Furthermore, r_2 is a projective semilinear involution relative to an involution automorphism of K_2 and $r_2^2 = \epsilon$ with ϵ in $Z(\lambda,\mu)$. So by Lemma 1.2, $r_2(x) = x\delta$ with δ in K_2 for all x in E_2. Moreover, r_2 must be semilinear with respect to an inner automorphism of K_2. Since $\bar{\varkappa}$

$\overline{\psi}$ and $\overline{\varkappa}\,\overline{\psi}$ are not inner automorphisms, we have that r_2 must be linear and ϵ is in Z. Furthermore, δ is in $Z(\lambda,\mu)$ and $\delta^2 = \epsilon$. We now apply the theorem on cyclic fields twice. First either δ is in $Z(\lambda)$ of $\delta = \delta_1\mu$ with δ_1 in $Z(\lambda)$. Then δ or δ_1 is a square of an element in Z. This implies that δ is Z, $\delta = \delta_1\mu$, $\delta = \delta_2\lambda$, or $\delta = \delta_3\lambda\mu$ with δ_i in Z. Set $\delta_o = \delta$. Let u_2 and v_2 be the restrictions of u and v to E_2. Let r_2' be that one of r_2, $u_2^{-1}r_2$, $v_2^{-1}r_2$, or $u_2^{-1}v_2^{-1}r_2$ such that $r_2'(x) = x\delta_i$ for x in E_2 and some $i = 1,2,3,4$. Since r_2 is in $U_n(K_2,f_2)$, δ is in Z, and E_2 is nonisotropic, $\delta^T\delta = \delta^J\delta = 1$. Therefore, the extension r' of r_2' to E is in C_n and r is in one of the cosets $\overline{1}$, $\overline{u}$, $\overline{v}$, or $\overline{uv}$. This gives us the following result.

PROPOSITION 3.3 Let v be a projective involution in $U_n(K,f)$ commuting with u such that its restriction v, to E, is a projective involution of the third kind in $U_n(K_1,f_1)$. If $n > 2$, $\nu(\overline{u},\overline{v}) = 4$.

3.3. Commuting Projective Involution whose Restriction is of the Fourth Kind

Let v_1 be of the fourth kind in $U_n(K_1,f_1)$. Assume $n > 4$. Then $v_1^2 = \beta$ with β in C and β not a square in K_1. As in Section 1.8, extend the sfield K_1 to a sfield L_1 by adjoining a square root ρ of β. The center L_1 is then $Z(\lambda,\rho)$. Then E_1 can be considered as a right vector space over L_1 by setting $x\rho = v_1(x)$ and f_1 can be extended to a nondegenerate hermitian sesquilinear form g_1 defined on F_1. We then form the group $\Gamma U_m(L_1,g_1)$ acting on F_1.

Let t_1 be the restriction of a projective involution in $U_n(K,f)$ which commutes with u or the representation of a projective

involution which anticommutes with u, in which case t_1 is semilinear with respect to the automorphism $\bar{\theta}$ of K_1. If, furthermore, t commutes or anticommutes with v, t_1 commutes or anticommutes, respectively, with v_1. Then t_1 can be considered as a projective semilinear involution in $\Gamma U_m(L_1,g_1)$ relative to one of the automorphisms 1, $\bar{\theta}$, σ or $\bar{\theta}\sigma$ where σ is the automorphism of L_1 defined by $\sigma\colon \rho \to -\rho$. It is easily verified that σ and $\bar{\theta}$ commute ; so all these automorphisms are involutive. The projective involutions u, v, and uv are represented in $U_m(L_1,g_1)$ by the homothetic transformations h_λ, h_ρ, and $h_{\lambda\rho}$, respectively.

Every 1-involution w_1 of $U_m(L_1,g_1)$ extends to a 2-involution w of $U_n(K,f)$ in c(u,v). If r_1 in $\Gamma U_m(L_1,g_1)$ represents a projective involution r in c(c(u,v)), then r must commute with w as $n > 2$ and r_1 must commute with w_1. So r_1 leaves invariant every 1-dimensional nonisotropic subspaces of F_1. Furthermore, r_1 is a projective semilinear transformation relative to an involutive automorphism of L_1 and $r_1^2 = \epsilon$ with ϵ in $Z(\lambda)$. So by Lemma 1.2, $r_1(x) = x\delta$ with δ in L_1 for all x in F_1. However, r_1 cannot be semilinear with respect to the automorphisms, $\bar{\theta}$, σ or $\bar{\theta}\sigma$ as none of these are inner automorphisms. Consequently, r_1 is linear and must be the representation of a projective involution r commuting with u and v. Therefore, ϵ is in Z and δ is in $Z(\lambda,\rho)$. Also $\delta^2 = \epsilon$. So applying the theorem on cyclic fields twice as in Section 3.2, we have that $\delta = \delta_0$, $\delta = \delta_1\rho$, $\delta = \delta_2\lambda$ or $\delta = \delta_3\lambda\rho$ where δ_1 is in Z and $\delta_i{}^J\delta_i = 1$. Thus r_1 is the representation of a projective involution in $\bar{1}$, $\bar{u}$, $\bar{v}$, or $\overline{uv}$. This yields the following result.

PROPOSITION 3.4 Let v be a projective involution in $U_n(K,f)$ commuting with u such that its restriction v_1 to E_1 is a projective involution of the fourth kind in $U_n(K_1,f_1)$. If $n > 4$, $\nu(\bar{u},\bar{v}) = 4$.

Now we must consider the case that v anticommutes with u. Then set $v' = h_\theta v$. Now v' is a semilinear projective involution in $\Gamma U_n(K,f)$ which commutes with $u' = \lambda^{-1}u$. So the restriction v_1 of v' to E_1 is a semilinear projective involution in $\Gamma U_n(K_1,f_1)$ relative to the automorphism θ and $v_1^2 = \beta$ with β in Z. First note that if $v_1(x) = x\mu$ for x in E_1, then $v_1^2(x) = x\mu\mu^{\bar{\theta}}$. Therefore $\beta = \mu\mu^{\bar{\theta}}$. So now we distinguish between two cases: Case I where such an element μ of K_1 exists and β is of the form $\mu\mu^{\bar{\theta}}$ and Case II where no such element exists in K_1 and β is not of the form $\mu\mu^{\bar{\theta}}$.

3.4 Anticommuting Projective Involutions. Case I

Let v be a projective involution in $U_n(K,f)$ anticommuting with u such that v is represented by a projective semilinear involution v_1 in $\Gamma U_n(K_1,f_1)$ where $v_1^2 = \beta$ and $\beta = \mu\mu^{\bar{\theta}}$ with μ in K_1. Assume $n > 2$. Then let K_2 be the fixed subfield of K_1 by $\bar{\theta}\mu^{-1}$. Since $\lambda^{\bar{\theta}\mu^{-1}} = -\lambda$, $K_1 = K_2 \oplus K_2\lambda$. Let Z_2 be the center of K_2; then an element in Z_2 commutes not only with every element of K_2 but also with λ and therefore with every element of K_1. So $Z_2 \subseteq K_2 \cap Z(\lambda) = Z$. Since Z is left elementwise invariant by $\bar{\theta}\mu^{-1}$, $Z \subseteq Z_2$. Therefore $Z = Z_2$. In the manner of Section 1.7, form the group $\Gamma U_n(K_2,f_2)$ acting on $E_2 = V^+$.

Now an element t of $c(u,v)$ which commutes with u and v commutes also with $h_\theta v$ and so can be restricted to an element t_1 of $\Gamma U_n(K_1,f_1)$ which commutes with v_1. Then t_1 can be restricted to

an element t_2 of $\Gamma U_n(K_2,f_2)$. Since t is linear in $U_n(K,f)$, t_1 will be linear in $U_n(K_2,f_2)$. If an element t commutes with u and anticommutes with v, then ut will commute with both u and $h_\theta v$ and hence ut can be restricted to a linear transformation in $U_n(K_2,f_2)$. Similarly, if t anticommutes with u and commutes with v, form vt; if t anticommutes with both u and v, form uvt.

Every 1-involution w_2 of $U_n(K_2,f_2)$ can be extended to a 1-involution w of $U_n(K,f)$ which commutes with u and v. Hence r is in $c(c(u,v))$ and must commute with w. Denote by r' that one of r, ur, vr, or uvr which commutes with u and v. Then r' will still commute with w and its restriction r_2 will be a linear projective involution of $U_n(K_2,f_2)$ which leaves every 1-dimensional nonisotropic subspace of E_2 invariant. Hence by Lemma 1.2, $r_2(x) = x\delta$ with δ in Z for every vector x in E_2. Furthermore, $\delta^H\delta = 1$ where H is the anti-automorphism corresponding to f_2. But since δ is in Z, $\delta^H = \delta^J$. So r_2 can be extended to a homothetic transformation of E. That is, r' is in $\overline{1}$; so r is in one of $\overline{1}$, $\overline{u}$, $\overline{v}$, or $\overline{uv}$. This yields the following result.

PROPOSITION 3.5. Let v be a projective involution of $U_n(K,f)$ anticommuting with u such that v is represented by a projective semilinear involution v_1 of $\Gamma U_n(K_1,f_1)$ where $v_1^2 = \beta$ and $\beta = \lambda\lambda^{\overline{\theta}}$. If $n > 2$, $\nu(\overline{u},\overline{v}) = 4$.

3.5 Anticommuting Projective Involutions. Case II

Now consider the case that v anticommutes with u and is represented by a projective semilinear involution v_1 acting on E_1 such that $v_1^2 = \beta$ where β is not of the form $\mu\mu^{\overline{\theta}}$ for in K_1. We then adjoin a square root ρ of β to K_1 forming a sfield $L_1 = K_1(\rho)$ and

form the group $\Gamma U_m(L_1,g_1)$ acting on the space F_1 as in Section 1.8. Now ρ satisfies the relation $\phi\rho = \rho\phi^{\bar{\theta}}$ for all ϕ in K_1. In particular if ϕ is in Z_1, the center of K_1, then $\phi^{\bar{\theta}} = \phi$. On the other hand $\phi(\xi + \rho\eta) = (\xi + \rho\eta)\phi$ for all ξ and η in K_1. This implies that $\phi\xi - \xi\phi = \rho(\eta\phi - \phi\eta)$. So ϕ must commute with ξ and hence $Z_1 \subseteq Z(\lambda)$. This implies that $Z_1 = Z$.

Again an element t in $U_n(K,f)$ which commutes with u and v commutes with $h_\theta v$ and hence can be restricted to a projective involution t_1 in $U_n(K_1,f_1)$ which commutes with v_1. Hence t_1 can be considered to be in $U_m(L_1,g_1)$. If t anticommutes with one or both of u, v, or uv, then one of ut, vt, or uvt will commute with both u and v.

Now let w_1 be an 1-involution of $U_m(L_1,g_1)$. Then w_1 can be extended to a 2-involution w in $U_n(K,f)$ which commutes with u and v. Since $n > 4$, r in c(c(u,v)) must commute with w. Denote by r' that one of r, ur, vr, or uvr which commutes with u and v. Then r' commutes with w and is represented by a linear projective involution r_1' of $\Gamma U_m(L_1,g_1)$ commuting with w_1. Hence r_1' leaves invariant every 1-dimensional nonisotropic subspace of F_1. Therefore, by Lemma 1.2, $r_1'(x) = x\delta$ with δ in Z for all x in F_1. Again as in Section 3.4, this implies that $\delta^J\delta = 1$ and thus that r' is in $\bar{1}$. Hence r is in one of $\bar{1}$, $\bar{u}$, $\bar{v}$, or $\overline{uv}$. This yields the following result.

PROPOSITION 3.6 *Let* v *be a projective involution of* $U_n(K,f)$ *anticommuting with* u *such that* v *is represented by a*

projective semilinear involution v_1 of $\Gamma U_n(K_1,f_1)$ where $v_1^2 = \beta$ is not of the form $\mu\mu^{\bar{\theta}}$ for μ in K_1. If $n > 4$, $\nu(\bar{u},\bar{v}) = 4$.

Now Propositions 3.1 through 3.6 yield the following theorem.

THEOREM 3.1. If the dimension of E is greater than 4, and if $\bar{u}$ is an involution of the third kind in $PU_n(K,f)$, then $\nu_{\bar{u}} = 4$.

So all involutions of the third kind are in $\bar{\mathfrak{D}}$.

CHAPTER IV

INVOLUTIONS OF THE FOURTH KIND

In this chapter, $\bar{u}$ will represent an involution of the fourth kind in $PU_n(K,f)$. Pick a representative u in $\bar{u}$. Then u is characterized by the relation $u^2 = \gamma$ in C_n where γ is not a square in K. As in Section 1.8, we form the sfield L obtained by adjoining a square root ρ of γ to K. The center of L is $Z(\rho)$. Then E can be considered as a right vector space F over L. Form the group $\Gamma U_m(L,g)$. Let σ be the automorphism of L defined by σ: $\rho \to -\rho$. An element t of $U_n(K,f)$ which commutes or anticommutes with t can be considered as an element of $\Gamma U_m(L,g)$ which is, respectively, linear or semilinear with respect to the automorphism σ.

Throughout this chapter, we assume that $n > 4$. We shall show that $\nu_{\bar{u}} = 4$. Let v be a projective involution of $U_n(K,f)$ which commutes or anticommutes with u. We consider v to be in $\Gamma U_m(L,g)$ and distinguish various cases. Since u is in c(u,v), this set can be considered to be a set of projective semilinear involutions in $\Gamma U_m(L,g)$ which are linear or semilinear with respect to the automorphism σ of L. Similarly, the set c(c(u,u) can be considered to be in $\Gamma U_m(L,g)$.

4.1 Projective Involutions of the First Kind in $U_m(L,g)$

Let v be a projective involution of the first kind in $U_m(L,g)$. Then $v^2 = \beta$ with β in C, $\beta = \mu^2$ where μ is in $Z(\rho)$ and $\mu^J\mu = 1$. As in Section 1.5, $F = V^+ \oplus V^-$ where V^+ and V^- are the orthogonal nonisotropic eigenspaces. Since $m > 2$, one of V^+ and V^-, say V^+, must be

of L-dimension greater than 1. Let x be a nonisotropic vector in V^+ and w the 1-involution of $U_m(L,g)$ with xK as its 1-dimensional subspace W^+. Since $W^+ \subseteq V^+$ and v is in $U_m(L,g)$, v leaves invariant W^+ and $W^- = (W^+)^\perp$. Consequently, w commutes with v, and thus, w is in c(u,v). Let r be in c(c(u,v)). Then as an element of $\Gamma U_m(L,g)$, r is linear or semilinear with respect to the automorphism σ. Furthermore, r must commute with w as $m > 2$; hence $r(x) = x\delta_x$ with δ_x in L for each nonisotropic vector x in V^+. So by Lemma 1.2, $r(x) = x\delta$ with δ in L for x in V^+. Also, r must be semilinear with respect to an inner automorphism of L. But σ does not leave the center $Z(\rho)$ of L elementwise invariant and, therefore, is not an inner automorphism of L. So r is linear and δ is in $Z(\rho)$. Also $r^2 = \epsilon$ with ϵ in Z; so $\delta^2 = \epsilon$. It then follows from the theorem on cyclic fields that δ is in Z or $\delta = \delta_0\rho$ with δ_0 in Z.

If $\dim_L V^- > 1$, it follows in a similar manner that $r(x) = x\delta^-$ with δ^- in L for x in V^-; and as $r^2 = \epsilon$, we have $(\delta^-)^2 = \delta^2 = \epsilon$. So $\delta^- = \pm\delta$. If $\dim_L V^- = 1$, then $r(x) = x\delta_x$ for x in V^-. Again, $\delta_x^2 = \delta^2$ with $\delta_x = \pm\delta$. So δ_x is in $Z(\rho)$ and hence is independent of x in V^-. Set $\delta_x = \delta^-$. Since g is sesquilinear with respect to the anti-automorphism J extended to L and V^+ is nonisotropic, we have that $\delta^J\delta = 1$. Thus if δ is in Z, δ is in C. If $\delta = \delta_0\rho$, $\delta_0{}^J\delta_0 = 1$ as $\rho^J = \rho^{-1}$; so δ_0 is in C. In any case r as an element of $U_n(K,f)$ is in one of the cosets $\bar{1}$, $\bar{u}$, $\bar{v}$, or $\bar{w}$. So we have proved the following result.

PROPOSITION 4.1 Let v be a projective involution of the first kind in $U_m(L,g)$. If $n > 4$, $\nu(\bar{u},\bar{v}) = 4$.

4.2 Projective Involutions of the Second Kind in $U_m(L,g)$

Let v be a projective involution of the second kind in $U_m(L,g)$. Then $v^2 = \beta$ with β in C, $\beta = \mu^2$ with μ in $Z(\rho)$, and $\mu^J\mu = -1$. As in Section 1.6, $F = V^+ \oplus V^-$ where V^+ and V^- are totally isotropic subspaces of L-dimension m/2. Now let x be an arbitrary vector in V^+ and pick y in V^- such that $g(x,y) \neq 0$. Then $W^+ = xL \oplus yL$ is nonisotropic and the 2-involution w in $U_m(L,g)$ with W^+ and $W^- = (W^+)^\perp$ as eigenspaces commutes with v. As w is in $U_m(L,g)$, it can also be considered to be a 4-involution in $U_n(K,f)$ which commutes with u; that is, w is in c(u,v). Also $W^- = W^- \cap V^+ \oplus W^- \cap V^-$. Both $W^- \cap V^+$ and $W^- \cap V^-$ are totally isotropic; and, as W^- is a nonisotropic subspace of L-dimension m/2, $\dim_L W^- \cap V^+ = \dim_L W^- \cap V^- = m/2 - 1$. Since $m > 2$, these subspaces are not zero and so there exists a vector b in $W^- \cap V^-$. Set $y' = y+b$; then $g(x,y') = g(x,y) \neq 0$. Consequently, the involution w' with subspaces $W'^+ = xL \oplus y'L$ and $W'^- = (W'^+)^\perp$ is also in c(u,v) when considered as a transformation in $U_n(K,f)$.

First consider the case that r in c(c(u,v)) commutes with w. (We know that this must happen when $n \neq 8$.) Then we can conclude that r commutes with w'. For, should r anticommute with w', then r(x) would not be in $V^+ \cap W^+ = W^+ \cap W'^+$ and so would be in $V^- \cap W^+$, as r must commute or anticommute with v. We can then take y in $V^- \cap W^+$ such that $r(x) = y$ and y would be in W'^- and hence orthogonal to x in W'^+, which is a contradiction. Consequently, r commutes with both w and w' and r(x) is in $W^+ \cap W'^+$; that is, $r(x) = x\delta_x$ with δ in L.

Now consider the possibility that r commutes with v but anticommutes with w. Then $n = 8$ and $r(x) = a$ with a in $V^+ \cap W^-$. Pick b in $V^- \cap W^-$ as above. Then $\dim_L V^+ \cap W^- = \dim_L V^- \cap W^- = 1$; so a and b generate W^-, which is nonisotropic. Consequently $g(a,b) \neq 0$. Furthermore,

$r(x) = a$ is not in $W'^{+} = xK \oplus y'K$ as $V^{+} \cap W'^{+} = xK$; therefore, r must anticommute with w' and a must be in $V^{+} \cap W'^{-}$. But a is not orthogonal to $y' = y+b$, as $g(a,y') = g(a,b) \neq 0$ while y' is in W'^{+}. Thus a cannot be in W'^{-}; and, consequently, r cannot anticommute with w'. But as w' is in $c(u,v)$ and r is in $c(c(u,v))$, this is a contradiction.

Finally consider the possibility that r anticommutes with both v and w. Again $n = 8$ and $r(x) = b$ with b in $V^{-} \cap W^{-}$. Since $r(V^{+} \cap W^{-}) = V^{-} \cap W^{+}$, there exists a vector a in $V^{+} \cap W^{-}$ such that $r(a) = y$. Set $z = x+a$ and , as before, $y' = y+b$; then $r(z) = y'$. On the other hand, $r^2 = \epsilon$; so $r(y) = a\epsilon$ and $r(b) = r^2(x) = x\epsilon$. From this it follows that $g(z,y') = g(x+a,y+b) = g(x,y) + g(a,b) = g(x,y) + g(y,x)\epsilon$. Since y was chosen arbitrarily to form W^{+}, it may be replaced by $y\phi$ with ϕ in L. Then there always exists a vector y' such that $g(z,y') \neq 0$. Indeed, if $g(x,y)\phi + \phi^{J}g(y,x)\epsilon = 0$ for all ϕ in L, we obtain upon setting $g(x,y) = \xi \neq 0$,

$$\xi\phi + \phi^{J}\xi^{J}\epsilon = 0.$$

Setting $\phi = 1$, we have $\xi^{-J} = -\epsilon\xi^{-1}$ and then that $\phi^{J} = \xi\phi\xi^{-1}$. Thus K must be commutative as J is an anti-automorphism, but then J must leave $Z(\rho) = K$ elementwise invariant. However, even in this case where $U_8(K,f)$ has to be the orthogonal group $O_8(K,f)$, J, which is the identity on K, extends to $L = K(\rho) = K(i)$ as the automorphism $J: i \to -i$ (cf. Section 1.8). This is in contradiction due to the assumption $g(z,y') = 0$ for all ϕ in K. So we can take $g(z,y') \neq 0$. Then the subspace Q^{+} generated by z and y' must be nonisotropic and r must commute with the 2-involution q with Q^{+} as its positive subspace. But now we are reduced to the case at the beginning of this discussion where we had r commuting with w instead of q as

here. The argument there tells us that $r(z) = z\delta_z$; but $r(z) = y'$. So we have a contradiction and r must commute with w and w'. That is, $r(x) = x\delta_x$ with δ_x in L.

Now by Lemma 1.3, it follows that $r(x) = x\delta$ with δ in L for all x in V^+. Furthermore, r cannot be semilinear with respect to σ as σ is not an inner automorphism of L for the reason that it does not leave $Z(\rho)$ elementwise invariant. Consequently, r is linear and δ is in $Z(\rho)$. Furthermore, $\delta^2 = \epsilon$ in Z. So by the theorem on cyclic fields (Section 1.9), δ is in Z or $\delta = \delta_0\rho$ with δ_0 in Z. A similar argument shows that $r(x) = x\delta^-$ with δ^- in $Z(\rho)$ for all x in V^-. Since $r^2 = \epsilon$, it follows that $\delta^- = \pm\,\delta$. But now if δ is in Z and $\delta^- = \delta$, r is the homothetic mapping h_δ on F. Since F is nonisotropic and r is in $U_m(L,g)$, $\delta^J\delta = 1$; that is, r is in $\overline{1}$. If δ is in Z and $\delta^- = -\delta$, then vr is a homothetic mapping $h_{\mu\delta}$ on F; for, $v(r(x)) = v(x)\delta = x\mu\delta$ if x is in V^+ and $v(r)x)) = -v(x)\delta = x\mu\delta$ if x is in V^-. So vr is in $\overline{1}$ and r is in $\overline{v}$. Similarly if $\delta = \delta_0\rho$ and $\delta^- = \delta$, ur is a homothetic mapping in C_n and r is in $\overline{u}$; and if $\delta = \delta_0\rho$ and $\delta^- = -\delta$, uvr is a homothetic mapping in C_n and r is in $\overline{uv}$. So $\overline{r}$ must be one of $\overline{1}$, $\overline{u}$, $\overline{v}$, or $\overline{uv}$. Therefore we have the following result.

PROPOSITION 4.2 Let v be a projective involution of the second kind in $U_m(L,g)$. If $n > 4$, $\nu(\overline{u},\overline{v}) = 4$.

4.3 Projective Involutions of the Third Kind in $U_m(L,g)$

Let v be a projective involution of the third kind in $U_m(L,g)$. Then $v^2 = \beta$, $\beta = \mu^2$ where μ is in L, but not in $Z(\rho)$. As in Section 1.7, let L_1 be the fixed subsfield of K_1 by $\overline{\mu}$ and let θ be an element of L such that $\theta^\Pi = -\theta$ and θ^2 is in $Z(\rho)$. Then $L = L_1 \oplus L_1\theta$. The center of L_1 is

$Z(\rho,\mu)$. Then F can be considered as a right vector space F* over L_1, and $F^* = V^+ \oplus V^-$. Set $F_1 = V^+$, and take the form g_1 as in Section 1.7. Then form the group $\Gamma U_m(L_1,g_1)$ acting on the space F_1.

If t is a projective involution of $U_m(L,g)$ which commutes with v, then by Section 1.7, t can be restricted to a projective involution t_1 of $U_m(L_1,g_1)$. Conversely, if t_1 is a projective involution of $U_m(L_1,g_1)$, t_1 can be extended to a projective involution t of $U_m(L,g)$. If t anticommutes with v, then t is represented by a projective semilinear involution t_1 of $\Gamma U_m(L_1,g_1)$ which is semilinear relative to the automorphism $\bar{\theta}$ of L_1.

The situation is not quite so simple if t is a projective semilinear involution of $\Gamma U_m(L,g)$ relative to the automorphism σ. If we can show that $\mu^\sigma = \mu$, then by Section 1.7, we can restrict t to a semilinear projective involution t_1 of $\Gamma U_m(L_1,g_1)$ which is semilinear with respect to the automorphism σ. Indeed, as we shall see, this is true if t is in c(c(u,v)) and t commutes with v. Furthermore, if t is such a projective semilinear involution in c(c(u,v)) which anticommutes with v, then $\mu^\sigma = -\mu$; but then we show that t has a restriction t_1 to F_1 which is a projective semilinear involution of $\Gamma U_m(L_1,g_1)$ relative to the automorphism σ of L.

Therefore, let t be a projective semilinear involution of $\Gamma U_m(L_1,g_1)$ relative to the automorphism σ of L which is in c(c(u,v)). Let x be a vector of F_1 which is nonisotropic with respect to the form g_1. Set $W_1^+ = xL_1$ and $W_1^- = (W_1^+)^\perp$, and let w_1 be the 1-involution of $U_m(L_1,g_1)$ with W_1^+ and W_1^- as its eigenspaces. As $m > 2$, a projective semilinear involution r in c(c(u,v)) must commute with the extension w of w_1 to F. In particular t commutes with w. As a transformation

of F, w has positive eigenspace $W^+ = xL_1 \oplus xL_1\theta = xL$. Therefore $r(x) = x\delta_x$ with δ_x in L. Set $\delta_x = \delta$. Since $r^2 = \epsilon$ in Z, we have $\delta\delta^\sigma = \epsilon$. Then it is easily verified that $\bar{\delta}\sigma = \sigma\overline{\delta^\sigma}$; so the automorphism $\pi = \bar{\delta}\sigma$ is an involution. Let L_0 be the fixed subsfield of L by π. Then L_0 is of index 2 in L. Furthermore, $\rho^\pi = \rho^{\bar{\delta}\sigma} = \rho^\sigma = -\rho$. So $L = L_0 \oplus L_0\rho$.

We next show that $L_0 = K$. Let ϕ be in K; then $\phi = \phi_1 + \rho\phi_2$ where ϕ_1 and ϕ_2 are in L_0. But $\phi^\sigma = \phi$; so $\phi_1{}^\sigma - \rho\phi_2{}^\sigma = \phi_1 + \rho\phi_2$. This implies that $\phi_1{}^\sigma = \phi_1$ and $\phi_2{}^\sigma = -\phi_2$. So ϕ_1 is in K and $\phi_2 = \rho\phi_3$ with ϕ_3 in K. Thus $\phi = \phi_1 + \phi_3\gamma = \rho(\phi_1\gamma^{-1}\rho+\phi_2) = \phi_1 + \rho\phi_2$, and the representation of ϕ is unique only if $\phi_2 = 0$ and $L_0 = K$ or if $\phi_1 = 0$ and $L_0 = K\rho$. The latter case is excluded as $K\rho$ is not a field. Therefore $L_0 = K$.

But then the restriction of $\pi = \bar{\delta}\sigma$ to $L_0 = K$ coincides with the restriction of $\bar{\delta}$ to L_0. But also π is the identity on L_0; so δ commutes with every element of K. This means that δ is in $Z(\rho)$. Indeed, if $\delta = \delta_1 + \delta_2\rho$ with δ_1 and δ_2 in K, $\phi^{\bar{\delta}} = \phi$ for ϕ in K implies that $\phi\delta_1 = \delta_1\phi$ and $\phi\delta_2 = \delta_2\phi$. Hence δ_1 and δ_2 are in Z and δ is in $Z(\rho)$. But then $t(v(x)) = t(x)\mu^\sigma = x\delta\mu^\sigma$ and $v(t(x)) = v(x)\delta = x\mu\delta$. So if t commutes with v, $\mu^\sigma = \mu$ and the method of Section 1.7 applies and we can restrict t to F_1.

If t anticommutes with v, then $\mu^\sigma = -\mu$. Therefore, $\bar{\mu}\sigma = \sigma\bar{\mu}$ and σ leave L_1 invariant. Furthermore, t also anticommutes with $h_{\mu^{-1}}$; so t commutes with $v' = h_{\mu^{-1}}v$ and consequently can be restricted to a semilinear projective involution of $\Gamma U_m(L_1,g_1)$ acting on F_2 relative to the automorphism σ of L_1.

So in general, a projective semilinear involution r in c(c(u,v)) can be restricted to or is represented by a projective semilinear involution r_1 in $\Gamma U_m(L_1,g_1)$ which is linear or semilinear with respect to the

involutive automorphism $\bar{\theta}$ or σ of L_1. Since the choice of a nonisotropic vector x in F_1 was arbitrary, r_1 must commute with every 1-involution in $U_m(L_1,g_1)$. Hence r_1 leaves invariant every 1-dimensional nonisotropic subspace of F_1. Since $r_1^2 = \epsilon$ with ϵ in $Z(\rho,\mu)$, Lemma 1.2 yields that $r(x) = x\delta$ with δ in L_1 for all x in F_1. Furthermore, r_1 must be semi-linear with respect to an inner automorphism of L_1. However, neither $\bar{\theta}$ nor σ is an inner automorphism. Consequently, r_1 must be linear; δ is in $Z(\rho,\mu)$; and $\delta^2 = \epsilon$ with ϵ in Z. Then, as it was argued in Section 3.3, r is in one of $\bar{1}$, $\bar{u}$, $\bar{v}$, or $\overline{uv}$. This yields the following result.

PROPOSITION 4.3 Let v be a projective involution of the third kind in $U_m(L,g)$. If $n > 4$, $\nu(\bar{u},\bar{v}) = 4$.

4.4 Projective Involutions of the Fourth Kind in $U_m(L,g)$

Let v be a projective involution of the fourth kind in $U_m(L,g)$. Then $v^2 = \beta$ in C_n and β is not a square in L. We adjoin a square root ρ' of β to L forming the sfield $M = L(\rho') = K(\rho,\rho')$. The center of L is $Z(\rho,\rho')$. As in Section 1.8, the space F can be considered as a right vector space G over M of dimension $q = m/2 = n/4$. The form g can be extended to a nondegenerate hermitian sesquilinear form h defined on G relative to the extension of the anti-automorphism J to M. We then form the group $\Gamma U_q(M,h)$. An element in $U_n(K,f)$ which commutes with u and v can be considered to be in $U_q(M,h)$ and conversely. An element in $U_n(K,f)$ which anticommutes with u and commutes with v, which commutes with u and anticommutes with v, or which anticommutes with both u and v can be considered to be in $\Gamma U_q(M,h)$ relative to the automorphism σ, σ', or $\sigma\sigma'$, respectively, where $\sigma\colon \rho \to -\rho$ and $\sigma'\colon \rho' \to -\rho'$.

Now let w be a 1-involution in $U_q(M,h)$; w can be considered as a 4-involution in $U_n(K,f)$ which commutes with both u and v. If r is in

$c(c(u,v))$ then r commutes or anticommutes with w. If $n > 8$, we can conclude immediately that r commutes with w. Let $n = 8$ and assume r anticommutes with w; then r, as an element of $\Gamma U_q(M,h)$, is such that $r(x) = y$ where x is in W^+, the positive eigenspace of w, and y is in W^-, the negative eigenspace of w; and y is orthogonal to x and is nonisotropic as $h(y,y) = h(x,x) \neq 0$. Also $r^2 = \epsilon$ with ϵ in Z, so $r(y) = r^2(x) = x\epsilon$. By Lemma 1.1, for every element ϕ of M, there exists an element $\psi = \phi$, $\phi + 1$, or $\phi - 1$ such that $z_\psi = x+y\psi$ is nonisotropic; and r commutes or anticommutes with the 1-involution w_ψ in $U_n(M,h)$ with $z_\psi M$ as its positive eigenspace. In the first case $r(z_\psi) = z_\psi\delta_\psi$. This yields $y + x\psi\epsilon = x\delta_\psi + y\psi\delta_\psi$. Consequently, $\delta_\psi = \psi^{-1}$ and, as $\delta_\psi{}^2 = \epsilon$, $\psi^{-2} = \epsilon$. If r anticommutes with w_ψ, then $r(x_\psi) = r(x+y\psi) = x\psi\epsilon + y$ is orthogonal to z_ψ. Then $h(z\psi, r(z\psi)) = \xi\psi\epsilon+\psi^J\xi = 0$ where $\xi = h(x,x) \neq 0$. This yields that $\epsilon = -\psi^{-1}\xi^{-1}\psi^J\xi$.

But suppose now that $\phi = \rho + \rho'$; and choose $\psi = \rho + \rho' + k$ where $k = 0, \pm 1$ in accordance with Lemma 1.1. In any event, the square of ψ is not in Z and hence $\psi^{-2} \neq \epsilon$; so r does not commute with w_ψ in this case. Also $\psi^{-1}\xi^{-1}\psi^J\xi = \psi^{-1}\psi^J \neq -\epsilon$. For, suppose that $\psi\epsilon = -\psi^J$. By Section 1.8, $\rho^J = \rho^{-1}$ and $\rho'^J = \rho^{-1}$. So we have $-(\rho^{-1} + \rho'^{-1} + k) = \rho\epsilon + \rho'\epsilon + k\epsilon$. Multiplying by ρ, we obtain $-1 = \beta\epsilon + \rho\rho'\epsilon + \rho\rho'^{-1}\epsilon + k\rho(\epsilon+1)$, which is a contradiction. So r can neither commute nor anticommute with w_ψ, which is in $c(u,v)$. Since r is in $c(c(u,v))$, this is a contradiction to the assumption that r anticommutes with w in the case $n = 8$.

This means that r commutes with every 1-involution in $U_q(M,h)$ and hence leaves every 1-dimensional nonisotropic subspace of G invariant. Then by Lemma 1.2, $r(x) = x\delta$ with δ in M for all x in G. If r is linear,

then δ is in $Z(\rho,\rho')$. If r is semilinear with respect to an automorphism $\pi = \sigma$, σ', or $\sigma\sigma'$ of M, then π, to be an inner automorphism, must leave $Z(\rho,\rho')$ elementwise invariant, which it does not. Hence δ is in $Z(\rho,\rho')$ and $\delta^2 = \epsilon$ in Z. Again applying the theorem on cyclic fields twice, we obtain that $\delta = \delta_0$, $\delta = \delta_1\rho$, $\delta = \delta_2\rho'$, or $\delta = \delta_3\rho\rho'$ with δ_i in Z. As r is a homothetic mapping in $U_q(M,h)$, it follows that $\delta^J\delta = 1$ and hence that $\delta_i{}^J\delta_i = 1$. Then one of r, ur, vr, or uvr is a homothetic transformation h_η in $U_q(M,h)$ where η is in C. So h_η can be considered in C_n and r is in one of the cosets $\overline{1}$, $\overline{u}$, $\overline{v}$, or $\overline{uv}$. Therefore, we have the following result.

PROPOSITION 4.4 Let v be a projective involution of the fourth kind in $U_m(L,g)$. If $n > 4$, $\nu(u,v) = 4$.

Now we treat the case that v considered as a projective involution of $U_n(K,f)$ anticommutes with u. Then v considered as an element of $\Gamma U_m(L,g)$ is a projective semilinear involution relative to the automorphism σ of L. If $v^2 = \beta$ where β is in C and if there exists a vector x in F such that $v(x) = x\mu$, then $v^2(x) = x\mu\mu^\sigma$. We therefore distinguish the two cases: β is of the form $\mu\mu^\sigma$ with $\rightarrow$ in L (Case I) and β is not of the form $\mu\mu^\sigma$ with μ in L (Case II).

4.5 Projective Semilinear Involutions in $\Gamma U_m(L,g)$. Case I

Let v be a projective semilinear involution in $\Gamma U_m(L,g)$ relative to the automorphism σ of L such that $v^2 = \beta$ where $\beta = \mu\mu^\sigma$ with μ in L. Then as in Section 1.7, form the subsfield L_1 of L consisting of elements of L left invariant by the automorphism $\sigma' = \overline{\mu}^{-1}\sigma$. Since $\sigma'^2 = 1$, L_1 is of degree 2 in L and $L = L_1 \oplus L_1\theta$ where

Θ is an arbitrary element in L such that $\theta^{\sigma'} = -\theta$. In particular we can choose $\rho = \Theta$. Then every element ϕ of Z_1, the center of L_1, commutes with ρ and every element of L_1. So if $\xi + \rho\eta$ is in L with ξ and η in L_1, $\phi(\xi + \rho\eta) = (\xi + \rho\eta)\phi$ and ϕ is in $Z(\rho)$, the center of L. Furthermore, σ' leaves Z elementwise invariant so $Z \subseteq Z_1 \subseteq Z(\rho) \cap L_1$. But ρ is not in L_1, so $Z = Z_1$.

Now F when considered as an L_1 vector space decomposes into the direct sum $F = V^+ \oplus V^-$. Set $F_1 = V^+$ and form the group $\Gamma U_m(L_1,g_1)$ as in Section 1.7. An element t of $U_n(K,f)$ which commutes with both u and v can be restricted to be in $U_m(L_1,g_1)$ and conversely. If t anticommutes with u and commutes with v, then vt will commute with both u and v and can be restricted to $U_m(L_1,g_1)$. Finally if t anticommutes with both u and v, then uvt commutes with both u and v and can be restricted to $U_m(L_1,g_1)$.

Let w_1 be a 1-involution in $U_m(L_1,g_1)$, then w_1 can be extended to a 2-involution w in $U_n(K,f)$ which commutes with both u and v. Hence r in c(c(u,v)) will commute with w. Let r' be that one of r, ur, vr, or uvr which commutes with u and v and let r_1 be the restriction of r' to F_1. Then r' commutes with w; so r_1 commutes with w_1. Thus r_1 is in $U_m(L_1,g_1)$ and r_1 leaves invariant every 1-dimensional nonisotropic subspace of F_1. Consequently, by Lemma 1.2, $r_1(x) = x\delta$ with δ in Z for all x in F_1. Furthermore, $\delta^T\delta = 1$ where T is the anti-automorphism of L_1 corresponding to g_1. Since δ is in Z, $\delta^T = \delta^J$; so δ is in C and the extension r' of r_1 to E is in C_n. Thus r is in one of $\bar{1}$, $\bar{u}$, $\bar{v}$, or $\overline{uv}$.

PROPOSITION 4.6 Let v be a projective semilinear involution in $\Gamma U_m(L,g)$ relative to the automorphism σ of L such that $v^2 = \beta$ is of the form $\beta = \mu\mu^\sigma$ with μ in L. If $n > 4$, then $\nu(\bar{u},\bar{v}) = 4$.

4.6 Projective Semilinear Involution in $\Gamma U_m(L,g)$. Case II

Let v be a projective semilinear involution of $\Gamma U_m(L,g)$ relative to the automorphism σ of L such that $v^2 = \beta$ where β is not of the form $\beta = \mu\mu^\sigma$ with μ in L. Then, as in Section 1.8, adjoin a square root ρ' of β to L forming the sfield $M = L(\rho') = K(\rho,\rho')$. If $\emptyset$ is in L, the relation $\emptyset\,\rho' = \rho'\emptyset^\sigma$ holds; in particular $\rho\rho' = -\rho'\rho$. Let Z_M be the center of M. Then clearly $Z \subseteq Z_M \subseteq Z(\rho)$ as $Z(\rho)$ is the center of L. Furthermore ρ is not in Z_M; and since $[Z(\rho):Z] = 2$, $Z_M = Z$.

Now form the M-space G by defining $x\rho' = v(x)$ and extend the anti-automorphism J to M by defining $\rho'^J = \rho'^{-1}$. Then, as in Section 1.8, extend the form g to a hermitian sesquilinear form h defined on G and form the group $\Gamma U_q(M,h)$ where $q = m/2 = n/4$. An element t of $U_n(K,f)$ which commutes with both u and v can be considered to be in $U_q(M,h)$ and conversely. If an element t of $U_n(K,f)$ anticommutes with u but commutes with v, then vt commutes with both u and v and so can be considered in $U_q(M,h)$. If t commutes with u and anticommutes with v, then ut commutes with both u and v and so can be restricted to an element in $U_q(M,h)$. If t anticommutes with u and commutes with v, then ut can be restricted to an element in $U_q(M,h)$. If t anticommutes with both u and v, then uvt can be restricted to an element of $U_q(M,h)$.

A 1-involution w_o in $U_q(M,h)$ can then be extended to a 4-involution in $U_n(K,f)$ which commutes with u and v. Then r in c(c(u,v)) commutes or anticommutes with w. If $n > 8$, it follows immediately that r must commute with w. If $n = 8$, the argument of Section 4.4 shows that r must commute with w. Let r' be that one of r, ur, vr, or uvr which commutes with both u and v. Then r' commutes with w and its restriction r_o to G is a linear projective involution commuting with w_o. That is, r_o commutes with

every 1-involution of $U_q(M,h)$ and so must leave invariant every 1-dimensional nonisotropic subspace of G. Then by Lemma 1.2, $r_o(x) = x\delta$ with δ in Z for every vector x of G. Since r_o is in $U_q(M,h)$, $\delta^J\delta = 1$ and the extension r' of r_o to E is a homothetic mapping of E. Thus r' is in $\bar{1}$, and r is in one of $\bar{1}$, $\bar{u}$, $\bar{v}$, or $\overline{uv}$.

PROPOSITION 4.7 Let v be a projective semilinear involution in $\Gamma U_m(L,g)$ relative to the automorphism σ of L such that $v^2 = \beta$ where β is not of the form $\beta = \mu\mu^\sigma$ with μ in L. If $n > 4$, then $\nu(\bar{u},\bar{v}) = 4$.

Propositions 4.1 through 4.7 yield the following theorem.

THEOREM 4.1 If the dimension n of E is greater than 4 and if $\bar{u}$ is an involution of the fourth kind in $PU_n(K,f)$, then $\nu_{\bar{u}} = 4$.

This means that all involutions of the fourth kind are in $\mathcal{J}$.

CHAPTER V

CHARACTERIZATION OF EXTREMAL INVOLUTIONS

In this chapter we characterize the extremal involutions of $PU_n(K,f)$ principally by the group-theoretical invariant $\omega_{\bar{u}}$ defined in Section 1.2. Thus we must determine the maximum number of elements in the set $c(c(\bar{u},\bar{v},\bar{w}))$ where $\bar{u}$, $\bar{v}$ and $\bar{w}$ are distinct mutually commuting involutions in $\bar{\mathcal{D}}$.

5.1 Extremal Involutions

We first show that $\omega_{\bar{u}} = 8$ when $\bar{u}$ is an extremal involution and the dimension n of E is greater than 4. Let u be the 1-involution in $\bar{u}$. Then every projective involution of $c(u,v,w)$ and $c(c(u,v,w))$ must commute with u and be of the first or third kinds by Lemma 2.1. Furthermore if v and w are of the first kind, then they must be contained in the coset of an extremal involution by Theorems 2.1 and 2.2. Any element which commutes with u can be restricted to a transformation of U^-. Since U^- is nonisotropic, the form f can be restricted to a non-degenerate hermitian sesquilinear form $\tilde{f}$ defined on U^-. Let $\mathcal{G}$ be the subgroup of $U_n(K,f)$ formed of elements t which commute with u and let $\tilde{\mathcal{G}}$ be the group of transformations $\tilde{t}$ consisting of elements t of $\mathcal{G}$ restricted to U^-. Then $\tilde{\mathcal{G}}$ is contained in a Unitary Group $U_{n-1}(K,\tilde{f})$ acting on U^-. But every element $\tilde{t}$ of $U_{n-1}(K,\tilde{f})$ can be extended to an element t of $\mathcal{G}$ whose restriction is $\tilde{t}$. So $\tilde{\mathcal{G}}$ is the group $U_{n-1}(K,\tilde{f})$. The sets $c(u,v,w)$ and $c(c(u,v,w))$ of projective involutions of $U_n(K,f)$ reduce to the sets $c(\tilde{v},\tilde{w})$ and $c(c(\tilde{v},\tilde{w}))$ of projective involutions of $\tilde{\mathcal{G}}$. As elements of $\tilde{\mathcal{G}}$, $\tilde{v}$ and $\tilde{w}$ are the same kind of projective

involutions that v and w are. Since n-1 > 3, it follows from Theorem 2.1 and Propositions 3.1, 3.3, and 3.5 that $\nu(\bar{v},\bar{w}) = 4$. In other words, $c(c(\bar{v},\bar{w}))$ contains only the involutions $\bar{1}$, $\bar{v}$, $\bar{w}$ and $\overline{vw}$.

Let $\bar{r}$ be an involution in $c(c(\bar{u},\bar{v},\bar{w})$. Then r in $\bar{r}$ must commute with u and hence is in $\mathcal{U}$. Therefore r can be chosen so that its restriction to U^- is one of $\tilde{1}$, $\tilde{v}$, $\tilde{w}$, or $\widetilde{vw}$. Denote by r' that one of 1, v, w, or vw whose restriction to U^- coincides with that of r. Then $r'^2(x) = r^2(x) = x\epsilon$ for all x in U^- and hence for all x in E. Furthermore ϵ is in C and $\epsilon = \delta^2$ where δ is in Z', the center of a subsfield K' of index 1 or 2 in K according as r is of the first or third kind. Let R^+ and R^- be the eigenspaces over K' of r and R'^+ and R'^- be the eigenspaces over K' of r'. Since r and r' coincide on U^-, we have $U^- \cap R^+ = U^- \cap R'^+$ and $U^- \cap R^- = U^- \cap R'^-$. Hence $r(x) = x\delta$ for x in R^+ while $r'(x) = x\delta$ for x in R'^+; $r(x) = -x\delta$ for x in R^- and $r'(x) = -x\delta$ for x in R'^-. Furthermore $U^+ = U^+ \cap R^+ \oplus U^+ \cap R^- = U^+ \cap R'^+ \oplus U^+ \cap R'^-$ as both r and r' commute with u. One of these summands may be zero. Indeed, if r and r' are of the first kind, then $U^+ = U^+ \cap R^+$ or $U^+ \cap R^-$ and also $U^+ = U^+ \cap R'^+$ or $U^+ \cap R'^-$. Therefore for x in U^+, $r(x) = \pm x\delta$. There are thus two possibilities for r(x) when x is in U^+: $r(x) = r'(x)$ or $r(x) = -r'(x)$.

If r and r' are of the third kind, then $U^+ = U^+ \cap R^+ \oplus U^+ \cap R^- = U^+ \cap R'^+ \oplus U^+ \cap R'^-$. Here K' is a sfield K_1 of index 2 in K and $\dim_{K_1} U^+ \cap R^{\pm} = \dim_{K_1} U^+ \cap R'^{\pm} = 1$. Since r is in c(c(u,v,w)) while r' is in c(u,v,w), r must commute or anticommute with r'. But since they both coincide on U^-, r commutes with r'. Therefore we have that $U^+ \cap R^+$ is equal to $U^+ \cap R'^+$ or $U^+ \cap R'^-$ and $U^+ \cap R^-$ is equal to

$U^+ \cap R'^-$ or $U^+ \cap R'^+$, respectively. That is, for x in $U^+ \cap R^+$ or $U^+ \cap R^-$, $r(x) = r'(x)$ or $r(x) = -r'(x)$. Since $U^+ = U^+ \cap R^+ \oplus U^+ \cap R^-$, this also holds for U^+. Consequently the four involutions $\bar{\bar{1}}$, $\bar{\bar{v}}$, $\bar{\bar{w}}$ and $\bar{\bar{v}}\bar{\bar{w}}$ are restrictions of at most eight involutions of $c(c(\bar{u},\bar{v},\bar{w}))$. Since eight is the minimal possible order of $c(c(\bar{u},\bar{v},\bar{w}))$, we have $\omega(\bar{u},\bar{v},\bar{w}) = 8$ in all cases.

THEOREM 5.1 If the dimension n of E is greater than 4 and if $\bar{u}$ is an extremal involution of $PU_n(K,f)$, then $\omega_{\bar{u}} = 8$.

5.2 Involutions of the Second Kind

Now let $\bar{u}$ be an involution of the second kind in $PU_n(K,f)$. Then n is even; assume $n > 6$. If $\bar{v}$ is another involution of the second kind, then $\bar{v}$ is conjugate to $\bar{u}$. This is shown by Dieudonné in [4], p.7 for symplectic groups, but the same proof holds for unitary groups.

Because two elements of $PU_n(K,f)$ commute if and only if their transforms by an element $\bar{t}$ in $PU_n(K,f)$ commute, we can say that $c(\bar{t}^{-1}\mathcal{J}\bar{t}) = \bar{t}^{-1}c(\mathcal{J})\bar{t}$ where $\bar{t}^{-1}\mathcal{J}\bar{t}$ is the set of transforms by $\bar{t}$ of elements of a set $\mathcal{J}$ in $PU_n(K,f)$. From this it follows that $\nu_{\bar{u}} = \nu_{\bar{v}}$ for all involutions $\bar{u}$ and $\bar{v}$ of the second kind. So if one such involution is in $\bar{\mathcal{D}}$, they all are.[8] Thus if $\bar{u}$ is not in $\bar{\mathcal{D}}$, we have already distinguished group-theoretically involutions of the second kind from extremal involutions. So consider that $\bar{u}$ and hence all involutions of the second kind are in $\bar{\mathcal{D}}$.

[8]Actually it can be shown that all involutions of the second kind are in $\bar{\mathcal{D}}$, but, as we see here, it is not necessary to do this. I am indebted to Dieudonné for pointing this out and consequently shortening the proof.

Let u be in $\bar{u}$. Then $u^2 = \gamma$ in C_n with $\gamma = \lambda^2$ where λ is in Z and $\lambda^J\lambda = -1$. Associated with u are its positive and negative eigenspaces, U^+ and U^-, which are totally isotropic. As Dieudonné shows ([4], p.7), we can pick a basis $x_1, x_2, \ldots, x_m$ for U^+ and $x_{m+1}, x_{m+2}, \ldots, x_n$ for U^- such that $f(x_i,x_j) = 0$ except when $|j-i| = m$, in which case $f(x_i,x_j) = 1$. Then the planes $Q_i = x_iK \oplus x_{m+i}K$, $i = 1,2,\ldots, m$, are all nonisotropic and orthogonal (cf. Section 1.4).

As n must be even and $n > 6$, $m = 4$. Let $P_1 = x_1K$, $P_2 = x_2K \oplus x_3K$, $P_3 = x_4K \oplus \ldots + x_mK$, $P_4 = x_{m+1}K$, $P_5 = x_{m+2}K + x_{m+3}K$, and $P_6 = x_{m+4}K \oplus \ldots \oplus x_nK$. Let $V^+ = P_1 \oplus P_2 \oplus P_6$, $V^- = P_3 \oplus P_4 \oplus P_5$, $W^+ = P_1 \oplus P_3 \oplus P_5$, and $W^- = P_2 \oplus P_4 \oplus P_6$. Since P_6 is orthogonal to P_1 and P_2 and P_1, P_2, and P_6 are totally isotropic, V^+ must be totally isotropic. Similarly V^-, W^+ and W^- are totally isotropic. Let v be the projective involution such that $v(x) = x\lambda$ for x in V^+ and $v(x) = -x\lambda$ for x in V^-. Let w be the projective involution such that $w(x) = x\lambda$ for x in W^+ and $w(x) = -x\lambda$ for x in W^-. Then v and w are of the second kind and hence in $\bar{\mathfrak{I}}$. Since u, v and w leave all the subspaces P_i invariant, they leave U^+, U^-, V^+, V^-, W^+, and W^- invariant; hence they all commute. We shall show that $\bar{\omega}(\bar{u},\bar{v},\bar{w}) > 8$.

Let r_1 be the 2-involution in $U_n(K,f)$ with subspaces $R_1^+ = Q_1 = P_1 \oplus P_4$ and $R_1^- = Q_2 \oplus Q_3 \oplus \ldots \oplus Q_m$. Let r_2 be the 4-involution with subspaces $R_2^+ = Q_2 \oplus Q_3 = P_2 \oplus P_5$ and $R_2^- = Q_1 \oplus Q_4 \oplus \ldots \oplus Q_m$. By choice of the vectors x_i, it follows that both R_1^+ and R_1^- and both R_2^+ and R_2^- are pairs of orthogonal nonisotropic subspaces. Since all the projective involutions u, v, w, uv, vw, wu, and uvw have eigenspaces which are each the direct sum of three of the P_i, r_1 and

r_2 are not in any of their respective cosets. So if we can show that R_1 or r_2 is in $c(c(u,v,w))$, then it follows that $\omega_{\bar{u}} > 8$.

First consider that $n \geqq 10$. We show that r_1 is in $c(c(u,v,w)$. Then dim $P_3 \geqq 2$ and dim $P_6 \geqq 2$. Let t be a projective involution in $c(u,v,w)$. If t commutes with u, v, and w, it leaves each P_i invariant and consequently R_1^+ and R_1^- invariant; so t commutes with r_1. If t anticommutes with some but not all of u, v and w, then t must map P_1 onto P_2, P_3, P_5, or P_6. But this is impossible for dim $P_1 = 1$ while $P_i > 1$ for i = 2, 3, 5, or 6. So the only remaining case is that t anticommutes with all of u, v, and w. Then $t(P_1) = P_4$, $t(P_2) = P_5$ and $t(P_3) = P_6$. Hence $t(R_1^+) = R_1^+$ and $t(R_1^-) = R_1^-$ and t commutes with r_1. Therefore r_1 is in $c(c(u,v,w))$ and $\omega_{\bar{u}} > 8$.

Now consider that $n = 8$. Then dim P_3 = dim $P_6 = 1$. Again any projective involution commuting with u, v, and w must leave R_2^+ and R_2^- invariant and consequently commutes with r_2. If a projective involution commutes with some but not all of u, v, and w, then $t(P_2) = P_i$ where i = 1, 3, 4, or 6. But dim $P_i = 1$ for i = 1, 3, 4, or 6 while dim $P_2 = 2$. Hence t cannot be in $c(u,v,w)$. If t anticommutes with u, v, and w, then, as before, $t(R_2^+) = R_2^+$ and $t(R_2^-) = R_2^-$. Consequently, t commutes with r_2 and r_2 is in $c(c(u,v,w))$. This again means that $\omega_{\bar{u}} > 8$. We have thus

THEOREM 5.2 If the dimension n of E is even and greater than 6 and if $\bar{u}$ is an involution of the second kind in $PU_n(K,f)$, then $\omega_{\bar{u}} > 8$.

5.3 Involutions of the Third Kind

Let now $\bar{u}$ be an involution of the third kind in $PU_n(K,f)$ and assume that $n > 4$ and $n \neq 6$. We shall show that $\omega_{\bar{u}} > 8$. Let u be a representative of $\bar{u}$. The $u^2 = \gamma$ with γ in C and $\gamma = \lambda^2$ where λ is not in Z. As in Section 1.7, let K_1 be the fixed subsfield of K by $\bar{\lambda}$. Then $K = K_1 \oplus K_1\theta$ where θ is an element of K which anticommutes with λ. Then E can be considered as a K_1-space E^*, and $E = U^+ \oplus U^-$ where U^+ and U^- are the positive and negative eigenspaces of u. Set $E_1 = U^+$ and form the group $\Gamma U_n(K_1,f_1)$ acting on $E_1\mu$

If $\mathcal{S}$ is a set of involutions in $PU_n(K_1,f_1)$, let $c^*(\mathcal{S})$ denote those involutions of $PU_n(K_1,f_1)$ which commute with every involution of $\mathcal{S}$. If $\bar{v}_1$ and $\bar{w}_1$ are distinct commuting involutions in $PU_n(K_1,f_1)$, denote the order of $c^*(c^*(\bar{v}_1,\bar{w}_1))$ by $\nu^*(\bar{v}_1,\bar{w}_1)$.

In the proof of Theorem 2.2 in Section 2.3, we showed that there exists two involutions of the first kind, which we here denote by $\bar{v}_1$ and $\bar{w}_1$, such that $\nu^*(\bar{v}_1,\bar{w}_1) > 4$. In other words, there is an involution $\bar{r}_1$ of the first kind (which is one of the $\bar{r}_i$ in Section 2.3) in $c^*(c^*(\bar{v}_1,\bar{w}_1))$ which is not one of $\bar{1}$, $\bar{v}_1$, $\bar{w}_1$ or $\bar{v}_1\bar{w}_1$. The restriction of u to E_1 is a homothetic mapping h_λ in the center C_n^* of $U_n(K_1,f_1)$. Let v_1', w_1' and r_1' be the p-involutions ($p \leq [n/2]$) contained in $\bar{v}_1$, $\bar{w}_1$, and $\bar{r}_1$, respectively; and set $v_1 = h_\lambda v_1'$ and $w_1 = h_\lambda w_1'$. We still have v_1 in $\bar{v}_1$ and w_1 in $\bar{w}_1$; so r_1 is in $c^*(c^*(v_1,w_1))$.

We can extend r_1 to an involution r of $U_n(K,f)$ by setting $r(x\theta) = r_1(x)\theta$ for x in $E_1 = U^+$. This defines $r(x)$ on $U^- = U^+\theta$ and consequently on E as $E^* = U^+ \oplus U^-$. Similarly, we can extend v_1 and w_1

to projective involutions v and w in $U_n(K,f)$. But now $v^2 = w^2 = \gamma$; so v and w are projective involutions of the third kind in $U_n(K,f)$. Therefore, by Theorem 3.3, v and w are in $\bar{\Theta}$.

Since h_λ, v_1 and w_1 all commute, their extensions u, v, and w must all commute. So form c(c(u,v,w)). We shall show that r is in c(c(u,v,w)). It will then follow that $\nu(\bar{u},\bar{v},\bar{w}) > 8$ as the restriction r_1 of r to E_1 does not coincide with the restriction of any of the elements of $\bar{I}$, $\bar{u}$, $\bar{v}$, $\bar{w}$, $\overline{uv}$, $\overline{vw}$, $\overline{wu}$, or $\overline{uvw}$, which are in the cosets $\bar{I}$, $\bar{v}_1$, $\bar{w}_1$, and $\bar{v}_1\bar{w}_1$.

If t is a projective involution in c(u,v,w) which commutes with u, then the restriction t_1 of t to E_1 will be a projective involution in $c(v_1,w_1)$. By Section 2.3, $t_1r_1 = r_1t_1$. Let x be in U^+ so that $x\Theta$ is in U^-. Then $t(r(x\Theta)) = t_1(r_1(x))\Theta = r_1(t_1(x)\Theta = r(t(x\Theta))$. Since $E = U^+ \oplus U^-$, t commutes with r_1. If t is a projective involution in c(u,v,w) which anticommutes with u, then set $t' = h_\Theta t$. Thus t' will commute with u and commute or anticommute with v and w. The restriction t_1 of t' to E_1 will then commute or anticommute with v_1 and w_1. But the subspaces P_i were chosen in Section 2.3 so that no transformation could anticommute with v_1 or w_1. So t_1 commutes with v_1 and w_1 and hence leaves the eigenspaces of r_i invariant. Thus $t_1r_1 = r_1t_1$. Again let x be in U^+ so that $x\Theta$ is in U^-. Then $t'(r(x\Theta)) = -t_1(r_1(x)\Theta = -r_1(t_1(x)\Theta = r(t'(x\Theta))$. This means that t' commutes with r and so t commutes with r. Thus r is in c(c(u,v,w)) and $\omega(\bar{u},\bar{v},\bar{w}) > 8$.

THEOREM 5.3 If the dimension of E is greater than 4 and not 6 and if $\bar{u}$ is an involution of the third kind in $PU_n(K,f)$, $\omega_{\bar{u}} > 8$.

5.4 Involutions of the Fourth Kind

Let $\bar{u}$ be an involution of the fourth kind in $PU_n(K,f)$. As mentioned in Section 1.8, n must be even. Furthermore assume $n > 6$. If we also assume that $n \neq 8$ and $n \neq 12$, we can conclude that $\omega_{\bar{u}} > 8$ in this case by methods similar to those employed in the previous sections of this chapter. However, we can obtain the more general result having $n > 6$ by the following argument.

Let $\bar{\mathcal{Q}}$ be the set of involutions $\bar{u}$ of $\bar{\mathcal{G}}$ such that $\omega_{\bar{u}} = 8$. By Theorems 5.1, 5.2, and 5.3, we have that $\bar{\mathcal{Q}}$ contains only extremal and possibly involutions of the fourth kind. Let $\mathcal{M}(\bar{u})$ denote a maximal set of involutions of $\bar{\mathcal{G}}$ commuting with $\bar{u}$. By Lemma 2.1, if $\bar{u}$ is extremal, $\mathcal{M}(\bar{u})$ contains only extremal involutions; and if $\bar{u}$ is of the fourth kind, $\mathcal{M}(\bar{u})$ contains only involutions of the fourth kind. Let $\kappa(\bar{u})$ denote the order of $\mathcal{M}(\bar{u})$ and $\kappa_{\bar{u}}$ the maximum of $\kappa(\bar{u})$. We shall use the group-theoretical invariant $\kappa_{\bar{u}}$ in addition to $\omega_{\bar{u}}$ to distinguish involutions of the fourth kind from extremal involutions.

Let $\bar{u}$ be an extremal involution and u a 1-involution in $\bar{u}$. The representatives of involutions commuting with $\bar{u}$ commute with u by Lemma 2.1. Consequently, in order to determine $\kappa_{\bar{u}}$, it suffices to determine the maximal number of distinct 1-involutions commuting with u. This has been shown to be n by Dieudonné ([5], p.5). Therefore, $\kappa_{\bar{u}} = n$ for extremal involutions. Furthermore, we note that the product of any distinct two or three mutually commuting 1-involutions is a 2- or 3-involution, respectively, and is thus not in $\bar{\mathcal{G}}$ by Theorem 2.2.

Now in order to distinguish involutions of the fourth kind from extremal involutions, we need only consider the case that $\bar{u}$ is of the fourth kind in $\mathcal{G}$ with $\kappa_{\bar{u}} = n$. Let $\bar{u}_1 = \bar{u}, \bar{u}_2, \ldots, \bar{u}_n$ be the elements of a set $\mathcal{M}(\bar{u})$ and pick representatives u_i of $\bar{u}_i$. We then

have $u_iu_j = \pm u_ju_i$. If $u_i^2 = \gamma_i$ in C_n where γ_i is not a square in K, $(u_iu_j)^2 = \gamma_{ij} = \pm\gamma_i\gamma_j$. First consider that there exists a pair i and j $(i \neq j)$ such that γ_{ij} is not a square in Z. Then u_iu_j is a projective involution of the third of fourth kind and hence in $\bar{\mathfrak{I}}$ by Theorems 3.1 and 4.1. This distinguishes $\bar{u}$ from extremal involutions in this case.

Otherwise $\gamma_{ij} = \lambda_{ij}^2$ with λ_{ij} in Z for all i and j where $i \neq j$. In particular, if there exists a projective involution u_k within $\bar{\mathfrak{I}}$, $k = 3,4,\ldots, n$, such that u_k commutes with u_1u_2, then $u_1u_2u_k$ is of the third or fourth kind, because $(u_1u_1u_k)^2 = \lambda_{12}^2\gamma_k$ cannot be a square in Z without γ_k being a square in Z. So in this case, $u_1u_2u_k$ is in $\bar{\mathfrak{I}}$. Finally suppose that all $u_k (k = 3,4,\ldots,n)$ anticommute with u_1u_2. Then, as $n > 6$, there exists at least two, u_k say u_3 and u_4, which commute with one, say u_1, and anticommute with the other, say u_2. But then u_1 commutes with u_3u_4 and so by the above argument $u_1u_3u_4$ is in $\bar{\mathfrak{I}}$.

This yields the following theorem.

THEOREM 5.4 If the dimension n of E is even and greater than 6, and if $\bar{u}$ is an involution of the fourth kind in $PU_n(K,f)$, $\bar{u}$ can be distinguished group-theoretically from extremal involutions by one of the three conditions:

(i) $\omega_{\bar{u}} > 8$

(ii) $\omega_{\bar{u}} = 8$ and $\kappa_{\bar{\mu}} \neq n$

(iii) $\omega_{\bar{\mu}} = 8$, $\kappa_{\bar{\mu}} = n$ and there exists two or three involutions in a maximal commuting set $\mathfrak{m}(\bar{u})$ of $\bar{\mathfrak{g}}$ such that their product is in $\bar{\mathfrak{I}}$.

5.5 Principal Theorem

Theorems 5.1 through 5.4 imply that extremal involutions can be characterized group-theoretically in $PU_n(K,f)$. We summarize this result in the following theorem.

PRINCIPAL THEOREM 5.5 Let E be a right vector space of dimension $n > 4$ and $n \neq 6$ over a sfield K which possesses an involutive anti-automorphism J, which is not of characteristic 2, and which is not the finite field GF(3). Let f be a nondegenerate hermitian sesquilinear form defined on E with respect to J and let $PU_n(K,f)$ be the projective unitary group with respect to this form. Then an extremal involution $\bar{u}$ of $PU_n(K,f)$ is characterized by the conditions

(i) $\nu_{\bar{u}} = 4$, $\omega_{\bar{u}} = 8$, and $\kappa_{\bar{u}} = n$.

(ii) Any product of two or three involutions of $\bar{\mathcal{G}}$ in a maximal commuting set $\bar{\mathcal{M}}(\bar{u})$ of involutions of $\bar{\mathcal{G}}$ is not in $\bar{\mathcal{D}}$.

APPENDIX

DETERMINATION OF THE AUTOMORPHISMS

This appendix is included in order to state the final results that Theorem 5.5 implies. The material here is a summary of that contained in Dieudonné [5], p. 76ff., and Rickart [10]. Details of their proofs will be omitted. Dieudonné stated his result for fields but his argument extends to sfields without alteration. He required that n be finite and $n \geq 4$ while Rickart required merely that $n \geq 6$.

According to Theorem 5.5, every automorphism ϕ of $PU_n(K,f)$ maps 1-involutions onto 1-involutions in a one-one manner. Thus associated with ϕ is a mapping Φ of P(E) which sends those elements which are 1-dimensional nonisotropic subspaces of E onto elements which are 1-dimensional nonisotropic subspaces. The problem is to show that Φ is a collineation of P(E). In fact, we have the following theorem.

Theorem A.1 (Rickart's Theorem 2.6). *There exists a one-one mapping Φ of the projective space P(E) onto itself with the following properties:*

(i) *if $\bar{u}$ is an extremal involution in $PU_n(K,f)$ with associated 1-dimensional subspace xK, then $\Phi(xK)$ is the 1-dimensional subspace associated with $\phi(\bar{u})$.*

(ii) *$\Phi(xK)$ is orthogonal to $\Phi(yK)$ if and only if xK is orthogonal to yK.*

(iii) *$\Phi(zK) \subseteq \Phi(xK) \oplus \Phi(yK)$ if and only if $zK \subseteq xK \oplus yK$.*

Proof: If xK is a nonisotropic subspace and $\bar{u}$ is the 1-involution with xK as its positive eigenspace, define $\Phi(xK)$ to be the positive

eigenspace of $\phi(\bar{u})$. If xK is isotropic, then it can be characterized as the intersection of two nonisotropic planes, P_1 and P_2. These planes are the subspaces of two 2-involutions $\bar{v}_1$ and $\bar{v}_2$. Two such 2-involutions are called intersecting 2-involutions if the subspace generated by their 2-dimensional eigenspaces is 3-dimensional. Not only 2-involutions but 3-involutions can be characterized group-theoretically (that is, in terms of 1-involutions). Those intersecting 2-involutions for which the common part of their 2-dimensional subspaces is isotropic can be characterized group-theoretically by having no 1-involution commuting with both of them. So we can define $\Phi(xK)$ to be the intersection of the 2-dimensional subspaces of $\phi(\bar{v}_1)$ and $\phi(\bar{v}_2)$.

The next step is to show that Φ is single valued. Clearly $\Phi(xK)$ is uniquely defined if xK is nonisotropic. If xK is isotropic, then it must be shown that $\Phi(xK)$ is defined independently of the pairs of intersecting 2-involutions whose 2-dimensional subspaces intersect in xK. This is the crucial part of the proof and the principal place where the proofs of Dieudonné and Rickart differ. The details of these proofs will be omitted here.

Property (i) is immediate. Property (ii) is trivial for nonisotropic subspaces because orthogonal subspaces belong to commuting involutions. For isotropic subspaces, this is proved by taking appropriate linear combinations of vectors involved so that the problem is reduced to the former case. Property (iii) is proved by contradiction. Suppose that $\Phi(zK) \not\subseteq \Phi(xK) \oplus \Phi(yK)$ while $zK \subseteq xK \oplus yK$. Then there exists a vector Z' such that $\Phi(z'K)$ is orthogonal to $\Phi(xK)$ and $\Phi(yK)$ but not to $\Phi(zK)$. This implies that z'K is orthogonal to xK and yK but not to zK by (ii), which is a contradiction.

Now apply the fundamental theorem of projective geometry to obtain the existence of a semilinear transformation r relative to an automorphism ρ of K such that $\Phi(xK) = r(x)K$. We show that r is in $\Gamma U_n(K,f)$. Let x and y be arbitrary nonzero vectors in E and pick z such that $f(x,z)=1$. Set $f(x,y) = \xi$ and form $y' = y - z\xi$. Then $f(x,y') = 0$ and so $f(r(x),r(y'))=0$. Hence $f(r(x),r(y)) = f(r(x),r(z))\cdot f(x,y)^{\rho}$. Set $e_x = f(r(x),r(z))$. Then e_x is shown to be independent of x by using the additivity of f(x,y). Thus $e_x = e$ is the multiplier of r in $\Gamma U_n(K,f)$.

Let $\bar{r}$ be the image of r under the natural homomorphism of $\Gamma U_n(K,f)$ onto $P\Gamma U_n(K,f)$. Then we have

Theorem A.2 When $PU_n(K,f)$ is subject to the conditions of Theorem 5.5, every automorphism ϕ of $PU_n(K,f)$ is of the form $\phi: \bar{t} \to \bar{r}\bar{t}\bar{r}^{-1}$ where $\bar{r}$ is an element of $P\Gamma U_n(K,f)$.

Indeed, if $\bar{u}$ is an extremal involution, we have $\phi(\bar{u}) = \bar{r}\bar{u}\bar{r}^{-1}$. If $\bar{t}$ is in $PU_n(K,f)$, then $\bar{t}\bar{u}\bar{t}^{-1}$ is also an extremal involution. Therefore $\phi(\bar{t}\bar{u}\bar{t}^{-1}) = \bar{r}\bar{t}\bar{u}\bar{t}^{-1}\bar{r}^{-1} = \phi(\bar{t})\bar{r}\bar{u}\bar{r}^{-1}\phi(\bar{t})^{-1}$. So $\bar{h}\bar{u}\bar{h}^{-1} = \bar{u}$ where $\bar{h} = \bar{t}^{-1}\bar{r}^{-1}\phi(\bar{t})\bar{r}$. On the other hand, $\bar{r}\phi(\bar{t})\bar{r}^{-1}$ is easily verified to be in $PU_n(K,f)$; so $\bar{h}$ is in $PU_n(K,f)$. Let h be in $\bar{h}$ and u be the 1-involution in $\bar{u}$. Then h is an element of $U_n(K,f)$ which commutes with every 1-involution. Thus by Lemma 1.2, h is a homothetic mapping. Hence $\bar{h} = \bar{1}$. This implies that $\phi(\bar{t}) = \bar{r}\bar{t}\bar{r}^{-1}$. Thus we have characterized the automorphisms of $PU_n(K,f)$.

As Rickart points out, the automorphism ϕ can be considered just as well to be an isomorphism between two projective unitary groups $PU_n(K,f)$ and $PU_m(L,g)$ acting on two projective spaces P(E) and P(F). Then r is a semilinear transformation of E onto F relative to an isomorphism σ of K onto L such that $g(r(x),r(y)) = e\cdot f(x,y)^{\sigma}$ for x and y in E. Finally $\bar{r}$ is

the induced collineation between P(E) and P(F). Furthermore, he notes that $PU_n(K,f)$ and $PU_m(L,g)$ can be replaced by subgroups which contain all the involutions of these groups since to characterize ϕ in the manner of Theorem A.2 only requires knowledge of its effect on the extremal involutions of $PU_n(K,f)$, which in turn need to be distinguished only from other involutions of $PU_n(K,f)$. This interpretation has interesting applications in the infinite dimensional case.

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