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THE RIEMANN PROBLEM IN GAS DYNAMICS.

The University of Michigan, Ph.D., 1976
Mathematics

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THE RIEMANN PROBLEM IN GAS DYNAMICS

by
Randolph Gordon Smith

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
1976

Doctoral Committee:

Professor Joel A. Smoller, Chairman
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CHAPTER 1

INTRODUCTION

In this paper we investigate existence and uniqueness of solutions of the Riemann problem (R.P.), with arbitrary initial data, for the equations of compressible fluid flow in one space variable. The results depend on the energy function $e = e(\tau, S)$ (τ = specific volume, S = specific entropy). In addition to the assumptions normally made (see (1.2) - (1.7)), we always assume some physically reasonable limiting behaviors of the energy and pressure functions near the boundaries of the domains on which they are defined (see (1.8) -(1.10)); ideal gases, for example, always satisfy these assumptions. We prove (Chapter 3) that the R.P. always has a solution. However, solutions need not always be unique, even if they satisfy the entropy condition and the Lax shock conditions. Our analysis is based on the behavior of the Hugoniot curve H (see (1.20) ff.) in the (τ, p)-plane (p = pressure). It is sometimes assumed (see [1]) that H is monotone in the sense that along H , one of the variables τ, p is a decreasing function of the other. This holds for polytropic gases, but need not hold for ideal gases. (See Ideal gases, Chapter 6.) It turns out that whether uniqueness fails, and if so how drastically, depends on how much H deviates from this monotonicity. We introduce three conditions on e : WEAK, MEDIUM and STRONG (see Def. 1.4, Prop. 1.5), which we relate to the behavior of H (see Lemmas 4.1, 4.2, 4.10, 4.11, 4.12 and Conventions 4.6, 4.7). WEAK and STRONG are already known ([2], [4]). The following implications hold:

(*) polytropic gas $\Rightarrow$ ideal gas $\Rightarrow$ STRONG $\Rightarrow$ MEDIUM $\Rightarrow$ WEAK ,

where all implications are proper. In Chapter 4 we prove that whenever MEDIUM holds, solutions of the R.P. are unique. However, whenever MEDIUM is violated, it is possible to construct initial data (U_ℓ, U_r) for which the R.P. has three distinct solutions; we call this particular situation type I nonuniqueness. We show that whenever WEAK fails, one can not only obtain type I nonuniqueness, but also get nonuniqueness a different way, by a different con-

struction; this is called type II nonuniqueness. The initial data that exhibit type II nonuniqueness also exhibit type I, and we then obtain five distinct solutions of the R. P. In Chapter 5 we show that for both types of nonuniqueness the data can always be chosen so that the nonuniqueness is stable under perturbation of the initial data. In Chapter 6, we show by constructing various examples of energy functions, that all the possible situations abstractly considered in Chapters 1-5 do indeed occur, e.g., that the implications (*) are proper, and that it is possible to have five solutions of one R. P. For all solutions considered, the shocks always satisfy the entropy condition (see (1.18)) and the Lax shock conditions (see (1.19)).

The equations of compressible flow in one space variable (in Eulerian coordinates) are

$$\rho_t + (\rho u)_x = 0 \tag{1.1}$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

$$\left[\rho\left(\frac{1}{2}u^2 + e\right)\right]_t + \left[\rho u\left(\frac{1}{2}u^2 + e\right) + up\right]_x = 0$$

where $t \geq 0$ is time, x is position, ρ is density, p is pressure, e is specific internal energy, and u is flow velocity. We also let S = specific entropy, T = temperature, and τ = specific volume ($= \rho^{-1}$). Here e is assumed to be a C^3 function $e(\tau, S)$ of τ and S, $p = -e_\tau$, and $T = e_S$. The letter g will always denote the function $-e_\tau$, i.e., $p = g(\tau, S)$. The function e is assumed to satisfy the following

Standard Assumptions:

$$e > 0 \tag{1.2}$$

$$p > 0 \qquad (\text{i.e., } g > 0) \tag{1.3}$$

$$T > 0 \qquad (\text{i.e., } e_S > 0) \tag{1.4}$$

$$g_\tau < 0 \tag{1.5}$$

$$g_{\tau\tau} > 0 \tag{1.6}$$

$$g_S > 0\ . \tag{1.7}$$

In addition, we shall always assume that e satisfies the following

Limit Assumptions:

$$\lim_{S \to -\infty} g(\tau, S) = 0 \ , \qquad \lim_{S \to +\infty} g(\tau, S) = +\infty \tag{1.8}$$

$$\lim_{S \to -\infty} e(\tau, S) = 0 \ , \qquad \lim_{S \to +\infty} e(\tau, S) = +\infty \tag{1.9}$$

$$\lim_{\tau \to 0^+} e(\tau, p) = 0 \ . \tag{1.10}$$

Here the expression "$e(\tau, p)$" requires definition. Suppose $\tau, p > 0$. Then by (1.8), there is an S such that $g(\tau, S) = p$, and by (1.7) S is unique; we denote $S \equiv S(\tau, p)$. Then we define

$$e(\tau, p) \equiv e(\tau, S(\tau, p)) \tag{1.11}$$

(whenever p is understood to be a pressure rather than an entropy; cf. [1], §64). We also define $g_\tau(\tau, p) \equiv g_\tau(\tau, S(\tau, p))$, etc.

Analogously, by (1.9) and (1.4) we can use ρ, e (i.e., τ, e) as independent variables so that S and p are functions of (ρ, e). If we let $U = (\rho, u, e)$, then the equations (1.1) can be written as

$$U_t + A(U)U_x = 0 \ , \tag{1.12}$$

for a 3 x 3 matrix A(U) with eigenvalues $u - c$, u and $u + c$, where c (the sound speed) is defined by

$$c^2 = f_\rho \tag{1.13}$$

where $f(\rho, S) \equiv g(\rho^{-1}, S)$.

The Riemann Problem (R. P.) (see [1], [4], [5]) for (1.1) or (1.12) is the initial-value problem for (1.1) with data

$$U(x, 0) = \begin{cases} U_\ell \ , & \text{if } x < 0 \\ U_r \ , & \text{if } x > 0 \ , \end{cases} \tag{1.14}$$

where $U_\ell = (\rho_\ell, u_\ell, e_\ell)$, $U_r = (\rho_r, u_r, e_r)$ are arbitrary constant vectors.

By a solution of (1.1), (1.12) we mean a function U(x/t) consisting of at most four constant states, U_ℓ, U_1, U_3, U_r. Here U_ℓ (on the left) is connected to U_1 (on the right) by either a 1-shock or a 1-rarefaction wave, U_1 (left) is connected to U_3 (right) by a contact discontinuity, and U_3 (left) is connected to U_r (right) by a 3-shock or a 3-rarefaction wave. (See [1] or [3] for the definition of these terms.)

From now on, we shall use the notation "$U_1 \overset{\text{1-shock}}{\longrightarrow} U_2$" to express the fact that U_1 (on the left) is, or can be, connected to U_2 (on the right) by a 1-shock. We define similarly the expressions "$U_1 \overset{\text{1-wave}}{\longrightarrow} U_2$", "$U_1 \overset{\text{contact}}{\longrightarrow} U_2$", "$U_1 \overset{\text{3-shock}}{\longrightarrow} U_2$" and "$U_1 \overset{\text{3-wave}}{\longrightarrow} U_2$", where "1-wave" (resp. "3-wave") is just an abbreviation for "1-rarefaction wave" (resp. "3-rarefaction wave"). We also abbreviate the sentence "$U_1 \overset{\text{1-shock}}{\longrightarrow} U_2$ or $U_1 \overset{\text{1-wave}}{\longrightarrow} U_2$" by "$U_1 \overset{1}{\longrightarrow} U_2$", etc. Thus for a solution of the R.P. with initial data (U_ℓ, U_r) we must have U_1 and U_3 such that

$$U_\ell \overset{1}{\longrightarrow} U_1 \overset{\text{contact}}{\longrightarrow} U_3 \overset{3}{\longrightarrow} U_r \ .$$

Finally, since (ρ, e) and (τ, p) determine each other, we can, and shall, describe states $U_1, U_2, \ldots$ by triples (τ_1, p_1, u_1), $(\tau_2, p_2, u_2), \ldots$.

By definition, $U_1 \overset{\text{contact}}{\longrightarrow} U_2$ iff $u_1 = u_2$ and $p_1 = p_2$. For shocks, we have the following well-known result (see [1]):

Proposition 1.1. U_0 (left) can be connected to U (right) by a shock with shock speed s iff the following hold.

Jump Conditions:

$$\rho v = \rho_0 v_0 \equiv m \neq 0 \tag{1.15}$$

$$(\rho u)v + p = (\rho_0 u_0)v_0 + p_0 \tag{1.16}$$

$$\rho(\tfrac{1}{2}u^2 + e)v + pu = \rho_0(\tfrac{1}{2}u_0^2 + e_0)v_0 + p_0 u_0 \ , \tag{1.17}$$

where $v \equiv u - s$, $v_0 \equiv u_0 - s$;

Entropy Condition:

(1.18) $$mS_0 \leq mS \ .$$

Moreover, the shock is a 1-shock (resp. 3-shock) iff $s < u, u_0$ (resp. $s > u, u_0$).

In this paper, shocks will also be required to satisfy the

Lax Shock Conditions:

(1.19) $$U_0 \xrightarrow{1\text{-shock}} U \Rightarrow v_0^2 > c_0^2 \text{ and } v^2 < c^2 \ ;$$

$$U \xrightarrow{3\text{-shock}} U_0 \Rightarrow v_0^2 > c_0^2 \text{ and } v^2 < c^2 \ .$$

(See (1.13).)

The Hugoniot function with center (τ_0, p_0) is defined by

(1.20) $$H(\tau, p) \equiv e(\tau, p) - e(\tau_0, p_0) + (\tau - \tau_0)\frac{p + p_0}{2} \qquad (\tau, p > 0) \ .$$

We define the Hugoniot curve H with center (τ_0, p_0) by

$$H \equiv \left\{(\tau, p) \,\middle|\, H(\tau, p) = 0\right\} \ .$$

Let U, U_0 be constant states, and let H have center (τ_0, p_0). For $a > 0$, let $\mathcal{H}_a$ be the upper branch of the hyperbola,

$$(\tau_0 - \tau)(p - p_0) = a \ ;$$

Let $\mathcal{H}'_a$ be the lower branch. (See Fig. 1.) The importance of the Hugoniot curve lies in the following two facts.

Proposition 1.2. $U_0 \xrightarrow{1\text{-shock}} U$ iff $u < u_0$ and $(\tau, p) \in \mathcal{H}_{(u_0 - u)^2} \cap H$.

Proposition 1.3. $U \xrightarrow{3\text{-shock}} U_0$ iff $u > u_0$ and $(\tau, p) \in \mathcal{H}_{(u - u_0)^2} \cap H$.

These will be proved in Chapter 2.

Definition 1.4. $e = e(\tau, S)$ is an energy function iff it satisfies the Standard Assumptions (1.2) - (1.7) and the Limit Assumptions (1.8) - (1.10).

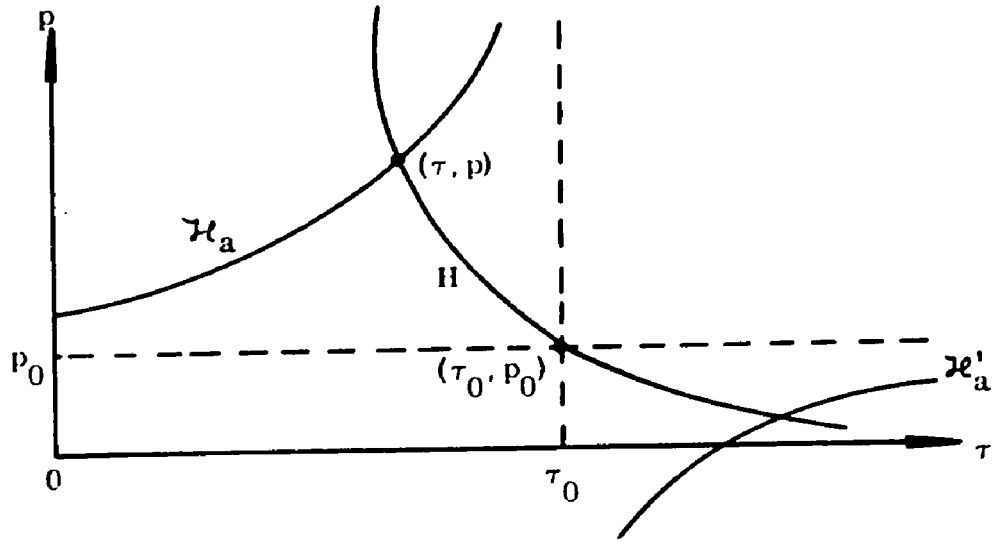


Fig. 1: The hyperbola $\mathcal{H}_a \cup \mathcal{H}'_a$ and a Hugoniot curve H.

We introduce the following conditions on an energy function e (see (1.11)):

STRONG $\quad \frac{\partial}{\partial \tau} e(\tau, p) \geq 0 \qquad (\tau, p > 0)$

MEDIUM $\quad \left(\frac{\partial p}{\partial \tau}\right)_e \leq \frac{p^2}{2e} \qquad (\tau, e > 0)$

WEAK $\quad \frac{\partial}{\partial \tau} e(\tau, p) \geq -\frac{p}{2} \qquad (\tau, p > 0)$.

Proposition 1.5. The conditions STRONG, MEDIUM and WEAK have the following equivalent formulations:

STRONG $\quad \frac{\partial}{\partial p} e(\tau, p) \geq \frac{p}{-g_\tau(\tau, p)} \qquad (\tau, p > 0)$

MEDIUM $\quad \frac{\partial}{\partial p} e(\tau, p) \geq \frac{2e(\tau, p)}{-g_\tau(\tau, p) \cdot \frac{2e(\tau, p)}{p} + p} \qquad (\tau, p > 0)$

WEAK $$\frac{\partial}{\partial p}\, e(\tau, p) \geq \frac{p}{-2g_\tau(\tau, p)} \qquad (\tau, p > 0) \ .$$

Proof: We have

(1.21) $$\frac{\partial}{\partial p}\, e(\tau, p) = \frac{\partial}{\partial p}\, e(\tau, S(\tau, p)) = e_S \frac{\partial}{\partial p}\, S(\tau, p) = \frac{e_S}{g_S} \ .$$

Here we have abbreviated $e_S(\tau, S(\tau, p))$ $(= e_S(\tau, p))$ by e_S; we shall do this often, e.g., $g_\tau = g_\tau(\tau, p)$. In particular

(1.22) $$\frac{\partial}{\partial p}\, e(\tau, p) > 0 \ .$$

Moreover

$$\frac{\partial}{\partial \tau}\, e(\tau, p) = \frac{\partial}{\partial \tau}\, e(\tau, S(\tau, p)) = e_\tau + e_S \frac{\partial}{\partial \tau}\, S(\tau, p) \ .$$

But $p = g(\tau, S(\tau, p))$, so taking $\partial/\partial\tau$ of both sides yields $0 = g_\tau + g_S \frac{\partial}{\partial \tau}\, S(\tau, p)$; hence

$$\frac{\partial}{\partial \tau}\, e(\tau, p) = e_\tau + e_S\left(-\frac{g_\tau}{g_S}\right) \ ,$$

so

(1.23) $$\frac{\partial}{\partial \tau}\, e(\tau, p) = -p + (-g_\tau) \frac{\partial}{\partial p}\, e(\tau, p) \ ,$$

by (1.21).

From (1.23) and $-g_\tau > 0$, the formulations for STRONG and WEAK are obvious. For MEDIUM, writing $\frac{\partial e}{\partial \tau} = \frac{\partial}{\partial \tau}\, e(\tau, p)$, etc., we have

$$-g_\tau = \frac{\frac{\partial e}{\partial \tau} + p}{\frac{\partial e}{\partial p}} \ ,$$

so that

$$\frac{\partial e}{\partial p} \geq \frac{2e}{-g_\tau \cdot \frac{2e}{p} + p} \iff \frac{\partial e}{\partial p} \geq \frac{2ep}{\frac{\frac{\partial e}{\partial \tau} + p}{\frac{\partial e}{\partial p}} \cdot 2e + p^2}$$

$$\iff \left(\frac{\partial e}{\partial \tau} + p\right)2e + \frac{\partial e}{\partial p} p^2 \geq 2ep \iff \frac{p^2}{2e} \geq \frac{-\frac{\partial e}{\partial \tau}}{\frac{\partial e}{\partial p}} = \left(\frac{\partial p}{\partial \tau}\right)_e$$

$\iff$ MEDIUM. ■

Lemma 1.6. Let ϕ be defined on $(0, +\infty)$ with $\phi > 0$, $\phi' < 0$ and $\phi'' > 0$. Then for every $\tau > 0$,

$$\phi(\tau)^2 < -2\phi'(\tau) \int_\tau^{+\infty} \phi \ .$$

Proof: Let $C = (\tau, \phi(\tau))$. Let L be the line tangent to ϕ at C; then L has slope $\phi'(\tau) < 0$ and intersects the τ-axis at a point $B = (\tau + x, 0)$ where $x > 0$. (See Fig. 2.) Let $A = (\tau, 0)$. Since the line-segment $\overline{CB}$ has slope $\phi'(\tau)$,

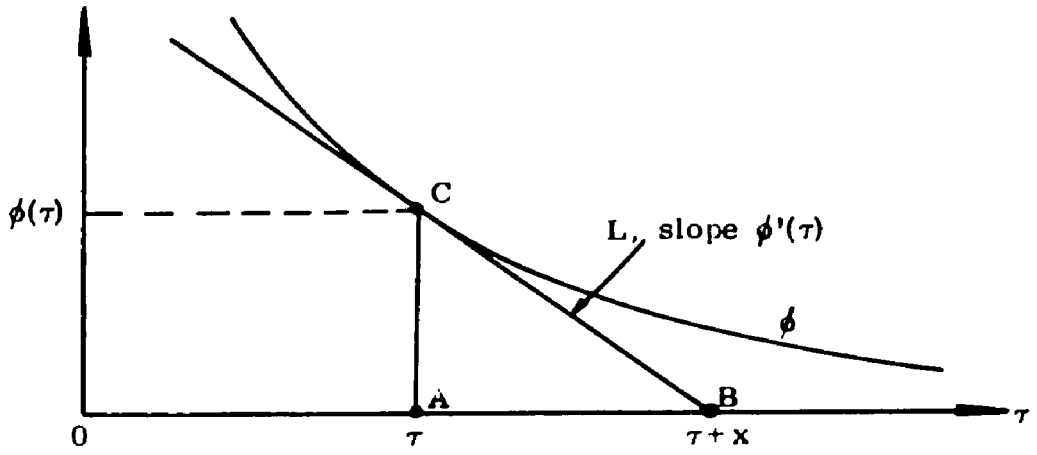


Fig. 2: The area of ΔABC is less than $\int_\tau^{+\infty} \phi$.

$x = -\frac{\phi(\tau)}{\phi'(\tau)}$. Since $\phi'' > 0$, ϕ lies above L; hence

$$\frac{1}{2}\left(-\frac{\phi(\tau)}{\phi'(\tau)}\right)\phi(\tau) = \frac{1}{2}x\phi(\tau) = \text{area } \Delta ABC < \int_{\tau}^{+\infty} \phi \ .$$

Then since $-\phi'(\tau) > 0$, the desired conclusion follows. ■

Proposition 1.7. STRONG $\Rightarrow$ MEDIUM $\Rightarrow$ WEAK.

Proof: By Prop. 1.5, it suffices to prove

$$\frac{p}{-g_\tau} \geq \frac{2ep}{-g_\tau \cdot 2e + p^2} \geq \frac{p}{-2g_\tau}$$

for arbitrary τ, $p > 0$. Now

$$\frac{2ep}{-g_\tau \cdot 2e + p^2} < \frac{2ep}{-g_\tau \cdot 2e} = \frac{p}{-g_\tau} \ ,$$

proving the first inequality. For the other, note that $p = g(\tau, S)$, where $S = S(\tau, p)$. By (1.3), (1.5) and (1.6), the function $g(\cdot, S)$ satisfies the hypothesis of Lemma 1.6; therefore

$$g(\tau, S)^2 < -2g_\tau(\tau, S)\int_{\tau}^{+\infty} g(v, S)dv \ . \tag{1.24}$$

But for every $\tau_1 > \tau$,

$$\int_{\tau}^{\tau_1} g(v, S)dv = \int_{\tau}^{\tau_1} -e_\tau(v, S)dv = e(\tau, S) - e(\tau_1, S) < e(\tau, S) \ ;$$

thus

$$\int_{\tau}^{+\infty} g(v, S)dv < e(\tau, S) \ .$$

Then (1.24) yields $g(\tau, S)^2 < -2g_\tau(\tau, S)\, e(\tau, S)$, i.e., $p^2 < -2g_\tau e$. Then

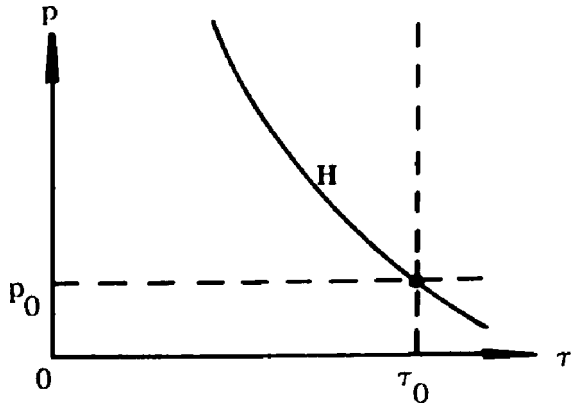
$$\frac{2ep}{-g_\tau \cdot 2e + p^2} > \frac{2ep}{-g_\tau \cdot 2e - 2g_\tau e} = \frac{p}{-2g_\tau} \quad . \qquad \blacksquare$$

We conclude with a rough summary, in table form, of what will be proved in the remainder of this paper regarding the connection between: (1), the conditions MEDIUM and WEAK; (2), the behavior of the Hugoniot curve H; and (3), existence and uniqueness of solutions of the R.P. The properties marked with a (*) are proved to hold for <u>some</u> choice of (τ_0, p_0) as center of H, or of (U_ℓ, U_r) as initial data; the others hold in general.

<u>polytropic</u>

H is monotone

existence and uniqueness



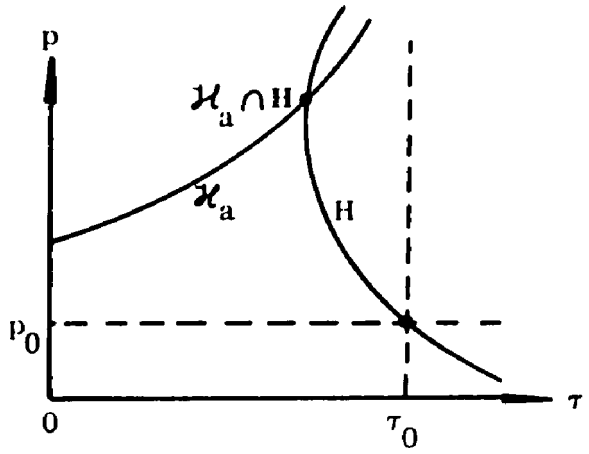
<u>MEDIUM or ideal</u>

τ is a function of p along H

$\forall a > 0$, $\mathcal{H}_a \cap H$ consists of one point

(H may not be monotone)

existence and uniqueness



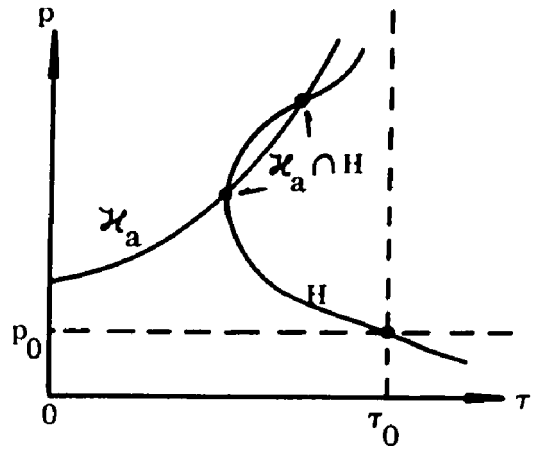
WEAK and not-MEDIUM

τ is a function of p along H

(*) $\exists a > 0 \ni \mathcal{H}_a \cap H$ contains more than one point

existence

(*) type I nonuniqueness

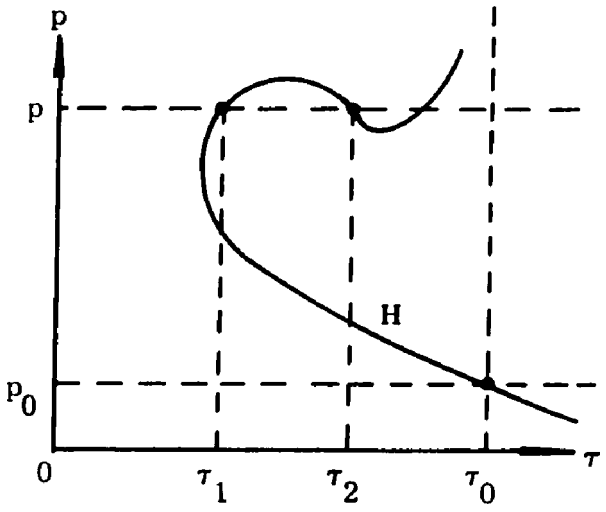


not-WEAK

(*) $\exists p > 0, \exists \tau_1, \tau_2 \in (0, \tau_0) \ni$
$(\tau_1, p), (\tau_2, p) \in H$

existence

(*) both type I and type II nonuniqueness



CHAPTER 2

PROPERTIES OF THE HUGONIOT CURVE

This chapter establishes some properties of H that hold for all energy functions e, under no additional assumptions. We first summarize some known results due to Weyl; these are proved in [1], §65.

Proposition 2.1. (Weyl) Let H have center (τ_0, p_0) and let $(\tau_1, p_1) \in H \setminus \{(\tau_0, p_0)\}$. Let $\mathcal{R}$ be the line through (τ_0, p_0) and (τ_1, p_1), parametrized by

$$p(s) = p_0 + as$$

$$\tau(s) = \tau_0 + bs$$

where $a = p_1 - p_0$, $b = \tau_1 - \tau_0$. Then

$$\frac{d}{ds} S(\tau(s), p(s))\Big|_{s=0} > 0 \ ;$$

$$\frac{d}{ds} S(\tau(s), p(s))\Big|_{s=1} < 0 \ .$$

Corollary 2.2. Let H have center (τ_0, p_0) and let $(\tau_1, p_1) \in H \setminus \{(\tau_0, p_0)\}$. Let L be the line through (τ_0, p_0) and (τ_1, p_1). Then L cannot be tangent to H at (τ_1, p_1); i.e., if the C^1 functions $\tau(t)$, $p(t)$, t in a neighborhood of t_1, satisfy

$$(\tau(t_1), p(t_1)) = (\tau_1, p_1) \ ,$$

$$(\tau(t), p(t)) \in H \text{ for all } t \ ,$$

then there cannot exist $\gamma \neq 0$ such that

$$(\tau'(t_1), p'(t_1)) = \gamma(\tau_1 - \tau_0, \ p_1 - p_0) \ .$$

Corollary 2.3. $dS \neq 0$ along H except at (τ_0, p_0); i.e., if τ_1, p_1, t_1, $\tau(t)$, $p(t)$ are as in Cor. 2.2. and $(\tau'(t_1), p'(t_1)) \neq 0$, then

$$\frac{d}{dt}\, S(\tau(t),\, p(t))\Big|_{t=t_1} \neq 0 \ .$$

Corollary 2.4. Whenever $\tau_1 < \tau_0$ and $(\tau_1, p_1) \in H$,

$$-g_\tau(\tau_0, S_0) < \frac{p_1 - p_0}{\tau_0 - \tau_1} < -g_\tau(\tau_1, S_1) \ ;$$

thus if $U_0 \overset{1\text{-shock}}{\underline{\qquad}} U_1$ or $U_1 \overset{3\text{-shock}}{\underline{\qquad}} U_0$, then $v_0^2 > c_0^2$ and $v_1^2 < c_1^2$. (Cf. (1.19) and Props. 1.1 - 1.3.)

The proofs that follow are long and tedious; hence we preface the formal proofs with an informal introduction. We define two subsets, Q and Q', of the (τ, p)-plane by

$$Q = \left\{(\tau, p) \,\middle|\, 0 < \tau < \tau_0 \text{ and } p > p_0\right\} \ , \tag{2.1}$$

$$Q' = \left\{(\tau, p) \,\middle|\, \tau > \tau_0 \text{ and } 0 < p < p_0\right\} \ ,$$

where (τ_0, p_0) is the center of the Hugoniot curve H. We prove that grad H $\neq 0$ everywhere on H, and that $H \subset Q \cup Q' \cup \{(\tau_0, p_0)\}$.

Definition 2.5. For all $\tau, p > 0$, define

$$\theta = \theta(\tau, p) = \frac{p - p_0}{\tau_0 - \tau} \qquad \text{if } \tau \neq \tau_0 \ ;$$

$$\theta_0 = \theta(\tau_0, p_0) = -\frac{dp}{d\tau}\Big|_{(\tau_0, p_0)} \qquad \text{along H.}$$

(See Fig. 3.) Then we find that the function θ is C^1 on H; we plan to use θ as parameter along H. We shall be concerned only with the "compression branch" $H \cap Q^{c\ell}$ of H. For all θ, let L_θ be the line through (τ_0, p_0) with slope $-\theta$. We want to prove that along H, (τ, p) is a C^1 function of θ; i.e., as L_θ sweeps through Q in the clockwise direction, starting at θ_0, there is always exactly one $(\tau(\theta), p(\theta)) \in H \cap L_\theta$. (See Fig. 4.) This is true locally by Cor. 2.2 and the fact that grad H $\neq 0$ on H, since then $d\theta \neq 0$ along H. In particular, H is repre-

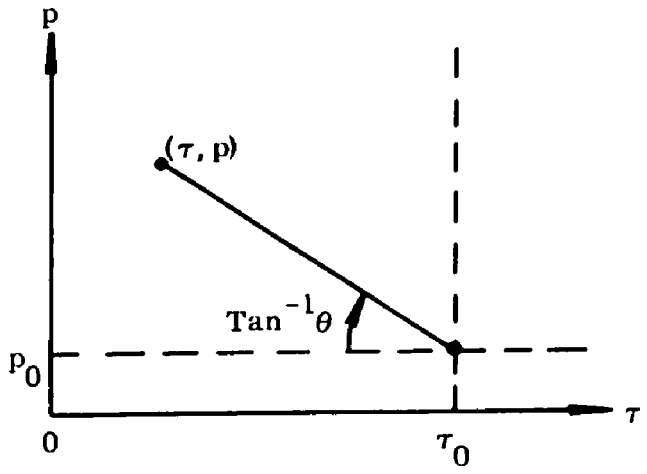


Fig. 3: The definition of θ.

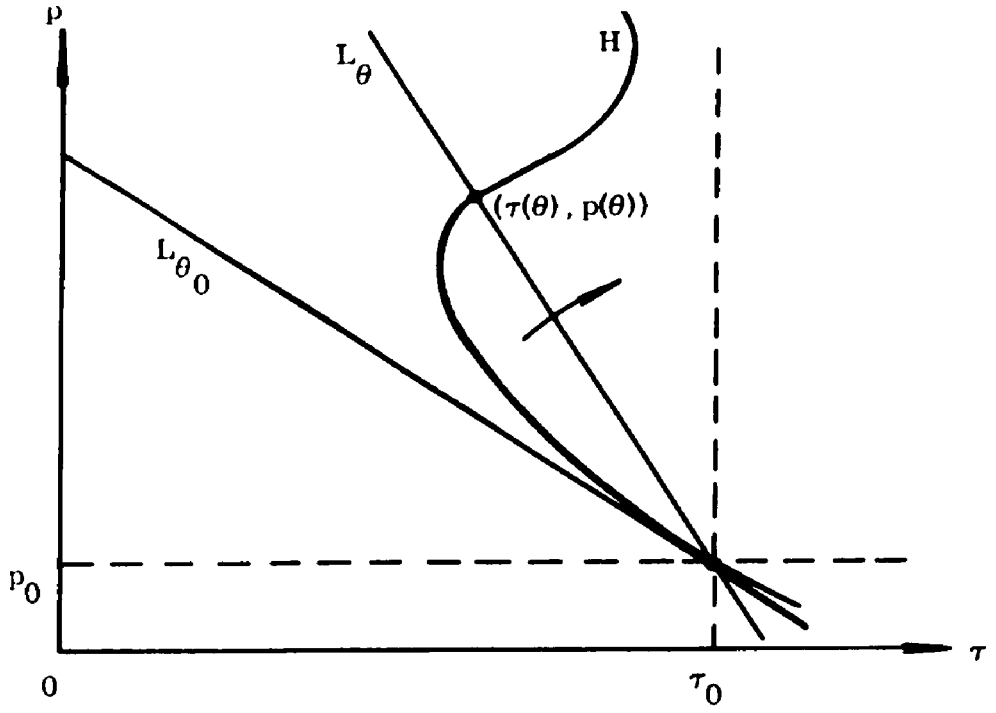


Fig. 4: (τ, p) is a function of θ along H.

sented by $(\tau(\theta), p(\theta))$ uniquely in a neighborhood of (τ_0, p_0). Here $(\tau(\theta), p(\theta))$ is defined for θ in some neighborhood of θ_0. We try to define $(\tau(\theta), p(\theta))$ on the largest possible interval $[\theta_0, \theta_1)$. Any two such functions $(\tau_1(\theta), p_1(\theta))$, $(\tau_2(\theta), p_2(\theta))$ on $[\theta_0, \theta_1)$ must be the same, since otherwise there would be a first point $(\bar{\tau}, \bar{p})$ along H at which the two curves begin to separate, which

contradicts grad H $\neq 0$ at $(\bar{\tau}, \bar{p})$. (See Fig. 5.) Hence there is a maximal interval $[\theta_0, \theta_{max})$ on which such a function $(\tau(\theta), p(\theta))$ can be defined. Then we

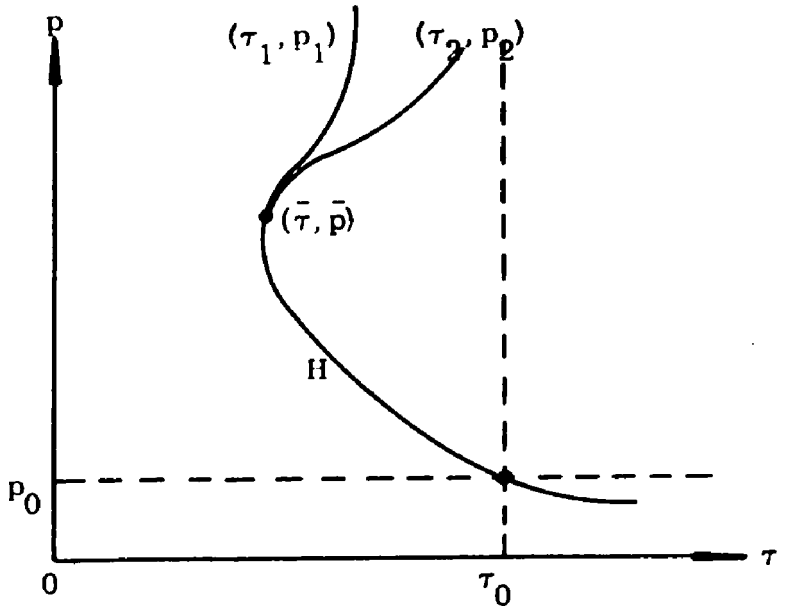


Fig. 5: The images of (τ_1, p_1) and (τ_2, p_2) begin to separate at $(\bar{\tau}, \bar{p})$.

prove $\theta_{max} = +\infty$ by showing that if not, then we can obtain an accumulation point in Q of $(\tau(\theta), p(\theta))$ as $\theta \uparrow \theta_{max}$, and using this, extend $(\tau(\theta), p(\theta))$ to some $[\theta_0, \theta_1)$, $\theta_1 > \theta_{max}$. Thus we have a C^1 branch $H' \subset H$ along which (τ, p) is a function of $\theta \in [\theta_0, +\infty)$, and we also prove that along H', $p \longrightarrow +\infty$ as $\theta \longrightarrow +\infty$; moreover, for every $a > 0$, $\mathcal{H}_a \cap H' \neq \phi$. (Cf. Fig. 1.) Finally we show $H' = H \cap Q^{c\ell}$ as follows: we pick any $(\bar{\tau}, \bar{p}) \in H \cap Q$; we start with a representation $(\tau_1(\theta), p_1(\theta))$ (θ near $\bar{\theta} \equiv \theta(\bar{\tau}, \bar{p})$) of H near $(\bar{\tau}, \bar{p})$; we extend (τ_1, p_1) to a maximal left interval of existence $(\theta_{min}, \bar{\theta}]$; we then prove that this curve $J = \{(\tau_1(\theta), p_1(\theta)) \mid \theta \in (\theta_{min}, \bar{\theta}]\}$ must intersect H', so $J \subset H'$; hence $(\bar{\tau}, \bar{p}) \in H'$.

We hope the above will aid the reader in understanding and ignoring various tedious details of the proofs that follow.

Lemma 2.6. (a) $H_\tau(\tau, p) > 0$ for $0 < p \leq p_0$; (b) $H_p(\tau, p) > 0$ for $\tau \geq \tau_0$; (c) grad H $\neq 0$ everywhere on H.

Proof: We have (see (1.20))

$$H_\tau(\tau, p) = \frac{\partial}{\partial \tau} e(\tau, p) + \frac{p + p_0}{2} = (-g_\tau) \frac{\partial}{\partial p} e(\tau, p) - \frac{p - p_0}{2} \tag{2.2}$$

by (1.23);

$$H_p(\tau, p) = \frac{\partial}{\partial p} e(\tau, p) - \frac{\tau_0 - \tau}{2} \quad . \tag{2.3}$$

These, with (1.5) and (1.22), yield (a) and (b). For (c), it suffices to prove grad $H(\tau, p) \neq 0$ for $(\tau, p) \in H \cap Q$. Suppose $H_p(\tau, p) = 0$. Then by (2.3),

$$\frac{\partial}{\partial p} e(\tau, p) = \frac{\tau_0 - \tau}{2} \quad ,$$

so by (2.2),

$$H_\tau(\tau, p) = (-g_\tau) \frac{\tau_0 - \tau}{2} - \frac{p - p_0}{2} > 0$$

by Cor. 2.4, since $(\tau, p) \in Q$ (see (2.1)). ■

Corollary 2.7. $H \subset Q \cup Q' \cup \{(\tau_0, p_0)\}$.

Proof: By Lemma 2.6(a) and (b),

$$H(\tau, p) > 0 \text{ for } \tau \geq \tau_0, \ p \geq p_0, \ (\tau, p) \neq (\tau_0, p_0) \ ;$$

$$H(\tau, p) < 0 \text{ for } \tau \leq \tau_0, \ p \leq p_0, \ (\tau, p) \neq (\tau_0, p_0) \ . \quad ■$$

Notation. For $x \in (-\infty, +\infty)$, $\epsilon > 0$, let

$$N_\epsilon(x) = (x - \epsilon, \ x + \epsilon) \ . \tag{2.4}$$

We know ([1], § 65) that there exist $\delta, \sigma > 0$ and a C^2 function G on $N_\delta(\tau_0)$ such that

$$G(\tau_0) = p_0 \ , \quad G'(\tau_0) = g_\tau(\tau_0, S_0) \ , \quad G''(\tau_0) = g_{\tau\tau}(\tau_0, S_0) \ , \tag{2.5}$$

and for all (τ, p),

$$(\tau, p) \in H \cap (N_\delta(\tau_0) \times N_\sigma(p_0)) \quad \text{iff} \quad p = G(\tau) \ .$$

Then Def. 2.5 implies

$$\theta_0 \equiv \theta(\tau_0, p_0) = -G'(\tau_0) > 0 \ , \tag{2.6}$$

by (2.5) and (1.5). Cor. 2.7 then yields

$$\theta(\tau, p) > 0 \quad \text{for every } (\tau, p) \in H \ . \tag{2.7}$$

Lemma 2.8. Define

$$\theta(\tau) = \theta(\tau, G(\tau)) \qquad \text{for } \tau \in N_\delta(\tau_0) \ .$$

Then θ is C^1 and $\theta'(\tau_0) < 0$.

Proof: Clearly θ is C^1 on $N_\delta(\tau_0) \setminus \{\tau_0\}$. We have

$$\theta(\tau) = \frac{G(\tau) - p_0}{\tau_0 - \tau} \qquad (\tau \neq \tau_0) \ .$$

Then since $p_0 = G(\tau_0)$,

$$\lim_{\substack{\tau \to \tau_0 \\ \tau \neq \tau_0}} \theta(\tau) = -G'(\tau_0) = \theta(\tau_0, p_0) = \theta(\tau_0) \ ;$$

hence θ is C^0 on $N_\delta(\tau_0)$.

To find $\theta'(\tau_0)$, observe that

$$\lim_{\tau \to \tau_0} \frac{\theta(\tau) - \theta(\tau_0)}{\tau - \tau_0} = \lim_{\tau \to \tau_0} \frac{\dfrac{G(\tau) - p_0}{\tau_0 - \tau} + G'(\tau_0)}{\tau - \tau_0}$$

$$= -\lim_{\tau \to \tau_0} \frac{G(\tau) - p_0 + (\tau_0 - \tau)G'(\tau_0)}{(\tau_0 - \tau)^2} \ .$$

By L'Hôpital's rule,

$$\lim_{\tau \to \tau_0} \frac{\theta(\tau) - \theta(\tau_0)}{\tau - \tau_0} = -\lim_{\tau \to \tau_0} \frac{G'(\tau) - G'(\tau_0)}{2(\tau - \tau_0)} = -\frac{1}{2} G''(\tau_0) ;$$

thus $\theta'(\tau_0)$ exists and $\theta'(\tau_0) = -\frac{1}{2} G''(\tau_0) < 0$. For $\tau \neq \tau_0$ we have

$$\theta'(\tau) = \frac{d}{d\tau} \frac{G(\tau) - p_0}{\tau_0 - \tau} = \frac{(\tau_0 - \tau)G'(\tau) + G(\tau) - p_0}{(\tau_0 - \tau)^2} ;$$

then by L'Hôpital's rule,

$$\lim_{\tau \to \tau_0} \theta'(\tau) = \lim_{\tau \to \tau_0} \frac{(\tau_0 - \tau)G''(\tau) - G'(\tau) + G'(\tau)}{-2(\tau_0 - \tau)} = -\frac{1}{2} G''(\tau_0) = \theta'(\tau_0) ;$$

hence θ is C^1 on $N_\delta(\tau_0)$. ■

Lemma 2.9. $d\theta \neq 0$ along H.

Proof: $d\theta \neq 0$ at (τ_0, p_0) by Lemma 2.8. Let $(\tau_1, p_1) \in H \setminus \{(\tau_0, p_0)\}$, and let $(\tau(t), p(t))$ be defined and C^1 on a neighborhood of t_1, with $(\tau(t_1), p(t_1)) = (\tau_1, p_1)$, $(\tau'(t_1), p'(t_1)) \neq 0$, and $(\tau(t), p(t)) \in H$ for all t. Then

$$\frac{d}{dt} \theta(\tau(t), p(t))\Big|_{t=t_1} = \frac{d}{dt} \frac{p(t) - p_0}{\tau_0 - \tau(t)}\Bigg|_{t=t_1}$$

$$= \frac{(\tau_0 - \tau(t_1))p'(t_1) + (p(t_1) - p_0)\tau'(t_1)}{(\tau_0 - \tau(t_1))^2} .$$

Thus $\frac{d}{dt} \theta(\tau(t), p(t))\Big|_{t=t_1} = 0$ implies

$$(p_1 - p_0, -(\tau_1 - \tau_0)) \perp (\tau'(t_1), p'(t_1)) ,$$

so that $(\tau'(t_1), p'(t_1))$ is a nonzero linear combination of $(\tau_1 - \tau_0, p_1 - p_0)$, which contradicts Cor. 2.2; hence $\frac{d}{dt}\theta(\tau(t), p(t))\Big|_{t=t_1} \neq 0$. Thus $d\theta \neq 0$ at (τ_1, p_1). ■

Definition 2.10. A θ-parametrization $(\theta - p)$ on an interval I is a C^1 vector function (τ^0, p^0) defined on I such that

(a) $$(\tau^0(\theta),\ p^0(\theta)) \in H \qquad (\theta \in I)\ ;$$

(b) $$\theta(\tau^0(\theta),\ p^0(\theta)) = \theta \qquad (\theta \in I)\ ;$$

(c) if $\theta \in I$ and $\theta > \theta_0$, then $(\tau^0(\theta),\ p^0(\theta)) \in Q$.

(See Def. 2.5 for θ_0.) A $\theta - p$ is centered $(c - \theta - p)$ if $\theta_0 \in I$ implies $(\tau^0(\theta_0),\ p^0(\theta_0)) = (\tau_0, p_0)$.

Remark: The superscript "0" of (τ^0, p^0) refers to the subscript in (τ_0, p_0), the center of H. Later, this superscript will be varied to distinguish functions arising from Hugoniot curves with different centers.

Lemma 2.11. Let $(\bar\tau, \bar p) \in H \cap Q^{c\ell}$, $\bar\theta = \theta(\bar\tau, \bar p)$. Then there exist $\epsilon_1, \epsilon_2 > 0$, a $\theta - p$ (τ^*, p^*) on $(\bar\theta - \epsilon_1, \bar\theta + \epsilon_2)$, and an open rectangle R about $(\bar\tau, \bar p)$, such that $(\tau^*(\bar\theta), p^*(\bar\theta)) = (\bar\tau, \bar p)$, and $\forall (\tau, p) \in H \cap R$, $\theta(\tau, p) \in (\bar\theta - \epsilon_1, \bar\theta + \epsilon_2)$ and $(\tau, p) = (\tau^*(\theta(\tau, p)),\ p^*(\theta(\tau, p)))$.

Proof: By Lemma 2.6(c), grad H $\neq 0$ at $(\bar\tau, \bar p)$.

Case 1. $H_p(\bar\tau, \bar p) \neq 0$ or $(\bar\tau, \bar p) = (\tau_0, p_0)$. Then $H_p(\bar\tau, \bar p) \neq 0$, so by the implicit function theorem, $\exists \delta,\ \sigma > 0$ and a C^1 function G on $N_\delta(\bar\tau)$ such that for all (τ, p),

(2.8) $$(\tau, p) \in H \cap (N_\delta(\bar\tau) \times N_\sigma(\bar p)) \quad \text{iff} \quad p = G(\tau)\ .$$

Define

(2.9) $$\theta^*(\tau) = \theta(\tau, G(\tau)) \qquad (\tau \in N_\delta(\bar\tau))\ .$$

Then by Lemma 2.9 if $\bar\tau < \tau_0$, or by Lemma 2.8 if $\bar\tau = \tau_0$, θ^* is C^1 on a neighborhood of $\bar\tau$ and $(\theta^*)'(\bar\tau) \neq 0$. Thus there exist $\delta_1 \in (0, \delta]$ and $\epsilon_1, \epsilon_2 > 0$ such that

(2.10) θ^* is a C^1 diffeomorphism of $N_{\delta_1}(\bar\tau)$ onto $(\bar\theta - \epsilon_1, \bar\theta + \epsilon_2)$.

Let

(2.11) $$\tau^* = (\theta^*)^{-1}\ ;$$

then τ^* is a C^1 diffeomorphism of $(\bar\theta-\epsilon_1, \bar\theta+\epsilon_2)$ onto $N_{\delta_1}(\bar\tau)$. Let $p^* = G\circ\tau^*$; then p^* is C^1 on $(\bar\theta-\epsilon_1, \bar\theta+\epsilon_2)$. Since $\bar p = G(\bar\tau)$, $\bar\theta = \theta^*(\bar\tau)$, so $(\tau^*(\bar\theta), p^*(\bar\theta)) = (\bar\tau, \bar p)$. Clearly for $\theta \in (\bar\theta-\epsilon_1, \bar\theta+\epsilon_2)$ we have $(\tau^*(\theta), p^*(\theta)) = (\tau^*(\theta), G(\tau^*(\theta))) \in H$ and $\theta(\tau^*(\theta), p^*(\theta)) = \theta^*(\tau^*(\theta)) = \theta$. We claim that $\theta > \theta_0$ implies $(\tau^*(\theta), p^*(\theta)) \in Q$. If $(\bar\tau, \bar p) = (\tau_0, p_0)$, then θ^* is decreasing on $N_{\delta_1}(\bar\tau)$ by Lemma 2.8 and the fact that it is a diffeomorphism; hence τ^* is decreasing on $(\bar\theta-\epsilon_1, \bar\theta+\epsilon_2)$ and we are done by Lemma 2.7. If $(\bar\tau, \bar p) \neq (\tau_0, p_0)$, then $(\bar\tau, \bar p) \in Q$. Thus we can make the above δ and σ (for (2.8)) sufficiently small so that $N_\delta(\bar\tau) \times N_\sigma(\bar p) \subset Q$, and the claim is proved. Hence (τ^*, p^*) is a θ-p on $(\bar\theta-\epsilon_1, \bar\theta+\epsilon_2)$. Let $R = N_{\delta_1}(\bar\tau) \times N_\sigma(\bar p)$. Let $(\tau, p) \in H \cap R$. Then $(\tau, p) \in N_\delta(\bar\tau) \times N_\sigma(\bar p)$, so $p = G(\tau)$. Since $\tau \in N_{\delta_1}(\bar\tau)$, we have

$$\theta(\tau, p) = \theta^*(\tau) \in (\bar\theta-\epsilon_1, \bar\theta+\epsilon_2)$$

and

$$\tau^*(\theta(\tau, p)) = \tau \ ,$$

$$p^*(\theta(\tau, p)) = G(\tau^*(\theta(\tau, p))) = p \ .$$

Case 2. $H_\tau(\bar\tau, \bar p) \neq 0$ and $(\bar\tau, \bar p) \neq (\tau_0, p_0)$. Then $(\bar\tau, \bar p) \in Q$, and there exist $\delta, \sigma > 0$ and a C^1 function V on $N_\sigma(\bar p)$ such that $N_\delta(\bar\tau) \times N_\sigma(\bar p) \subset Q$ and for all (τ, p),

$$(\tau, p) \in H \cap (N_\delta(\bar\tau) \times N_\sigma(\bar p)) \text{ iff } \tau = V(p) \ .$$

Let $\theta^*(p) = \theta(V(p), p)$ $(p \in N_\sigma(\bar p))$. Then θ^* is C^1 and by Lemma 2.9, $(\theta^*)'(\bar p) \neq 0$, so there exist $\sigma_1 \in (0, \sigma]$ and $\epsilon_1, \epsilon_2 > 0$ such that θ^* is a C^1 diffeomorphism of $N_{\sigma_1}(\bar p)$ onto $(\bar\theta-\epsilon_1, \bar\theta+\epsilon_2)$. Let $p^* = (\theta^*)^{-1}$, $\tau^* = V \circ p^*$, $R = N_\delta(\bar\tau) \times N_{\sigma_1}(\bar p)$, and proceed as in Case 1. ■

Corollary 2.12. Let (τ_1^0, p_1^0), (τ_2^0, p_2^0) be θ-p's on an interval I; let $\bar\theta \in I$ with $(\tau_1^0(\bar\theta), p_1^0(\bar\theta)) = (\tau_2^0(\bar\theta), p_2^0(\bar\theta)) \in Q^{cl}$. Then $(\tau_1^0, p_1^0) = (\tau_2^0, p_2^0)$ on some neighborhood (in I) of $\bar\theta$.

Proof: Let $(\bar{\tau}, \bar{p}) = (\tau_1^0(\bar{\theta}), p_1^0(\bar{\theta}))$. Then $\theta(\bar{\tau}, \bar{p}) = \bar{\theta}$ by Def. 2.10(b), and $(\bar{\tau}, \bar{p}) \in H \cap Q^{c\ell}$ by Def. 2.10(a). Hence Lemma 2.11 applies; let ϵ_1, ϵ_2, (τ^*, p^*) and R be as in Lemma 2.11 for $(\bar{\tau}, \bar{p})$, $\bar{\theta}$. $\exists \epsilon > 0$ such that $\forall \theta \in N_\epsilon(\bar{\theta}) \cap I$, $(\tau_1^0(\theta), p_1^0(\theta))$, $(\tau_2^0(\theta), p_2^0(\theta)) \in R$. Then $\forall \theta \in N_\epsilon(\bar{\theta}) \cap I$,

$$\theta(\tau_1^0(\theta), p_1^0(\theta)) = \theta = \theta(\tau_2^0(\theta), p_2^0(\theta))$$

by Def. 2.10(b), so

$$(\tau_1^0(\theta), p_1^0(\theta)) = (\tau^*(\theta), p^*(\theta)) = (\tau_2^0(\theta), p_2^0(\theta))$$

by Lemma 2.11. ■

Lemma 2.13. Let (τ_1^0, p_1^0), (τ_2^0, p_2^0) be c-θ-p's on $[\theta_0, \theta_1)$. Then $(\tau_1^0, p_1^0) = (\tau_2^0, p_2^0)$.

Proof: Suppose not. Let $\theta_{min} = \inf\{\theta > \theta_0 \mid (\tau_1^0(\theta), p_1^0(\theta)) \neq (\tau_2^0(\theta), p_2^0(\theta))\}$. Then $\theta_{min} \in [\theta_0, \theta_1)$ and $(\tau_1^0(\theta_{min}), p_1^0(\theta_{min})) = (\tau_2^0(\theta_{min}), p_2^0(\theta_{min}))$. But by Cor. 2.12 (see Def. 2.10(c)), $(\tau_1^0, p_1^0) = (\tau_2^0, p_2^0)$ on a neighborhood (in $[\theta_0, \theta_1)$) of θ_{min}, a contradiction. ■

Proposition 2.14. There exists a maximal right-open interval $[\theta_0, \theta_{max})$ on which a c-θ-p can be defined; i.e., there is a c-θ-p (τ^0, p^0) on $[\theta_0, \theta_{max})$, and there cannot be a c-θ-p on any larger interval $[\theta_0, \theta_1)$, $\theta_1 > \theta_{max}$. Here $\theta_{max} \in (\theta_0, +\infty]$.

Proof: Let $\Theta = \{\theta_1 > \theta_0 \mid \exists$ a c-θ-p on $[\theta_0, \theta_1)\}$. Let $\theta_{max} = \sup \Theta$. With ϵ_2 as in Lemma 2.11 for $(\bar{\tau}, \bar{p}) = (\tau_0, p_0)$, $\theta_0 + \epsilon_2 \in \Theta$; hence $\theta_{max} > \theta_0$. We define (τ^0, p^0) on $[\theta_0, \theta_{max})$ as follows:

> Let $\theta \in [\theta_0, \theta_{max})$. Choose some $\theta_1 \in (\theta, +\infty) \cap \Theta$. $\exists$ a c-θ-p (τ_1^0, p_1^0) on $[\theta_0, \theta_1)$. Let $(\tau^0(\theta), p^0(\theta)) = (\tau_1^0(\theta), p_1^0(\theta))$.

By Lemma 2.13, (τ^0, p^0) is well-defined by this process and is a c-θ-p on $[\theta_0, \theta_{max})$. ■

Proposition 2.15. $\theta_{max} = +\infty$.

Proof: Suppose $\theta_{max} < +\infty$. Then $L_{\theta_{max}}$ has p-intercept $p_{max} = \tau_0 \theta_{max} + p_0$. (See Fig. 6.) Since $e(\tau_0, p_0) > 0$, and

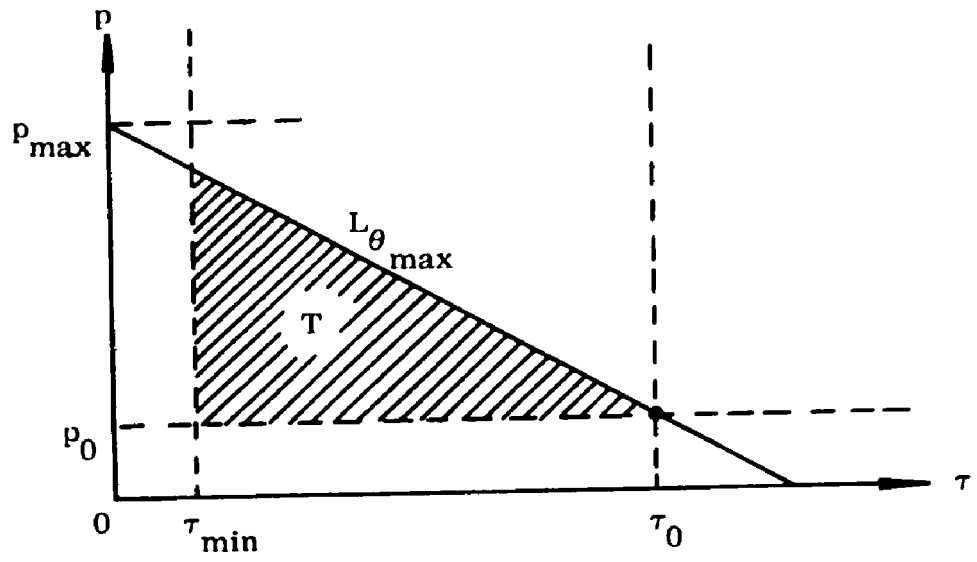


Fig. 6: The region T.

$\lim_{\tau \to 0^+} e(\tau, p_{max}) = 0$ by (1.10), $\exists \tau_{min} > 0$ such that $\forall \tau > 0$, $\tau \leq \tau_{min}$ implies $e(\tau, p_{max}) < e(\tau_0, p_0)$; we may assume $\tau_{min} \leq \tau_0$. Now suppose $0 < \tau \leq \tau_{min}$ and $0 < p \leq p_{max}$. Then from (1.20) and (1.22), we get

$$H(\tau, p) \leq e(\tau, p_{max}) - e(\tau_0, p_0) + (\tau - \tau_0) \frac{p + p_0}{2} < (\tau - \tau_0) \frac{p + p_0}{2} \leq 0 \ ;$$

thus

$$(2.12) \qquad \left[\tau \leq \tau_{min} \text{ and } p \leq p_{max}\right] \Longrightarrow (\tau, p) \notin H \ .$$

Let T be the open triangular region bounded by $L_{\theta_{max}}$ and the lines $\{p = p_0\}$ and $\{\tau = \tau_{min}\}$. (See Fig. 6.) We assert that $(\tau^0(\theta), p^0(\theta)) \in T$ $\forall \theta \in (\theta_0, \theta_{max})$. For we know (Def. 2.10) that $(\tau^0(\theta), p^0(\theta)) \in H \cap Q$ and $\theta(\tau^0(\theta), p^0(\theta)) = \theta < \theta_{max}$. Hence $(\tau^0(\theta), p^0(\theta))$ lies below $L_{\theta_{max}}$, which then

implies $p^0(\theta) \leq p_{max}$. From (2.12) we conclude $\tau^0(\theta) > \tau_{min}$, and the assertion is proved.

Now choose a sequence $\{\theta_n\}$ from $(\theta_0, +\infty)$ such that $\theta_n \uparrow \theta_{max}$. Then $(\tau^0(\theta_n), p^0(\theta_n)) \in T^{c\ell} Vn$, so there exist a subsequence of $\{\theta_n\}$, which we call $\{\theta_n\}$ again, and a point $(\bar{\tau}, \bar{p}) \in T^{c\ell}$ such that

$$(\tau^0(\theta_n), p^0(\theta_n)) \longrightarrow (\bar{\tau}, \bar{p}) \quad . \tag{2.13}$$

Suppose $(\bar{\tau}, \bar{p}) = (\tau_0, p_0)$; let $\epsilon_1, \epsilon_2, R$ be as in Lemma 2.11. Let N be so large that $(\tau^0(\theta_n), p^0(\theta_n)) \in R$ whenever $n \geq N$; then from Lemma 2.11 we conclude that $Vn \geq N$, $\theta_n = \theta(\tau^0(\theta_n), p^0(\theta_n)) \in (\theta_0 - \epsilon_1, \theta_0 + \epsilon_2)$. But by Lemma 2.13 and (2.9) - (2.11) of the proof of Lemma 2.11, τ^0 is a C^1 diffeomorphism on $(\theta_0 - \epsilon_1, \theta_0 + \epsilon_2)$, hence (strictly) monotone. Then since $\{\theta_n\}$ is increasing and $(\tau^0(\theta_0), p^0(\theta_0)) = (\tau_0, p_0)$, we cannot have $\tau^0(\theta_n) \longrightarrow \tau_0$; this contradicts (2.13). Therefore $(\bar{\tau}, \bar{p}) \neq (\tau_0, p_0)$. But since H is closed, $(\bar{\tau}, \bar{p}) \in H$; then Cor. 2.7 gives $(\bar{\tau}, \bar{p}) \in Q$. Since θ is continuous on Q,

$$\theta(\bar{\tau}, \bar{p}) = \lim_{n \to \infty} \theta(\tau^0(\theta_n), p^0(\theta_n)) = \lim_{n \to \infty} \theta_n = \theta_{max} \; ;$$

hence Lemma 2.11 applies with $\bar{\theta} = \theta_{max}$; let $\epsilon_1, \epsilon_2, R, (\tau^*, p^*)$ be as in Lemma 2.11 for $(\bar{\tau}, \bar{p}), \bar{\theta}$. Then $(\tau^0(\theta_N), p^0(\theta_N)) \in R \cap Q$ for some N, and we have

$$\theta_N = \theta(\tau^0(\theta_N), p^0(\theta_N)) \in (\theta_{max} - \epsilon_1, \theta_{max} + \epsilon_2) \; ,$$

$$(\tau^0(\theta_N), p^0(\theta_N)) = (\tau^*(\theta_N), p^*(\theta_N)) \; .$$

(See Fig. 7.) Then (τ^0, p^0) and (τ^*, p^*) are $c\text{-}\theta\text{-}p$'s on a neighborhood of θ_N that agree at θ_N, so by Cor. 2.12, $(\tau^0, p^0) = (\tau^*, p^*)$ on some neighborhood of θ_N. Thus we may extend (τ^0, p^0) to $[\theta_0, \theta_{max} + \epsilon_2)$ by defining $(\tau^0, p^0) = (\tau^*, p^*)$ on $(\theta_N, \theta_{max} + \epsilon_2)$ and obtain a C^1 function, hence a $c\text{-}\theta\text{-}p$, on $[\theta_0, \theta_{max} + \epsilon_2)$, which contradicts the maximality of θ_{max}. Hence $\theta_{max} = +\infty$.

■

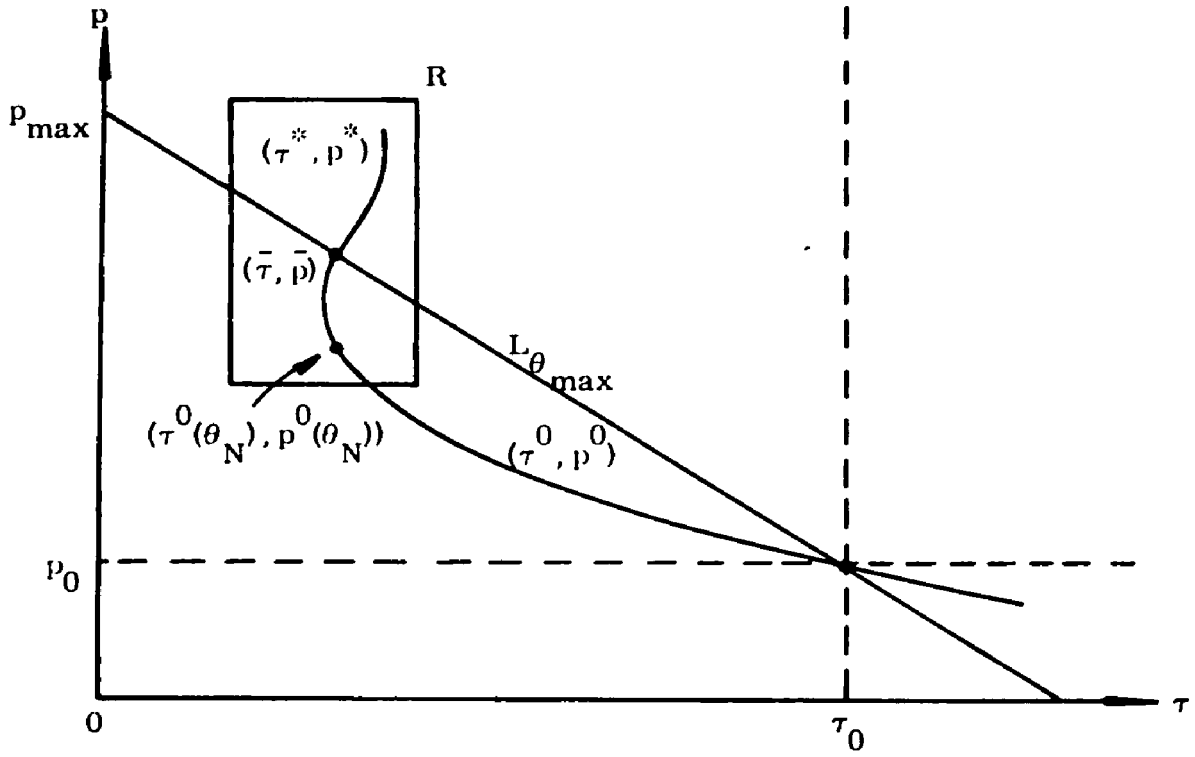


Fig. 7: Finding θ_N at which to join (τ^0, p^0) and (τ^*, p^*).

Proposition 2.16. Let $(\bar{\tau}, \bar{p}) \in H \cap Q$. Then $(\bar{\tau}, \bar{p}) = (\tau^0(\theta), p^0(\theta))$ for some $\theta > \theta_0$.

Proof: Suppose

$$(\bar{\tau}, \bar{p}) \notin \left\{(\tau^0(\theta), p^0(\theta)) \mid \theta > \theta_0\right\} . \tag{2.14}$$

Let $\bar{\theta} = \theta(\bar{\tau}, \bar{p})$. Let $\epsilon_1^0, \epsilon_2^0, R_0$ (and $(\tau^0, p^0)\Big|_{(\theta_0 - \epsilon_1^0, \theta_0 + \epsilon_2^0)}$) be as in Lemma 2.11 for (τ_0, p_0), θ_0. Let R_1 be the closed rectangle concentric with R_0 and having half the length and width of R_0. (See Fig. 8.) Then we assert

Claim 1. For any θ-p (τ_1^0, p_1^0) on $(\theta_1, \bar{\theta}]$ with $(\tau_1^0(\bar{\theta}), p_1^0(\bar{\theta})) = (\bar{\tau}, \bar{p})$, we have

$$(\tau_1^0(\theta), p_1^0(\theta)) \in Q \setminus R_1 \qquad \forall \theta \in (\theta_1, \bar{\theta}] .$$

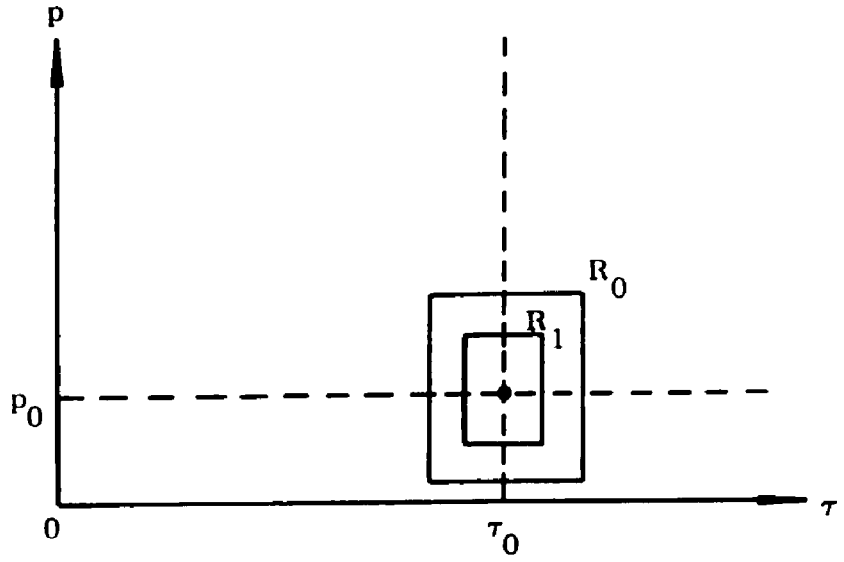


Fig. 8: The rectangles R_0 and R_1.

To prove this, suppose $\exists \theta_2 \in (\theta_1, \bar{\theta}]$ such that $(\tau_1^0(\theta_2), p_1^0(\theta_2)) \in (Q \setminus R_1)^c$.

Case 1. $(\bar{\tau}, \bar{p}) \in (Q \setminus R_1)^c$. Thus $(\bar{\tau}, \bar{p}) \in Q^c \cup R_1$, so $(\bar{\tau}, \bar{p}) \in R_0$, i.e., $(\tau_1^0(\bar{\theta}), p_1^0(\bar{\theta})) \in R_0$.

Case 2. $(\bar{\tau}, \bar{p}) \in Q \setminus R_1$. Then either $(\tau_1^0(\theta_2), p_1^0(\theta_2)) \in \partial(Q \setminus R_1)$, or $(\tau_1^0(\theta_2), p_1^0(\theta_2)) \in \mathrm{Int}[(Q \setminus R_1)^c]$, in which case $\exists \theta_3 \in (\theta_2, \bar{\theta})$ such that $(\tau_1^0(\theta_3), p_1^0(\theta_3)) \in \partial(Q \setminus R_1)$. (See Fig. 9.) Thus we may assume $(\tau_1^0(\theta_2), p_1^0(\theta_2)) \in \partial(Q \setminus R_1)$. Then by Cor. 2.7, $(\tau_1^0(\theta_2), p_1^0(\theta_2)) \in R_0 \cap Q$.

Thus in either case, $\exists \theta_2 \in (\theta_1, \bar{\theta}]$ such that $(\tau_1^0(\theta_2), p_1^0(\theta_2)) \in R_0 \cap Q$. Then by Lemma 2.11, $(\tau_1^0(\theta_2), p_1^0(\theta_2)) = (\tau^0(\theta_2), p^0(\theta_2))$. By Lemma 2.8 and (2.9) - (2.11), we must have $\theta_2 > \theta_0$. Thus by Cor. 2.12, $(\tau_1^0, p_1^0) = (\tau^0, p^0)$ on some neighborhood, in $(\max\{\theta_0, \theta_1\}, \bar{\theta}]$, of θ_2. Define

$$(\tau_2^0(\theta), p_2^0(\theta)) = \begin{cases} (\tau^0(\theta), p^0(\theta)), & \text{if } \theta_0 \le \theta \le \theta_2 \\ (\tau_1^0(\theta), p_1^0(\theta)), & \text{if } \theta_2 \le \theta \le \bar{\theta} \end{cases} .$$

Then (τ_2^0, p_2^0) is a c-θ-p on $[\theta_0, \bar{\theta}]$, so $(\tau_2^0, p_2^0) = (\tau^0, p^0)$ on $[\theta_0, \bar{\theta}]$ by

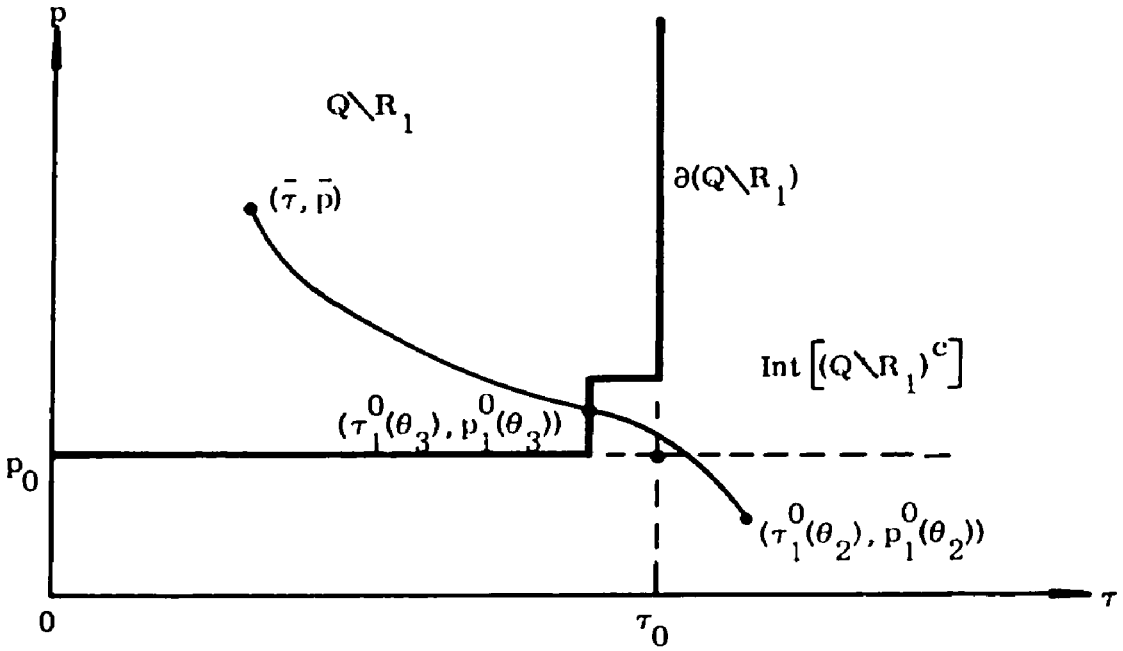


Fig. 9: The image of (τ_1^0, p_1^0) must cross $\partial(Q\backslash R_1)$.

Lemma 2.13 and continuity. Thus $(\bar{\tau}, \bar{p}) = (\tau_1^0(\bar{\theta}), p_1^0(\bar{\theta})) = (\tau_2^0(\bar{\theta}), p_2^0(\bar{\theta})) = (\tau^0(\bar{\theta}), p^0(\bar{\theta}))$, which contradicts (2.14). The claim is thus proved.

Next, in analogy to Lemma 2.13, we have

Claim 2. Let (τ_1^0, p_1^0), (τ_2^0, p_2^0) be θ-p's on $(\theta_1, \bar{\theta}]$ with $(\tau_1^0(\bar{\theta}), p_1^0(\bar{\theta})) = (\bar{\tau}, \bar{p}) = (\tau_2^0(\bar{\theta}), p_2^0(\bar{\theta}))$. Then $(\tau_1^0, p_1^0) = (\tau_2^0, p_2^0)$.

We prove this like Lemma 2.13. Suppose $(\tau_1^0, p_1^0) \neq (\tau_2^0, p_2^0)$. Let $\theta_2 = \sup\{\theta \in (\theta_1, \bar{\theta}] \mid (\tau_1^0(\theta), p_1^0(\theta)) \neq (\tau_2^0(\theta), p_2^0(\theta))\}$; then $\theta_1 < \theta_2 \leq \bar{\theta}$ and $(\tau_1^0(\theta_2), p_1^0(\theta_2)) = (\tau_2^0(\theta_2), p_2^0(\theta_2)) \in Q\backslash R_1$, by Claim 1. Cor. 2.12 then yields $(\tau_1^0, p_1^0) = (\tau_2^0, p_2^0)$ on some neighborhood (in $(\theta_1, \bar{\theta}]$) of θ_2, a contradiction.

Now define

$$\Theta = \left\{\theta_1 \in (0, \bar{\theta}] \mid \exists a \ \theta\text{-p}\,(\tau_1^0, p_1^0) \text{ on } (\theta_1, \bar{\theta}] \text{ with } (\tau_1^0(\bar{\theta}), p_1^0(\bar{\theta})) = (\bar{\tau}, \bar{p})\right\}.$$

$\Theta \neq \phi$ by Lemma 2.11. Let $\theta_{\min} = \inf\Theta$; then $0 \leq \theta_{\min} < \bar{\theta}$. By Claim 2, we can define a θ-p (τ_1^0, p_1^0) on $(\theta_{\min}, \bar{\theta}]$ as follows:

Let $\theta \in (\theta_{min}, \bar{\theta}]$. Choose some $\theta_1 \in (\theta_{min}, \theta)$ such that $\theta_1 \in \Theta$. $\exists$a θ-p (τ_2^0, p_2^0) on $(\theta_1, \bar{\theta}]$ with $(\tau_2^0(\bar{\theta}), p_2^0(\bar{\theta})) = (\bar{\tau}, \bar{p})$. Define $(\tau_1^0(\theta), p_1^0(\theta)) = (\tau_2^0(\theta), p_2^0(\theta))$.

Moreover, $(\tau_1^0(\bar{\theta}), p_1^0(\bar{\theta})) = (\bar{\tau}, \bar{p})$. Claim 1 gives

(2.15) $$(\tau_1^0(\theta), p_1^0(\theta)) \in Q \setminus R_1 \qquad \forall \theta \in (\theta_{min}, \bar{\theta}] .$$

Then Def. 2.10(b) implies

(2.16) $$(\tau_1^0(\theta), p_1^0(\theta)) \text{ is never above } L_{\bar{\theta}} \text{ for } \theta \in (\theta_{min}, \bar{\theta}] .$$

Let p_{max} be the p-intercept of $L_{\bar{\theta}}$. Then as in the proof of Prop. 2.15, we can find $\tau_{min} \in (0, \tau_0]$ such that (2.12) holds. It follows from (2.15) and (2.16) that

(2.17) $$\tau_1^0(\theta) > \tau_{min} \qquad \forall \theta \in (\theta_{min}, \bar{\theta}] .$$

(See Fig. 10.) So by (2.15) -(2.17) we have

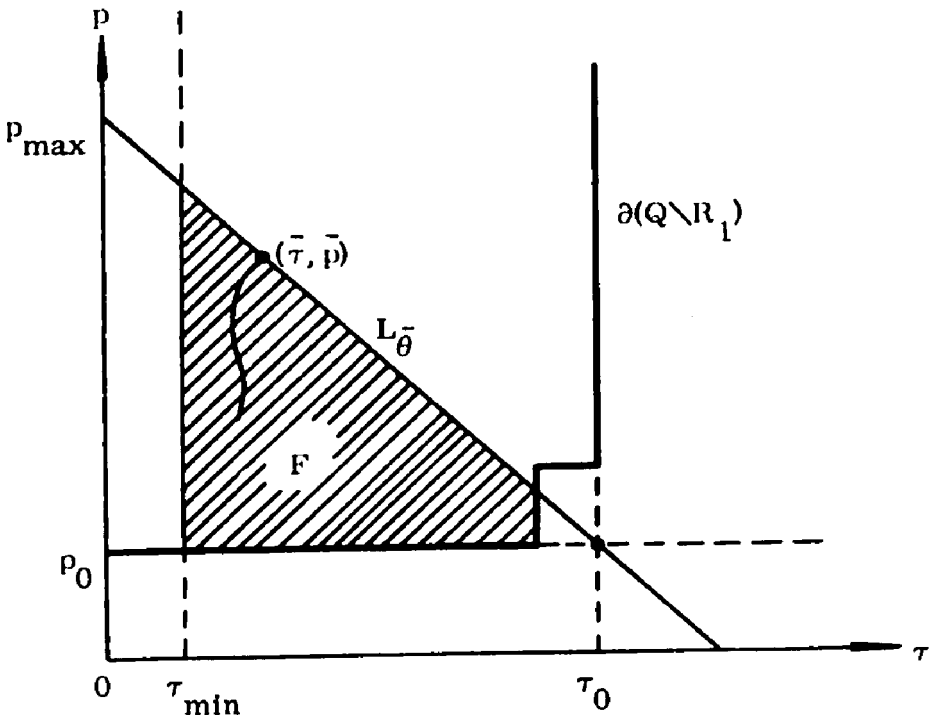


Fig. 10: The region F.

$$(\tau_1^0(\theta),\ p_1^0(\theta)) \in F \qquad\qquad \forall \theta \in (\theta_{\min},\ \bar{\theta}\,] \ ,$$

where F is the shaded closed region in Fig. 10. Thus (as before) $\exists$ a sequence $\theta_n \downarrow \theta_{\min}$ such that $(\tau_1^0(\theta_n),\ p_1^0(\theta_n)) \longrightarrow (\bar{\tau}_1, \bar{p}_1) \in F$, and we must have $(\bar{\tau}_1, \bar{p}_1) \in H \cap Q$; hence $\theta_{\min} = \theta(\bar{\tau}_1, \bar{p}_1)$. Let ϵ_1, ϵ_2, R and (τ^*, p^*) be as in Lemma 2.11 for $(\bar{\tau}_1, \bar{p}_1)$, $\theta_{\min}$. Then for some N, $(\tau_1^0(\theta_N),\ p_1^0(\theta_N)) \in R$, and as before, we can construct a θ-p on $(\theta_{\min} - \epsilon_1,\ \bar{\theta}\,]$ by joining (τ^*, p^*) and $(\tau_1^0,\ p_1^0)$ together at θ_N, contradicting the minimality of $\theta_{\min}$. ■

Thus, combining Props. 2.14 - 2.16, we see that the entire curve $H \cap Q^{c\ell}$ is given as a function of θ by the function (τ^0, p^0) on $[\theta_0, +\infty)$.

Lemma 2.17. Along $H \cap Q^{c\ell}$, S increases with increasing θ.

Proof: We know ([1], § 65) that near (τ_0, p_0), S increases along H with decreasing τ. Then by Lemma 2.8, S increases along $H \cap Q^{c\ell}$ with increasing θ for θ near θ_0. Lemma 2.11 and its proof imply $((\tau^0)'(\theta),\ (p^0)'(\theta)) \neq 0$ for $\theta > \theta_0$; hence

$$\frac{d}{d\theta}\, S(\tau^0(\theta),\ p^0(\theta)) \neq 0 \qquad\qquad \text{for } \theta > \theta_0 \ ,$$

by Cor. 2.3. Then we must have

$$\frac{d}{d\theta}\, S(\tau^0(\theta),\ p^0(\theta)) > 0 \qquad\qquad \text{for } \theta > \theta_0 \ ;$$

hence $\theta \longmapsto S(\tau^0(\theta),\ p^0(\theta))$ is increasing on $[\theta_0, +\infty)$. ■

Lemma 2.18. $H \cap (Q')^{c\ell}$ is given by a decreasing C^1 function $\tau = V(p)$ from $(0, p_0]$ onto $[\tau_0, \tau_{\max})$, where

$$\tau_{\max} = \tau_0 + \frac{2e(\tau_0, p_0)}{p_0} \ .$$

Proof: Let $0 < p \leq p_0$. Using (1.2), (1.10) and (1.20) we get

$$\lim_{\tau \to 0^+} H(\tau, p) < 0 \ , \qquad\qquad \lim_{\tau \to +\infty} H(\tau, p) = +\infty \ .$$

By Lemma 2.6(a), $H_\tau(\tau, p) > 0$ $\forall \tau > 0$. Hence there is exactly one $\tau \in (0, +\infty)$ for which $(\tau, p) \in H$; let $V(p) = \tau$. V is C^1 on $(0, p_0]$ by the implicit function theorem. Moreover, $V(p) \geq \tau_0$ for $0 < p \leq p_0$ by Cor. 2.7, so Lemma 2.6(b) gives $H_p(V(p), p) > 0$ for $0 < p \leq p_0$; hence $V'(p) = -H_p(V(p), p)/H_\tau(V(p), p) < 0$, so V is decreasing. (See Fig. 11.)

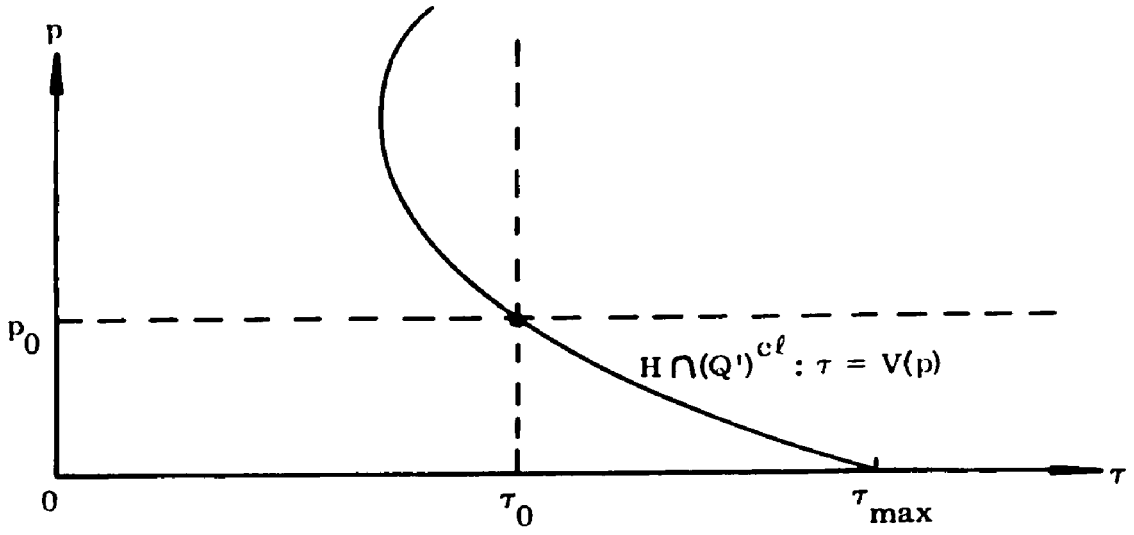


Fig. 11: H for $p < p_0$.

For $0 < p \leq p_0$ we have, by (1.20),

$$H(\tau_{max}, p) = e(\tau_{max}, p) - e(\tau_0, p_0) + \frac{2e(\tau_0, p_0)}{p_0} \cdot \frac{p + p_0}{2}$$

$$> -e(\tau_0, p_0) + \frac{2e(\tau_0, p_0)}{p_0} \cdot \frac{p_0}{2} = 0 \ ;$$

then since $H(V(p), p) = 0$ and $H_\tau(\tau, p) > 0$ $\forall \tau$, we must have $V(p) < \tau_{max}$. Hence $V((0, p_0]) \subset [\tau_0, \tau_{max})$.

Now let $\tau_1 \in [\tau_0, \tau_{max})$. Let us remark that our assumptions (1.8) and (1.9) imply

$$\lim_{p \to 0^+} e(\tau, p) = 0 \qquad \forall \tau > 0 . \tag{2.18}$$

Then

$$\lim_{p \to 0^+} H(\tau_1, p) = -e(\tau_0, p_0) + (\tau_1 - \tau_0)\frac{p_0}{2} < -e(\tau_0, p_0) + (\tau_{max} - \tau_0)\frac{p_0}{2} = 0 ,$$

and $H(\tau_1, p_0) \geq H(\tau_0, p_0) = 0$ by Lemma 2.6(a); hence $\exists p \in (0, p_0]$ such that $H(\tau_1, p) = 0$, i.e., $\tau_1 = V(p)$. Thus V maps $(0, p_0]$ onto $[\tau_0, \tau_{max})$. ■

Corollary 2.19. Along $H \cap (Q')^{c\ell}$, S increases with increasing p.

Proof: Similar to that of Lemma 2.17, using Cor. 2.3, the function G of Lemma 2.8, and Lemma 2.18.

We can now give the

Proof of Propositions 1.2 and 1.3: We prove Prop. 1.2; Prop. 1.3 is similar.

Suppose $U_0 \overset{1\text{-shock}}{\longrightarrow} U$ with shock speed s. By Prop. 1.1, the conditions (1.15) -(1.18) hold. It is known ([1], §§54, 55, 59, 64) that (1.15) - (1.17) imply

$$(\tau, p) \in (\mathcal{H}_{(u_0-u)^2} \cup \mathcal{H}'_{(u_0-u)^2}) \cap H ,$$

where H, $\mathcal{H}$, $\mathcal{H}'$ have center (τ_0, p_0). Prop. 1.1 also gives $v, v_0 > 0$. Then (1.18) yields $S > S_0$ (where $S_0 = S(\tau_0, p_0)$, $S = S(\tau, p)$). From Cor. 2.19 we then conclude $(\tau, p) \notin (Q')^{c\ell}$; hence $(\tau, p) \in \mathcal{H}_{(u_0-u)^2} \cap H$. In particular $(\tau, p) \in Q$, so $\tau < \tau_0$; then (1.15) yields $v < v_0$, so $u < u_0$.

Now suppose $u < u_0$ and $(\tau, p) \in \mathcal{H}_{(u_0-u)^2} \cap H$. If s is to be the shock speed, then (1.15) is satisfied iff

$$s = \frac{\rho u - \rho_0 u_0}{\rho - \rho_0} .$$

Using this value of s, one shows easily, by reversing the arguments of [1] cited above, that U_0, U and s also satisfy (1.16) and (1.17). Since $(\tau, p) \in \mathcal{H}_{(u_0-u)^2}$, $\rho > \rho_0$ and $p > p_0$. Then

$$s = \frac{\rho u - \rho_0 u_0}{\rho - \rho_0} < \frac{\rho u - \rho_0 u}{\rho - \rho_0} = u \ .$$

So $s < u, u_0$, and $m > 0$. But $S > S_0$ by Lemma 2.17 (and Props. 2.14 - 2.16, etc.), since $(\tau, p) \in H \cap Q$; thus (1.18) holds, and $U_0 \xrightarrow{1\text{-shock}} U$.

Now recall that in this paper, all shocks are required to satisfy the Lax shock conditions (1.19). Thus for the purposes of this paper, the proof is not complete until (1.19) is proved. But (1.19) is a simple consequence of Cor. 2.4 and the fact that $\rho > \rho_0$ and $(\tau, p) \in H$. ■

Observe that the above proof also establishes

Proposition 2.20. Shocks always satisfy (1.19).

Lemma 2.21. $\lim_{\theta \to +\infty} p^0(\theta) = +\infty$.

Proof: Suppose not. Then we can find $M > 0$ and a sequence $\{\theta_n\}$ from $(\theta_0, +\infty)$ such that $\theta_n \to +\infty$ and $p^0(\theta_n) \leq M \ \forall n$. We have, for each n,

$$\theta_n = \theta(\tau^0(\theta_n), p^0(\theta_n)) = \frac{p^0(\theta_n) - p_0}{\tau_0 - \tau^0(\theta_n)} \leq \frac{M - p_0}{\tau_0 - \tau^0(\theta_n)} \ ,$$

since $(\tau^0(\theta_n), p^0(\theta_n)) \in Q$. (See Def. 2.10(c).) Since $\theta_n \to +\infty$, we must have $\tau^0(\theta_n) \to \tau_0$. There exist a subsequence of $\{\theta_n\}$, which we call $\{\theta_n\}$ again, and a number $\bar{p} \in [p_0, M]$ such that $p^0(\theta_n) \to \bar{p}$. Suppose $\bar{p} = p_0$. Let R be as in Lemma 2.11 for (τ_0, p_0), θ_0. Then since $(\tau^0(\theta_n), p^0(\theta_n)) \to (\tau_0, \bar{p}) = (\tau_0, p_0)$, $(\tau^0(\theta_n), p^0(\theta_n)) \in R$ for arbitrarily large θ_n, a contradiction. Hence $\bar{p} > p_0$. But since H is closed, it follows that $(\tau_0, \bar{p}) \in H$, which contradicts Cor. 2.7. ■

Lemma 2.22. For every $a > 0$, $\mathcal{H}_a \cap H \neq \phi$. In other words, the function $a^0(\theta)$, $\theta \in [\theta_0, +\infty)$ defined by

$$a^0(\theta) = \sqrt{(\tau_0 - \tau^0(\theta))(p^0(\theta) - p_0)}$$

assumes all nonnegative values.

Proof: Suppose $\exists a > 0$ such that $\forall \theta \geq \theta_0$, $(\tau^0(\theta), p^0(\theta))$ lies to the right of $\mathcal{H}_a$, i.e.,

$$\tau^0(\theta) \geq \tau_0 - \frac{a}{p^0(\theta) - p_0} \qquad (\theta \geq \theta_0) \ . \tag{2.19}$$

(See Fig. 12.) Then from (1.20) and (2.19),

$$e(\tau^0(\theta), p^0(\theta)) - e(\tau_0, p_0) = (\tau_0 - \tau^0(\theta)) \frac{p^0(\theta) + p_0}{2} \leq \frac{a}{2} \frac{p^0(\theta) + p_0}{p^0(\theta) - p_0} \to \frac{a}{2} \text{ as } \theta \to +\infty,$$

by Lemma 2.21. Hence $e(\tau^0(\theta), p^0(\theta))$ is bounded above for $\theta \geq \theta_0$.

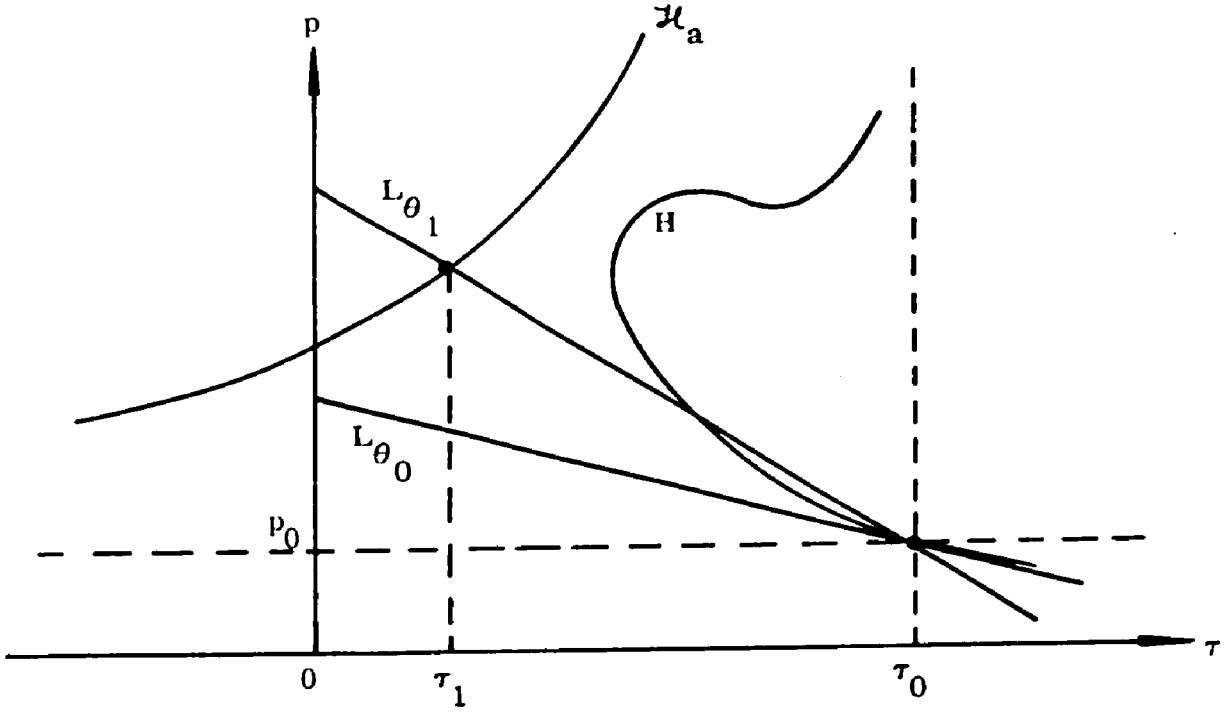


Fig. 12: H lies to the right of $\mathcal{H}_a$.

Choose $\theta_1 \geq \theta_0$ so large that $L_{\theta_1} \cap \mathcal{H}_a$ is a point $(\tau_1, p_1) \in Q$, i.e., $\tau_1 > 0$. Then $\forall \theta \geq \theta_1$, $(\tau^0(\theta), p^0(\theta))$ lies above L_{θ_1} and to the right of $\mathcal{H}_a$; thus

(2.20) $$\tau^0(\theta) \geq \tau_1 \qquad (\theta \geq \theta_1) \ .$$

Let $S^0(\theta) = S(\tau^0(\theta), p^0(\theta))$ $(\theta \geq \theta_0)$ and suppose $\exists M > 0$ such that $\forall \theta \geq \theta_0$, $S^0(\theta) \leq M$. By Lemma 2.17, $S^0(\theta) \geq S(\tau_0, p_0)$ $\forall \theta \geq \theta_0$. Thus $\forall \theta \geq \theta_1$, $(\tau^0(\theta), S^0(\theta)) \in [\tau_1, \tau_0] \times [S(\tau_0, p_0), M]$. But g is bounded above on $[\tau_1, \tau_0] \times [S(\tau_0, p_0), M]$, so $p^0(\theta) = g(\tau^0(\theta), S^0(\theta))$ is bounded above for $\theta \geq \theta_1$, which contradicts Lemma 2.21. Therefore $S^0(\theta)$ assumes arbitrarily large values for $\theta \geq \theta_0$.

Finally, observe that $\forall \theta \geq \theta_0$,

(2.21) $$e(\tau^0(\theta), p^0(\theta)) = e(\tau^0(\theta), S^0(\theta)) \geq e(\tau_0, S^0(\theta)) \ ,$$

by (1.3). But the left member of (2.21) is bounded above, whereas by (1.9) the right member is not, yielding a contradiction.

Therefore $\forall a > 0$, $\exists \theta_1 > \theta_0$ such that

$$\tau^0(\theta_1) < \tau_0 - \frac{a}{p^0(\theta_1) - p_0} \ .$$

But for $\theta > \theta_0$, θ near θ_0,

$$\tau^0(\theta) > \tau_0 - \frac{a}{p^0(\theta) - p_0} \ .$$

Hence $\exists \bar{\theta} \in (\theta_0, \theta_1)$ such that

$$\tau^0(\bar{\theta}) = \tau_0 - \frac{a}{p^0(\bar{\theta}) - p_0} \ ,$$

i.e., $(\tau^0(\bar{\theta}), p^0(\bar{\theta})) \in \mathcal{K}_a$. ■

CHAPTER 3

EXISTENCE OF SOLUTIONS

In this chapter we prove that for any energy function e (see Def. 1.4), the R.P. with arbitrary initial data (U_ℓ, U_r) always has a solution. In order to do this, it is sometimes necessary, as usual, to allow a "void" constant state $(\rho, e, p = 0)$, corresponding to a region in which there is no gas. (See [5], and Theorem 3.5, below.)

We first use the results of Chapter 2 to investigate shocks. Throughout this chapter we assume that we are given arbitrary fixed initial data (U_ℓ, U_r). We define

$$S_1 = \left\{U \middle| U_\ell \xrightarrow{1\text{-shock}} U\right\} .$$

$$S_3 = \left\{U \middle| U \xrightarrow{3\text{-shock}} U_r\right\} .$$

Recall $U_\ell = (\tau_\ell, p_\ell, u_\ell)$, $U_r = (\tau_r, p_r, u_r)$. Let $H^\ell, Q^\ell, \theta^\ell, \theta_\ell, L^\ell_\theta, (\tau^\ell, p^\ell), \mathcal{H}^\ell_a$ and a^ℓ be the $H, Q, \theta, \theta_0, L_\theta, (\tau^0, p^0), \mathcal{H}_a$ and a^0 of Chapter 2 referred to the center (τ_ℓ, p_ℓ) instead of (τ_0, p_0); thus, for example, H^ℓ is the Hugoniot curve with center (τ_ℓ, p_ℓ), $\theta^\ell(\tau, p) = (p - p_\ell)/(\tau_\ell - \tau)$ for $\tau \neq \tau_\ell$, and $a^\ell(\theta) = \sqrt{(\tau_\ell - \tau^\ell(\theta))(p^\ell(\theta) - p_\ell)}$ $(\theta \in [\theta_\ell, +\infty))$. Define $H^r, Q^r, \theta^r, \theta_r, L^r_\theta$, (τ^r, p^r), $\mathcal{H}^r_a$ and a^r similarly relative to (τ_r, p_r). Then we have

Proposition 3.1. $S_1 = \left\{U_1(\theta) \middle| \theta > \theta_\ell\right\}$ where $U_1(\theta) = (\tau^\ell(\theta), p^\ell(\theta), u^\ell(\theta))$ $(\theta \geq \theta_\ell)$, and $u^\ell(\theta) = u_\ell - a^\ell(\theta)$ $(\theta \geq \theta_\ell)$. Thus (τ^ℓ, p^ℓ) is C^1 on $[\theta_\ell, +\infty)$ with $(\tau^\ell(\theta_\ell), p^\ell(\theta_\ell)) = (\tau_\ell, p_\ell)$, and

$$(\tau^\ell(\theta), p^\ell(\theta)) \in H^\ell \cap Q^\ell \cap L^\ell_\theta \qquad \text{for } \theta > \theta_\ell . \tag{3.1}$$

Moreover, (p^ℓ, u^ℓ) is continuous and injective on $[\theta_\ell, +\infty)$, $(p^\ell(\theta_\ell), u^\ell(\theta_\ell)) = (p_\ell, u_\ell)$, $\lim_{\theta \to +\infty} p^\ell(\theta) = +\infty$, and $u^\ell(\theta)$ is not bounded below as $\theta \to +\infty$.

Proof: Props. 2.14 and 2.15 imply the existence of a $c\text{-}\theta\text{-}p$ (τ^ℓ, p^ℓ) on $[\theta_\ell, +\infty)$. Thus by Def. 2.10, (τ^ℓ, p^ℓ) is C^1, $(\tau^\ell(\theta_\ell), p^\ell(\theta_\ell)) = (\tau_\ell, p_\ell)$, and

(3.1) holds. By Lemma 2.21, $\lim_{\theta \to +\infty} p^\ell(\theta) = +\infty$; by Lemma 2.22, $u^\ell(\theta)$ is not bounded below as $\theta \to +\infty$. Also u^ℓ is clearly continuous; hence so is (p^ℓ, u^ℓ).

Let us prove

$$S_1 = \left\{ U_1(\theta) \,\middle|\, \theta > \theta_\ell \right\} .$$

Let $U \in S_1$. Then by Prop. 1.2, $u < u_\ell$ and $(\tau, p) \in \mathcal{H}^\ell_{(u_\ell - u)^2} \cap H^\ell$. Thus $(\tau, p) \in H^\ell \cap Q^\ell$, so by Prop. 2.16, $(\tau, p) = (\tau^\ell(\theta), p^\ell(\theta))$ for some $\theta > \theta_\ell$. Then $a^\ell(\theta) = u_\ell - u$, so $u = u^\ell(\theta)$. Hence $U = U_1(\theta)$.

Next, let $\theta > \theta_\ell$. Then $u^\ell(\theta) = u_\ell - a^\ell(\theta) < u_\ell$, and by (3.1), $(\tau^\ell(\theta), p^\ell(\theta)) \in \mathcal{H}^\ell_{a^\ell(\theta)^2} \cap H^\ell = \mathcal{H}^\ell_{(u_\ell - u^\ell(\theta))^2} \cap H^\ell$, so $U_1(\theta) = (\tau^\ell(\theta), p^\ell(\theta), u^\ell(\theta)) \in S_1$ by Prop. 1.2.

Finally, we show (p^ℓ, u^ℓ) is injective. Let $\theta_1, \theta_2 \geq \theta_\ell$ and suppose $(p^\ell(\theta_1), u^\ell(\theta_1)) = (p^\ell(\theta_2), u^\ell(\theta_2))$. Then $p^\ell(\theta_1) = p^\ell(\theta_2)$ and $a^\ell(\theta_1) = a^\ell(\theta_2)$. If $p^\ell(\theta_1) = p_\ell$, then $\theta_1 = \theta_\ell = \theta_2$ by (3.1). If $p^\ell(\theta_1) \neq p_\ell$, then (3.1) gives $p^\ell(\theta_1) > p_\ell$ and we conclude $\tau^\ell(\theta_1) = \tau^\ell(\theta_2)$. Hence $(\tau^\ell(\theta_1), p^\ell(\theta_1)) = (\tau^\ell(\theta_2), p^\ell(\theta_2))$, so Def. 2.10(b) gives $\theta_1 = \theta^\ell(\tau^\ell(\theta_1), p^\ell(\theta_1)) = \theta^\ell(\tau^\ell(\theta_2), p^\ell(\theta_2)) = \theta_2$. ∎

Similarly we have

Proposition 3.2. $S_3 = \left\{ U_3(\theta) \,\middle|\, \theta > \theta_r \right\}$ where $U_3(\theta) = (\tau^r(\theta), p^r(\theta), u^r(\theta))$, $u^r(\theta) = u_r + a^r(\theta)$ $(\theta \geq \theta_r)$. Thus (τ^r, p^r) is C^1 on $[\theta_r, +\infty)$ with $(\tau^r(\theta_r), p^r(\theta_r)) = (\tau_r, p_r)$, and

$$(\tau^r(\theta), p^r(\theta)) \in H^r \cap Q^r \cap L^r_\theta \qquad \text{for } \theta > \theta_r . \tag{3.2}$$

Moreover, (p^r, u^r) is continuous and injective on $[\theta_r, +\infty)$, $(p^r(\theta_r), u^r(\theta_r)) = (p_r, u_r)$, $\lim_{\theta \to +\infty} p^r(\theta) = +\infty$, and $u^r(\theta)$ is not bounded above as $\theta \to +\infty$.

We now study the rarefaction wave curves. Let

$$W_1 = \left\{ U \,|\, U_\ell \xrightarrow{\text{1-wave}} U \right\},$$

$$W_3 = \left\{ U \,|\, U \xrightarrow{\text{3-wave}} U_r \right\}.$$

Proposition 3.3. Along W_1, U is a continuous function $U_1(p) = (\tau_1(p), p, u_1(p))$ of $p \in (0, p_\ell]$; u_1 is continuous and decreasing and maps $(0, p_\ell]$ onto $[u_\ell, u_{max})$ for some $u_{max} \in (u_\ell, +\infty]$.

Proof: Suppose v_1 and v_2 are independent 1-Riemann invariants. Then by [3], $U \in W_1$ iff

$$v_1(U) = v_1(U_\ell) \tag{3.3}$$

$$v_2(U) = v_2(U_\ell)$$

$$u - c \geq u_\ell - c_\ell \ .$$

Moreover, using equations equivalent to (1.1) with unknowns ρ, u, S, we see ([3]) that we may take

$$v_1(U) = S \tag{3.4}$$

$$v_2(U) = u + h(\rho, S) \ ,$$

where

$$h(\rho, S) = \int_{\rho_\ell}^{\rho} \bar{\rho}^{-1} \sqrt{f_\rho(\bar{\rho}, S)}\ d\bar{\rho} \ ,$$

so that

$$\frac{\partial}{\partial \rho} h(\rho, S) = \frac{c}{\rho} \ . \tag{3.5}$$

(Here $f(\rho, S) \equiv g(\rho^{-1}, S)$; see (1.13).) Then by (3.3) and (3.4), $U \in W_1$ iff

$$S = S_\ell \tag{3.6}$$

$$u + h(\rho, S_\ell) = u_\ell + h(\rho_\ell, S_\ell) \tag{3.7}$$

(3.8) $$u - c \geq u_\ell - c_\ell \ .$$

By (3.7) and (3.5), u is a decreasing function of ρ along W_1. Its domain is restricted by (3.8) as follows. Since $g_{\tau\tau}(\tau, S) = \rho^3[\rho f_{\rho\rho}(\rho, S) + 2f_\rho(\rho, S)]$, (1.7) implies

(3.9) $$\rho f_{\rho\rho} + 2f_\rho > 0 \ .$$

Then along W_1 we have

$$\frac{d}{d\rho}(u - c) = -\frac{\partial}{\partial\rho} h(\rho, S_\ell) - \frac{\partial}{\partial\rho}\sqrt{f_\rho(\rho, S_\ell)} = -\frac{1}{2\rho c}(2f_\rho + \rho f_{\rho\rho}) < 0 \ .$$

So assuming (3.6) and (3.7), (3.8) is equivalent to $\rho \in (0, \rho_\ell]$; thus along W_1, u is a continuous, decreasing function of ρ, defined for $\rho \in (0, \rho_\ell]$. Since $p = g(\rho^{-1}, S_\ell)$ along W_1, (1,2), (1.3) and (1.5) imply that along W_1, p is a continuous, increasing function of ρ mapping $(0, \rho_\ell]$ onto $(0, p_\ell]$. It follows that along W_1, τ is a continuous, decreasing function $\tau_1(p)$ of $p \in (0, p_\ell]$, and u is a continuous, decreasing function $u_1(p)$ of $p \in (0, p_\ell]$, with range $[u_\ell, u_{max})$ for some $u_{max} \in (u_\ell, +\infty]$. Thus along W_1, U is a continuous function $U_1(p) = (\tau_1(p), p, u_1(p))$ of $p \in (0, p_\ell]$ as desired. ■

Similarly we have

Proposition 3.4. Along W_3, U is a continuous function $U_3(p) = (\tau_3(p), p, u_3(p))$ of $p \in (0, p_r]$; u_3 is continuous and increasing and maps $(0, p_r]$ onto $(u_{min}, u_r]$ for some $u_{min} \in [-\infty, u_r)$.

Theorem 3.5. If $u_{min} < u_{max}$, then the R.P. (with initial data (U_ℓ, U_r)) has a solution. If $u_{min} \geq u_{max}$, then the R.P. has no solution; however, if we allow $\rho, p, e = 0$, then the R.P. has a solution.

Proof: Let $C_1 = S_1 \cup W_1$, $C_3 = S_3 \cup W_3$. Thus

$$C_1 = \{U \mid U_\ell \overset{1}{\text{---}} U\} \ ,$$

$$C_3 = \{U \mid U \overset{3}{\text{---}} U_r\} \ .$$

Case 1. $u_{max} > u_{min}$. Our plan is to find $U_1 \in C_1$ and $U_3 \in C_3$ such that $p_1 = p_3$ and $u_1 = u_3$. Then we will have a solution,

$$U_\ell \overset{1}{\text{---}} U_1 \overset{\text{contact}}{\text{---}} U_3 \overset{3}{\text{---}} U_r \ .$$

(Cf. (3.22), below.) We shall do this by looking at the projections of C_1 and C_3 onto the (p,u)-plane and proving that they intersect. By Props. 3.3 and 3.4, $\lim_{p \to 0^+} u_1(p) = u_{max}$ and $\lim_{p \to 0^+} u_3(p) = u_{min}$. Hence $\exists \delta > 0$ with $\delta < p_\ell, p_r$ such that

$$u_1(\delta) > u_3(\delta) \ ; \tag{3.10}$$

we also have

$$u_1(\delta) > u_\ell \ , \quad u_3(\delta) < u_r \ . \tag{3.11}$$

(See Fig. 13.) Let

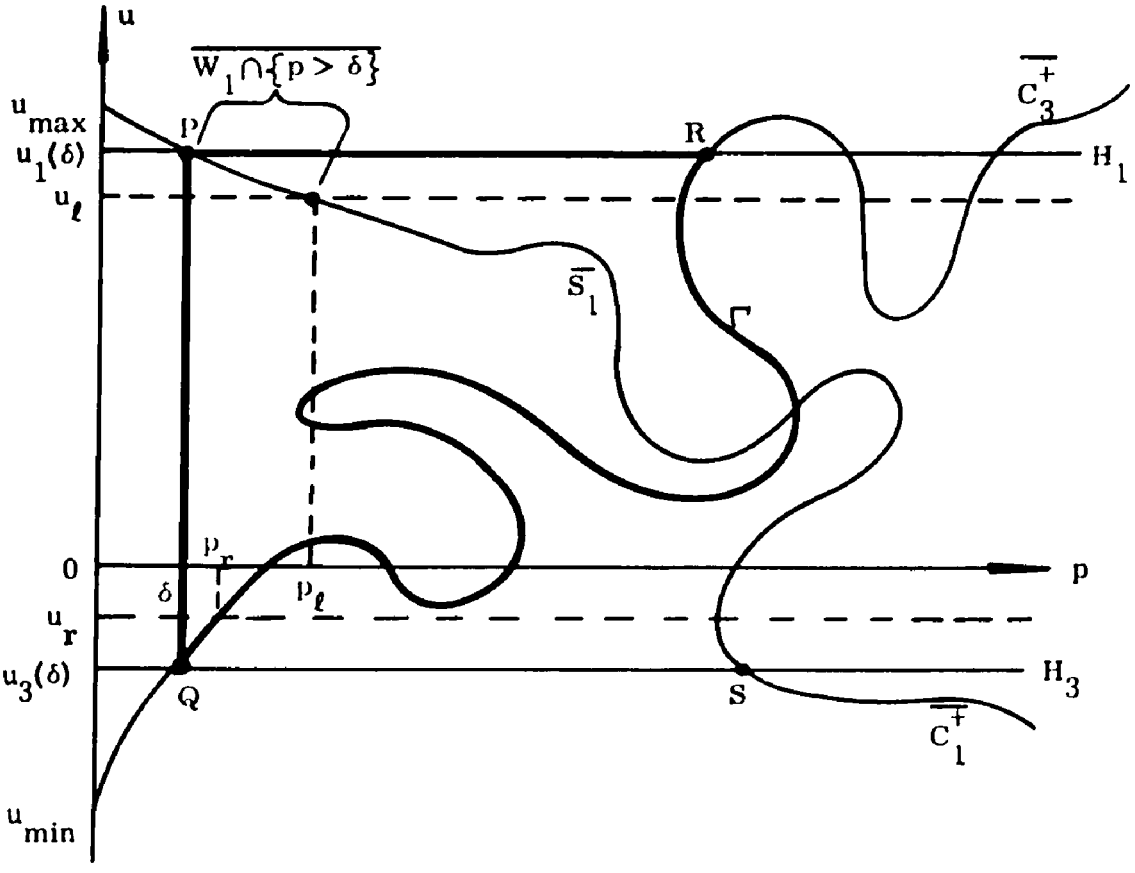


Fig. 13: $\overline{C_1}$ and $\overline{C_3}$ must intersect.

$$C_1^+ = S_1 \cup \left[W_1 \cap \{p > \delta\}\right] ,$$

$$C_3^+ = S_3 \cup \left[W_3 \cap \{p > \delta\}\right] .$$

Then we have, from Props. 3.1 - 3.4,

(3.12) $$p \in (\delta, p_\ell] \Longrightarrow u_1(p) < u_1(\delta) ,$$

$$p \in (\delta, p_r] \Longrightarrow u_3(p) > u_3(\delta) ;$$

(3.13) $$\theta \geq \theta_\ell \Longrightarrow p^\ell(\theta) \geq p_\ell \quad \text{and} \quad u^\ell(\theta) \leq u_\ell ,$$

$$\theta \geq \theta_r \Longrightarrow p^r(\theta) \geq p_r \quad \text{and} \quad u^r(\theta) \geq u_r .$$

Let $\overline{C_1^+}$ (resp. $\overline{C_3^+}$) be the projection onto the (p, u)-plane of C_1^+ (resp. C_3^+). Then from (3.11) - (3.13) we get

(3.14) $$(p, u) \in \overline{C_1^+} \Longrightarrow p > \delta \quad \text{and} \quad u < u_1(\delta) ,$$

$$(p, u) \in \overline{C_3^+} \Longrightarrow p > \delta \quad \text{and} \quad u > u_3(\delta) .$$

By adjoining the curve $W_1 \cap \{p > \delta\}$ (resp. $W_3 \cap \{p > \delta\}$) to the curve S_1 (resp. S_3), we can, in an obvious fashion, extend the domain of the function U_1 (resp. U_3) to a larger interval $(\theta_1, +\infty)$, $\theta_1 < \theta_\ell$ (resp. $(\theta_3, +\infty)$, $\theta_3 < \theta_r$), so that we have

(3.15) $$C_1^+ = U_1((\theta_1, +\infty)) , \qquad C_3^+ = U_3((\theta_3, +\infty)) ,$$

$$\lim_{\theta \to \theta_1^+} (p^\ell(\theta), u^\ell(\theta)) = (\delta, u_1(\delta)) ,$$

(3.16) $$\lim_{\theta \to \theta_3^+} (p^r(\theta), u^r(\theta)) = (\delta, u_3(\delta)) ,$$

and (p^ℓ, u^ℓ), (p^r, u^r) are continuous and injective. Let $P = (\delta, u_1(\delta))$, $Q = (\delta, u_3(\delta))$; Let H_1 (resp. H_3) be the horizontal line through P (resp. Q). (See Fig. 13.) By (3.10) and (3.16), for $\theta > \theta_3$, sufficiently near θ_3, $(p^r(\theta), u^r(\theta))$ lies below H_1. But since $u^r(\theta)$ is not bounded above as $\theta \to +\infty$, $\exists \theta > \theta_r$ such that $(p^r(\theta), u^r(\theta))$ lies above H_1. Hence $\exists \theta \in (\theta_3, +\infty)$ such that $(p^r(\theta), u^r(\theta)) \in H_1$; let θ_3^* be the smallest such θ. Let $R = (p^r(\theta_3^*), u^r(\theta_3^*))$. By (3.14) and (3.15),

$$p^r(\theta_3^*) > \delta \ . \tag{3.17}$$

Similarly, $\exists \theta_1^* > \theta_1$ such that $(p^\ell(\theta_1^*), u^\ell(\theta_1^*)) \in H_3$. Let $S = (p^\ell(\theta_1^*), u^\ell(\theta_1^*))$. We have

$$p^\ell(\theta_1^*) > \delta \ . \tag{3.18}$$

Let $\overline{PR}$ (resp. $\overline{PQ}$) be the line-segment joining P and R (resp. P and Q), and let $\overline{QR}$ be the portion of C_3^+ strictly between Q ($\theta = \theta_3$) and R ($\theta = \theta_3^*$). Let $\Gamma = \overline{PR} \cup \overline{PQ} \cup \overline{QR}$. By (3.14), (3.17) and the minimality of θ_3^*,

$$\begin{aligned} &(p,u) \in \overline{PR} \Longrightarrow p \geq \delta \quad \text{and} \quad u = u_1(\delta) \ ; \\ &(p,u) \in \overline{PQ} \Longrightarrow p = \delta \quad \text{and} \quad u_3(\delta) \leq u \leq u_1(\delta) \ ; \\ &(p,u) \in \overline{QR} \Longrightarrow p > \delta \quad \text{and} \quad u_3(\delta) < u < u_1(\delta) \ . \end{aligned} \tag{3.19}$$

By (3.19), (3.16) and the injectivity and continuity of (p^r, u^r), Γ is a Jordan curve. By (3.18), $S \notin \Gamma$. Let D be an open disk centered at P such that $D \subset \{p > 0\}$. Let

$$D_1 = D \cap \{p < \delta \text{ or } u > u_1(\delta)\} \ ,$$

$$D_2 = D \cap \{p > \delta \text{ and } u < u_1(\delta)\} \ .$$

(See Fig. 14.) By (3.19), $\Gamma \subset [\delta, +\infty) \times [u_3(\delta), u_1(\delta)]$. Hence $D_1 \cap \Gamma = \phi$, and any $T \in D_1$ can be joined to S by a path γ with $\gamma \cap \Gamma = \phi$. But $S \in \mathrm{Ext}\Gamma$, since the ray $\mathcal{R} = \{p^\ell(\theta_1^*)\} \times (-\infty, u_3(\delta)]$ is unbounded, $S \in \mathcal{R}$, and $\mathcal{R} \cap \Gamma = \phi$;

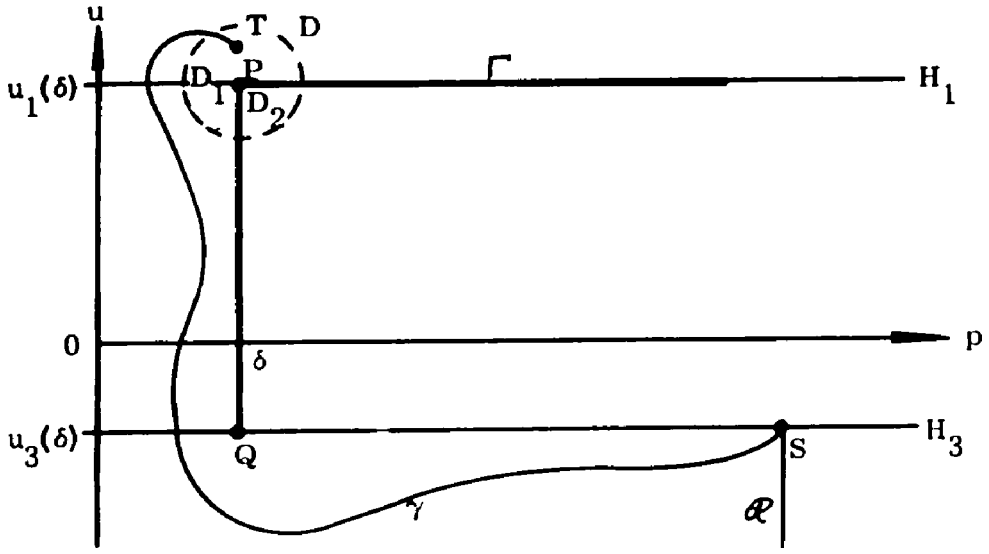


Fig. 14: $D_1 \subset \text{Ext}\,\Gamma$.

hence $T \in \text{Ext}\Gamma$. Thus $D_1 \subset \text{Ext}\Gamma$. Now suppose $(\tau, p, u) \in S_3$. Then $p = p^r(\theta)$ for some $\theta > \theta_r$, whence $p \geq p_r$ by (3.13). Thus

$$(\tau, p, u) \in S_3 \Longrightarrow p \geq p_r \ . \tag{3.20}$$

Next (see (3.10)), pick any $u_0 \in (u_3(\delta), u_1(\delta))$. Then by Prop. 3.4, $\exists p_0 \in (\delta, p_r]$ such that $\forall p \in (\delta, p_0)$, $u_3(p) \leq u_0$. Now suppose $(\tau, p, u) \in W_3 \cap \{p > \delta\}$. Then $u = u_3(p)$ by Prop. 3.4, so by the preceding, $p < p_0$ implies $u \leq u_0$. Thus we have

$$(\tau, p, u) \in W_3 \cap \{p > \delta\} \Longrightarrow [p \geq p_0 \text{ or } u \leq u_0] \ . \tag{3.21}$$

Combining (3.20) and (3.21), we conclude

$$(p, u) \in \overline{QR} \Longrightarrow [p \geq p_0 \text{ or } u \leq u_0] \ .$$

Hence we may choose D so small that $D \cap \overline{QR} = \phi$. Then $D_2 \cap \Gamma = \phi$. Now $P \in \Gamma = \partial(\text{Int}\Gamma)$, so $\exists$ a sequence $\{P_n\}$ from $\text{Int}\,\Gamma$ such that $P_n \longrightarrow P$. Then

for some n, $P_n \in D$. But also $P_n \notin \Gamma$, so $P_n \in D_1 \cup D_2$; hence $P_n \in D_2$ since $D_1 \subset \text{Ext}\Gamma$. Hence $D_2 \subset \text{Int}\,\Gamma$. But by Prop. 3.3, for all $\theta > \theta_1$ sufficiently near θ_1, $(p^\ell(\theta), u^\ell(\theta)) \in D_2$ (since u_1 is decreasing), so $(p^\ell(\theta), u^\ell(\theta)) \in \text{Int}\,\Gamma$; whereas $(p^\ell(\theta_1^*), u^\ell(\theta_1^*)) = S \in \text{Ext}\,\Gamma$. Hence $\exists \bar{\theta}_1 \in (\theta_1, \theta_1^*)$ such that $(p^\ell(\bar{\theta}_1), u^\ell(\bar{\theta}_1)) \in \Gamma$. Then by (3.14), (3.15) and (3.19), $(p^\ell(\bar{\theta}_1), u^\ell(\bar{\theta}_1)) \in \overline{QR}$, so $\exists \bar{\theta}_3 \in (\theta_3, \theta_3^*)$ such that $(p^r(\bar{\theta}_3), u^r(\bar{\theta}_3)) = (p^\ell(\bar{\theta}_1), u^\ell(\bar{\theta}_1))$. Then the R.P. has the solution,

$$U_\ell \overset{1}{\text{---}} U_1(\bar{\theta}_1) \overset{\text{contact}}{\text{---}} U_3(\bar{\theta}_3) \overset{3}{\text{---}} U_r \,. \tag{3.22}$$

Case 2. $u_{max} \leq u_{min}$. If the problem had a solution, then there would exist states U_1 and U_3 such that

$$U_\ell \overset{1}{\text{---}} U_1 \overset{\text{contact}}{\text{---}} U_3 \overset{3}{\text{---}} U_r \,.$$

These relations imply $U_1 \in C_1$, $u_1 = u_3$ and $U_3 \in C_3$; then, using Props. 3.3 and 3.4 and (3.14), also $u_1 < u_{max}$ and $u_3 > u_{min}$, a contradiction. Hence there is no solution. But if we allow $\rho, p, e = 0$ (corresponding to a vacuum), then we can connect U_ℓ and U_r each by a (complete) rarefaction wave to a constant "void" state ($\rho, e = 0$, u undefined). (See [5], and Fig. 15.) To see that

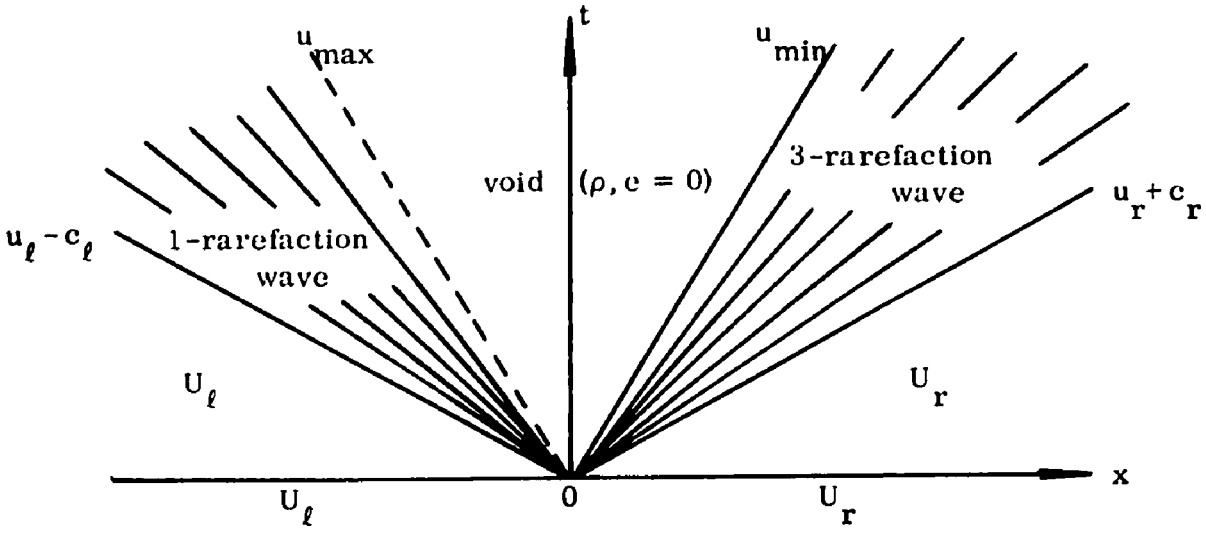


Fig. 15: The case of the vacuum.

these two waves do not overlap, we note that $u - c < u_{max}$ on W_1, and $u + c > u_{min}$ on W_3. Thus our "solution" is

$$U_\ell \xrightarrow{1\text{-wave}} \text{void} \xrightarrow{3\text{-wave}} U_r \ . \qquad \blacksquare$$

CHAPTER 4

UNIQUENESS AND NONUNIQUENESS OF SOLUTIONS

In this section we show that the condition MEDIUM is necessary and sufficient for uniqueness of solutions of the R. P. (See Th. 4.5 and Th. 4.13.) We first prove the sufficiency. To obtain uniqueness, we replace Props. 3.1 and 3.2 by Props. 4.3 and 4.4, which assert that when MEDIUM holds, the curves S_1 and S_3 have a simpler form: along $\overline{S_1}$ (the projection of S_1 onto the (p, u)-plane), u is a decreasing function of p; along $\overline{S_3}$, u is an increasing function of p. Then the curves $\overline{C_1}$ and $\overline{C_3}$ can intersect in at most one point. (Cf. Figs. 13 and 17.)

Lemma 4.1. Suppose e satisfies WEAK. Then along H, $\tau = V(p)$, where V is a C^1 function defined on $(0, +\infty)$.

Proof: For any $(\tau, p) \in H$,

$$H_\tau(\tau, p) = \frac{\partial}{\partial \tau} e(\tau, p) + \frac{p + p_0}{2} \quad ,$$

by (2.2). But by WEAK,

$$\frac{\partial}{\partial \tau} e(\tau, p) \geq -\frac{p}{2} \quad .$$

Hence

$$H_\tau(\tau, p) \geq \frac{p_0}{2} > 0 \qquad \forall (\tau, p) \in H. \tag{4.1}$$

Let $p > 0$. As before, using (1.2), (1.10) and (1.20), we have

$$\lim_{\tau \to 0^+} H(\tau, p) < 0 \quad , \qquad \lim_{\tau \to +\infty} H(\tau, p) = +\infty \quad ;$$

hence $H(\tau, p) = 0$ for some τ, and by (4.1) there can be only one such τ, which we denote by $V(p)$. V is C^1 by (4.1) and the implicit function theorem. ■

When WEAK holds, we define

(4.2) $$a_0(p) = \sqrt{(\tau_0 - V(p))(p - p_0)} \qquad (p > 0) \ .$$

Lemma 4.2. Suppose e satisfies MEDIUM. Then

$$\frac{d}{dp}\, a_0(p) > 0 \qquad (p > p_0) \ .$$

Proof: Let H^0 denote the Hugoniot curve with center (τ_0, p_0). Since WEAK holds, we have $\tau = V^0(p)$ along H^0 by Lemma 4.1. As in (4.2), we define a_0 by

$$a_0(p) = \sqrt{(\tau_0 - V^0(p))(p - p_0)} \qquad (p > 0) \ .$$

Then

$$\frac{d}{dp}\, V^0(p) = - \frac{H_p(V^0(p), p)}{H_\tau(V^0(p), p)} \ .$$

We write $\tau = V^0(p)$; then

$$\frac{d}{dp}\, V^0(p) = - \frac{\frac{\partial}{\partial p}\, e(\tau, p) - \frac{\tau_0 - \tau}{2}}{(-g_\tau(\tau, p))\, \frac{\partial}{\partial p}\, e(\tau, p) - \frac{p - p_0}{2}}$$

by (2.2) and (2.3). Hence

$$\frac{d}{dp}\, a_0(p) = \frac{1}{2a_0(p)} \left[\tau_0 - V^0(p) - (p - p_0)\, \frac{d}{dp}\, V^0(p)\right]$$

$$= \frac{1}{2a_0(p)} \left[(\tau_0 - \tau) + (p - p_0)\, \frac{\frac{\partial}{\partial p}\, e(\tau, p) - \frac{\tau_0 - \tau}{2}}{(-g_\tau)\, \frac{\partial}{\partial p}\, e(\tau, p) - \frac{p - p_0}{2}}\right] \ .$$

But $(-g_\tau)\, \frac{\partial}{\partial p}\, e(\tau, p) - \frac{p - p_0}{2} > 0$ by (4.1) and (2.2), so

(4.3) $\frac{d}{dp}\, a_0(p) > 0$

$$\Longleftrightarrow (p - p_0)\left[-\frac{\partial}{\partial p}\, e(\tau, p) + \frac{\tau_0 - \tau}{2}\right] < (\tau_0 - \tau)\left[(-g_\tau)\frac{\partial}{\partial p}\, e(\tau, p) - \frac{p - p_0}{2}\right]$$

$$\Longleftrightarrow \frac{\partial}{\partial p}\, e(\tau, p) > \frac{(p - p_0)(\tau_0 - \tau)}{(-g_\tau)(\tau_0 - \tau) + (p - p_0)} \quad .$$

We also have

(4.4) $$\frac{d}{dp}\, a_0(p) < 0 \Longleftrightarrow \frac{\partial}{\partial p}\, e(\tau, p) < \frac{(p - p_0)(\tau_0 - \tau)}{(-g_\tau)(\tau_0 - \tau) + (p - p_0)} \quad ,$$

which we shall use later.

Let $(\tau_1, p_1) \in H^0 \cap Q$ (see (2.1)); then $\tau_1 = V^0(p_1)$. We must prove $\frac{d}{dp}\, a_0(p)\Big|_{p = p_1} > 0$. By (4.3), it suffices to prove

(4.5) $$\frac{\partial}{\partial p}\, e(\tau_1, p_1) > \frac{(p_1 - p_0)(\tau_0 - \tau_1)}{(-g_\tau(\tau_1, p_1))(\tau_0 - \tau_1) + (p_1 - p_0)} \quad .$$

Let H^1 be the Hugoniot curve with center (τ_1, p_1), and let $\tau = V^1(p)$ along H^1 as in Lemma 4.1. Then $(\tau_0, p_0) \in H^1$, so $\tau_0 = V^1(p_0)$. Since $\tau_0 > \tau_1$ and $p_0 < p_1$, Lemma 2.18 gives $\tau_0 = V^1(p_0) \in [\tau_1, \tau^1_{max})$, where $\tau^1_{max} = \tau_1 + \left[2e(\tau_1, p_1)/p_1\right]$. Hence

(4.6) $$0 < \tau_0 - \tau_1 < \frac{2e(\tau_1, p_1)}{p_1} \quad .$$

Also,

(4.7) $$0 < p_1 - p_0 < p_1 \quad .$$

Now observe that

$$(4.8)\qquad \frac{(p_1 - p_0)(\tau_0 - \tau_1)}{(-g_\tau(\tau_1, p_1))(\tau_0 - \tau_1) + (p_1 - p_0)} = f(\tau_0 - \tau_1, p_1 - p_0) \ ,$$

where f is the function

$$f(x, y) = \frac{xy}{ax + by} \qquad (x, y > 0) \ ,$$

where $a = -g_\tau(\tau_1, p_1) > 0$, $b = 1 > 0$. But f satisfies $\frac{\partial f}{\partial x} > 0$, $\frac{\partial f}{\partial y} > 0$; hence from (4.6) and (4.7) we get

$$f(\tau_0 - \tau_1, p_1 - p_0) < f\left(\frac{2e(\tau_1, p_1)}{p_1}, p_1\right)$$

$$= \frac{2e(\tau_1, p_1)}{(-g_\tau(\tau_1, p_1))\ \dfrac{2e(\tau_1, p_1)}{p_1} + p_1} \leq \frac{\partial}{\partial p}\, e(\tau_1, p_1) \ ,$$

by MEDIUM and Prop. 1.5. This, together with (4.8), proves (4.5). ■

Proposition 4.3. Suppose e satisfies MEDIUM. Then along S_1, U is a continuous function $U_1(p) = (\tau_1(p), p, u_1(p))$ of $p > p_\ell$, and u_1 is a continuous, decreasing function mapping $(p_\ell, +\infty)$ onto $(-\infty, u_\ell)$.

Proof: Let H^ℓ be the Hugoniot curve with center (τ_ℓ, p_ℓ); let $\tau = V^\ell(p)$ along H^ℓ as in Lemma 4.1, and define $a_\ell(p) = \sqrt{(\tau_\ell - V^\ell(p))(p - p_\ell)}$ $(p > 0)$. Define $\tau_1 = V^\ell$ and $u_1(p) = u_\ell - a_\ell(p)$ $(p > p_\ell)$.

Let $p > p_\ell$. Then $(V^\ell(p), p) \in H^\ell$, $a_\ell(p) > 0$ and $(V^\ell(p), p) \in \mathcal{H}^\ell_{a_\ell(p)^2}$. (See Fig. 1.) So $u_1(p) < u_\ell$ and $(V^\ell(p), p) \in \mathcal{H}^\ell_{(u_\ell - u_1(p))^2}$. Then since $\tau_1(p) = V^\ell(p)$, Prop. 1.2 gives $U_1(p) \equiv (\tau_1(p), p, u_1(p)) \in S_1$. (Recall $S_1 = \left\{U \,\middle|\, U_\ell \overset{1\text{-shock}}{\longrightarrow} U\right\}$.)

Now let $U \in S_1$. Then by Prop. 1.2, $u < u_\ell$ and $(\tau, p) \in H^\ell \cap \mathcal{H}^\ell_{(u_\ell - u)^2}$. By Lemma 4.1, $\tau = V^\ell(p)$. By (4.8), $u_\ell - u = a_\ell(p)$. Hence $U = (\tau_1(p), p, u_1(p)) = U_1(p)$. Thus we have shown $S_1 = \left\{U_1(p) \,\middle|\, p > p_\ell\right\}$. u_1 is decreasing by Lemma 4.2; it maps $(p_\ell, +\infty)$ onto $(-\infty, u_\ell)$ by Lemma 2.22. ■

Similarly we have

Proposition 4.4. Suppose e satisfies MEDIUM. Then along S_3, U is a continuous function $U_3(p) = (\tau_3(p), p, u_3(p))$ of $p > p_r$, and u_3 is a continuous, increasing function mapping $(p_r, +\infty)$ onto $(u_r, +\infty)$.

Theorem 4.5. Suppose e satisfies MEDIUM. Then the R.P. always has a unique solution (in the sense of Th. 3.5).

Proof: We are given arbitrary initial data (U_ℓ, U_r). As in Th. 3.5, let $C_1 = S_1 \cup W_1$, $C_3 = S_3 \cup W_3$. Combining Props. 3.3 and 4.3, we have

(4.9) Along C_1, U is a continuous function $U_1(p) = (\tau_1(p), p, u_1(p))$ of $p > 0$; u_1 is a continuous, decreasing function mapping $(0, +\infty)$ onto $(-\infty, u_{max})$, for some $u_{max} \in (u_\ell, +\infty]$.

(See Fig. 16.) Similarly, Props. 3.4 and 4.4 give

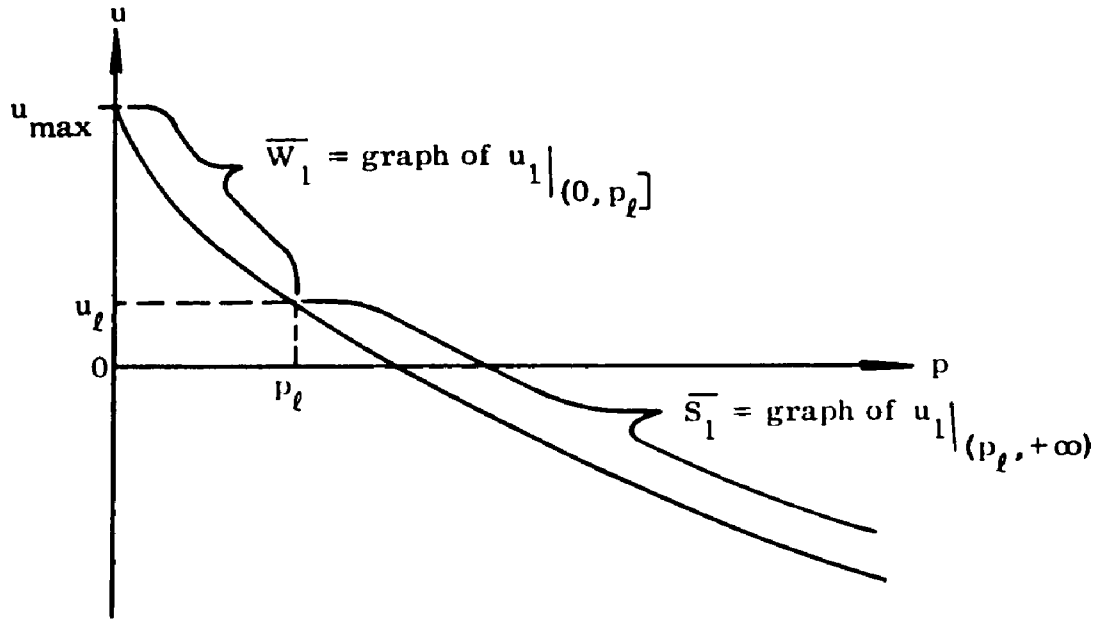


Fig. 16: u is a decreasing function of p along $\overline{C_1}$.

(4.10) Along C_3, U is a continuous function $U_3(p) = (\tau_3(p), p, u_3(p))$ of $p > 0$; u_3 is a continuous, increasing function mapping $(0, +\infty)$ onto $(u_{min}, +\infty)$, for some $u_{min} \in [-\infty, u_r)$.

Case 1. $u_{max} > u_{min}$. Let $\overline{C_1}$ (resp. $\overline{C_3}$) be the projection of C_1 (resp. C_3) onto the (p,u)-plane. Then $\overline{C_1}$ is the graph of u_1, and $\overline{C_3}$ is the graph of u_3. A solution of the R.P. is uniquely determined by states $\overline{U_1}$, $\overline{U_3}$ such that

$$U_\ell \overset{1}{-\!\!-} \overline{U_1} \overset{\text{contact}}{\text{---}} \overline{U_3} \overset{3}{-\!\!-} U_r \ .$$

For such states $\overline{U_1}$, $\overline{U_3}$ we must have $\overline{U_1} \in C_1$, $\overline{U_3} \in C_3$ and $(\overline{p_1}, \overline{u_1}) = (\overline{p_3}, \overline{u_3})$. In this case, denote $\overline{p_1}$ and $\overline{p_3}$ by $\bar{p}$; then by (4.9) and (4.10), $\bar{p}$ uniquely determines $\overline{U_1}$ and $\overline{U_3}$, and $u_1(\bar{p}) = u_3(\bar{p})$. But u_1 is decreasing and u_3 is increasing, so $u_1 - u_3$ is decreasing; hence $u_1(\bar{p}) = u_3(\bar{p})$ can hold for at most one $\bar{p}$. (See Fig. 17.) Thus there can be at most one solution of the R.P.

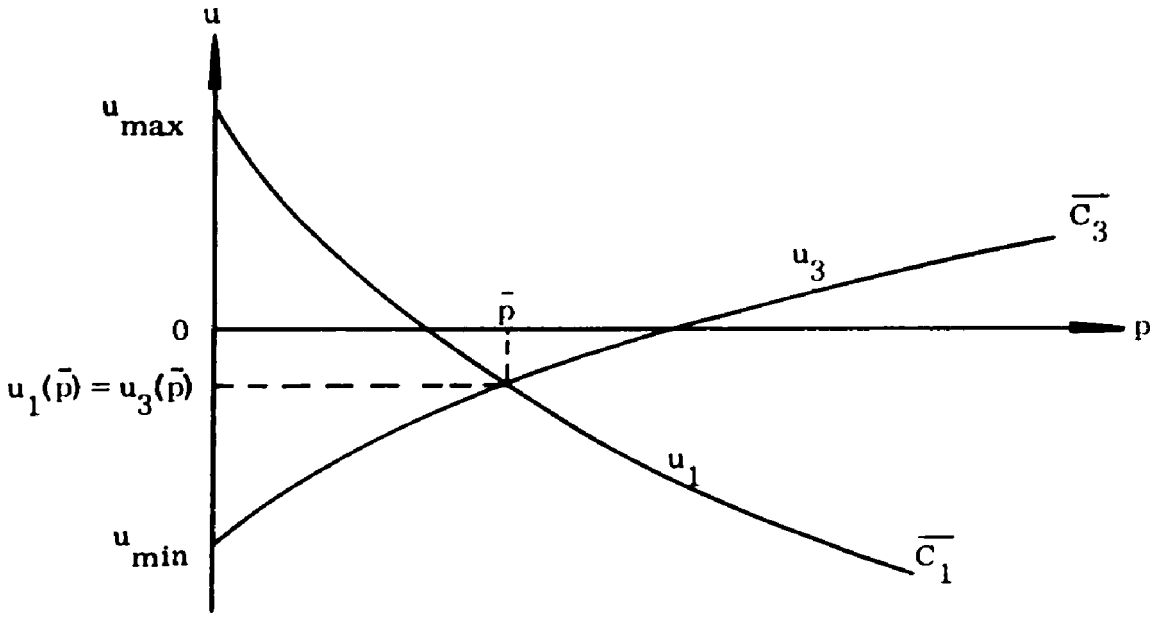


Fig. 17: The strict monotonicity of $\overline{C_1}$ and $\overline{C_3}$ assures uniqueness.

Here it is necessary to stipulate that in the case $u_{max} > u_{min}$, we do not admit the void state ($\rho, e = 0$, u undefined). (See Th. 3.5, Case 2.)

Case 2. $u_{max} \leq u_{min}$. Clearly there is only one way to connect U_ℓ (left) to the void state on the right by a complete 1-rarefaction wave; similarly for

$$\text{void} \xrightarrow{\text{3-wave}} U_r \ . \qquad \blacksquare$$

We turn now to nonuniqueness. We begin by showing two different ways in which a property of the Hugoniot curve allows one to construct initial data (U_ℓ, U_r) for which the R. P. has more than one solution.

Convention 4.6. H will always denote the Hugoniot curve with center (τ_0, p_0). Similarly, we shall always denote

$$Q = (0, \tau_0) \times (p_0, +\infty) \ ,$$

$$\theta = \frac{p - p_0}{\tau_0 - \tau} \qquad (\tau \neq \tau_0) \ ,$$

$\mathcal{H}_a$ = upper branch of hyperbola $(\tau_0 - \tau)(p - p_0) = a$,

$L_{\bar{\theta}}$ = line through (τ_0, p_0) with slope $-\bar{\theta}$ $(\bar{\theta} > 0)$.

(See, for example, Lemma 4.10, below.)

First, suppose we have two distinct points (τ_1, p_1), (τ_2, p_2) and $a > 0$ such that

$$(\tau_i, p_i) \in \mathcal{H}_{a_2} \cap H \tag{4.11}$$

for $i = 1, 2$. (See Fig. 18.) Let u_ℓ be arbitrary and let $U_\ell = (\tau_0, p_0, u_\ell)$, $U_r = (\tau_0, p_0, u_\ell - 2a)$. Then (4.11) and Props. 1.2 and 1.3 yield the following two solutions of the R. P. with data (U_ℓ, U_r):

$$U_\ell \xrightarrow{\text{1-shock}} (\tau_1, p_1, u_\ell - a) \xrightarrow{\text{3-shock}} U_r \ , \tag{4.12}$$

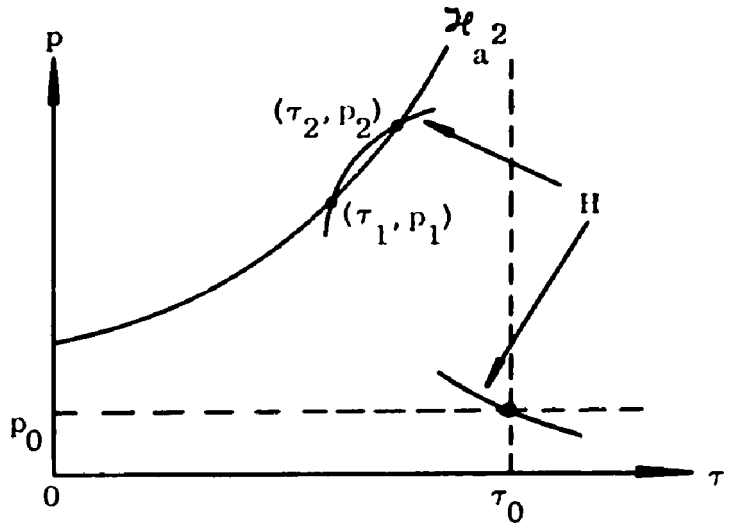


Fig. 18: The behavior of H that gives rise to type I nonuniqueness.

$$U_\ell \xrightarrow{1\text{-shock}} (\tau_2, p_2, u_\ell - a) \xrightarrow{3\text{-shock}} U_r \,. \tag{4.13}$$

Clearly, if we have n distinct points $(\tau_1, p_1), \ldots, (\tau_n, p_n)$ that satisfy (4.11) for $i = 1, \ldots, n$, then we obtain n distinct solutions of the R. P. with data (U_ℓ, U_r). Motivated by the above, we make

Definition 4.7. We say that type I nonuniqueness holds for (U_ℓ, U_r) if U_ℓ and U_r have the same thermodynamic variables (τ_0, p_0) and there exist distinct (τ_1, p_1) and (τ_2, p_2) that satisfy (4.11), where $2a = u_\ell - u_r$; then the R. P. with data (U_ℓ, U_r) has the two distinct solutions (4.12) and (4.13), which we call type I solutions. The (type I) nonuniqueness is n-fold if there exist n (type I) solutions.

Now suppose we have $\bar{p} > p_0$ and $0 < \tau_1 < \tau_2 < \tau_0$ such that

$$(\tau_1, \bar{p}), (\tau_2, \bar{p}) \in H. \tag{4.14}$$

(See Fig. 19.) Let a and b satisfy

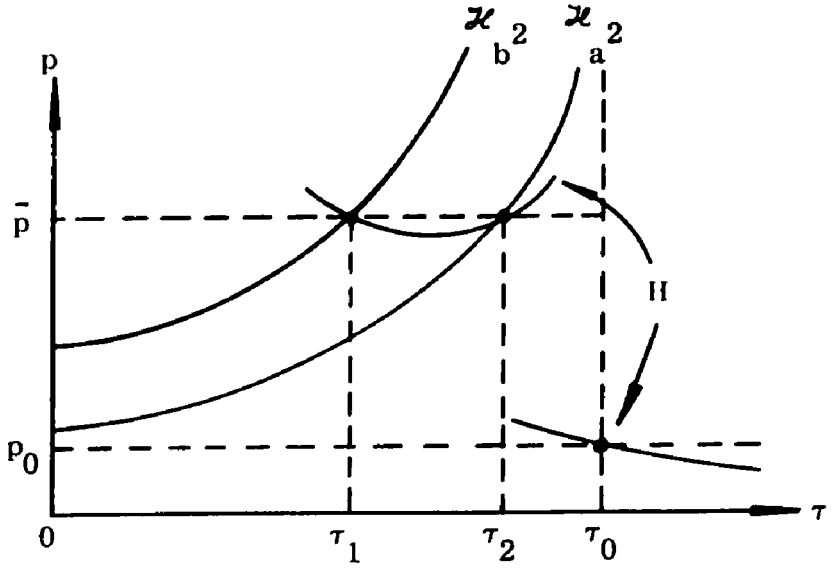


Fig. 19: The behavior of H that gives rise to type II nonuniqueness.

(4.15) $$(\tau_1, \bar{p}) \in \mathcal{H}_{b^2}, \qquad (\tau_2, \bar{p}) \in \mathcal{H}_{a^2}$$

(e.g., $a = \sqrt{(\tau_0 - \tau_2)(\bar{p} - p_0)}$); then $b > a > 0$. Let u_ℓ be arbitrary and define $U_\ell = (\tau_0, p_0, u_\ell)$, $U_r = (\tau_0, p_0, u_\ell - (a+b))$. Then by (4.14), (4.15), Props. 1.2 and 1.3, and the meaning of contact discontinuity, the R.P. with data (U_ℓ, U_r) has the following two solutions:

(4.16) $$U_\ell \xrightarrow{1\text{-shock}} (\tau_1, \bar{p}, u_\ell - b) \xrightarrow{\text{contact}} (\tau_2, \bar{p}, u_\ell - b) \xrightarrow{3\text{-shock}} U_r .$$

(4.17) $$U_\ell \xrightarrow{1\text{-shock}} (\tau_2, \bar{p}, u_\ell - a) \xrightarrow{\text{contact}} (\tau_1, \bar{p}, u_\ell - a) \xrightarrow{3\text{-shock}} U_r .$$

Definition 4.8. We say that type II nonuniqueness holds for (U_ℓ, U_r) if U_ℓ and U_r have the same thermodynamic variables (τ_0, p_0) and there exist $\bar{p} > p_0$, $\tau_1, \tau_2 \in (0, \tau_0)$ with $\tau_1 < \tau_2$, $a > 0$ and $b > 0$ such that $u_\ell - u_r = a + b$ and (4.14) and (4.15) hold; then the R.P. with data (U_ℓ, U_r) has the two solutions (4.16) and (4.17), which we call type II solutions.

We now prove that MEDIUM is necessary for uniqueness. Some of the lemmas below are stronger than necessary; these strengthened statements will be used in Chapter 5 to obtain stability.

Convention 4.9. We shall henceforth often use a more informal notation, in which all derivatives are assumed to be taken along H. Here are two examples:

$$\left.\frac{d\tau}{d\theta}\right|_{(\tau_1,\bar{p})} = \left.\frac{d}{d\theta}\,\tau^0(\theta)\right|_{\theta=\theta_1} .$$

where $\tau_1, \bar{p}$ are as above and $\theta_1 = \theta(\tau_1, \bar{p})$;

$$\frac{d\tau}{dp} = V'(p)$$

when WEAK holds and V is as in Lemma 4.1.

Lemma 4.10. Suppose e satisfies WEAK, but violates MEDIUM. Then there exist (τ_0, p_0), and $(\tau_1, p_1) \in H \cap Q$, such that

$$\left.\frac{da}{dp}\right|_{(\tau_1,p_1)} < 0 . \qquad \left.\frac{dp}{d\theta}\right|_{(\tau_1,p_1)} > 0 .$$

(See Conventions 4.6 and 4.9; here, and often in what follows, we also denote $a = \sqrt{(\tau_0 - \tau)(p - p_0)}$.)

Proof: Since e violates MEDIUM, then by Prop. 1.5 there exists (τ_1, p_1) such that

$$\frac{\partial}{\partial p}\, e(\tau_1, p_1) < \frac{2e(\tau_1, p_1)}{-g_\tau(\tau_1, p_1) \cdot \dfrac{2e(\tau_1, p_1)}{p_1} + p_1} . \tag{4.18}$$

Let H^1 be the Hugoniot curve with center (τ_1, p_1), and let $\tau = V^1(p)$ along H^1 as in Lemma 4.1. By Lemma 2.18,

$$\lim_{p \to 0^+} V^1(p) = \tau^1_{\max} \equiv \tau_1 + \frac{2e(\tau_1, p_1)}{p_1} \quad .$$

Hence, writing $\tau = V^1(p)$, we have

$$\lim_{p \to 0^+} \frac{(p_1 - p)(\tau - \tau_1)}{(-g_\tau(\tau_1, p_1))(\tau - \tau_1) + (p_1 - p)} = \frac{2e(\tau_1, p_1)}{-g_\tau(\tau_1, p_1) \cdot \dfrac{2e(\tau_1, p_1)}{p_1} + p_1} \quad .$$

Therefore by (4.18), $\exists p_0 \in (0, p_1)$ such that, writing $\tau_0 = V^1(p_0)$, we have

$$(4.19) \qquad \frac{(p_1 - p_0)(\tau_0 - \tau_1)}{(-g_\tau(\tau_1, p_1))(\tau_0 - \tau_1) + (p_1 - p_0)} > \frac{\partial}{\partial p} e(\tau_1, p_1) \quad .$$

But also $\tau_1 = V(p_1)$, where $\tau = V(p)$ along H as in Lemma 4.1; hence by (4.4) and (4.19),

$$\frac{d}{dp} a_0(p)\Big|_{p = p_1} > 0 \quad .$$

For the other inequality, observe that since $\theta = (p - p_0)/(\tau_0 - \tau)$ for $\tau \neq \tau_0$, we have

$$(4.20) \quad \frac{d\theta}{dp} = \frac{(\tau_0 - \tau) + (p - p_0)\dfrac{d\tau}{dp}}{(\tau_0 - \tau)^2}$$

$$= (\tau_0 - \tau)^{-2} \left[(\tau_0 - \tau) - (p - p_0) \frac{\dfrac{\partial}{\partial p} e(\tau, p) - \dfrac{\tau_0 - \tau}{2}}{(-g_\tau)\dfrac{\partial}{\partial p} e(\tau, p) - \dfrac{p - p_0}{2}} \right] \qquad (p \neq p_0),$$

by WEAK, Cor. 2.7, (2.2) and (2.3). But by WEAK, $(-g_\tau)\dfrac{\partial}{\partial p} e(\tau, p) - \dfrac{p - p_0}{2} > 0$ $(p > 0)$, so

$$\frac{d\theta}{dp} > 0 \iff \left[(-g_\tau)\frac{\partial}{\partial p}e(\tau, p) - \frac{p - p_0}{2}\right](\tau_0 - \tau) > (p - p_0)\left[\frac{\partial}{\partial p}e(\tau, p) - \frac{\tau_0 - \tau}{2}\right]$$

$$\iff \frac{\partial}{\partial p}e(\tau, p)\left[(-g_\tau)(\tau_0 - \tau) - (p - p_0)\right] > 0 \qquad (p \neq p_0) ,$$

which is true for $p > p_0$ by Cor. 2.4 and (1.22). (Here $(\tau, p) \in H$, since the derivative $d\theta/dp$, along H by Convention 4.9, is involved.) Hence also $dp/d\theta > 0$ for $p > p_0$. In particular,

$$\left.\frac{dp}{d\theta}\right|_{(\tau_1, p_1)} > 0 . \qquad \blacksquare$$

Now suppose (τ, p) satisfies $H(\tau, p) = 0$ and $H_\tau(\tau, p) \neq 0$. Then τ is a function of p along H, near (τ, p); hence $d\tau/dp$ exists and (4.20) holds if $\tau \neq \tau_0$; by Lemma 2.9, $d\theta/dp \neq 0$; thus $dp/d\theta$ exists and is nonzero. In particular, if $(\tau, p) \in H$, $p > p_0$ and $H_\tau(\tau, p) \neq 0$, then by (4.20) the following hold:

(4.21) $$\frac{dp}{d\theta} \neq 0 ,$$

(4.22) $$\frac{dp}{d\theta} > 0 \iff \frac{d\tau}{dp} > \frac{\tau - \tau_0}{p - p_0} ,$$

(4.23) $$\frac{dp}{d\theta} < 0 \iff \frac{d\tau}{dp} < \frac{\tau - \tau_0}{p - p_0} \Longrightarrow \frac{d\tau}{dp} < 0 .$$

Lemma 4.11. Suppose e violates WEAK. Then there exist (τ_0, p_0), and $(\tau_2, \bar{p}) \in H \cap Q$, such that

$$\left.\frac{d\tau}{dp}\right|_{(\tau_2, \bar{p})} < 0 , \qquad \left.\frac{dp}{d\theta}\right|_{(\tau_2, \bar{p})} < 0 .$$

Proof: Since WEAK fails, $\exists(\tau_2, \bar{p})$ such that

(4.24) $$\frac{\partial}{\partial p}e(\tau_2, \bar{p}) < \frac{\bar{p}}{-2g_\tau(\tau_2, \bar{p})} .$$

Let $\bar{H}$ be the Hugoniot curve with center $(\tau_2, \bar{p})$, and let $\tau = \bar{V}(p)$ along $\bar{H}$ for $p \in (0, \bar{p}]$, as in Lemma 2.18. By (4.24), $(-g_\tau(\tau_2, \bar{p}))\frac{\partial}{\partial p} e(\tau_2, \bar{p}) - \frac{\bar{p}}{2} < 0$; hence we can find $p_0 \in (0, \bar{p})$ such that

$$(-g_\tau(\tau_2, \bar{p}))\frac{\partial}{\partial p} e(\tau_2, \bar{p}) - \frac{\bar{p} - p_0}{2} < 0 \ . \tag{4.25}$$

Write $\tau_0 = \bar{V}(p_0)$; then $(\tau_0, p_0) \in \bar{H}$, so $(\tau_2, \bar{p}) \in H$. But (4.20) holds for any $(\tau, p) \in H$ for which $H_\tau(\tau, p) \neq 0$ and $p > p_0$; in particular (4.20) holds for $(\tau, p) = (\tau_2, \bar{p})$, by (4.25) and (2.2). Then by (4.25) we see that

$$\left.\frac{d\theta}{dp}\right|_{(\tau_2, \bar{p})} < 0 \iff \left[(-g_\tau(\tau_2, \bar{p}))\frac{\partial}{\partial p} e(\tau_2, \bar{p}) - \frac{\bar{p} - p_0}{2}\right](\tau_0 - \tau_2)$$

$$> (\bar{p} - p_0)\left[\frac{\partial}{\partial p} e(\tau_2, \bar{p}) - \frac{\tau_0 - \tau_2}{2}\right]$$

$$\iff \frac{\partial}{\partial p} e(\tau_2, \bar{p})\left[(-g_\tau(\tau_2, \bar{p}))(\tau_0 - \tau_2) - (\bar{p} - p_0)\right] > 0 \ ,$$

which is true by Cor. 2.4 and (1.22). Then also

$$\left.\frac{dp}{d\theta}\right|_{(\tau_2, \bar{p})} < 0 \ .$$

It follows from (4.23) that

$$\left.\frac{d\tau}{dp}\right|_{(\tau_2, \bar{p})} < 0 \ . \qquad \blacksquare$$

Combining Lemmas 4.10 and 4.11 gives

Lemma 4.12. Suppose e violates MEDIUM. Then there exist (τ_0, p_0), and $(\tau_1, p_1) \in H \cap Q$, such that

$$\left.\frac{da}{d\theta}\right|_{(\tau_1, p_1)} < 0 \ , \qquad \left.\frac{dp}{d\theta}\right|_{(\tau_1, p_1)} \neq 0 \ .$$

Proof: If e satisfies WEAK, then by Lemma 4.10 there exist (τ_0, p_0), and $(\tau_1, p_1) \in H \cap Q$, such that

$$\left.\frac{da}{dp}\right|_{(\tau_1, p_1)} < 0 \ , \qquad \left.\frac{dp}{d\theta}\right|_{(\tau_1, p_1)} > 0 \ .$$

But since $\frac{da}{d\theta} = \frac{da}{dp} \cdot \frac{dp}{d\theta}$, the above also yield

$$\left.\frac{da}{d\theta}\right|_{(\tau_1, p_1)} < 0 \ .$$

If e violates WEAK, then by Lemma 4.11 there exist (τ_0, p_0), and $(\tau_1, p_1) \in H \cap Q$, such that

$$\left.\frac{d\tau}{dp}\right|_{(\tau_1, p_1)} < 0 \ , \qquad \left.\frac{dp}{d\theta}\right|_{(\tau_1, p_1)} < 0 \ .$$

Since $a = \sqrt{(\tau_0 - \tau)(p - p_0)}$ (see Convention 4.6 and Lemma 4.10),

$$\frac{da}{dp} = \frac{1}{2a}\left[(\tau_0 - \tau) - (p - p_0)\frac{d\tau}{dp}\right] \qquad (p \neq p_0) \ .$$

Hence

$$\left.\frac{da}{dp}\right|_{(\tau_1, p_1)} > 0 \ . \qquad \blacksquare$$

Theorem 4.13. Suppose e violates MEDIUM. Then uniqueness fails; there exist initial data (U_ℓ, U_r) for which 3-fold type I nonuniqueness holds.

Proof: By Lemma 4.12, there exist (τ_0, p_0), and $(\tau_2, p_2) \in H \cap Q$, such that

$$\left.\frac{da}{d\theta}\right|_{(\tau_2, p_2)} < 0 \ .$$

Let $\theta_2 = \theta(\tau_2, p_2)$ (and, as always, $\theta_0 = \theta(\tau_0, p_0) = -\left.\frac{dp}{d\tau}\right|_{(\tau_0, p_0)}$). Then

$$\frac{d}{d\theta}\, a^0(\theta)\Big|_{\theta=\theta_2} < 0 \ .$$

Thus $a^0(\theta) > a^0(\theta_2)$ for $\theta < \theta_2$, θ near θ_2; $a^0(\theta) < a^0(\theta_2)$ for $\theta > \theta_2$, near θ_2. (See Fig. 20.) But $a^0(\theta_0) = 0$, so $\exists \theta_1 \in (\theta_0, \theta_2)$ such that $a^0(\theta_1) = a^0(\theta_2)$.

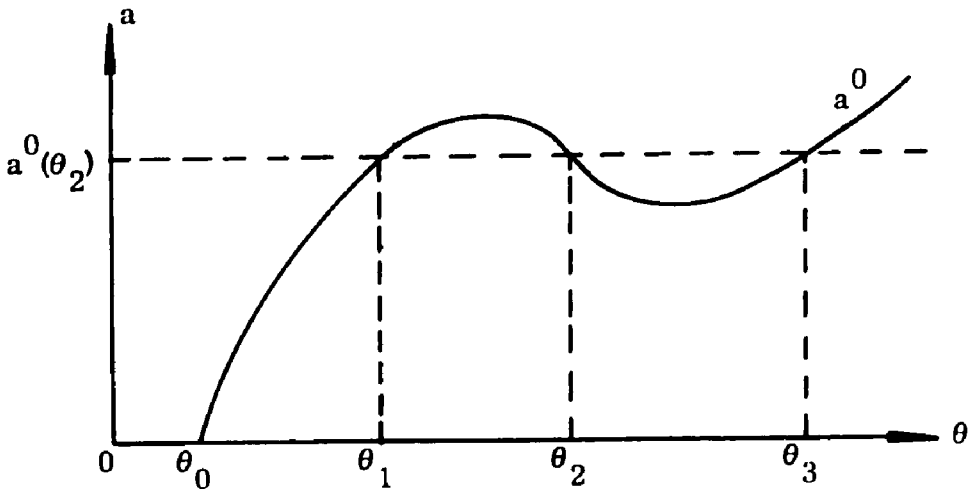


Fig. 20: The behavior of a^0.

Moreover, by Lemma 2.22, we can find arbitrarily large θ such that $a^0(\theta) > a^0(\theta_2)$, so $\exists \theta_3 \in (\theta_2, +\infty)$ such that $a^0(\theta_3) = a^0(\theta_2)$. Let $a = a^0(\theta_2)$, and let $(\tau_i, p_i) = (\tau^0(\theta_i), p^0(\theta_i))$ for $i = 1, 2, 3$. Then the (τ_i, p_i)'s are distinct and satisfy (4.11). Let $U_\ell = (\tau_0, p_0, u_\ell)$, $U_r = (\tau_0, p_0, u_\ell - 2a)$, where u_ℓ is arbitrary. Then 3-fold type I nonuniqueness holds for (U_ℓ, U_r). (See Fig. 21.) ■

Thus we see from Th. 4.5 and Th. 4.13 that the condition MEDIUM is necessary and sufficient for uniqueness of solutions of the R. P.

We conclude this chapter by showing that when WEAK fails, type II nonuniqueness occurs.

Lemma 4.14. Suppose e violates WEAK. Then there exist a point (τ_0, p_0), an open interval $N \subset (p_0, +\infty)$ and three C^1 functions V_1, V_2, V_3 on N such that

$$(V_i(p), p) \in \Pi \qquad (p \in N,\ i = 1, 2, 3) \ ,$$

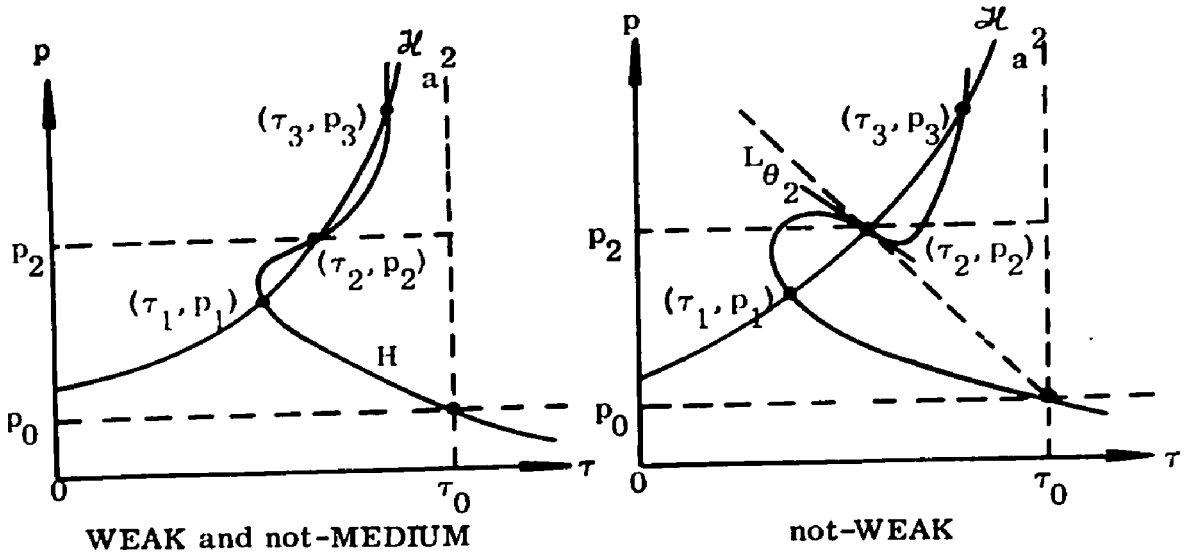


Fig. 21: 3-fold type I nonuniqueness occurs whenever MEDIUM fails.

(4.26) $$V_1(p) < V_2(p) < V_3(p) \qquad (p \in N),$$

(4.27) $$V_2'(p) < 0 \qquad (p \in N),$$

(4.28) $$H_\tau(V_i(p), p) \neq 0 \qquad (p \in N,\ i = 1, 2, 3).$$

(See Fig. 22.)

Proof: By Lemma 4.11, there exist (τ_0, p_0), and $(\tau_2, p_2) \in H \cap Q$, such that

(4.29) $$\left.\frac{d\tau}{dp}\right|_{(\tau_2, p_2)} < 0 \ , \quad \left.\frac{dp}{d\theta}\right|_{(\tau_2, p_2)} < 0 \ .$$

We know by the meaning of $\left.\frac{d\tau}{dp}\right|_{(\tau_2, p_2)}$ (and the proof of Lemma 4.11) that $H_\tau(\tau_2, p_2) \neq 0$; thus $\exists$ a C^1 function V_2 defined on some neighborhood $N_\delta(p_2)$ of p_2 $(\delta > 0)$ such that $V_2(p_2) = \tau_2$ and

(4.30) $$(V_2(p), p) \in H \ , \quad H_\tau(V_2(p), p) \neq 0 \quad \forall p \in N_\delta(p_2) \ .$$

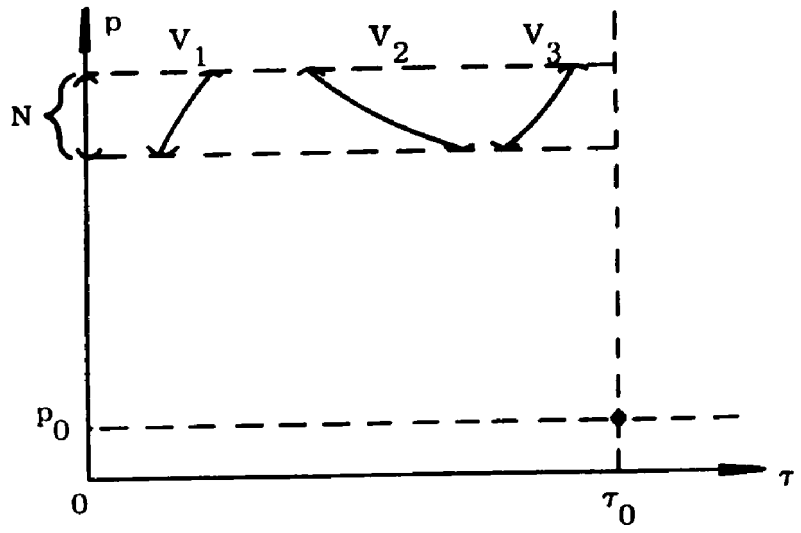


Fig. 22: The objects (τ_0, p_0), N, V_1, V_2 and V_3 to be constructed.

Also by (4.29), $(p^0)'(\theta_2) < 0$, where $\theta_2 = \theta(\tau_2, p_2)$. Thus $p^0(\theta) < p_2$ for all $\theta > \theta_2$ sufficiently near θ_2. (See Fig. 23.) But by Lemma 2.21, $p^0(\theta) > p_2$ for θ sufficiently large. Hence $p^0(\theta) = p_2$ for some $\theta > \theta_2$; let θ_3 be the smallest such θ, and let $\tau_3 = \tau^0(\theta_3)$. Then $\tau_3 > \tau_2$, since $\theta_3 > \theta_2$ and $p^0(\theta_3) = p^0(\theta_2)$. Now pick any $\tau_4 \in (\tau_2, \tau_3)$. By (4.30), $V_2'(p_2) < 0$. Since $V_2'(p) = -H_p(V_2(p), p)/H_\tau(V_2(p), p)$ $(p \in N_\delta(p_2))$, we may assume, by making δ smaller if necessary, that $\forall p \in N_\delta(p_2)$,

$$V_2(p) < \tau_4 \ , \tag{4.31}$$

$$V_2'(p) < 0 \ . \tag{4.32}$$

By continuity, $\exists \theta_4 \in (\theta_2, \theta_3)$ such that $\forall \theta \in [\theta_4, \theta_3)$,

$$p_2 > p^0(\theta) > p_2 - \delta \ , \tag{4.33}$$

$$\tau^0(\theta) > \tau_4 \ . \tag{4.34}$$

Since (4.33) holds for $\theta = \theta_4$, we can find $\bar{\theta} \in (\theta_4, \theta_3)$ such that

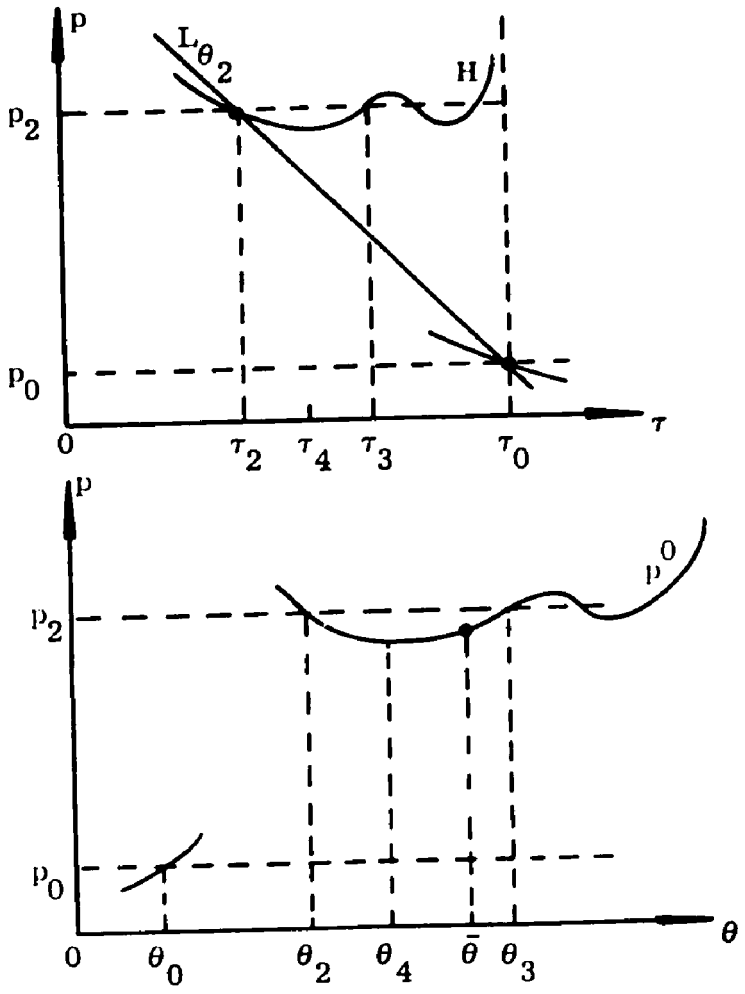


Fig. 23: Obtaining (v_3, p_3).

$$\frac{d}{d\theta}\, p^0(\theta)\Big|_{\theta=\bar{\theta}} = \frac{p^0(\theta_3) - p^0(\theta_4)}{\theta_3 - \theta_4} > 0 \ , \tag{4.35}$$

since $p^0(\theta_3) = p_2$. Let $p_3 = p^0(\bar{\theta})$, $v_3 = \tau^0(\bar{\theta})$. We assert that $H_\tau(v_3, p_3) \neq 0$.

In general, suppose we have a point $(\tau, p) \in H$ for which $dp/d\theta \neq 0$. By Lemma 2.6(c), either $H_\tau \neq 0$ or $H_p \neq 0$. But if $H_p \neq 0$, then $p = p(\tau)$ locally (near (τ, p)), so $\frac{dp}{d\tau} \cdot \frac{d\tau}{d\theta} = \frac{dp}{d\theta} \neq 0$, whence $\frac{dp}{d\tau} \neq 0$; hence $H_\tau \neq 0$. Thus we have proved

$$\frac{dp}{d\theta} \neq 0 \Longrightarrow H_\tau \neq 0 \ . \tag{4.36}$$

Now by (4.35), $\left.\frac{dp}{d\theta}\right|_{(v_3, p_3)} \neq 0$, so our assertion $H_\tau(v_3, p_3) \neq 0$ follows from (4.36). Next, observe that by (4.31), (4.33) and (4.34), we have $p_3 \in N_\delta(p_2)$ and $V_2(p_3) < \tau_4 < \tau^0(\bar{\theta}) = v_3$. Thus there exist $\sigma > 0$ and a C^1 function V_3 defined on $N_\sigma(p_3)$ such that $N_\sigma(p_3) \subset N_\delta(p_2)$, $V_3(p_3) = v_3$, and $\forall p \in N_\sigma(p_3)$,

(4.37) $\quad (V_3(p), p) \in H\ , \quad H_\tau(V_3(p), p) \neq 0 \quad \text{and} \quad V_2(p) < V_3(p)\ .$

(See Fig. 24.)

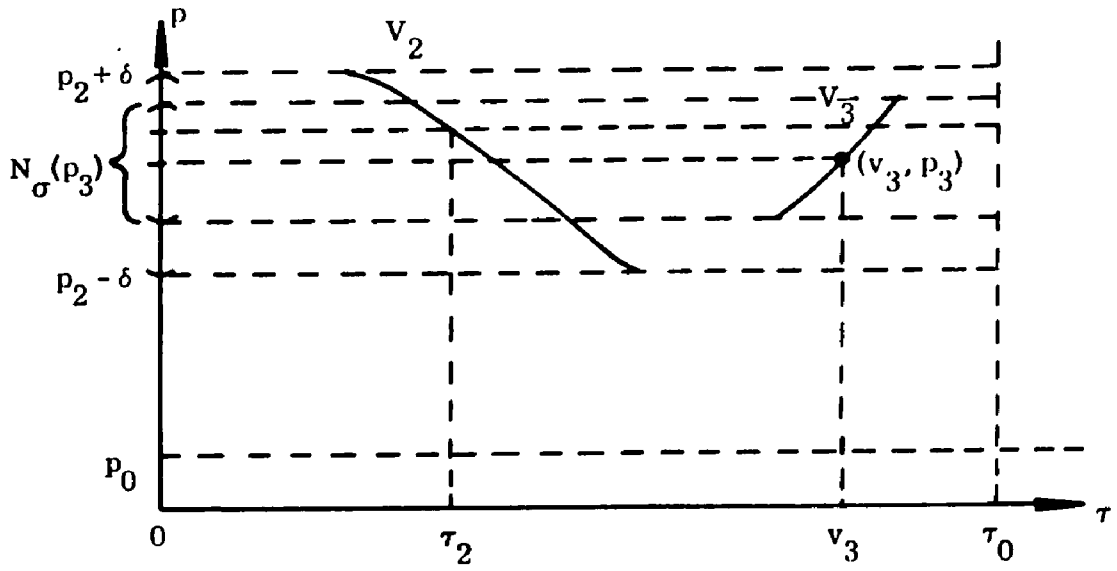


Fig. 24: Defining V_3.

We next argue analogously to get V_1. By (4.33), $p_2 > p_3$. Since $(p^0)'(\theta_2) < 0$, $p^0(\theta) > p_2 > p_3$ for all $\theta < \theta_2$ sufficiently near θ_2. But $p^0(\theta_0) = p_0 < p_3$ (because $(v_3, p_3) \in Q$ by Def. 2.10(c), since $\bar{\theta} > \theta_4 > \theta_2 > \theta_0$). Thus $p^0(\theta) = p_3$ for some $\theta \in (\theta_0, \theta_2)$; let θ_1 be the largest such θ, and let $\tau_1 = \tau^0(\theta_1)$. Then $\tau_1 < \tau_2$. Pick any $\tau_5 \in (\tau_1, \tau_2)$. (See Fig. 25.) By making σ smaller if necessary, we may assume that $\forall p \in N_\sigma(p_3)$,

(4.38) $\quad V_2(p) > \tau_5\ .$

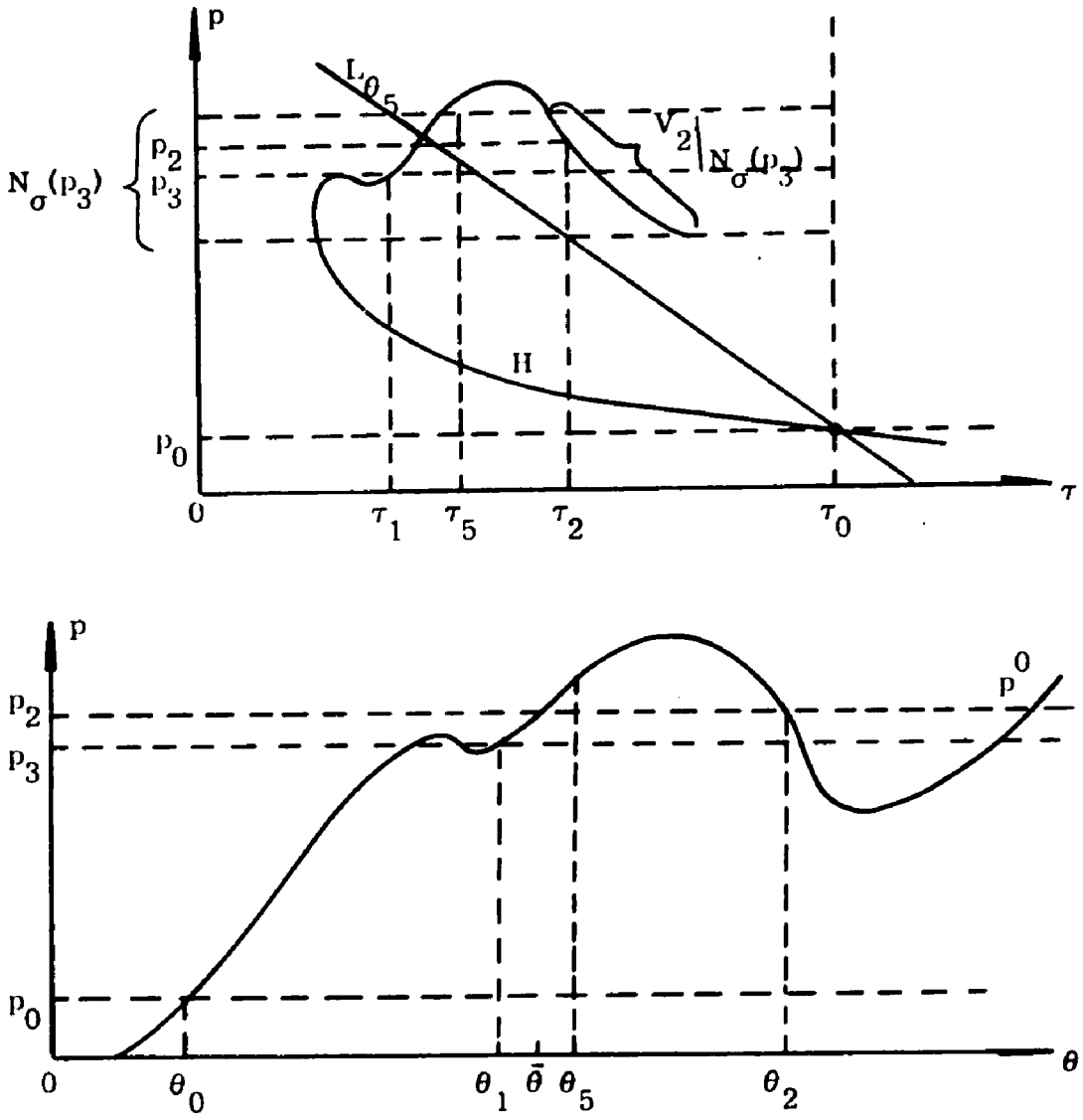


Fig. 25: Obtaining (v_1, p_1).

$\exists \theta_5 \in (\theta_1, \theta_2)$ such that $\forall \theta \in (\theta_1, \theta_5]$,

(4.39) $$p_3 < p^0(\theta) < p_3 + \sigma$$

(4.40) $$\tau^0(\theta) < \tau_5 .$$

Then as before, $\exists \bar{\theta} \in (\theta_1, \theta_5)$ such that $(p^0)'(\bar{\theta}) \neq 0$. Let $p_1 = p^0(\bar{\theta})$, $v_1 = \tau^0(\bar{\theta})$; then by (4.36), $H_\tau(v_1, p_1) \neq 0$. Also, by (4.38) - (4.40), $v_1 < \tau_5 < V_2(p_1)$. Hence there exist an open interval N and a C^1 function V_1 on N such that

$p_1 \in N \subset N_\sigma(p_3)$, $V_1(p_1) = v_1$, and $\forall p \in N$,

(4.41) $\quad (V_1(p), p) \in H$, $H_\tau(V_1(p), p) \neq 0$ and $V_1(p) < V_2(p)$.

(Cf. Figs. 22 and 24.) Since $N \subset N_\sigma(p_3) \subset N_\delta(p_2)$, if we restrict V_2 and V_3 to N, then V_1, V_2 and V_3 satisfy the desired conditions by (4.30), (4.32), (4.37) and (4.41). ■

Theorem 4.15. Suppose e violates WEAK. Then there exist initial data (U_ℓ, U_r) for which type II nonuniqueness holds.

Proof: Let (τ_0, p_0), N, V_1, V_2 and V_3 be as in Lemma 4.14. Choose $\bar{p} \in N$, and let $\tau_1 = V_1(\bar{p})$, $\tau_2 = V_2(\bar{p})$; then $(\tau_1, \bar{p}), (\tau_2, \bar{p}) \in H$. Also, $\bar{p} > p_0$, so by Cor. 2.7, $\tau_1, \tau_2 < \tau_0$; by (4.26), $\tau_1 < \tau_2$. (See Fig. 26.) It is now clear from

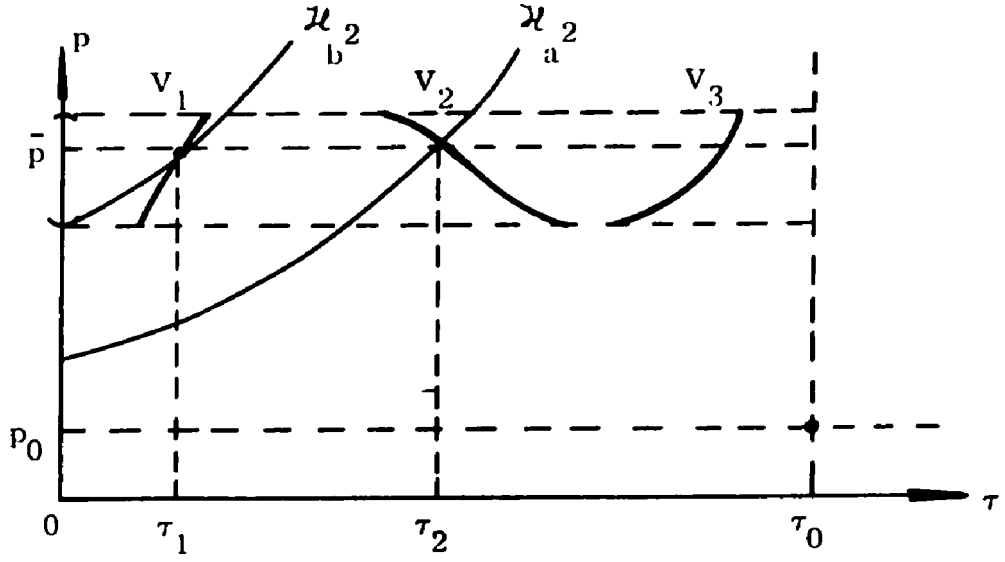


Fig. 26: Type II nonuniqueness.

Def. 4.8 and the remarks preceding it that data (U_ℓ, U_r) can be found for which type II nonuniqueness holds. ■

Observe that, although the violation of WEAK implies the occurrence of both type I and type II nonuniqueness, the data (U_ℓ, U_r) given by Theorems 4.13 and 4.15 need not be the same. In the next chapter we shall prove that (when WEAK fails) there always exists one set of data (U_ℓ, U_r) for which type I and type II nonuniqueness hold simultaneously, and for which the resulting 5-fold nonuniqueness is stable.

CHAPTER 5

STABILITY OF THE NONUNIQUE SOLUTIONS

We begin by providing a meaning for the title of this chapter.

Suppose we have initial data (U_ℓ^0, U_r^0) for which n-fold nonuniqueness (type I, type II, both, or neither) holds, i.e., the R.P. with data (U_ℓ^0, U_r^0) has n distinct solutions,

$$U_\ell^0 \overset{1}{\text{---}} U_1^{i,0} \underline{\text{contact}} \; U_3^{i,0} \overset{3}{\text{---}} U_r^0 \qquad (1 \le i \le n) . \tag{5.1}$$

Definition 5.1. We say the n-fold nonuniqueness is stable if there exists a neighborhood $\mathcal{N}$ of (U_ℓ^0, U_r^0) such that

(a) for any $(U_\ell, U_r) \in \mathcal{N}$, the R.P. with data (U_ℓ, U_r) has n solutions;

(b) these n solutions can be expressed,

$$U_\ell \overset{1}{\text{---}} U_1^i(U_\ell, U_r) \underline{\text{contact}} \; U_3^i(U_\ell, U_r) \overset{3}{\text{---}} U_r$$

$$((U_\ell, U_r) \in \mathcal{N}, \; i = 1, \ldots, n),$$

in such a way that

$$U_j^i(U_\ell, U_r) \longrightarrow U_j^{i,0} \qquad (j = 1, 3; \; i = 1, \ldots, n)$$

as $(U_\ell, U_r) \longrightarrow (U_\ell^0, U_r^0)$.

In this chapter we shall prove that for all types of n-fold nonuniqueness considered, the data can be chosen so that the n-fold nonuniqueness is stable. Thus when MEDIUM fails, there exist data that exhibit 3-fold type I nonuniqueness, and for which the 3-fold nonuniqueness is stable. (See Th. 5.4.) When WEAK fails, we can find data that exhibit simultaneously 3-fold type I nonuniqueness and (2-fold) type II nonuniqueness, and for which the 5-fold nonuniqueness is stable. (See Th. 5.6.)

We begin with type I nonuniqueness; the following two lemmas provide sufficient conditions for stability.

Lemma 5.2. Suppose $\theta_0 < \theta_1 < \theta_2 < \theta_3$, $a > 0$ and

$$\left.\frac{da}{d\theta}\right|_{\theta=\theta_i} \neq 0 \qquad (i = 1,2,3), \tag{5.2}$$

$$\left.\frac{dp}{d\theta}\right|_{\theta=\theta_i} \neq 0 \qquad (i = 1,2,3), \tag{5.3}$$

$$a^0(\theta_i) = a \qquad (i = 1,2,3). \tag{5.4}$$

Let u_0 be arbitrary and let $U_\ell^0 = (\tau_0, p_0, u_0)$, $U_r^0 = (\tau_0, p_0, u_0 - 2a)$. Then the R.P. with data (U_ℓ^0, U_r^0) has three type I solutions for which the 3-fold non-uniqueness is stable.

Proof: Let $(\tau_i, p_i) = (\tau^0(\theta_i), p^0(\theta_i))$ $(i = 1,2,3)$. Then by (5.4), the R.P. with data (U_ℓ^0, U_r^0) has the three type I solutions,

$$U_\ell^0 \xrightarrow{1\text{-shock}} (\tau_i, p_i, u_0 - a) \xrightarrow{\text{contact}} (\tau_i, p_i, u_0 - a) \xrightarrow{3\text{-shock}} U_r^0 \qquad (i = 1,2,3), \tag{5.5}$$

which we have expressed in the form of (5.1). By (4.36), $H_\tau(\tau_i, p_i) \neq 0$ $(i = 1,2,3)$. For any point (τ_c, p_c) as center, we define

$$H(\tau_c, p_c; \tau, p) = e(\tau, p) - e(\tau_c, p_c) + (\tau - \tau_c)\,\frac{p + p_c}{2} \qquad (\tau_c, p_c, \tau, p > 0). \tag{5.6}$$

Then $H_\tau(\tau_0, p_0; \tau_i, p_i) \neq 0$ $(i = 1,2,3)$, so by the implicit function theorem there exist neighborhoods N_i of (τ_0, p_0, p_i) and C^1 functions V_i on N_i such that

$$V_i(\tau_0, p_0; p_i) = \tau_i \qquad (i = 1,2,3), \tag{5.7}$$

$$H(\tau_c, p_c; V_i(\tau_c, p_c; p), p) = 0 \qquad ((\tau_c, p_c, p) \in N_i,\ i = 1,2,3), \tag{5.8}$$

and we may assume

$$N_i \cap N_j = \phi \qquad (1 \leq i < j \leq 3), \tag{5.9}$$

$$(\tau_c, p_c, p) \in N_i \Longrightarrow p > p_c \qquad (i = 1, 2, 3).$$

For brevity, we shall usually write

$$V_i^c(p) = V_i(\tau_c, p_c; p) \qquad ((\tau_c, p_c, p) \in N_i). \tag{5.10}$$

We also define

$$\eta(\tau_c, p_c; \tau, p) = \sqrt{(\tau_c - \tau)(p - p_c)}\ , \tag{5.11}$$

and we abbreviate $\eta^c(\tau, p) = \eta(\tau_c, p_c; \tau, p)$. In similar fashion, we define

$$a_c(p) = \eta^c(V_i^c(p), p) \qquad \text{when } (\tau_c, p_c, p) \in N_i\ . \tag{5.12}$$

By Lemma 2.9, (5.2), and the fact that $H_\tau(\tau_i, p_i) \neq 0$, we have

$$\left.\frac{da}{dp}\right|_{(\tau_i, p_i)} = \left.\frac{da}{d\theta}\right|_{(\tau_i, p_i)} \cdot \left.\frac{d\theta}{dp}\right|_{(\tau_i, p_i)} \neq 0 \qquad (i = 1, 2, 3),$$

which, in the notation (5.10) - (5.12), can be expressed,

$$a_0'(p_i) \neq 0 \qquad (i = 1, 2, 3). \tag{5.13}$$

Define

$$\Phi(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, p) = (\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, u_\ell - (a_\ell(p) + a_r(p)))\ ,$$

for all $(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, p)$ for which (τ_ℓ, p_ℓ, p), $(\tau_r, p_r, p) \in N_i$ for some i. (Here, for example, $a_\ell(p) = \eta^\ell(V_i^\ell(p), p)$ when $(\tau_\ell, p_\ell, p) \in N_i$; see (5.10) - (5.12).) Φ is a C^1 function defined on an open set containing $\{(\tau_0, p_0, u_0, \tau_0, p_0, p_i) \mid i = 1, 2, 3\}$, and has Jacobian matrix of the form

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & 1 & & & & \\ & 0 & & 1 & & & \\ & & & & 1 & & \\ * & * & * & * & * & \frac{\partial}{\partial p}(u_\ell - (a_\ell(p) + a_r(p))) \end{bmatrix};$$

hence its Jacobian determinant is given by

$$J_{\Phi}(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, p) = \frac{\partial}{\partial p}(u_\ell - (a_\ell(p) + a_r(p))) = -(a_\ell'(p) + a_r'(p)) ,$$

whence

$$J_{\Phi}(\tau_0, p_0, u_0, \tau_0, p_0, p_i) = -(a_0'(p_i) + a_0'(p_i)) = -2a_0'(p_i) \neq 0 \qquad (i = 1,2,3) .$$

Moreover,

$$\Phi(\tau_0, p_0, u_0, \tau_0, p_0, p_i) = (\tau_0, p_0, u_0, \tau_0, p_0, u_0 - (a_0(p_i) + a_0(p_i))) \tag{5.14}$$

$$= (\tau_0, p_0, u_0, \tau_0, p_0, u_0 - 2a) \qquad (i = 1,2,3) ,$$

since $a_0(p_i) = \eta^0(\tau_i, p_i) = a^0(\theta_i) = a$ by (5.4) and (5.7). Hence there exist neighborhoods $\mathcal{N}_i$ of $(\tau_0, p_0, u_0, \tau_0, p_0, p_i)$ and η_i of $(\tau_0, p_0, u_0, \tau_0, p_0, u_0 - 2a)$ such that $\Phi|_{\mathcal{N}_i}$ is a C^1 diffeomorphism of $\mathcal{N}_i$ onto η_i; we may assume

$$\mathcal{N}_i \cap \mathcal{N}_j = \phi \qquad (1 \leq i < j \leq 3) . \tag{5.15}$$

In addition, (5.9) implies

$$(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, p) \in \mathcal{N}_i \Longrightarrow p > p_\ell, p_r \qquad (i = 1,2,3) . \tag{5.16}$$

Observe that $(U_\ell^0, U_r^0) = (\tau_0, p_0, u_0, \tau_0, p_0, u_0 - 2a)$. Let $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2 \cap \mathcal{N}_3$; $\mathcal{N}$ is a neighborhood of (U_ℓ^0, U_r^0). Let $(U_\ell, U_r) \in \mathcal{N}$. Then $\forall i$, $(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, u_r) = (U_\ell, U_r) \in \mathcal{N}_i$, so there is a $\overline{p_i} > 0$ with $(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, \overline{p_i}) \in \mathcal{N}_i$ such that $\Phi(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, \overline{p_i}) = (\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, u_r)$. Since also $\Phi(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, \overline{p_i}) = (\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, u_\ell - (a_\ell(\overline{p_i}) + a_r(\overline{p_i})))$, we have

$$u_r = u_\ell - (a_\ell(\overline{p_i}) + a_r(\overline{p_i})) \qquad (i = 1,2,3) ; \tag{5.17}$$

moreover $\overline{p_i} > p_\ell, p_r$ by (5.16). Here $\overline{p_i}$ is a C^1 function of (U_ℓ, U_r); in fact, it is the last coordinate function of the C^1 diffeomorphism $(\Phi|_{\mathcal{N}_i})^{-1}|_{\mathcal{N}}$. Note that by (5.15), the $\overline{p_i}$'s are distinct. Thus from (5.8) and (5.17) we see that the R. P. with data (U_ℓ, U_r) has the following three distinct solutions, for $i = 1,2,3$:

$$(5.18)\quad U_\ell \xrightarrow{\text{1-shock}} (V_i^\ell(\bar{p}_i), \bar{p}_i, u_\ell - a_\ell(\bar{p}_i))$$

$$\xrightarrow{\text{contact}} (V_i^r(\bar{p}_i), \bar{p}_i, u_r + a_r(\bar{p}_i)) \xrightarrow{\text{3-shock}} U_r \ ,$$

which establishes condition (a) of Def. 5.1.

By (5.14) and the fact that $(\Phi|_{\mathcal{H}_i})^{-1}$ is continuous on $\bar{\eta}_i$, $\bar{p}_i \to p_i$ as $(U_\ell, U_r) \to (U_\ell^0, U_r^0)$. Letting

$$U_1^i(U_\ell, U_r) = (V_i^\ell(\bar{p}_i), \bar{p}_i, u_\ell - a_\ell(\bar{p}_i)) \ ,$$

$$U_3^i(U_\ell, U_r) = (V_i^r(\bar{p}_i), \bar{p}_i, u_r + a_r(\bar{p}_i)) \ ,$$

we see that the solutions (5.18) are expressible as

$$U_\ell \xrightarrow{1} U_1^i(U_\ell, U_r) \xrightarrow{\text{contact}} U_3^i(U_\ell, U_r) \xrightarrow{3} U_r \qquad (i = 1, 2, 3) \ ,$$

and $U_j^i(U_\ell, U_r) \to (\tau_i, p_i, u_0 - a)$ as $(U_\ell, U_r) \to (U_\ell^0, U_r^0)$ $(i = 1, 2, 3;\ j = 1, 3)$. (See (5.17).) Referring to (5.1) and (5.5), we see that condition (b) of Def. 5.1 holds. ■

Lemma 5.3. Let $\theta_0 < \bar{\theta}_1 < \bar{\theta} < \bar{\theta}_2$, and suppose $a^0(\bar{\theta}_1) > a^0(\bar{\theta}) > a^0(\bar{\theta}_2)$, $(a^0)'(\bar{\theta}) \neq 0$ and $(p^0)'(\bar{\theta}) \neq 0$. Then for any open interval I containing $a^0(\bar{\theta})$, there exist $\theta_1, \theta_2, \theta_3$ and $a \in I$ with $\theta_0 < \theta_1 < \theta_2 < \theta_3$, such that $a^0(\theta_i) = a$, $(a^0)'(\theta_i) \neq 0$ and $(p^0)'(\theta_i) \neq 0$ $(i = 1, 2, 3)$.

Proof: By Lemma 2.22, $\exists \theta > \bar{\theta}_2$ such that $a^0(\theta) = a^0(\bar{\theta})$; let $\bar{\theta}_3$ be the smallest such θ. (See Fig. 27.) Choose $\epsilon > 0$ such that $\theta \in N_\epsilon(\bar{\theta})$ implies $\theta \in (\bar{\theta}_1, \bar{\theta}_2)$, $(a^0)'(\theta) \neq 0$ and $(p^0)'(\theta) \neq 0$. a^0 is a C^1 diffeomorphism of a neighborhood of $\bar{\theta}$ onto a neighborhood of $a^0(\bar{\theta})$; hence there is an open interval J with $a^0(\bar{\theta}) \in J \subset I \cap (0, +\infty)$ such that $J \subset a^0(N_\epsilon(\bar{\theta}))$. Then since $a^0(\bar{\theta}_3) = a^0(\bar{\theta})$, $a^0(\bar{\theta}_3) \in J$; so $\exists \sigma > 0$ such that $\bar{\theta}_3 - \sigma > \bar{\theta}_2$ and

$$(5.19)\qquad a^0(\theta) \in J \quad \text{whenever} \quad \theta \in [\bar{\theta}_3 - \sigma, \bar{\theta}_3) \ .$$

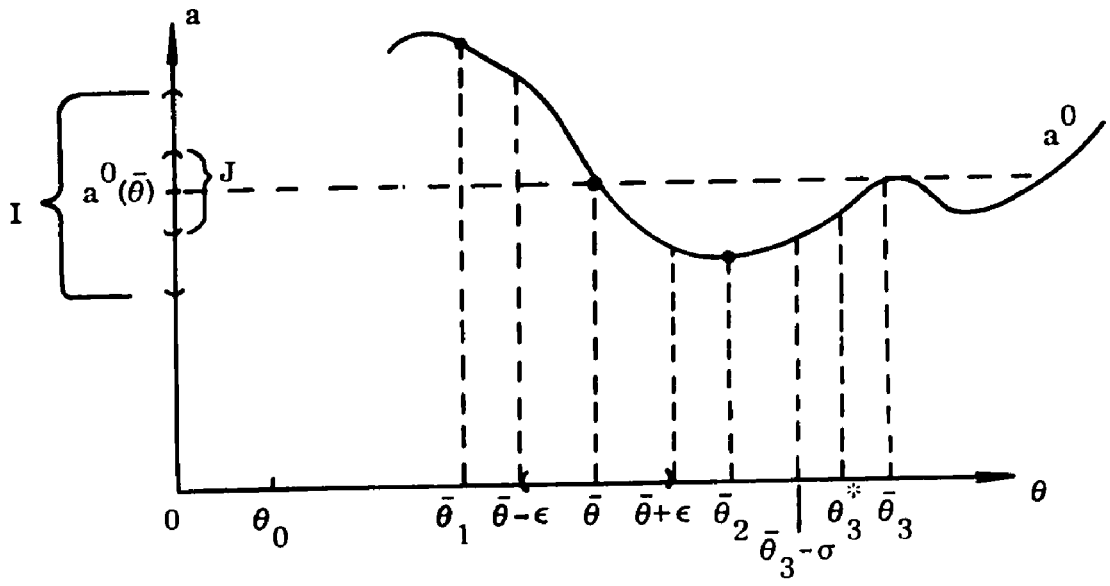


Fig. 27: Constructing θ_3^*.

Thus $a^0(\overline{\theta_3}-\sigma) < a^0(\overline{\theta_3})$. $\exists \theta_3^* \in (\overline{\theta_3}-\sigma, \overline{\theta_3})$ such that

$$(5.20) \qquad (a^0)'(\theta_3^*) = \frac{a^0(\overline{\theta_3}) - a^0(\overline{\theta_3}-\sigma)}{\overline{\theta_3} - (\overline{\theta_3}-\sigma)} > 0 \ .$$

Also, $\theta_3^* \in (\overline{\theta_2}, \overline{\theta_3})$, so $0 < a^0(\theta_3^*) < a^0(\overline{\theta_3})$. By (5.19), $a^0(\theta_3^*) \in J$.

For $a, \theta > 0$, a and θ determine p; the function $p = p(\theta, a)$ is C^1 and satisfies

$$(5.21) \qquad \frac{\partial}{\partial\theta} p(\theta, a) > 0 \ , \qquad \frac{\partial}{\partial a} p(\theta, a) > 0 \qquad (\theta, a > 0) \ .$$

Thus $p^0(\theta) = p(\theta, a^0(\theta))$ $(\theta > \theta_0)$, and

$$(5.22) \qquad (p^0)'(\theta) = \frac{\partial p}{\partial\theta}(\theta, a^0(\theta)) + \frac{\partial p}{\partial a}(\theta, a^0(\theta)) \cdot (a^0)'(\theta) \qquad (\theta > \theta_0) \ .$$

By (5.20) - (5.22), $(p^0)'(\theta_3^*) > 0$.

Since also $\theta_3^* > \overline{\theta_3} - \sigma > \overline{\theta_2}$, $\exists \delta > 0$ such that

$$(5.23) \qquad \theta \in N_\delta(\theta_3^*) \Longrightarrow \theta > \overline{\theta_2} \text{ and } (p^0)'(\theta), (a^0)'(\theta) > 0 \ .$$

As before, by (5.19), there is an open interval K with $a^0(\theta_3^*) \in K \subset J$ such that $K \subset a^0(N_\delta(\theta_3^*))$. Since $0 < a^0(\theta_3^*) < a^0(\overline{\theta_3}) = a^0(\bar{\theta}) < a^0(\overline{\theta_1})$, and $a^0(\theta_0) = 0$, $\exists \theta \in (\theta_0, \overline{\theta_1})$ such that $a^0(\theta) = a^0(\theta_3^*)$; let θ_1^* be the smallest such θ. (See Fig. 28.) $\exists \eta > 0$ such that $\theta_1^* - \eta > \theta_0$ and

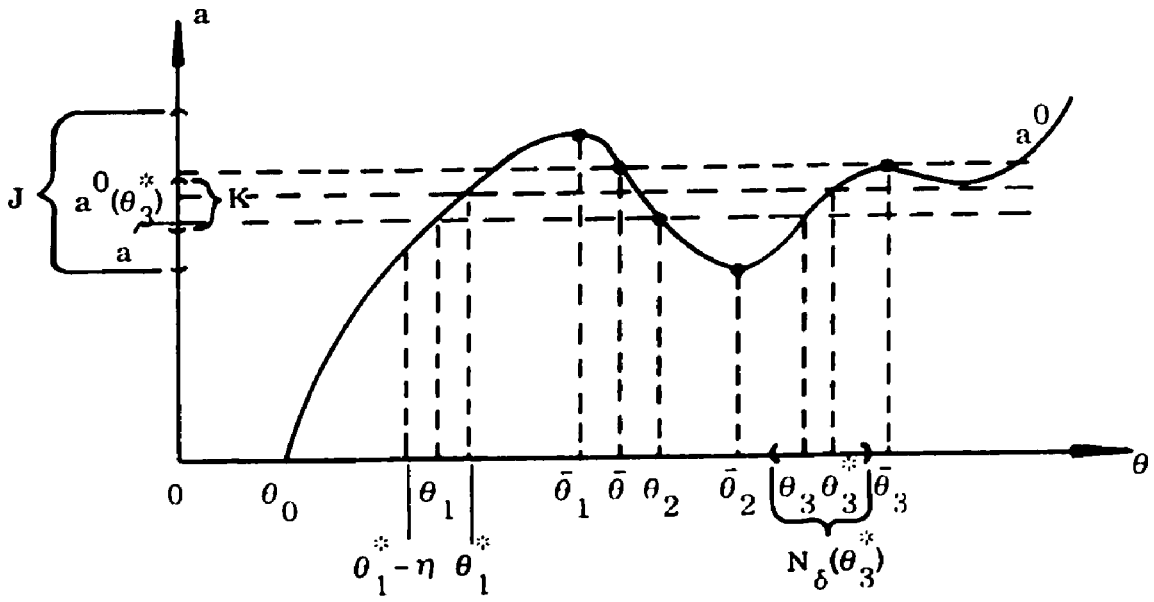


Fig. 28: Constructing $\theta_1, \theta_2, \theta_3$ and a.

(5.24) $$a^0(\theta) \in K \quad \text{whenever } \theta \in [\theta_1^* - \eta, \theta_1^*) \ .$$

Then $a^0(\theta_1^* - \eta) < a^0(\theta_1^*)$. $\exists \theta_1 \in (\theta_1^* - \eta, \theta_1^*)$ such that

$$(a^0)'(\theta_1) = \frac{a^0(\theta_1^*) - a^0(\theta_1^* - \eta)}{\theta_1^* - (\theta_1^* - \eta)} > 0 \ ;$$

moreover, $a^0(\theta_1) \in K$ by (5.24). Using (5.21) and (5.22), we see that $(p^0)'(\theta_1) > 0$. Since $K \subset a^0(N_\delta(\theta_3^*))$, $a^0(\theta_1) \in a^0(N_\delta(\theta_3^*))$, so $\exists \theta_3 \in N_\delta(\theta_3^*)$ such that $a^0(\theta_3) = a^0(\theta_1)$. Then by (5.23), $\theta_3 > \overline{\theta_2}$, $(p^0)'(\theta_3) > 0$ and $(a^0)'(\theta_3) > 0$. But since $K \subset J \subset a^0(N_\epsilon(\bar{\theta}))$, $a^0(\theta_1) \in a^0(N_\epsilon(\bar{\theta}))$, so $\exists \theta_2 \in N_\epsilon(\bar{\theta})$ such that $a^0(\theta_2) = a^0(\theta_1)$; then also $\theta_2 \in (\overline{\theta_1}, \overline{\theta_2})$, $(a^0)'(\theta_2) \neq 0$ and $(p^0)'(\theta_2) \neq 0$. Let $a = a^0(\theta_1)$. Since $K \subset J \subset I$, $a \in I$. Finally, $\theta_0 < \theta_1^* - \eta < \theta_1 < \theta_1^* < \overline{\theta_1} < \theta_2 < \overline{\theta_2} < \theta_3$. ■

As a corollary to the preceding two lemmas and Lemma 4.12, we have

Theorem 5.4. Suppose e violates MEDIUM. Then there exist data (U_ℓ^0, U_r^0) for which 3-fold type I nonuniqueness holds and the 3-fold nonuniqueness is stable.

Proof: By Lemma 4.12, there exist a center (τ_0, p_0), and a point $(\bar\tau, \bar p) \in H \cap Q$ (see (2.1)), such that

$$\left.\frac{da}{d\theta}\right|_{(\bar\tau, \bar p)} < 0 , \qquad \left.\frac{dp}{d\theta}\right|_{(\bar\tau, \bar p)} \neq 0 .$$

Let $\bar\theta = \theta(\bar\tau, \bar p)$; then $\bar\theta > \theta_0$, $(a^0)'(\bar\theta) < 0$ and $(p^0)'(\bar\theta) \neq 0$. Then there exist $\overline{\theta_1} \in (\theta_0, \bar\theta)$ and $\overline{\theta_2} > \bar\theta$ such that $a^0(\overline{\theta_1}) > a^0(\bar\theta) > a^0(\overline{\theta_2})$. By Lemma 5.3, there exist $\theta_1, \theta_2, \theta_3$ and $a > 0$ with $\theta_0 < \theta_1 < \theta_2 < \theta_3$, such that $a^0(\theta_i) = a$, $(a^0)'(\theta_i) \neq 0$ and $(p^0)'(\theta_i) \neq 0$ $(i = 1, 2, 3)$. Let u_0 be arbitrary and let $U_\ell^0 = (\tau_0, p_0, u_0)$, $U_r^0 = (\tau_0, p_0, u_0 - 2a)$. By Lemma 5.2 the R. P. with data (U_ℓ^0, U_r^0) has three type I solutions for which the 3-fold nonuniqueness is stable. ∎

In preparation for Th. 5.6, we prove the following lemma, which gives a sufficient condition for stability in the case of type II nonuniqueness.

Lemma 5.5. Suppose $(\tau_1, \bar p), (\tau_2, \bar p) \subset H \cap Q$ with $\tau_1 < \tau_2$ and

$$H_\tau(\tau_1, \bar p), H_\tau(\tau_2, \bar p) \neq 0 , \tag{5.25}$$

$$\left.\frac{da}{dp}\right|_{(\tau_1, \bar p)} + \left.\frac{da}{dp}\right|_{(\tau_2, \bar p)} \neq 0 . \tag{5.26}$$

Let u_0 be arbitrary and let $U_\ell^0 = (\tau_0, p_0, u_0)$, $U_r^0 = (\tau_0, p_0, u_0 - (a_0 + b_0))$, where $a_0 = \eta^0(\tau_2, \bar p)$, $b_0 = \eta^0(\tau_1, \bar p)$. Then type II nonuniqueness holds for the data (U_ℓ^0, U_r^0), and the (2-fold) nonuniqueness is stable.

Proof: First recall that the R. P. with data (U_ℓ^0, U_r^0) has the two type II solutions,

$$U_\ell^0 \xrightarrow{1\text{-shock}} (\tau_1, \bar p, u_\ell - b_0) \xrightarrow{\text{contact}} (\tau_2, \bar p, u_\ell - b_0) \xrightarrow{3\text{-shock}} U_r^0 , \tag{5.27}$$

$$U_\ell^0 \xrightarrow{\text{1-shock}} (\tau_2, \bar{p}, u_\ell - a_0) \xrightarrow{\text{contact}} (\tau_1, \bar{p}, u_\ell - a_0) \xrightarrow{\text{3-shock}} U_r^0 .$$

By (5.25), $H_\tau(\tau_0, p_0; \tau_1, \bar{p})$, $H_\tau(\tau_0, p_0; \tau_2, \bar{p}) \neq 0$ (see (5.6)); hence we can find a neighborhood N of $(\tau_0, p_0, \bar{p})$ and C^1 functions V_1 and V_2 on N such that, abbreviating $V_i(\tau_c, p_c; p) = V_i^c(p)$, we have

$$V_i^0(\bar{p}) = \tau_i \qquad (i = 1, 2) , \tag{5.28}$$

$$H(\tau_c, p_c; V_i^c(p), p) = 0 \qquad ((\tau_c, p_c, p) \in N, \ i = 1, 2), \tag{5.29}$$

$$(\tau_c, p_c, p) \in N \Longrightarrow p > p_c , \tag{5.30}$$

$$(\tau_c, p_c, p), (\tau_d, p_d, q) \in N \Longrightarrow V_1^c(p) < V_2^d(q) . \tag{5.31}$$

Let $\mathcal{V} = \{(\tau_\ell, p_\ell, \tau_r, p_r, p, q) \mid (\tau_\ell, p_\ell, p), (\tau_\ell, p_\ell, q), (\tau_r, p_r, p), (\tau_r, p_r, q) \in N\}$; $\mathcal{V}$ is an open set containing $(\tau_0, p_0, \tau_0, p_0, \bar{p}, \bar{p})$. For $(\tau_\ell, p_\ell, \tau_r, p_r, p, q) \in \mathcal{V}$, we define

$$a_r = \eta^r(V_2^r(p), p) ,$$

$$b_r = \eta^r(V_1^r(q), q) ,$$

$$a_\ell = \eta^\ell(V_2^\ell(q), q) ,$$

$$b_\ell = \eta^\ell(V_1^\ell(p), p) .$$

(See Fig. 29; $\mathcal{H}_a^c$ is the upper branch of the hyperbola $(\tau_c - \tau)(p - p_c) = a$.) Suppose u_ℓ and u_r satisfy

$$u_r = u_\ell - (a_r + b_\ell) ; \tag{5.32}$$

then by (5.29) and (5.30), the R. P. with data $(U_\ell, U_r) = (\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, u_r)$ has the solution,

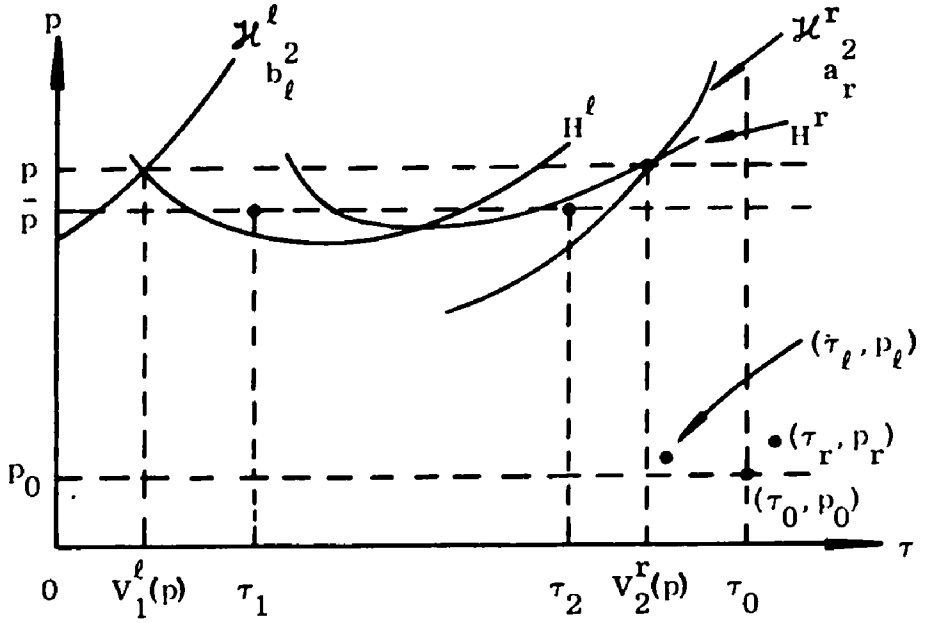


Fig. 29: One of the solutions.

(5.33) $$U_\ell \xrightarrow{1\text{-shock}} (V_1^\ell(p), p, u_\ell - b_\ell) \xrightarrow{\text{contact}} (V_2^r(p), p, u_r + a_r) \xrightarrow{3\text{-shock}} U_r ,$$

since $u_\ell - b_\ell = u_r + a_r$ by (5.32). (See Fig. 29.) If, on the other hand, u_ℓ and u_r satisfy

(5.34) $$u_r = u_\ell - (a_\ell + b_r) ,$$

then $u_\ell - a_\ell = u_r + b_r$, and the R. P. with data (U_ℓ, U_r) has the solution,

(5.35) $$U_\ell \xrightarrow{1\text{-shock}} (V_2^\ell(q), q, u_\ell - a_\ell) \xrightarrow{\text{contact}} (V_1^r(q), q, u_r + b_r) \xrightarrow{3\text{-shock}} U_r .$$

Thus if u_ℓ and u_r satisfy (5.32) and (5.34) simultaneously, then the R. P. with data (U_ℓ, U_r) has the two solutions (5.33) and (5.35), which are distinct by (5.31). But (5.32) and (5.34) hold iff

(5.36) $$a_\ell + b_r = a_r + b_\ell .$$

Define $\Phi(\tau_\ell, p_\ell, \tau_r, p_r, p, q) = a_\ell + b_r - a_r - b_\ell$ $((\tau_\ell, p_\ell, \tau_r, p_r, p, q) \in \mathcal{V})$. (Note

that, for example, a_ℓ is defined as a number depending on the variables τ_ℓ, $p_\ell, q.$) Then $\Phi(\tau_0, p_0, \tau_0, p_0, \bar{p}, \bar{p}) = 0$ (since, for example, $a_\ell(\tau_0, p_0, \tau_0, p_0, \bar{p}, \bar{p}) = \eta^0(V_2^0(\bar{p}), \bar{p}) = \eta^0(\tau_2, \bar{p}) = a_0$). Moreover,

$$\frac{\partial}{\partial q}\,\Phi(\tau_\ell, p_\ell, \tau_r, p_r, p, q) = \frac{\partial}{\partial q}\,(a_\ell + b_r - a_r - b_\ell) = \frac{\partial a_\ell}{\partial q} + \frac{\partial b_r}{\partial q} - 0 - 0$$

$$= \frac{d}{dq}\,\eta^\ell(V_2^\ell(q), q) + \frac{d}{dq}\,\eta^r(V_1^r(q), q)\ ;$$

then

$$\frac{\partial \Phi}{\partial q}\,(\tau_0, p_0, \tau_0, p_0, \bar{p}, \bar{p}) = \frac{d}{dq}\,\eta^0(V_2^0(q), q)\Big|_{q=\bar{p}} + \frac{d}{dq}\,\eta^0(V_1^0(q), q)\Big|_{q=\bar{p}}$$

$$= \frac{da}{dp}\Big|_{(\tau_2, \bar{p})} + \frac{da}{dp}\Big|_{(\tau_1, \bar{p})} \neq 0\ ,$$

by (5.6), (5.28), (5.29), Conventions 4.6 and 4.9, and (5.26). Thus by the implicit function theorem, there exist a neighborhood $\mathcal{W}$ of $(\tau_0, p_0, \tau_0, p_0, \bar{p})$ and a C^1 function $q = q(\tau_\ell, p_\ell, \tau_r, p_r, p)$ $((\tau_\ell, p_\ell, \tau_r, p_r, p) \in \mathcal{W})$ such that $q(\tau_0, p_0, \tau_0, p_0, \bar{p}) = \bar{p}$ and

(5.37) $$\Phi(\tau_\ell, p_\ell, \tau_r, p_r, p, q(\tau_\ell, p_\ell, \tau_r, p_r, p)) = 0 \qquad ((\tau_\ell, p_\ell, \tau_r, p_r, p) \in \mathcal{W})\ .$$

Henceforth we shall abbreviate $q(\tau_\ell, p_\ell, \tau_r, p_r, p) = q$. Thus q is a number depending continuously on the variables $\tau_\ell, p_\ell, \tau_r, p_r$ and p. Using this q in the definitions of a_ℓ and b_r above, we see that a_r, b_r, a_ℓ and b_ℓ likewise are numbers depending continuously on the variables $\tau_\ell, p_\ell, \tau_r, p_r$ and p. Here q, a_r, b_r, a_ℓ and b_ℓ are defined whenever $(\tau_\ell, p_\ell, \tau_r, p_r, p) \in \mathcal{W}$. Thus (5.37) becomes

(5.38) $$a_\ell + b_r = a_r + b_\ell \qquad ((\tau_\ell, p_\ell, \tau_r, p_r, p) \in \mathcal{W})\ .$$

Now for $(\tau_\ell, p_\ell, \tau_r, p_r, p) \in \mathcal{W}$ and u_ℓ arbitrary, we define

(5.39) $$\Psi(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, p) = (\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, u_\ell - (a_r + b_\ell))\ .$$

Clearly Ψ has Jacobian determinant given by

$$J_\Psi(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, p) = \frac{\partial}{\partial p}(u_\ell - (a_r + b_\ell)) = -\left(\frac{\partial}{\partial p} a_r + \frac{\partial}{\partial p} b_\ell\right)$$

$$= -\left(\frac{d}{dp}\eta^r(V_2^r(p), p) + \frac{d}{dp}\eta^\ell(V_1^\ell(p), p)\right)$$

$$((\tau_\ell, p_\ell, \tau_r, p_r, p) \in \mathcal{W}).$$

Then

$$-J_\Psi(\tau_0, p_0, u_0, \tau_0, p_0, \bar{p}) = \frac{d}{dp}\eta^0(V_2^0(p), p)\Big|_{p=\bar{p}} + \frac{d}{dp}\eta^0(V_1^0(p), p)\Big|_{p=\bar{p}}$$

$$= \frac{da}{dp}\Big|_{(\tau_2, \bar{p})} + \frac{da}{dp}\Big|_{(\tau_1, \bar{p})} \neq 0 ;$$

moreover $\Psi(\tau_0, p_0, u_0, \tau_0, p_0, \bar{p}) = (\tau_0, p_0, u_0, \tau_0, p_0, u_0 - (a_0 + b_0)) = (U_\ell^0, U_r^0)$. Hence there exist neighborhoods $\mathcal{N}$ of $(\tau_0, p_0, u_0, \tau_0, p_0, \bar{p})$ and $\mathcal{N}$ of (U_ℓ^0, U_r^0) such that

$$(\tau_\ell, p_\ell, u_\ell, \tau_r, p_r, p) \in \mathcal{N} \Rightarrow (\tau_\ell, p_\ell, \tau_r, p_r, p) \in \mathcal{W},$$

and $\Psi|_{\mathcal{N}}$ is a C^1 diffeomorphism of $\mathcal{N}$ onto $\mathcal{N}$. Thus, for any data $(U_\ell, U_r) \in \mathcal{N}$, there exists a number p such that

$$(\tau_\ell, p_\ell, \tau_r, p_r, p) \in \mathcal{W}, \tag{5.40}$$

$$u_r = u_\ell - (a_r + b_\ell) \tag{5.41}$$

(the latter by (5.39)); p depends continuously on (U_ℓ, U_r), and $p = \bar{p}$ when $(U_\ell, U_r) = (U_\ell^0, U_r^0)$. Moreover, by (5.40) there is a number q, also depending continuously on $(U_\ell, U_r) \in \mathcal{N}$, such that (5.36) holds; $q = \bar{p}$ when $(U_\ell, U_r) = (U_\ell^0, U_r^0)$. Therefore whenever $(U_\ell, U_r) \in \mathcal{N}$, numbers p and q exist such that the R. P. with data (U_ℓ, U_r) has the two distinct solutions (5.33) and (5.35); as $(U_\ell, U_r) \to (U_\ell^0, U_r^0)$, the constant states of the solutions (5.33) and (5.35) approach the respective constant states of the solutions (5.27), since $p, q \to \bar{p}$. ■

Lemmas 5.2, 5.3 and 5.5 give sufficient conditions for the occurrence of type I and type II nonuniqueness and their stability. Thus for our final

result—that when WEAK fails we can get type I and type II nonuniqueness simultaneously, as well as their stability, for the same initial data—we shall have to construct data (U_ℓ^0, U_r^0) that at once satisfy the hypotheses of Lemmas 5.2-5.3 and those of Lemma 5.5.

Theorem 5.6. Suppose e violates WEAK. Then there exist data (U_ℓ^0, U_r^0) for which both 3-fold type I nonuniqueness and type II nonuniqueness hold, and the 5-fold nonuniqueness is stable.

Proof: Let (τ_0, p_0), N, V_1, V_2 and V_3 be as in Lemma 4.14. Choose a $\bar{p} \in N$. Define

$$a_i(p) = \eta^0(V_i(p), p) \qquad (p \in N,\ i = 1,2,3)\ . \tag{5.42}$$

By (4.27), $\left.\frac{d\tau}{dp}\right|_{(V_2(\bar{p}),\,\bar{p})} < 0$; hence $\left.\frac{da}{dp}\right|_{(V_2(\bar{p}),\,\bar{p})} > 0$, i.e., $a_2'(\bar{p}) > 0$. It follows that for some i, j with $1 \le i < j \le 3$, $a_i'(\bar{p}) + a_j'(\bar{p}) \neq 0$. By making N smaller if necessary, keeping $\bar{p} \in N$, we may assume

$$\frac{d}{dp}\left[a_i(p) + a_j(p)\right] \neq 0 \qquad (p \in N)\ . \tag{5.43}$$

To simplify notation, we replace "i" by "1" and "j" by "2"; then the following still hold:

$$N \text{ is an open interval; } N \subset (p_0, +\infty)\ ; \tag{5.44)*}$$

$$V_1 \text{ and } V_2 \text{ are } C^1 \text{ on } N\ ; \tag{5.45)*}$$

$$(V_i(p), p) \in H,\ H_\tau(V_i(p), p) \neq 0 \qquad (p \in N,\ i = 1,2)\ ; \tag{5.46)*}$$

$$V_1(p) < V_2(p) \qquad (p \in N)\ . \tag{5.47)*}$$

(The asterisk is here used to single out a class of important conditions, as will become apparent shortly.) Define the function b by

(5.48) $$b = \frac{1}{2}(a_1 + a_2) \ ;$$

then

(5.49)* $$b'(p) \neq 0 \qquad (p \in N) \ .$$

Consequently

(5.50) b(N) is an open interval.

Let $\bar{\theta}_i = \theta(V_i(\bar{p}), \bar{p})$ $(i = 1, 2)$; then $\bar{\theta}_1 < \bar{\theta}_2$. (See Fig. 30.) Moreover,

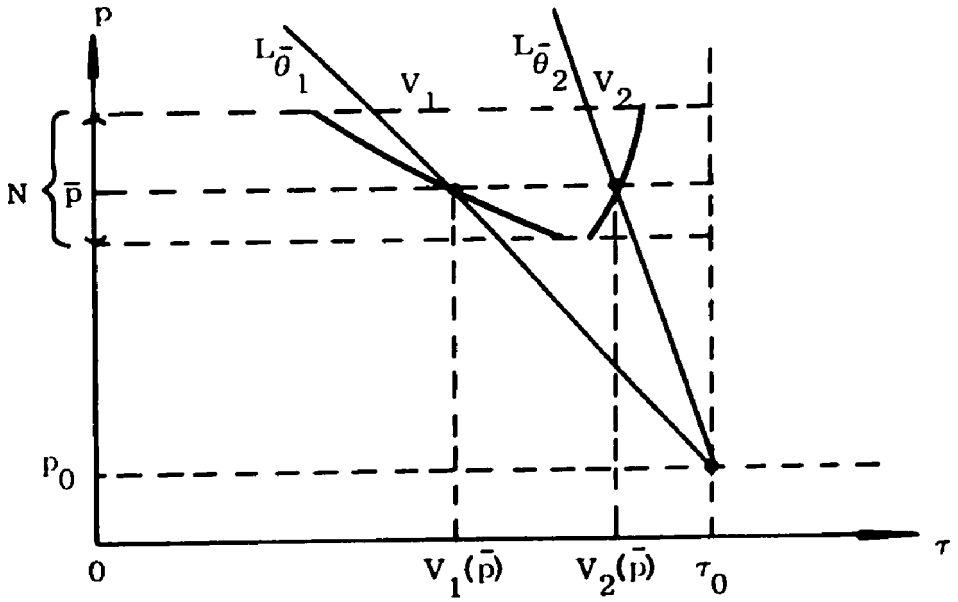


Fig. 30: The functions V_1 and V_2.

$a^0(\bar{\theta}_i) = a_i(\bar{p})$ $(i = 1, 2)$, so by (5.47) and (5.48),

(5.51) $$a^0(\bar{\theta}_2) < a^0(\bar{\theta}_1) \ ,$$

$$b(\bar{p}) = \frac{1}{2}\left[a^0(\bar{\theta}_1) + a^0(\bar{\theta}_2)\right] \ .$$

Thus $a^0(\bar{\theta}_2) < b(\bar{p}) < a^0(\bar{\theta}_1)$, so $\exists \bar{\theta} \in (\bar{\theta}_1, \bar{\theta}_2)$ such that $a^0(\bar{\theta}) = b(\bar{p})$. (See Fig. 31.) Thus $\bar{\theta}$ satisfies

(5.52)* $$\theta_0 < \bar{\theta}_1 < \bar{\theta} < \bar{\theta}_2 \ ,$$

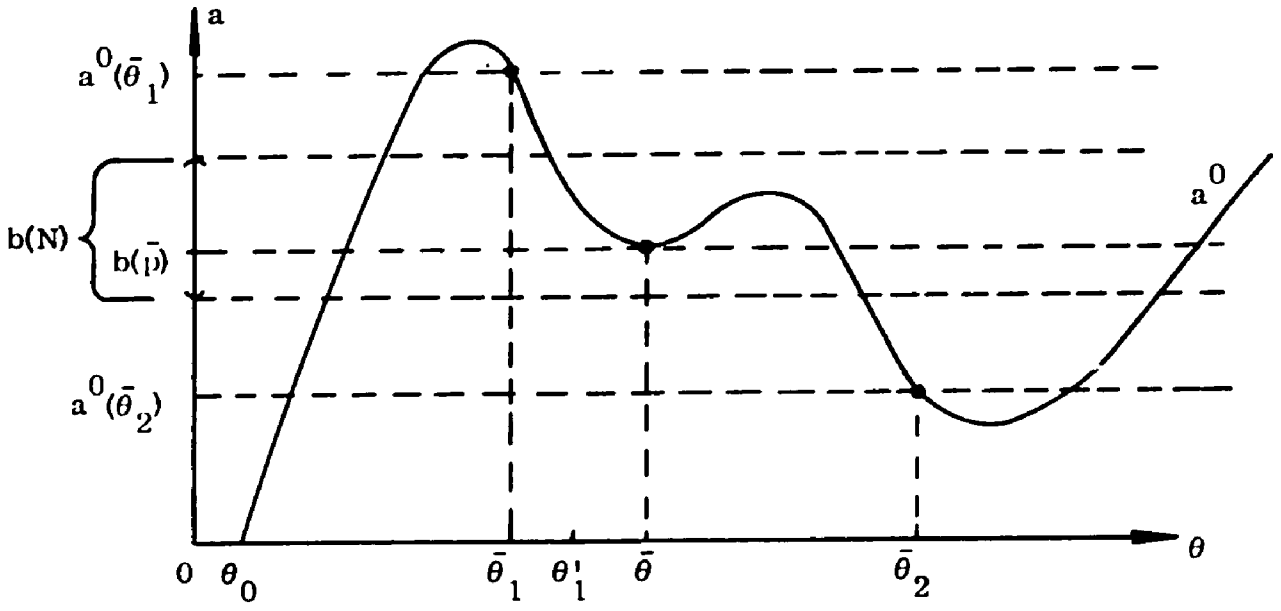


Fig. 31: The behavior of a^0.

(5.53)* $$a^0(\overline{\theta_1}) > a^0(\bar{\theta}) > a^0(\overline{\theta_2}) \ ,$$

(5.54)* $$a^0(\bar{\theta}) \in b(N) \ .$$

Notice that by (5.44), (5.46), (5.47) and (5.49), any $p \in N$ yields points $(V_1(p), p)$, $(V_2(p), p)$ to which Lemma 5.5 can be applied. But in order to apply Lemma 5.3 to $\overline{\theta_1}$, $\bar{\theta}$, $\overline{\theta_2}$ and b(N), we must have

(5.55) $$(a^0)'(\bar{\theta}) \neq 0 \ ,$$

(5.56) $$(p^0)'(\bar{\theta}) \neq 0 \ ,$$

either of which may fail. (See Fig. 31.) We denote the conditions (5.44) -(5.47), (5.49), (5.52) -(5.54) together by (*). We have constructed objects (τ_0, p_0), N, $V_1, V_2, \overline{\theta_1}, \bar{\theta}$ and $\overline{\theta_2}$ that satisfy (*); from these, we shall proceed to construct corresponding objects that satisfy (5.55) and (5.56) as well as (*).

We begin with (5.55). In the above, $\bar{\theta}$ can be taken to be the smallest $\theta \in (\overline{\theta_1}, \overline{\theta_2})$ for which $a^0(\theta) = b(\bar{p})$. (See Fig. 31.) Next, by (5.52) we may pick θ_1' such that

(5.57) $$\overline{\theta_1} < \theta_1' < \bar{\theta} \ .$$

Then, by making N smaller if necessary, keeping $\bar{p} \in N$, we may assume

(5.58) $$\theta(V_1(p), p) < \theta_1' ,$$

$$\theta(V_2(p), p) > \bar{\theta} \qquad (p \in N) .$$

Now by (5.50), the minimality of $\bar{\theta}$, and the usual mean value theorem argument, we can find $\theta_2 \in (\theta_1', \bar{\theta})$ such that

(5.59) $$a^0(\theta_2) \in b(N) ,$$

(5.60) $$(a^0)'(\theta_2) < 0 .$$

By (5.59), $\exists p' \in N$ such that

(5.61) $$a^0(\theta_2) = b(p') .$$

Then by (5.58),

(5.62) $$\theta_0 < \theta(V_1(p'), p') < \theta_1' < \theta_2 < \bar{\theta} < \theta(V_2(p'), p') .$$

Let us now substitute into (*) and (5.55): θ_2 for $\bar{\theta}$, $\theta(V_1(p'), p')$ for $\overline{\theta_1}$, $\theta(V_2(p'), p')$ for $\overline{\theta_2}$. Clearly (5.44) -(5.47) and (5.49), being unaffected, remain true. (5.52) follows from (5.62). But (5.62) also implies $a_1(p') > a_2(p')$. Then (5.53) is a consequence of (5.61) and the definition of b. (5.54) and (5.55) follow from (5.59) and (5.60). Thus (*) and (5.55) hold under this substitution.

Finally, we must find (τ_0, p_0), N, V_1, V_2, $\overline{\theta_1}$, $\bar{\theta}$ and $\overline{\theta_2}$ that satisfy (*), (5.55) and (5.56). If (5.56) holds under the above substitution (using $\theta(V_1(p'), p')$ for $\overline{\theta_1}$, etc.), we are done. Hence we may assume (5.56) is false under this substitution, i.e., $(p^0)'(\theta_2) = 0$. Let $(\tau_2, p_2) = (\tau^0(\theta_2), p^0(\theta_2))$; then

(5.63) $$\theta_2 = \theta(\tau_2, p_2) ,$$

(5.64) $$\left.\frac{dp}{d\theta}\right|_{(\tau_2, p_2)} = 0 .$$

If $H_\tau(\tau_2, p_2) \neq 0$, then τ is a C^1 function of p along H near (τ_2, p_2), so $\left.\frac{d\theta}{dp}\right|_{(\tau_2, p_2)}$ exists, which contradicts (5.64). Hence $H_\tau(\tau_2, p_2) = 0$, so by Lemma 2.6(c), $H_p(\tau_2, p_2) \neq 0$. Then p is a C^1 function of τ along H near (τ_2, p_2). Thus $\left.\frac{da}{d\tau}\right|_{(\tau_2, p_2)}$ exists, and $\left.\frac{da}{d\tau}\right|_{(\tau_2, p_2)} \cdot \left.\frac{d\tau}{d\theta}\right|_{(\tau_2, p_2)} = \left.\frac{da}{d\theta}\right|_{(\tau_2, p_2)} \neq 0$, by (5.60) and the chain rule; hence

$$\left.\frac{da}{d\tau}\right|_{(\tau_2, p_2)} \neq 0 \ . \tag{5.65}$$

We now return to our expanded notation, defining $H(\tau_c, p_c; \tau, p)$ and $\eta(\tau_c, p_c; \tau, p)$ by (5.6) and (5.11), and also defining

$$\theta(\tau_c, p_c; \tau, p) = \frac{p - p_c}{\tau_c - \tau} \qquad (\tau \neq \tau_c) \ . \tag{5.66}$$

By (5.46), we have

$$H(\tau_0, p_0; V_i(p'), p') = 0 \ , \quad H_\tau(\tau_0, p_0; V_i(p'), p') \neq 0 \qquad (i = 1, 2) \ . \tag{5.67}$$

Hence there exist a neighborhood $\mathcal{U}$ of (τ_0, p_0, p') and C^1 functions V_1 and V_2 defined on $\mathcal{U}$ such that

$$V_i(\tau_0, p_0; p') = V_i(p') \qquad (i = 1, 2) \ , \tag{5.68}$$

$$H(\tau_c, p_c; V_i(\tau_c, p_c; p), p) = 0 \qquad ((\tau_c, p_c, p) \in \mathcal{U}, \ i = 1, 2) \ . \tag{5.69}$$

(Note that we do not assert $V_i(\tau_0, p_0; p) = V_i(p)$ $(p \in N, \ i = 1, 2)$.) Define

$$a(\tau_c, p_c) = \eta(\tau_c, p_c; \tau_2, p_2) \equiv \eta^c(\tau_2, p_2) \ , \tag{5.70}$$

$$\begin{aligned} b(\tau_c, p_c; p) &= \frac{1}{2}\left[\eta(\tau_c, p_c; V_1(\tau_c, p_c; p), p) + \eta(\tau_c, p_c; V_2(\tau_c, p_c; p), p)\right] \\ &= \frac{1}{2}\left[\eta^c(V_1^c(p), p) + \eta^c(V_2^c(p), p)\right] \qquad ((\tau_c, p_c, p) \in \mathcal{U}) \ , \end{aligned} \tag{5.71}$$

where we have abbreviated $V_i^c(p) = V_i(\tau_c, p_c; p)$, $\eta^c(\tau, p) = \eta(\tau_c, p_c; \tau, p)$. Then

$$\frac{\partial}{\partial p}\, b(\tau_c, p_c; p) = \frac{1}{2} \sum_{i=1}^{2} \Big[\eta_\tau(\tau_c, p_c; V_i(\tau_c, p_c; p), p) \frac{\partial}{\partial p} V_i(\tau_c, p_c; p) + \eta_p(\tau_c, p_c; V_i(\tau_c, p_c; p), p)\Big]$$

$$= \frac{1}{2} \sum_{i=1}^{2} \Big[-\eta_\tau(\tau_c, p_c; V_i^c(p), p) \frac{H_p(\tau_c, p_c; V_i^c(p), p)}{H_\tau(\tau_c, p_c; V_i^c(p), p)} + \eta_p(\tau_c, p_c; V_i^c(p), p)\Big] ;$$

Hence

$$\frac{\partial b}{\partial p}(\tau_0, p_0; p') = \frac{1}{2} \sum_{i=1}^{2} \Big[-\eta_\tau^0(V_i(p'), p') \frac{H_p(V_i(p'), p')}{H_\tau(V_i(p'), p')} + \eta_p^0(V_i(p'), p')\Big]$$

$$= \frac{1}{2} \sum_{i=1}^{2} \Big[\eta_\tau^0(V_i(p'), p') V_i'(p') + \eta_p^0(V_i(p'), p')\Big] = b'(p') ,$$

by (5.48), (5.42) and (5.46). Then since $b'(p') \neq 0$ by (5.49),

(5.72) $$\frac{\partial b}{\partial p}(\tau_0, p_0; p') \neq 0 .$$

Moreover, from (5.68), (5.47), (5.46), (5.62) and (5.63) we get

(5.73) $$V_1(\tau_0, p_0; p') < V_2(\tau_0, p_0; p') ,$$

(5.74) $$H_\tau(\tau_0, p_0; V_i^0(p'), p') \neq 0 \qquad (i = 1, 2) ,$$

(5.75) $$\theta(\tau_0, p_0; V_1^0(p'), p') < \theta(\tau_0, p_0; \tau_2, p_2) < \theta(\tau_0, p_0; V_2^0(p'), p') .$$

Hence $\exists$ a neighborhood $\mathcal{W}$ of (τ_0, p_0, p') with $\mathcal{W} \subset \mathcal{U}$ such that $\forall (\tau_c, p_c, p) \in \mathcal{W}$, we have

(5.76) $$p > p_c ,$$

(5.77) $$H_\tau(\tau_c, p_c; V_i^c(p), p) \neq 0 \qquad (i = 1, 2) ,$$

(5.78) $$\frac{\partial b}{\partial p}(\tau_c, p_c; p) \neq 0 ,$$

(5.79) $$\theta(\tau_c, p_c; V_1^c(p), p) < \theta(\tau_c, p_c; \tau_2, p_2) < \theta(\tau_c, p_c; V_2^c(p), p) \quad ,$$

(5.80) $$V_1^c(p) < V_2^c(p) \quad .$$

Define $\Phi(\tau_c, p_c, p) = (\tau_c, p_c, b(\tau_c, p_c; p))$ $((\tau_c, p_c, p) \in \mathcal{W})$; then $J_\Phi(\tau_0, p_0, p') = \frac{\partial b}{\partial p}(\tau_0, p_0; p') \neq 0$ by (5.72), so $\exists$ an open set $\mathcal{V}$ such that

(5.81) $$\Phi(\tau_0, p_0, p') \in \mathcal{V} \subset \Phi(\mathcal{W}) \quad .$$

Now define $\Psi(\tau_c, p_c) = (\tau_c, p_c, a(\tau_c, p_c))$; Ψ is continuous. By (5.61), (5.63) and the definitions, $a(\tau_0, p_0) = \eta^0(\tau_2, p_2) = a^0(\theta_2) = b(p') = b(\tau_0, p_0; p')$; hence $\Psi(\tau_0, p_0) = (\tau_0, p_0, a(\tau_0, p_0)) = (\tau_0, p_0, b(\tau_0, p_0; p')) = \Phi(\tau_0, p_0, p') \in \mathcal{V}$ by (5.81), so

(5.82) $$\Psi(\tau_0, p_0) \in \mathcal{V} \quad .$$

Let H^2 be the Hugoniot curve with center (τ_2, p_2). Then if $(\tau_c, p_c) \in H^2$ and H^c is the Hugoniot curve with center (τ_c, p_c), we have

(5.83) $$\left.\frac{da}{d\tau}\right|_{(\tau_2, p_2)} \text{(along } H^c\text{)}$$

$$= \frac{1}{2\eta(\tau_c, p_c; \tau_2, p_2)} \left[(\tau_c - \tau_2) \cdot \left(\left.\frac{dp}{d\tau}\right|_{(\tau_2, p_2)} \text{(along } H^c\text{)}\right) + (p_2 - p_c)\right]$$

$$= \frac{1}{2\eta^c(\tau_2, p_2)} \left[-(\tau_c - \tau_2) \frac{H_\tau(\tau_c, p_c; \tau_2, p_2)}{H_p(\tau_c, p_c; \tau_2, p_2)} + (p_2 - p_c)\right] \quad ,$$

whenever $H_p(\tau_c, p_c; \tau_2, p_2) \neq 0$. (Here $a = \sqrt{(\tau_c - \tau)(p - p_c)}$.) We know that $H_\tau(\tau_2, p_2) = 0$ and $H_p(\tau_2, p_2) \neq 0$. Thus

(5.84) $$H_p(\tau_0, p_0; \tau_2, p_2) \neq 0 \quad ,$$

(5.85) $$H_\tau(\tau_0, p_0; \tau_2, p_2) = 0 \quad .$$

Also by (2.2), $H_\tau(\tau_c, p_c; \tau_2, p_2) = (-g_\tau(\tau_2, p_2)) \frac{\partial}{\partial p} e(\tau_2, p_2) - \frac{p_2 - p_c}{2}$ $((\tau_c, p_c) \in H^2)$; this and (5.85) imply

$$H_\tau(\tau_c, p_c; \tau_2, p_2) \neq 0 \qquad ((\tau_c, p_c) \in H^2,\ p_c \neq p_0) \ . \tag{5.86}$$

Also by (5.65),

$$\left.\frac{da}{d\tau}\right|_{(\tau_2, p_2)} (\text{along } H^0) \neq 0 \ . \tag{5.87}$$

Combining (5.83) and (5.87), and using (5.84) and (5.82), we see that we may choose $(\tau_c, p_c) \in H^2$ sufficiently near (τ_0, p_0), with $p_c \neq p_0$, so that

$$\left.\frac{da}{d\tau}\right|_{(\tau_2, p_2)} (\text{along } H^c) \neq 0 \ , \tag{5.88}$$

$$H_p(\tau_c, p_c; \tau_2, p_2) \neq 0 \ , \tag{5.89}$$

$$\Psi(\tau_c, p_c) \in \mathcal{V} \ , \tag{5.90}$$

and then it also follows from (5.86) that

$$H_\tau(\tau_c, p_c; \tau_2, p_2) \neq 0 \ . \tag{5.91}$$

By (5.81) and (5.90), $\Psi(\tau_c, p_c) = \Phi(\tau_d, p_d, \bar{p})$ for some $(\tau_d, p_d, \bar{p}) \in \mathcal{W}$; i.e., $(\tau_c, p_c, a(\tau_c, p_c)) = (\tau_d, p_d, b(\tau_d, p_d; \bar{p}))$, whence $(\tau_d, p_d) = (\tau_c, p_c)$ and

$$a(\tau_c, p_c) = b(\tau_c, p_c; \bar{p}) \ . \tag{5.92}$$

Since $(\tau_c, p_c, \bar{p}) \in \mathcal{W}$, $\exists$ an open interval N' such that

$$\bar{p} \in N' \ , \tag{5.93}$$

$$\{(\tau_c, p_c)\} \times N' \subset \mathcal{W} \ . \tag{5.94}$$

Now in (*), (5.55) and (5.56), let us replace the symbols on the left, in the table below, by the objects on the right.

(τ_0, p_0)	(τ_c, p_c)
N	N'
V_1	V_1^c
V_2	V_2^c
$\overline{\theta_1}$	$\theta(\tau_c, p_c; V_1^c(\bar{p}), \bar{p})$
$\bar{\theta}$	$\theta(\tau_c, p_c; \tau_2, p_2)$
$\overline{\theta_2}$	$\theta(\tau_c, p_c; V_2^c(\bar{p}), \bar{p})$

We assert that (*), (5.55) and (5.56) are true under this replacement. By (5.76) and (5.94), (5.44) holds. Since $\mathcal{W} \subset \mathcal{U}$ and V_1 and V_2 are C^1 on $\mathcal{U}$, (5.94) also yields (5.45). (5.46) follows from (5.69) and (5.77). (Note that under our replacement, "H" in (5.46) is replaced by H^c.) (5.80) implies (5.47). (5.49) follows from (5.48), (5.42), (5.71) and (5.78). (5.52) follows from (5.79). Now in (5.53) and (5.54), a^0 is replaced by the function a^c defined by

$$(5.95) \qquad a^c(\theta') = \eta^c(\tau', p') \,, \qquad \text{where } \theta(\tau_c, p_c; \tau', p') = \theta' \,,$$

and b is replaced by $b(\tau_c, p_c; \cdot)$. Since $a(\tau_c, p_c) = \eta^c(\tau_2, p_2)$, (5.54) follows from (5.92) and (5.93). Moreover, (5.92), (5.71) and (5.95) together yield $a^c(\theta(\tau_c, p_c; \tau_2, p_2)) = \frac{1}{2}[a^c(\theta(\tau_c, p_c; V_1^c(\bar{p}), \bar{p})) + a^c(\theta(\tau_c, p_c; V_2^c(\bar{p}), \bar{p}))]$; this implies (5.53). By (5.88), (5.89) and (5.91), the following also hold under our replacement:

$$(5.96) \qquad H_p(\tau_2, p_2) \neq 0 \,,$$

$$(5.97) \qquad \left.\frac{da}{d\tau}\right|_{(\tau_2, p_2)} \neq 0 \,,$$

$$(5.98) \qquad H_\tau(\tau_2, p_2) \neq 0 \,.$$

By (5.96), p is a C^1 function of τ along H near (τ_2, p_2), so $\left.\frac{d\theta}{d\tau}\right|_{(\tau_2, p_2)}$

exists. Then $\left.\frac{da}{d\theta}\right|_{(\tau_2, p_2)} \cdot \left.\frac{d\theta}{d\tau}\right|_{(\tau_2, p_2)} = \left.\frac{da}{d\tau}\right|_{(\tau_2, p_2)} \neq 0$ by (5.97); hence $\left.\frac{da}{d\theta}\right|_{(\tau_2, p_2)} \neq 0$. Thus (5.55) holds under our replacement. Moreover, by (5.98) $\left.\frac{d\theta}{dp}\right|_{(\tau_2, p_2)}$ exists, so $\left.\frac{dp}{d\theta}\right|_{(\tau_2, p_2)} \neq 0$, i.e., (5.56) holds under our replacement. Thus our assertion is proved, and we have constructed (τ_0, p_0), N, V_1, V_2, $\bar{\theta}_1$, $\bar{\theta}$ and $\bar{\theta}_2$ that satisfy (*), (5.55) and (5.56).

By (5.49), b(N) is an open interval. Hence by (5.52) - (5.56) and Lemma 5.3, there exist θ_1, θ_2, θ_3 and $a \in b(N)$ with $\theta_0 < \theta_1 < \theta_2 < \theta_3$, such that $a^0(\theta_i) = a$, $(a^0)'(\theta_i) \neq 0$ and $(p^0)'(\theta_i) \neq 0$ $(i = 1, 2, 3)$. Let u_0 be arbitrary, and let $U_\ell^0 = (\tau_0, p_0, u_0)$, $U_r^0 = (\tau_0, p_0, u_0 - 2a)$. Then by Lemma 5.2 the R. P. with data (U_ℓ^0, U_r^0) has three type I solutions, and $\exists$ a neighborhood η_I of (U_ℓ^0, U_r^0) such that conditions (a) and (b) of Def. 5.1 hold for $\eta = \eta_I$, $n = 3$; i.e., for any data $(U_\ell, U_r) \in \eta_I$ the R. P. has three solutions, whose constant states approach the corresponding constant states of the three type I solutions of the R. P. with data (U_ℓ^0, U_r^0) as $(U_\ell, U_r) \to (U_\ell^0, U_r^0)$. But since $a \in b(N)$, $a = b(\bar{p})$ for some $\bar{p} \in N$. Let $\tau_1 = V_1(\bar{p})$, $\tau_2 = V_2(\bar{p})$, $a_0 = \eta^0(\tau_2, \bar{p})$, $b_0 = \eta^0(\tau_1, \bar{p})$. Then $a_0 = a_2(\bar{p})$ and $b_0 = a_1(\bar{p})$, so $a_0 + b_0 = 2b(\bar{p}) = 2a$; thus $U_r^0 = (\tau_0, p_0, u_0 - (a_0 + b_0))$. By (5.44) and (5.46), $(\tau_i, \bar{p}) \in H \cap Q$ $(i = 1, 2)$ and (5.25) holds. (5.47) implies $\tau_1 < \tau_2$; (5.49) and (5.46) imply (5.26). Hence by Lemma 5.5, type II nonuniqueness holds for the data (U_ℓ^0, U_r^0), and $\exists$ a neighborhood η_{II} of (U_ℓ^0, U_r^0) such that for any data $(U_\ell, U_r) \in \eta_{II}$, the R. P. has two solutions, whose constant states approach those of the two type II solutions for data (U_ℓ^0, U_r^0) as $(U_\ell, U_r) \to (U_\ell^0, U_r^0)$. Since type I solutions are always different from type II solutions, we can find a neighborhood η such that $(U_\ell^0, U_r^0) \in \eta \subset \eta_I \cap \eta_{II}$ and the R. P. with data $(U_\ell, U_r) \in \eta$ always has five distinct solutions; clearly η also satisfies condition (b) of Def. 5.1 for $n = 5$. ■

CHAPTER 6

EXAMPLES

6.1 Energy functions that distinguish the conditions WEAK, MEDIUM and STRONG

All energy functions e considered in this section are assumed to satisfy

$$\lim_{\tau \to +\infty} e(\tau, S) = 0 \ . \tag{6.1}$$

We shall focus throughout this section on the one-parameter family of functions $g(\cdot, S)$, $-\infty < S < +\infty$; each function is defined on $(0, +\infty)$ with τ as independent variable. To see what properties these functions have in general, let us fix $S \in (-\infty, +\infty)$ and define $\phi(\tau) = g(\tau, S)$ $(\tau > 0)$. We see immediately from (1.3), (1.5) and (1.6) that

$$\phi > 0 \ , \quad \phi' < 0 \ , \quad \phi'' > 0 \ . \tag{6.2}$$

Now let $\tau, V > 0$. Since $g = -e_\tau$, we have

$$\int_\tau^V \phi = \int_\tau^V g(v, S)dv = -\int_\tau^V e_\tau(v, S)dv = e(\tau, S) - e(V, S) < e(\tau, S) < +\infty \ .$$

For $\tau = 1$ we conclude

$$\int_1^{+\infty} \phi < +\infty \ ; \tag{6.3}$$

moreover, letting $V \to +\infty$, (6.1) gives

$$e(\tau, S) = \int_\tau^{+\infty} g(v, S)dv \ . \tag{6.4}$$

Finally, we assert that

$$\lim_{\tau \to 0^+} \phi(\tau) = +\infty \ . \tag{6.5}$$

For suppose not. Then since ϕ is decreasing, $\exists p_1 > 0$ such that $g(\tau, S) = \phi(\tau) \leq p_1$ $(\tau > 0)$. (See Fig. 32.) Let $\tau \in (0, 1)$. By (1.8), $\exists S_1$ such that

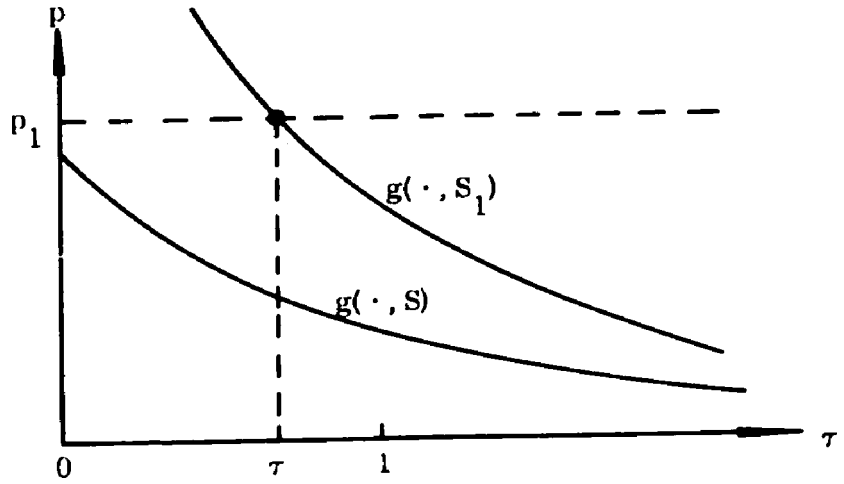


Fig. 32: $g(\cdot, S)$ cannot be bounded above.

$g(\tau, S_1) = p_1$. Then $g(\tau, S) \leq g(\tau, S_1)$, so by (1.7), $S \leq S_1$. Then by (1.3) and (1.4), $e(\tau, p_1) = e(\tau, S_1) \geq e(\tau, S) \geq e(1, S)$. Thus $e(\tau, p_1) \geq e(1, S) > 0$ $\forall \tau \in (0, 1)$, which contradicts (1.10). Hence (6.5) holds. Note that ϕ is C^2.

Motivated by the fact that the functions $g(\cdot, S)$ always satisfy (6.2), (6.3) and (6.5), we make

Definition 6.1. Let $\mathcal{P}$ be the set of all C^2 functions f defined on $(0, +\infty)$ for which $f > 0$, $f' < 0$, $f'' > 0$, $\int_1^{+\infty} f < +\infty$ and $\lim_{\tau \to 0^+} f(\tau) = +\infty$.

"$\mathcal{P}$" signifies pressure. We have just proved

(6.6) $$g(\cdot, S) \in \mathcal{P} \qquad (-\infty < S < +\infty),$$

for every energy function e. (Remember that e always satisfies (6.1) in this section.) In what follows, we shall construct and analyze one-parameter families $g(\cdot, S)$, $-\infty < S < +\infty$, and get our energy functions via (6.4).

Lemma 6.2. Suppose e satisfies (1.2) - (1.9) and (6.1). Suppose also that $g(\cdot, S) \in \mathcal{P}$ for all S. Then e satisfies (1.10); hence e is an energy function.

Proof: Fix $p > 0$. We first prove

(6.7) $$\lim_{\tau \to 0^+} S(\tau, p) = -\infty \ .$$

(See (1.7) - (1.11) ff.) Let $S_0 \in (-\infty, +\infty)$. Since $g(\cdot, S_0) \in \mathcal{P}$, we may choose $\tau_0 > 0$ such that $g(\tau_0, S_0) = p$. Let $\tau \in (0, \tau_0)$ and suppose $S(\tau, p) \geq S_0$. Then $p = g(\tau, S(\tau, p)) > g(\tau_0, S(\tau, p)) \geq g(\tau_0, S_0) = p$, a contradiction. Hence $S(\tau, p) < S_0$. Thus (6.7) is proved.

Now let $\epsilon > 0$. Choose $\tau_1 > 0$ so that $\tau_1 p \leq \frac{1}{2}\epsilon$. Fix any S_1 and note that since $g(\cdot, S_1) \in \mathcal{P}$,

$$\int_{\tau_1}^{+\infty} g(\cdot, S_1) < +\infty \ .$$

By (1.7), $g(\cdot, S) < g(\cdot, S_1)$ whenever $S < S_1$. By (1.8), $g(\cdot, S) \to 0$ pointwise on $[\tau_1, +\infty)$ as $S \to -\infty$. Hence by the Lebesgue convergence theorem,

$$\lim_{S \to -\infty} \int_{\tau_1}^{+\infty} g(\cdot, S) = 0 \ .$$

Thus $\exists M > 0$ such that

(6.8) $$\int_{\tau_1}^{+\infty} g(\cdot, S) < \frac{1}{2}\epsilon \quad \text{whenever} \quad S < -M \ .$$

By (6.7), $\exists \delta \in (0, \tau_1)$ such that $S(\tau, p) < -M$ whenever $\tau \in (0, \delta)$. Let $\tau \in (0, \delta)$. By (1.5), $g(v, S(\tau, p)) \leq g(\tau, S(\tau, p)) = p$ whenever $v \geq \tau$. Then from (6.1) and (6.8) we get

$$e(\tau, p) = e(\tau, S(\tau, p)) = \int_{\tau}^{+\infty} g(v, S(\tau, p))\, dv$$

cont.

$$= \int_{\tau}^{\tau_1} g(v, S(\tau, p))\,dv + \int_{\tau_1}^{+\infty} g(v, S(\tau, p))\,dv < \int_{\tau}^{\tau_1} p\,dv + \frac{1}{2}\epsilon$$

$$= (\tau_1 - \tau)p + \frac{1}{2}\epsilon < \tau_1 p + \frac{1}{2}\epsilon \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \ .$$

Thus (1.10) is proved. ■

Proposition 6.3. Let $\phi \in \mathcal{P}$. Let θ be a C^3 function mapping $(-\infty, +\infty)$ onto $(0, +\infty)$ such that $\theta' > 0$. Define

$$e(\tau, S) = \theta(S) \int_{\tau}^{+\infty} \phi \qquad (\tau > 0, \ -\infty < S < +\infty) \ .$$

Then e is an energy function satisfying MEDIUM. ϕ can be chosen so that e violates STRONG.

Proof: Clearly e is C^3 in (τ, S) and $e > 0$. Define $g = -e_\tau$. Then for all $\tau > 0$ and $S \in (-\infty, +\infty)$,

$$g(\tau, S) = \theta(S)\phi(\tau) > 0 \ ,$$

$$g_\tau(\tau, S) = \theta(S)\phi'(\tau) < 0 \ ,$$

$$g_{\tau\tau}(\tau, S) = \theta(S)\phi''(\tau) > 0 \ ,$$

$$g_S(\tau, S) = \theta'(S)\phi(\tau) > 0 \ ,$$

$$e_S(\tau, S) = \theta'(S) \int_{\tau}^{+\infty} \phi > 0 \ .$$

Thus the Standard Assumptions hold. To prove the Limit Assumptions (1.8) and (1.9), observe that

$$\lim_{S \to -\infty} g(\tau, S) = \lim_{S \to -\infty} \theta(S)\phi(\tau) = \left(\lim_{S \to -\infty} \theta(S) \right) \phi(\tau) = 0 \cdot \phi(\tau) = 0 \ ;$$

the other three relations follow similarly. Clearly $g(\cdot, S) \in \mathcal{P}$ $\forall S$. Then (6.1)

also holds. Hence by Lemma 6.2 , e is an energy function.

Let $\tau, p > 0$. Let S satisfy $g(\tau, S) = p$; then

$$\theta(S)\phi(\tau) = p \ . \tag{6.9}$$

Moreover, $e(\tau, p) = e(\tau, S) = \theta(S) \int_\tau^{+\infty} \phi$; this, together with (6.9), gives

$$e(\tau, p) = \frac{p}{\phi(\tau)} \int_\tau^{+\infty} \phi \qquad (\tau, p > 0) \ . \tag{6.10}$$

We also have

$$g_\tau(\tau, p) = g_\tau(\tau, S) = \theta(S)\phi'(\tau) \ , \tag{6.11}$$

$$\frac{\partial}{\partial p} e(\tau, p) = \frac{1}{\phi(\tau)} \int_\tau^{+\infty} \phi \ . \tag{6.12}$$

Then, using Prop. 1.5 and (6.9) - (6.12),

$$\text{MEDIUM} \iff \frac{\partial}{\partial p} e(\tau, p) \geq \frac{2e(\tau, p)}{-g_\tau(\tau, p) \cdot \frac{2e(\tau, p)}{p} + p}$$

$$\iff \frac{1}{\phi(\tau)} \int_\tau^{+\infty} \phi \geq \frac{2 \frac{p}{\phi(\tau)} \int_\tau^{+\infty} \phi}{-\theta(S)\phi'(\tau) \cdot \frac{2 \frac{p}{\phi(\tau)} \int_\tau^{+\infty} \phi}{p} + p}$$

$$\iff 1 \geq \frac{2p}{-\theta(S)\phi'(\tau) \cdot \frac{2 \frac{p}{\phi(\tau)} \int_\tau^{+\infty} \phi}{p} + p}$$

$$\iff 1 \geq \frac{2}{-\frac{\phi'(\tau)}{\phi(\tau)} \cdot \frac{2}{\phi(\tau)} \int_{\tau}^{+\infty} \phi + 1} \iff -\frac{2\phi'(\tau)}{\phi(\tau)^2} \int_{\tau}^{+\infty} \phi + 1 \geq 2 \ ,$$

which holds by Lemma 1.6. Thus e satisfies MEDIUM.

Finally,

$$\text{STRONG} \iff \frac{\partial}{\partial p} e(\tau, p) \geq \frac{p}{-g_{\tau}} \iff \frac{1}{\phi(\tau)} \int_{\tau}^{+\infty} \phi \geq \frac{\theta(S)\phi(\tau)}{-\theta(S)\phi'(\tau)}$$

$$\iff \int_{\tau}^{+\infty} \phi \geq \phi(\tau) \left(\frac{\phi(\tau)}{-\phi'(\tau)}\right) \ .$$

Referring to Fig. 33, we see that $\phi(\tau)\left(\frac{\phi(\tau)}{-\phi'(\tau)}\right)$ is the area of the rectangle

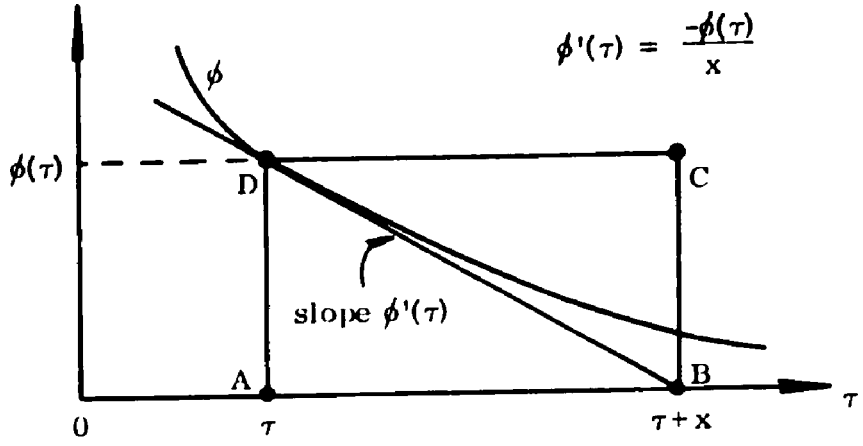


Fig. 33: ϕ can be chosen so that STRONG fails.

ABCD, whereas ϕ can be chosen so as to make $\int_{\tau}^{+\infty} \phi$ arbitrarily close to the area of ΔABD. In particular

$$\int_{\tau}^{+\infty} \phi < \phi(\tau)\left(\frac{\phi(\tau)}{-\phi'(\tau)}\right)$$

for suitable choice of ϕ; STRONG fails for such ϕ. ■

As a corollary we have

Theorem 6.4. There exists an energy function that satisfies MEDIUM but violates STRONG.

To get an example of an energy function that satisfies WEAK but violates MEDIUM, we shall begin by establishing a criterion for WEAK in terms of the one-parameter family $g(\cdot, S)$, $-\infty < S < +\infty$, of elements of $\mathcal{P}$.

Definition 6.5. Let $f, g \in \mathcal{P}$. We say $f \overset{W}{\sim} g$ iff $f \leq g$ and for any $\tau_1, \tau_2, p > 0$ such that $\tau_1 \leq \tau_2$ and $f(\tau_1) = g(\tau_2) = p$, we have

$$\int_{\tau_2}^{+\infty} g \geq \int_{\tau_1}^{+\infty} f - (\tau_2 - \tau_1)\frac{1}{2}p \ .$$

Proposition 6.6. e satisfies WEAK iff for all $S_1, S_2 \in (-\infty, +\infty)$, $S_1 < S_2$ implies $g(\cdot, S_1) \overset{W}{\sim} g(\cdot, S_2)$.

Proof: (1.7) implies that $g(\cdot, S_1) < g(\cdot, S_2)$ whenever $S_1 < S_2$. Now suppose there exist S_1, S_2 with $S_1 < S_2$ such that $\left[g(\cdot, S_1) \overset{W}{\sim} g(\cdot, S_2)\right]$ is false. Then there must be $\tau_1, \tau_2, p > 0$ with $\tau_1 \leq \tau_2$ such that $g(\tau_1, S_1) = g(\tau_2, S_2) = p$ and

$$\int_{\tau_2}^{+\infty} g(v, S_2)dv < \int_{\tau_1}^{+\infty} g(v, S_1)dv - (\tau_2 - \tau_1)\frac{1}{2}p \ . \tag{6.13}$$

But by (6.4),

$$\int_{\tau_i}^{+\infty} g(v, S_i)dv = e(\tau_i, S_i) = e(\tau_i, p) \qquad (i = 1, 2) \ ; \tag{6.14}$$

thus (6.13) becomes

$$e(\tau_2, p) < e(\tau_1, p) - (\tau_2 - \tau_1)\frac{1}{2}p \ . \tag{6.15}$$

If $\tau_1 = \tau_2$, then (6.14) and (6.15) imply $e(\tau_1, S_2) < e(\tau_1, S_1)$, which is false by (1.4). Hence $\tau_1 < \tau_2$, so (6.15) yields

$$\frac{e(\tau_2, p) - e(\tau_1, p)}{\tau_2 - \tau_1} < -\frac{1}{2}p \ . \tag{6.16}$$

$\exists \bar{\tau} \in (\tau_1, \tau_2)$ such that

$$\frac{\partial}{\partial \tau} e(\tau, p)\Big|_{(\bar{\tau}, p)} = \frac{e(\tau_2, p) - e(\tau_1, p)}{\tau_2 - \tau_1} \ ;$$

this and (6.16) imply WEAK is false.

For the reverse implication, suppose $\forall S_1, S_2 \in (-\infty, +\infty)$, $S_1 < S_2$ implies $g(\cdot, S_1) \overset{W}{\sim} g(\cdot, S_2)$. Let $p > 0$ and let $0 < \tau_1 < \tau_2$. Let S_1 and S_2 satisfy $g(\tau_1, S_1) = p = g(\tau_2, S_2)$. Then $S_1 < S_2$; for otherwise $p = g(\tau_1, S_1) \geq g(\tau_1, S_2) > g(\tau_2, S_2) = p$, a contradiction. By assumption, $g(\cdot, S_1) \overset{W}{\sim} g(\cdot, S_2)$; hence

$$\int_{\tau_2}^{+\infty} g(v, S_2)dv \geq \int_{\tau_1}^{+\infty} g(v, S_1)dv - (\tau_2 - \tau_1)\frac{1}{2}p \ . \tag{6.17}$$

But (6.14) holds in this situation, so (6.17) becomes

$$\frac{e(\tau_2, p) - e(\tau_1, p)}{\tau_2 - \tau_1} \geq -\frac{1}{2}p \ . \tag{6.18}$$

Since (6.18) holds for all $\tau_1, \tau_2, p > 0$ with $\tau_1 \neq \tau_2$,

$$\frac{\partial}{\partial \tau} e(\tau, p) \geq -\frac{1}{2}p \qquad (\tau, p > 0) \ ;$$

thus WEAK holds. ■

Proposition 6.7. The relation $\overset{W}{\sim}$ has the following properties.

(a) $f \overset{W}{\sim} g$ and $\alpha > 0 \Rightarrow \alpha f \overset{W}{\sim} \alpha g$.

(b) $f \in \mathcal{P}$ and $0 < \alpha \leq \beta \Rightarrow \alpha f \overset{W}{\sim} \beta f$.

(c) $f \overset{W}{\sim} g$ and $g \overset{W}{\sim} h \Rightarrow f \overset{W}{\sim} h$.

Proof:

(a) Clearly $\alpha f \leq \alpha g$. Let $0 < \tau_1 \leq \tau_2$ with $(\alpha f)(\tau_1) = (\alpha g)(\tau_2) = p$. Then $f(\tau_1) = g(\tau_2) = p/\alpha$, so

$$\int_{\tau_2}^{+\infty} g \geq \int_{\tau_1}^{+\infty} f - (\tau_2 - \tau_1) \frac{1}{2} \frac{p}{\alpha} ,$$

whence

$$\int_{\tau_2}^{+\infty} \alpha g \geq \int_{\tau_1}^{+\infty} \alpha f - (\tau_2 - \tau_1) \frac{1}{2} p .$$

(b) If $\alpha = \beta$, the result is obvious. So assume $\alpha < \beta$. Pick any S_1 and S_2 such that $S_1 < S_2$. Let θ be as in Prop. 6.3 with

$$\theta(S_1) = \alpha , \quad \theta(S_2) = \beta . \tag{6.19}$$

Then by Prop. 6.3, the function e defined by

$$e(\tau, S) = \theta(S) \int_{\tau}^{+\infty} f \qquad (\tau > 0, \ -\infty < S < +\infty)$$

is an energy function satisfying MEDIUM; hence e satisfies WEAK. Then by Prop. 6.6,

$$g(\cdot, S_1) \overset{W}{\sim} g(\cdot, S_2) , \tag{6.20}$$

where $g = -e_\tau$. But $g(\tau, S) = \theta(S) f(\tau)$ $(\tau > 0, \ -\infty < S < +\infty)$, i.e., $g(\cdot, S) = \theta(S) f$ $(-\infty < S < +\infty)$. Hence $\alpha f \overset{W}{\sim} \beta f$ by (6.19) and (6.20).

(c) Since $f \le g$ and $g \le h$, $f \le h$. Let $0 < \tau_1 \le \tau_2$ with $f(\tau_1) = h(\tau_2) = p$. Let $\tau_0 > 0$ satisfy $g(\tau_0) = p$. Then $\tau_1 \le \tau_0 \le \tau_2$. (For if, say, $\tau_1 > \tau_0$, then $p = f(\tau_1) < f(\tau_0) \le g(\tau_0) = p$.) By Def. 6.5, we have

$$\int_{\tau_0}^{+\infty} g \ge \int_{\tau_1}^{+\infty} f - (\tau_0 - \tau_1)\frac{1}{2}p ;$$

$$\int_{\tau_2}^{+\infty} h \ge \int_{\tau_0}^{+\infty} g - (\tau_2 - \tau_0)\frac{1}{2}p \ge \int_{\tau_1}^{+\infty} f - (\tau_0 - \tau_1)\frac{1}{2}p - (\tau_2 - \tau_0)\frac{1}{2}p$$

$$= \int_{\tau_1}^{+\infty} f - (\tau_2 - \tau_1)\frac{1}{2}p \ .$$

Hence $f \overset{w}{\sim} h$. ■

Lemma 6.8. Let $f, g \in \mathcal{P}$ with $f < g$, and define $\pi(\cdot, \lambda) = (1-\lambda)f + \lambda g$. Suppose for all $\tau > 0$,

$$\frac{2}{g(\tau) - f(\tau)} \int_{\tau}^{+\infty} (g - f) \ge \begin{cases} \dfrac{f(\tau)}{-f'(\tau)} \\ \\ \dfrac{g(\tau)}{-g'(\tau)} \end{cases} .$$

Then $\pi(\cdot, \lambda_1) \overset{w}{\sim} \pi(\cdot, \lambda_2)$ whenever $0 \le \lambda_1 \le \lambda_2 \le 1$.

Proof: Since $f, g \in \mathcal{P}$, we may define

$$\epsilon(\tau, \lambda) = \int_{\tau}^{+\infty} \pi(v, \lambda)\,dv = (1-\lambda)\int_{\tau}^{+\infty} f + \lambda \int_{\tau}^{+\infty} g \ . \tag{6.21}$$

Then ϵ is C^3 in (τ, λ). We have

$$\pi_{\lambda}(\tau, \lambda) = g(\tau) - f(\tau) > 0 \ ,$$

so we can define for any $\tau, p > 0$,

$$\epsilon(\tau, p) = \epsilon(\tau, \lambda) \quad \text{where } \lambda \text{ satisfies } \pi(\tau, \lambda) = p.$$

(We then denote $\lambda = \lambda(\tau, p)$.) Similarly, define $\pi_\tau(\tau, p) = \pi_\tau(\tau, \lambda(\tau, p))$ $(\tau, p > 0)$. Now let $\tau, p > 0$, and let $\lambda = \frac{p - f(\tau)}{g(\tau) - f(\tau)}$. Then $1 - \lambda = \frac{g(\tau) - p}{g(\tau) - f(\tau)}$, and $\pi(\tau, \lambda) = (1 - \lambda) f(\tau) + \lambda g(\tau) = p$, whence

$$\lambda(\tau, p) = \frac{p - f(\tau)}{g(\tau) - f(\tau)} \qquad (\tau, p > 0) \ . \tag{6.22}$$

Also note that

$$\pi_\tau(\tau, \lambda) = (1 - \lambda) f'(\tau) + \lambda g'(\tau) \ . \tag{6.23}$$

We assert that

$$\frac{\partial}{\partial p} \epsilon(\tau, p) \geq \frac{-p}{2\pi_\tau(\tau, p)} \qquad (f(\tau) \leq p \leq g(\tau)) \ . \tag{6.24}$$

For by (6.21) and (6.22),

$$\begin{aligned} \epsilon(\tau, p) &= \epsilon(\tau, \lambda(\tau, p)) \\ &= \frac{g(\tau) - p}{g(\tau) - f(\tau)} \int_\tau^{+\infty} f + \frac{p - f(\tau)}{g(\tau) - f(\tau)} \int_\tau^{+\infty} g \ , \end{aligned}$$

whence

$$\frac{\partial}{\partial p} \epsilon(\tau, p) = \frac{1}{g(\tau) - f(\tau)} \int_\tau^{+\infty} (g - f) \ . \tag{6.25}$$

Let us temporarily hold τ constant; define

$$\phi(p) = \frac{-p}{(g(\tau) - p) f'(\tau) + (p - f(\tau)) g'(\tau)} \ .$$

ϕ is defined on an open interval I containing $[f(\tau), g(\tau)]$. For $p \in I$, (6.22) and (6.23) yield

$$(6.26)\qquad -\frac{p}{2\pi_\tau(\tau,p)} = \frac{-p}{2\left[\frac{g(\tau)-p}{g(\tau)-f(\tau)}\,f'(\tau) + \frac{p-f(\tau)}{g(\tau)-f(\tau)}\,g'(\tau)\right]} = \frac{1}{2}(g(\tau)-f(\tau))\,\Phi(p)\ .$$

We have

$$\Phi'(p) = \frac{f(\tau)g'(\tau) - g(\tau)f'(\tau)}{\left[(g(\tau)-p)f'(\tau) + (p-f(\tau))g'(\tau)\right]^2} \qquad (p\in I)\ ;$$

thus, depending on the sign of $f(\tau)g'(\tau) - g(\tau)f'(\tau)$, either $\Phi' \geq 0$ on I or $\Phi' \leq 0$ on I. Hence Φ attains its maximum on $[f(\tau), g(\tau)]$ at an endpoint, $f(\tau)$ or $g(\tau)$. But

$$\Phi(f(\tau)) = \frac{-f(\tau)}{(g(\tau)-f(\tau))f'(\tau)}\ ,$$

$$\Phi(g(\tau)) = \frac{-g(\tau)}{(g(\tau)-f(\tau))g'(\tau)}\ .$$

Using (6.26), we conclude that $-p/2\pi_\tau(\tau,p)$ attains its maximum value for $p\in[f(\tau), g(\tau)]$ either at $p = f(\tau)$ or $p = g(\tau)$, and its maximum value is either $\frac{g(\tau)}{-2g'(\tau)}$ or $\frac{f(\tau)}{-2f'(\tau)}$. But by (6.25) and the hypothesis, $\frac{\partial}{\partial p}\epsilon(\tau,p)$ is always greater than or equal to each of these values; hence (6.24) holds.

From here we proceed by analogy with the proofs of Prop. 1.5 and Prop. 6.6. We denote

$$\epsilon_\tau(\tau,p) = \frac{\partial}{\partial\tau}\epsilon(\tau,\lambda)\Big|_{(\tau,\lambda)\,=\,(\tau,\lambda(\tau,p))}\ ,$$

etc. Then

$$\frac{\partial}{\partial p}\epsilon(\tau,p) = \frac{\partial}{\partial p}\epsilon(\tau,\lambda(\tau,p)) = \epsilon_\lambda\cdot\frac{\partial}{\partial p}\lambda(\tau,p) = \frac{\epsilon_\lambda}{\pi_\lambda}\ ;$$

$$\frac{\partial}{\partial\tau}\epsilon(\tau,p) = \epsilon_\tau + \epsilon_\lambda\frac{\partial}{\partial\tau}\lambda(\tau,p)\ ;$$

$$p = \pi(\tau,\lambda(\tau,p))\ ;$$

$$0 = \pi_\tau + \pi_\lambda\frac{\partial}{\partial\tau}\lambda(\tau,p)\ .$$

Hence

$$\frac{\partial}{\partial\tau}\,\epsilon(\tau,p) = \epsilon_{\tau}(\tau,p)+\epsilon_{\lambda}(\tau,p)\left(\frac{-\pi_{\tau}(\tau,p)}{\pi_{\lambda}(\tau,p)}\right) = -p+(-\pi_{\tau}(\tau,p))\,\frac{\partial}{\partial p}\,\epsilon(\tau,p)\ ,$$

by (6.21). Then by (6.24) and the fact that $\pi_{\tau}(\tau,p)<0$ for $p\in[f(\tau),g(\tau)]$,

$$\frac{\partial}{\partial\tau}\,\epsilon(\tau,p) \geq -\frac{p}{2} \qquad (f(\tau)\leq p\leq g(\tau))\ . \tag{6.27}$$

Now suppose we have $0\leq\lambda_1\leq\lambda_2\leq 1$ such that $\left[\pi(\cdot,\lambda_1)\overset{w}{\sim}\pi(\cdot,\lambda_2)\right]$ is false. Since $f<g$, we know $\pi(\cdot,\lambda_1)\leq\pi(\cdot,\lambda_2)$; thus there exist $\tau_1,\tau_2,p>0$ with $\tau_1\leq\tau_2$ and $\pi(\tau_1,\lambda_1)=\pi(\tau_2,\lambda_2)=p$ such that

$$\int_{\tau_2}^{+\infty}\pi(v,\lambda_2)dv < \int_{\tau_1}^{+\infty}\pi(v,\lambda_1)dv-(\tau_2-\tau_1)\,\frac{1}{2}\,p\ ,$$

i.e.,

$$\epsilon(\tau_2,p) < \epsilon(\tau_1,p)-(\tau_2-\tau_1)\,\frac{1}{2}\,p\ . \tag{6.28}$$

(See (6.21) ff.) But for $\tau\in[\tau_1,\tau_2]$ we have

$$p=\pi(\tau_1,\lambda_1)\geq\pi(\tau,\lambda_1)\geq f(\tau)\ ,$$

$$p=\pi(\tau_2,\lambda_2)\leq\pi(\tau,\lambda_2)\leq g(\tau)\ .$$

(See Fig. 34.) Thus $f(\tau)\leq p\leq g(\tau)$ $(\tau_1\leq\tau\leq\tau_2)$, so (6.27) yields

$$\frac{\partial}{\partial\tau}\,\epsilon(\tau,p)\geq-\frac{p}{2} \qquad (\tau_1\leq\tau\leq\tau_2)\ . \tag{6.29}$$

On the other hand, (6.28) gives

$$\frac{\epsilon(\tau_2,p)-\epsilon(\tau_1,p)}{\tau_2-\tau_1} < -\frac{p}{2}\ ,$$

which contradicts (6.29). Hence $\pi(\cdot,\lambda_1)\overset{w}{\sim}\pi(\cdot,\lambda_2)$ whenever $0\leq\lambda_1\leq\lambda_2\leq 1$. ■

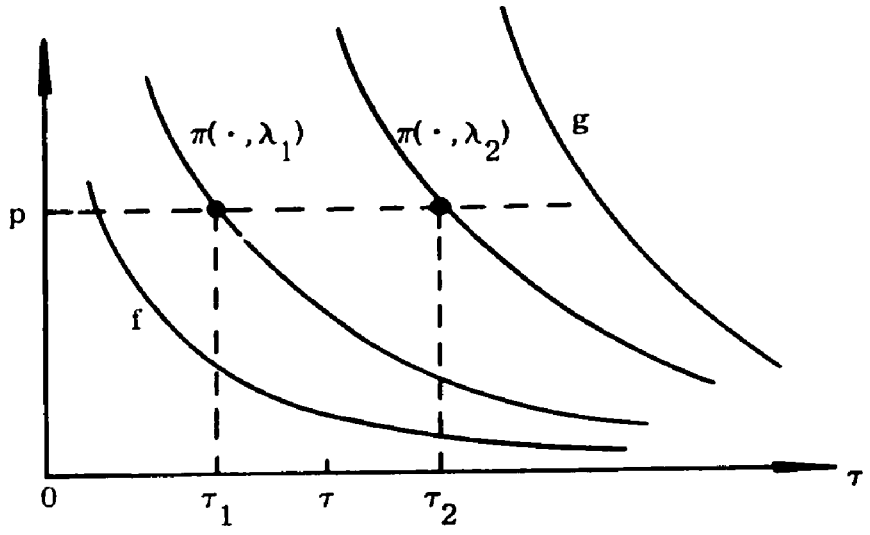


Fig. 34: $\pi(\cdot,\lambda)$ curves $(0 < \lambda_1 < \lambda_2 < 1)$.

Theorem 6.9. There exists an energy function e that satisfies WEAK but violates MEDIUM.

Proof: Choose any $\phi_0 \in \mathcal{P} \cap C^3(0,+\infty)$ that satisfies $\phi_0''' < 0$. Define

$$\phi(\tau,S) = \phi_0(\tau)(1 - S\phi_0'(\tau)) \qquad (\tau > 0,\ -\infty < S < +\infty)\ . \tag{6.30}$$

Then

$$|\phi(\tau,S)| \le \phi_0(\tau) + |S|\phi_0(\tau)(-\phi_0'(\tau)) \qquad (\tau > 0)\ . \tag{6.31}$$

Now

$$\int_\tau^M -\phi_0(v)\phi_0'(v)dv = -\frac{1}{2}\phi_0(v)^2\Big|_{v=\tau}^{v=M} = \frac{1}{2}\left[\phi_0(\tau)^2 - \phi_0(M)^2\right] \qquad (\tau, M > 0). \tag{6.32}$$

With $\tau = 1$, (6.32) yields

$$\int_1^{+\infty} -\phi_0(v)\phi_0'(v)dv < +\infty\ .$$

Since also

$$\int_1^{+\infty} \phi_0 < +\infty \ ,$$

we obtain from (6.31)

$$\int_1^{+\infty} |\phi(\cdot, S)| < +\infty \ . \tag{6.33}$$

Thus we may define

$$\epsilon(\tau, S) = \int_\tau^{+\infty} \phi(v, S)dv \qquad (\tau > 0) \ ; \tag{6.34}$$

in fact, by (6.30) and (6.32) we have

$$\epsilon(\tau, S) = \int_\tau^{+\infty} \phi_0 + \frac{S}{2}\phi_0(\tau)^2 \qquad (\tau > 0) \ , \tag{6.35}$$

which shows ϵ is C^3 in (τ, S). By (6.34), $\epsilon_\tau(\tau, S) = -\phi(\tau, S)$ $(\tau > 0)$. By (6.30), $\phi_S(\tau, S) = -\phi_0(\tau)\phi_0'(\tau) > 0$ $(\tau > 0)$; thus we may, as usual, define

$$\epsilon(\tau, p) = \epsilon(\tau, S) \qquad (\tau > 0, \ -\infty < p < +\infty) \ ,$$

where $S = \hat{S}(\tau, p)$ is the unique solution of $\phi(\tau, S) = p$.

We assert that

$$\phi(\cdot, S) \in \mathcal{P} \qquad (S \geq 0) \ . \tag{6.36}$$

Clearly $\phi(\tau, S) > 0$ $(\tau > 0, \ S \geq 0)$. Differentiating (6.30) with respect to τ gives

$$\phi_\tau(\tau, S) = \phi_0'(\tau) - S\left[\phi_0(\tau)\phi_0''(\tau) + \phi_0'(\tau)^2\right] \qquad (\tau > 0) \ , \tag{6.37}$$

whence $\phi_\tau(\cdot, S) < 0$ $(S \geq 0)$. Differentiating (6.37), we get

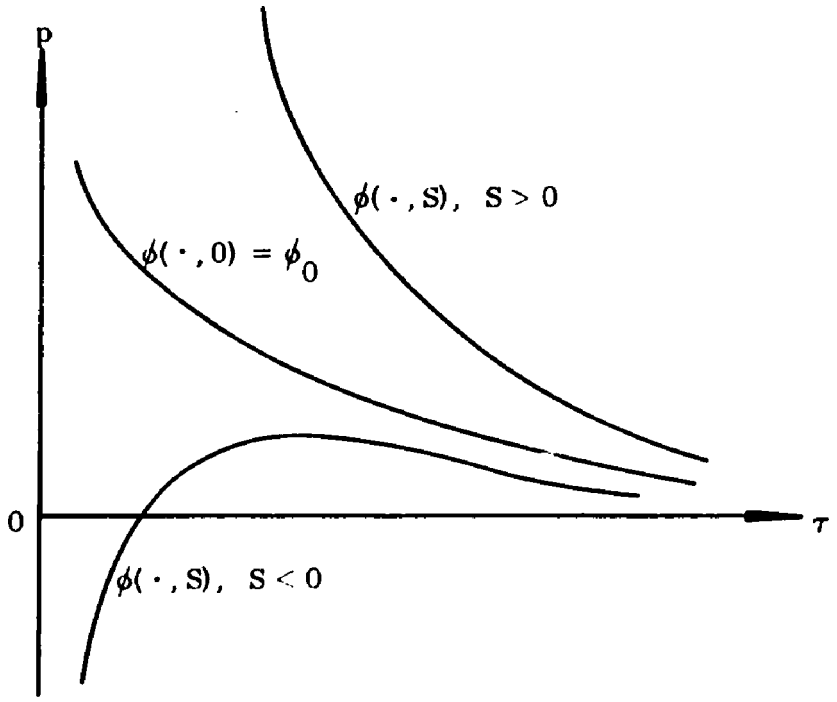


Fig. 35: The family $\phi(\cdot, S)$, $-\infty < S < +\infty$.

$$\phi_{\tau\tau}(\tau, S) = \phi_0''(\tau) - S\left[\phi_0(\tau)\phi_0'''(\tau) + 3\phi_0'(\tau)\phi_0''(\tau)\right] > 0 \qquad (\tau > 0,\ S \geq 0) \ .$$

By (6.33), $\int_1^{+\infty} \phi(\cdot, S) < +\infty$ $(S \geq 0)$. Since $\phi(\cdot, S) \geq \phi_0$ $(S \geq 0)$,

$\lim_{\tau \to 0^+} \phi(\tau, S) = +\infty$ $(S \geq 0)$. Thus (6.36) is proved.

Now for $\tau > 0$ and all S, we have

$$\int_\tau^{+\infty} \phi(\cdot, S) - \int_\tau^{+\infty} \phi_0 = \frac{S}{2}\phi_0(\tau)^2$$

by (6.34) and (6.35); and by (6.30),

$$\phi(\tau, S) - \phi_0(\tau) = -S\phi_0(\tau)\phi_0'(\tau) \ .$$

Thus

$$(6.38)\qquad \frac{\int_\tau^{+\infty} \phi(\cdot, S) - \int_\tau^{+\infty} \phi_0}{\phi(\tau, S) - \phi_0(\tau)} = -\frac{1}{2}\frac{\phi_0(\tau)}{\phi_0'(\tau)} \qquad (\tau > 0,\ S \neq 0)\ .$$

For $S = 1$, (6.38) becomes

$$(6.39)\qquad \frac{2}{\phi(\tau, 1) - \phi_0(\tau)} \int_\tau^{+\infty} (\phi(\cdot, 1) - \phi_0) = \frac{\phi_0(\tau)}{-\phi_0'(\tau)} \qquad (\tau > 0)\ .$$

Now observe that for $\tau, S > 0$,

$$\frac{\phi_0(\tau)}{-\phi_0'(\tau)} \geq \frac{\phi(\tau, S)}{-\phi_\tau(\tau, S)} \iff -\phi_0(\tau)\phi_\tau(\tau, S) \geq -\phi(\tau, S)\phi_0'(\tau)$$

$$\iff -\phi_0(\tau)\left[\phi_0'(\tau) - S(\phi_0(\tau)\phi_0''(\tau) + \phi_0'(\tau)^2)\right] \geq -\phi_0(\tau)(1 - S\phi_0'(\tau))\phi_0'(\tau)$$

$$\iff S\phi_0(\tau)(\phi_0(\tau)\phi_0''(\tau) + \phi_0'(\tau)^2) \geq S\phi_0(\tau)\phi_0'(\tau)^2 \iff \phi_0(\tau)\phi_0''(\tau) \geq 0\ ,$$

by (6.30) and (6.37). Hence

$$(6.40)\qquad \frac{\phi_0(\tau)}{-\phi_0'(\tau)} \geq \frac{\phi(\tau, S)}{-\phi_\tau(\tau, S)} \qquad (\tau, S > 0)\ .$$

It follows from (6.39) and (6.40) that $\forall \tau > 0$,

$$\frac{2}{\phi(\tau, 1) - \phi_0(\tau)} \int_\tau^{+\infty} (\phi(\cdot, 1) - \phi_0) \geq \begin{cases} \dfrac{\phi_0(\tau)}{-\phi_0'(\tau)} \\ \\ \dfrac{\phi(\tau, 1)}{-\phi_\tau(\tau, 1)} \end{cases}\ .$$

Note that $\phi(\cdot, 1) > \phi_0$, so Lemma 6.8 applies. Define

$$\pi(\tau, \lambda) = (1 - \lambda)\phi_0(\tau) + \lambda\phi(\tau, 1) \qquad (\tau > 0,\ -\infty < \lambda < +\infty)\ ;$$

then by Lemma 6.8,

$$\pi(\cdot,\lambda_1) \overset{w}{\sim} \pi(\cdot,\lambda_2) \qquad (0 \le \lambda_1 \le \lambda_2 \le 1) \ .$$

But

$$\pi(\tau,\lambda) = (1-\lambda)\phi_0(\tau) + \lambda\phi_0(\tau)(1-\phi_0'(\tau))$$
$$= \phi_0(\tau)(1-\lambda+\lambda(1-\phi_0'(\tau))) = \phi_0(\tau)(1-\lambda\phi_0'(\tau)) = \phi(\tau,\lambda) \ ;$$

hence

$$\phi(\cdot,S_1) \overset{w}{\sim} \phi(\cdot,S_2) \qquad (0 \le S_1 \le S_2 \le 1) \ . \tag{6.41}$$

Now let $\tau > 0$; we shall compute $\frac{\partial}{\partial p}\epsilon(\tau,p)\Big|_{(\tau,\phi_0(\tau))}$. Since $\phi(\tau,0) = \phi_0(\tau)$, $\hat{S}(\tau,\phi_0(\tau)) = 0$, so

$$\epsilon(\tau,\phi_0(\tau)) = \epsilon(\tau,0) = \int_\tau^{+\infty} \phi(v,0)dv = \int_\tau^{+\infty} \phi_0 \ .$$

Also, with $p = \phi(\tau,S)$, i.e., $S = \hat{S}(\tau,p)$ (τ fixed), $S \to 0$ as $p \to \phi_0(\tau)$ and vice-versa. Thus

$$\frac{\partial}{\partial p}\epsilon(\tau,p)\Big|_{(\tau,\phi_0(\tau))} = \lim_{p\to\phi_0(\tau)} \frac{\epsilon(\tau,p)-\epsilon(\tau,\phi_0(\tau))}{p-\phi_0(\tau)} \tag{6.42}$$
$$= \lim_{p\to\phi_0(\tau)} \frac{\epsilon(\tau,\hat{S}(\tau,p))-\epsilon(\tau,\phi_0(\tau))}{\phi(\tau,\hat{S}(\tau,p))-\phi_0(\tau)} = \lim_{S\to 0} \frac{\epsilon(\tau,S)-\int_\tau^{+\infty}\phi_0}{\phi(\tau,S)-\phi_0(\tau)}$$
$$= \lim_{S\to 0} \frac{\int_\tau^{+\infty}\phi(\cdot,S)-\int_\tau^{+\infty}\phi_0}{\phi(\tau,S)-\phi_0(\tau)} = \lim_{S\to 0} -\frac{1}{2}\frac{\phi_0(\tau)}{\phi_0'(\tau)} = -\frac{1}{2}\frac{\phi_0(\tau)}{\phi_0'(\tau)} \ ,$$

by (6.38). We define $\phi_\tau(\tau,p) = \phi_\tau(\tau,\hat{S}(\tau,p))$; then, with $\phi_0(\tau)$ understood to be p, we have

$$\phi_\tau(\tau,\phi_0(\tau)) = \phi_\tau(\tau,0) = \phi_0'(\tau) \ . \tag{6.43}$$

By (6.34), (6.36) and Lemma 1.6,

$$\phi(\tau, S)^2 < -2\phi_\tau(\tau, S)\epsilon(\tau, S) \qquad (S \geq 0) \ ;$$

by (6.43), this becomes for $S = 0$,

$$\phi_0(\tau)^2 < -2\phi_\tau(\tau, \phi_0(\tau))\epsilon(\tau, \phi_0(\tau)) \ . \tag{6.44}$$

Using (6.43), (6.42) may be written

$$\frac{\partial}{\partial p}\epsilon(\tau, p)\Big|_{(\tau, \phi_0(\tau))} = \frac{\phi_0(\tau)}{-2\phi_\tau(\tau, \phi_0(\tau))}$$

$$= \frac{2\epsilon(\tau, \phi_0(\tau))\phi_0(\tau)}{-\phi_\tau(\tau, \phi_0(\tau)) \cdot 2\epsilon(\tau, \phi_0(\tau)) - \phi_\tau(\tau, \phi_0(\tau)) \cdot 2\epsilon(\tau, \phi_0(\tau))} \ ;$$

then using (6.44) and dividing numerator and denominator by $\phi_0(\tau)$, we get

$$\frac{\partial}{\partial p}\epsilon(\tau, p)\Big|_{(\tau, \phi_0(\tau))} < \frac{2\epsilon(\tau, \phi_0(\tau))}{-\phi_\tau(\tau, \phi_0(\tau)) \cdot \dfrac{2\epsilon(\tau, \phi_0(\tau))}{\phi_0(\tau)} + \phi_0(\tau)} \qquad (\tau > 0) \ . \tag{6.45}$$

Now fix $\tau_1 > 0$. Since $\epsilon(\tau, p)$, $\phi_\tau(\tau, p)$ and $\frac{\partial}{\partial p}\epsilon(\tau, p)$ are continuous in (τ, p), it follows from (6.45) with $\tau = \tau_1$ that

$$\frac{\partial}{\partial p}\epsilon(\tau, p)\Big|_{(\tau_1, p_1)} < \frac{2\epsilon(\tau_1, p_1)}{-\phi_\tau(\tau_1, p_1) \cdot \dfrac{2\epsilon(\tau_1, p_1)}{p_1} + p_1} \tag{6.46}$$

for some p_1 with $\phi_0(\tau_1) < p_1 < \phi(\tau_1, 1)$. Let S_1 satisfy $\phi(\tau_1, S_1) = p_1$; then $0 < S_1 < 1$. Choose $\delta > 0$ such that $\delta < S_1 < 1 - \delta$. Let λ be a C^3 function defined on $(-\infty, +\infty)$ such that

$$\lambda'(S) > 0 \qquad (-\delta < S < 1 + \delta) \ , \tag{6.47}$$

$$\lambda(S) = 0 \qquad (S \leq -\delta) \ , \tag{6.48}$$

$$\lambda(S) = 1 \qquad (S \geq 1 + \delta) \ , \tag{6.49}$$

(6.50) $\lambda(S) = S \qquad (\delta \leq S \leq 1-\delta)$.

(See Fig. 36.) Let α be a C^3 function defined on $(-\infty,+\infty)$ such that $\alpha > 0$

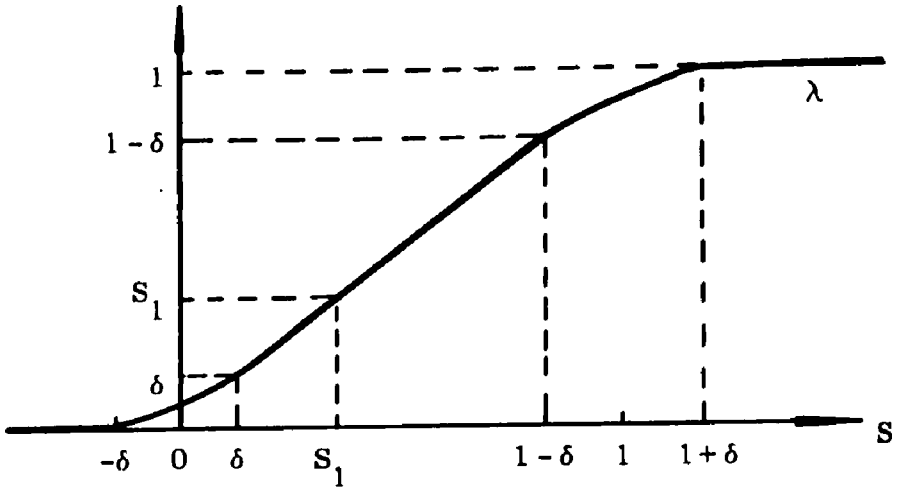


Fig. 36: The function λ.

and

(6.51) $\alpha'(S) > 0 \qquad (S < 0,\ S > 1)$,

(6.52) $\alpha(S) = 1 \qquad (0 \leq S \leq 1)$,

(6.53) $\lim_{S \to -\infty} \alpha(S) = 0$,

(6.54) $\lim_{S \to +\infty} \alpha(S) = +\infty$.

(See Fig. 37.) Define

(6.55) $g(\tau, S) = \alpha(S)\phi(\tau, \lambda(S)) \qquad (\tau > 0,\ -\infty < S < +\infty)$,

(6.56) $e(\tau, S) = \alpha(S)\epsilon(\tau, \lambda(S)) \qquad (\tau > 0,\ -\infty < S < +\infty)$.

(See (6.30) and (6.34).) Then clearly e is C^3 in (τ, S), and $e > 0$ by (6.35) since $\lambda \geq 0$. We have

(6.57) $e_S(\tau, S) = \alpha'(S)\epsilon(\tau, \lambda(S)) + \alpha(S)\epsilon_S(\tau, \lambda(S))\lambda'(S) \qquad (\tau > 0,\ -\infty < S < +\infty)$.

By (6.35), $\epsilon_S > 0$. Since $\lambda \geq 0$, we know $\epsilon(\tau, \lambda(S)) > 0$ by (6.35). Moreover,

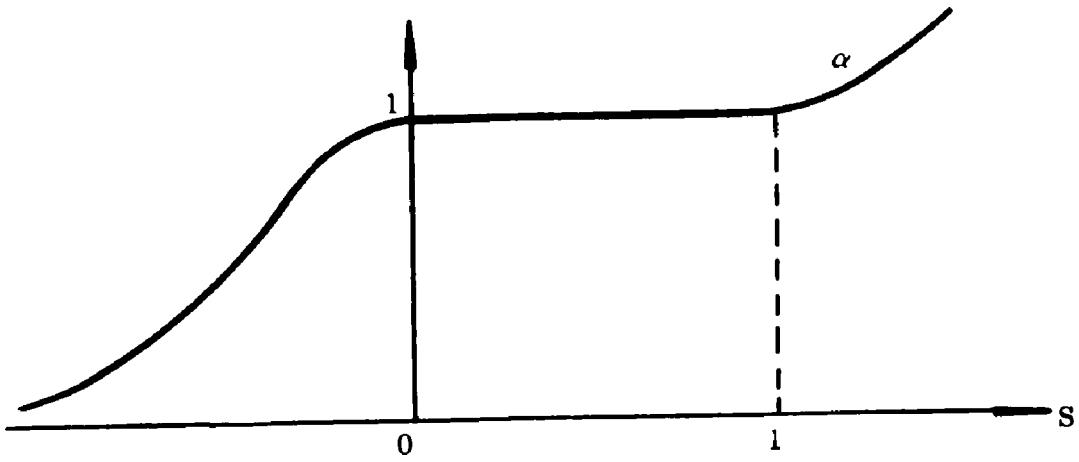


Fig. 37: The function α.

$\alpha > 0$, $\lambda' \geq 0$ and $\alpha' \geq 0$. By (6.47) and (6.51), for all values of S, either $\alpha' > 0$ or $\lambda' > 0$; it now follows from (6.57) that $e_S > 0$. Again because $\lambda \geq 0$,

$$g(\cdot, S) \in \mathcal{P} \qquad (-\infty < S < +\infty) \tag{6.58}$$

by (6.36) and (6.55). We have $e_\tau(\tau, S) = \alpha(S)\epsilon_\tau(\tau, \lambda(S)) = \alpha(S)(-\phi(\tau, \lambda(S))) = -g(\tau, S)$. By (6.58), $g > 0$, $g_\tau < 0$ and $g_{\tau\tau} > 0$. Finally

$$g_S(\tau, S) = \alpha'(S)\phi(\tau, \lambda(S)) + \alpha(S)\phi_S(\tau, \lambda(S))\lambda'(S) \qquad (\tau > 0,\ -\infty < S < +\infty)\ ;$$

then $\phi(\tau, \lambda(S)) > 0$ by (6.30) and the fact that $\lambda \geq 0$, and since also $\phi_S > 0$, we conclude as before that $g_S > 0$. Thus e satisfies the Standard Assumptions (1.2)-(1.7). Let $S \to -\infty$. Then by (6.48), for all $S < -\delta$, $\lambda(S) = 0$, so $\phi(\tau, \lambda(S)) = \phi(\tau, 0)$, whereas $\alpha(S) \downarrow 0$. Thus $\lim_{S \to -\infty} g(\tau, S) = 0$. Similarly, $\lim_{S \to -\infty} e(\tau, S) = 0$. Let $S \to +\infty$. By (6.49), for all $S > 1 + \delta$, $\lambda(S) = 1$, so $\phi(\tau, \lambda(S)) = \phi(\tau, 1) > 0$ and $\epsilon(\tau, \lambda(S)) = \epsilon(\tau, 1) > 0$, whereas $\alpha(S) \to +\infty$. Thus $\lim_{S \to +\infty} g(\tau, S) = +\infty$ and $\lim_{S \to +\infty} e(\tau, S) = +\infty$. Thus (1.8) and (1.9) hold. By

(6.34), (6.36) and (6.56), (6.1) holds. Thus by (6.58) and Lemma 6.2, e is an energy function.

To establish WEAK, we shall apply Prop. 6.6. Let $S_1, S_2 \in (-\infty, +\infty)$ with $S_1 < S_2$. Then certainly $0 \leq \lambda(S_1) \leq \lambda(S_2) \leq 1$, so by (6.41),

$$\phi(\cdot, \lambda(S_1)) \overset{w}{\sim} \phi(\cdot, \lambda(S_2)) \ .$$

Moreover, $0 < \alpha(S_1) \leq \alpha(S_2)$, so by Prop. 6.7(a),

(6.59) $$\alpha(S_1)\phi(\cdot, \lambda(S_1)) \overset{w}{\sim} \alpha(S_1)\phi(\cdot, \lambda(S_2)) \ ;$$

and by Prop. 6.7(b),

(6.60) $$\alpha(S_1)\phi(\cdot, \lambda(S_2)) \overset{w}{\sim} \alpha(S_2)\phi(\cdot, \lambda(S_2)) \ .$$

Applying Prop. 6.7(c) to (6.59) and (6.60), we obtain

$$\alpha(S_1)\phi(\cdot, \lambda(S_1)) \overset{w}{\sim} \alpha(S_2)\phi(\cdot, \lambda(S_2)) \ ,$$

i.e.,

$$g(\cdot, S_1) \overset{w}{\sim} g(\cdot, S_2) \ .$$

Since S_1 and S_2 were arbitrary, e satisfies WEAK by Prop. 6.6.

By (6.50) and (6.52), $\lambda(S) = S$ and $\alpha(S) = 1$ whenever $\delta < S < 1 - \delta$; hence

(6.61) $$g(\tau, S) = \phi(\tau, S) \ ,$$

(6.62) $$e(\tau, S) = \epsilon(\tau, S) \ ,$$

whenever $(\tau, S) \in (0, +\infty) \times (\delta, 1 - \delta)$. Since $(\tau_1, S_1) \in (0, +\infty) \times (\delta, 1 - \delta)$, we see that $g = \phi$ and $e = \epsilon$ on a neighborhood N of (τ_1, S_1). Define $\Phi(\tau, S) = (\tau, g(\tau, S))$, $\Psi(\tau, S) = (\tau, \phi(\tau, S))$; then

(6.63) $$\Phi = \Psi \text{ on } N,$$

and the Jacobian determinant $J_\Psi(\tau_1, S_1) = \phi_S(\tau_1, S_1) > 0$. Hence by making N smaller we may assume Φ maps N (injectively) onto an open set M containing

(τ_1, p_1). (Recall that $\phi(\tau_1, S_1) = p_1$.) For $(\tau, p) \in M$, we have

$$\Phi^{-1}(\tau, p) = (\tau, S(\tau, p)) \ ,$$

$$\Psi^{-1}(\tau, p) = (\tau, \hat{S}(\tau, p)) \ ;$$

hence by (6.63), $S = \hat{S}$ on M. Then by (6.62), $\forall (\tau, p) \in M$, $e(\tau, p) = e(\tau, S(\tau, p)) = \epsilon(\tau, S(\tau, p)) = \epsilon(\tau, \hat{S}(\tau, p)) = \epsilon(\tau, p)$. From this it follows that

$$\frac{\partial}{\partial p} e(\tau, p)\Big|_{(\tau_1, p_1)} = \frac{\partial}{\partial p} \epsilon(\tau, p)\Big|_{(\tau_1, p_1)} \ . \tag{6.64}$$

Also, by (6.61), $g_\tau(\tau_1, p_1) = g_\tau(\tau_1, S_1) = \phi_\tau(\tau_1, S_1) = \phi_\tau(\tau_1, p_1)$. From this, (6.64) and (6.46), we obtain

$$\frac{\partial}{\partial p} e(\tau, p)\Big|_{(\tau_1, p_1)} < \frac{2e(\tau_1, p_1)}{-g_\tau(\tau_1, p_1) \cdot \dfrac{2e(\tau_1, p_1)}{p_1} + p_1} \ ;$$

thus e violates MEDIUM. ■

6.2. An energy function that violates WEAK.

Theorem 6.10. There exists an energy function that violates WEAK.

Proof: For convenience, we shall set $v = 1 - \tau$. Fix $\alpha \in (0, 1/2)$, and define

$$f(v, S) = e^v + \alpha(1 - e^{-S}) \exp\left[\frac{1}{\alpha}(v + S)\right] \ . \tag{6.65}$$

After some computation, we shall almost get the energy function we want by replacing "v" by "$1 - \tau$". We have

$$f_v(v, S) = e^v + (1 - e^{-S}) \exp\left[\frac{1}{\alpha}(v + S)\right] \ , \tag{6.66}$$

$$f_{vv}(v, S) = e^v + \frac{1}{\alpha}(1 - e^{-S}) \exp\left[\frac{1}{\alpha}(v + S)\right] \ ,$$

$$f_{vvv}(v,S) = e^{v} + \frac{1}{\alpha^2}(1-e^{-S})\exp\left[\frac{1}{\alpha}(v+S)\right].$$

From (6.65) and (6.66) we see that f, f_v, f_{vv} and f_{vvv} have the form

(6.67) $$h(v,S) = e^{v} + \beta(1-e^{-S})\exp\left[\frac{1}{\alpha}(v+S)\right],$$

where $0 < \beta \leq \alpha^{-2}$. But $h(v,S) > 0$ iff

(6.68) $$\exp\left[(1-\frac{1}{\alpha})v\right] > \beta(e^{-S}-1)e^{S/\alpha}.$$

Also, since $1-\frac{1}{\alpha} < 0$,

(6.69) $$\exp\left[(1-\frac{1}{\alpha})v\right] \geq \exp\left(1-\frac{1}{\alpha}\right) \text{ whenever } v \leq 1.$$

Choose $\delta_1 > 0$ such that

(6.70) $$\frac{1}{\alpha^2}(e^{-S}-1)e^{S/\alpha} < \exp\left(1-\frac{1}{\alpha}\right) \text{ whenever } S \geq -\delta_1.$$

Combining (6.67) - (6.70), we conclude

$$h(v,S) > 0 \quad \text{whenever} \quad v \leq 1,\ S \geq -\delta_1,$$

so that

(6.71) $$f, f_v, f_{vv}, f_{vvv} > 0 \qquad (v \leq 1,\ S \geq -\delta_1).$$

We also have

$$f_S(v,S) = \exp\left[\frac{1}{\alpha}(v+S)\right]\left[1-(1-\alpha)e^{-S}\right],$$

$$f_{vS}(v,S) = \frac{1}{\alpha} f_S(v,S).$$

Choose $\delta_2 > 0$ such that $(1-\alpha)e^{-S} < 1$ whenever $S \geq -\delta_2$. Then

(6.72) $$f_S, f_{vS} > 0 \qquad (v \leq 1,\ S \geq -\delta_2).$$

Let δ be the smaller of δ_1 and δ_2. Then by (6.71) and (6.72),

$$f, f_v, f_{vv}, f_{vvv}, f_S, f_{vS} > 0 \qquad (v \leq 1,\ S \geq -\delta). \tag{6.73}$$

Now define

$$\epsilon(\tau, S) = f(1-\tau, S) \qquad (\tau \geq 0,\ S > -\delta). \tag{6.74}$$

Since $\tau \geq 0$ iff $1-\tau \leq 1$, we see that (6.73) yields

$$\epsilon > 0,\ \epsilon_\tau < 0,\ \epsilon_{\tau\tau} > 0,\ \epsilon_{\tau\tau\tau} < 0,\ \epsilon_S > 0,\ \epsilon_{\tau S} < 0 \qquad (\tau \geq 0,\ S > -\delta), \tag{6.75}$$

which, except for the restriction on S, is equivalent to (1.2) -(1.7) for ϵ in place of e. Let ξ be a C^3 function defined on $(-\infty, +\infty)$ such that

$$\xi' > 0\ , \tag{6.76}$$

$$\xi(S) = S \qquad (S > -\delta/2)\ , \tag{6.77}$$

$$\lim_{S \to -\infty} \xi(S) = -\delta\ . \tag{6.78}$$

(See Fig. 38.) Let θ be a C^3 function defined on $(-\infty, +\infty)$ such that $\theta > 0$

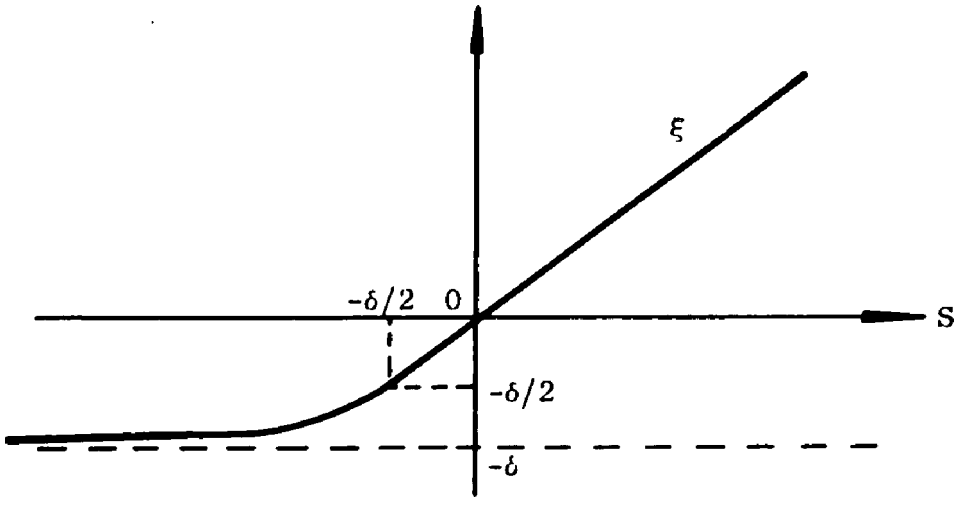


Fig. 38: The function ξ.

and

(6.79) $$\theta' \geq 0 \ ,$$

(6.80) $$\theta(S) = 1 \qquad (S > -\delta/2) \ ,$$

(6.81) $$\lim_{S \to -\infty} \theta(S) = 0 \ .$$

(See Fig. 39.) Let γ be a C^3 function defined on $(0, +\infty)$ such that

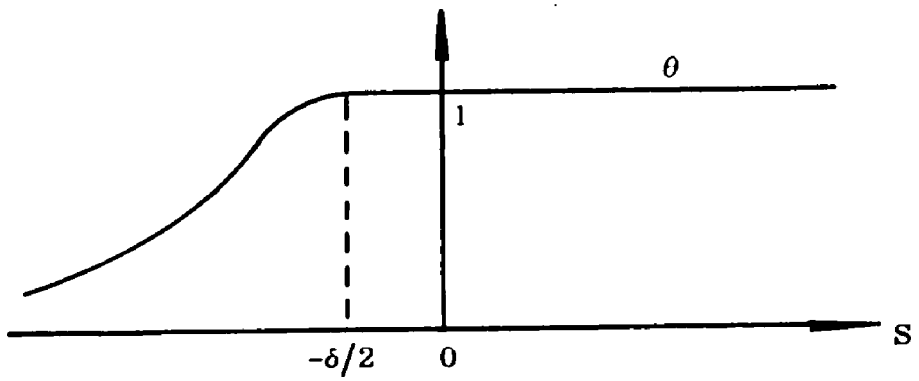


Fig. 39: The function θ.

(6.82) $$\gamma' \leq 0, \quad \gamma'' \geq 0, \quad \gamma''' \leq 0,$$

(6.83) $$\gamma(\tau) = 1 \qquad (\tau > 1 - \tfrac{\delta}{2}) \ ,$$

(6.84) $$\lim_{\tau \to 0^+} \gamma(\tau) = +\infty \ .$$

(See Fig. 40.) Define

(6.85) $$e(\tau, S) = \gamma(\tau)\theta(S)\epsilon(\tau, \xi(S)) \qquad (\tau > 0, \ -\infty < S < +\infty) \ .$$

Since $\gamma, \theta > 0$ and $\xi > -\delta$, we see from (6.75) that $e > 0$. Let $g = -e_\tau$. Then

(6.86) $$\begin{aligned} g(\tau, S) &= -e_\tau(\tau, S) \\ &= -\theta(S)\left[\gamma'(\tau)\epsilon(\tau, \xi(S)) + \gamma(\tau)\epsilon_\tau(\tau, \xi(S))\right] > 0 \qquad (\tau > 0) \end{aligned}$$

by (6.75), (6.82) and (6.85) (since $\xi > -\delta$); moreover

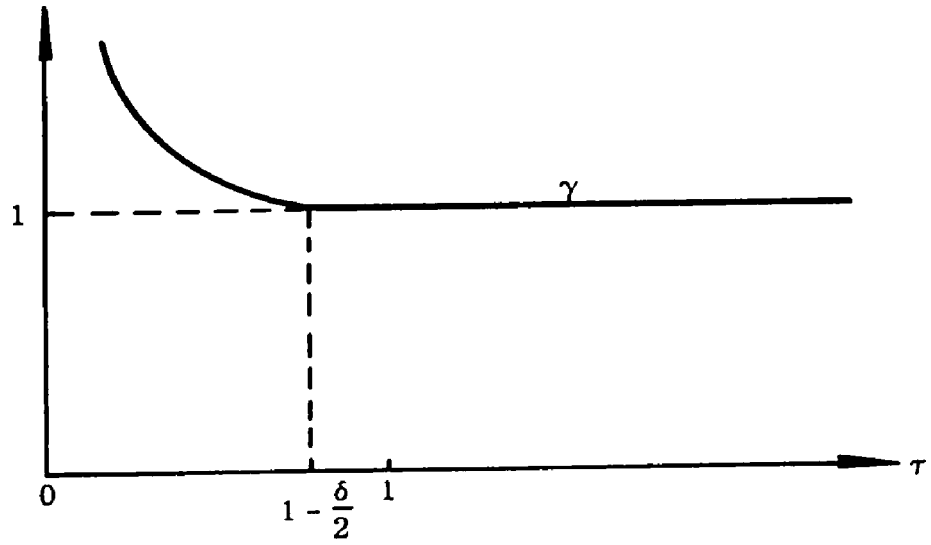


Fig. 40: The function γ.

$$(6.87)\qquad g_\tau(\tau, S) = -\theta(S)\left[\gamma''(\tau)\epsilon(\tau, \xi(S)) + 2\gamma'(\tau)\epsilon_\tau(\tau, \xi(S)) + \gamma(\tau)\epsilon_{\tau\tau}(\tau, \xi(S))\right] < 0 ,$$

$$(6.88)\qquad g_{\tau\tau}(\tau, S) = -\theta(S)\left[\gamma'''(\tau)\epsilon(\tau, \xi(S)) + 3\gamma''(\tau)\epsilon_\tau(\tau, \xi(S)) + 3\gamma'(\tau)\epsilon_{\tau\tau}(\tau, \xi(S)) + \gamma(\tau)\epsilon_{\tau\tau\tau}(\tau, \xi(S))\right] > 0 \qquad (\tau > 0) .$$

Also, by (6.75), (6.76), (6.79), (6.85) and (6.86),

$$e_S(\tau, S) = \gamma(\tau)\left[\theta'(S)\epsilon(\tau, \xi(S)) + \theta(S)\epsilon_S(\tau, \xi(S))\xi'(S)\right] > 0 ,$$

$$g_S(\tau, S) = \theta'(S)\,\frac{g(\tau, S)}{\theta(S)} - \theta(S)\left[\gamma'(\tau)\epsilon_S(\tau, \xi(S)) + \gamma(\tau)\epsilon_{\tau S}(\tau, \xi(S))\right]\xi'(S) > 0 \qquad (\tau > 0) .$$

Thus e satisfies (1.2) - (1.7).

The following are clear from (6.65) and (6.66):

$$\lim_{v \to -\infty} f(v, S) = 0 , \qquad \lim_{S \to +\infty} f(v, S) = +\infty , \qquad \lim_{S \to +\infty} f_v(v, S) = +\infty .$$

Then using (6.74), (6.77), (6.78), (6.80), (6.81), (6.85) and (6.86), we get:

$$\lim_{S\to+\infty} e(\tau,S) = \lim_{S\to+\infty} \gamma(\tau)\theta(S)\epsilon(\tau,\xi(S)) = \gamma(\tau) \lim_{S\to+\infty} \epsilon(\tau,S)$$

$$= \gamma(\tau) \lim_{S\to+\infty} f(1-\tau,S) = +\infty ;$$

$$\lim_{S\to-\infty} e(\tau,S) = \lim_{S\to-\infty} \gamma(\tau)\theta(S)\epsilon(\tau,\xi(S))$$

$$= \gamma(\tau) \lim_{S\to-\infty} \theta(S) \cdot \lim_{S\to-\infty} \epsilon(\tau,\xi(S))$$

$$= \gamma(\tau) \lim_{S\to-\infty} \theta(S) \cdot \lim_{S\to-\delta} \epsilon(\tau,S) = \gamma(\tau)\cdot 0\cdot\epsilon(\tau,-\delta) = 0 ;$$

$$\lim_{S\to+\infty} g(\tau,S) = \lim_{S\to+\infty} -\theta(S)\left[\gamma'(\tau)\epsilon(\tau,\xi(S)) + \gamma(\tau)\epsilon_\tau(\tau,\xi(S))\right]$$

$$= -\lim_{S\to+\infty}\left[\gamma'(\tau)f(1-\tau,S) - \gamma(\tau)f_v(1-\tau,S)\right] = +\infty$$

(since $\gamma' \le 0$) ;

$$\lim_{S\to-\infty} g(\tau,S) = -\lim_{S\to-\infty} \theta(S) \cdot \lim_{S\to-\delta}\left[\gamma'(\tau)\epsilon(\tau,S) + \gamma(\tau)\epsilon_\tau(\tau,S)\right]$$

$$= 0\cdot\left[\gamma'(\tau)\epsilon(\tau,-\delta) + \gamma(\tau)\epsilon_\tau(\tau,-\delta)\right] = 0 .$$

Thus (1.8) and (1.9) hold for e. Moreover, for any S,

$$\lim_{\tau\to+\infty} e(\tau,S) = \lim_{\tau\to+\infty} \gamma(\tau)\theta(S)\epsilon(\tau,\xi(S)) = \theta(S) \lim_{\tau\to+\infty} f(1-\tau,\xi(S))$$

$$= \theta(S) \lim_{v\to-\infty} f(v,\xi(S)) = 0 ,$$

by (6.74), (6.83) and (6.85). Thus e satisfies (6.1). Since the proof of (6.3) does not require (1.10), we conclude

$$\int_1^{+\infty} g(\cdot,S) < +\infty \qquad (-\infty < S < +\infty) . \tag{6.89}$$

Fix $S\in(-\infty,+\infty)$, and observe that by (6.75), $\epsilon_\tau(0,\xi(S)) < 0$. Then using (6.75),

(6.82),(6.84), (6.86) and the fact that $\theta(S) > 0$, we get

$$\lim_{\tau \to 0^+} g(\tau, S) = \theta(S) \lim_{\tau \to 0^+} \left[(-\gamma'(\tau))\epsilon(\tau, \xi(S)) + \gamma(\tau)(-\epsilon_\tau(\tau, \xi(S)))\right]$$

$$\geq \theta(S) \lim_{\tau \to 0^+} \gamma(\tau)(-\epsilon_\tau(\tau, \xi(S)))$$

$$= \theta(S) \lim_{\tau \to 0^+} \gamma(\tau) \cdot (-\epsilon_\tau(0, \xi(S))) = +\infty \ .$$

This, together with (6.86) -(6.89), shows that $g(\cdot, S) \in \mathcal{P}$ for all S. Hence by Lemma 6.2, e is an energy function.

We shall prove that WEAK fails at $\tau = 1$, $p = 1$. For clarity, we let $\bar{p} = 1$ throughout. Now let $\tau > 1 - \frac{\delta}{2}$. Then $\tau - 1 > -\frac{\delta}{2}$, and from (6.77), (6.80) and (6.83) we conclude $\xi(\tau - 1) = \tau - 1$, $\theta(\tau - 1) = 1$, $\gamma(\tau) = 1$ and $\gamma'(\tau) = 0$. Then from (6.85) we see that

$$e(\tau, \tau - 1) = \gamma(\tau)\theta(\tau - 1)\epsilon(\tau, \xi(\tau - 1)) = \epsilon(\tau, \tau - 1) \ . \tag{6.90}$$

Similarly, we have by (6.74), (6.66) and (6.86),

$$g(\tau, \tau - 1) = -\epsilon_\tau(\tau, \tau - 1) = f_v(1 - \tau, \tau - 1)$$

$$= e^{1-\tau} + (1 - e^{-(\tau - 1)})\exp\left[\frac{1}{\alpha}(1 - \tau + (\tau - 1))\right] = \bar{p} \ .$$

This means that $S(\tau, \bar{p}) = \tau - 1$, so

$$e(\tau, \bar{p}) = e(\tau, S(\tau, \bar{p})) = e(\tau, \tau - 1) = \epsilon(\tau, \tau - 1) = f(1 - \tau, \tau - 1) \tag{6.91}$$

$$= e^{1-\tau} + \alpha(1 - e^{-(\tau - 1)})\exp\left[\frac{1}{\alpha}(1 - \tau - (\tau - 1))\right] = \alpha + (1 - \alpha)e^{1-\tau}.$$

Since (6.91) holds $\forall \tau > 1 - \frac{\delta}{2}$,

$$\frac{\partial}{\partial \tau} e(\tau, p)\Big|_{(\tau, \bar{p})} = (\alpha - 1)e^{1-\tau} \qquad (\tau > 1 - \tfrac{\delta}{2}) \ ;$$

in particular,

$$\frac{\partial}{\partial \tau} e(\tau, p)\Big|_{(1,\bar{p})} = \alpha - 1 < -\ddot{p}/2 \ ,$$

so WEAK fails at $(1, \bar{p})$. ■

6.3. Ideal gases.

In this section we construct an example of an ideal gas e for which no Hugoniot curve is monotone.

An ideal gas is a medium that satisfies (1.2) -(1.7) and

$$p\tau = RT \ , \tag{6.92}$$

where $R > 0$ is a constant. In [1], (6.92), together with (1.2) -(1.7), is shown to be equivalent to

$$e(\tau, S) = h(\tau H) \ , \tag{6.93}$$

where $H = \exp(-S/R)$. Here h is a function defined on $(0, +\infty)$ satisfying:

$$h \text{ maps } (0,+\infty) \text{ onto } (0,+\infty) \ , \tag{6.94}$$

$$h > 0, \quad h' < 0, \quad h'' > 0, \quad h''' < 0 \ , \tag{6.95}$$

$$k \text{ maps } (0,+\infty) \text{ onto } (0,+\infty) \ , \tag{6.96}$$

$$k' < 0 \ , \tag{6.97}$$

where $k(x) \equiv -h'(x)x$. In this situation we have

$$p \equiv g(\tau, S) = -h'(\tau H)H \ , \tag{6.98}$$

$$T = -\frac{1}{R} h'(\tau H)\tau H \ .$$

By (6.98),

$$p\tau = -h'(\tau H)\tau H = k(\tau H) \ ,$$

so that

$$e(\tau, p) = e(\tau, S) = \psi(p\tau) \ , \tag{6.99}$$

where $\psi = h \circ k^{-1}$. Thus ψ maps $(0, +\infty)$ onto $(0, +\infty)$ and $\psi' > 0$. From this it is clear that e for an ideal gas is an energy function that satisfies STRONG.

Note that (6.99) can also be expressed,

(6.100) $$e(\tau, p) = h(x), \text{ where } p\tau = -h'(x)x \ .$$

We now give an example of an ideal gas for which every Hugoniot curve H is not monotone, i.e., along H, τ is not a nonincreasing function of p. Let

(6.101) $$h(x) = e^{1/x} - 1 \ .$$

Then

(6.102) $$h'(x) = -\frac{e^{1/x}}{x^2} \ ,$$

$$h'(x)x = -\frac{e^{1/x}}{x} \ ,$$

and h satisfies (6.94) -(6.97); therefore (6.100) defines an ideal gas. Fix $\tau_0, p_0 > 0$ and let H be the Hugoniot curve with center (τ_0, p_0). Then $(\tau, p) \in H$ iff

$$e(\tau, p) - e(\tau_0, p_0) = \frac{1}{2}(\tau_0 - \tau)\left[p + p_0\right] .$$

By (6.100), $(\tau, p) \in H$ iff

(6.103) $$h(x) - h(x_0) = \frac{1}{2}(\tau_0 - \tau)\left[p + p_0\right]$$

where

(6.104) $$p\tau = -h'(x)x \ , \quad p_0\tau_0 = -h'(x_0)x_0 \ .$$

We show that along H, τ comes arbitrarily near τ_0 as $p \to +\infty$. (See Lemma 4.1.) Suppose not. Then $\exists \delta$, $P > 0$ such that along H,

(6.105) $$\tau_0 - \tau > \delta \text{ whenever } p \geq P \ .$$

Then we must have $x \to 0$ as $p \to +\infty$ along H. For otherwise, there would exist $\sigma > 0$ and a sequence $\{p_n\}$ such that $p_n \to +\infty$ and, with $(\tau_n, p_n) \in H$ and $p_n\tau_n = -h'(x_n)x_n$ $(n = 1, 2, \ldots)$, $x_n \geq \sigma$ $\forall n$. Then by (6.94) and (6.95), for the

values x_n, τ_n and p_n the left member of (6.103) would be bounded as $n \to +\infty$ while the right member would approach $+\infty$ as $n \to +\infty$, a contradiction. Using (6.104), (6.103) becomes

$$h(x) - h(x_0) = \frac{1}{2}(\tau_0 - \tau)\left[-\frac{h'(x)x}{\tau} - \frac{h'(x_0)x_0}{\tau_0}\right].$$

Then by (6.101), (6.102) and the above, we get

$$e^{1/x} \geq e^{1/x} - e^{1/x_0} = (\tau_0 - \tau)\left[\frac{e^{1/x}}{2x\tau} + \frac{e^{1/x_0}}{2x_0\tau_0}\right] > (\tau_0 - \tau)\frac{e^{1/x}}{2x\tau} \qquad (p > p_0).$$

Thus

$$\tau > \frac{\tau_0}{2x+1} \to \tau_0 \text{ as } p \to +\infty,$$

which contradicts (6.105). Hence along H, τ comes arbitrarily near τ_0 as $p \to +\infty$. In particular, H is not monotone.

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