DOCTORAL DISSERTATION SERIES
Publication No.: 12,564

AUTHOR: Donald Lee Dean, Ph. D., 1955
University of Michigan

TITLE: A STATIC AND DYNAMIC ANALYSIS OF TALL FLEXIBLE TOWERS

University Microfilms, Ann Arbor, Michigan

A STATIC AND DYNAMIC ANALYSIS OF TALL FLEXIBLE TOWERS

by
Donald L. Dean

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1955

Doctoral Committee:

Professor Lawrence C. Maugh, Chairman
Professor Bruce G. Johnston
Associate Professor Leo M. Legatski
Professor Paul M. Naghdi
Assistant Professor John R. Sellars

ABSTRACT

OF

A STATIC AND DYNAMIC ANALYSIS OF TALL FLEXIBLE TOWERS

By

Donald LEE Dean

This dissertation presents and illustrates methods which can be used for the design and analysis of very tall guyed flexible towers, including those with sections cantilevered above the top guy connections, and unguyed or free standing towers whose stiffness can be assumed constant over large segments of the tower. The chief contribution of new material is in the chapters dealing with the static and dynamic analyses of the guy cables and of members with distributed axial loads.

In part one of the study, methods are presented which are suitable for design office use in carrying out an accurate static analysis of a guyed tower subjected to any form of transverse loading. The formulas which are developed are of a "slope deflection" form and are based on the assumption of a linear moment-curvature relation. Their use enables one to easily account for the following "secondary" effects: the non-linear relation between the horizontal resistance and the tower motion at support points; variation in axial loads with tower movement; variation, with tower movement, of the external moments applied by the cable forces at the support points; and moments caused by the distributed tower

weight in a deflected position. The properties of the catenary are adhered to throughout the derivation of the formulas associated with the guy cables.

In part two, a method is presented for finding the natural frequencies of vibration of a system composed of a very tall flexible tower and its guys. Some of the effects accounted for are as follows: distributed and concentrated axial loads; external joint moments due to the eccentricity of cable connections; mass inertia of guy cables; and variation of tension throughout the length of the guys. However, the assumption of small vibrations is made so that the governing equations are linear. The method is lengthy, for hand method computations, as the natural frequencies appear as the roots of a $(2n)^{th}$ order determinate, where "n" is the number of guy support points. For this reason, some of the methods presented in part two may not be satisfactory for design office use.

ACKNOWLEDGMENTS

The writer wishes to express his appreciation for the guidance and assistance rendered by the members of his special committee during the preparation of this paper. He is especially grateful to Professor L. C. Maugh, who suggested the topic for the study and maintained an active interest in its progress.

TABLE OF CONTENTS

Page

ACKNOWLEDGMENTS ii

TABLE OF CONTENTS iii

LIST OF TABLES vi

LIST OF ILLUSTRATIONS vii

NOTATIONS ix

Chapter

I. INTRODUCTION 1

Failures 2

Scope of Problem 3

Literature Review 5

II. GUY CABLES - STATIC ANALYSIS 7

Curve Equations and Properties 8

Effects of Changes in Cable Variables 14

Numerical Examples 21

Resultant of Several Cable Forces at One Level 24

Numerical Example 27

III. STATIC ANALYSIS OF TOWER BELOW LEVEL OF TOP GUYS 31

Derivation of Elastic Curve Equations for Span "i" 33

General Loading 39

Equilibrium Equations 43

EI Not Constant Between Joints 44

Page

IV. THE STATIC ANALYSIS OF UNGUYED TOWERS AND THAT PORTION OF GUYED TOWERS ABOVE THE TOP GUY CONNECTIONS 46

Development of Elastic Curve Equations for the Uniformly Loaded Vertical Cantilever 47

Development of Equations for Elastic Curve of Uniformly Loaded Vertical Cantilever with End Loads 52

Illustrative Example 57

Elastic Curve Equations for Vertical Cantilever with End Loads and General Transverse Span Loading .. 60

V. SUMMARY OF STATIC ANALYSIS OF GUYED FLEXIBLE TOWERS 63

Calculation of "I" to Account for Shear Deformation .. 64

Initial Cable Tension 70

Example of Static Analysis of Oklahoma City TV Tower Subjected to Simultaneous Wind and Ice Loads 71

VI. GUY CABLES - DYNAMIC ANALYSIS 84

Forced Vibration of Guy Cable 85

Illustrative Example 90

VII. DYNAMIC ANALYSIS OF TOWER BELOW LEVEL OF TOP GUY CONNECTIONS 93

Development of Equations for Normal Mode Shapes .. 94

Equilibrium Equations 98

VIII. DYNAMIC ANALYSIS OF UNGUYED TOWERS AND THAT PORTION OF GUYED TOWERS ABOVE TOP GUY CONNECTIONS 100

Dynamic Analysis of Uniform Vertical Member with Fixed Base 101

Page

Illustrative Example 106

Dynamic Analysis of Vertical Member Whose Base is Subjected to Harmonic Translation and Rotation 108

Dynamic Analysis of Vertical Member with End Loads and a Rotating and Translating Base .. 114

Illustrative Example 118

IX. SUMMARY OF PROCEDURE FOR COMPUTING NATURAL FREQUENCIES OF GUYED TOWER SYSTEM 120

X. CONCLUSIONS 125

APPENDIX A. The End Slopes of Guy Cables 128

APPENDIX B. Evaluation of the Definite Integral Involved in Dynamic Cable Analysis 130

BIBLIOGRAPHY 132

LIST OF TABLES

Table		Page
5.1	Values of C for Uniform Load with Various End Conditions	69
5.2	Physical Constants for Tower	73
5.3	Physical Constants for Guy Cables	76
5.4	Equations for Unknown Joint Slopes and Deflections	81
9.1	Equilibrium Equations for Vibrating Tower	122

LIST OF ILLUSTRATIONS

Figure		Page
2.1	Cable Element	8
2.2	Hanging Cable	9
2.3	Three Common Guy Arrangements	26
2.4	Symmetrical Guy Load	27
3.1	Deflected Tower	31
3.2	Beam Element	34
3.3	Fixed End Beam-Column with Arbitrarily Placed Concentrated Load	40
3.4	Fixed End Beam-Column with Mid-Span Load Change	40
3.5	Span with Arbitrarily placed Concentrated Load and General End Conditions	42
4.1	Vertical Cantilever	47
4.2	Vertical Cantilever with End Loads	52
4.3	Vertical Cantilever with Change in Cross-Section	57
4.4	Vertical Cantilever - General Loading	60
5.1	Section and Cross-Section of Triangular Tower	65
5.2	Fixed End Tower Segment Subjected to Uniform Transverse Load	67
5.3	Tower Loads and Approximate Moments	72
5.4	Span "5" in Deflected Position	78
5.5	True Tower Moments	83

Figure		Page
6.1	Cable with Vibrating Support	86
6.2	Symmetrical Vibration of Tower at Support Point	90
8.1	Vibration of Heavy Vertical Cantilever	101
8.2	Vibration of Heavy Vertical Member with Rotating and Translating Base	108
8.3	Vibration of Heavy Vertical Member with End Loads and a Rotating and Translating Base	114
8.4	Vibration of Member Consisting of Two Antennas	118

NOTATIONS

All of the symbols, employed in the work to follow, are defined where they first appear. However, many appear quite frequently and sometimes several pages from where they are defined. Therefore, for ease of reference, they are listed below:

a_1, a_2 - integration constants in Chapter 1

$A_c E_c$ and $A_l E_l$ - area and modulus of elasticity of cable and tower leg, respectively

α - $L\sqrt{\frac{P}{EI}}$

b - width of tower face

$b_1 \cdots b_4, b_1' \cdots b_4', \bar{b}_1 \cdots \bar{b}_4$ - integration constants in Chapter 2

β_0, β_1, β - $\frac{wL^3}{EI}$, $\frac{WL^2}{EI}$, and $\frac{qL^3}{EI}$ respectively

$C_1 \cdots C_7$ - slope deflection coefficients

$d_1 \cdots d_3, d_1' \cdots d_3', \bar{d}_1 \cdots \bar{d}_6$ - integration constants in Chapter 4

η, ξ - coordinates of cable in Chapter 6

$f_n(y_n)$ - function expressing static reactive force developed by guys in terms of y_n at n^{th} support level

$\bar{f}_n(y_n)$ - same as $f_n(y_n)$ except for dynamic instead of static case

g - $\frac{1}{\sqrt{2}}\left(\sqrt{\alpha^4 + 4\lambda} - \alpha^2\right)^{\frac{1}{2}}$

$\gamma_0, \bar{\gamma}, \gamma_1$ - arc tan. of slope of cable at ground, arc tan. of slope of cable chord, and arc tan. of slope of cable at tower, respectively

$G_n(y_n)$ - function expressing external moment caused by guys at n^{th} support level

$\bar{G}_n(\gamma_n)$ - same as $G_n(\gamma_n)$ except for dynamic case

h - vertical projection of cable length

H - horizontal component of cable tension

I - effective moment of inertia of cross-section

J - Bessel functions

k - $\frac{1}{\sqrt{2}}\left(\sqrt{\alpha^4+4\lambda}+\alpha^2\right)^{\frac{1}{2}}$

K - dimensionless quantity used in computing change in cable tension, and defined on page 18.

λ - $\frac{\rho L^4 \omega^2}{EI}$

ℓ - horizontal projection of cable length

L - length of tower spans and total length of cable

$\bar{L}$ - length of cable chord

μ - $\ell\sqrt{\frac{\rho}{H}}\,\omega$ dimensionless cable parameter in Chapter 6

M - bending moment

m - $\frac{M}{PL}$

$\bar{m}$ - $\frac{ML}{EI}$

ω - circular frequency of vibration

P - axial load

$\bar{P}_1$ - dimensionless term for column load = $\frac{P_1}{qL}$

Q - shear force

q - weight per foot of tower or cable

r - dimensionless cable parameter = $\frac{ql}{2H}$

ρ - mass per unit length of tower or cable

S - Lommel functions

t - time in seconds

T - cable tension in kips

u - dependent variable =

$w W$ - uniform and concentrated transverse loads

$\bar{w}$ - dimensionless term for transverse load = $\frac{w}{q}$

$x, y, \bar{x}, \bar{y}$ - coordinates and dimensionless expressions for coordinates

CHAPTER I

INTRODUCTION

In the past 5 to 10 years television has enjoyed a rapidly increasing popularity along with other information media that are transmitted at line-of-sight frequencies. Apparently this popularity has not reached its peak and it will be financially feasible to build a large number of very high transmitting antennas in the future. In many instances it will be economical, from the standpoint of increased population coverage, to build antenna towers two or three times as high as is the general practice in the construction of radio towers. For instance, a 1,572 foot television tower has been in use for several months at Oklahoma City, and construction will soon begin on a 1,873 foot tower to be used for television transmission by the Deep South Broadcasting Company at Montgomery, Alabama. The latter will be 400 foot higher than the antenna atop the world's tallest building, the Empire State Building in New York City.

These towers, built very high so that the antennas they support can radiate to large areas by line-of-sight transmission, are unique as structures in that their main function is not to carry loads, but merely to attain height. They are, of course, subjected to transverse loads when the wind blows against the projected surfaces of the structure, but these loads are generally small as compared with the vertical loads due to the dead weight of the

structure, and ice on the members. Due to the many factors involved, it would be difficult to predict the maximum height to which it would be practical to build a braced tower. However, it is quite easy to establish the absolute maximum theoretical height possible by assuming the tower is to be braced in a perfectly efficient manner so that no stress is introduced by either the transverse loads or the bracing system. In this case, the maximum height, h_m, is equal to the yield stress of the material divided by its density. For example, for mild steel, with a yield stress of 33 ksi. and a density of 490 lbs/ft^3, h_m = 9,680 ft. and for aluminum, with a yield stress of 40 ksi. and a density of 165 lbs/ft^3, h_m = 35,000 ft.

The Oklahoma City tower, referred to above, was built at a cost of over $500,000. This seems to indicate that transmitting towers are coming into a class of structures which warrant a more thorough and rigorous analysis than is customarily employed in the design of radio towers. Inquiries, from companies which have been asked to insure towers, and replies to inquiries made by the writer to organizations currently engaged in the design of high towers, indicate that there is both a need and a desire for more accurate methods to be used in the analysis of flexible towers. It is the purpose of this study to present the formulas and methods needed to carry out this more accurate and thorough analysis.

Failures

The relatively high incidence of failure occurring in the higher

radio towers serves as an additional argument for the need of more thorough methods of analysis for flexible towers. This argument is especially strong as all, or nearly all, tower failures occur during wind storms which have maximum wind velocities considerably below those specified in the design of towers. This could indicate that the distribution of the static wind loads has not been given the proper consideration, or possibly that the design should not be based on a static analysis. Also, since ice is involved in a large number of tower failures, there is a good chance that the current methods for the analysis of these structures do not properly include the effects of the vertical loads involved.

The failures of these structures are given very little attention in engineering literature and the majority of tower failures are not even reported in engineering periodicals. A rather thorough search for information of this nature netted the writer the following references to tower failures in the Engineering News Record:

Vol. 149, December 11, 1952, page 33
Vol. 140, January 15, 1948, page 9
Vol. 125, December 19, 1940, page 74.

For additional references to tower failures it has been necessary to rely upon personal accounts and newspaper reports.

Scope of Problem

It is proposed in this study to deal with the primary problem involved in the analysis of guyed towers. (An unguyed tower, whose rigidity is not continuously changing, can be handled in the same way

as the portion of a guyed tower which cantilevers above the top guy connections.) The static problem is that of developing formulas and methods for the analysis of a continuous beam-column which is loaded with transverse loads that are small as compared with the weight of the member. The formulas should give accurate results which take into account the following factors: the non-linear relation betweenthe horizontal resistance offered by guys and the tower movement; the variation ofthe axial loads during tower movement due to the vertical components of guy forces; the application of external moments at each set of guy connections during tower movement as a result of the eccentricity of the cable connections; the importance of the moments caused by the distributed tower weight in its deflected position. In addition, the stiffness of the tower must be calculated in such a way as to account for the effect of the deformation of web members in the trussed sections of the tower. One of the goals of this dissertation is the presentation of a practical method, for the static analysis of flexible towers, which accurately accounts for the above factors.

A complete dynamic analysis, of a system comprised of a very tall antenna and its guys, would be extremely difficult to carry out in a rigorous manner. If one were to assume that the wind pressure, or a certain portion of the wind pressure, could vary in a harmonic manner for a long enough period of time so that a steady state existed, then the forcing frequency could be assumed equal to

one of the natural frequencies of the tower. In this case, it would be necessary to know more of the damping properties of these structures than is known at present. On the other hand, if tower failures occur due to large tower vibrations of a transient nature, resulting from a small number of adversely timed gusts, it would be necessary to know the nature of the time history of gusts to an accuracy that is small as compared with the periods of natural vibration of the system (probably to 1/100 of a second). Of course, the need for information on damping and wind gusts for dynamic analyses is not restricted to structures of the type considered in this study.

In lieu of information of a complete nature concerning wind gusts and damping, it is proposed in this study to develop formulas for the first step in any dynamic analysis, namely, the computation of the natural frequencies of vibration for the system being studied. A method will be presented for computing the natural frequencies of a guyed tower system where the tower is treated as a distributed mass beam-column and the dynamic effect of the mass of the guys is considered.

Literature Review

Of the relatively few published papers on the subject considered in this work, only two discuss any but the most elementary concepts involved in the analysis of very tall flexible tower systems.

In a paper published in 1946 (6) Mr. Harding presents a rather complete discussion of the static resistance-motion relationship

for guys, but assumes that the guy cables hang as parabolas. The tower is not analyzed as a beam-column and dynamic effects are not investigated.

The most thorough paper published to date, on the subject of guyed flexible towers, was written by Dr. Kolousek (11) in 1947 and contained a discussion of the static and the dynamic analysis of these structures. Like the authors of all existing papers on the subject, he assumed that the cable guys hang as parabolas instead of catenaries.

Dr. Kolousek discusses the eigenvalue stability problem of the buckling of a guyed tower subjected to an axial load applied at the top, a condition that could be realized by excessive tightening of the upper guy cables. However, in his solution of the stress problem, where the antenna is subjected to transverse loads, the structure is not treated as a beam-column and so the effects of axial loads, change of axial loads with tower movement, and external moments at guy connections, are ignored.

In finding the natural frequencies of the tower, formulas are presented which take into account the axial forces, but which are valid only for towers with two sets of guys, and with no portion cantilevered above the upper guy connection.

CHAPTER II

GUY CABLES - STATIC ANALYSIS

In the design and construction of the type of towers considered in this study, the lateral support is furnished by guys which are usually made up of steel strand cables. Since these cables are also responsible for a major portion of the tower's axial loads, it is important to have available an accurate and reliable set of working formulae which relate the variables associated with hanging cables. An important example would be a formula which expresses the change in cable tension as a function of the tower movement. In some special cases, the formulae developed by simply considering the cables as straight two-force members would be sufficient. However, for very tall towers with heavy cables, it is necessary to take into consideration the sag of the cables due to their own weight.

To account for the deflection of the guy cables, authors of existing papers have referred to the similarity between flat catenaries and parabolas, and then proceeded to develop all formulae associated with the cables on the basis that they hang as parabolas. Such an assumption will not be made in the work to follow. It is felt that the formulae derived by adhering to the properties of the catenary are as easy to use as those based on the properties of a parabola. Therefore, the inaccuracies involved in the latter are not warranted.

Curve Equations and Properties

The cable is considered to be perfectly flexible so that the differential equation of the cable curve can be written by summing the vertical forces which act on an element. Some of the details of the derivation of the equation for the cable curve are shown below.

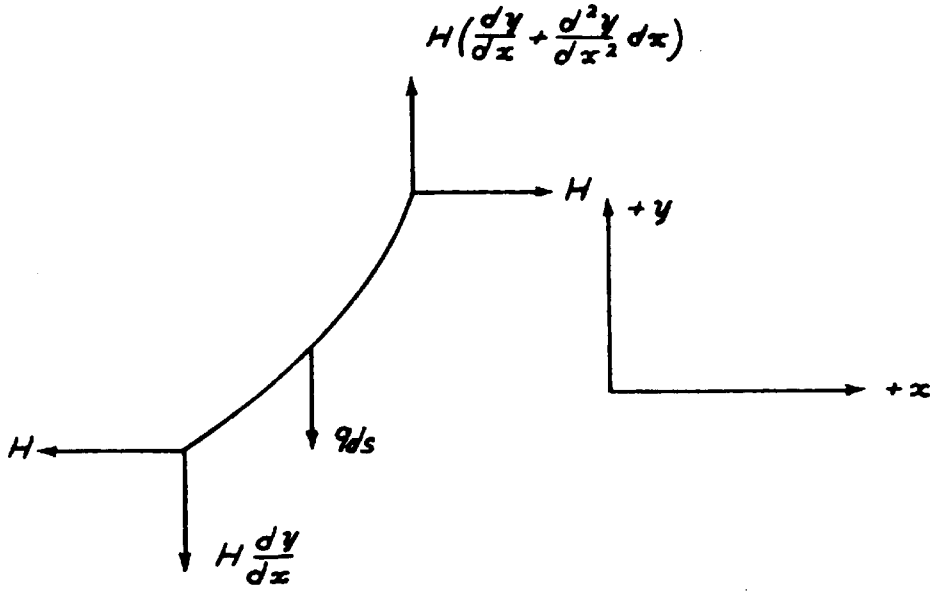


Fig. 2.1
Cable Element

H - horizontal component of cable tension.

q - weight per unit length of cable.

$$H\frac{d^2y}{dx^2}dx = q\,ds$$

$$\frac{d^2y}{dx^2} = \frac{q}{H}\frac{ds}{dx} = \frac{q}{H}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \sinh\left(\frac{q}{H}x + a_1\right) \qquad (2.1)$$

$$y = \frac{H}{q}\cosh\left(\frac{q}{H}x + a_1\right) + a_2 \qquad (2.2)$$

The integration constants a_1 and a_2 are evaluated from the boundary conditions, $y\Big|_{x=0} = 0$ and $y\Big|_{x=\ell} = h$.

$$a_2 = -\frac{H}{q}\cosh a_1 \tag{2.3}$$

$$a_1 = \sinh^{-1}\left[\frac{qh}{2H\sinh r}\right] - r \tag{2.4}$$

where $r \equiv \frac{q\ell}{2H}$

Equation 2.2 can be rewritten

$$Y = \frac{2H}{q}\sinh(r\bar{x} + a_1)\sinh r\bar{x}$$

where $\bar{x} = \frac{x}{\ell}$ (2.2a)

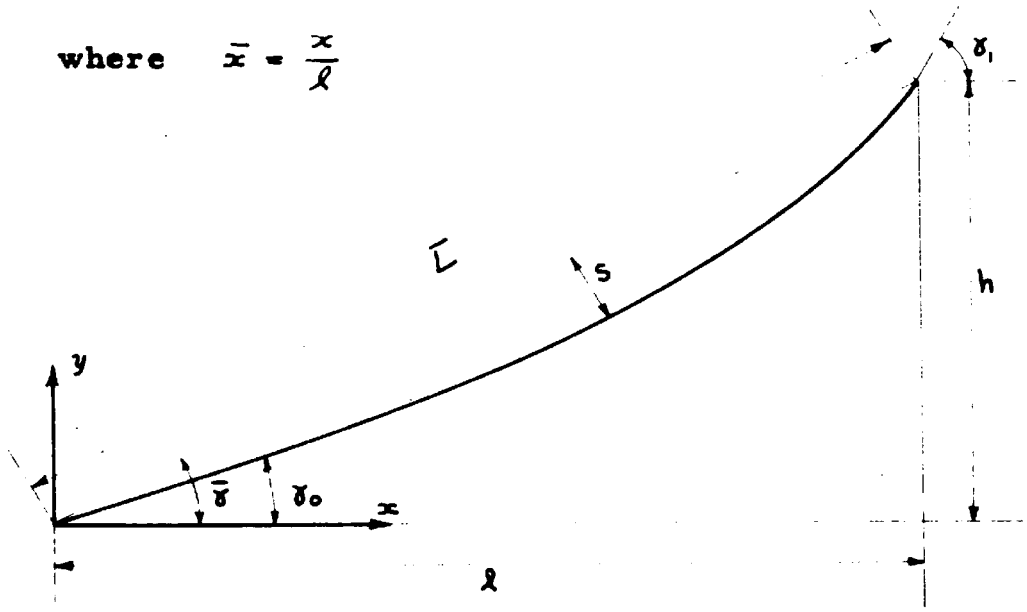


Fig. 2.2
Hanging Cable

The cable tension, T, at any point, x, is equal to $H\frac{ds}{dx}$.

From equation 2.1 then, it can be seen that

$$T = H\cosh\left(\frac{qx}{H} + a_1\right). \tag{2.1.a}$$

Equation 2.1 above can be used to obtain the total length of the cable, shown in Fig. 2.2, in terms of the other cable variables H, l, h, q. Such an expression will furnish the key relations needed to express changes in one variable in terms of changes in the other variables, e.g., $\Delta l = f(\Delta H)$.

$$L = \int_0^{\ell} \frac{ds}{dx}\, dx = \int_0^{\ell} \cosh\left(\frac{q}{H}x + a_1\right) dx$$

Carrying out the integration and substituting in the limits, one obtains, after simplification:

$$L = \left[h^2 + \frac{4H^2}{q^2}\sinh^2 r\right]^{\frac{1}{2}} \quad \text{or} \quad \sqrt{L^2 - h^2} = \frac{2H}{q}\sinh r \tag{2.5}$$

Two other quantities frequently needed in the design and erection of guy cables are the slopes of the cable curve at the upper and lower extremities. The algebra involved in the derivation of the final form of the expressions for these two quantities is presented in Appendix A, and serves as an example of the type of term rearrangement often omitted in presenting the derivations which appear in this chapter.

The slope of the cable at the ground, $\tan \gamma_o$, is a quantity which is used in many instances, in conjunction with a sighting apparatus, for setting the correct initial cable tension.

$$\text{Tan}\, \gamma_o = \left.\frac{dy}{dx}\right|_{x=0} = \sinh a_1 \tag{2.6}$$

As shown in Appendix A, this may be rewritten,

$$\text{Tan}\, \gamma_o = \frac{q}{2H}\left[h \coth r - L\right] \tag{2.6a}$$

The slope of the cable at the tower, tan γ_1, is a quantity needed for the computation of tower loads resulting from the guy forces.

$$Tan\ \gamma_1 = \left.\frac{dy}{dx}\right|_{x=1} = \sinh\left[2r + a_1\right] \qquad (2.7)$$

Rewriting, as above, one obtains:

$$Tan\ \gamma_1 = \frac{q}{2H}\left[h \coth r + L\right] \qquad (2.7a)$$

The maximum sag, denoted "s" and measured perpendicular to the chord (Fig. 2.2), is usually desired for the various loading conditions to which the guy is subjected. In fact, at least one large organization *, engaged in the design of guyed towers, makes use of the maximum sag as a criterion for establishing the initial erection stress to be placed in the guys. An example of this is as follows: "The initial tension shall be such that at 32°F with a 1/2 inch coating of ice, "s" shall not exceed 1-1/2 percent of the chord length, $\overline{L}$."

"s" may be determined as follows:

Let x_s be the horizontal distance to the point of maximum sag, then: $\left.\frac{dy}{dx}\right|_{x_s} = \frac{h}{\ell}$

$$x_s = \frac{H}{q}\left(\sinh^{-1}\frac{h}{\ell} - a_1\right) \qquad (2.8)$$

$$Y_{(x_s)} = \frac{H}{q}\left(\cosh \sinh^{-1}\frac{h}{\ell} - \cosh a_1\right) = \frac{H}{q}\left(\sec \bar{\gamma} - \cosh a_1\right) \qquad (2.9)$$

$$S = \left(x_s\frac{h}{\ell} - Y_{(x_s)}\right)\cos\bar{\gamma} = x_s \sin\bar{\gamma} - Y_{(x_s)}\cos\bar{\gamma} \qquad (2.10)$$

* The Canadian Bridge Company, Ltd., Walkerville, Ontario.

The dimensionless parameter, $r=\frac{q\ell}{2H}$, and hyperbolic functions involving "r" have appeared quite often in the formulae presented thus far and will occur even more frequently in the work to follow. "r" can be given physical significance by speaking of it as the ratio between the vertical force necessary to support one-half the weight of the horizontal projection of a cable and the total horizontal force acting on the cable. In guy cables, or cables functioning as guys, "r" will usually be quite small, rarely exceeding 0.5. Therefore, sinh r, cosh r, and tanh r can be replaced by a few terms of their respective series without a meaningful loss of accuracy. It will often be convenient and helpful to do this in the developments to follow, and many expressions will be simplified considerably by making such a substitution. A few examples are given below which illustrate the error involved in "cutting off" the series after a few terms. "e" is an abbreviation for error.

$$\sinh r = r + \frac{r^3}{3!} + \frac{r^5}{5!} + \frac{r^7}{7!} + \cdots \qquad (2.11)$$

$$\cosh r = 1 + \frac{r^2}{2!} + \frac{r^4}{4!} + \frac{r^6}{6!} + \cdots \qquad (2.12)$$

$$\tanh r = r - \frac{r^3}{3} + \frac{2r^5}{15} - \frac{17r^7}{315} + \cdots \qquad (2.13)$$

$$\sinh r \simeq r\left(1+\frac{r^2}{6}\right) \qquad |e| < .1\% \text{ if } r < .6 \qquad (2.11.a)$$

$$\cosh r \simeq \left(1+\frac{r^2}{2}\right) \qquad |e| < .1\% \text{ if } r < .4 \qquad (2.12.a)$$

$$\tanh r \simeq r\left(1-\frac{r^2}{3}\right) \qquad |e| < .1\% \text{ if } r < .3 \qquad (2.13.a)$$

$$\sinh r \simeq r\left(1+\frac{r^2}{6}+\frac{r^4}{120}\right) \qquad |e| < .01\% \text{ if } r < .9 \qquad (2.11.b)$$

$$\cosh r \simeq \left(1+\frac{r^2}{2}+\frac{r^4}{24}\right) \qquad |e| < .05\% \text{ if } r < .9 \qquad (2.12.b)$$

$$\tanh r \simeq r\left(1-\frac{r^2}{3}+\frac{2r^4}{15}\right) \qquad |e| < .08\% \text{ if } r < .5 \qquad (2.13.b)$$

By using equation 2.11.a, it is possible to rewrite expression 2.4 of page 9 in a form that is easier to use numerically.

$$a_1 = \sinh^{-1}\left[\frac{h}{\ell\left(1+\frac{r^2}{6}\right)}\right] - r \qquad (2.4.a)$$

Equation 2.5, repeated below, can also be changed to forms which make calculations easier by using 2.11.a and 2.11.b. This equation, relating the cable variables L, H, l, h, and q, serves as a basis for most of the work to follow in this chapter, and will be used in its exact form, 2.5, in derivations. However, when it is to be used for calculating L in terms of known values of H, l, h, and q, it is much more convenient to work with a modified form that can be derived as follows:

$$\sqrt{L^2-h^2} = \frac{2H}{q}\sinh r \simeq \ell\left(1+\frac{r^2}{6}\right)$$

$$L^2 \simeq h^2+\ell^2+\frac{\ell^2r^2}{3} = \bar{L}^2+\frac{\ell^2r^2}{3}$$

$$L \simeq \bar{L}+\frac{\ell^2r^2}{6\bar{L}} \qquad (2.5.a)$$

Equation 2.5.a was derived by consistently dropping all powers of "r" above the third. It is very easy to use and is highly accurate for most cables that are functioning as guys. In fact, it is identical in form with a formula presented for the same

purpose in a text on structural analysis (16). However, since the authors of this text were using properties of the parabola instead of the catenary, their formula, expressed in the symbols of this study, $L = \bar{L} + \frac{\ell^4 r^2}{6\bar{L}^3}$, is seen to be in error by a factor $\cos^2\bar{\sigma}$ for the amount added to the chord length.

Although it will be seen that 2.5.a is quite accurate for normal use in carrying out calculations with guys, it may not be sufficently accurate to use in computing "L" for cables with excessive sags. In this case it may be necessary to retain more terms in the expansion of sinh r. By using 2.11.b and retaining powers of "r" including the fifth, it is seen that:

$$L = \bar{L} + \frac{\ell^2 r^2}{6\bar{L}} + \frac{\ell^2 r^4}{45\bar{L}} - \frac{\ell^4 r^4}{288\bar{L}^3} \qquad (2.5.6)$$

Effects of Changes in Cable Variables

Each time the tower is loaded or undergoes a change of dimensions, some of the cable variables must change to keep the tower in equilibrium. Therefore, to carry out a rational analysis of a cable guyed tower, one must have available formulae which accurately relate these variables and their changes. The three main classes of these changes are:

a) l changes due to a transverse tower load, causing changes in H, L, and h.

b) h and L change due to a variation in temperature, causing a change in H.

c) q increases due to ice on the cable, causing H, L, and h to change.

Formula 2.5, page 10, is the relation between variables which will furnish the information necessary to calculate quantitative values for the three main cases described above. Cases (a) and (b) may be derived from the differential relation:

$$dH = \frac{\partial H}{\partial \ell} d\ell + \frac{\partial H}{\partial L} dL + \frac{\partial H}{\partial h} dh \qquad (2.14)$$

The partial derivatives needed in 2.14 are obtained from 2.5, $\sqrt{L^2 - h^2} = \frac{2H}{q} \sinh r$, as follows:

$$\frac{\partial H}{\partial \ell} = \frac{1}{\frac{\ell}{H} - \frac{2}{q} \tanh r} \qquad (2.15)$$

$$\frac{\partial H}{\partial L} = \frac{L}{\sqrt{L^2 - h^2}\left(\frac{2}{q} \sinh r - \frac{\ell}{H} \cosh r\right)} \qquad (2.16)$$

$$\frac{\partial H}{\partial h} = \frac{-h}{\sqrt{L^2 - h^2}\left(\frac{2}{q} \sinh r - \frac{\ell}{H} \cosh r\right)} \qquad (2.17)$$

It is apparent that if one attempted to derive working formulae by using the three equations above, the results would be too complex. In addition, due to the fact that differences of quantities of the same order of magnitude are involved, very accurate tables of hyperbolic functions would have to be used to obtain results that are correct to 0.1 percent. These two objections are largely overcome if the three term series 2.11.b, 2.12.b, and 2.13.b are substituted for the hyperbolic functions.

$$\frac{dH}{d\ell} \simeq \frac{1}{\frac{\ell}{H} - \frac{\ell}{H}\left(1 - \frac{r^2}{3} + \frac{2r^4}{15}\right)} \simeq \frac{12H^3}{q^2\ell^3}\left(1 + .4r^2\right) \qquad (2.15.a)$$

Note that in deriving 2.15.a, tanh r was replaced by three terms of its series and, in addition, the approximation,

$$\frac{1}{1 - Cr^2} = 1 + Cr^2 + C^2r^4 + C^3r^6 + \cdots \simeq 1 + Cr^2$$

was introduced. If care is exercised in handling these two approximations, they will not be accumulative and in many instances an increase in accuracy may be obtained by causing one to partially cancel the other. For instance, if four terms of the series were used instead of three, 2.15.a would be written:

$$\frac{dH}{d\ell} = \frac{12H^3}{q^2\ell^3} \frac{1}{\left(1 - .4r^2 + .162r^4\right)}$$

The final form presented as 2.15.a can be rewritten:

$$\frac{dH}{d\ell} = \frac{12H^3}{q^2\ell^3} \frac{1}{\left(1 - .4r^2 + .160r^4 - \cdots\right)}$$

In other words, in this particular case, cancellation of the errors involved in the two approximations yielded four term accuracy when only three terms were used. When available, advantage will be taken of this increase in accuracy throughout the work to follow.

Continuing with the simplification of equations 2.16 and 2.17, it is seen that:

$$\frac{dH}{dL} \simeq \frac{L}{\sqrt{L^2 - h^2}\frac{\ell}{H}\left[\left(1 + \frac{r^2}{6} + \frac{r^4}{120}\right) - \left(1 + \frac{r^2}{2} + \frac{r^4}{24}\right)\right]} =$$

$$\frac{-12H^3L}{q^2\ell^3\sqrt{L^2 - h^2}\left(1 + .1r^2\right)} \qquad (2.16.a)$$

$$\text{and } \frac{dH}{dh} \simeq \frac{12H^3h}{q^2\ell^3\sqrt{L^2-h^2}\,(1+.1r^2)} \qquad (2.17.a)$$

Expressions 2.15.a, 2.16.a, and 2.17.a all contain H^3 as a factor so that the integration of 2.14 can be easily carried out as follows:

$$\int_{H_o}^{H}\frac{dH}{H^3} = \frac{12}{q^2}\int_{\ell_o}^{\ell}\frac{(1+.4r^2)\,d\ell}{\ell^3} - \frac{12}{q^2\ell^3}\int_{L_o}^{L}\frac{L\,dL}{\sqrt{L^2-h^2}\,(1+.1r^2)} + \frac{12}{q^2\ell^3}\int_{h_o}^{h}\frac{h\,dh}{\sqrt{L^2-h^2}\,(1+.1r^2)}$$

Where the variables with the subscript "o" indicate the initial values before change, or $\Delta H = H - H_o$; $\Delta l = l - l_o$; $\Delta h = h - h_o$; and $\Delta L = L - L_o$.

Since "r" contains "H", it is not constant during integration of the above expressions. However, according to the mean value theorem for integrals, there exists an intermediate value of "r", denoted "r_i", such that "r" can be replaced by "r_i" and treated as a constant during integration. The values of the integrals will not be sensitive to small changes in "r" as $r^2 \ll 1$. If the initial and final values of "r" are known, "r_i" should be made the arithmetical mean of the two values, but when the initial and final values are not both known, it will be necessary to guess the value of "r_i" from practical considerations. In case an extremely poor guess is made and the results are affected, the calculation can be repeated using the first results as a guide to improve the first guess.

Carrying out the integration of 2.14 indicated above, the

following result is obtained after some algebraic simplifications.

$$\frac{H^2-H_o^2}{H^2H_o^2} = \frac{24}{q^2\ell^3}\left[(1+.4r_i^2) - \frac{L\Delta L - h\Delta h}{\sqrt{L^2-h^2(1+.1r_i^2)}}\right]$$

This may be rearranged to:

$$H = \frac{H_o}{\sqrt{1-K}} \qquad (2.18)$$

where

$$K = \frac{6}{r_o^2\ell}\left[(1+.4r_i^2)\Delta\ell - \frac{L\Delta L - h\Delta h}{\ell(1+\frac{1}{6}r_o^2)(1+.1r_i^2)}\right]$$

and $\sqrt{L^2-h^2}$ is replaced by $\ell(1+\frac{1}{6}r_o^2)$.

When K<<1, 2.18 is easier to use in the form:

$$\Delta H = H_o\,\frac{K}{2}\left[1+\frac{3}{4}K+\frac{5}{8}K^2+\frac{35}{64}K^3+\cdots\right] \qquad (2.18.a)$$

Δ L and Δ h are seldom known. Even in case (b), page 14, Δ L and Δ h are not completely known, as any change in H causes a change in L and h due to the fact that the cable and tower are elastic. So it is necessary to be able to express ΔL and Δ h as functions of Δ H in equation 2.18, before it can be used. Some of the steps in the derivation of these expressions are as follows:

$$\Delta L = \frac{1}{A_c E_c}\int_L (T-T_o)\,ds = \int_0^\ell \left[H\cosh^2\left(\frac{qx}{H}+a_1\right) - H_o\cosh^2\left(\frac{qx}{H_o}+a_1\right)\right]dx$$

A_c, E_c - Area and Modulus of Elasticity of Cable.

T, T_o - Final and Initial Cable Tension.

$$A_c E_c \Delta L = \frac{L^2+h^2}{2\sqrt{L^2-h^2}}\left(H\cosh r - H_o\cosh r_o\right) + \Delta H\,\frac{\ell}{2} \qquad (2.19)$$

or, replacing $\sqrt{L^2-h^2}$ by $\ell(1+\frac{1}{6}r^2)$ and cosh r by two terms of its

series, 2.19 can be transformed to the following convenient form:

$$\Delta L = \frac{L^2 \Delta H}{\ell A_c E_c}\left[1 - \frac{r^2}{6}\left(4 - \frac{\ell^2}{L^2}\right)\right]. \qquad (2.19.a)$$

Note that as $r \to 0$, the result of 2.19.a approaches the result obtained by considering the cable to be a simple straight two-force member.

An expression for Δh could include many factors other than temperature, such as: compression of leg of tower due to change in cable stress; effect of change in stress of other cables attached to the same leg at different levels; slope of elastic curve representing tower's center; decrease in tower's vertical projection due to lateral bending; and the effect of fiber stress as calculated from elementary beam theory. All of these factors could be considered, but, since the effect of Δh is usually small as compared with other effects, it does not warrant computations detailed enough to account for all the above effects. Therefore, it is the writer's recommendation that the first two factors be approximated by assuming that the change in cable stress compresses or permits elongation of a member which has a cross-section equal to the area of the leg at the level of the cable connection and a length equal to h, the vertical projection of the cable.

From equation 2.7.a, it is seen that the vertical component of cable force, V, is:

$$V = \frac{q}{2}(h \coth r + L) \qquad (2.20)$$

$$dV = \frac{q^2 h \ell}{4H^2} \operatorname{csch}^2 r\, dH \qquad (2.21)$$

Equation 2.21 is transformed to a simpler form by substituting for the hyperbolic function two terms of its series, giving:

$$dV = \frac{h}{\ell}\left(1 - \frac{1}{3} r^2\right) dH \qquad (2.21.a)$$

Equation 2.21.a is integrated to give:

$$\Delta V = \frac{h}{\ell}\left(1 - \frac{1}{3} r_i^2\right) \Delta H \qquad (2.21.b)$$

Note that, as $r_i \to 0$, equation 2.21.b gives results which approach those obtained by treating the cable as a two-force member, but the approach is from below instead of from above as one may expect.

Now that ΔV is known in terms of ΔH, one can make use of the relation $\Delta h = \frac{-h \Delta V}{A_\ell E_\ell}$ (A_ℓ and E_ℓ are the area and elastic modulus of the tower leg at the level of the cable connection) to obtain,

$$\Delta h = \frac{-h^2}{\ell A_\ell E_\ell}\left(1 - \frac{1}{3} r_i^2\right) \Delta H \qquad (2.21.c)$$

which can be used together with equation 2.19.a to complete an expression for "K" that will account for cable stretch and compression of the tower leg. For example, ΔH can be determined for a given value of $\Delta 1$ by using equation 2.18 (repeated below) where "K" is given as:

$$(2.18)\quad H = \frac{H_0}{\sqrt{1-K}};$$

$$K = \frac{6}{r_0^2 \ell}\left\{(1 + .4 r_i^2)\Delta \ell - \frac{L^3\left[1 - \frac{r_i^2}{6}\left(4 - \frac{\ell^2}{L^2}\right)\right]\Delta H}{\ell^2 A_c E_c \left(1 + \frac{r_0^2}{6}\right)(1 + .1 r_i^2)} - \frac{h^3\left(1 - \frac{r_i^2}{3}\right)\Delta H}{\ell^2 A_\ell E_\ell \left(1 + \frac{r_0^2}{6}\right)(1 + .1 r_i^2)}\right\}$$

As will be shown in the next section, this is easy to work with numerically.

Numerical Examples

Three examples will be worked out in this section to illustrate the use of the cable formulas developed in previous sections of this chapter. The physical constants used in the examples below are those of the top cables which support the Oklahoma City tower and the results obtained here will be used in the complete static analysis of the tower in Chapter 5.

Constants: $l = 950'$; $h = 1349'$; $A_c = 2.42$ in.^{2}; $E_c = 24 \cdot 10^3$ksi;

$q = 9.0$ lb/ft; $A_1 = 26.4$ in.^{2}; $E_1 = 30 \cdot 10^3$ksi; $H_o = 35.3$k;

$r_o = \frac{ql}{2H_o} = .1211$; $r_o^2 = .0147$; $\bar{L} = \sqrt{l^2 + h^2} = 1649.94'$;

$$L_o = \bar{L} + \frac{l^2 r_o^2}{6\bar{L}} + \frac{l^2 r^4}{45\bar{L}} - \frac{l^4 r^4}{288\bar{L}^3} = 1649.94 + 1.34 + .003 - .0001 = 1651.28$$

1) Find effect on H_o of a temperature drop of 40^o. Equation 2.18.a,

$$\Delta H = H_o \frac{K}{2}\left[1 + \frac{3}{4}K + \frac{5}{8}K^2 + \frac{35}{64}K^3 + \cdots\right],$$

can be used for the trial solution to this problem, where

$$K = \frac{6}{r_o^2 l}\left[(1 + .4r_i^2)\Delta l - \frac{L\Delta L - h\Delta h}{l\left(1 + \frac{1}{6}r_o^2\right)(1 + .1r_i^2)}\right]$$

The change in H will be small as the decrease in L and h are canceling effects, so r_i^2 will be taken equal to r_o^2, i.e., $r_i^2 = .0147$. Taking the coefficient of expansion for steel, α, equal to $6.7 \cdot 10^{-6}/F^o$, and using the equations 2.19.a and 2.21.c, it is seen that:

$$\Delta L = \alpha L(-40^o) + \frac{L^2}{l A_c E_c}\left[1 - \frac{r_i^2}{6}\left(4 - \frac{l^2}{L^2}\right)\right]\Delta H = -.442 + .0489\,\Delta H$$

$$\Delta h = \alpha h(-40^\circ) - \frac{2h^2}{\ell A_\ell E_\ell}\left(1 - \frac{1}{3}H_1^2\right)\Delta H = -.362 - .0048\Delta H$$

In the expression for Δh the factor "2" was inserted to account for the fact that two cables are attached to each leg.

With $\Delta \ell = 0$ the working formula for K is

$$K = 4.53 \cdot 10^{-4}\left[-L\Delta L + h\Delta h\right] = .109\left[1 - .361\,\Delta H\right]$$

Note that the equation being solved for ΔH also contains ΔH so a trial solution is called for. However, this may easily be obtained in three trials as follows:

Guess $\Delta H = 1.0k$ then $K = .0697$

and $\Delta H = (35.3)(\frac{.0697}{2})(1.0553) = 1.30k.$

Guess $\Delta H = 1.20k$ then $K = .0616$ and $\Delta H = 1.14k.$

Now interpolating to make the difference between the guess and the result zero, one obtains $\Delta H = 1.167$, which gives $\Delta H = 1.167$ when substituted in the working formula.

2) Find the additional effect of an increase of q due to 1/2 inch of ice on whole structure.

Here the increase in q will be dealt with separately and then the result will be corrected to account for cable stretch and tower compression.

$$dH = \frac{\partial H}{\partial q}dq \quad \text{or} \quad dH = \frac{H}{q}dq$$

This linear relationship may be integrated to give, $\Delta H = H\frac{\Delta q}{q}$

where L, h, and l are held constant.

For 1/2 inch of ice $\Delta q = 1.55$ or $\Delta H = (36.5)(\frac{1.55}{9}) = 6.3k$.

This result must be decreased to account for the amount of cable stretch and tower compression resulting from the net change in H. In addition, there is a negative effect from the amount the tower compresses due to its own ice load. This was calculated separately as $\Delta h = -.0176'$.

$r_o = .1171$; $K = -4.84 \cdot 10^{-4}\left[L\Delta L - h\Delta h\right]$; $\Delta L = .0489\ H$;

$\Delta h = -.0176 - .0048\Delta H$; or $K = -.0422\left[\Delta H + .272\right]$

Guess a net $\Delta H = 3.5k$, then $K = -.149$ and

$$\Delta H = (36.5 + 6.3)(\frac{-.149}{2})(1 - .112 + .014 - .002) = -2.8k,$$

or the net $\Delta H = 6.3 - 2.8 = 3.5k$ as assumed.

So, with ice on the cable $H = 40k$ and $r = .125$. Then the vertical component of cable stress V, is obtained from 2.20 as:

$$V = \frac{q}{2}(h \coth r + L) = 66.0k.$$

3) Find additional effect of $\Delta l = 2.5$ ft.

Here the change in H is probably going to be of the order of magnitude of H so that equation 2.18, $H = \frac{H_o}{\sqrt{1-K}}$, should be used. Since all the changes in h and L are functions of ΔH only, the following expression for K should be used.

$$K = \frac{6}{r_o^2 l}\left\{(1+.4r_i^2)\Delta l - \frac{L^3\left[1-\frac{1}{6}r_i^2(4-\frac{l^2}{L^2})\right]\Delta H}{l^2 A_c E_c (1+\frac{1}{6}r_o^2)(1+.1r_i^2)} - \frac{h^3(1-\frac{1}{3}r_i^2)\Delta H}{l^2 A_l E_l (1+\frac{1}{6}r_o^2)(1+.1r_i^2)}\right\}$$

$$K = .404\left[(1+.4r_1^2)\Delta l - .0892(1-.713r_1^2)\Delta H\right]$$

Guess $\Delta H = 15.1$ then $r_1^2 = r_0^2\left(\frac{40}{40+7.6}\right)^2 = .011$

and $$K = .404\left[(1.0044)(2.5) - (.0892)(.9921)(15.1)\right] = .4746$$

and $$H = \frac{40}{\sqrt{1-.4746}} = 55.18 \text{ or } \Delta H = 15.2K$$

Resultant of Several Cable Forces at One Level

The purpose of the work in this section is to extend the formulae developed thus far so that they can be used to compute the total resistance offered, by all guys attached to the tower at one level, to motion in any direction. Inspection of the formulae presented for the solution of a single guy, shows that the relation between guy resistance and tower motion is not linear. Also, it is apparent that, to evaluate the resistance corresponding to a given motion, one must employ a "trial and error" method of solving the given equations. This "trial and error" approach is simplified if one has available a set of formulae giving Δl as a function of ΔH for each cable instead of ΔH as a function of Δl as in previous sections. These formulae are developed below.

$$dl = \frac{\partial l}{\partial H}dH + \frac{\partial l}{\partial L}dL + \frac{\partial l}{\partial h}dh \qquad (2.22)$$

$$\frac{\partial l}{\partial H} = \frac{q^2 l^3}{12H^3(1+.4r^2)} \text{ from } (2.15.a)$$

$$\frac{\partial \ell}{\partial L} = \frac{L}{\sqrt{L^2 - h^2 \cosh r}} \simeq \frac{L}{\ell(1+\frac{1}{6}r_o^2)(1+.5r^2)} \qquad (2.23)$$

$$\frac{\partial \ell}{\partial h} = \frac{-h}{\sqrt{L^2 - h^2 \cosh r}} \simeq \frac{-h}{\ell(1+\frac{1}{6}r_o^2)(1+.5r^2)} \qquad (2.24)$$

Substituting the last three expressions into 2.22 and integrating, one obtains:

$$\Delta \ell = \frac{r_o^2 \ell}{6(1+.4r_i^2)}\left[1-\left(\frac{H_o}{H}\right)^2\right] + \frac{L\Delta L - h\Delta h}{\ell(1+\frac{1}{6}r_o^2)(1+.5r_i^2)} \qquad (2.25)$$

By substituting the equations which express ΔL and Δh in terms of ΔH into 2.25, the following working equation is obtained:

$$\Delta \ell = \frac{r_o^2 \ell}{6(1+.4r_i^2)}\left[1-\left(\frac{H_o}{H}\right)^2\right] + \frac{\Delta H}{\ell^2(1+\frac{1}{6}r_o^2)(1+.5r_i^2)}\left\{\frac{L^3}{A_c E_c}\left[1-\frac{1}{6}r_i^2\left(4-\frac{\ell^2}{L^2}\right)\right] + \frac{h^3}{A_\ell E_\ell}\left(1-\frac{1}{3}r_i^2\right)\right\} \qquad (2.25.a)$$

Numerical examples of the use of equation 2.25.a will be presented later in this section. They will be used to determine the tower movement necessary for a group of guys to develop a given total resistance in a specified direction.

In a detailed study of the efficiency of various cable arrangements, it is necessary to investigate the action of the tower resulting from wind forces acting at different angles. Such an investigation is complicated by the fact that the relation between Δl and ΔH is not linear for cable guys. Therefore, the movement of the tower is not along the line of action of the applied force as would be the case with a tower guyed by straight two-force members.

Figure 2.3 below shows three of the most common guy arrangements employed in the construction of very tall towers. The

loading is at a general angle, θ.

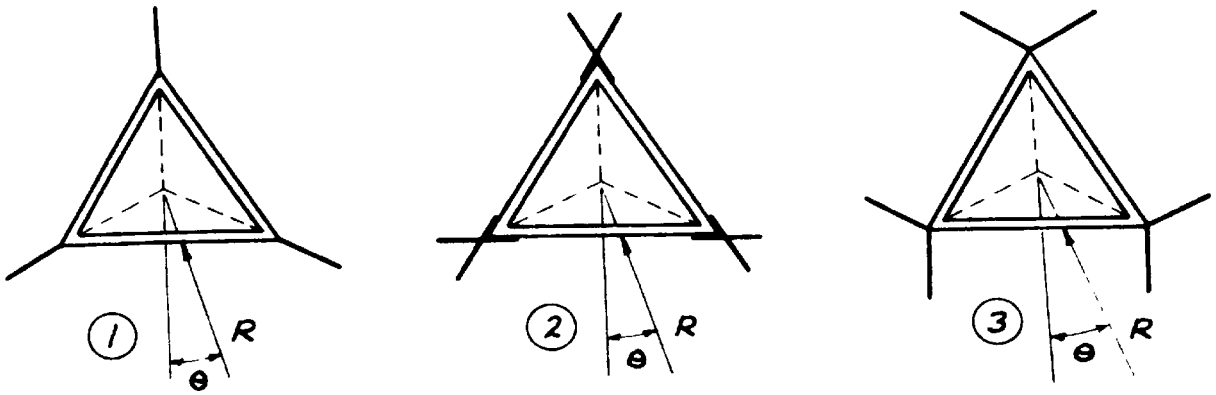


Fig. 2.3
Three Common Guy Arrangements

In none of these examples would the movement of the center of the tower be along the line of action of "R" as "R" is increased in magnitude for a general angle, θ. However, in any of the three cases shown, if θ equals 0° or 60°, a condition of symmetry exists and the tower movement will be along the line of action "R". Thus, since any force can be represented by its components along $\theta = 0^{\circ}$ and $\theta = 60^{\circ}$, a force whose line of action is θ, can be studied by investigating two symmetrical cases. For this reason, the symmetrical case worked out later in this section is not narrow in its applicability.

Before proceeding with a numerical example where $\theta = 60^{\circ}$ in case three (Fig. 2.3), it is necessary to discuss a method for determining Δl for a guy whose horizontal projection is not parallel to the tower movement. If the cable projection and line of movement intersect at some angle ϕ then for a tower movement, y, that is very small as compared with the horizontal cable projection, l, one

can usually write $\Delta\ell = y \cos\phi$ i.e., Δl is taken as the component of y along the cable projection. However, the component of y perpendicular to the cable projection, $y \sin\phi$, does cause a small increase in the projected length of the cable, $\frac{1}{2}\frac{(y\sin\phi)^2}{\ell}$ and for precise calculations it may be necessary to consider this addition, i.e.,

$$\Delta\ell = y\cos\phi + \frac{1}{2}\frac{(y\sin\phi)^2}{\ell}$$

Numerical Example

The upper guys of the Oklahoma City tower will be used again to illustrate the numerical use of the cable formulas presented in this chapter. The physical constants for each of the six cables comprising the top guy system are listed on page 21 . Here the motion ofthe tower, necessary for the guys to develop a given net horizontal reactive force, will be studied and linear equations, relating the forces and moments to the motion, will be calculated.

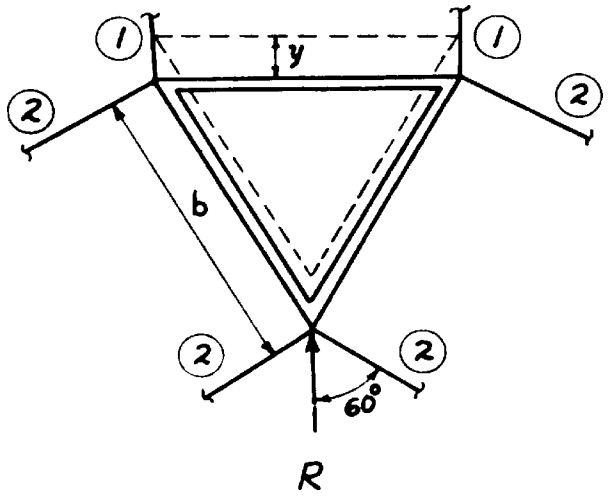


Fig. 2.4
Symmetrical Guy Load

Fig. 2.4 shows a plan view of the guy system employed in the construction of the Oklahoma City tower. Here the system is being subjected to a symmetrical loading at a support point corresponding to $\theta = 60^{\circ}$ in drawing 3 of Fig. 2.3. All six cables have identical properties, but they are numbered above as "1" or "2" to distinguish between the cables whose horizontal projections are parallel to the line of action of the force and those whose horizontal projections intersect it at 60°. The subscripts "1" and "2" in the equations below refer to cables "1" and "2" in Fig. 2.4 above.

$$R = 4(\Delta H_2 \cos 60^{\circ}) - 2(\Delta H_1) = 2(\Delta H_2 - \Delta H_1)$$

$$\Delta \ell_1 = -y \; ; \; \Delta \ell_2 = \left[y \cos 60^{\circ} + \frac{1}{2} \frac{(y \sin 60^{\circ})^2}{\ell} \right] = \frac{y}{2}\left(1 + \frac{3}{4}\frac{y}{\ell}\right)$$

The procedure that the writer recommends, for finding y and the cable stresses caused by R, is as follows:

1) Guess ΔH_2 and calculate corresponding ΔH_1,

   i.e., $\Delta H_1 = \Delta H_2 - \frac{R}{2}$

2) Calculate $\Delta \ell_1$ and $\Delta \ell_2$ and check for consistency,

   i.e., $\frac{2\Delta \ell_2}{\left(1 + \frac{3}{4}\frac{y}{\ell}\right)} + \Delta \ell_1 = 0$

This is numerically illustrated as follows:

$$\Delta \ell = \frac{r_0^2 \ell}{6(1 + .1 r_1^2)}\left[1 - \left(\frac{H_0}{H}\right)^2\right] + \frac{\Delta H}{\ell^2\left(1 + \frac{r_0^2}{6}\right)(1 + .5 r_1^2)}\left\{\frac{L^3}{A_c E_c}\left[1 - \frac{r_1^2}{6}\left(4 - \frac{\ell^2}{L^2}\right)\right] + \frac{h^3\left(1 - \frac{1}{3} r_1^2\right)}{A_\ell E_\ell}\right\} \tag{2.25a}$$

The case to be worked out here is the determination of cable stresses and tower movement where ice is on the cables and

$R = 60.1.$ $H_o = 40.$ $r_o^2 = .01562.$

$$\Delta \ell = \frac{2.47}{(1+.4r_i^2)}\left[1-\left(\frac{H_o}{H}\right)^2\right]+(.0891)(1-1.1r_i^2)\Delta H$$

from substitution in equation 2.25.a.

Guess $\Delta H_2 = 14.5$; then $\Delta H_1 = -15.55$

$$r_i^2)_1 = (.01562)\left(\frac{40}{40-7.8}\right)^2 = .024 \; ; \; r_i^2)_2 = (.01562)\left(\frac{40}{40+7.3}\right) = .011$$

$$\Delta \ell_1 = \frac{2.47}{1.0096}\left[1-\left(\frac{40}{24.45}\right)^2\right] + (.0891)(.9736)(-15.55) = -4.09-1.35 = -5.44 \text{ ft.}$$

$$\Delta \ell_2 = \frac{2.47}{1.0044}\left[1-\left(\frac{40}{54.5}\right)^2\right] + (.0891)(.9878)(14.5) = 1.13+1.28 = 2.41 \text{ ft.}$$

Checking for consistency $\dfrac{(2)(2.41)}{1+\frac{3}{4}\frac{5.4}{950}} - 5.44 = (-.64) \neq 0$

A similar trial with $\Delta H_2 = 15.3$ nets a residual of $(+.06)$.

Interpolating to make the residual zero, one obtains the correct results: $\Delta H_2 = 15.23k$; $\Delta H_1 = -14.82k$; $y = 5.00$ ft. for $R = 60.1$.

As a result of the above tower movement, the vertical components of stress in cables "2" increased and the vertical components of stress in cables "1" decreased. This causes a moment to be applied to the tower which may be computed as follows:

$$M = \frac{b\sqrt{3}}{6}\left[(2)(2\Delta V_2) - 2(\Delta V_1 + \Delta V_2)\right] = \frac{b\sqrt{3}}{3}(\Delta V_2 - \Delta V_1)$$

or here $M = 4\sqrt{3}\,(\Delta V_2 - \Delta V_1)$

From 2.21.b $\Delta V = \frac{h}{l}\left(1 - \frac{1}{3}r_i^2\right)\Delta H$ or $\Delta V_1 = -20.85$;

$\Delta V_2 = 21.53$ and $M = 293.5$ k ft.

In order to obtain information on the tangent to the curve relating R and y, the above calculations are repeated for R = 62.0, giving $\Delta H_2 = 15.91$, $\Delta H_1 = -15.09$, M = 302.5, and y = 5.20 ft. So, in the range near R = 60.1, the function of R vs y for guy system four, denoted $f_4(y_4)$ may be written: $f_4(y_4) = 9.5\,y_4 + 12.6$

and in this range the external moment may be written as a function of y_4 as follows: $M = G_4(y_4) = 45\,y_4 + 68.5$

In addition, near R = 60.1, the axial load below guy system four has been increased as follows: $\Delta P = 4\Delta V_2 + 2\Delta V_1 = +44.42$

Similar calculations should be carried out for each set of guys in order to carry out a complete static analysis of a guyed tower.

CHAPTER III

STATIC ANALYSIS OF TOWER BELOW LEVEL OF TOP GUY CONNECTIONS

Fig. 3.1, to the right, shows the external loads which act on the tower as a result of its being deflected from a vertical position by a static wind load. If the deflection of each support point is proportional to its height so that a straight line can be passed through all support points of the tower when in a deflected position, and if the vertical loads and moments due to the vertical loads can be ignored, then the problem involved in analyzing the tower is simply that of drawing the shear and bending moment diagrams for a continuous beam. Such assumptions are quite common in practice and several published discussions (6, 16) of the analysis of these structures recommend this type of solution. While the writer feels that such a solution is inadequate for the final review of very tall towers, it may

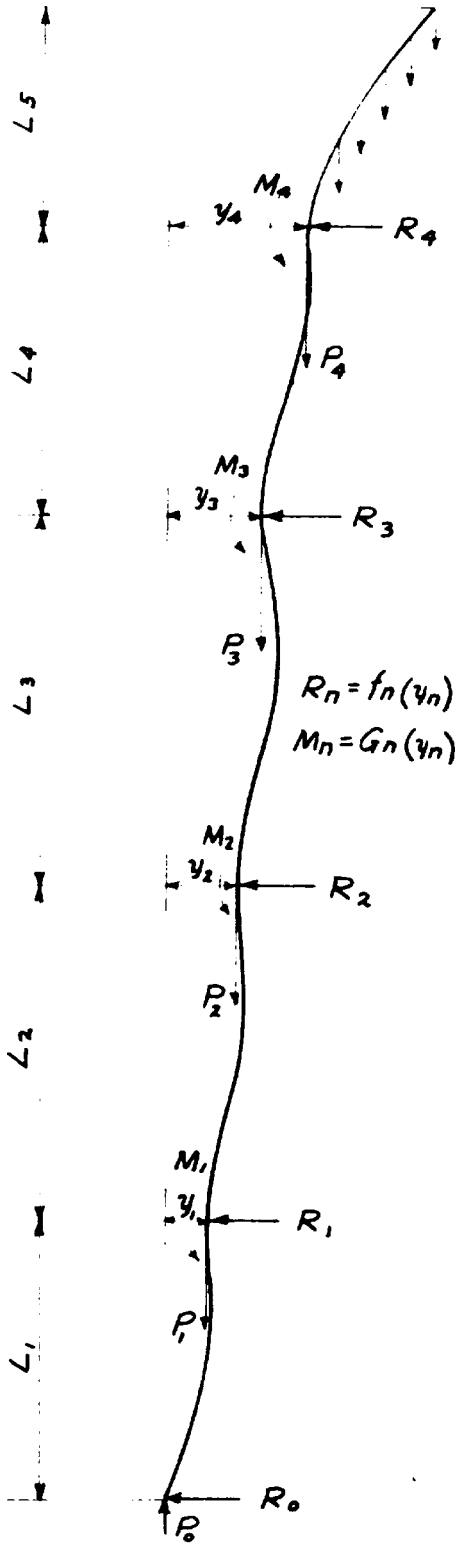


Fig. 3.1
Deflected Tower

suffice for some preliminary calculations and it will be used in this study as a "first guess" in carrying out a more thorough static analysis.

Essentially, it is the recommendation of this study that the analysis of that portion of the tower below the level of the top guy connections be carried out as if the tower were a continuous beam column on elastic supports, and most of the work in this chapter is devoted to the derivation of the formulas necessary for such an analysis. These formulas are available in existing literature (13) and, except for the method of derivation and some suggestions in regard to their numerical application, the work in Chapter 3 does not constitute a contribution of new material. For this reason, the derivations to follow will be more brief than in the other chapters of this study.

In the analysis of vertical members that are designed for transverse loads which are small as compared with the member's weight, it is sometimes necessary to take into account the moments caused by this distributed weight in its deflected position. A television tower is such a member, i.e., it's weight, even without ice, is large as compared with the transverse loads due to wind blowing on the open structure. Furthermore, the axial load in each span of the tower is, to some extent, a function of the deflection of all support points above that span. Therefore, representation of the vertical loads below the top guy connections as "constant" axial forces



in the various spans must be defended.

The action of a vertical member, with an axial load at the top that is large as compared with the weight of the member, can be satisfactorily approximated by adding one-half of the member's weight to the axial load and otherwise ignoring the distributed weight of the member. This approach is indicated in existing literature on the corresponding stability problem (20) and will be discussed in Chapter 4 of this study.

The effect of the variation of the axial load in each span with the deflection of the tower at the guy connection points is not usually an important effect. For small movements, the relief in pressure from the slackening of some of the guys very nearly cancels the increase in pressure resulting from the other guys becoming more taut. Also, since the axial load will always be considerably below the Euler buckling load, it is not necessary to know the value of the axial load to a high degree of accuracy to get satisfactory results. However, the method of analysis recommended in this study will use the final value of P_n for the axial load so that the results will be correct even for large variations of P_n with Y_n.

Derivation of Elastic Curve Equations for Span "i"

Generally span "i" will be considered between points n-1 and n, but to simplify notation in the derivations to follow, "i" will be considered 1, or the end points of the span will be 0 and 1.

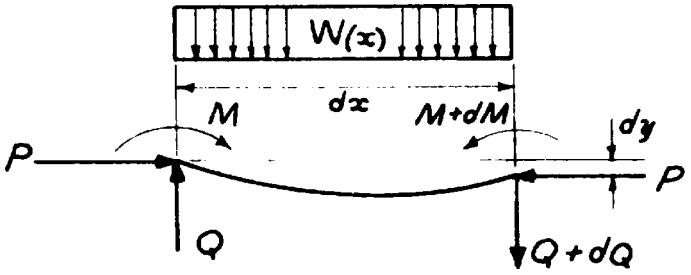


Fig. 3.2
Beam Element

Setting the summation of moments about the left end of the element in Fig. 3.2 equal to zero and omitting second order terms, one obtains:

$$dM - P\,dy - Q\,dx = 0$$

$$\frac{dM}{dx} - P\frac{dy}{dx} = Q$$

$$\frac{d^2M}{dx^2} - P\frac{d^2y}{dx^2} = \frac{dQ}{dx} = -W(x) \qquad (3.1)$$

Substituting $M = -EI\frac{d^2y}{dx^2}$, the moment-curvature relation from elementary beam theory, and taking EI to be constant, the following forms of the basic differential equation are obtained:

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = W(x) = -\frac{d^2M_s}{dx^2} \qquad (3.1.a)$$

Where M_s is the bending moment when no axial load is acting, i.e., P = 0.

Throughout the work to follow in this study, the advantages to be gained by adhering to dimensionless parameters will be exploited whenever possible. Generally, structural engineers do

not make use of dimensionless equations as extensively as do engineers in other fields, but the arguments in favor of their use apply to the structural field. Some arguments in favor of the use of equations with dimensionless terms are: the equations and the algebra involved in their manipulation are simplified; there is less chance for error due to units; more experience is gained by working a specific example, since results apply not only to the structure solved, but to a whole class of structures having the same dimensionless parameters; the selection of the governing parameters is made a more natural and direct process.

The following transformations and definitions will make equation 3.1.a dimensionless:

$$\bar{x} = \frac{x}{L} \; ; \left(\frac{d}{dx} = \frac{1}{L}\frac{d}{d\bar{x}}\right) ; \; \bar{y} = \frac{y}{L} \; ; \; W_{(x)} = W_o \, \varphi_{(\bar{x})} \; ;$$

$$\alpha^2 = \frac{PL^2}{EI} \; ; \; \beta_o = \frac{W_o L^3}{EI} \; ; \; m = \frac{M}{PL} \; ; \left(\frac{d^2\bar{y}}{d\bar{x}^2} = -\alpha^2 m\right)$$

Substituting in 3.1.a

$$EI\frac{L}{L^4}\bar{y}^{IV} + P\frac{L}{L^2}\bar{y}'' = W_o \phi = -\frac{PL}{L^2} m_s''$$

Prime (') denotes differentation with respect to $\bar{x}$.

$$\bar{y}^{IV} + \alpha^2 \bar{y}'' = \beta_o \varphi = -\alpha^2 m_s'' \qquad (3.1.b)$$

The solution of 3.1.b for a general ϕ is: $\bar{y} = \bar{y}_h + \bar{y}_p$, where $\bar{y}_h$ is the solution to the homogeneous equation ($\phi = 0$) and $\bar{y}_p$ is a particular solution of 3.1.b.

$$\bar{y}_h = b_1 \cos \alpha \bar{x} + b_2 \sin \alpha \bar{x} + b_3 \bar{x} + b_4 \qquad (3.2)$$

$$\bar{y}_p = \frac{\beta_0}{\alpha^4}\left[\alpha^2 \iint \phi - \phi + \frac{1}{\alpha^2}\phi'' - \frac{1}{\alpha^4}\phi^{IV} + \cdots\right] \qquad (3.3)$$

$$\text{or } \bar{y}_p = \left[-m_s + \frac{1}{\alpha^2} m_s'' - \frac{1}{\alpha^4} m_s^{IV} + \frac{1}{\alpha^6} m_s^{VI} - \cdots\right] \qquad (3.3.a)$$

Here the important special case of $\phi = 1$, i.e., a uniform load of w_o, will be worked out in detail. Later in the chapter, results will be presented which make the calculation of any desired quantity a simple matter for any form of ϕ.

When $\phi = 1$,

$$\bar{y} = b_1 \cos \alpha \bar{x} + b_2 \sin \alpha \bar{x} + b_3 \bar{x} + b_4 + \frac{\beta_0}{2\alpha^2} \bar{x}^2 \qquad (3.4)$$

$$\theta \equiv \frac{dy}{dx} = -b_1 \alpha \sin \alpha \bar{x} + b_2 \alpha \cos \alpha \bar{x} + b_3 + \frac{\beta_0}{\alpha^2} \bar{x} \qquad (3.5)$$

$$m = -\frac{1}{\alpha^2}\frac{d^2\bar{y}}{d\bar{x}^2} = b_1 \cos \alpha \bar{x} + b_2 \sin \alpha \bar{x} - \frac{\beta_0}{\alpha^4} \qquad (3.6)$$

$$\text{Max. } m \text{ at } \alpha \bar{x} = \tan^{-1}\frac{b_2}{b_1} \qquad (3.7)$$

Denoting the moments and deflections at the beginning and end of the span M_o, y_o, and M_1, y_1, it is seen that the boundary conditions to which equation 3.4 above must be fitted, in order to evaluate the constants of integration, are:

$$\bar{y}\Big|_{\bar{x}=0} = \bar{y}_o \; ; \; \bar{y}\Big|_{\bar{x}=1} = \bar{y}_1 \; ; \; \frac{d^2\bar{y}}{d\bar{x}^2}\Big|_{\bar{x}=0} = -\frac{L}{EI} M_o \equiv -\alpha^2 m_o \; ; \; \frac{d^2\bar{y}}{d\bar{x}^2} = \frac{L}{EI} M_1 = \alpha^2 m_1$$

The sign convention employed here is the so-called slope deflection convention in which a moment acting on the beam at a joint is considered positive if acting in a clockwise direction and negative if

acting in a counterclockwise direction.

Substituting the above boundary conditions into equation 3.4, the following values are obtained for the integration constants:

$$b_1 = m_0 + \frac{\beta_0}{\alpha^4} \; ; \quad b_2 = \frac{1}{\sin\alpha}\left[\frac{\beta_0}{\alpha^4} - m_1 - \left(m_0 + \frac{\beta_0}{\alpha^4}\right)\cos\alpha\right]$$

$$b_3 = \bar{Y}_1 - \bar{Y}_0 + m_1 + m_0 - \frac{\beta_0}{2\alpha^2} \; ; \quad b_4 = \bar{Y}_0 - m_0 - \frac{\beta_0}{\alpha^4}$$

Thus, equation 3.4 and its derivatives can be used to obtain the deflection, slope, moment, etc., at any point in the span of a beam-column in terms of the end moments and deflections.

Letting $\bar{x}$ assume values of 0 and 1, and denoting the corresponding slopes θ_0, and θ_1, it is seen that:

$$\theta_0 = Am_0 - Bm_1 + \bar{y}_1 - \bar{y}_0 + D \qquad (3.8)$$

and

$$\theta_1 = -Bm_0 + Am_1 + \bar{y}_1 - \bar{y}_0 - D \qquad (3.9)$$

where $A \equiv (1 - \alpha\cot\alpha)$; $B \equiv (\alpha\csc\alpha - 1)$; $D \equiv \frac{\beta_0}{2\alpha^2}\left(\frac{2}{\alpha}\tan\frac{\alpha}{2} - 1\right)$

using 3.8 and 3.9 to express the fact that $\theta_{n,\,n-1} = \theta_{n,\,n+1}$,

one obtains: (span "i" from n-1 to n)

$$-\left(\frac{B}{PL}\right)_i M_{n-1,n} + \left(\frac{A}{PL}\right)_i M_{n,n-1} - \left(\frac{A}{PL}\right)_{i+1} M_{n,n+1} + \left(\frac{B}{PL}\right)_{i+1} M_{n+1,n} =$$

$$\frac{Y_{n-1}}{L_i} - \left(\frac{1}{L_i} + \frac{1}{L_{i+1}}\right) Y_n + \frac{Y_{n+1}}{L_{i+1}} + D_i + D_{i+1} \qquad (3.10)$$

Equation 3.10 is of the "three moment" form and could be used, together with equations expressing equilibrium of forces on each joint, to obtain the static solution for a guyed tower. Although this approach yields the joint moments directly, without the intermediate step of finding the joint slopes, it has been the writer's

experience that it is not as easy to use numerically as equations of the slope deflection form. Therefore, equation 3.10 will not be given further consideration in this study.

By solving equations 3.8 and 3.9 simultaneously for m_o and m_1 and inserting the general subscripts, one obtains the following slope deflection equations:

$$M_{n,n+1} = \frac{1}{\alpha^2}\left[C_1\theta_n + C_2\theta_{n+1} + (C_1+C_2)(\bar{Y}_n - \bar{Y}_{n+1})\right] + m_{F_{n,n+1}}$$

$$m_{n+1,n} = \frac{1}{\alpha^2}\left[C_2\theta_n + C_1\theta_{n+1} + (C_1+C_2)(\bar{Y}_n - \bar{Y}_{n+1})\right] + m_{F_{n+1,n}}$$

Now returning to dimensional terms, the slope deflection equations are placed in the standard form:

$$M_{n,n+1} = \frac{EI}{L}\left[C_1\theta_n + C_2\theta_{n+1} + (C_1+C_2)\frac{1}{L}(Y_n - Y_{n+1})\right] + M_{F_{n,n+1}} \quad (3.11)$$

$$M_{n+1,n} = \frac{EI}{L}\left[C_1\theta_n + C_2\theta_{n+1} + (C_1+C_2)\frac{1}{L}(Y_n - Y_{n+1})\right] + M_{F_{n+1,n}} \quad (3.12)$$

where $C_1 = \frac{A\alpha^2}{A^2-B^2} = \frac{1-\alpha\cot\alpha}{\frac{2}{\alpha}\tan\frac{\alpha}{2}-1}$; $C_2 = \frac{B\alpha^2}{A^2-B^2} = \frac{\alpha\csc\alpha-1}{\frac{2}{\alpha}\tan\frac{\alpha}{2}-1}$ (3.13)(3.14)

and $M_{F_{n,n+1}}$ and $M_{F_{n+1,n}}$ are the "fixed end" moments of the span defined as $M_{n,n+1}$ and $M_{n+1,n}$ when $\theta_n = \theta_{n+1} = Y_n - Y_{n+1} = 0$

$$M_{F_{n,n+1}} = -M_{F_{n+1,n}} = -wL^2\frac{\frac{2}{\alpha}\tan\frac{\alpha}{2}-1}{2\alpha\tan\frac{\alpha}{2}} \quad (3.15)$$

Formulas 3.13, 3.14, and 3.15 are somewhat awkward to use and for this reason they have been plotted in Professor Maugh's text (13) so that all three quantities can be read directly from curves to an accuracy of about 2-3 percent. Where greater accuracy is required and the column loads are relatively low, i.e., $\alpha \leq 1$, as compared with the Euler buckling load, i.e., $\alpha = \pi$, it has been the

writer's experience that the formulas are more convenient to use in open form. The open forms are obtained by replacing the trigonometric functions by their infinite series and carrying out the indicated algebra. The results of these operations are:

$$C_1 = 4\left(1 - \frac{\alpha^2}{30} - \frac{11\alpha^4}{25,200} - \frac{7\alpha^6}{756,000} - \cdots\right) \qquad (3.13.a)$$

$$C_2 = 2\left(1 + \frac{\alpha^2}{60} + \frac{13\alpha^4}{25,200} + \frac{11\alpha^6}{756,000} + \cdots\right) \qquad (3.14.a)$$

$$M_{Fn,n+1} = -M_{Fn+1,n} = -\frac{w_o L^2}{12}\left[1 + \frac{\alpha^2}{60} + \frac{\alpha^4}{2520} + \frac{\alpha^6}{100,800} + \cdots\right] \qquad (3.15.a)$$

Note that the forms of these series are such that they serve to modify the usual beam values to account for the axial load acting on the beam. They appear to converge quite rapidly for the values of α usually encountered in the design of members that are to be subjected to transverse loads.

General Loading

Although the slope deflection equations 3.11 and 3.12 were developed for a uniform load distribution ($\phi = 1$), inspection of the development of these equations shows that a more general value of ϕ will change only the "fixed end" moments, $M_{Fn,n+1}$ and $M_{Fn+1,n}$. Since the governing differential equation, 3.1.a, is linear and variations in loading affect only the particular solution of this linear equation, superposition is permitted. Therefore, the "fixed end" moments for any form of load distribution may be easily computed by summing results obtained from using the formulas for these moments when the beam is loaded with an arbitrarily placed

concentrated load, as shown in Fig. 3.3.

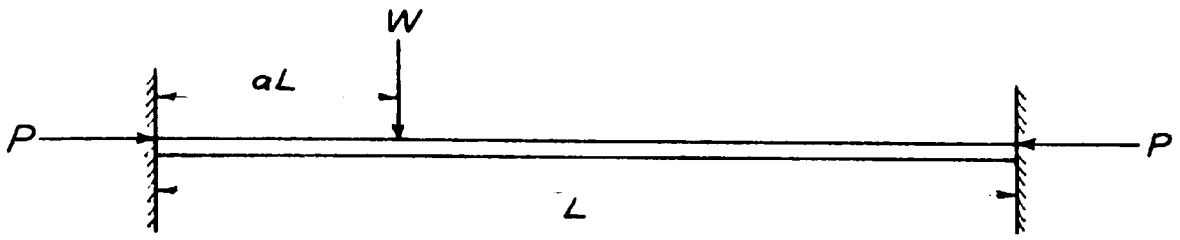


Fig. 3.3
Fixed End Beam-Column with Arbitrarily Placed Concentrated Load

The end moments for the beam shown in Fig. 3.3 are as follows:

$$M_{Fn,n+1} = \frac{WL}{\alpha^2\left(\frac{2}{\alpha}\tan\frac{\alpha}{2}-1\right)}\left[\left(\tan\frac{\alpha}{2}\right)\left(\alpha a+\sin\alpha a\right)-\alpha\sin\alpha a-\left(1-\alpha\cot\alpha\right)\left(1-\cos\alpha a\right)\right] \tag{3.16}$$

$$M_{Fn+1,n} = \frac{WL}{\alpha^2\left(\frac{2}{\alpha}\tan\frac{\alpha}{2}-1\right)}\left[\left(\alpha\csc\alpha-1\right)\left(1-\cos\alpha a\right)-\tan\frac{\alpha}{2}\left(\alpha a-\sin\alpha a\right)\right] \tag{3.17}$$

The following example will illustrate the use of formulas 3.16 and 3.17 for finding fixed end moments for general loadings.

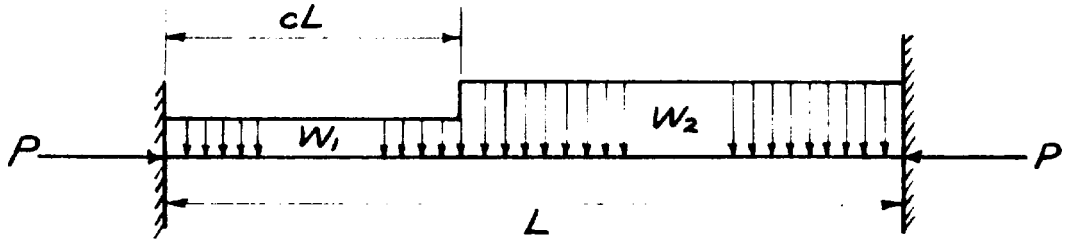


Fig. 3.4
Fixed End Beam-Column with Mid-Span Load Change

Span three of the Oklahoma City tower is represented above where: $w_1 = .13$ k/ft; $w_2 = .194$ k/ft; $\alpha^2 = 2.001$, $\alpha = 1.4145$, $L = 390'$, and $c = .3077$. So, $\tan\frac{\alpha}{2} = .85479$, $\frac{2}{\alpha}\tan\frac{\alpha}{2} - 1 = .208613$; $\alpha\csc\alpha - 1 = .43197$; $1 - \alpha\cot\alpha = .77709$. To obtain the fixed end moments for span 2-3, a small concentrated load $(wL)da$ is substituted for W in the above formulas and the summation of these loads is taken over the entire span as follows:

$$M_{F23} = -\frac{(390)^2}{(2.001)(.208613)}\left[.13\int_0^{.3077} F_{(a)}\,da + .194\int_{.3077}^{1} F_{(a)}\,da\right].$$

where $F_{(a)} = \left[.85479(1.4145a + \sin 1.4145a) - 1.4145\sin 1.4145a - .77709(1-\cos 1.4145a)\right]$

and $\int F_{(a)}\,da = \left[.60455a^2 - .77709a + .395687\cos 1.4145a + .549374\sin 1.4145a\right]$

Substitution of the indicated limits gives $M_{F23} = -2251$k', which is about 4 percent higher than the figure obtained by ignoring the axial load.

In a similar manner:

$$M_{F32} = +\frac{(390)^2}{(2.001)(.20861)}\left[.13\int_0^{.3077} G_{(a)}\,da + .194\int_{.3077}^{1} G_{(a)}\,da\right].$$

where $G_{(a)} = \left[.43197(1-\cos 1.4145a) - .85479(1.4145a - \sin 1.4145a)\right]$

and $\int G_{(a)}\,da = \left[-.60455a^2 + .43197a - .60431\cos 1.4145a - .30539\sin 1.4145a\right]$

Substitution of the limits gives: $M_{32} = +2463$ k/ft. This is about 3-1/2 percent higher than the figure obtained by ignoring the axial load.

The needed properties of the elastic curve between supports can be calculated for a general loading by summing results given by the equations for the beam shown in Fig. 3.5

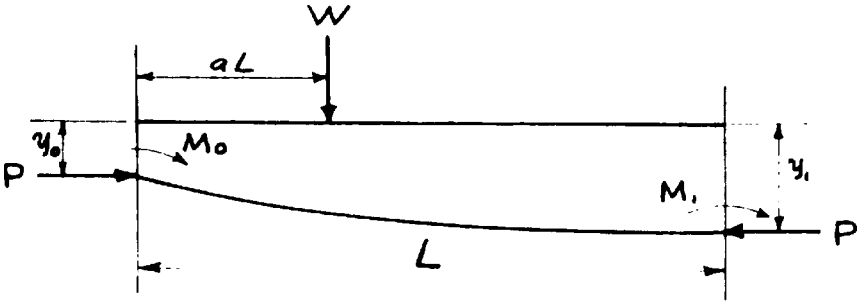


Fig. 3.5
Span with Arbitrarily Located Concentrated Load and General End Conditions

As in equation 3.4, let:

$$\bar{y} = \frac{y}{L}\ ,\quad \bar{x} = \frac{x}{L}\ ;\quad \alpha^2 = \frac{PL^2}{EI}\ ;\quad \beta_1 = \frac{WL^2}{EI}\ ;\quad m = \frac{M}{PL}\ .$$

The elastic curve properties of the span shown in Fig. 3.5 are as follows:

$$0 \le \bar{x} \le a \qquad \bar{y} = \frac{y}{L} = b_1' \sin\alpha\bar{x} + b_2' \cos\alpha\bar{x} + b_3'\bar{x} + b_4' \qquad (3.18)$$

$$a \le \bar{x} \le 1 \qquad \bar{y} = \frac{y}{L} = \bar{b}_1 \sin\alpha\bar{x} + \bar{b}_2 \cos\alpha\bar{x} + \bar{b}_3\bar{x} + \bar{b}_4 \qquad (3.19)$$

$$0 \le \bar{x} \le a \qquad \frac{d\bar{y}}{d\bar{x}} = \theta = b_1'\alpha\cos\alpha\bar{x} - b_2'\alpha\sin\alpha\bar{x} + b_3' \qquad (3.20)$$

$$a \le \bar{x} \le 1 \qquad \frac{d\bar{y}}{d\bar{x}} = \theta = \bar{b}_1\alpha\cos\alpha\bar{x} - \bar{b}_2\alpha\sin\alpha\bar{x} + \bar{b}_3 \qquad (3.21)$$

$$0 \le \bar{x} \le a \qquad -\frac{1}{\alpha^2}\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{M}{PL} = b_1'\sin\alpha\bar{x} + b_2'\cos\alpha\bar{x} \qquad (3.22)$$

$$a \le \bar{x} \le 1 \qquad -\frac{1}{\alpha^2}\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{M}{PL} = \bar{b}_1\sin\alpha\bar{x} + \bar{b}_2\cos\alpha\bar{x} \qquad (3.23)$$

Where $b_1' = \frac{\beta_1}{\alpha_3}\cos\alpha a - \frac{\beta_1}{\alpha^3}\cot\alpha \sin\alpha a - m_0\cot\alpha - m_1\csc\alpha$;

$b_2' = m_0$; $b_3' = \bar{y}_1 - \bar{y}_0 + m_1 + m_0 - (1-a)\frac{\beta_1}{\alpha_2}$; $b_4' = \bar{y}_0 - m_0$

and $\bar{b}_1 = -\frac{\beta_1}{\alpha^3}\cot\alpha \sin\alpha a - m_0\cot\alpha - m_1\csc\alpha$; $\bar{b}_2 = \frac{\beta_1}{\alpha^3}\sin\alpha a + m_0$

$\bar{b}_3 = \bar{y}_1 - \bar{y}_0 + m_1 + m_0 + \frac{\beta_1}{\alpha_2}a$; $\bar{b}_4 = \bar{y}_0 - m_0 - \frac{\beta_1}{\alpha_2}a$.

Equilibrium Equations

Equations 3.11 and 3.12 (repeated below) express the joint moments in terms of the unknown slopes and deflections at the joints. In some cases, i.e., a hinged end or a cantilevered end, the moment at the joint may be known. In such cases, it is possible to eliminate the slope at that joint as an unknown, thus reducing the numerical work necessary for the analysis of the structure.

(3.11) $M_{n,n+1} = \frac{EI}{L}\left[C_1\theta_n + C_2\theta_{n+1} + \frac{1}{L}(C_1 + C_2)(y_n - y_{n+1})\right] + M_{Fn,n+1}$

(3.12) $M_{n+1,n} = \frac{EI}{L}\left[C_2\theta_n + C_1\theta_{n+1} + \frac{1}{L}(C_1 + C_2)(y_n - y_{n+1})\right] + M_{Fn+1,n}$

If $M_{n,n+1} = M_0$ (a known value), then elimination of θ_n between 3.11 and 3.12 yields:

$$M_{n+1,n} = \frac{EI}{L}\left[C_1 - \frac{C_2^2}{C_1}\right]\left[\theta_{n+1} + \frac{1}{L}(y_n - y_{n+1})\right] + M_{Fn+1,n} + \frac{C_2}{C_1}(M_0 - M_{Fn,n+1}) \qquad (3.24)$$

The equations needed to solve for the unknown slopes and deflections, y_n and θ_n, are obtained by inserting the above expressions in the equilibrium equations. The equilibrium equations are those equations which express the facts that the summation of moments and the summation of transverse forces at each joint are zero.

The first class of equilibrium equations is obtained by summing the moments which act on joint "n" as follows:

$$M_{n,n-1} + M_{n,n+1} + G_n(y_n) = 0 \qquad (3.25)$$

where $G_n(y_n)$ is the magnitude of the external moment at joint "n" caused by the eccentricity of the cable forces. This moment is counterclockwise when y_n is to the right

The second class of equilibrium equations is obtained by summing the forces which act on joint "n" as follows:

$$\frac{1}{L_i}\left[P_i(y_{n-1} - y_n) - (M_{n-1,n} + M_{n,n-1})\right] + \frac{1}{L_{i+1}}\left[P_{i+1}(y_{n+1} - y_n) + (M_{n,n+1} + M_{n+1,n})\right] + f_n(y_n) = \frac{1}{2}wL)_i + \frac{1}{2}wL)_{i+1} \qquad (3.26)$$

where $f_n(y_n)$ is the magnitude of the external horizontal reactive force developed by the guys attached to joint "n". The direction of this force is opposed to y_n. If the load distribution in the span is other than uniform, the right-hand side of equation 3.26 would have to be changed.

EI Not Constant Between Joints

Except near the tower bearing, guyed towers are usually constructed so that the distance between the main tower members remains constant and a change in EI is brought about when the size of these main members is changed. If the change occurs between joints and it is a large enough change to appreciably affect the joint moments, the point of change should be treated as a joint. The regular slope

deflection equations would then be written including this point as a joint, but it should be remembered that $f_n(y_n)$ and $G_n(y_n)$ will be zero for any joint not supported by guys.

CHAPTER IV

THE STATIC ANALYSIS OF UNGUYED TOWERS AND THAT PORTION OF GUYED TOWERS ABOVE THE TOP GUY CONNECTIONS

In the analysis of any vertical member whose weight is large as compared with the transverse loads acting on it, the effects of the distributed axial loads due to the member's weight should be given consideration. It is the purpose of the work in this chapter to develop formulae which will include the effects of the distributed axial loads on the deflection, slopes, moments, etc., of any member whose EI and weight distribution is constant or changes abruptly at only a few points. It is intended that the formulae should be general enough to handle any type of transverse loading or any form of support system. Some examples of the forms of support that may be encountered are: a fixed support, a rotational spring support, a simple support, and a series of deflectional spring supports.

A set of formulae which satisfies the above requirements could be used to review the whole tower. However, when a tower segment is subjected to an external axial load that is large as compared with the weight of that segment, accurate and satisfactory results can be obtained more easily by adding one-half the weight to the external axial load and treating the span as a beam-column. This idea was introduced in Chapter 3, and will be justified later in this chapter.

Since the tower segments below the level of the top guy connections are usually subjected to heavy axial loads by the guys, it

is very seldom necessary to analyze them as members with distributed axial loads. Therefore, the main use for the formula presented in Chapter 4 will be for the review of that portion of guyed towers which projects above the top guy connections. Of course, they could also be used for the review of unguyed towers whose variations in EI and weight could be made to fulfill the conditions stipulated in the first paragraph of this chapter.

Throughout this study, use will be made of equations of the slope deflection form. Thus, it is desirable to be able to express the moment at the lower end of the tower cantilever as a function of the slope at that point. To bring this about easily, the coordinates are selected in a different manner from that in the spans discussed in the previous chapter. Note that in Fig. 4.1 the total deflection at a point is $y + \theta_o x$.

Development of Elastic Curve Equations for the Uniformly Loaded Vertical Cantilever

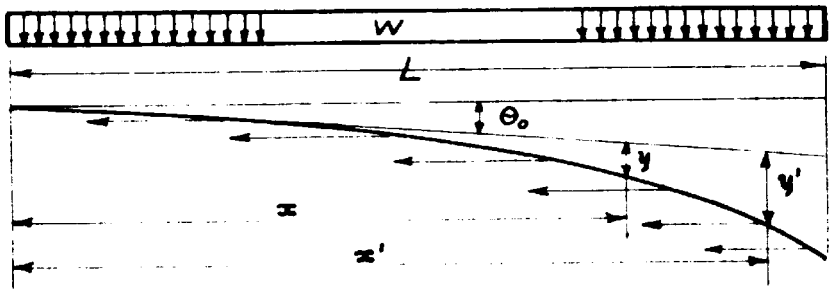


Fig. 4.1
Vertical Cantilever

Making use of the linear moment-curvature relation,

$\frac{d^2y}{dx^2} = -\frac{M(x)}{EI}$ it is seen that:

$$EI\frac{d^2y}{dx^2} = \frac{w}{2}(L-x)^2 + \int_x^L q(y' + \theta_o x' - y - \theta_o x)\,dx' \qquad (4.1)$$

Differentiating equation 4.1 with respect to x and assuming EI constant,

$$EI\frac{d^3y}{dx^3} = -w(L-x) - q\int_x^L \left(\frac{dy}{dx} - \theta_o\right) dx'$$

or

$$EI\frac{d^3y}{dx^3} + q(L-x)\frac{dy}{dx} = -(L-x)(w + q\theta_o) \qquad (4.2)$$

Here, as in Chapter 3, the basic equation will be transformed to dimensionless terms. The identities used in this transformation are, $\bar{x} = \frac{x}{L}$; $\bar{y} = \frac{y}{L}$; $\beta = \frac{qL^3}{EI}$; and $\bar{w} = \frac{w}{q}$, and the transformed equation is:

$$\frac{d^3\bar{y}}{d\bar{x}^3} + \beta(1-\bar{x})\frac{d\bar{y}}{d\bar{x}} = -\beta(\bar{w} + \theta_o)(1-\bar{x}) \qquad (4.2.a)$$

The solution of 4.2.a is available, but the presentation is improved if the independent variable, $\bar{x}$, is transformed in the following manner: $u = \int \beta^{\frac{1}{2}}(1-\bar{x})^{\frac{1}{2}}\,d\bar{x} = \frac{2}{3}\beta^{\frac{1}{2}}(1-\bar{x})^{\frac{3}{2}}$ (This is the transformation normally used to make the coefficient of the dependent variable a constant in the corresponding second order equation.) The transformed equation is:

$$\frac{d^2\bar{y}'}{du^2} + u^{-1}\frac{d\bar{y}'}{du} + \left(1 - \frac{1}{9u^2}\right)\bar{y}' = \left(\frac{3}{2}\beta u\right)^{-\frac{1}{3}}(\bar{w} + \theta_o) \qquad (4.2.b.)$$

where $\bar{y}' \equiv \frac{d\bar{y}}{du}$.

The left-hand side of 4.2.b is in a standard form, and the complete solution (including a particular solution to the inhomogeneous equation) is: $\bar{y}' = d_1 J_{\frac{1}{3}}(u) + d_2 J_{-\frac{1}{3}}(u) + (\bar{w} + \theta_o)\left(\frac{3}{2}\beta u\right)^{-\frac{1}{3}}$ (4.3)

where $J_{\frac{1}{3}}$ and $J_{-\frac{1}{3}}$ are the well known and extensively tabulated Bessel functions of the first kind of order, 1/3 and -1/3 respectively. These functions are accurately and conveniently tabulated in reference (1).

In order to determine the integration constants d_1 and d_2, the solution of equation 4.3 must be fitted to the boundary conditions, which are: $\left.\frac{d^2\bar{y}}{d\bar{x}^2}\right|_{\bar{x}=1} = 0 \; ; \; \left.\frac{d\bar{y}}{d\bar{x}}\right|_{\bar{x}=0} = 0$.

This is carried out below:

$$\left.\frac{d^2\bar{y}}{d\bar{x}^2}\right|_{\bar{x}=1} = \left(\frac{3}{2}\beta\right)^{\frac{2}{3}} u^{-\frac{1}{3}}\left(u\frac{d\bar{y}}{du} + \frac{1}{3}\bar{y}'\right)\Big|_{u=0} = 0$$

$$\lim_{u\to 0} u^{-\frac{1}{3}}\left(u\frac{d\bar{y}'}{du} + \frac{1}{3}\bar{y}'\right) = \lim_{u\to 0} u^{\frac{2}{3}}\left[d_1 J_{-\frac{1}{3}}(u) - d_2 J_{\frac{2}{3}}(u)\right] = .592\, d_1$$

Therefore $d_1 = 0$.

$$\left.\frac{d\bar{y}}{d\bar{x}}\right|_{\bar{x}=0} = -\left(\frac{3}{2}\beta u\right)^{\frac{1}{3}}\bar{y}'\Big|_{u=\frac{2}{3}\beta^{\frac{1}{2}}} = 0$$

$$d_2 J_{-\frac{1}{3}}\left(\frac{2}{3}\beta^{\frac{1}{2}}\right) + (\bar{w}+\theta_0)\beta^{-\frac{1}{2}} = 0$$

so $$d_2 = \frac{-\beta^{-\frac{1}{2}}(\bar{w}+\theta_0)}{J_{-\frac{1}{3}}\left(\frac{2}{3}\beta^{\frac{1}{2}}\right)}$$

$$\bar{y}' \equiv \frac{d\bar{y}}{du} = -\left(\frac{3}{2}\beta u\right)^{-\frac{1}{3}}\frac{d\bar{y}}{d\bar{x}} = \beta^{-\frac{1}{2}}(\bar{w}+\theta_0)\left[\left(\frac{3}{2}\beta^{-\frac{1}{2}}u\right)^{-\frac{1}{3}} - \frac{J_{-\frac{1}{3}}(u)}{J_{-\frac{1}{3}}\left(\frac{2}{3}\beta^{\frac{1}{2}}\right)}\right] \quad (4.3.a)$$

It is interesting to note here that the solution of the corresponding stability problem can be obtained from equation 4.3.a as

a special case. The corresponding stability problem is that of a member fixed vertically at the base, $\theta_o = 0$, with no transverse load, $\bar{w} = 0$. In this case, the slope would be zero at all points in the member unless $J_{-\frac{1}{3}}\left(\frac{2}{3}\beta^{\frac{1}{2}}\right)$ equals zero. Thus, the roots of $J_{-\frac{1}{3}}$ will furnish information for the various modes of buckling. The first three zeros of $J_{-\frac{1}{3}}$ are $r_1 = 1.866$, $r_2 = 4.988$, and $r_3 = 8.124$. These roots correspond to values of β as follows: $\beta_1 = 7.834$, $\beta_2 = 55.98$, $\beta_3 = 148.5$, or the corresponding beam lengths for the first three buckling modes are: $L_1 = 1.986\left(\frac{EI}{q}\right)^{\frac{1}{3}}$, $L_2 = 3.825\left(\frac{EI}{q}\right)^{\frac{1}{3}}$, and $L_3 = 5.295\left(\frac{EI}{q}\right)^{\frac{1}{3}}$. Of course, any member subjected to even a small transverse load will have to have a β value considerably less than β_1 above.

To return to the problem of presenting formulae that may be used for the analysis of a flexible vertical cantilever, it can be shown that the moment at any point on the member will be as follows:

$$M = -\frac{EI}{L}\frac{d^2\bar{y}}{d\bar{x}^2} = -\frac{EI}{L}\,\frac{\beta^{\frac{1}{6}}(\bar{w}+\theta_o)}{J_{-\frac{1}{3}}\left(\frac{2}{3}\beta^{\frac{1}{2}}\right)}\left(\frac{3}{2}u\right)^{\frac{2}{3}}J_{\frac{2}{3}}(u) \qquad (4.4)$$

The deflection, obtained by integrating equation 4.3.a, can be expressed in the following manner:

$$\bar{y} = \frac{y}{L} = \beta^{-\frac{1}{2}}(\bar{w}+\theta_o)\left[\beta^{\frac{1}{6}}\left(\frac{3}{2}u\right)^{\frac{2}{3}} - \frac{1}{J_{-\frac{1}{3}}\left(\frac{2}{3}\beta^{\frac{1}{2}}\right)}\sum_{n=0}^{\infty}\frac{(-1)^n u^{2n+\frac{2}{3}}}{2^{2n-\frac{1}{3}}\left(2n+\frac{2}{3}\right)n!\,\Gamma\left(n+\frac{2}{3}\right)}\right] + d_3 \qquad (4.5)$$

(Γ is the gamma, or the generalized factorial function $\Gamma(n+1) = n!$).

By fitting equation 4.5 to the boundary condition, $\bar{y}\Big|_{\bar{x}=o} = \bar{y}\Big|_{u=\frac{2}{3}\beta^{\frac{1}{2}}} = 0$,

it can be shown that:

$$d_3 = (\bar{w} + \theta_0)\frac{\beta}{8}\left[1 + \frac{9\beta}{70} + \frac{23\beta^2}{1400} + \frac{1679\beta^3}{(143)(5600)} + \cdots\right]$$

$$\bar{y}\Big|_{\bar{x}=1} = \bar{y}\Big|_{u=0} = d_3 = \frac{(w+9\theta_0)L^2}{8EI}\left[1 + .1286\beta + .0164\beta^2 + .0021\beta^3 + \cdots\right] \qquad (4.5.a)$$

Note that equation 4.5.a gives the end deflection in the form of a modification of the statically determinate case. (It should be remembered that the actual deflection from the vertical position is $L\bar{y} + \theta_0 L\bar{x}$).

The form represented in equation 4.5.a is quite easy to work with and, by expanding $\int_{-\frac{1}{3}}$ into its series, it is possible to obtain two more useful quantities in a similar form. One of these quantities is the moment at the base of the vertical cantilever.

$$M\Big|_{\bar{x}=0} = M\Big|_{u=\frac{2}{3}\beta^{\frac{1}{2}}} = -(w+9\theta_0)\frac{L^2}{2}\left(1 + \frac{1}{10}\beta + \frac{1}{80}\beta^2 + \frac{7}{4400}\beta^3 + \cdots\right) \qquad (4.4.a)$$

The second is the slope at the top of the cantilever.

$$\frac{d\bar{y}}{d\bar{x}}\Big|_{\bar{x}=1} = -\left(\frac{3}{2}\beta u\right)^{\frac{1}{3}}\frac{d\bar{y}}{du}\Big|_{u=0} = \frac{(w+9\theta_0)L^3}{6EI}\left(1 + \frac{2\beta}{15} + \frac{37\beta^2}{2160} + \frac{3119\beta^3}{1,425,600} + \cdots\right) \qquad (4.3.b)$$

Thus, for any vertical cantilever whose base slope is known, equations 4.3.a, 4.5.a, and 4.4.a may be used to determine easily the effect that the member's weight has on its upper slope and deflection, and on its base moment. As was mentioned before, the form in which these formulas have been presented is convenient to use, because the desired quantity is calculated in the usual manner and then is modified by a factor depending upon the dimensionless parameter $\beta \equiv \frac{9L^3}{EI}$.

Development of Equations for Elastic Curve of Uniformly Loaded Vertical Cantilever with End Loads

It is proposed in this section to develop formulas similar to those presented in the last section, but applicable to members with more general boundary conditions. The type of member to be considered (see Fig. 4.2 below) is the uniformly loaded vertical cantilever with a general base slope and general loading conditions at its upper end. In fact, the end conditions are general enough so that the elastic curve equations which satisfy them could be used to analyze continuous members of this type. Also, it will be possible, in this section, to demonstrate the validity of the assumptions made in Chapter 3 in regard to adding one-half the weight of a member to the external axial load.

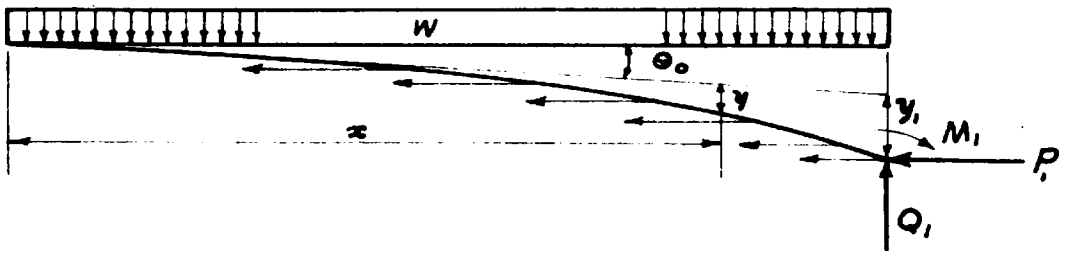


Fig. 4.2
Vertical Cantilever with End Loads

By reasoning similar to that employed in writing equation 4.1, one obtains here:

$$EI\frac{d^2y}{dx^2} = M_1 + \frac{w}{2}(L-x)^2 - Q_1(L-x) + P_1(y_1 - y + \theta_0 L - \theta_0 x)$$

$$+ \int_x^L q(y' + \theta_0 x' - y - \theta_0 x)\,dx' \qquad (4.6)$$

Taking EI as constant and differentiating with respect to x,

$$EI\frac{d^3y}{dx^3}+\left[P_1+q(L-x)\right]\frac{dy}{dx}=Q_1-w(L-x)-\left[P_1+q(L-x)\right]\theta_0 \qquad (4.7)$$

Again the basic equation is transposed to dimensionless terms. The identities to be used in this operation are,

$$\bar{y}=\frac{y}{L}\;;\;\bar{x}=\frac{x}{L}\;;\;\beta=\frac{qL^3}{EI}\;;\;\bar{w}=\frac{w}{q}\;;\;\bar{P}_1=\frac{P}{qL}\;;\;\bar{Q}_1=\frac{Q_1}{qL}\;;\;\bar{m}=\frac{ML}{EI}\;;\;\left(\frac{d^2\bar{y}}{d\bar{x}^2}=-\bar{m}\right).$$

Substituting in equation 4.7,

$$\frac{d^3\bar{y}}{d\bar{x}^3}+\beta(\bar{P}_1+1-\bar{x})\frac{d\bar{y}}{d\bar{x}}=\beta(\bar{Q}_1+\bar{w}\bar{P}_1)-\beta(\bar{w}+\theta_0)(\bar{P}_1+1-\bar{x}) \qquad (4.7a)$$

Before proceeding with the solution to this equation, it will be studied for the purpose of justifying the assumptions made in Chapter 3; namely, for a member subjected to a large axial load the weight of the member may be accounted for by merely adding one-half of the weight to the external axial load. The basic equation for the beam column with a uniform transverse load, as given by equation 3.1.b, is $\bar{y}^{IV}+\alpha^2\bar{y}''=\beta_0$ where $\alpha^2=\frac{PL^2}{EI}=\frac{(P_1+\frac{1}{2}qL)L^2}{EI}$ and $\beta_0=\frac{wL^3}{EI}$.

When the member, for which equation 4.7.a was written, is subjected to a large axial load, $\bar{P}_1>>1$ so that the quantity ($\bar{P}_1+1-\bar{x}$) varies by only a small amount as $\bar{x}$ takes on values from 0 to 1. In this case, it can be taken as a constant equal to its median value of ($\bar{P}_1+\frac{1}{2}$). Now rewriting the right-hand side of equation 4.7.a as $\beta\bar{Q}_1+\beta\bar{w}(\bar{x}-1)-\beta\theta_0(\bar{P}_1+1-\bar{x})$ and differentiating both sides with respect to $\bar{x}$, taking ($\bar{P}_1+1-\bar{x}$) as a

constant equal to ($\bar{P}_1 + \frac{1}{2}$), one obtains: $\bar{y}'''' + \beta(\bar{P}_1 + \frac{1}{2})\bar{y}'' = \beta\bar{w}$. Since $\alpha^2 = \frac{(\bar{P}_1 + \frac{1}{2}qL)L^2}{EI} = \beta(\bar{P}_1 + \frac{1}{2})$ and $\beta_0 = \frac{wL^3}{EI} = \beta\bar{w}$ it is seen that equation 4.7.a may be written identical to equation 3.1.b, the beam-column equation, when the external axial load is large as compared with the weight of the member, i.e., $\bar{P}_1 >> 1$

Returning now to the solution of equation 4.7.a, it is convenient to make a transformation of independent variables similar to that made in the last section, namely,

$$u = \int \beta^{\frac{1}{2}}(\bar{P}_1 + 1 - \bar{x})^{\frac{1}{2}}\,dx = \frac{2}{3}\beta^{\frac{1}{2}}(\bar{P}_1 + 1 - \bar{x})^{\frac{3}{2}}$$

When this transformation is carried out, equation 4.7.a appears in the form:

$$\frac{d^2\bar{y}'}{du^2} + u^{-1}\frac{d\bar{y}'}{du} + \left(1 - \frac{1}{9u^2}\right)\bar{y}' = (\bar{w} + \theta_0)\left(\frac{3}{2}\beta u\right)^{-\frac{1}{3}} - \left(\frac{3}{2}u\right)^{-1}(\bar{Q}_1 + \bar{w}\bar{P}_1) \tag{4.7.b}$$

where $\bar{y}' \equiv \frac{d\bar{y}}{du}$.

The solution of equation 4.7.b is:

$$\bar{y}' = -\left(\frac{3}{2}\beta u\right)^{-\frac{1}{3}}\frac{d\bar{y}}{d\bar{x}} = d_1' J_{\frac{1}{3}}(u) + d_2' J_{-\frac{1}{3}}(u) + (\bar{w} + \theta_0)\left(\frac{3}{2}\beta u\right)^{-\frac{1}{3}} - \frac{2}{3}(\bar{Q}_1 + \bar{w}\bar{P}_1)\, S_{0,\frac{1}{3}}(u) \tag{4.8}$$

where $J_{\frac{1}{3}}$ and $J_{-\frac{1}{3}}$ are Bessel's functions of the first kind as before, and $S_{0,\frac{1}{3}}$ is one of Lommel's functions of one variable, as defined in Watson's treatise (24). Although Lommel's functions are well known and have several recurrence relations similar to those for Bessel's functions, they do not occur nearly as frequently as do Bessel's functions, and the writer was unable to find reference to their having been tabulated.

The recurrence relations and series for the Lommel functions employed in this study are as follows:

Series for Lommel's Function of one Variable

$$S_{u\nu}(z) = z^{u-1}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2}z\right)^{2n+2}\Gamma\left(\frac{1}{2}u-\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}u+\frac{1}{2}\nu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}u-\frac{1}{2}\nu+n+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}u+\frac{1}{2}\nu+n+\frac{3}{2}\right)} \qquad (4.9)$$

$$S_{0,\frac{1}{3}}(z) = \Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right)\left(\tfrac{z}{4}\right)\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{2^{2n}\Gamma\left(n+\frac{4}{3}\right)\Gamma\left(n+\frac{5}{3}\right)} \qquad (4.9.a)$$

$$S_{0,\frac{1}{3}}(z) = \frac{9}{8}z\left[1-\frac{(3z)^2}{(8)(10)}+\frac{(3z)^4}{(8\cdot14)(10\cdot16)}-\frac{(3z)^6}{(8\cdot14\cdot20)(10\cdot16\cdot22)}+\cdots\right] \qquad (4.9.b)$$

$$S_{-1,-\frac{2}{3}}(z) = \frac{1}{4}\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(-\tfrac{1}{3}\right)\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{2^{2n}\Gamma\left(n+\frac{4}{3}\right)\Gamma\left(n+\frac{2}{3}\right)} \qquad (4.9.c)$$

$$S_{-1,-\frac{2}{3}}(z) = -\frac{9}{4}\left[1-\frac{(3z)^2}{(8)(4)}+\frac{(3z)^4}{(8\cdot14)(4\cdot10)}-\frac{(3z)^6}{(8\cdot14\cdot20)(4\cdot10\cdot16)}+\cdots\right] \qquad (4.9.d)$$

Recurrence relations

$$\frac{d}{dz}\left[S_{u\nu}(z)\right]+\frac{\nu}{z}S_{u\nu}(z) = (u+\nu-1)\,S_{u-1,\nu-1}(z) \qquad (4.10)$$

The boundary conditions to be used in determining the integration constants in equation 4.8 are: $\left.\frac{d\bar{y}}{d\bar{x}}\right|_{\bar{x}=0}=0$ **and** $\left.\frac{d^2\bar{y}}{d\bar{x}^2}\right|_{\bar{x}=1}=\bar{m}_1$. **Substituting in these conditions, it is seen that:**

$$\left.\frac{d\bar{y}}{d\bar{x}}\right|_{\bar{x}=0} = -\left(\frac{3}{2}\beta u\right)^{\frac{1}{3}}\bar{y}'\Big|_{u=u_0} = -\left(\frac{3}{2}\beta u_0\right)^{\frac{1}{3}}\left[d_1'J_{\frac{1}{3}}(u_0)+d_2'J_{-\frac{1}{3}}(u_0)+(\bar{w}+\theta_0)\left(\frac{3}{2}\beta u_0\right)^{-\frac{1}{3}}\right.$$
$$\left.-\frac{2}{3}(\bar{Q}_1+\bar{w}\bar{P}_1)\,S_{0,\frac{1}{3}}(u_0)\right] = 0$$

where $u_0 = u\Big|_{\bar{x}=0} = \frac{2}{3}\beta^{\frac{1}{2}}(\bar{P}_1+1)^{\frac{3}{2}}$ **and**

$$\left.\frac{d^2\bar{y}}{d\bar{x}^2}\right|_{\bar{x}=1} = \left(\frac{3}{2}\beta\right)^{\frac{2}{3}} u^{-\frac{1}{3}}\left(u\frac{d\bar{y}'}{du} + \frac{1}{3}\bar{y}'\right)\Big|_{u=u_1} =$$

$$\left(\frac{3}{2}\beta u_1\right)^{\frac{2}{3}}\left[d_1' \int_{-\frac{1}{3}}(u_1) - d_2' \int_{\frac{2}{3}}(u_1) + \frac{4}{9}(\bar{Q}_1 + \bar{w}\bar{P}_1) S_{-1,-\frac{2}{3}}(u_1)\right] = \bar{m}_1$$

where $$u_1 = u\Big|_{\bar{x}=1} = \frac{2}{3}\beta^{\frac{1}{2}}\bar{P}_1^{\frac{3}{2}}.$$

These two equations are solved for d_1' and d_2' giving:

$$d_1' = \frac{d_{22}\,d_{13} - d_{12}\,d_{23}}{d_{11}\,d_{22} - d_{12}\,d_{21}} \quad ; \quad d_2' = \frac{d_{11}\,d_{23} - d_{21}\,d_{13}}{d_{11}\,d_{22} - d_{12}\,d_{21}}$$

where,

$$d_{11} = \int_{-\frac{2}{3}}(u_1); \; d_{12} = -\int_{\frac{2}{3}}(u_1); \; d_{21} = \int_{\frac{1}{3}}(u_0); \; d_{22} = \int_{-\frac{1}{3}}(u_0);$$

$$d_{13} = \frac{\bar{m}_1}{\beta\bar{P}_1} - \frac{4}{9}(\bar{Q}_1 + \bar{w}\bar{P}_1) S_{-1,-\frac{2}{3}}(u_1); \; d_{23} = -(\bar{w}+\theta_0)\beta^{-\frac{1}{2}}(\bar{P}_1+1)^{-\frac{1}{2}} + \frac{2}{3}(\bar{Q}_1+\bar{w}\bar{P}_1) S_{0,\frac{1}{3}}(u_0).$$

The moment and deflection may be calculated at any point in the span by substituting in the following equations:

$$\bar{m} = -\frac{d^2\bar{y}}{d\bar{x}^2} = -\left(\frac{3}{2}\beta u\right)^{\frac{2}{3}}\left[d_1' \int_{-\frac{2}{3}}(u) - d_2' \int_{\frac{2}{3}}(u) + \frac{4}{9}(\bar{Q}_1 + \bar{w}\bar{P}_1) S_{-1,-\frac{2}{3}}(u)\right] \quad (4.11)$$

$$\bar{y} = d_1' \iint_{\frac{1}{3}}(u) + d_2' \iint_{-\frac{1}{3}}(u) + (\bar{w}+\theta_0)\left(\frac{3}{2}\beta^{-\frac{1}{2}}u\right)^{\frac{2}{3}} - \frac{2}{3}(\bar{Q}_1+\bar{w}\bar{P}_1) \iint S_{0,\frac{1}{3}}(u) + d_3' \quad (4.12)$$

or

$$\bar{y} = d_1' \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+\frac{4}{3}}}{2^{2n+\frac{4}{3}}\left(n+\frac{2}{3}\right) n! \sqrt{(n+\frac{4}{3})}} + d_2' \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+\frac{2}{3}}}{2^{2n+\frac{2}{3}}(n+\frac{1}{3}) n! \sqrt{(n+\frac{2}{3})}}$$

$$+ (\bar{w}+\theta_0)\left(\frac{3}{2}\beta^{-\frac{1}{2}}u\right)^{\frac{2}{3}} - \frac{1}{12}(\bar{Q}_1+\bar{w}\bar{P}_1) \iint_{\frac{1}{3}} \iint_{\frac{2}{3}} u^2 \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{2^{2n}(n+1)\sqrt{(n+\frac{4}{3})}\sqrt{(n+\frac{5}{3})}} + d_3' \quad (4.12.a)$$

or in a convenient working form

$$\bar{y} = d_1' \frac{3\left(\frac{1}{2}u\right)^{\frac{4}{3}}}{2\sqrt{\frac{4}{3}}}\left[1 - \frac{3u^2}{8\,5} + \frac{3^2u^4}{(4)(8\cdot14)(8)} - \frac{3^3u^6}{(4\cdot6)(8\cdot14\cdot20)11} + \frac{3^4u^8}{(4\cdot6\cdot8)(8\cdot14\cdot20\cdot26)14} - \cdots\right]$$

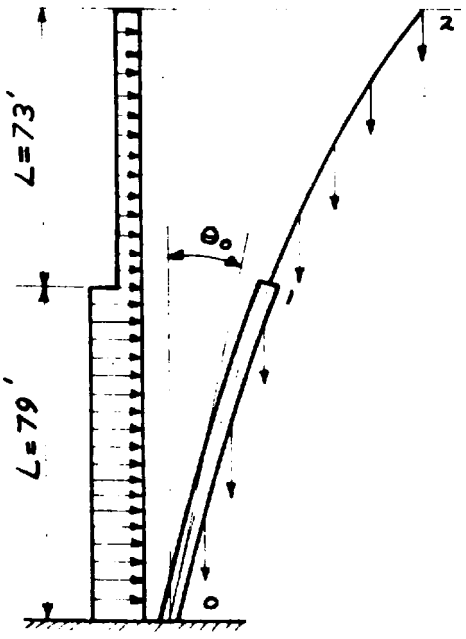
$$+ d_2' \frac{3\left(\frac{1}{2}u\right)^{\frac{2}{3}}}{\sqrt{\frac{2}{3}}}\left[1 - \frac{3u^2}{2\cdot4\cdot4} + \frac{3^2u^4}{(2\cdot4)(4\cdot10)7} - \frac{3^3u^6}{(2\cdot4\cdot6)(4\cdot10\cdot16)10} + \cdots\right] + (\bar{w}+\theta_0)\left(\frac{3}{2}\beta^{-\frac{1}{2}}u\right)^{\frac{2}{3}}$$

$$- \frac{3}{8}(\bar{Q}_1 + \bar{w}\bar{P}_1)u^2\left[1 - \frac{(3u)^2}{2\cdot8\cdot10} + \frac{(3u)^4}{3(8\cdot14)(10\cdot16)} - \frac{(3u)^6}{4(8\cdot14\cdot20)(10\cdot16\cdot22)} + \cdots\right] + d_3' \qquad (4.12.b)$$

In the expressions 4.12, 4.12.a, and 4.12.b, d_3' can be evaluated from the condition $\bar{y}(u_0) = 0$.

Illustrative Example

A typical problem, illustrating the use of the formulas developed so far in this chapter, would be that of expressing the moment at the lower end of the bottom antenna of the Oklahoma City tower as a function of the tower's slope at that point.



$w = .09\ ^k/_{ft}$.
$q = .13\ ^k/_{ft}$. $\beta_{12}\frac{qL^3}{EI} = .891926$
$I = 1.89\ in^2ft^2$

$w = .12\ ^k/_{ft}$.
$q = .41\ ^k/_{ft}$. $\beta_{01} = \frac{(.41)(79)^3}{(3\cdot10^4)(41.4)} = .162756$
$I = 41.4\ in^2ft^2$

Fig. 4.3
Vertical Cantilever with Change in Cross-Section

The procedure for finding M_{01} in terms of the given physical quantities and θ_o, is as follows:

1) Obtain M_{12} interms of load in span 1-2 and θ_1 by use of 4.4.a.

2) Compute constants for elastic curve of span 0-1 in terms of M_{10} and θ_o by use of expressions for d_1' and d_2' shown on page 56.

3) Write expression for θ_1, in terms of constants computed in step 2, by use of equation 4.8.

4) Use results of steps 3 and 1 and fact that $M_{12} = -M_{10}$ to eliminate M_{10} from constants computed in step 2.

5) Find M_{01} in terms of θ_o by use of 4.11.

These steps are numerically illustrated as follows:

1) $M_{12} = -(W_{12} + q_{12}\theta_1)\frac{L^2}{2}\left(1 + \frac{1}{10}\beta + \frac{1}{80}\beta^2 + \frac{7}{4400}\beta^3 + \cdots\right)$

$= -(239.805 + 346.3850_1)(1.09991)$

or $\theta_1 = -2.62472 \cdot 10^{-3} M_{1-2} - .692308$

2) For span 0-1, $\bar{P}_1 = \frac{P_1}{qL} = .292992$; $\bar{Q}_1 = \frac{Q_1}{qL} = -.202840$

$\bar{w} = w/q = .292683$; $u_1 = \frac{2}{3}\beta^{\frac{1}{2}}(\bar{P}_1)^{\frac{3}{2}} = .0426539$; and

$u_o = \frac{2}{3}\beta^{\frac{1}{2}}(\bar{P}_1 + 1)^{\frac{3}{2}} = .395434$

$d_1' = \frac{d_{22}d_{13} - d_{12}d_{23}}{d_{11}d_{22} - d_{12}d_{21}}$; $d_2' = \frac{d_{11}d_{23} - d_{21}d_{13}}{d_{11}d_{22} - d_{12}d_{21}}$;

$d_{11} = \int_{-\frac{2}{3}} (u_1) = 4.84718$; $d_{12} = -\int_{\frac{2}{3}} (u_1) = -.0851668$;

$d_{22} = \int_{-\frac{1}{3}} (u_o) = 1.19419$; $d_{21} = \int_{\frac{1}{3}} (u_o) = .633420$;

$$d_{13} = \frac{M_{10}}{P_1 L} - \frac{4}{9}(\bar{Q} + \bar{W}\bar{P}_1)\, S_{-1,-\frac{2}{3}}(u_1) = 1.33386 \cdot 10^{-3} M_{10} - .117046$$

$$d_{23} = \frac{-(\bar{W}+\theta_0)}{\sqrt{3(\bar{P}_1+1)}} + \frac{2}{3}(\bar{Q} + \bar{W}\bar{P}_1)\, S_{0,\frac{1}{3}}(u_0) = -2.17988\theta_0 - .671952$$

$$d_1' = .272641 \cdot 10^{-3} M_{10} - .0317768\theta_0 - .0508358$$

$$d_2' = -.141641 \cdot 10^{-3} M_{10} - 1.80855\theta_0 - .544798$$

3)

$$\left.\frac{d\bar{y}}{d\bar{x}}\right|_{\bar{x}=1} = \theta_1 - \theta_0 = -\left(\frac{3}{2}\beta u_1\right)^{\frac{1}{3}} \left[d_1' \int_{\frac{1}{3}}(u_1) + d_2' \int_{-\frac{1}{3}}(u_1) - \frac{2}{3}(\bar{Q}_1 + \bar{W}\bar{P}_1) S_{0,\frac{1}{3}}(u_1)\right] - \bar{W} - \theta_0$$

$$\int_{\frac{1}{3}}(u_1) = .310446 \; ; \quad \int_{-\frac{1}{3}}(u_1) = 2.66116$$

$$\text{or } \theta_1 = 63.8274 \cdot 10^{-6} M_{10} + 1.05313\,\theta_0 + .031102$$

4) Combining the expression for θ_1 in "3" with the expression for θ_1 in "1", namely, $\theta_1 = 2.62472 \cdot 10^{-3} M_{10} - .692308$, $M_{10} = 282.483 + 411.2360_0$ and this changes the expressions for d_1' and d_2'

$d_1' = .0261806 + .0803430\theta_0$; $d_2' = -.584809 - 1.866800_0$

5) Substituting in 4.11, and rearranging terms

$$M_{01} = -9L^2(\bar{P}_1+1)\left[d_1' \int_{-\frac{2}{3}}(u_0) - d_2' \int_{\frac{2}{3}}(u_0) + \frac{4}{9}(\bar{Q}_1 + \bar{W}\bar{P}_1) S_{-1,-\frac{2}{3}}(u_0)\right] \qquad (4.11.a)$$

$$\int_{-\frac{2}{3}}(u_0) = .972763 \; ; \quad \int_{\frac{2}{3}}(u_0) = .367197$$

$$\frac{4}{9}(\bar{Q}_1 + \bar{W}\bar{P}_1) S_{-1,-\frac{2}{3}}(u_0) = -.112008$$

$$M_{01} = -1165.31 - 2526.51\,\theta_0$$

From static considerations only $M_{01} = -1133.30 - 2375.50_o$, so that the effect of the distributed axial load increased the first term by 3 percent and the second by 6 percent.

Elastic Curve Equations for Vertical Cantilever with End Loads and General Transverse Span Loading

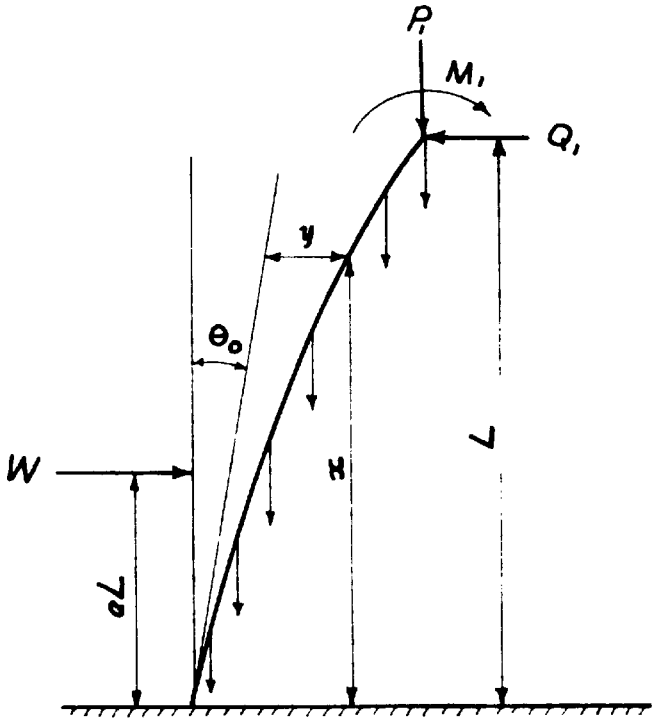


Fig. 4.4
Vertical Cantilever - General Loading

In this section, formulas will be presented which extend the generality of those developed in the section ending on page 57 . In that section, formulas were developed which can be used for the analysis of any vertical member whose transverse loading is uniform. The formulas presented below can be used for the analysis of any vertical member with any form of transverse loading.

As was mentioned in Chapter 3, for members whose governing differential equation is linear, superposition of loads is permitted.

In other words, the solution for the case of an arbitrarily located transverse concentrated load can be used to obtain results for any form of transverse load by the proper summation process. Therefore, since equation 4.7.a (page 53) is linear, its solution, applied to the member shown in Fig. 4.4, can be used for the analysis of a vertical member with any form of transverse loading.

The symbols in the solution presented below are defined as follows:

$$\bar{x} = \frac{x}{L} \; ; \; \bar{y} = \frac{y}{L} \; ; \; \beta = \frac{qL^3}{EI} \; ; \; \bar{Q}_1 = \frac{Q_1}{qL} \; ; \; \bar{W} = \frac{w}{qL} \; ; \; \bar{P}_1 = \frac{P_1}{qL} \; ; \; u = \frac{3}{2}\beta^{\frac{1}{2}}\left(\bar{P}_1 + 1 - \bar{x}\right)^{\frac{2}{3}} \; ; \; \bar{m} = \frac{ML}{EI} \; ;$$

$$u_0 = \frac{3}{2}\beta^{\frac{1}{2}}\left(\bar{P}_1 + 1\right)^{\frac{2}{3}} \; ; \; u_1 = \frac{3}{2}\beta^{\frac{1}{2}}\bar{P}_1^{\frac{2}{3}} \; ; \; u_a = \frac{3}{2}\beta^{\frac{1}{2}}\left(\bar{P}_1 + 1 - a\right)^{\frac{2}{3}} .$$

The solution is as follows:

$$0 \le \bar{x} \le a \qquad \bar{y} = \bar{d}_1 \int J_{\frac{1}{3}}(u) + \bar{d}_2 \int J_{-\frac{1}{3}}(u) + \theta_0\left(\frac{3}{2}\beta^{-\frac{1}{2}}u\right)^{\frac{2}{3}} + \frac{2}{3}\left(\bar{w} - \bar{Q}_1\right) \int S_{0,\frac{1}{3}}(u) + \bar{d}_3 \tag{4.13}$$

$$a \le \bar{x} \le 1 \qquad \bar{y} = \bar{d}_4 \int J_{\frac{1}{3}}(u) + \bar{d}_5 \int J_{-\frac{1}{3}}(u) + \theta_0\left(\frac{3}{2}\beta^{-\frac{1}{2}}u\right)^{\frac{2}{3}} - \frac{2}{3}\bar{Q}_1 \int S_{0,\frac{1}{3}}(u) + \bar{d}_6 \tag{4.13.a}$$

$$0 \le \bar{x} \le a \qquad \frac{d\bar{y}}{du} = -\left(\frac{3}{2}\beta u\right)^{-\frac{1}{3}} \frac{d\bar{y}}{d\bar{x}} = \bar{d}_1 J_{\frac{1}{3}}(u) + \bar{d}_2 J_{-\frac{1}{3}}(u) + \theta_0\left(\frac{3}{2}\beta u\right)^{-\frac{1}{3}} + \frac{2}{3}\left(\bar{w} - \bar{Q}_1\right) S_{0,\frac{1}{3}}(u) \tag{4.14}$$

$$a \le \bar{x} \le 1 \qquad \frac{d\bar{y}}{du} = -\left(\frac{3}{2}\beta u\right)^{-\frac{1}{3}} \frac{d\bar{y}}{d\bar{x}} = \bar{d}_4 J_{\frac{1}{3}}(u) + \bar{d}_5 J_{-\frac{1}{3}}(u) + \theta_0\left(\frac{3}{2}\beta u\right)^{-\frac{1}{3}} - \frac{2}{3}\bar{Q}_1 S_{0,\frac{1}{3}}(u) \tag{4.14.a}$$

$$0 \le \bar{x} \le a \qquad \frac{d^2\bar{y}}{d\bar{x}^2} = -\bar{m} = \left(\frac{3}{2}\beta u\right)^{\frac{2}{3}}\left[\bar{d}_1 J_{-\frac{2}{3}}(u) - \bar{d}_2 J_{\frac{2}{3}}(u) - \frac{4}{9}\left(\bar{w} - \bar{Q}_1\right) S_{-1,-\frac{2}{3}}(u)\right] \tag{4.15}$$

$$a \le \bar{x} \le 1 \qquad \frac{d^2\bar{y}}{d\bar{x}^2} = -\bar{m} = \left(\frac{3}{2}\beta u\right)^{\frac{2}{3}}\left[\bar{d}_4 J_{-\frac{2}{3}}(u) - \bar{d}_5 J_{\frac{2}{3}}(u) + \frac{4}{9}\bar{Q}_1 S_{-1,-\frac{2}{3}}(u)\right] \tag{4.15.a}$$

where $\bar{d}_1$, $\bar{d}_2$, $\bar{d}_3$, $\bar{d}_4$, $\bar{d}_5$, and $\bar{d}_6$ are evaluated as follows:

$\bar{d}_1$, $\bar{d}_2$, $\bar{d}_3$, and $\bar{d}_4$ can be found by solving simultaneously the four equations below. These equations were written by making use of the boundary conditions, $\left.\frac{d^2\bar{y}}{d\bar{x}^2}\right|_{u_1} = \bar{m}_1$; $\left.\frac{d\bar{y}}{d\bar{x}}\right|_{u_0} = 0$ and the

continuity conditions at $\bar{x} = a$ on $\frac{d\bar{y}}{d\bar{x}}$ and $\frac{d^2\bar{y}}{d\bar{x}^2}$.

$$\bar{d}_4 \int_{-\frac{2}{3}}(u_1) - \bar{d}_5 \int_{\frac{2}{3}}(u_1) = \bar{m}_1 \left(\frac{3}{2}\beta u_1\right)^{\frac{2}{3}} - \frac{4}{9}\bar{Q}_1 S_{-1,-\frac{2}{3}}(u_1) \qquad (4.16)$$

$$\bar{d}_1 \int_{\frac{1}{3}}(u_0) + \bar{d}_2 \int_{-\frac{1}{3}}(u_0) = -\theta_0 \left(\frac{3}{2}\beta u_0\right)^{-\frac{1}{3}} - \frac{3}{2}(\bar{W} - \bar{Q}_1) S_{0,\frac{1}{3}}(u_0) \qquad (4.17)$$

$$\bar{d}_1 \int_{\frac{1}{3}}(u_a) + \bar{d}_2 \int_{-\frac{1}{3}}(u_a) - \bar{d}_4 \int_{\frac{1}{3}}(u_a) - \bar{d}_5 \int_{-\frac{1}{3}}(u_a) = \frac{2}{3}\bar{W} S_{0,\frac{1}{3}}(u_a) \qquad (4.18)$$

$$\bar{d}_1 \int_{-\frac{2}{3}}(u_a) - \bar{d}_2 \int_{\frac{2}{3}}(u_a) - \bar{d}_4 \int_{-\frac{2}{3}}(u_a) + \bar{d}_5 \int_{\frac{2}{3}}(u_a) = \frac{4}{9}\bar{W} S_{-1,-\frac{2}{3}}(u_a) \qquad (4.19)$$

$\bar{d}_3$ is evaluated by making use of the boundary condition $\bar{y}\big|_{u_c} = 0$, then $\bar{d}_6$ may be found from the condition of continuity of $\bar{y}$ at u_a.

CHAPTER V

SUMMARY OF STATIC ANALYSIS OF GUYED FLEXIBLE TOWERS

The purpose of the work in this chapter is to further illustrate the use of the formulas developed in Chapters 2, 3, and 4 as applied to the analysis of flexible guyed towers. A method of accounting for shear deformation by modifying the calculations for "I" is also discussed.

The steps listed below form an outline of the procedure that the writer recommends for the static analysis of a guyed tower subjected to a given transverse loading.

1. Obtain an approximate solution by analyzing the tower as an ordinary continuous beam whose supports continue to lie along a straight line during loading. This is conveniently accomplished by use of the "Moment Distribution" method.
2. By use of the cable formulas developed in Chapter 2, determine the tower movement necessary at each set of guy connections to develop the reactions found in step 1. This movement can be used to find the axial load in each section of the tower when it is in static equilibrium.
3. Replace the resistance-motion curve for each set of cables by a straight line tangent to the curve at the reaction, or resistance, found in step 1. This makes it possible to write both the horizontal resistance and the external moment

applied at each set of guy connections in the form $a\,y_n + b$, where y_n is the deflection at the n^{th} guy connection level.

4. Substitute in the slope deflection and equilibrium equations (3.11, 3.12, 3.24, 3.25, and 3.26) and, where needed, the equations developed in Chapter 4 to find the deflection, slopes, and moments at the guy attachment points.

5. Use the equations of Chapters 3 and 4 to find the needed properties of the elastic curve between the points of support.

The above steps will be numerically illustrated later in this chapter by carrying through a complete static analysis for the television tower recently erected for station KWTV at Oklahoma City.

Calculation of "I" to Account for Shear Deformation

In all of the formulas developed in the preceeding chapters, the tower was treated as if it were a bar and the effects of shear deformation were ignored. In practice though, the tower section is usually made up of 3 or 4 rather heavy legs held together by a lattice system of battens and diagonals as shown in Fig. 5.1 below: (Dotted lines indicate slack, or inoperative tension members.)

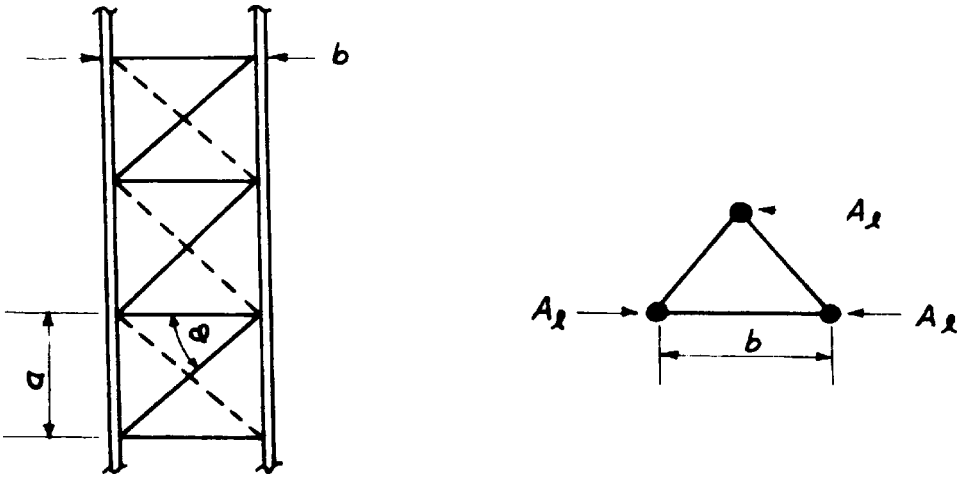


Fig. 5.1
Section and Cross-Section of Triangular Tower

The members of the lattice system deform when the section is subjected to shearing forces, causing the section to be less stiff than a solid section which has the same "I". Therefore, it is desirable to modify the value one obtains by computing the moment of inertia in the usual manner, denoted I_s, to get the quantity "I" used throughout Chapters 3 and 4.

The moment of inertia of the cross section shown in Fig. 5.1, for any orientation of axes, can be computed from: $I_s = \frac{1}{2} A_\ell b^2$ or the radius of gyration from $r_g^2 = \frac{b^2}{6}$.

In order to modify I_s to account for shear deformation, there are actually two effects to take into consideration. One is the shear due to the axial load when the member is in a deflected position, and the other is the shear due to transverse loads. The first effect is dealt with by S. Timoshenko on page 141 of reference (20) and his final formula can be used for a triangular tower section without

modification. To account for shear deformation due to an axial load P, I_s is multiplied by the factor $1 - \frac{P}{P_d}$ where:

$$\frac{1}{P_d} = \frac{1}{A_d E \sin\varphi \cos^2\varphi} + \frac{1}{A_b E \tan\varphi} \qquad (5.1)$$

and $\tan\varphi = \frac{a}{b}$; A_d = area of two diagonals; A_b = area of two battens. (See Fig. 5.1)

The effect of shear due to transverse loading can be accounted for, as indicated in reference (9), by adjusting the I of the equivalent bar so that the strain energy due to bending in the bar is equal to the strain energy in the actual latticed member, i.e., $\int_L \frac{M^2 dx}{I} = \sum_{i=1}^{n} \frac{S_i^2 l_i}{A_i}$ However, for members with a large number of panels between supports, e.g., 20 +, the energy equation can be written: $\int_L \frac{M^2 dx}{I} = \int_L \frac{M^2 dx}{I_s} + \sum_{i=1}^{n_v} \frac{S_i^2 l_i}{A_i}$ where the summation is taken only over the shear or web members, i.e., battens and diagonals. Thus, to account for shear deformation due to transverse loading, I_s is multiplied by the factor $\frac{1}{1+K_v}$ where K_v is a parameter depending upon the distribution of shears and moments along the member.

$$K_v = \frac{I_s \sum_{i=1}^{n_v} \frac{S_i^2 l_i}{A_i}}{\int_L M^2 dx} \qquad (5.2)$$

S = force in member; l = length of member; A = area of member;

n_v - indicates summation taken for web members only.

The distribution of moments and shears found in step one of the outline for static analysis can be used in 5.2 to determine K_v. The numerical work involved in evaluating the term $\sum_{i=1}^{n_v} \frac{s_i^2 \ell_i}{A_i}$ can usually be shortened by calculating the summation as two integrals, one for the battens and one for the diagonals. Such a procedure is permissible whenever the number of panels between supports is large. $\frac{\ell_i}{A_i}$ is treated as a constant equal to the length divided by the area for an average member so that only the variation in shear need be left under the integral sign. To illustrate this, a tower segment, whose members are arranged as shown in Fig. 5.1, will be worked out for a uniform transverse load and fixed ends.

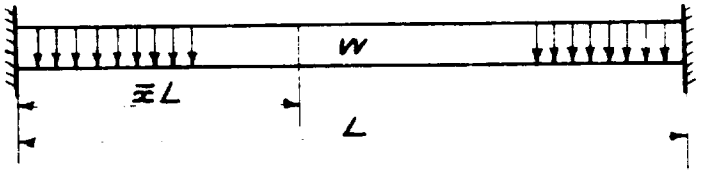


Fig. 5.2
Fixed End Tower Segment Subjected to Uniform Transverse Load

$$M(\bar{x}) = \frac{wL^2}{2}\left(\bar{x} - \bar{x}^2 - \frac{1}{6}\right)$$

$$Q(\bar{x}) = \frac{wL}{2}(1 - 2\bar{x})$$

The force in two battens at any point is equal to $\frac{2}{\sqrt{3}}Q$, and the force in two diagonals at any point is equal to $Q\left(\frac{2}{\sqrt{3}\cos\varphi}\right)$ so that from 5.2:

$$K_v = \frac{I_s\left[\frac{\ell_b}{A_b}\int_0^1 \left(\frac{2}{\sqrt{3}}Q\right)^2 d\bar{x} + \frac{\ell_d}{A_d}\int_0^1 \left(\frac{2Q}{\sqrt{3}\cos\varphi}\right)^2 d\bar{x}\right]}{\int_0^1 M^2 L\, d\bar{x}}$$

$$K_v = \frac{4 I_s}{3 L^3} \frac{\left(\frac{l_b}{A_b} + \frac{l_d}{\cos^2\alpha A_d}\right)\int_0^1 (1-2\bar{x})^2 dx}{\int_0^1 \left(\bar{x}-\bar{x}^2-\frac{1}{6}\right)^2 d\bar{x}} = 80\,\frac{I_s}{L^3}\left(\frac{l_b}{A_b} + \frac{l_d}{\cos^2\alpha A_d}\right)$$

Summary: In order to account for shear deformation in a latticed member when using the formulae presented in this study, "I" should be calculated in the following manner: $I = I_s \frac{1 - \frac{P}{P_d}}{1 + K_v}$ where: I_s is the moment of inertia computed in the regular way as if the member had a solid cross-section, i.e., $\Sigma y^2 dA$; P is the axial load acting on the member; P_d is a quantity corresponding to the shear rigidity of the member given by $\frac{1}{P_d} = \frac{1}{A_d E \sin\alpha \cos^2\alpha} + \frac{1}{A_b E \tan\alpha}$; and $K_v = C \frac{I_s}{L^3}\left(\frac{l_d}{\cos^2\alpha A_d} + \frac{l_b}{A_b}\right)$; $\alpha = \tan^{-1}\frac{a}{b}$ l_d and l_b are the lengths of the individual diagonal and batten members respectively; A_d and A_b are the areas of two diagonals and two batten members, respectively; and C is a constant depending upon the shear and moment distribution on the tower segment under consideration.

While the quantity "C" can be easily calculated for any given distribution of shears and moments, e.g., C = 80 in the example worked out above, the approximation involved in selecting the typical or average diagonal and batten does not seem to warrant such computations. The writer feels that, by careful use of the table below, "C" can be estimated with sufficient accuracy for most cases.

Table 5.1

Values of C for Use in $K_v = C\frac{I_s}{L^3}\left(\frac{l_d}{\cos^2\theta A_d} + \frac{l_b}{A_b}\right)$ for Uniform Span Load

End Conditions	"C"
Fixed-Fixed	80.0
Fixed-Hinged	42.2
Hinged-Hinged	13.3
Fixed-Free	8.9

In order to furnish further data to be used in estimating "C", the writer calculated "C" for each span of the Oklahoma City tower for the shear and moment distribution shown in Fig. 5.2.

End Conditions	"C"
Span 1	26.0
Span 2	49.2
Span 3	77.0
Span 4	29.2
Span 5	.95
Span 6	2.5

The above values are for a tower of triangular cross-section. To obtain "C" for a tower of rectangular cross-section the values above should be multiplied by 3/4.

Initial Cable Tension

The nominal initial cable tension used in the erection of the Oklahoma City tower was 1/8 of the cable breaking strength, or $H_o = \frac{B.S.}{8} \cos \bar{\gamma}$ where $\bar{\gamma}$ is the cable chord angle shown in Fig. 2.2, page 9. The specified factor of safety, for the guy cables supporting this structure, is 3, so that the initial tension is approximately 3/8 of the working stress. This criterion is very widely used in the erection of guy cables in general and it seems to be the most popular method for determining the initial tension to be used in the guys supporting antenna towers. It results in slightly lower values than the sag criterion referred to in Chapter 2.

The initial tension placed in the guy cables is a very important parameter in the design of guyed towers. Too much tension places the tower in danger of buckling as a column when additional vertical loads are added, and too little tension permits excessive tower movement when wind subjects the tower to transverse loads. Along with other undesirable effects, large tower movements result in high dynamic stresses. In view of the impertance of obtaining the optimim value for initial guy tension, the use of 1/8 B.S. as a criterion seems rather arbitrary. There is an apparent need for numerical research aimed at establishing a criterion for the initial tension of guy cables which would take into account the structural properties of the tower system as well as the breaking strength of the cables. The formulas developed and illustrated in this study

could serve as a basis for this type of research.

The initial horizontal components of tension shown in the illustrative example below were computed by changing H_o above, i.e., $H_o = \frac{B.S.}{\beta} \cos \bar{\gamma}$, to account for the fact that the whole system had been subjected to a temperature drop of 40°F and an ice load of 1/2 inch radius. The illustrative examples in Chapter 2 show the calculation of these changes for the cables attached at point 4.

Example of Static Analysis of Oklahoma City TV Tower Subjected to Simultaneous Wind and Ice Loads

The sketches in Fig. 5.3 below show the transverse wind loads on the tower and the moment diagram as computed according to the assumptions made in step 1 of the analysis, page 63. The wind loads were computed according to the specifications used for the design of this tower; namely, 60 psf. above the 780' level and 40 psf. below this level for flat surfaces and 2/3 of these values for round surfaces. The wind was considered to be acting on the iced structure in the direction shown in Fig. 2.4 so as to cause symmetrical guy stresses.

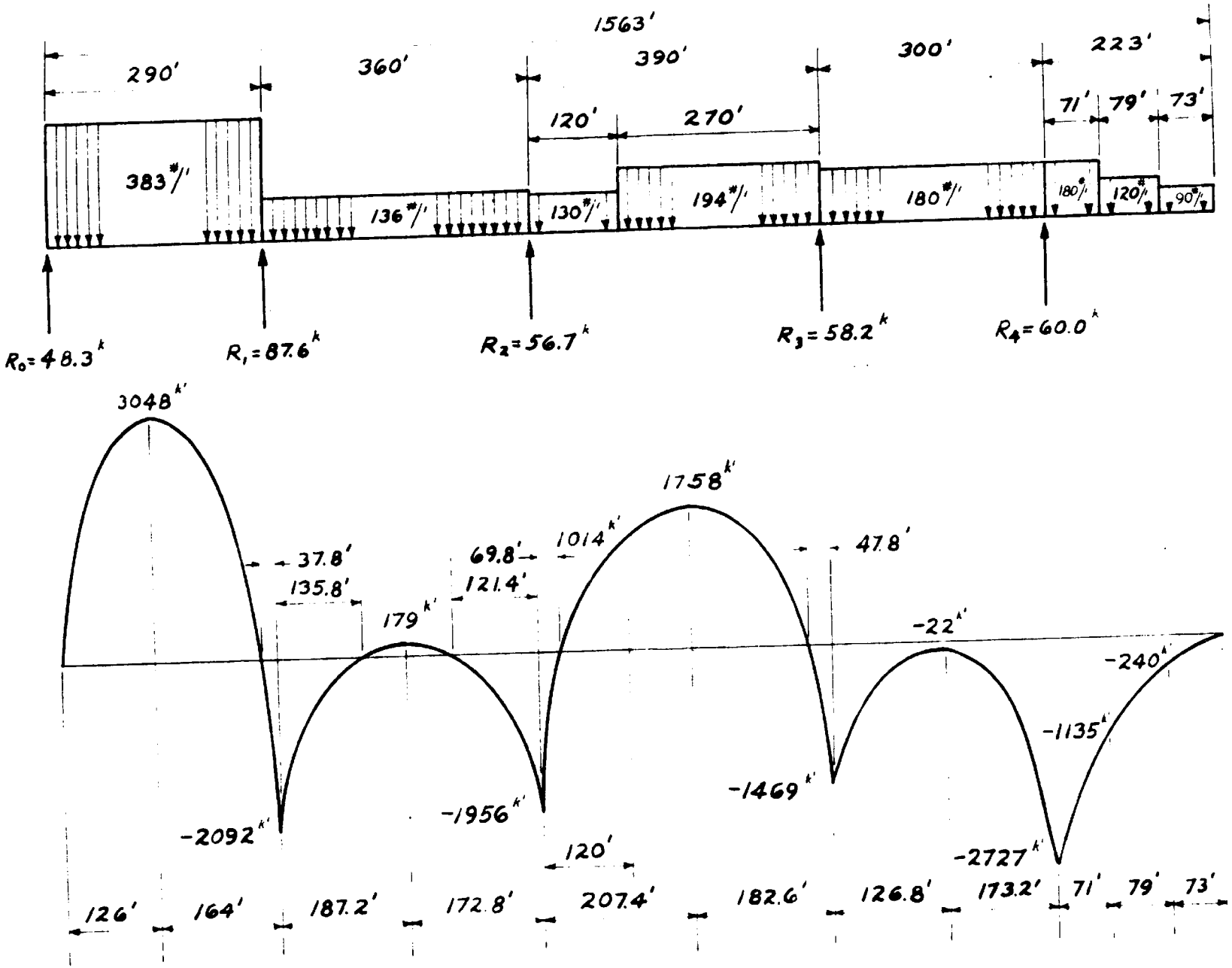


Fig. 5.3
Tower Loads and Approximate Moments

Table 5.2

Physical Constants for Tower

Span	1 (0-1)	2 (1-2)	3 (2-3)	4 (3-4)	5 (4-5)	6 (5-6)
L	290	360	390	300	71	79
A_l	86.3	48.0	43.5	26.4	21.65	15
I_s	6214	3456	3132	1901	1560	41.4
A_d	4.50	2.88	3.02	3.20	4.76	7.40
A_b	7.94	9.30	7.36	6.74	5.64	19.4
C	26.0	49.2	77.0	29.2	.95	2.46
$K_v x10^{-3}$	48.8	38.1	42.2	20.7	31.9	.18
$P_d x10^{-3}$	40.5	28.6	28.8	29.8	38.8	68.5
P_o	1899	1496	1065	530	56.4	25.3
I	5628	3144	2886	1826	1520	39.9
q	1.07	.65	.60	.42	.41	.41
P	2024	1589	1139	574.4		
α^2	1.008	2.183	2.001	.9437		
C_1	3.864	3.700	3.726	3.873		
C_2	2.035	2.078	2.071	2.032		
$M_{F_{n-1,n}}$	-2730	-1525	-2251	-1372		
$M_{F_{n,n-1}}$	+2730	+1525	+2472	+1372		
$\frac{EI}{L} x10^{-3}$	582.2	262.0	222.0	182.6		

The quantities listed in Table 5.2 are defined as follows:

A_1 (in.^{2}) - Average leg area in span. In spans 1 through 5 there are 3 such legs at the vertices of a 12 foot equilateral triangle.

I_s (in.^{2}ft.^{2}) - Moment of inertia calculated in usual manner, e.g., $1/2\ A_1\ (12)^2$ for main tower.

A_d (in.^{2}) - Double the area of average diagonal member.

A_b (in.^{2}) - Double the area of average batten member.

C - Coefficient depending on shear and moment distribution. The calculation of this quantity was discussed earlier in this chapter.

K_v - Coefficient used in correcting I_s to account for shear deformation due to transverse loads.

P_d - Coefficient used in correcting I_s to account for shear due to axial load P_o.

P_o (kip) - Axial load with ice on system, but no transverse loads.

I (in.^{2}ft.^{2}) - The corrected moment of inertia accounting for shear deformation $I = I_s \frac{1 - \frac{P}{P_d}}{1 + K_v}$

q (units k/ft.) - Weight per foot of tower section.

P (units k) - Approximate final axial load in span when tower is in equilibrium with transverse loads. The difference $P - P_o$ is caused by the increase in vertical components of guy tension as the tower deflects.

α^2 - Parameter depending on column load on section, e.g., α^2 for span 4 $= \frac{PL^2}{EI} = \frac{(574.4)(300)^2}{3 \cdot 10^4 (1826)} = .9437$

C_1 and C_2 - Slope deflection coefficients for beam-column given in Chapter 3 as,

$$C_1 = 4\left(1 - \frac{\alpha^2}{30} - \frac{11\alpha^4}{25,200} - \frac{7\alpha^6}{756,000} - \cdots\right).$$

and $$C_2 = 2\left(1 + \frac{\alpha^2}{60} + \frac{13\alpha^4}{25,200} + \frac{11\alpha^6}{756,000} + \cdots\right).$$

e.g., for span 4, $C_1 = 4(1 - .0315 - .0004 - \cdots) = 3.873$

and $C_2 = 2(1 + .0157 + .0005 + \cdots) = 2.032$

$M_{F_{n-1,n}}$ and $M_{F_{n,n-1}}$ (units k-ft.) - Fixed end moments in beam-column given in Chapter 3 as $M_{F_{n-1,n}} = -M_{F_{n,n-1}}$

$$= -\frac{wL^2}{12}\left(1 + \frac{\alpha^2}{60} + \frac{\alpha^4}{2520} + \frac{\alpha^6}{100,800} + \cdots\right)$$ for a uniform load

e.g., for span 4 $M_{F_{34}} = -M_{F_{43}} = \frac{-(.18)(300)^2}{12}(1 + .0157 + .0004 + \cdots)$

$= -1372k'$. The fixed end moments in span 3, where there is a mid-span load change, were calculated as a special problem in Chapter 3.

$\frac{EI}{L}$ (units k-ft.) - Stiffness coefficient needed in slope deflection approach.

Table 5.3

Physical Constants for Guy Cables

Guy	Level 1	2	3	4
l	520	950	950	950
h	299	659	1049	1349
A_c	1.36	1.60	2.13	2.42
B.S.	276	324	432	490
q	6.65	7.52	9.38	10.55
H_o	34.2	40.8	43.4	40.0
r_o	.0503	.0875	.103	.125
r_o^2	.00253	.00767	.0106	.0156
R_e	87.6	56.7	58.2	60.1
Y_e	1.44	2.32	3.08	5.00
f_n	40.8Y+27.9	22.7Y+4.2	20Y-3.6	9.5Y+12.6
G_n	77.5Y+61.2	51.5Y+16.7	72.3-1.0	45Y+68.5

The quantities listed in Table 5.3 are defined as follows:

l and h (units ft.) - Lengths of horizontal and vertical projections of cable, respectively.

A_c (in.^{2}) - Cross sectional area for cable.

B.S. (kips) - Breaking strength of cable. The cables have been prestretched and have a modulus of elasticity, E, equal to $24 \cdot 10^3$ ksi.

q (units lbs/ft) - Weight per unit length of cable.

H_o (kips) - Horizontal component of cable stress after a temperature drop of 40^o and application of 1/2 inch of ice on tower system.

r_o - Cable parameter $= \frac{9\ell}{2H_o}$

R_e (kips) - Approximate guy reaction when tower is in equilibrium with the transverse wind load. This was computed in step 1 of the analysis.

Y_e (units ft.) - Deflection of guys corresponding to R_e.

$f_{n(y_n)}$ (kips) - Approximate function relating resistance to motion at n^{th} guy level. This function is written so as to be tangent to the true resistance-motion curve at R_e.

$G_{n(y_n)}$ (units k-ft.) - Approximate function relating moment applied by guys to the tower motion at n^{th} guy level. This function is written so as to be tangent to the exact function at Y_e.

The illustrative example at the end of Chapter 2 shows the computations involved in finding Y_e, $f_{n(y_n)}$, and $G_{n(y_n)}$ for the 4th guy level.

Before proceeding with the slope deflection solution (by substituting the data contained in tables 5.2 and 5.3 into the slope deflection equations presented in Chapter 3) it is necessary to obtain an expression for M_{45} in terms of θ_4.

An important step toward obtaining this expression was completed in the illustrative example worked out in Chapter 4,

namely, that of obtaining M_{54} as a function of θ_5. Of course, it would be possible to continue in the manner employed in Chapter 4 to obtain M_{45} as a function of θ_4 by treating member 4-5 or 5 as a member with a distributed axial load. However, this will not usually be necessary for a section loaded at its upper end by two additional sections, and one can use the simpler equations presented in Chapter 3. In this specific case though, member 5 is so stiff, i.e., $\beta = .0032$ or $\alpha^2 = .0063$, that the additional deflection due to vertical loads is negligible. In other words, the elementary deflection formulas hold. This is illustrated briefly below.

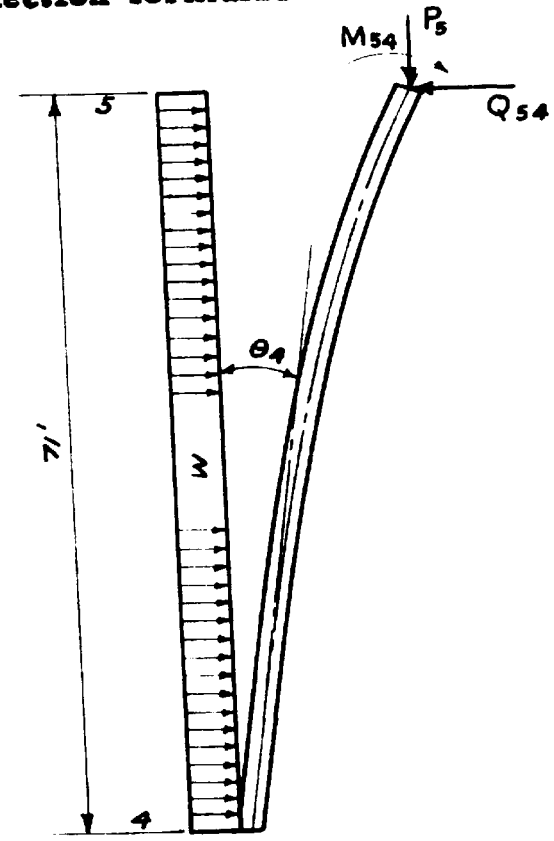


Fig. 5.4
Span 5 in Deflected Position

$$M_{54} = 1165.31 + 2526.510_5$$

(from Chapter 4)

$$Q_5 = -16.1k$$

$$P_5 = 41.88k$$

$$\theta_5 = \theta_4 + \frac{WL^3}{6EI} + \frac{M_{54}L}{EI}$$

$$\theta_5 = 1.00323\ \theta_4 + 1.68277 \cdot 10^{-3}$$

so

$$M_{54} = 1169.56 + 2534.67\ \theta_4$$

$$-M_{45} = M_{54} + \frac{WL^2}{2} + P_5(y_5 - y_4) - Q_5 L$$

$$y_5 - y_4 = L\theta_4 + \frac{WL^4}{8EI} + \frac{M_{54}}{2}\frac{L^2}{EI}$$

$$= .0771848 + 71.1401\ \theta_4$$

so

$$-M_{45} = 2783.21 + 6549.46\ \theta_4$$

Now proceeding with the slope deflection method of solution, it is seen that:

From 3.24, i.e.,

$$M_{n+1,n} = \frac{EI}{L}\left[C_1 - \frac{C_2^2}{C_1}\right]\left[\theta_{n+1} + \frac{1}{L}(y_n - y_{n+1})\right] + MF_{n+1,n} + \frac{C_2}{C_1}(M_0 - MF_{n,n+1})$$

$$M_{10} = 582.2 \cdot 10^3\left[3.864 - \frac{\overline{2.035}^2}{3.864}\right]\left[\theta_1 - \frac{y_1}{290}\right] + 2730 +$$

$$\left(\frac{2.035}{3.864}\right)(2730) = 1626\,\theta_1' - 5606\,y_1 + 4168\,;\ \theta_1' = 1000\,\theta_1$$

And from 3.11 and 3.12, i.e.,

$$M_{n,n+1} = \frac{EI}{L}\left[C_1\theta_n + C_2\theta_{n+1} + \frac{1}{L}(C_1 + C_2)(y_n - y_{n+1})\right] + MF_{n+1,n}.$$

and

$$M_{n+1,n} = \frac{EI}{L}\left[C_1\theta_n + C_1\theta_{n+1} + \frac{1}{L}(C_1 + C_2)(y_n - y_{n+1})\right] + MF_{n+1,n}.$$

$$M_{12} = 262.0 \cdot 10^3\left[3.700\,\theta_1 + 2.078\,\theta_2 + \frac{5.778}{360}(y_1 - y_2)\right] - 1525$$

$$= 969.4\,\theta_1' + 544.5\,\theta_2' + 4205\,y_1 - 4205\,y_2 - 1525 \qquad \theta' = 1000\,\theta$$

$$M_{21} = 262.0 \cdot 10^3\left[2.078\,\theta_1 + 3.700\,\theta_2 + 5.778(y_1 - y_2)\right] + 1525$$

$$= 544.5\,\theta_1' + 969.4\,\theta_2' + 4205\,y_1 - 4205\,y_2 + 1525$$

and in a like manner still letting $\theta' = 1000\theta$

$$M_{23} = 827.3\,\theta_2' + 459.8\,\theta_3' + 3300\,y_2 - 3300\,y_3 - 2251$$

$$M_{32} = 459.8\,\theta_2' + 827.3\,\theta_3' + 3300\,y_2 - 3300\,y_3 + 2472$$

$$M_{34} = 707.3\,\theta_3' + 371.2\,\theta_4' + 3595\,y_3 - 3595\,y_4 - 1372$$

$$M_{43} = 371.2\,\theta_3' + 707.3\,\theta_4' + 3595\,y_3 - 3595\,y_4 + 1372$$

The unknowns θ_n' and y_n above are obtained by solving simultaneously the equations obtained by substituting the above moment expressions into the equilibrium equation, 3.25, $M_{n,n-1} + M_{n,n+1} + G_n(y_n) = 0$ and 3.26 $\frac{1}{L_i}\left[P_i(y_{n-1} - y_n) - (M_{n-1,n} + M_{n,n-1})\right] + \frac{1}{L_{i+1}}\left[P_{i+1}(y_{n+1} - y_n) + (M_{n,n+1} + M_{n+1,n})\right]$

$$+ f_n(y_n) = \frac{1}{2} wL)_i + \frac{1}{2} wL)_{i+1}$$

In solving this particular tower problem, 3.26 is modified slightly in two instances as follows: 1) in dealing with span 2-3 or 3, where the transverse load is not uniform, the term wL/2 should be replaced by the reactions which would exist if span 3 were simply supported; 2) in dealing with the cantilever span $M_{n,n+1}$, $M_{n+1,n}$ and P_{i+1} are taken to be zero and $wL)_{i+1}$ is used instead of $\frac{wL}{2})_{i+1}$

Substitution of joints n = 1, 2, 3, and 4 into 3.25 and 3.26, in that order, gives the following equations for the unknown slopes and deflections. (For ease in reading, the equations are written in tabular form with the coefficients of each variable listed in a column headed by the variable.)

Table 5.4

Equations for Unknown Joint Slopes and Deflections

Equation	θ_1'	θ_2'	θ_3'	θ_4'	y_1	y_2	y_3	y_4	C
I	2595	544.5	0	0	-1323	-4205	0	0 =	-2704
II	544.5	1797	459.8	0	4205	-853.7	-3300	0 =	709.3
III	0	459.8	1535	371.2	0	3300	367.1	-3595=	-1099
IV	0	0	371.2	700.7	0	0	3595	-3550=	1343
V	-140.1	420.5	0	0	7210	-1895	0	0 =	6649
VI	-420.5	-90.51	330.0	0	-1895	5565	-1400	0 =	5105
VII	0	-330.0	29.48	359.5	0	-1400	5635	-2205=	6782
VIII	0	0	-359.5	-359.5	0	0	-2205	3155 =	4323

Note: $\theta' \cdot 10^{-3} = \theta$ in radians, the y's are in feet.

The equations listed in table 5.4 are best handled by an iteration type solution. Two excellent iteration methods for solving linear algebraic equations are the Gauss-Seidel method and the relaxation method. (Both are clearly described in reference 18). It has been the writer's experience that the Gauss-Seidel method is the most convenient method if a desk computer is available, but if all calculations must be carried out by slide-rule, the relaxation method is less susceptible to arithmetical errors. For either method though, a temporary transformation of variables should be carried out so that the coefficients of the unknowns have a "strong" diagonal, i.e., coefficients on the diagonal are larger than any other

coefficients in the equation. For example, this may be carried out for the equations in table 5.4 by making the substitution $Y' = 8Y$ and dividing all coefficients of Y by 8. The Y_e's listed in table 5.3 can be used as first guesses for the unknown Y's in the iteration process to speed convergence.

The equations in table 5.4 are solved simultaneously to give:

$\theta_1' = 3.000$; $\theta_2' = 2.892$; $\theta_3' = 1.764$; $\theta_4' = 8.672$

$Y_1 = 1.446'$; $Y_2 = 2.414'$; $Y_3 = 3.319'$; $Y_4 = 4.879'$

These results are then substituted into the moment equations shown on page 79 to give the final joint moments. For spans 1, 2, and 4 equations 3.6 and 3.7 of page 36 are used to find the moments throughout the span, but for span 3, which has a mid-span load change, it is necessary to integrate equations 3.22 and 3.23 of page 42 to find the span moments. The final results of these operations are shown in the moment diagram in Fig. 5.5. The approximate moment diagram of Fig. 5.3 is superimposed as a dotted line for ease of comparison. The numerical values refer to the results of the thorough analysis, i.e., solid lines.

It is apparent from inspection of Fig. 5.5 that, even for the relatively stiff Oklahoma City tower, considerable error is introduced if the effects of the vertical loads are ignored in computing the tower moments.

———— Thorough Analysis
- - - - - - - - - Approximate Analysis

Note: Numbers refer to thorough analysis.

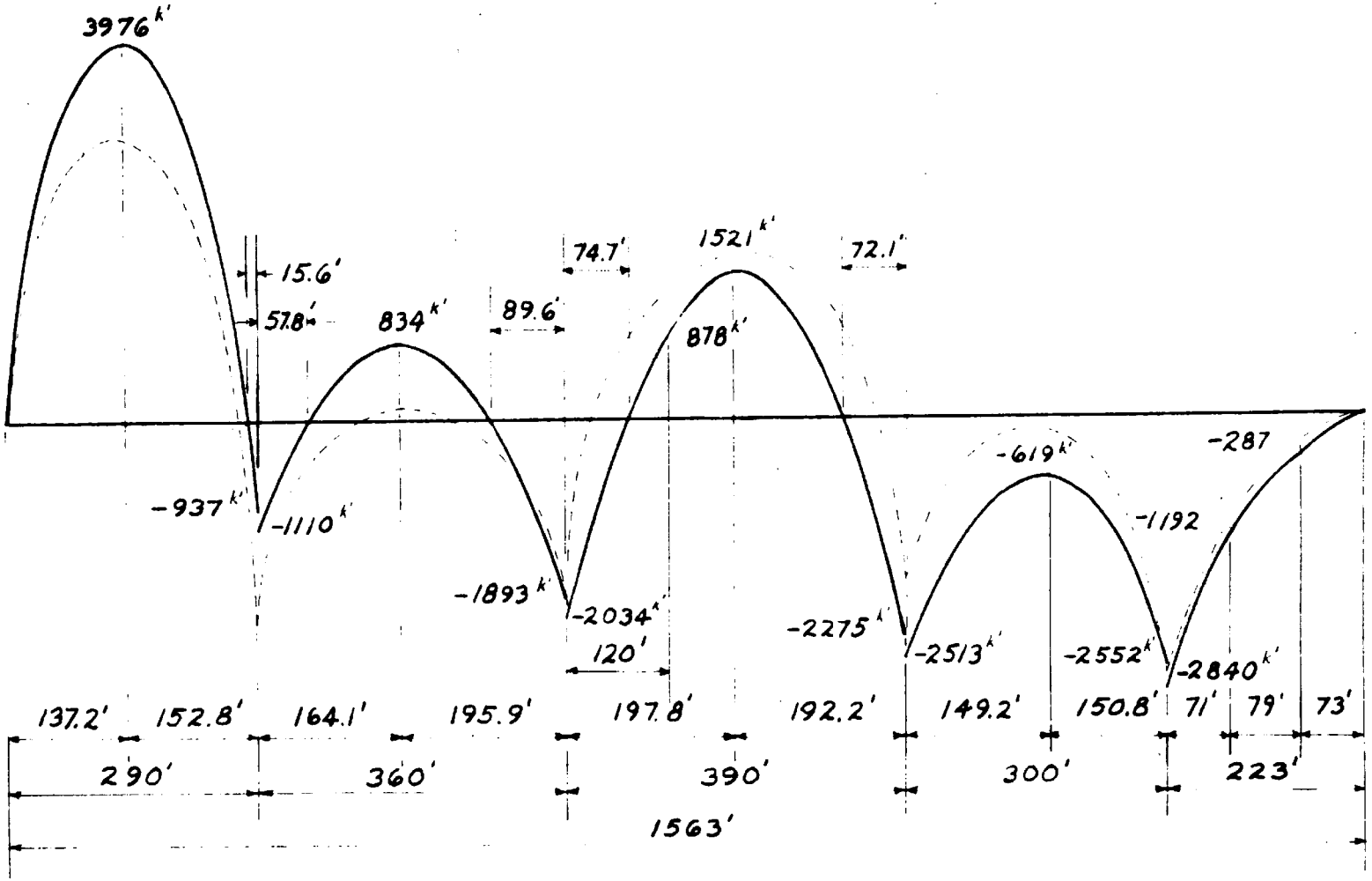


Fig. 5.5
Diagram of Moments Obtained
by Thorough Analysis

CHAPTER VI

GUY CABLES - DYNAMIC ANALYSIS

In Chapter 2, formulae were presented which can be used to compute the total horizontal resistance developed by the guys at a support point for any given tower motion at that point. These formulae, which were illustrated in Chapters 2 and 5, were correct for large tower movements, but they were based on static considerations where it is assumed that the loads and movements take place over a long period of time. Obviously, such an analysis does not take into account the inertia of the mass of the cables and ice on the cables. Since a dynamic solution, at least to the extent of finding natural frequencies for small vibrations, is one of the goals of this study, it will be necessary to develop cable formulae which account for the effects of inertia on the resistance which the guys offer to tower motion. Such formulae, for small cable motion, will be derived in this chapter.

When the support point of a tower is vibrating horizontally, the guys attached at that point are experiencing a longitudinal and transverse forced amplitude vibration equal to the longitudinal and transverse components of the horizontal movement. When the tower movement is not in the plane of the guy, there are two components of transverse cable vibration. Preliminary calculations indicate that the order of magnitude of the frequencies of the lower modes of lateral vibration of tower guys is the same as for the tower

itself. Thus, the lateral resistance the guys offer to tower movement must be calculated as a problem in dynamics where the inertia of the guys is taken into consideration. However, the frequencies of even the lower modes of longitudinal vibration of tower guys are not of the same order of magnitude as those of the tower. Therefore, only the elastic properties of the cable need be considered in computing the longitudinal resistance the guys offer to tower movement.

For instance, it can easily be shown that the ratio of the fundamental frequencies of longitudinal vibration to lateral vibration, for a uniform stretched cable, is $\frac{1}{\sqrt{\epsilon}}$ where ϵ is the cable strain. This ratio is approximately 40 for cables which are functioning as guys, so that the tower movement would be slow enough, as compared with the fundamental frequencies of vibration, for a static analysis to be sufficient for longitudinal motion.

Forced Vibration of Guy Cable

By solving for the equations of motion for a vibrating guy cable, it is proposed to determine the variations in horizontal force a guy exerts on the tower when the support point is executing harmonic motion. For small vibrations, out-of-plane lateral vibrations have the same equations as in-plane vibrations; however, for convenience of presentation, the work to follow will be devoted to in-plane lateral vibrations.

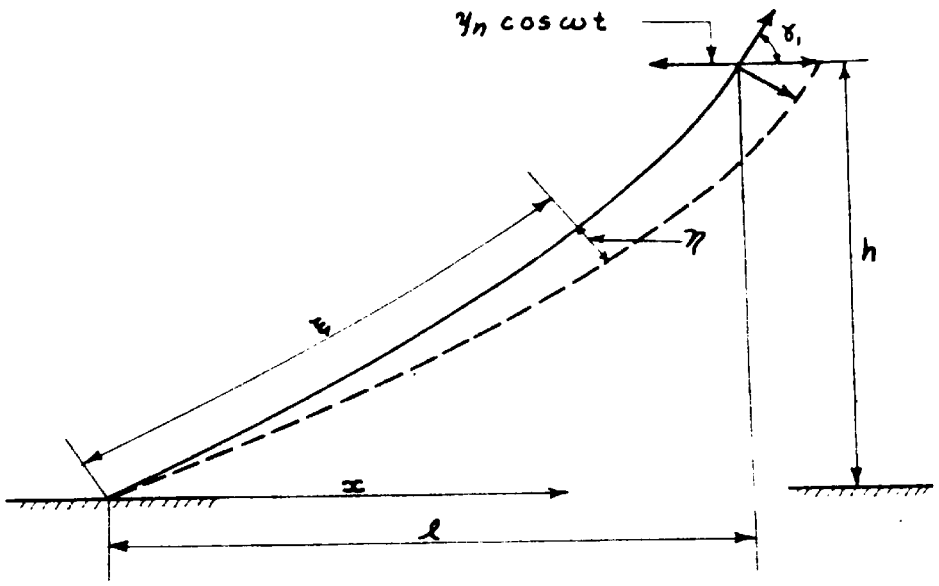


Fig. 6.1
Cable with Vibrating Support

The coordinates ξ and η are chosen along the arc and perpendicular to the arc of the cable in its static equilibrium position.

Summing forces on an element of the cable in a direction perpendicular to its equilibrium position, i.e., in the direction η, it is seen that:

$$-T\frac{\partial \eta}{\partial \xi} + \left(T + \frac{\partial T}{\partial \xi} d\xi\right)\left(\frac{\partial \eta}{\partial \xi} + \frac{\partial^2 \eta}{\partial \xi^2} d\xi\right) = \rho\, d\xi \frac{\partial^2 \eta}{\partial t^2} \qquad (6.1)$$

where ξ and η are the coordinates shown in Fig. 6.1, ρ is the mass per unit length of cable, and T is the tension on the element when the cable is in its equilibrium position. Here only small cable vibrations are dealt with, so the second order terms on the left side of equation 6.1 can be ignored.

Thus,
$$\frac{\partial}{\partial \xi}\left(T\frac{\partial \eta}{\partial \xi}\right) = \rho \frac{\partial^2 \eta}{\partial t^2} \qquad (6.1.a)$$

Remembering from Chapter 2 that $\frac{d\xi}{dx} = \cosh\left(\frac{qx}{H} + a_1\right)$, it is apparent that $\xi = \int_0^x \cosh\left(\frac{qx}{H} + a_1\right) dx = \frac{H}{q}\left[\sinh\left(\frac{qx}{H} + a_1\right) - \sinh a_1\right]$ and according to equation 2.1.a $T = H\cosh\left(\frac{qx}{H} + a_1\right)$ (As in Chapter 2, H = horizontal component of cable tension, q = weight per unit length of cable, $a_1 = \sinh^{-1}\left[\frac{qh}{2H\sinh r}\right] - r$, and $r = \frac{ql}{2H}$).

Substitution of the above quantities for ξ and T into equation 6.1.a, yields the following.

$$\frac{H}{\cosh\left(\frac{qx}{H} + a_1\right)} \frac{\partial^2 \eta}{\partial x^2} = \rho \frac{\partial^2 \eta}{\partial t^2} \qquad (6.1.b)$$

It is assumed that all points on the cable are executing harmonic motion so that the variables in equation 6.1.b can be separated as follows: $\eta = \bar{\eta}_{(\bar{x})}\cos\omega t$, where $\bar{x} = x/l$, and $\bar{\eta}$ is a function of $\bar{x}$ only.

$$\frac{d^2\bar{\eta}}{d\bar{x}^2} + \mu^2 \cosh(2r\bar{x} + a_1)\bar{\eta} = 0 \qquad (6.1.c)$$

$$\mu^2 = \frac{l^2\rho\omega^2}{H}$$

By expanding the hyperbolic function into its series, it is possible to obtain a series solution to the ordinary differential equation, equation 6.1.c. However, the series solution is quite difficult to work with and such a general solution is not actually needed here. What is needed here is the solution for a vibrating cable which is functioning as a guy, i.e., a cable whose tension varies only slightly along its length. The problem of the vibration of a cable whose tension and/or density varies slowly along its length can be solved by the W-K-B (Wentzel-Kramers-Brillouin) method which is used in wave mechanics and other branches of physics (19). In equation 6.1.c, the variable coefficient appears

in front of the dependent variable, $\bar{\eta}$, so that the form is that of the equation for a varying density cable.

In general, the governing differential equation for the vibration of a varying density cable is in the form: $y'' + \lambda^2 \rho^2 y = 0$ where ρ^2 is a function of x representing the variation in density. If ρ^2 does not vary rapidly in the range under consideration, the solution, according to the W-K-B method is:

$$y = C_{1,2}\, \rho^{-\frac{1}{2}} \exp. \pm i\lambda \int \rho\, dx$$

Adapting this solution to equation 6.1.c, and making use of the boundary condition, $\bar{\eta}\Big|_{\bar{x}=0} = 0$, one obtains:

$$\bar{\eta}_{(x)} = \frac{a_1'}{\left[\cosh(2r\bar{x} + a_1)\right]^{\frac{1}{4}}} \sin\left[\mu \int_0^{\bar{x}} \sqrt{\cosh(2r\bar{x} + a_1)}\, d\bar{x}\right] \qquad (6.2)$$

where a_1' is an integration constant to be evaluated by the boundary condition at $\bar{x} = 1$, or

$$\eta_{(\bar{x},t)} = \frac{a_1'}{\left[\cosh(2r\bar{x} + a_1)\right]^{\frac{1}{4}}} \sin\mu\, D_{(\bar{x})} \cos\omega t \qquad (6.2.a)$$

where

$$D_{(\bar{x})} \equiv \int_0^{\bar{x}} \sqrt{\cosh(2r\bar{x} + a_1)} = \frac{1}{r\sqrt{2}} z \left(1 + \frac{z^2}{12} - \frac{z^4}{32} + \frac{13z^6}{16\cdot 56} - \frac{47z^8}{16\cdot 16\cdot 24} + \frac{399z^{10}}{32\cdot 32\cdot 88} - \cdots\right)\Bigg|_{z_{(0)}}^{z_{(\bar{x})}}$$

and $z_0 \equiv \sqrt{\cosh a_1 - 1}$, $z_{(\bar{x})} \equiv \sqrt{\cosh(2r\bar{x} + a_1) - 1}$

See Appendix B

The boundary condition at $\bar{x} = 1$ is $\eta_{(1,t)} = y_n \sin\gamma_1 \cos\omega t$ where $y_n \cos\omega t$ is the tower motion at the support point and $\tan\gamma_1$, is the slope of the guy at the tower when it is in its equilibrium position. Substitution of this boundary condition into equation 6.2.a yields:

$$a_1' = \frac{y_n \sin\gamma_1 \left[\cosh(2r + a_1)\right]^{\frac{1}{4}}}{\sin\mu\, D_{(1)}}$$

Equation 6.2.a is the equation of motion for a guy cable which is being vibrated about its equilibrium position by a forced harmonic

motion at its upper end. While this result is of interest, it is not the goal of the work in this chapter. What is sought here is the force "felt" by the tower as it imparts the transverse motion, $y_n \sin\gamma_1 \cos\omega t$, at the upper end of the cable. In general, this force is equal to $T \frac{\partial \eta}{\partial \xi}$ evaluated at the point in question.

$$T \frac{\partial \eta}{\partial \xi} = \frac{H}{\ell} \frac{\partial \eta}{\partial \bar{x}}$$

and from equation 6.2.a

$$\frac{H}{\ell}\frac{\partial \eta}{\partial \bar{x}} = \frac{\frac{H}{\ell} a_1' \sin\mu D_{\bar{x}}}{\left[\cosh(2r\bar{x}+a_1)\right]^{\frac{1}{4}}}\left[\mu\sqrt{\cosh(2r\bar{x}+a_1)}\cot\mu D_{(\bar{x})} - \frac{r}{2}\tanh(2r\bar{x}+a_1)\right]\cos\omega t \qquad (6.4)$$

$$\frac{H}{\ell}\frac{\partial \eta}{\partial \bar{x}}\bigg|_{\bar{x}=1} = \frac{H}{\ell} y_n \sin\gamma_1\left[\mu(\cot\mu D_{(1)})\sqrt{\cosh(2r+a_1)} - \frac{r}{2}\tanh(2r+a_1)\right]\cos\omega t \qquad (6.4.a)$$

The expression $\frac{H}{\ell}\frac{\partial \eta}{\partial \bar{x}}\Big|_{\bar{x}=1}$ represents the transverse force that the tower must exert on the cable to impart a lateral motion of $y_n \sin\gamma_1 \cos\omega t$. The tower also imparts a longitudinal cable motion of $y_n \cos\gamma_1 \cos\omega t$ which can be accounted for by using the static equations developed in Chapter 2. The amplitude of the longitudinal motion can be represented by its horizontal and vertical components, i.e., $\Delta\ell = y_n \cos^2\gamma_1$, $\Delta h = y_n \cos\gamma_1 \sin\gamma_1$. By making use of the expressions 2.15.a and 2.17.a, it is seen that the amplitude of the horizontal force that the tower must exert on the cable to impart the longitudinal component of its motion is:

$$\frac{\partial H}{\partial \ell}\Delta\ell + \frac{\partial H}{\partial h}\Delta h = \frac{3H}{r^2\ell} y_n\left[(1+.4r^2)\cos^2\gamma_1 + \frac{h}{\ell}(1-.267r^2)\cos\gamma_1\sin\gamma_1\right]$$

Combining this with the horizontal component of the transverse force above, one obtains the following expression for the additional horizontal force that the tower must exert on the guy to impart a horizontal motion, $y_n \cos\omega t$, at its upper end:

$$\Delta H = \left\{ \frac{H}{\ell} \sin^2\gamma_1 \left[\mu (\cot \mu D_{(1)}) \sqrt{\cosh(2r+a_1)} - \frac{r}{2} \tanh(2r+a_1) \right] + \frac{3H\cos^2\gamma_1}{r^2\ell} \left[(1+.4r^2) + \frac{h}{\ell}(1-.267r^2)\tan\gamma_1 \right] \right\} y_n \cos\omega t \qquad (6.5)$$

Illustrative Example

The following example will serve to illustrate the numerical use of equation 6.5 and its application to groups of guys for finding the total dynamic resistance they offer to tower movement at a support point.

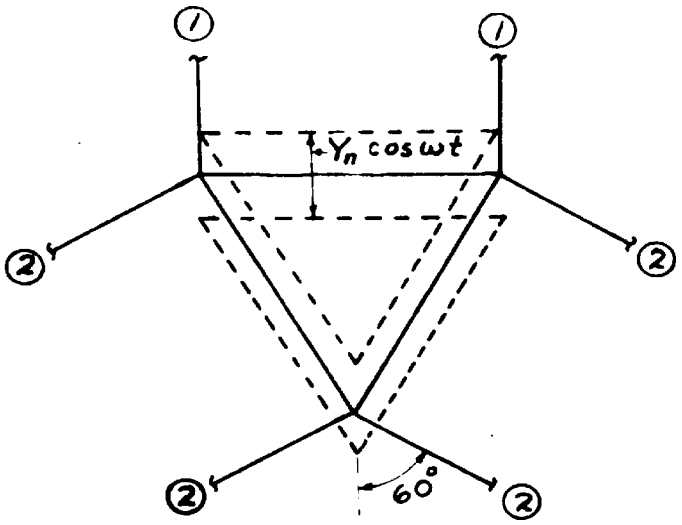


Fig. 6.2
Symmetrical Vibration of Tower
at Support Point

Here again the guy system of the Oklahoma City tower will be used for illustrative purposes. As seen in Fig. 6.2, the tower is assumed to be oscillating so as to cause symmetrical changes in the guy stresses. The line of action of the resultant forces is parallel to the horizontal projection of the cables denoted "1" and intersects the projection of the cables denoted "2" at 60°. So the "1" cables are experiencing only an in-plane forced horizontal

vibration of $Y_n \cos\omega t$ and the "2" cables are experiencing both an in-plane motion of $Y_n \cos 60^\circ \cos\omega t$ and an out-of-plane motion of $Y_n \sin 60^\circ \cos\omega t$.

The first term in equation 6.5 accounts for the transverse component of cable motion, $T_{(r,\omega)}$ below, and the second term gives the static force necessary to impart the longitudinal component of motion, $L_{(r)}$ below. Since only the first term is used in expressing the force necessary to impart out-of-plane motion, equation 6.5 should be written:

$$\Delta H = \left\{ T_{(r,\omega)} + L_{(r)} \right\} Y_n \cos\omega t \qquad (6.5.a)$$

where

$$T_{(r,\omega)} \equiv \frac{H}{\ell} \sin^2\gamma_1 \left[\mu \left(\cot\mu D_{(1)} \right) \sqrt{\cosh(2r + a_1)} - \frac{r}{2} \tanh(2r + a_1) \right]$$

and

$$L_{(r)} \equiv \frac{3H}{r^2 \ell} \cos^2\gamma_1 \left[(1 + .4r^2) + \frac{h}{\ell} (1 - .267r^2) \tan\gamma_1 \right]$$

Then it may be seen that the resultant force in the direction of the line of movement of the tower is:

$$R = \left\{ 2 \left[T_{(r,\omega)} + L_{(r)} \right]_1 + 4 \left[T_{(r,\omega)} + L_{(r)} \right]_2 \cos^2 60^\circ + 4 \left(T_{(r,\omega)} \right)_2 \sin^2 60^\circ \right\} Y_n \cos\omega t \qquad (6.6)$$

When the tower is vibrating about a vertical equilibrium position, then $T_{(r,\omega)}$ and $L_{(r)}$ are the same for cables "1" and "2" and this expression may be simplified to

$$R \equiv \bar{f}_n(y_n) = \left[6\, T_{(r,\omega)} + 3\, L_{(r)} \right] Y_n \cos\omega t \qquad (6.6.a)$$

A similar expression also may be obtained for the dynamic moment caused by the changes in the vertical components of the guy forces as the support point oscillates. Only in-plane vibrations will cause changes in the vertical components of the guy forces,

so by use of 2.21.b, i.e., $\Delta V = \frac{h}{\ell}(1-\frac{1}{3}r_i^2)\Delta H$ it is seen that:

$$\Delta V_1 = \frac{h}{\ell}(1-\tfrac{1}{3}r^2)(T_{(r,\omega)}+L_{(r)})\,Y_n\cos\omega t \qquad \text{and}$$

$$\Delta V_2 = \frac{h}{\ell}(1-\tfrac{1}{3}r^2)(T_{(r,\omega)}+L_{(r)})\,\frac{Y_n}{2}\cos\omega t \quad \text{or the moment, } \bar{G}_{n(y_n)},$$

is expressed as follows:

$$\bar{G}_{n(y_n)} = b\,\frac{\sqrt{3}}{2}\,\frac{h}{\ell}(1-\tfrac{1}{3}r^2)\left\{T_{(r,\omega)}+L_{(r)}\right\}Y_n\cos\omega t \qquad (6.7)$$

For a numerical example, the number 4 guys of the example in Chapter 5 will be used where: H = 40.0k; r = .125; h = 1349'; l = 950'; γ_1 = 1.0255; and q = .01055k/ft.

$$a_1 = \sinh^{-1}\frac{h}{\ell(1+\frac{1}{6}r^2)} - r = 1.02242$$

$$\mu = \ell\sqrt{\frac{q}{g_a H}}\,\omega = 2.7206\,\omega\;; \qquad z_0 = \sqrt{\cosh a_1 - 1} = .75497$$

$$z_1 = \sqrt{\cosh(2r+a_1)-1} = .96169.$$

$$D_{(1)} = \frac{1}{\sqrt{2}\,r}\left[z\left(1+\frac{z^2}{12}-\frac{z^4}{32}+\frac{13z^6}{32\cdot 28}-\frac{47z^8}{32\cdot 32\cdot 6}+\frac{399z^{10}}{32\cdot 32\cdot 88}-\cdots\right)\right]_{z_0}^{z_1}$$

$$D_{(1)} = 1.3186$$

$$T_{(r,\omega)} = \frac{H}{\ell}\sin^2\gamma_1\left[\mu\cot(\mu D_{(1)})\sqrt{\cosh(2r+a_1)}-\frac{r}{2}\tanh(2r+a_1)\right]$$

$$= .1161\,\omega\cot(3.5873\,\omega) - .00164$$

$$L_{(r)} = \frac{3H\cos^2\gamma_1}{\ell r^2}\left[(1+.4r^2)+\frac{h}{\ell}(1-.267r^2)\tan\gamma_1\right] = 7.259$$

$$\bar{f}_{n(y_n)} = \left[6\,T_{(r,\omega)}+3L_{(r)}\right]Y_n\cos\omega t$$

$$\bar{f}_{4(y_4)} = \left[.6966\,\omega\cot 3.5873\,\omega + 21.766\right]Y_4\cos\omega t$$

$$\bar{G}_{n(y_n)} = b\,\frac{\sqrt{3}}{2}\,\frac{h}{\ell}(1-\tfrac{1}{3}r^2)\left\{T_{(r,\omega)}+L_{(r)}\right\}Y_n\cos\omega t$$

$$\bar{G}_{4(y_4)} = \left[1.704\,\omega\cot 3.5873\,\omega + 106.53\right]Y_4\cos\omega t$$

CHAPTER VII

DYNAMIC ANALYSIS OF TOWER BELOW LEVEL OF TOP GUY CONNECTIONS

In Chapter 3 of the static analysis, equations were developed which were valid for tower segments below the level of the top guy connections. This portion of the tower was considered separately because the segments below the top guy connections had external axial loads which were large as compared with their weights. This, as was proved in Chapter 4, made it possible to analyze each segment as if it were weightless and subjected to an external axial load which included one-half the weight of the segment. This section of the tower is also considered separately in the dynamic analysis and for exactly the same reason. As will be proved in Chapter 8, in the dynamic analysis of a heavy vertical member subjected to an external axial load that is large as compared with its weight, the member may be treated as an ordinary beam-column by including one-half of its weight in the external axial load.

As was stated in the introduction, this study does not have as its goal the complete dynamic analysis of a guyed tower system. Only the initial phase of a complete dynamic analysis will be dealt with here; namely, the development of an accurate method for computing the natural frequencies of the system for small vibrations about some equilibrium position. It is proposed to treat the tower

as a distributed mass problem so the frequency equation will be a transcendental equation with an infinite number of roots corresponding to the natural frequencies.

Finally, the purpose of the work in this chapter is to develop the equations which can be used to find the natural frequencies and mode shapes of a continuous beam-column on elastic supports, i.e., the tower below the top guy connections. This will be carried out in a manner analagous to that employed in solving the corresponding static problem in Chapter 3. That is, the differential equation, expressing the fact that the beam elements are in dynamic equilibrium, will be solved to give the elastic curve equation for each tower segment in terms of the unknown slopes and deflections at the ends. It is then possible to determine these slopes and deflections by solving the equations which express the fact that the tower is continuous and that each joint is in equilibrium.

Only small vibrations are considered so that the axial loads may be considered constant with respect to time and equal to the axial loads corresponding to the equilibrium position about which the tower is oscillating.

Development of Equations for Normal Mode Shapes

By use of d'Alembert's principle, equation 3.1.a may be modified to give the differential equation governing the free vibration of a uniform beam-column.

$$(3.1.a)\quad EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = w(x)$$

$w_{(x)}$ is replaced by the inertia force which acts when the beam is vibrating, resulting in the expression,

$$EI\frac{d^4y}{dx^4} + \rho\frac{d^2y}{dx^2} = -\rho\frac{\partial^2 y}{\partial t^2} \qquad (7.1)$$

where ρ = mass per unit length.

The following transformations: $\bar{x} = \frac{x}{L}$; $\bar{y} = \frac{y}{L}$; $\alpha^2 = \frac{\rho L^2}{EI}$; $\lambda = \frac{\rho L^4 \omega^2}{EI}$ (ω = circular frequency of vibration in radians per second) are carried out to place equation 7.1 in the dimensionless form:

$$\frac{\partial^4 \bar{y}}{\partial \bar{x}^4} + \alpha^2\frac{\partial^2 \bar{y}}{\partial \bar{x}^2} = -\lambda\frac{\partial^2 \bar{y}}{\partial(\omega t)^2} \qquad (7.1.a)$$

The solutions which are sought here are the periodic natural modes of vibration where damping is ignored. Therefore, the variables of equation 7.1.a can be separated by assuming a solution of the form, $\bar{y} = \bar{Y}_{(\bar{x})}\cos\omega t$, which, when substituted in 7.1.a yields:

$$\bar{Y}'' + \alpha^2\bar{Y}'' - \lambda\bar{Y} = 0 \qquad (7.1.b)$$

Prime (') denotes differentation with respect to $\bar{x}$. The solution to 7.1.b for $\bar{Y}$, the "amplitude" function, is:

$$\bar{Y} = e_1\cosh q\bar{x} + e_2\sinh q\bar{x} + e_3\cos k\bar{x} + e_4\sin k\bar{x} \qquad (7.2)$$

where $$q \equiv \frac{1}{\sqrt{2}}\left(\sqrt{\alpha^4+4\lambda} - \alpha^2\right)^{\frac{1}{2}}; \quad k = \frac{1}{\sqrt{2}}\left(\sqrt{\alpha^4+4\lambda} + \alpha^2\right)^{\frac{1}{2}}$$

The other curve equations are:

$$\bar{y}_{(\bar{x},t)} = \frac{y}{L} = \left(e_1\cosh q\bar{x} + e_2\sinh q\bar{x} + e_3\cos k\bar{x} + e_4\sin k\bar{x}\right)\cos\omega t \qquad (7.3)$$

$$\theta_{(\bar{x},t)} = \frac{d\bar{y}}{d\bar{x}} = \left(e_1 q\sinh q\bar{x} + e_2 q\cosh q\bar{x} - e_3 k\sin k\bar{x} + e_4 k\cos k\bar{x}\right)\cos\omega t \qquad (7.4)$$

$$-\frac{ML}{EI} = \frac{d^2\bar{y}}{d\bar{x}^2} = (e_1 g^2 \cosh g\bar{x} + e_2 g^2 \sinh g\bar{x} - e_3 k^2 \cos k\bar{x} - e_4 k^2 \sin k\bar{x})\cos\omega t \qquad (7.5)$$

$$-\frac{QL^2}{EI} = \frac{d^3\bar{y}}{d\bar{x}^3} = (e_1 g^3 \sinh g\bar{x} + e_2 g^3 \cosh g\bar{x} + e_3 k^3 \sin k\bar{x} - e_4 k^3 \cos k\bar{x})\cos\omega t \qquad (7.6)$$

The constants of integration are evaluated by making use of the following boundary conditions on the amplitude of motion:

$$\bar{Y}\Big|_{\bar{x}=0} = \bar{Y}_0 \; ; \; \bar{Y}\Big|_{\bar{x}=1} = \bar{Y}_1 \; ; \; \frac{d\bar{Y}}{d\bar{x}}\Big|_{\bar{x}=0} = \theta_0 \; ; \; \frac{d\bar{Y}}{d\bar{x}}\Big|_{\bar{x}=1} = \theta_1$$

Although the solution is general and may be thought of as describing span "i" between end points n-1 and n, the presentation is for span 1 between end points 0 and 1.

The constants are as follows:

$$e_1 = \frac{1}{e_{(\alpha,\lambda)}}\Big[\Big(\frac{1}{g}\sinh\cos k - \frac{1}{k}\cosh g \sin k\Big)\theta_0 + \Big(-\frac{1}{g}\sinh g + \frac{1}{k}\sin k\Big)\theta_1$$
$$+ (1 - \cosh g\cos k - \frac{k}{g}\sinh g \sin k)\bar{Y}_0 + (\cosh g - \cos k)\bar{Y}_1\Big]$$

$$e_2 = \frac{1}{e_{(\alpha,\lambda)}}\Big[\frac{1}{g}(1 - \cosh g\cos k + \frac{g}{k}\sinh g \sin k)\theta_0 + \frac{1}{g}(\cosh g - \cos k)\theta_1$$
$$+ \Big(\frac{k}{g}\cosh g \sin k + \sinh g \cos k\Big)\bar{Y}_0 - (\sinh g + \frac{k}{g}\sin k)\bar{Y}_1\Big]$$

$$e_3 = \frac{-1}{e_{(\alpha,\lambda)}}\Big[\Big(\frac{1}{g}\sinh g\cos k - \frac{1}{k}\cosh g \sin k\Big)\theta_0 + \Big(-\frac{1}{g}\sinh g + \frac{1}{k}\sin k\Big)\theta_1$$
$$+ (-1 + \cosh g\cos k - \frac{g}{k}\sinh g \sin k)\bar{Y}_0 + (\cosh g - \cos k)\bar{Y}_1\Big]$$

$$e_4 = \frac{-1}{e_{(\alpha,\lambda)}}\Big[\frac{1}{k}(-1 + \cosh g\cos k + \frac{k}{g}\sinh g \sin k)\theta_0 + \frac{1}{k}(\cosh g - \cos k)\theta_1$$
$$+ (\cosh g \sin k + \frac{g}{k}\sinh g\cos k)\bar{Y}_0 - \Big(\frac{g}{k}\sinh g + \sin k\Big)\bar{Y}_1\Big]$$

$$e_{(\alpha,\lambda)} = 2 - 2\cosh g\cos k - \frac{\alpha^2}{gk}\sinh g \sin k$$

The amplitudes of the end shears and moments in terms of the beam constants and the end slopes and deflections are as follows:

$$M_{01} = -\frac{EI}{L}\left(g^2 e_1 - k^2 e_3\right) = \frac{EI}{L}\left[C_1\theta_0 + C_2\theta_1 + \frac{1}{L}\left(-C_3 Y_0 + C_4 Y_1\right)\right] \quad (7.7)$$

$$M_{10} = \frac{EI}{L}\left(g^2 e_1 \cosh g + g^2 e_2 \sinh g - k^2 e_3 \cos k - k^2 e_4 \sin k\right)$$

$$= \frac{EI}{L}\left[C_2\theta_0 + C_1\theta_1 + \frac{1}{L}\left(-C_4 Y_0 + C_3 Y_1\right)\right] \quad (7.8)$$

$$Q_{01} = -\frac{EI}{L^2}\left(g^3 e_2 - k^3 e_4\right) = \frac{EI}{L^2}\left[C_5\theta_0 + C_4\theta_1 + \frac{1}{L}\left(C_6 Y_0 - C_7 Y_1\right)\right] \quad (7.9)$$

$$Q_{10} = -\frac{EI}{L^3}\left[\left(g^3 e_1 \sinh g + g^3 e_2 \cosh g + k^3 e_3 \sin k - k^3 e_4 \cos k\right)\right]$$

$$= \frac{EI}{L^2}\left[C_4\theta_0 + C_5\theta_1 + \frac{1}{L}\left(C_7 Y_0 - C_6 Y_1\right)\right] \quad (7.10)$$

Where

$$C_1 = C_{(\alpha,\lambda)}\left(k \sinh g \cos k - g \cosh g \sin k\right);$$

$$C_2 = C_{(\alpha,\lambda)}\left(-k \sinh g + g \sin k\right)$$

$$C_3 = C_{(\alpha,\lambda)} \frac{gk}{g^2 + k^2}\left[2gk \sinh g \sin k + (\alpha^2)(1 - \cosh g \cos k)\right]$$

$$C_4 = C_{(\alpha,\lambda)}\, gk(\cosh g - \cos k)$$

$$C_5 = C_{(\alpha,\lambda)} \frac{gk}{g^2 + k^2}\left[\frac{g^4 + k^4}{gk} \sinh g \cdot \sin k + (-\alpha^2)(1 - \cosh g \cos k)\right]$$

$$C_6 = C_{(\alpha,\lambda)}\, gk(g \sinh g \cos k + k \cosh g \sin k)$$

$$C_7 = C_{(\alpha,\lambda)}\, gk(g \sinh g + k \sin k)$$

$$C_{(\alpha,\lambda)} = \frac{g^2 + k^2}{\left[(\alpha^2) \sinh g \sin k - 2gk(1 - \cosh g \cos k)\right]}$$

It is interesting to note the similarity in form between the ordinary slope deflection equations 3.11 and 3.12 and their dynamic counterparts, equations 7.7 and 7.8. In the dynamic equations the movement of each end point is important; whereas in the static equations only the relative movement of the end points affects the end moments.

Also, (as in the static case) if one end moment is known, it is not necessary to solve for the slope at that point. For instance, if $M_{01} = 0$, then:

$$M_{10} = \frac{EI}{LC_1}\left[\left(C_1^2 - C_2^2\right)\theta_1 + \left(C_2C_3 - C_1C_4\right)\frac{Y_0}{L} + \left(C_1C_3 - C_2C_4\right)\frac{Y_1}{L}\right] \qquad (7.8.a)$$

$$Q_{01} = \frac{EI}{L^2C_1}\left[\left(C_1C_4 - C_2C_5\right)\theta_1 + \left(C_1C_6 + C_3C_6\right)\frac{Y_0}{L} - \left(C_1C_2 + C_4C_5\right)\frac{Y_1}{L}\right] \qquad (7.9.a)$$

$$Q_{10} = \frac{EI}{L^2C_1}\left[\left(C_1C_5 - C_2C_4\right)\theta_1 + \left(C_1C_7 + C_3C_4\right)\frac{Y_0}{L} - \left(C_1C_6 + C_4^2\right)\frac{Y_1}{L}\right] \qquad (7.10.a)$$

Equilibrium Equations

Equations 7.7, 7.8, 7.9, and 7.10 can be substituted in the equilibrium equations below to solve for the natural frequencies of a guyed tower that is entirely made up of tower segments having large external axial loads.

The equilibrium equations are:

$$M_{n,n-1} + M_{n,n+1} + \bar{G}_n(y_n) = 0 \qquad (7.11)$$

and

$$Q_{n,n-1} - Q_{n,n+1} + \bar{f}_n(y_n) = 0 \qquad (7.12)$$

where $\bar{G}_n(y_n)$ is the external moment exerted on the joint and $\bar{f}_n(y_n)$ is the external horizontal force exerted on the joint by

the cables as the tower vibrates. These quantities are found by using the formulas developed in Chapter 6.

When an attempt is made to solve the equations, which result from the substitution of 7.7, 7.8, 7.9, and 7.10 into the equilibrium equations 7.11 and 7.12, for the unknowns θ_n and Y_n, it is seen that the equations (to be solved simultaneously) contain no terms other than those made up of the unknowns multiplied by their various coefficients. This is due to the fact that the spans were not considered to be loaded with external loads in the development of equations 7.7 through 7.10. The non-trivial solutions of such a set of equations are obtained by setting the determinate of the coefficients equal to zero and finding its roots. These coefficients are in terms of the physical constants of the structure and ω, the circular frequency of vibration ($\lambda = \frac{\rho L^4 \omega^2}{EI}$). In other words, the determinate of the coefficients set equal to zero is the frequency equation and the roots of the determinate are the natural frequencies.

This approach will be discussed in more detail in Chapter 9.

CHAPTER VIII

DYNAMIC ANALYSIS OF UNGUYED TOWERS AND THAT PORTION OF GUYED TOWERS ABOVE THE TOP GUY CONNECTIONS

In Chapter 4, formulas were developed which can be used for the static analysis of a vertical member whose weight and stiffness are such that the moments, due to the distributed weight of the member being in a deflected position, are not negligible as compared with the other moments acting on the member. The purpose of the work in this chapter is to develop analagous formulae for the dynamic analysis of such a member. In connection with tower structures, such formulae are needed for the dynamic analysis of either an unguyed tower or that portion of a guyed tower above the top guy connections. The problem of the member with a continuously varying stiffness will not be treated here. However, formulae will be developed which can be used for the case where the stiffness makes only a few abrupt changes, or may be assumed to do so.

In the static problem, the coordinates of the elastic curve were selected so that the deflected position of the member was represented as a horizontal distance measured from a line drawn tangent to the curve at the base. Since the moments were not affected by the position of the base, Y_o, the slope of the base, θ_o, served to increase the transverse load, w, by a factor, $q\theta_o$. Therefore, the analysis of a member serving as an unguyed tower was merely a special case of the cantilever where $\theta_o = 0$. However, in the dynamic problem,

the movement and rotation of the base, $Y_o \cos \omega t$ and $\theta_o \cos \omega t$, both have important effects upon the moments and the stresses induced into the member. In addition, the analysis of a member cantilevered above a guyed tower is not, in itself, an eigenvalue problem, as is the analysis of a member serving as an unguyed tower; but rather it is a part of the larger eigenvalue problem, that of finding the natural frequencies of the whole guyed tower system. Therefore, it will be necessary to treat the analysis of the unguyed tower as a separate problem.

Dynamic Analysis of Uniform Vertical Member with Fixed Base

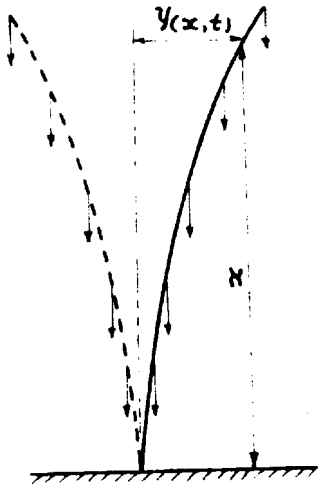


Fig. 8.1
Vibration of Heavy Vertical Cantilever

It should be noted that the method presented here was indicated in reference (14), where the same problem was dealt with, but the answer presented was only the first two terms of the infinite series that will be used to represent the answer here.

As before, the governing differential equation for the dynamic problem will be obtained from the equation for the static case.

$$(4.2)\quad EI\frac{d^3y}{dx^3} + q(L-x)\frac{dy}{dx} = -(L-x)(w + q\theta_o)$$

differentiating with respect to x,

$$EI\frac{d^4y}{dx^4} + q\frac{d}{dx}\left[(L-x)\frac{dy}{dx}\right] = w + q\theta_o \qquad (8.1)$$

Now setting $\theta_o = 0$ and replacing w by its dynamical equivalent, the following governing partial differential equation is obtained.

$$EI\frac{\partial^4 y}{\partial x^4} + q\frac{\partial}{\partial x}\left[(L-x)\frac{\partial y}{\partial x}\right] = -\rho\frac{\partial^2 y}{\partial t^2} \qquad (8.2)$$

Transforming to dimensionless terms as in previous chapters,

where $\bar{y} = \frac{y}{L}$; $\bar{x} = \frac{x}{L}$; $\beta = \frac{qL^3}{EI}$; $\lambda = \frac{\rho L^4\omega^2}{EI} \equiv \frac{\beta L\omega^2}{g_a}$

$$\frac{\partial^4\bar{y}}{\partial\bar{x}^4} + \beta\frac{\partial}{\partial\bar{x}}\left[(1-\bar{x})\frac{\partial\bar{y}}{\partial\bar{x}}\right] = -\lambda\frac{\partial^2 y}{\partial(\omega t)^2} \qquad (8.2.a)$$

The boundary conditions to which a solution to 8.2.a must be fitted are:

$$\left.\frac{\partial^2\bar{y}}{\partial\bar{x}^2}\right|_{\bar{x}=1} = \left.\frac{\partial^3\bar{y}}{\partial\bar{x}^3}\right|_{\bar{x}=1} = \left.\frac{\partial\bar{y}}{\partial\bar{x}}\right|_{\bar{x}=0} = \left.\bar{y}\right|_{\bar{x}=0} = 0$$

It is again assumed that the member is vibrating in one of its normal modes so that all points are executing harmonic motion, so the variables may be separated by assuming $\bar{y} = \bar{Y}_{(x)}\cos\omega t$. This is substituted in equation 8.2.a to obtain the following ordinary differential equation for the amplitudes of the normal modes.

$$\frac{d^4\bar{Y}}{d\bar{x}^4} + \beta\frac{d}{d\bar{x}}\left[(1-\bar{x})\frac{d\bar{Y}}{d\bar{x}}\right] - \lambda\bar{Y} = 0 \qquad (8.2.b)$$

A general closed form solution of 8.2.b does not seem to be available. Therfore, a perturbation theory, or approach, will be used to obtain a series solution that can be used to obtain the eigenfunctions, $\bar{Y}$, and the eigenvalues, λ . Actually, it would be proper to have the subscript "n" after each $\bar{Y}$ and λ , i.e., $\bar{Y}_n$ and

λ_n, in equation 8.2.b, to clarify the point that for each value of λ, λ_n, there is a different solution for $\bar{Y}$, $\bar{Y}_n$.

Of course, if β were equal to 0 in 8.2.b, the problem could easily be solved and, in fact, the solution for this problem is readily available in the literature, e.g., reference (25). Now it is apparent on physical grounds (or in using perturbation theory it is standard practice to assume) that as β is slowly increased from zero, the values obtained for $\bar{Y}_n$ and λ_n will change in a regular manner. Therefore, an expansion of each of these values into a Taylor's series in β, expanded about $\beta = 0$, is indicated. For the values of β normally encountered in tower structures, it will be practical to do so. It was pointed out in Chapter 4 that the value of β corresponding to the first mode of buckling was $\beta_1 = 7.834$; however, for practical structures subjected to transverse loads, the value of β will usually be less than 1.0. In other words, usually, only a very few terms of the series will be needed.

Continuing with the solution according to a perturbation theory, $\bar{Y}_n$ and λ_n are expressed as Taylor's series in β expanded about $\beta = 0$, i.e., the derivaties are evaluated at $\beta = 0$.

$$\bar{Y}_n = V_{n,0} + \sum_{i=1}^{\infty} \frac{\beta^i}{i!} \frac{\partial^i \bar{Y}_n}{\partial \beta^i} = V_{n,0} + \sum_{i=1}^{\infty} \beta^i V_{n,i} \qquad (8.3)$$

$$\lambda_n = \lambda_{n,0} + \sum_{i=1}^{\infty} \frac{\beta^i}{i!} \frac{\partial^i \lambda_n}{\partial \beta^i} = \lambda_{n,0} + \sum_{i=1}^{\infty} \beta^i \varepsilon_{n,i} \qquad (8.4)$$

where $V_{n,0}$ are functions coinciding with $\bar{Y}_n$, when $\beta = 0$ and $\lambda_{n,0}$ are eigenvalues coinciding with λ_n when $\beta = 0$. Thus,

$$V_{n,0}^{IV} - \lambda_{n,0} V_{n,0} = 0 \qquad (8.5)$$

(') denotes differentation with respect to $\bar{x}$.

The functions $V_{n,o}$ and $V_{n,i}$ must be fitted to the same boundary conditions as $\bar{Y}_n$, namely, $V_{n,i} = V'_{n,i} = 0$ at $\bar{x}=0$ and $V''_{n,i} = V'''_{n,i} = 0$ at $\bar{x}=1$.

The normalized solution to equation 8.5, then, is:

$$V_{n,o} = (\cos m_n \bar{x} - \cosh m_n \bar{x}) + b_n (\sin m_n \bar{x} - \sinh m_n \bar{x}) \qquad (8.6)$$

where $m_n = (\lambda_{n,o})^{\frac{1}{4}}$

$$b_n = -\frac{\cos m + \cosh m}{\sin m + \sinh m} = \frac{\sin m - \sinh m}{\cos m + \cosh m}$$

and the values for m are determined from the frequency equation,

$\cos m \cosh m = -1$, (8.6.a)

and are as follows: $m_1 = .597\pi$, $m_2 = 1.494\pi$, $m_3 = 2.50\pi$, $m_n = \frac{2n-1}{2}\pi$ if $n>2$

The functions $V_{n,o}$ are normal orthogonal functions, i.e.,

$$\int_o^1 V_{n,o}\, V_{k,o}\, d\bar{x} = \begin{matrix} 1 \text{ if } n=k \\ 0 \text{ if } n\neq k \end{matrix} \qquad (8.7)$$

If 8.3 and 8.4 are substituted into 8.2.b, and use is made of the fact that $V_o'' - \lambda_o V_o = 0$ (for brevity, the subscript "n" will be omitted in most of the work to follow) The following expression is the result:

$$\sum_{i=1}^{\infty} \beta^i V_i'' - \lambda_o \sum_{i=1}^{\infty} \beta^i V_i = V_o \sum_{i=1}^{\infty} \beta^i \varepsilon_i + \sum_{i=1}^{\infty} \beta^i \varepsilon_i \sum_{i=1}^{\infty} \beta^i V_i -$$

$$\beta \frac{d}{d\bar{x}} \left[(1-\bar{x}) \left(V_o' + \sum_{i=1}^{\infty} \beta^i V_i' \right) \right] \qquad (8.8)$$

Equating the coefficients of first powers of β in equation 8.8, the following inhomogeneous differential equation for V_i is obtained:

$$V_i'' - \lambda_o V_i = F_i \equiv \varepsilon_i V_o - \frac{d}{d\bar{x}} \left[(1-\bar{x}) V_o' \right] \qquad (8.8.a)$$

The solution to the homogeneous equation, i.e., $F_i = 0$, is the set of orthogonal functions V_o. It is a principle of differential equations, references (2) and (3), that for a solution to exist for such an equation, the right-hand side must be orthogonal to the

functions satisfying the homogeneous equation. That is, $\int_0^1 V_0 F_1 \, d\bar{x} = 0$ (8.9)

Substituting for F_1 in 8.9, it is seen that:

$$\varepsilon_1 = \int_0^1 \frac{d}{d\bar{x}}\left[(1-\bar{x})V_0'\right] V_0 \, d\bar{x} = -\int_0^1 (1-\bar{x})(V_0')^2 \, d\bar{x} \qquad (8.9.a)$$

Carrying out the integration and temporarily returning the subscripts "n", the following expression is obtained for ε_1 :

$$\varepsilon_{n,1} = -\left(2 + m_n b_n + \tfrac{1}{2} m_n^2 b_n^2\right) \qquad (8.9.b)$$

Now that ε_1 is known, it is possible to solve the inhomogeneous differential equation 8.8.a for V_1 . In general, this will be $K_1 V_0 + V_{1P}$, or some constant times the solution to the homogeneous equation plus a particular solution which satisfies the boundary conditions, e.g., a particular solution determined by variation of parameters. In such a case, one is free to assign any arbitrary value to K_1. If only one or two corrective terms are needed to give the desired accuracy, e.g., $\bar{Y} = V_0 + \beta V_1$ and $\lambda = \lambda_0 + \varepsilon_1 \lambda_1 + \varepsilon_2 \lambda_2$, the easiest course is probably to set $K_1 = 0$. However, if it is going to be necessary to consider several terms of the series, it is best to select K_1 so that $\int_0^1 V_1 V_0 \, d\bar{x} = 0$. Since $V_1 = K_1 V_0 + V_{1P}$, this can be accomplished by setting $K_1 = -\int_0^1 V_{1P} V_0 \, d\bar{x}$. In general, it is possible to make $\int_0^1 V_i V_0 \, d\bar{x} = 0$ by consistently setting $K_i = -\int_0^1 V_{iP} V_0 \, d\bar{x}$. This simplifies the general recurrence relations, and is worth-while if several terms of the indicated series are needed.

At this stage ε_1 and V_1 are determined and $\int_0^1 V_1 V_0 \, d\bar{x} = 0$. Now, equating the coefficients of the second powers of "β" in 8.8

$$V_2'^{V} - \lambda_0 V_2 = F_2 \equiv \varepsilon_2 V_0 + \varepsilon_1 V_1 - \frac{d}{d\bar{x}}\left[(1-\bar{x})V_1'\right] \qquad (8.8.b)$$

As before, $\int_0^1 F_2 V_0 \, d\bar{x} = 0$ (8.10)

Substituting in 8.10 and making use of the fact that $\int_0^1 V_1 V_0 \, d\bar{x} = 0$

it is seen that: $\varepsilon_2 = \int_0^1 \frac{d}{d\bar{x}}\left[(1-\bar{x})V_1'\right] V_0 \, d\bar{x} = -\int_0^1 (1-\bar{x}) V_1' V_0' \, d\bar{x}$ (8.10.a)

With ε_2 determined, 8.8.b can be solved for V_2, and this procedure can be carried on until the desired accuracy is obtained.

The general recurrence relations, obtained by equating the coefficients of β' in 8.8, are as follows:

$$V_i'^{V} - \lambda_0 V_i = F_i \qquad (8.8.c)$$

ε_i is obtained from $\int F_i V_0 \, d\bar{x} = 0$ (8.11)

as $\varepsilon_i = -\int_0^1 (1-\bar{x}) V_{i-1}' V_0' \, d\bar{x}$ (8.11.a)

provided $\int_0^1 V_i V_0 \, d\bar{x} = 0$ (8.12)

has been consistently assured by setting $K_i = -\int_0^1 V_{ip} V_0 \, d\bar{x}$ (8.13)

Then with ε_i determined, F_i is known and 8.8.c can be solved for V_i.

Illustrative Example

To illustrate the theory presented in this section, the first two natural frequencies of the top antenna of the Oklahoma City tower will be calculated. Only two terms of the series will be used here.

It is recalled from the examples worked in Chapter 4 that for this member L = 73' and $\beta = .892$.

$\lambda_n \cong (\lambda_{n,0} + \beta \varepsilon_{n,1}) \equiv (m_n^4 + \beta \varepsilon_{n,1})$ from 8.4

and $$\varepsilon_{n,1} = -(2 + m_n b_n + \tfrac{1}{2} m_n^2 b_n^2)$$ from 8.9.b

$$\lambda_n = \frac{\rho L^4}{EI}\omega_n^2 \equiv \frac{\beta L}{g_a}\omega_n^2$$

Frequency for first mode: $f_1 = \frac{\omega_1}{2\pi}$

$$m_1 = .597\pi\,;\; b_1 = \frac{\sin m_1 - \sinh m_1}{\cos m_1 + \cosh m_1} = -1.462$$

$$\varepsilon_{1,1} = -\left[2 + (.597\pi)(-1.462) + \tfrac{1}{2}(.597\pi)^2(-1.462)^2\right] = -3.02$$

$$\lambda_1 \cong \left[(.597\pi)^4 - (.892)(3.02)\right] = 12.36 - 2.69 = 9.67$$

$$\omega_1 = \sqrt{\frac{g_a\lambda_1}{\beta L}} = 2.19\,;\; f_1 = .35 \text{ cycles per second}$$

Frequency for second mode: $f_2 = \frac{\omega_2}{2\pi}$

$$m_2 = 1.494\pi\,;\; b_2 = -1.019$$

$$\varepsilon_{2,1} = -\left[2 + (1.494\pi)(-1.019) + \tfrac{1}{2}(1.494\pi)^2(-1.019)^2\right] = -8.65$$

$$\lambda_2 \cong \left[(1.494\pi)^4 + (.892)(-8.65)\right] = 487.5 - 7.7 = 479.8$$

$$\omega_2 = \sqrt{\frac{g_a\lambda_2}{\beta L}} = 15.4 \qquad f_2 = 2.45 \text{ cycles per second}$$

Dynamic Analysis of Vertical Member Whose Base is Subjected to Harmonic Translation and Rotation

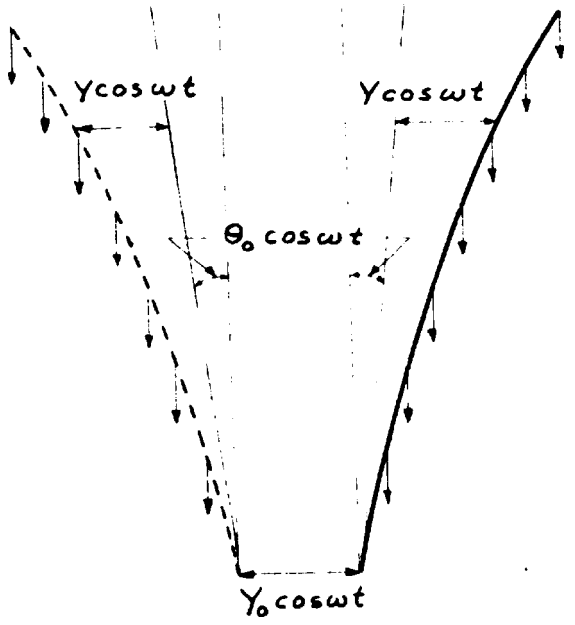


Fig. 8.2
Vibration of Heavy Vertical Member with
Rotating and Translating Base

The type of analysis described in the title of this section is needed for that portion of the tower or antenna which is cantilevered above the top guy connections. Since the frequency of rotation and translation of the base of this portion of the tower is equal to one of the natural frequencies of the tower system, and since the physical properties of the cantilever affect these natural frequencies, it is apparent that the problem here is not an isolated eigenvalue problem. Therefore, λ_n below can be treated as an ordinary known coefficient and will not have to be determined by summing a series.

Again, equation 8.1 is modified to obtain the governing partial differential equiation, but the modification differs from that

carried out in the previous section. Here θ is replaced by the amplitude of the base rotation multiplied by $\cos\omega t$, or $\theta_o \cos\omega t$. "w" is again replaced by the inertia forces, but here the inertia forces are $-\rho \frac{\partial^2}{\partial t^2}\left(Y_o \cos\omega t + x\theta_o \cos\omega t + y\right)$, due to the fact that "y" is measured from a tangent drawn at the base. Carrying out these modifications of equation 8.1, one obtains:

$$EI\frac{\partial^4 y}{\partial x^4} + q\frac{\partial}{\partial x}\left[(L-x)\frac{\partial y}{\partial x}\right] = -\rho\frac{\partial^2 y}{\partial t^2} + \left[\theta_o(q+\rho\omega^2 x)+\rho\omega^2 Y_o\right]\cos\omega t \qquad (8.14)$$

Again the governing equation is transformed to dimensionless terms as follows: $\bar{y} = \frac{y}{L}$; $\bar{x} = \frac{x}{L}$; $\beta\frac{qL^3}{EI}$; $\lambda = \frac{\rho L^4\omega^2}{EI} \equiv \frac{\beta L\omega^2}{g_a}$; $\bar{Y}_o = \frac{Y_o}{L}$;

$$\frac{\partial^4 \bar{y}}{\partial \bar{x}^4} + \beta\frac{\partial}{\partial \bar{x}}\left[(1-\bar{x})\frac{\partial \bar{y}}{\partial \bar{x}}\right] = -\lambda\frac{\partial^2 \bar{y}}{\partial(\omega t)^2} + \left[\theta_o(\beta+\lambda\bar{x})+\lambda\bar{Y}_o\right]\cos\omega t \qquad (8.14.a)$$

The substitution, $\bar{y} = \bar{Y}_{(x)}\cos\omega t$, is made to separate the variables, giving:

$$\frac{d^4\bar{Y}}{d\bar{x}^4} + \beta\frac{d}{d\bar{x}}\left[(1-\bar{x})\frac{d\bar{Y}}{d\bar{x}}\right] - \lambda\bar{Y} = \lambda\bar{Y}_o + (\beta+\lambda\bar{x})\theta_o \qquad (8.14.b)$$

whose general solution, $\bar{Y}$, must be fitted to the boundary conditions, $\bar{Y} = \frac{d\bar{Y}}{d\bar{x}} = 0$ at $x = 0$, and $\frac{d^2\bar{Y}}{d\bar{x}^2} = \frac{d^3\bar{Y}}{d\bar{x}^3} = 0$ at $\bar{x} = 1$ to describe the normal mode shapes.

This equation differs from the one considered in the previous section in two respects. First, λ , here, is not an eigenvalue in the usual sense and can be treated as an ordinary coefficient; this simplifies the solution. Second, the equation is not homogeneous, however, this does not complicate the solution to any extent, as a particular solution, $\bar{Y}_p$, is easy to obtain by inspection: $\bar{Y}_p = -(\bar{Y}_o + \theta_o\bar{x})$.

In the previous section the "unperturbed" problem, which served as the first term of the series solution, was selected as the ordinary

vibrating beam problem where the axial loads were equal to zero. The solutions for this problem were orthogonal and use was made of their orthogonality in the development of the recurrence relations for the series. In the development to follow, there will be no need to select an unperturbed problem whose solutions are orthogonal. Thus, one is allowed a greater freedom in the selection of the problem whose solution is to serve as the first term of a series solution for the main problem.

Axial loads, whether distributed or concentrated, affect only the lower modes of vibration. Thus, here, if one selects for the "unperturbed" problem a beam with a concentrated axial load equivalent to the distributed axial load for the first mode of vibration, very few corrective terms of the Taylor's series will be needed to give an accurate solution. It can easily be shown, by calculations based on an approximate energy method (page 367 reference 21), that the first mode of vibration of a cantilever beam with a uniformly distributed axial load is very closely approximated by the first mode of vibration of a cantilever beam with a concentrated axial load at its upper end equal to one-third the total distributed axial load. Therefore, the unperturbed problem in this section will be that of a uniform cantilever beam with a concentrated axial load equal to one-third the weight of the beam instead of a simple uniform beam as in the previous section.

Rewriting equation 8.14.b,

$$\frac{d^4\bar{y}}{d\bar{x}^4} + \frac{\beta}{3}\frac{d^2\bar{y}}{d\bar{x}^2} - \lambda\bar{Y} + \beta\frac{d}{d\bar{x}}\left[\left(\frac{2}{3} - \bar{x}\right)\frac{d\bar{y}}{d\bar{x}}\right] = \lambda\bar{Y}_o + (\beta + \lambda\bar{x})\theta_o \qquad (8.14.c)$$

The solution of equation 8.14.c will be taken as:

$$\bar{Y} = \bar{Y}_p + V_o + \sum_{i=1}^{\infty} \beta^i V_i \qquad (8.15)$$

where $\bar{Y}_p$ is a particular solution to the inhomogeneous equation, $\bar{Y}_p = -(\bar{Y}_o + \theta_o\bar{x})$. The V_i's represent a solution to the homogeneous equation, and V_o is defined as a solution to the following differential equation:

$$\frac{d^4V_o}{d\bar{x}^4} + \bar{\alpha}^2\frac{d^2V_o}{d\bar{x}^2} - \lambda V_o = 0 \quad \text{where } \bar{\alpha}^2 \equiv \frac{\beta}{3} \equiv \frac{9L^3}{3EI} \qquad (8.16)$$

The boundary conditions which 8.15 must fulfill are as follows:

$$\left[\bar{Y}_p + V_o\right]_{\bar{x}=0} = \frac{d}{d\bar{x}}\left[\bar{Y}_p + V_o\right]_{\bar{x}=0} = \frac{d^2}{d\bar{x}^2}\left[\bar{Y}_p + V_o\right]_{\bar{x}=1} = \frac{d^3}{d\bar{x}^3}\left[\bar{Y}_p + V_o\right]_{\bar{x}=1} = 0$$

and

$$V_i\Big|_{\bar{x}=0} = V_i'\Big|_{\bar{x}=0} = V_i''\Big|_{\bar{x}=1} = V_i'''\Big|_{\bar{x}=1} = 0$$

The general solution to 8.16 is:

$$V_o = b_1\cosh g\bar{x} + b_2\sinh g\bar{x} + b_3\cos k\bar{x} + b_4\sin k\bar{x} \qquad (8.17)$$

where $g = \frac{1}{\sqrt{2}}\left(\sqrt{\bar{\alpha}^4 + 4\lambda} - \bar{\alpha}^2\right)^{\frac{1}{2}}$; $k = \frac{1}{\sqrt{2}}\left(\sqrt{\bar{\alpha}^4 + 4\lambda} + \bar{\alpha}^2\right)^{\frac{1}{2}}$

By substituting 8.17 in the first boundary conditions above, it is seen that:

$$b_1 = \frac{1}{b_{(\bar{\alpha},\lambda)}}\Big[k^2\bar{Y}_o(k^2 + g^2\cosh g\cos k + kg\sinh g\sin k) + kg\theta_o(g\cosh g\sin k - k\sinh g\cos k)\Big].$$

$$b_2 = \frac{1}{gb_{(\bar{\alpha},\lambda)}}\Big[-\lambda\bar{Y}_o(k\cosh g\sin k + g\sinh g\cos k) + \theta_o(k^4 + \lambda\cosh g\cos k - kg^3\sinh g\sin k)\Big]$$

$$b_3 = \bar{Y}_o - b_1 \; ; \quad b_4 = \frac{1}{k}(\theta_o - gb_2).$$

$$b_{(\bar{\alpha},\lambda)} \equiv \left[k^4 + g^4 + 2k^2g^2\cosh g\cos k + kg(k^2 - g^2)\sinh g\sin k\right].$$

$$\equiv \left[\bar{\alpha}^4 + 2\lambda + 2\lambda\cosh g\cos k + \bar{\alpha}^2 kg\sinh g\sin k\right].$$

Now, substituting the assumed solution, 8.15, into 8.14.c, the equation to be solved, and making use of the fact that $\bar{Y}_p$ is a particular solution to 8.14.c and V_o is a solution to 8.16, it is seen that:

$$\sum \beta^i V_i'''' - \lambda \sum \beta^i V_i = -\beta \frac{d}{d\bar{x}}\left[\left(\frac{2}{3} - \bar{x}\right) V_o\right] - \beta \frac{d}{d\bar{x}}\left[(1-\bar{x}) \sum \beta^i V_i'\right] \quad (8.18)$$

The recurrence formulae for V_i are obtained from this equation by equating the coefficients of like powers of β. Thus, V_i is determined by solving the inhomogeneous differential equation 8.18.a, below, and fitting the solution to the boundary conditions stated on the previous page. "V_2" is determined by solving equation 8.18.b, and the general term V_i is determined by solving equation 8.18.c.

$$V_1'''' - \lambda V_1 = -\frac{d}{d\bar{x}}\left[\left(\frac{2}{3} - \bar{x}\right) V_o'\right] \quad (8.18.a)$$

$$V_2'''' - \lambda V_2 = -\frac{d}{d\bar{x}}\left[(1-\bar{x})\, V_1'\right] \quad (8.18.b)$$

$$V_i'''' - \lambda V_i = -\frac{d}{d\bar{x}}\left[(1-\bar{x})\, V_{i-1}'\right] \quad (8.18.c)$$

The other needed elastic curve properties are:

$$\bar{y} = \bar{Y} \cos \omega t = \left[Y_p + V_o + \sum_{i=1}^{\infty} \beta^i V_i\right] \cos \omega t \quad (8.15.a)$$

$$-\frac{ML}{EI} = \frac{d^2\bar{y}}{d\bar{x}^2} = \left[V_o'' + \sum_{i=1}^{\infty} \beta^i V_i''\right] \cos \omega t \quad (8.15.b)$$

$$-\frac{QL^2}{EI} = \frac{d^3\bar{y}}{d\bar{x}^3} = \left[V_o''' + \sum_{i=1}^{\infty} \beta^i V_i'''\right] \cos \omega t \quad (8.15.c)$$

The method outlined above can be used to study the properties of the elastic curve of a heavy vertical cantilever which is vibrating at any assumed frequency. By using enough terms of the series which serves as a solution, any desired accuracy can be obtained.

However, since the first term of the series very closely approximates the complete solution, this term plus the first correction is all that is usually needed for practical problems, i.e.,

$$\bar{y} = (\bar{Y}_p + V_0 + \beta V_1)\cos\omega t \qquad (8.15.d)$$

$$M = -\frac{EI}{L}(V_0'' + \beta V_1'')\cos\omega t \qquad (8.15.e)$$

$$Q = -\frac{EI}{L^2}(V_0''' + \beta V_1''')\cos\omega t \qquad (8.15.f)$$

Thus, it is worth-while to present an expression for V_1, the solution to the inhomogeneous equation 8.18.a.

$$V_1 = b_1'\cosh m\bar{x} + b_2'\sinh m\bar{x} + b_3'\cos m\bar{x} + b_4'\sin m\bar{x}$$
$$+ b_{11}\cosh g\bar{x} + b_{12}\bar{x}\cosh g\bar{x} + b_{21}\sinh g\bar{x} + b_{22}\bar{x}\cdot\sinh g\bar{x}$$
$$+ b_{31}\cos k\bar{x} + b_{32}\bar{x}\cos k\bar{x} + b_{41}\sin k\bar{x} + b_{42}\bar{x}\sin k\bar{x} \qquad (8.19)$$

Where: $b_1' \cdots b_4'$ **are integration constants to be determined from the boundary conditions,**

$$V_1\Big|_{\bar{x}=0} = V_1'\Big|_{\bar{x}=0} = V_1''\Big|_{\bar{x}=1} = V_1'''\Big|_{\bar{x}=1} = 0$$

$$m = \lambda^{\frac{1}{4}};\quad g = \frac{1}{\sqrt{2}}\left(\sqrt{\alpha^4 + 4\lambda} - \alpha^2\right)^{\frac{1}{2}};\quad k = \frac{1}{\sqrt{2}}\left(\sqrt{\alpha^4 + 4\lambda} + \alpha^2\right)^{\frac{1}{2}}$$

$$b_{11} = \frac{g}{g^4-\lambda}\left(b_2 - \frac{2}{3}b_1 g - \frac{4b_2 g^4}{g^4-\lambda}\right);\quad b_{12} = \frac{b_1 g^2}{g^4-\lambda}$$

$$b_{21} = \frac{g}{g^4-\lambda}\left(b_1 - \frac{2}{3}b_2 g - \frac{4b_1 g^4}{g^4-\lambda}\right);\quad b_{22} = \frac{b_2 g^2}{g^4-\lambda}$$

$$b_{31} = \frac{k}{k^4-\lambda}\left(b_4 + \frac{2}{3}b_3 k - \frac{4b_4 k^4}{k^4-\lambda}\right);\quad b_{32} = \frac{-b_3 k^2}{k^4-\lambda}$$

$$b_{41} = \frac{k}{k^4-\lambda}\left(-b_3 + \frac{2}{3}b_4 k + \frac{4b_3 k^4}{k^4-\lambda}\right);\quad b_{42} = \frac{-b_4 k^2}{k^4-\lambda}$$

and $b_1 \cdots b_4$ **are the integration constants evaluated on page 111.**

Detailed instructions for the use of the formulae developed in this section are contained in the discussion at the end of this chapter.

Dynamic Analysis of Vertical Member with End Loads and a Rotating and Translating Base

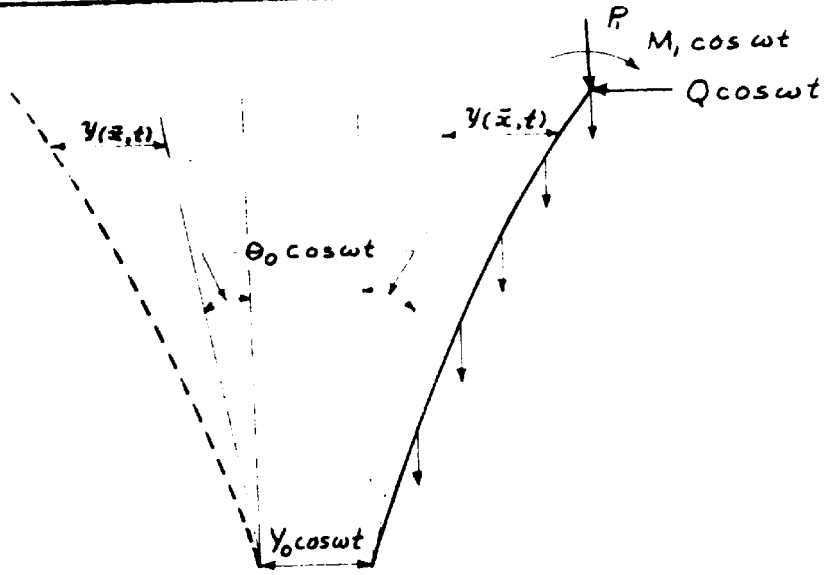


Fig. 8.3
Vibration of Heavy Vertical Member with End Loads and a Rotating and Translating Base

As in the static case, it is desirable to have a solution available for a more general set of boundary conditions than those presented in the previous section. The boundary conditions indicated above are the type that would be encountered in analyzing a tower whose cantilevered section contained an abrupt change in cross section.

In a manner analagous to that employed in the previous section, one may show that the governing partial differential equation, in dimensionless terms, is as follows:

$$\frac{\partial^4 \bar{y}}{\partial \bar{x}^4} + \beta \frac{\partial}{\partial \bar{x}}\left[(\bar{P}_1 + 1 - \bar{x})\frac{\partial \bar{y}}{\partial \bar{x}}\right] = -\lambda \frac{\partial^2 \bar{y}}{\partial (\omega t)^2} + \left[\theta_0 (\beta + \lambda \bar{x}) + \lambda \bar{Y}_0\right] \cos \omega t \quad (8.20)$$

where $\bar{y} = \frac{y}{L}$; $\bar{x} = \frac{x}{L}$; $\bar{Y}_0 = \frac{Y_0}{L}$; $\beta = \frac{qL^3}{EI}$; $\lambda = \frac{\rho L^4 \omega^2}{EI} \equiv \frac{\beta L \omega^2}{g_0}$; $\bar{P}_1 = \frac{P_1}{qL}$

Separating variables as before, by assuming $\bar{y} = \bar{Y}(\bar{x}) \cos \omega t$, the following ordinary differential equation is obtained:

$$\frac{d^4 \bar{Y}}{d \bar{x}^4} + \beta \frac{d}{d \bar{x}}\left[(\bar{P}_1 + 1 - \bar{x})\frac{d \bar{Y}}{d \bar{x}}\right] - \lambda \bar{Y} = \lambda \bar{Y}_0 + (\beta + \lambda \bar{x})\theta_0 \quad (8.20.a)$$

Before proceeding with the solution of equation 8.20.a, it should be noted that when the external axial load, P_1, is large as compared with the weight of the member, i.e., $\bar{P}_1 >> 1$, the quantity $(\bar{P}_1 + 1 - \bar{x})$ changes very little as $\bar{x}$ varies over its range from 0 to 1. Therefore, no appreciable loss in accuracy occurs when the quantity is taken to be a constant equal to its average value. The homogeneous equation may then be written: $\frac{d^4 \bar{y}}{d \bar{x}^4} + \alpha^2 \frac{d^2 \bar{y}}{d \bar{x}^2} - \lambda \bar{Y} = 0$ (8.21)

where $\alpha^2 \equiv \beta(\bar{P}_1 + \frac{1}{2}) \equiv \frac{(P_1 + \frac{1}{2} qL) L^2}{EI}$

In other words, the dynamic analysis of a vertical member with a large external axial load is equivalent to the dynamic analysis of a beam column where the axial load includes one-half the weight of the member. This is the assumption made in Chapter 7 concerning the analysis of tower sections below the level of the top guy connections, and the discussion above serves to justify this assumption.

When $\bar{p}_1$ is not large compared with 1.0, equation 8.21 may not serve as a sufficiently accurate substitution for the left side of equation 8.20.a. It is proposed here to solve the homogeneous portion of 8.20.a by a series, with the solution to 8.21 serving as

the first term and with subsequent terms serving as corrections to the first term.

The solution of equation 8.20.a will be assumed in the following form:

$$\bar{Y} = \bar{Y}_p + V_0 + \sum_{i=1}^{\infty} \beta^i V_i \qquad (8.22)$$

where $\bar{Y}_p$ is a particular solution to the inhomogeneous equation 8.20.a, $\bar{Y}_p = -(\theta_0 \bar{x} + \bar{Y}_0)$, and $V_0 + \sum_{i=1}^{\infty} \beta^i V_i$ is the solution to the homogeneous equation. The boundary conditions, to which 8.22 is fitted, are:

$$(\bar{Y}_p + V_0)\Big|_{\bar{x}=0} = \frac{d}{d\bar{x}}(\bar{Y}_p + V_0)\Big|_{\bar{x}=0} = 0 \; ; \; \frac{d^2}{d\bar{x}^2}(\bar{Y}_p + V_0)\Big|_{\bar{x}=1} = \frac{M_1 L}{EI} \equiv \bar{m}_1 \; ;$$

$$\frac{d^3}{d\bar{x}^3}(\bar{Y}_p + V_0)\Big|_{\bar{x}=1} = \frac{Q_1 L^2}{EI} = \bar{Q}_1 \; ; \; V_i\Big|_{\bar{x}=0} = V_i'\Big|_{\bar{x}=0} = V_i''\Big|_{\bar{x}=1} = V_i'''\Big|_{\bar{x}=1} = 0.$$

As indicated above, V_0 is the solution of the equation below:

$$V_0'''' + \alpha^2 V_0 - \lambda V_0 = 0 \qquad (8.21.a)$$

$$V_0 = d_1 \cosh g\bar{x} + d_2 \sinh g\bar{x} + d_3 \cos k\bar{x} + d_4 \sin k\bar{x} \qquad (8.23)$$

$$g = \frac{1}{\sqrt{2}}\left(\sqrt{\alpha^4 + 4\lambda} - \alpha^2\right)^{\frac{1}{2}} \; ; \; k = \frac{1}{\sqrt{2}}\left(\sqrt{\alpha^4 + 4\lambda} + \alpha^2\right)^{\frac{1}{2}}$$

d_1, d_2, d_3, and d_4 are determined by making use of the first set of boundary conditions above, and their values are:

$$d_1 = \frac{1}{d_{(\alpha,\lambda)}}\Big[\bar{Y}_0 k^4 + \bar{m}_1 \cos k - \bar{Q}_1 \sin k + g^2(\bar{m}_1 + \bar{Y}_0 k^2 \cos k + \theta_0 k \sin k)\cosh g$$

$$- g(\bar{Q}_1 - \bar{Y}_0 k^3 \sin k + \theta_0 k^2 \cos k)\sinh g\Big].$$

$$d_2 = \frac{1}{g\, d_{(\alpha,\lambda)}}\Big[\theta_0 k^4 + \bar{m}_1 k^3 \sin k + \bar{Q}_1 k^2 \cos k - g^3(\bar{m}_1 + \bar{Y}_0 k^2 \cos k + \theta_0 k \sin k)\sinh g$$

$$+ g^2(\bar{Q}_1 - \bar{Y}_0 k^3 \sin k + \theta_0 k^2 \cos k)\cosh g\Big].$$

$$d_3 = \bar{Y}_0 - d_1 \quad ; \quad d_4 = \frac{1}{k}(\theta_0 - g d_2).$$

$$d_{(\alpha,\lambda)} = \left[\alpha^4 + 2\lambda + 2\lambda \cosh g \cos k + \alpha^2 g k \sinh g \sin k\right].$$

Substitution of the proposed solution, $\bar{Y} = \bar{Y}_p + V_0 + \sum_{i=1}^{\infty} \beta^i V_i$ into 8.20.a yields equation 8.24 below when use is made of the facts that $\bar{Y}_p$ is a particular solution and $V_0'''' + \beta(\bar{P}_1 + \frac{1}{2})V_0'' - \lambda V_0 = 0$.

$$\sum_{i=1}^{\infty} \beta^i V_i'''' - \lambda \sum_{i=1}^{\infty} \beta^i V_i = -\beta \frac{d}{d\bar{x}}\left[\left(\frac{1}{2} - \bar{x}\right) V_0'\right] - \beta \frac{d}{d\bar{x}}\left[(\bar{P}_1 + 1 - \bar{x}) \sum_{i=1}^{\infty} \beta^i V_i'\right] \quad (8.24)$$

As in the previous two sections, the recurrence relations for V_i are obtained by equating coefficients of like powers of β in 8.24. The results are as follows:

$$V_1'''' - \lambda V_1 = -\frac{d}{d\bar{x}}\left[\left(\frac{1}{2} - \bar{x}\right) V_0'\right] \quad (8.24.a)$$

$$V_2'''' - \lambda V_2 = -\frac{d}{d\bar{x}}\left[(\bar{P}_1 + 1 - \bar{x}) V_1'\right] \quad (8.24.b)$$

$$V_i'''' - \lambda V_i = -\frac{d}{d\bar{x}}\left[(\bar{P}_1 + 1 - \bar{x}) V_{i-1}'\right] \quad (8.24.c)$$

Where the solutions for V_i are fitted to the conditions:

$$V_i\Big|_{\bar{x}=0} = V_i'\Big|_{\bar{x}=0} = V_i''\Big|_{\bar{x}=1} = V_i'''\Big|_{\bar{x}=1} = 0$$

As in the last section, the solution V_0 here is a good enough approximation to the final answer so that it is seldom necessary to add more than one correction. In other words, the following solutions are usually sufficient:

$$\bar{y} = (\bar{Y}_p + V_0 + \beta V_1) \cos \omega t \quad (8.22.a)$$

$$M = -\frac{EI}{L}(V_0'' + \beta V_1'') \cos \omega t \quad (8.22.b)$$

$$Q = -\frac{EI}{L^2}(V_0''' + \beta V_1''') \cos \omega t \quad (8.22.c)$$

Therefore, a solution for V_1 is needed in this section too. However, V_1 is obtained here in a manner identical to that employed in the previous section, so that the solution for V_1 here is very similar to the solution presented in the previous section, 8.19. The values of $b_{11}, b_{12}, \cdots b_{42}$ here are computed by replacing b_1, b_2, b_3, b_4 by d_1, d_2, d_3, d_4 above and 2/3 by 1/2. The constants of integration are evaluated in the same manner as before.

Illustrative Example

A use for the solutions presented in the last two sections will be illustrated here by discussing in detail their application in determining the moment and shear at the base of the two top sections in terms of the amplitudes of motion at that point.

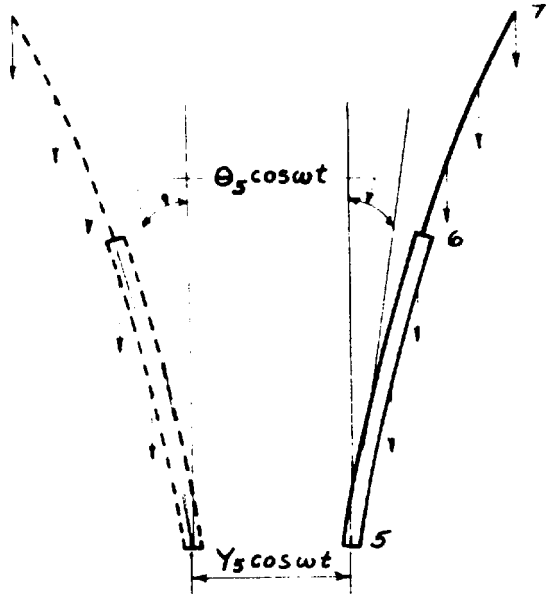


Fig. 8.4
Vibration of Vertical Cantilever
with Change in Cross-Section

In carrying through a dynamic analysis of the Oklahoma City tower, it would be necessary to express M_{56} and Q_{56} (Fig. 8.4) in terms of Y_5 and θ_5 for any assumed frequency of vibration, ω. If the effects of the distributed axial loads were to be taken into account, the formulas presented in the previous two sections would be used. As was mentioned previously, the two term series solutions 8.15.d, 8.15.e, 8.15.f, 8.22.a, 8.22.b, and 8.22.c will usually be sufficiently accurate.

The method for obtaining M_{56} and Q_{56} interms of Y_5 and θ_5, for an assumed value of ω, is very similar tothe method used for the corresponding static problem which is outlined on page 58.

The method is outlined as follows:

1. Obtainthe amplitude of M_{67} and Q_{67} in terms of θ_6 and Y_6 from $M_{67} = -\frac{EI}{L}(V_o'' + \beta V_1'')\Big|_{\bar{x}=0}$ and $Q_{67} = -\frac{EI}{L^2}(V_o''' + \beta V_1''')\Big|_{\bar{x}=0}$ where V_o is given by 8.17 and V_1 by 8.19.

2. By use of equation 8.23 for V_o and the modification of equation 8.19 for V_1, express elastic curve equations for span 5-6 in terms of Y_5, θ_5, Y_6, and θ_6. The results of step 1 are used to eliminate Q_{65} and M_{65} from these equations.

3. Eliminate Y_6 and θ_6 fromthe equations in step 2 by making use of the two following expressions:

$$Y_6 = L_{56}(V_o + \beta V_1)\Big|_{\bar{x}=1}$$
$$\theta_6 = (V_o' + \beta V_1')\Big|_{\bar{x}=1}$$

4. The simplified equations obtained in step 3 can then be used to get the following desired quantities in terms of Y_5 and θ_5:

$$M_{56} = -\frac{EI}{L}(V_o'' + \beta V_1'')\Big|_{\bar{x}=0}$$
$$Q_{56} = -\frac{EI}{L^2}(V_o''' + \beta V_1''')\Big|_{\bar{x}=0}$$

The significance of a single result will not warrant the numerical illustration of the above steps here.

CHAPTER IX

SUMMARY OF PROCEDURE FOR COMPUTING NATURAL FREQUENCIES OF GUYED TOWER SYSTEM

In Chapters 2, 3, and 4 of the static analysis the various components of a guyed tower were analyzed separately and the work in Chapter 5 served to demonstrate the procedure to be used in utilizing the previous developments for a complete static analysis. The purpose of the work in Chapter 9 is analagous to that of Chapter 5 in that here a procedure will be outlined for utilizing the dynamic analyses of Chapters 6, 7, and 8 for the computation of the natural frequencies of a system made up of a very tall flexible tower and its guys.

The work in Chapter 6 was devoted to the development and illustration of formulas which can be used to calculate $\overline{f}_n(y_n)$ and $\overline{G}_n(y_n)$ the dynamic net horizontal force and the external moment that the guys exert at the n^{th} support level. From the illustrative examples worked out in Chapter 6, it may be seen that $\overline{f}_n(y_n)$ and $\overline{G}_n(y_n)$ appear as coefficients of Y_n and are functions of ω, the circular frequency of vibration of the system.

In Chapter 7 formulas, of a slope deflection form, were presented which expressed the amplitudes of the joint moments and shears in terms of the amplitudes of the joint rotations and deflections, e.g.,

$$M_{12} = \frac{EI}{L}\left[C_1\theta_1 + C_2\theta_2 + \frac{1}{L}\left(-C_3 Y_1 + C_4 Y_2\right)\right] \qquad \text{and}$$

$$Q_{12} = \frac{EI}{L^2}\left[C_5\theta_1 + C_6\theta_2 + \frac{1}{L}\left(C_6 Y_1 - C_7 Y_2\right)\right]$$

In these expressions the coefficients $C_1 \ldots C_7$ are functions of the

physical constants of the span, including the axial loads, and ω . So, for any known or assumed value of ω it would appear that one could substitute the above expressions into equilibrium equations, e.g., $M_{10} + M_{12} + \bar{G}_1(y_1) = 0$ and $Q_{10} - Q_{12} + \bar{F}_1(y_1) = 0$, as in Chapter 5 and solve for the unknown θ's and Y's.

The formulas which were developed and illustrated in Chapter 8 are used to obtain the moment and shear at the base of the cantilever section. For example, if only the first corrective term is used to account for the distributed axial load, $M_{45} = \frac{-EI}{L}(V_0'' + \beta V_1'')$ and $Q_{45} = \frac{-EI}{L^2}(V_0''' + \beta V_1''')$. Both the moment and the shear at the base of the cantilever appear as coefficients of the amplitude of translation and rotation at the base, e.g., Y_4 and θ_4, and may be substituted in the equilibrium equation above in an attempt to solve for the unknown slopes and deflections.

The procedure, as outlined thus far, is similar to the procedure followed in Chapter 5. That is, the slope deflection coefficients are calculated in such a way as to account for distributed and concentrated axial loads and the resistance of the guys appears in the equilibrium equations. However, there are two important differences: 1) Here all the slope deflection coefficients contain an additional factor which is some function of ω , and 2) none of the equations resulting from substitution in the equilibrium equations have any constant terms, that is, they are entirely made up of the unknowns and coefficients of the unknowns. Their form would be as follows for the tower analyzed in Chapter 5.

Table 9.1

Equilibrium Equations for Vibrating Tower

Equation	θ_1	θ_2	θ_3	θ_4	Y_1	Y_2	Y_3	Y_4	C
I	λ_{11}	λ_{12}	0	0	λ_{15}	λ_{16}	0	0 =	0
II	λ_{21}	λ_{22}	λ_{23}	0	λ_{25}	λ_{26}	λ_{27}	0 =	0
III	0	λ_{32}	λ_{33}	λ_{34}	0	λ_{36}	λ_{37}	λ_{38} =	0
IV	0	0	λ_{43}	λ_{44}	0	0	λ_{47}	λ_{48} =	0
V	λ_{51}	λ_{52}	0	0	λ_{55}	λ_{56}	0	0 =	0
VI	λ_{61}	λ_{62}	λ_{63}	0	λ_{65}	λ_{66}	λ_{67}	0 =	0
VII	0	λ_{72}	λ_{73}	λ_{74}	0	λ_{76}	λ_{77}	λ_{78} =	0
VIII	0	0	λ_{83}	λ_{84}	0	0	λ_{87}	λ_{88} =	0

Therefore, the equations have a trivial solution, i.e., $\theta_1 = \theta_2 \ldots = Y_3 = Y_4 = 0$, unless the determinate of the coefficients is equal to zero. The values of the quantities $\lambda_{11}, \lambda_{12} \cdots \cdots \cdots \lambda_{88}$ depend on the physical properties of the tower, i.e., stiffness, axial loads, etc., and ω, the circular frequency of vibration of the tower. So, for a given set of physical properties of the tower system, the values of ω which make the determinate of the coefficients of the unknown slopes and deflections, $|\lambda_{ij}|$, zero are the natural frequencies of vibration for the tower system. This can be illustrated as follows:

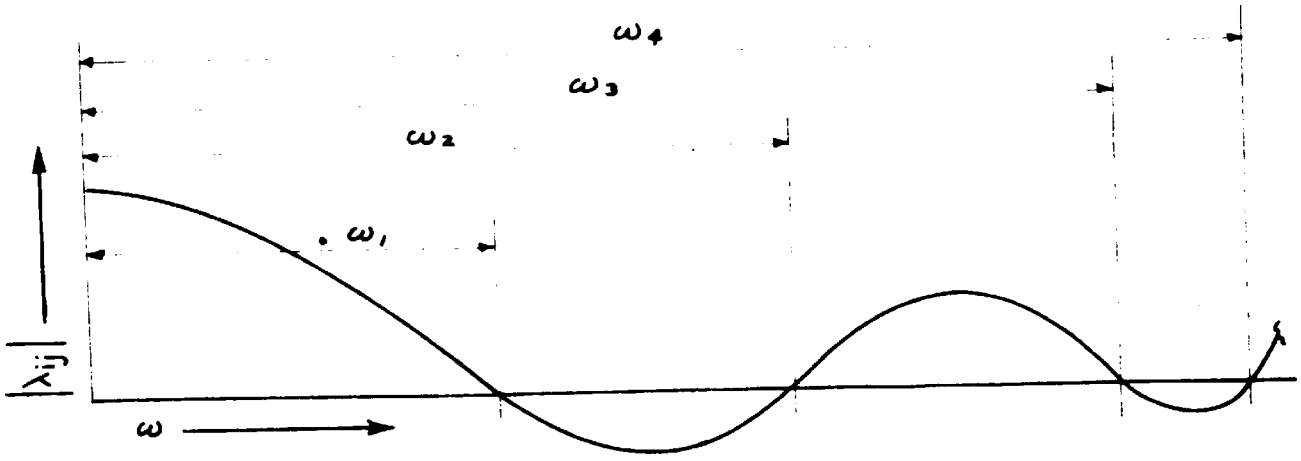


Fig. 9.1
Plot of Magnitude of Determinate of
Coefficients of Unknown Slopes and Deflections Versus ω

$\omega_1, \omega_2, \cdots \omega_n$ are the natural frequencies of the system.

In other words, the natural frequencies of a tower system are obtained as follows:

1) Determine slope deflection coefficients in terms of ω and physical properties of tower by use of formulas presented in Chapters 7 and 8.

2) Determine guy forces and the moments they cause in terms of ω by use of expressions presented in Chapter 6.

3) Substitute in the equilibrium equations 7.11 and 7.12, i.e.,

$$M_{n,n-1} + M_{n,n+1} + \bar{G}_n(\gamma_n) = 0 \text{ and } Q_{n,n-1} - Q_{n,n+1} + \bar{f}_n(\gamma_n) = 0$$

4) Plot values of the coefficient-determinate of the equations obtained in step 3 with variations in ω. The roots of the resulting curve are the natural frequencies of the system.

Numerical examples of the procedure outlined in step 4 are available in the literature, e.g., reference (11), so it will not

be necessary to numerically illustrate the process here.

It is felt that the above method is too lengthy for normal design office use except by firms which have available automatic computing devices. Since it will probably be some time before a large percentage of the design offices are in a position to avail themselves of such equipment, it is the writer's opinion that further research is required. The goal of this research should be the simplification of the above method to a procedure which is satisfactory for normal design office use.

CHAPTER X

CONCLUSIONS

In the preceding work, methods and formulas were presented which can be used for the static and dynamic analysis of very tall flexible guyed towers. The methods presented in the portion devoted to the static analysis (Chapters 2 through 5) were complete and suitable for design office use. However, the methods presented in the portion devoted to the dynamic analysis (Chapters 6 through 8) were carried only far enough so that the natural frequencies of the tower system could be computed and some of the methods appear to be too lengthy for normal design office practice.

In Chapter 2, a complete set of working formulae was presented for use in calculating any desired quantity associated with hanging cables which are functioning as guys. In addition, convenient equations were developed for use in computing the various effects of a change in one or more of the cable variables. While the purpose of developing these equations was for use in the thorough analysis of guyed towers, it is felt that they constitute an addition to the published literature on cables in general and that they would be useful in the analysis of other structures such as suspension bridges with unloaded back stays.

In Chapters 3 and 4, formulae were developed which furnish complete information on the action of beam columns with general end conditions and transverse loading for both distributed and

concentrated axial loads. The formulas are intended for use in carrying out a rigorous investigation of a guyed tower, but their form is such that the governing parameters and their effects are evident, and a few brief calculations will demonstrate the validity of a simplified investigation.

The use of the formulas presented in Chapters 2, 3, and 4 for the complete static analysis of a guyed tower was discussed in Chapter 5. By making use of the results of an approximate solution, it was possible to carry through the accurate analysis, which accounted for the non-linear resistance-motion relations of the guy supports and variations in the axial loads by solving simple linear slope deflection equations.

In Chapter 6, equations were derived which can be conveniently used to obtain the dynamic resistance a group of guys offers to tower motion at a support point. These equations can be used, together with the slope deflection equations for vibrating beam-columns developed in Chapter 7, for the dynamic analysis of a guyed continuous beam-column such as an antenna tower that is guyed at the top. The work in Chapter 8 extends the above equations so that they can be used to find the natural frequencies of a guyed tower with a top cantilever section. Here the effect of the distributed axial loads, due to dead weight and ice, were dealt with by use of a perturbation theory so that the result appears as a series of functions multiplied by ascending powers of a dimensionless parameter which relates the

axial load to the member's stiffness.

A method of using the equations developed in Chapters 6, 7, and 8 for finding the natural frequencies of vibration of the tower system, including the guys, is presented in Chapter 9. It consists of finding the roots of a $(2n)^{th}$ order determinate where "n" is the number of points that are supported by guys. The method is lengthy and will have to be simplified before it can be classified as satisfactory for normal design office use.

APPENDIX A

THE END SLOPES OF GUY CABLES

In order to illustrate the type of term rearrangement that is frequently necessary in the development of working formulae of the type presented in Chapter 2, some of the details of the derivation of equations 2.6.a and 2.7.a (page 10) are presented below:

I. The slope of the guy cable at its lower extremity is desired. This quantity, (Fig. 2.2, page 9), can be obtained from equation 2.1, repeated here, as follows:

$$\frac{dy}{dx} = \sinh\left(\frac{qx}{H} + a_1\right) \quad \text{where} \quad a_1 = \sinh^{-1}\left[\frac{qh}{2H\sinh r}\right] - r \qquad (2.1)$$

and $\quad r \equiv \frac{qL}{2H}\,; \quad \left.\frac{dy}{dx}\right|_{x=0} = \tan\gamma_0 = \sinh a_1$

by direct substitution of the identities

$$\sinh(\theta_1 - \theta_2) = \sinh\theta_1 \cosh\theta_2 - \sinh\theta_2 \cosh\theta_1,$$

and $\quad \cosh\theta_1 = \sqrt{1 + \sinh^2\theta_1}$ $\quad$ one obtains

$$\sinh a_1 = \frac{qh\cosh r}{2H\sinh r} - \sinh r\sqrt{1 + \left(\frac{qh}{2H\sinh r}\right)^2} = \frac{q}{2H}\left[h\coth r - \sqrt{\frac{4H^2}{q^2}\sinh^2 r + h^2}\right]$$

The second term in the brackets coincides with the exact expression for "L" presented on page 10. Making this substitution, one obtains the equation 2.6.a below.

$$\tan\gamma_0 = \sinh a_1 = \frac{q}{2H}\left[h\coth r - L\right] \qquad (2.6.a)$$

II. The slope of the guy cable at the tower, $\tan\gamma_1$, is desired for the computation of the axial loads in the tower. This quantity

is also obtained from 2.1.

$$\left.\frac{dy}{dx}\right|_{x=\ell} = \tan\gamma_1 = \sinh\left(\frac{q\ell}{H} + a_1\right) \equiv \sinh(2r + a_1).$$

Substitution of the expression shown on the previous page for a_1 makes it immediately apparent that the only difference between the development here and the one on the previous page is that here the "sinh" of the sum of two quantities is desired and there the "sinh" of the difference of the same two quantities was required. Thus, the only difference in the results will be in the sign between the two terms, or,

$$\tan\gamma_1 = \sinh(2r + a_1) = \frac{q}{2H}\left[h\coth r + L\right] \qquad (2.7.a)$$

APPENDIX B

EVALUATION OF THE DEFINITE INTEGRAL INVOLVED IN DYNAMIC CABLE ANALYSIS

As a result of applying the W-K-B method to the solution of the vibrating cable with varying tension, on page 88 of Chapter 6, it was necessary to evaluate, $\int_0^{\bar{x}} \sqrt{\cosh(2r\bar{x}+a_1)}\, d\bar{x}$, as part of the argument of the sine function. The details of the evaluation of this integral, denoted $D_{(\bar{x})}$ in Chapter 6, are shown below:

$$D_{(x)} = \int_0^{\bar{x}} \sqrt{\cosh(2r\bar{x}+a_1)}\, d\bar{x}$$

where $r \equiv \frac{9l}{2H}$; $a_1 \equiv \sinh^{-1}\left(\frac{9h}{2H\sinh r}\right) - r$ and $\bar{x}$ varies from 0 to 1

let $\cosh(2r\bar{x}+a_1) = z^2+1$; $z\Big|_{\bar{x}=0} = z_0$; $z\Big|_{\bar{x}} = z_{(\bar{x})}$.

$$\text{then } d\bar{x} = \frac{dz}{r\sqrt{z^2+2}}$$

$$\text{and } D_{(\bar{x})} = \frac{1}{r}\int_{z_0}^{z_{\bar{x}}} \frac{\sqrt{z^2+1}}{\sqrt{z^2+2}}\, dz = \frac{1}{\sqrt{2}\, r}\int_{z_0}^{z_{\bar{x}}} \frac{1+\frac{1}{2}z^2-\frac{1}{8}z^4+\frac{1}{16}z^6-\frac{5}{8\cdot16}z^8+\frac{21}{32\cdot24}z^{10}-\cdots}{1+\frac{1}{4}z^2-\frac{1}{32}z^4+\frac{1}{8\cdot16}z^6-\frac{5z^8}{8\cdot16\cdot16}+\frac{21z^{10}}{32\cdot32\cdot24}-\cdots}\, dz$$

$$\text{or } D_{(\bar{x})} = \frac{1}{\sqrt{2}\, r}\int_{z_0}^{z_{\bar{x}}} \left(1+\frac{1}{4}z^2-\frac{5}{32}z^4+\frac{13}{8\cdot16}z^6-\frac{141}{8\cdot16\cdot16}z^8+\frac{1197z^{10}}{32\cdot32\cdot24}-\cdots\right)dz$$

$$= \frac{1}{\sqrt{2}\, r}\, z\left(1+\frac{z^2}{12}-\frac{z^4}{32}+\frac{13z^6}{16\cdot56}-\frac{47z^8}{16\cdot16\cdot24}+\frac{399z^{10}}{32\cdot32\cdot88}-\cdots\right)\Bigg|_{z_0}^{z_{(\bar{x})}}$$

where $z_0 = \sqrt{\cosh a_1 - 1}$; $z_{(\bar{x})} = \sqrt{\cosh(2r\bar{x}+a_1)-1}$

A quantity that is frequently needed is $D_{(\bar{x})}$ evaluated at $\bar{x} = 1$, or $D_{(1)}$.

$$D_{(1)} = \frac{1}{\sqrt{2r}} z\left(1 + \frac{z^2}{12} - \frac{z^4}{32} + \frac{13z^6}{16\cdot 56} - \frac{47z^8}{16\cdot 16\cdot 24} + \frac{399z^{10}}{32\cdot 32\cdot 88} - \cdots\right)\Bigg|_{z_{(0)}}^{z_{(1)}}$$

where $$z_{(1)} = z\Big|_{\bar{x}=1} = \sqrt{\cosh(2r + a_1) - 1}$$

BIBLIOGRAPHY

1. The Computation Laboratory of the National Bureau of Standards, Tables of Bessel Functions of Fractional Order, Columbia University Press, New York, 1948.

2. Bieberbach, L. Theorie Der Differentialgleichungen, Dover Publications, New York, 1944.

3. Collatz, L. Eigenwert Probleme, Chelsea Publishing Co., New York, 1948.

4. Forsyth, A. R. A Treatise on Differential Equations, Macmillan and Co., London, 1885.

5. Hansen, H. M., and Chenea, P. F. Mechanics of Vibration, John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1952.

6. Harding, J. P. "The Reconstruction of Ten 305 Foot Tubular Steel Radio Masts in Reinforced Concrete", Institute of Civil Engineers Journal, Vol. 27, No. 2, p. 113-179, 1946.

7. Ince, E. L. Ordinary Differential Equations, Dover Publications, New York, 1944.

8. Jahnke, E., and Emde, F. Tables of Functions with Formulae and Curves, Dover Publications, New York, 1943.

9. Johnson J., Bryan C., and Turneaure, F. Modern Framed Structures, John Wiley and Sons, New York, Chapman and Hall Ltd., London, 1911.

10. Kamke, E. Differentialgleichungen Losungsmethoden und Losungen, Chelsea Publishing Co., New York, 1948.

11. Kolousek, V. "Solution Statique et Dynamique des Pylones D'Antenne Haubanes", International Association for Bridge and Structural Engineering, Vol. 8, p. 105-140, 1947.

12. McLachlan, N. W. Bessel Functions for Engineers, Oxford University Press, London, 1946.

13. Maugh, L. C. Statically Indeterminate Structures, John Wiley and Sons, Inc., New York; Chapman and Hall Ltd., London, 1946.

14. Meyer, W., zur Capellen, "Die Theorie des Frahmschen Schwingungsdaueranzeigers", Annalen der Physik, Vol. 407, p. 1-28, 1932.

15. Meyer, W., zur Capellen, "Methode zur Angenaherten Losung von Eigenwertproblemen mit Anwendungen auf Schwingungsprobleme", Annalen der Physik, Vol. 400, p. 297-352, 1931.

16. Morris, C. T., and Carpenter, S. T. Structural Frameworks, John Wiley and Sons, New York; Chapman and Hall Ltd., London, 1943.

17. Rayleigh, J. W. S. Theory of Sound, Dover Publications, New York, 2d edition, 1945.

18. Salvadori, M. G., and Baron, M. L. Numerical Methods in Engineering, Prentice-Hall, Inc., New York, 1952.

19. Slater, J. C., and Frank, N. H. Mechanics, McGraw-Hill Book Co., New York, 1st edition, 1947.

20. Timoshenko, S. Theory of Elastic Stability, McGraw-Hill Book Co., New York and London, 1936.

21. Timoshenko, S. Vibration Problems in Engineering, D. Van Nostrand Co. Inc., New York and London, 1953.

22. Veletsos, A. S., and Newmark, N. M., "A Method for Calculating the Natural Frequencies of Continuous Beams, Frames and Certain Types of Plates", Technical Report to Office of Naval Research, University of Illinois, Urbana, Illinois, June, 1953.

23. Walmsley, T. "Length, Tension, and Sag of Stay Ropes", Institute of Civil Engineering, selected engineering paper No. 19, 1924.

24. Watson, G. N. Theory of Bessel Functions, University Press, Cambridge; Macmillan Co., New York, 1945.

25. Wylie, C. R., Jr. Advanced Engineering Mathematics, McGraw-Hill Book Co., New York and London, 1951.