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TRANSIENT STRESSES IN A LAYERED THERMOELASTIC MEDIA GENERATED BY IMPULSIVE ENERGY DEPOSITION

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A B S T R A C T

TRANSIENT STRESSES IN A LAYERED THERMOELASTIC MEDIA GENERATED BY IMPULSIVE ENERGY DEPOSITION

by

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When high intensity electromagnetic energy impinges on an elastic media, energy is deposited internally increasing the mean kinetic energy of the microscopic particles of the solid. The transient stresses generated by such deposition in an initially undisturbed layered media is investigated. The internal energy deposition is considered to be equivalent to an internal heat generation varying exponentially with the distance from the exposed surface. Energy is deposited internally according to a different law in each layer. The energy deposition is assumed to occur in a time negligible in comparison to a characteristic mechanical response time.

It is assumed that the lateral dimensions of the layered media are large compared to the thickness and the deposition is uniform over the surface. Thus only spatial variations through the thickness are considered.

The equations of linear uncoupled thermoelasticity, assuming constant material properties, are applied to the layered media. A boundary value problem is formulated neglecting heat conduction because the thermal response time is large compared to the mechanical response time.

A Laplace transform with respect to time is used to reduce the governing partial differential equations to ordinary differential equations. The transform problem is solved and inverted exactly for all time. Only small time behavior is valid due to neglecting heat conduction.

Numerical examples are presented for a half-space, a single homogeneous slab (both special cases of the more general solution), and for several cases of a layered slab showing the effect of specific energy deposition functions, material thickness ratio and material parameters. A complete description of interior stress reflection is presented. The influence of energy deposition and other material parameters on the bond stress is shown in the examples considering two-layer slabs. Relatively small changes in layer parameters may have a marked effect on the maximum stress depending on the manner in which the generated stress waves combine.

To examine quantitatively the effect of heat conduction, the problem of a thermally conducting homogeneous slab subject to energy deposition is formulated and solved. For thermal conductivity values of common engineering materials, the effect of heat conduction is negligible in determining the initial transient stresses.

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NOMENCLATURE

$A, B, \cdots\cdots H,$ $I_1, \cdots\cdots I_4$	Coefficients in solutions of differential equations
E_0, E_1, E_2	Total energy distributions per unit volume
$\vec{F}$	Body force vector
$H(\ )$	Unit step function
K	Nondimensional ratio, $\frac{(3\lambda+2\mu)_2\alpha_2}{(3\lambda+2\mu)_1\alpha_1}\frac{E_2/(\rho S)_2}{E_1/(\rho S)_1}$
P	Summation index
$Q(x,t)$	Heat generation per unit volume per unit time
R_1, R_2	Coefficients in solution of differential equation
S	Specific heat at constant strain
Z_1, Z_2, Z_3	Functions of specific acoustic impedance ratio
Z	Function of $Z_1, Z_2, Z_3,$ material parameters and transform parameter S
a	Thickness of material layer
b	Total thickness of two-layer media
c_1, c_2	Dialatational wave speed in media 1 and 2, respectively
e_1, e_2	Energy deposition per unit time in media 1 and 2, respectively
$f(x,t)$	General function of space and time
$g_1(t), g_2(t)$	General functions of time
k	Thermal conductivity
i, j, ℓ, m, n, p	Summation indices
r	Wave speed ratio $\frac{c_1}{c_2}$
s	Laplace transform parameter
t	Time

NOMENCLATURE (Continued)

u, v, w	Cartesian components of displacement vector
x, y, z	Cartesian coordinates
$\mathcal{Z}_{21}$	Specific acoustic impedance ratio, $\mathcal{Z}_2/\mathcal{Z}_1$
α	Coefficient of linear thermal expansion
β	Thickness ratio, $\frac{b}{a}$
$\beta_0 \cdots\cdots \beta_3$	Functions of a, b, c_1 and c_2
γ	Energy absorption parameter
δ_{ij}	Knonecker delta
$\delta( )$	Dirac delta function
$\delta_{1p} \cdots\cdots \delta_{4p}$	Functions of χ and c arising in transform stress
e_{ij}	Components of strain tensor
λ, μ	Lamé elastic constants
κ	Thermal diffusivity
ρ_1, ρ_2	Mass density of media 1 and 2, respectively
σ_{ij}	Components of stress tensor
$\sigma_{\xi 1}, \sigma_{\xi 2}$	Nondimensional stresses
θ_0	Reference temperature
θ	Temperature excess above reference
ϕ	A defined function independent of the transform parameter
ξ	Nondimensional coordinate, $\frac{x}{a}$
τ	Nondimensional time, $\frac{c_1}{a} t$
ω	Nondimensional conduction parameter, $\frac{\kappa}{ac}$
$\Lambda_1, \Lambda_2, \Lambda_4$	Elasto-expansion constant
Λ_3	Elastic constant

NOMENCLATURE (Concluded)

Δ	Coefficient determinant
$\Delta_{1P}, \cdots \Delta_{4P}$	Dimensionless counterpart of $\delta_{1P}, \cdots \delta_{4P}$
Γ	Dimensionless absorption parameter
$\sum$	Summation symbol defined by Eqn.(3.56)
Φ	Dimensionless counterpart of ϕ

CHAPTER I

INTRODUCTION

Many problems in continuum mechanics require consideration of the interaction of the mechanical and thermodynamic states of the media. A small deformation theory accounting for this interaction is obtained when the theory of linear elasticity and irreversible thermodynamics are combined. The earliest systemic study of this interaction was recorded by Duhamel in 1837 [1]. He presented a theory for the effect of a nonuniform temperature distribution on the state of strain. The converse effect was recognized, but apparently no logical description could be advanced. Voigt [2], in 1910, appears to be the first to obtain coupled mechanical and thermal states. Boley and Weiner [3] furnish the most complete textbook to date on this subject, while Chadwick [4] provides an excellent summary of the research in the field until about 1958.

The technical importance of thermal and mechanical interaction increased with high speed flight vehicles of the 1940's. Nonuniform temperature distributions due to aerodynamic heating became significant. In general, most of the early work in thermal stress analysis was quasi-static, i.e. inertia effects are ignored. One of the first problems including inertia effects was considered by Danilovskaya [5] in 1950. She examined the problem of an initially undisturbed elastic half-space subjected to a step increase in temperature at the boundary. The thermomechanical coupling,

i.e. the effect of the state of mechanical strain on the temperature distribution, was neglected. Nariboli and Nyayadhish [6], Muki and Breuer [7], and Boley and Tolins [8] have re-examined the basic work of Danilovskaya. The latter include thermomechanical coupling for a step increase in strain at the boundary and a step increase in velocity at the boundary. To determine the inertia effects in slender bodies, Boley [9] investigated thermally induced vibrations in beams, and Boley and Barber [10] studied the dynamic response of homogeneous beams and plates subject to rapid surface heating.

Dillon [11] solved a class of thermoelastic problems for a particular value of the thermomechanical coupling parameter. Wilms [12] solved a coupled thermoelastic problem in which the surface of a spherical cavity in an infinite media is suddenly subjected to a step increase of normal pressure.

All of the above problems consider the effect of retaining the inertia in the mechanical equation. In addition the disturbing stress or temperature acts at a boundary of the body under consideration.

Although response of structural elements to effects applied at a boundary is a frequently encountered problem, there is a class of problems where the excitation is in effect imposed internally. The interaction of high energy electromagnetic radiation with an elastic solid is such an example. Most solids exhibit some opacity to electromagnetic radiation with energy being deposited in the media due

to scattering. In general this produces a spatially nonuniform heating with a resultant nonuniform temperature distribution. Because of the extremely short-time duration of deposition, elastic deformation waves are generated. In 1963 White [13] studied the effects of absorption of radiation created by high power, pulsed light sources. Another example of energy deposition is that generated by electromagnetic energy in the X-ray region.

The deposition of energy as a function of depth penetrated depends on the energy spectra of the radiation. The deposition (or absorption) law is an exponential decreasing function. In this study the absorption parameter is considered known. For a discussion of electromagnetic energy absorption see Guinier [14]. Zaker [15], in the study of elastic stress wave generation caused by absorption of energy at microwave frequencies, assumed for simplicity a simple exponential absorption law.

Energy is propagated into the media at velocities comparable to the speed of light, i.e. about 3×10^{10} cm/sec, with a time pulse width such that deposition is complete in times on the order of 10^{-8} sec. This input time is substantially less than the time for an elastic dialatation wave to travel through the medium a characteristic distance. For metallic structures, using a characteristic length of 1 cm, this typical mechanical response time is on the order of 10^{-6} sec.

Many modern aerospace structures are composed of plate

and shell-like sections formed as laminates with two or more layers of different materials. They are joined either by molecular (diffusion) bonding or by a bond with finite thickness and elasticity. Adhesion failure due to transient loading producing large interface bond stresses may be extremely important in causing structural failure.

Goland and Reissner [16] were early investigators in bonded joints. Payton [17,18] investigated dynamic bond stresses in composite shells and rods subjected to sudden application of external loading. Jones [19,20] considers steady state wave propagation in two finite thickness layers joined by an interface bond.

The purpose of the present work is to determine the transient mechanical response of an initially undisturbed composite media formed by two finite thickness plane layers with different material properties. These layers are joined by an interfacial molecular bond. The media is subjected to energy deposition uniformly over the lateral surface, the deposition occurring in a time small compared with a defined mechanical response time. It is assumed that the lateral dimensions of the layered structure are large compared to its thickness. Edge effects are neglected. Hence a one-dimensional representation is assumed accounting for variation through the thickness of the layered media. Considering the energy deposition as equivalent to a spatially varying internal heat source, the equations of linear thermoelasticity are applied to the layered

media. A boundary value problem is formulated neglecting thermomechanical coupling and heat conduction. Thermomechanical coupling is neglected because of its generally small effect, and heat conduction is neglected because of a large thermal response time compared with the mechanical response time. A Laplace transform with respect to time is used to reduce the governing linear partial differential equations to ordinary differential equations. The Laplace transform method was chosen because of the ease in which traveling discontinuities may be treated. The transform problem is solved for all values of time, with the realization that only small time behavior will generally be valid because heat conduction is neglected.

Numerical examples are presented for the limiting cases of a half-space and single homogeneous slab of finite thickness, as well as for several cases of a layered slab showing the effect of specific energy deposition functions, material thickness ratio and other material parameters. A complete description of interior stress reflection is presented, and the failure mechanism of surface spallation is shown to be possible. In the examples of layered slabs, the interfacial bond stress is calculated because of its importance in potential practical applications.

To examine the effect of heat conduction, the problem of a thermally conducting homogeneous slab subject to energy deposition is formulated and the solution is given in an Appendix. For thermal conductivities of common engineering

materials, the effect of heat conduction is negligible in the determination of the initial transient stresses.

CHAPTER II

MATHEMATICAL FORMULATION

A. Governing Equations of Linear Thermoelasticity

The equations of linear thermoelasticity are well known. Their derivation may be found in [3]. The body is assumed to be stress and strain free at a temperature θ_0. In terms of the displacement vector $\vec{u}$ and the temperature rise θ above the initial reference temperature, the equations for constant material properties are

$$\mu \nabla\cdot\nabla\vec{u} + (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) - (3\lambda+2\mu)\alpha\nabla\theta + \vec{F} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \tag{2.1}$$

and

$$k\nabla\cdot\nabla\theta + Q = \rho S\frac{\partial\theta}{\partial t} + (3\lambda+2\mu)\alpha\theta_0\frac{\partial}{\partial t}(\nabla\cdot\vec{u}) \tag{2.2}$$

The linearized strain-displacement relations are

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad (i,j = 1,2,3) \tag{2.3}$$

where u_i are the cartesian components of the displacement vector. The constitutive equations expressed in cartesian components are

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - (3\lambda+2\mu)\alpha\theta\delta_{ij} \tag{2.4}$$

where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 0, & i\neq j \\ 1, & i=j \end{cases} \tag{2.5}$$

In the above relations ∇ is the gradient operator, x_i are

the cartesian spatial coordinates, t is the time, $\vec{F}$ is the body force vector, Q is the internal heat generation per unit volume per unit time and the constants $\mu, \lambda, \rho, \alpha, k$ and S are, in that order, the shear modulus, a Lamé constant, the mass density, the coefficient of linear thermal expansion, the thermal conductivity and the specific heat at constant strain. The formulation is completed by imposing appropriate boundary and initial conditions.

In deriving Equation (2.2) the excess temperature θ is assumed to be small compared to the reference temperature. Thus in the fourth term θ is replaced by θ_0. If the temperature excess were sufficiently large as compared to the reference temperature, Equations (2.1) and (2.2) would form a set of non-linear equations due to the product $\theta \frac{\partial}{\partial t}(\nabla \cdot \vec{u})$. Chadwick [4] gives a clear description of the assumptions leading to the linear coupled equations.

B. Specialization of the General Equations to a Layered Media

For convenience in the particular problem considered here, let the cartesian components of the position vector be denoted by (x, y, z), the cartesian components of the displacement vector $\vec{u}$ by (u, v, w), the cartesian components of the stress tensor σ_{ij} by $(\sigma_{11} = \sigma_x, \sigma_{12} = \sigma_{xy}, \cdots)$. The layered media under consideration and its associated coordinate system are shown in Fig. 1. The first layer has thickness a with elastic constants λ_1, μ_1 and mass density ρ_1. It

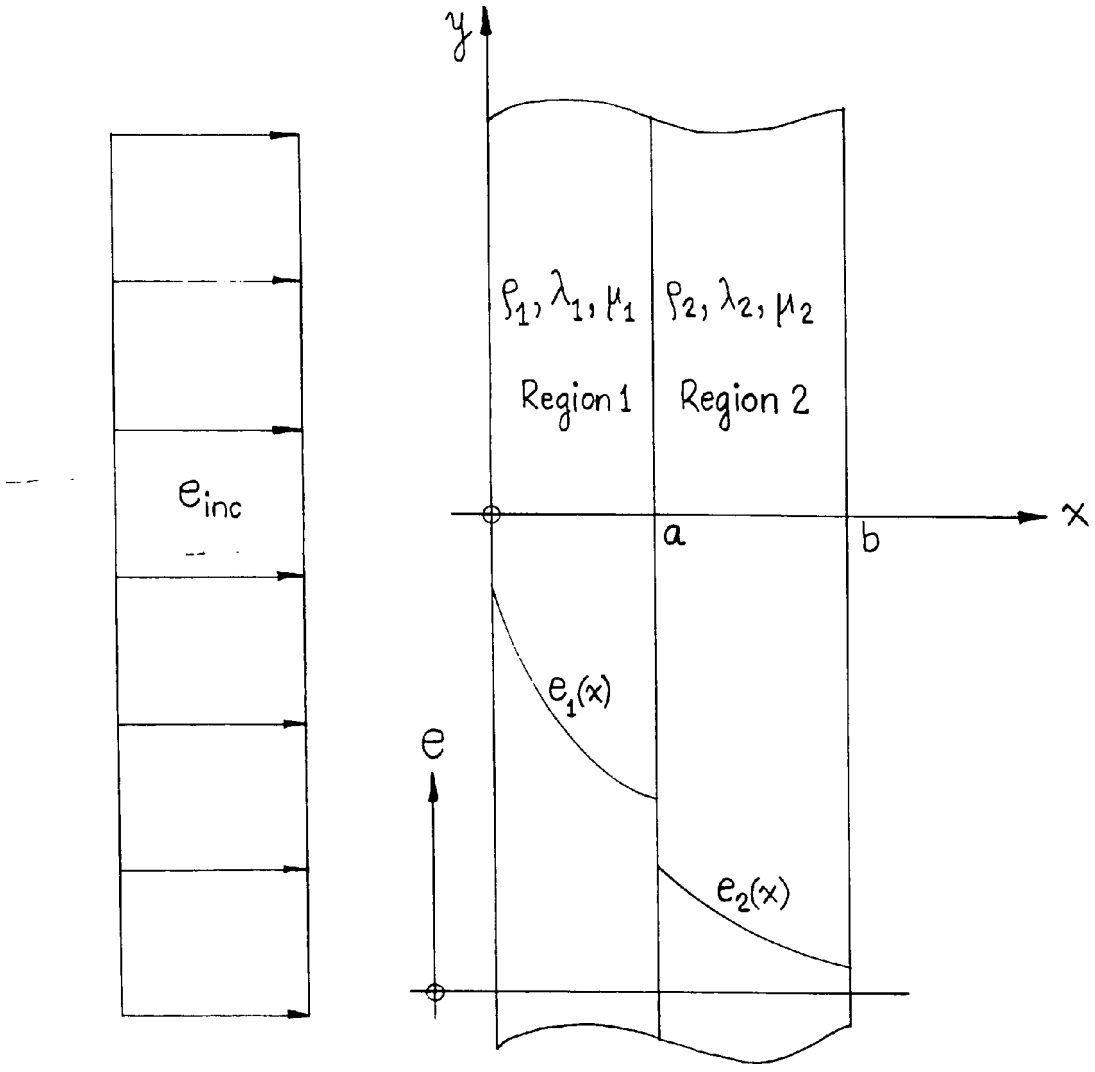


Fig. 1. Definition of Coordinate System and Energy Deposition.

is bonded to another of thickness $(b-a)$ with elastic constants λ_2, μ_2 and mass density ρ_2. The bond at $x=a$ has negligible thickness. The layered media is suddenly subjected to uniform normally incident electromagnetic energy flux e_{inc} per unit time impinging onto the surface at $x=0$. A portion of the incident energy is propagated into the material at the velocity of electromagnetic radiation for the particular media. In general, energy is deposited internally according to a different law in each layer resulting in a discontinuous deposition occurring at $x=a$ as shown in Fig. 1. The deposition of energy is considered equivalent to internal heat generation

$$Q_1(x,t) = e_1(x) g_1(t), \quad 0 \leq x < a, \quad t > 0 \tag{2.6}$$

and

$$Q_2(x,t) = e_2(x) g_2(t), \quad a < x \leq b, \quad t > 0 \tag{2.7}$$

As discussed in Chapter I, the time duration of deposition is small compared with a characteristic mechanical response time. Thus it will be assumed that the time dependence can be adequately described by the Dirac delta function. Physically this assumption means that a finite amount of energy is distributed in the media in a time small compared with the mechanical response time; hence the exact duration does not significantly influence the quantities of interest. In this context, the delta function has the properties

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{2.8}$$

Following the energy deposition constrained uniaxial motion of the layered media is considered. Thus

$$u = u(x,t)\,,\quad v = w \equiv 0 \tag{2.9}$$

From Eqn.(2.3) the only nonzero strain is

$$e_x(x,t) = \frac{\partial u}{\partial x}(x,t) \tag{2.10}$$

The corresponding nonzero stresses obtained from Eqn.(2.4) are

$$\sigma_x(x,t) = (\lambda+2\mu)\frac{\partial u}{\partial x}(x,t) - (3\lambda+2\mu)\alpha\theta(x,t) \tag{2.11}$$

and

$$\sigma_y(x,t) = \sigma_z(x,t) = \lambda\frac{\partial u}{\partial x}(x,t) - (3\lambda+2\mu)\alpha\theta(x,t) \tag{2.12}$$

where the appropriate values of the material constants are used for the two regions.

With Eqn.(2.9) and taking the body force to be zero, Eqns.(2.1) and (2.2) reduce to

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}(x,t) - (3\lambda+2\mu)\alpha\frac{\partial\theta}{\partial x}(x,t) = \rho\frac{\partial^2 u}{\partial t^2}(x,t) \tag{2.13}$$

and

$$k\frac{\partial^2\theta}{\partial x^2}(x,t) + Q(x,t) = \rho S\frac{\partial\theta}{\partial t}(x,t) + (3\lambda+2\mu)\alpha\theta_0\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) \tag{2.14}$$

Eqns.(2.13) and (2.14) hold in each layer shown in Fig.1 when the appropriate values of the material constants and heat generation function Q are used.

If thermomechanical coupling is neglected, i.e. the

fourth term in Eqn.(2.14) is taken to be zero, then Eqns. (2.13) and (2.14) uncouple, and the temperature field may be determined independently from the displacement field. Thermomechanical coupling will be neglected in this investigation. Studies which include this effect are given in [6] and [8].

The effect of heat conduction on the present problem can be inferred from the characteristic response times of the system. The characteristic mechanical response time is $t_m = \frac{a}{c_1}$. For most metallic structural materials with a characteristic length of 1 cm, $t_m = O(10^{-6}\,\text{sec})$. Similarly, a characteristic conduction time is $t_c = \frac{a^2}{\kappa}$, where thermal diffusivity κ is

$$\kappa = \frac{k}{\rho S} \tag{2.15}$$

For materials of interest here $t_c = O(10^{-1}\,\text{sec})$. In considering the response for times of the order of the mechanical response time, neglecting conduction is justified.

To obtain a quantitative estimate of the effect of neglecting heat conduction, a single layer subject to energy deposition with heat conduction retained is examined in Appendix A. For this example heat conduction is negligible on the time scale of interest. With conduction neglected, Eqns.(2.13) and (2.14) become

$$(\lambda+2\mu)\frac{\partial^2 u(x,t)}{\partial x^2} - (3\lambda+2\mu)\alpha\frac{\partial \theta(x,t)}{\partial x} = \rho\frac{\partial^2 u(x,t)}{\partial t^2} \tag{2.16}$$

and

$$Q(x,t) = \rho S \frac{\partial \theta(x,t)}{\partial t} \tag{2.17}$$

C. Boundary, Initial and Continuity Conditions

Since thermal conduction has been neglected, the determination of the temperature from (Eqn.(2.17) requires only the initial value of θ , i.e. $\theta(x,0)$. Thus the spatial distribution of temperature is specified completely by the heat generation. Here the initial temperature excess is taken as zero. Hence

$$\theta_1(x,0) = 0 , \quad 0 \leqslant x < a \tag{2.18}$$

and

$$\theta_2(x,0) = 0 , \quad a < x \leqslant b \tag{2.19}$$

The mechanical boundary conditions are taken to be those of traction free surfaces at $x = 0, b$. They are

$$\sigma_{x1}(0,t) = 0 , \quad t \geqslant 0 \tag{2.20}$$

and

$$\sigma_{x2}(b,t) = 0 , \quad t \geqslant 0 \tag{2.21}$$

where the subscripts 1 and 2 refer to the appropriate region. Using Eqn.(2.11), Eqns.(2.20) and (2.21) become

$$(\lambda+2\mu)_1 \frac{\partial u_1(0,t)}{\partial x} - (3\lambda+2\mu)_1 \alpha_1 \theta_1(0,t) = 0 , \quad t \geqslant 0 \tag{2.22}$$

and

$$(\lambda+2\mu)_2 \frac{\partial u_2(b,t)}{\partial x} - (3\lambda+2\mu)_2 \alpha_2 \theta_2(b,t) = 0 \,, \quad t \geq 0 \tag{2.23}$$

Continuity of normal stress and displacement must hold at the interface. Thus

$$(\lambda+2\mu)_1 \frac{\partial u_1(a,t)}{\partial x} - (3\lambda+2\mu)_1 \alpha_1 \theta_1(a,t) =$$
$$(\lambda+2\mu)_2 \frac{\partial u_2(a,t)}{\partial x} - (3\lambda+2\mu)_2 \alpha_2 \theta_2(a,t) \,, \quad t \geq 0 \tag{2.24}$$

and

$$u_1(a,t) = u_2(a,t) \,, \quad t \geq 0 \tag{2.25}$$

The mechanical initial conditions are those for a quiescent media, namely

$$u_1(x,0) = 0 \,, \quad 0 \leq x \leq a \tag{2.26}$$

$$u_2(x,0) = 0 \,, \quad a \leq x \leq b \tag{2.27}$$

$$\frac{\partial u_1(x,0)}{\partial t} = 0 \,, \quad 0 \leq x \leq a \tag{2.28}$$

and

$$\frac{\partial u_2(x,0)}{\partial t} = 0 \,, \quad a \leq x \leq b \tag{2.29}$$

Eqns. (2.16), (2.17) and (2.20) through (2.29) completely specify the problem.

CHAPTER III

SOLUTION OF THE BOUNDARY VALUE PROBLEM

A. The Laplace Transform Problem

The Laplace transform is (see [21] or [22])

$$f^*(x,s) = \mathcal{L}\{f(x,t)\} = \int_0^\infty f(x,t)\, e^{-st}\, dt \tag{3.1}$$

The corresponding inverse transform is

$$f(x,t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} f^*(x,s)\, e^{st}\, ds \tag{3.2}$$

where α is to the right of every singularity of $f^*(x,s)$ in the complex S plane. Applying Eqn.(3.1) to (2.17), which applies to both region 1 and 2, and using the operational properties for the derivative gives

$$s\,\theta_1^*(x,s) - \theta_1(x,0) = \frac{Q_1^*(x,s)}{(\rho S)_1}\,; \quad 0 \le x < a \tag{3.3}$$

and

$$s\,\theta_2^*(x,s) - \theta_2(x,0) = \frac{Q_2^*(x,s)}{(\rho S)_2}\,; \quad a \le x \le b \tag{3.4}$$

where Q_1^* and Q_2^* are the Laplace transforms of Q_1 and Q_2. In a similar manner the transform of Eqn.(2.16) is

$$\frac{d^2 u_1^*(x,s)}{dx^2} - \frac{1}{C_1^2}\left\{s^2 u_1^*(x,s) - s\,u_1(x,0) - \frac{\partial u_1(x,0)}{\partial t}\right\} = \Lambda_1 \frac{d\theta_1^*(x,s)}{dx}\,, \quad 0 \le x < a \tag{3.5}$$

and

$$\frac{d^2 u_2^*(x,s)}{dx^2} - \frac{1}{C_2^2}\left\{s^2 u_2^*(x,s) - s\,u_2(x,0) - \frac{\partial u_2(x,0)}{\partial t}\right\} = \Lambda_2 \frac{d\theta_2^*(x,s)}{dx}\,, \quad a < x \le b \tag{3.6}$$

Applying the initial conditions Eqns.(2.18), (2.19) and (2.26) through (2.29) yields

$$\theta_1^*(x,s) = \frac{Q_1^*(x,s)}{(\rho S)_1}\frac{1}{s} \quad ; \quad 0 \le x < a \tag{3.7}$$

$$\theta_2^*(x,s) = \frac{Q_2^*(x,s)}{(\rho S)_2}\frac{1}{s} \quad ; \quad a < x \le b \tag{3.8}$$

$$\frac{d^2u_1^*(x,s)}{dx^2} - \frac{s^2}{c_1^2}u_1^*(x,s) = \Lambda_1\frac{d\theta_1^*(x,s)}{dx} \quad . \quad 0 \le x < a \tag{3.9}$$

$$\frac{d^2u_2^*(x,s)}{dx^2} - \frac{s^2}{c_2^2}u_2^*(x,s) = \Lambda_2\frac{d\theta_2^*(x,s)}{dx} \quad , \quad a < x \le b \tag{3.10}$$

where

$$c_i^2 = \frac{(\lambda+2\mu)_i}{\rho_i} \qquad \Lambda_i = \frac{(3\lambda+2\mu)_i\alpha_i}{(\lambda+2\mu)_i} \quad , \quad (i=1,2) \tag{3.11}$$

Transforming the boundary and continuity conditions Eqns.(2.22) through (2.25) yields

$$\frac{du_1^*(0,s)}{dx} - \Lambda_1\theta_1^*(0,s) = 0 \tag{3.12}$$

$$\frac{du_2^*(b,s)}{dx} - \Lambda_2\theta_2(b,s) = 0 \tag{3.13}$$

$$\frac{du_1^*(a,s)}{dx} - \Lambda_1\theta_1^*(a,s) = \Lambda_3\frac{du_2^*(a,s)}{dx} - \Lambda_4\theta_2^*(a,s) \tag{3.14}$$

and

$$u_1^*(a,s) = u_2^*(a,s) \tag{3.15}$$

where

$$\Lambda_3 = \frac{(\lambda+2\mu)_2}{(\lambda+2\mu)_1} \qquad \Lambda_4 = \frac{(3\lambda+2\mu)_2\alpha_2}{(\lambda+2\mu)_1} \tag{3.16}$$

Delta function time dependence is taken in Eqn.(2.6)

and (2.7). Noting that

$$L\{\delta(t)\} = 1 \tag{3.17}$$

Eqns.(3.7) and (3.8) become

$$\theta_1^*(x,s) = \frac{E_1(x)}{(\rho S)_1}\frac{1}{S} \quad ; \quad 0 \leq x < a \tag{3.18}$$

and

$$\theta_2^*(x,s) = \frac{E_2(x)}{(\rho S)_2}\frac{1}{S} \quad ; \quad a < x \leq b \tag{3.19}$$

where $E_1(x)$ and $E_2(x)$ are the total energy distributions per unit volume deposited in layer 1 and 2, respectively. As indicated in Chapter I, the energy distribution will be taken to be exponentially decreasing with distance from the radiated surface. In general, for electromagnetic energy with a wavelength spectrum, the depth of penetration increases with decreasing wavelength. This effect can be approximated by taking the energy distribution functions $E_1(x)$ and $E_2(x)$ as a sum of exponential functions, i.e.

$$E_1(x) = E_{10}e^{-\gamma_{10}x} + E_{11}e^{-\gamma_{11}x} + \cdots = \sum_{m=0}^{M} E_{1m}e^{-\gamma_{1m}x} \quad ; \quad 0 \leq x < a \tag{3.20}$$

and

$$E_2(x) = E_{20}e^{-\gamma_{20}(x-a)} + E_{21}e^{-\gamma_{21}(x-a)} + \cdots = \sum_{n=0}^{N} E_{2n}e^{-\gamma_{2n}(x-a)} \quad ; \quad a < x \leq b \tag{3.21}$$

For convenience a solution will be obtained for a one-term representation of Eqns.(3.20) and (3.21). More general solutions are obtained by superposition. The total energy distributions are then

$$E_1(x) = E_1 e^{-\gamma_1 x} \quad , \quad 0 \leq x < a \tag{3.22}$$

and

$$E_2(x) = E_2 e^{-\gamma_2 (x-a)} \quad , \quad a < x \leq b \tag{3.23}$$

With this Eqns. (3.18) and (3.19) are

$$\theta_1^*(x,s) = \frac{E_1 e^{-\gamma_1 x}}{(\rho S)_1} \cdot \frac{1}{S} \quad , \quad 0 \leq x < a \tag{3.24}$$

and

$$\theta_2^*(x,s) = \frac{E_2 e^{-\gamma_2 (x-a)}}{(\rho S)_2} \cdot \frac{1}{S} \quad , \quad a < x \leq b \tag{3.25}$$

Performing the required differentiations and substituting into Eqns. (3.9) and (3.10) gives

$$\frac{d^2 u_1^*(x,s)}{dx^2} - \frac{s^2}{c_1^2} u_1^*(x,s) = - \frac{\Lambda_1 \gamma_1 E_1 e^{-\gamma_1 x}}{(\rho S)_1} \cdot \frac{1}{S} \quad , \quad 0 \leq x < a \tag{3.26}$$

and

$$\frac{d^2 u_2^*(x,s)}{dx^2} - \frac{s^2}{c_2^2} u_2^*(x,s) = - \frac{\Lambda_2 \gamma_2 E_2 e^{-\gamma_2 (x-a)}}{(\rho S)_2} \cdot \frac{1}{S} \quad , \quad a < x \leq b \tag{3.27}$$

Using Eqns. (3.24) and (3.25), Eqns. (3.12), (3.13), and (3.14) become, respectively

$$\frac{du_1^*(0,s)}{dx} - \frac{\Lambda_1 E_1}{(\rho S)_1} \cdot \frac{1}{S} = 0 \tag{3.28}$$

$$\frac{du_2^*(b,s)}{dx} - \frac{\Lambda_2 E_2 e^{-\gamma_2 (b-a)}}{(\rho S)_2} \cdot \frac{1}{S} = 0 \tag{3.29}$$

$$\frac{du_1^*(a,s)}{dx} - \frac{\Lambda_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1} \cdot \frac{1}{S} = \Lambda_3 \frac{du_2^*(a,s)}{dx} - \frac{\Lambda_4 E_2}{(\rho S)_2} \cdot \frac{1}{S} \tag{3.30}$$

Eqn. (3.15) is unaffected and is rewritten here for convenience

$$u_1^*(a,s) = u_2^*(a,s) \tag{3.31}$$

The problem is now reduced to solving Eqns.(3.26) and (3.27) subject to the boundary and continuity conditions Eqns.(3.28) through (3.31).

B. Solution of the Transform Problem

The solution of Eqns.(3.26) can be written as

$$u_1^*(x,s) = u_{1h}^*(x,s) + u_{1p}^*(x,s) \tag{3.32}$$

where $u_{1h}^*(x,s)$ is a solution of the homogeneous equation and $u_{1p}^*(x,s)$ is a particular integral. The solution is

$$u_1^*(x,s) = E(s)\, e^{\frac{s}{c_1}x} + F(s)\, e^{-\frac{s}{c_1}x} + \frac{\Lambda_1 E_1 \gamma_1 c_1^2 e^{-\gamma_1 x}}{(\rho S)_1} \cdot \frac{1}{s(s^2 - c_1^2\gamma_1^2)} \tag{3.33}$$

Similarly

$$u_2^*(x,s) = G(s)\, e^{\frac{s}{c_2}x} + H(s)\, e^{-\frac{s}{c_2}x} + \frac{\Lambda_2 E_2 \gamma_2 c_2^2 e^{-\gamma_2 (x-a)}}{(\rho S)_2} \cdot \frac{1}{s(s^2 - c_2^2\gamma_2^2)} \tag{3.34}$$

$E(s)$, $F(s)$, $G(s)$ and $H(s)$ are determined from the boundary and continuity conditions.

Eqns.(3.28) through (3.31) gives a system of four equations for the four unknowns in Eqns.(3.33) and (3.34). They are

$$\begin{aligned}
E \quad - F \quad & = I_1 \\
e^{\frac{s}{c_2}b} G \quad - e^{-\frac{s}{c_2}b} H \quad & = I_2 \\
e^{\frac{s}{c_1}a} E \; - e^{-\frac{s}{c_1}a} F - \Lambda_3 \frac{c_1}{c_2} e^{\frac{s}{c_2}a} G + \Lambda_3 \frac{c_1}{c_2} e^{-\frac{s}{c_2}} H \quad & = I_3 \\
e^{\frac{s}{c_1}a} E \; + e^{-\frac{s}{c_1}a} F - e^{\frac{s}{c_2}a} G \quad - e^{-\frac{s}{c_2}a} H \quad & = I_4
\end{aligned} \tag{3.35}$$

where

$$I_1 = \frac{\Lambda_1 c_1 E_1}{(\rho S)_1}\frac{1}{s^2} + \frac{\Lambda_1 c_1^3 E_1 \gamma_1^2}{(\rho S)_1}\frac{1}{s^2(s^2-c_1^2\gamma_1^2)}$$

$$I_2 = \frac{\Lambda_2 c_2 E_2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\cdot\frac{1}{s^2} + \frac{\Lambda_2 c_2^3 E_2 \gamma_2^2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\cdot\frac{1}{s^2(s^2-c_2^2\gamma_2^2)}$$

$$I_3 = \frac{\Lambda_1 c_1^3 E_1 \gamma_1^2 e^{-\gamma_1 a}}{(\rho S)_1}\frac{1}{s^2(s^2-c_1^2\gamma_1^2)} + \frac{\Lambda_1 c_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{1}{s^2} \qquad (3.36)$$
$$- \frac{\Lambda_2\Lambda_3 c_2^2 c_1 E_2 \gamma_2^2}{(\rho S)_2}\frac{1}{s^2(s^2-c_2^2\gamma_2^2)} - \frac{\Lambda_4 c_1 E_2}{(\rho S)_2}\frac{1}{s^2}$$

$$I_4 = -\frac{\Lambda_1 c_1^2 E_1 \gamma_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{1}{s(s^2-c_1^2\gamma_1^2)} + \frac{\Lambda_2 c_2^2 E_2 \gamma_2}{(\rho S)_2}\frac{1}{s(s^2-c_2^2\gamma_2^2)}$$

Solving for E, F, G and H yields

$$\Delta E = -I_1\left\{(1+\Lambda_3 r)e^{-\left[\frac{a}{c_1}+\frac{(b-a)}{c_2}\right]s} + (1-\Lambda_3 r)e^{-\left[\frac{a}{c_1}-\frac{(b-a)}{c_2}\right]s}\right\}$$
$$+ 2 I_2 \Lambda_3 r$$
$$+ (I_3 + I_4\Lambda_3 r)e^{-\left[-\frac{(b-a)}{c_2}\right]s}$$
$$+ (I_3 - I_4\Lambda_3 r)e^{-\left[\frac{(b-a)}{c_2}\right]s} \qquad (3.37)$$

$$\Delta F = -I_1\left\{(1+\Lambda_3 r)e^{-\left[-\frac{a}{c_1}-\frac{(b-a)}{c_2}\right]s} + (1-\Lambda_3 r)e^{-\left[-\frac{a}{c_1}+\frac{(b-a)}{c_2}\right]s}\right\}$$
$$+ 2 I_2 \Lambda_3 r$$
$$+ (I_3 + I_4\Lambda_3 r)e^{-\left[-\frac{(b-a)}{c_2}\right]s}$$
$$+ (I_3 - I_4\Lambda_3 r)e^{-\left[\frac{(b-a)}{c_2}\right]s} \qquad (3.38)$$

$$\Delta G = -2 I_1 e^{-\frac{b}{C_2}s} + I_2\left\{(1+\Lambda_3 r)e^{-\left[-\frac{a}{C_1}+\frac{a}{C_2}\right]s} - (1-\Lambda_3 r)e^{-\left[\frac{a}{C_1}+\frac{a}{C_2}\right]s}\right\} + (I_3+I_4)e^{-\left[\frac{a}{C_1}+\frac{b}{C_2}\right]s} + (I_3-I_4)e^{-\left[-\frac{a}{C_1}+\frac{b}{C_2}\right]s} \tag{3.39}$$

and

$$\Delta H = -2 I_1 e^{-\left[-\frac{b}{C_2}\right]s} + I_2\left\{(1+\Lambda_3 r)e^{-\left[\frac{a}{C_1}-\frac{a}{C_2}\right]s} - (1-\Lambda_3 r)e^{-\left[-\frac{a}{C_1}-\frac{a}{C_2}\right]s}\right\} + (I_3+I_4)e^{-\left[\frac{a}{C_1}-\frac{b}{C_2}\right]s} + (I_3-I_4)e^{-\left[-\frac{a}{C_1}-\frac{b}{C_2}\right]s} \tag{3.40}$$

The determinant of the coefficients is

$$\Delta = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & e^{\frac{s}{C_2}b} & e^{-\frac{s}{C_2}b} \\ e^{\frac{s}{C_1}a} & -e^{-\frac{s}{C_1}a} & -\Lambda_3\frac{C_1}{C_2}e^{\frac{s}{C_2}a} & \Lambda_3\frac{C_1}{C_2}e^{-\frac{s}{C_2}a} \\ e^{\frac{s}{C_1}a} & e^{-\frac{s}{C_1}a} & -e^{\frac{s}{C_2}a} & -e^{-\frac{s}{C_2}a} \end{vmatrix} \tag{3.41}$$

By suitably expressing Δ , the transformed solution may be rendered into a form in which repeated wave reflections may be recognized. Expanding Eqns.(3.41) gives

$$\Delta = (1+\Lambda_3 r)e^{\left[\frac{a}{C_1}+\frac{(b-a)}{C_2}\right]s} + (1-\Lambda_3 r)e^{\left[\frac{a}{C_1}-\frac{(b-a)}{C_2}\right]s} - (1+\Lambda_3 r)e^{-\left[\frac{a}{C_1}+\frac{(b-a)}{C_2}\right]s} - (1-\Lambda_3 r)e^{-\left[\frac{a}{C_1}-\frac{(b-a)}{C_2}\right]s} \tag{3.42}$$

With $r = \frac{c_1}{c_2}$ Eqn.(3.42) may be written as

$$\frac{1}{\Delta} = \frac{e^{-\beta_0 s}}{(1+\Lambda_3 r)} \cdot \frac{1}{1-Z} \tag{3.43}$$

where

$$Z = Z_1 e^{-\beta_1 s} + Z_2 e^{-\beta_2 s} - Z_3 e^{-\beta_3 s} \tag{3.44}$$

with

$$Z_1 = Z_3 = \frac{(1-\Lambda_3 r)}{(1+\Lambda_3 r)} \;,\; Z_2 = 1 \;,$$

$$\beta_0 = \frac{a}{c_1} + \frac{(b-a)}{c_2} \;,\; \beta_1 = \frac{2a}{c_1} \;;\; \beta_2 = \frac{2a}{c_1} + \frac{2(b-a)}{c_2} \tag{3.45}$$

and

$$\beta_3 = \frac{2(b-a)}{c_2}$$

The range of Z_1 is $-1 \leq Z_1 \leq 1$.

A simple calculation will show that

$$\Lambda_3 r = \frac{\rho_2 c_2}{\rho_1 c_1} = \frac{\gamma_2}{\gamma_1} \equiv \gamma_{21} \tag{3.46}$$

where γ_1 and γ_2 are the specific acoustic impedances of media 1 and media 2, respectively. If

$$|Z| = |Z_1 e^{-\beta_1 s} + Z_2 e^{-\beta_2 s} - Z_3 e^{-\beta_3 s}| < 1 \tag{3.47}$$

the expression $\frac{1}{1-Z}$ may be expanded in an absolutely and uniformly convergent geometric series

$$\frac{1}{1-Z} = \sum_{i=0}^{\infty} Z^i \tag{3.48}$$

Eqn.(3.48) is valid provided

$$Re(s) > 0 \tag{3.49}$$

It is desirable to express Eqn.(3.48) in a form explicitly showing the exponential factors defined by Eqn.(3.44). It is possible to show the following equivalence (see Appendix B)

$$\sum_{i=0}^{\infty} Z^i = \sum_{i=0}^{\infty} \sum_{\substack{j,k,l=0 \\ (j+k+l=i)}}^{i} \frac{i!}{j!k!l!} Z_1^j Z_2^k (-Z_3)^l e^{-(j\beta_1+k\beta_2+l\beta_3)s} \tag{3.50}$$

Hence Eqn.(3.43) may be written as

$$\frac{1}{\Delta} = \frac{1}{(1+\Lambda_3 r)} \sum_{i=0}^{\infty} \sum_{\substack{j,k,l=0 \\ (j+k+l=i)}}^{i} \frac{i!}{j!k!l!} Z_1^j Z_2^k (-Z_3)^l e^{-(\beta_0+j\beta_1+k\beta_2+l\beta_3)s} \tag{3.51}$$

Transforming Eqn.(2.11) gives

$$\sigma^*_{x1}(x,s) = (\lambda+2\mu)_1 \frac{du_1^*(x,s)}{dx} - (3\lambda+2\mu)_1 \alpha_1 \theta_1^*(x,s) \tag{3.52}$$

Using Eqns.(3.24) and (3.33) in Eqn.(3.52) gives an expression for the transformed stress

$$\sigma^*_{x1}(x,s) = (\lambda+2\mu)_1 \left[\left[\frac{s}{c_1} E e^{\frac{s}{c_1}x} - \frac{s}{c_1} F e^{-\frac{s}{c_1}x} - \frac{\Lambda_1 c_1^2 E_1 \gamma_1^2 e^{-\gamma_1 x}}{(\rho S)_1} \frac{1}{s(s^2 - c_1^2\gamma_1^2)} \right] \right]$$
$$- (3\lambda+2\mu)_1 \alpha_1 \left[\frac{E_1 e^{-\gamma_1 x}}{(\rho S)_1} \cdot \frac{1}{s} \right] \tag{3.53}$$

Similarily

$$\sigma^*_{x2}(x,s) = (\lambda+2\mu)_2 \frac{du_2^*(x,s)}{dx} - (3\lambda+2\mu)_2 \alpha_2 \theta_2^*(x,s) \tag{3.54}$$

and

$$\sigma^*_{x2}(x,s) = (\lambda+2\mu)_2 \left[\left[\frac{s}{c_2} G e^{\frac{s}{c_2}x} - \frac{s}{c_2} H e^{-\frac{s}{c_2}x} - \frac{\Lambda_2 c_2^2 E_2 \gamma_2^2 e^{-\gamma_2(x-a)}}{(\rho S)_2} \frac{1}{s(s^2 - c_2^2\gamma_2^2)} \right] \right]$$
$$- (3\lambda+2\mu)_2 \alpha_2 \left[\frac{E_2 e^{-\gamma_2(x-a)}}{} \cdot \frac{1}{s} \right] \tag{3.55}$$

Using Eqn.(3.51) with Eqns.(3.37) through (3.40) it is possible to write explicit expressions for E, F, G and H. Substituting these into Eqns.(3.53) and (3.55) and defining the symbol $\sum$ as

$$\sum = \sum(i,j,k,l,\gamma_{21}) = \sum_{i=0}^{\infty} \sum_{\substack{j,k,l=0 \\ (j+k+l=i)}}^{i} \frac{i!}{j!k!l!} Z_1^j Z_2^k (-Z_3)^l \qquad (3.56)$$

gives the transformed stresses $\sigma^*_{x1}(x,s)$ and $\sigma^*_{x2}(x,s)$ as

(See page 25)

$$\bar{\sigma}^{*}_{x1}(x,s) = (\lambda+2\mu)_1 \Bigg[\Bigg[- \sum e^{-\left[2\beta_0 + j\beta_1 + k\beta_2 + l\beta_3 - \frac{x}{c_1}\right]s} \left\{ \frac{\Lambda_1 E_1}{(\rho S)_1} \frac{1}{s} + \frac{\Lambda_1 c_1^2 E_1 \gamma_1^2}{(\rho S)_1} \frac{1}{s(s^2 - c_1^2\gamma_1^2)} \right\}$$

$$- \frac{(1-\Lambda_3 r)}{(1+\Lambda_3 r)} \sum e^{-\left[\beta_0 + \left(\frac{2j+1}{2}\right)\beta_1 + k\beta_2 + \left(\frac{2l-1}{2}\right)\beta_3 - \frac{x}{c_1}\right]s} \left\{ \frac{\Lambda_1 E_1}{(\rho S)_1} \frac{1}{s} + \frac{\Lambda_1 c_1^2 E_1 \gamma_1^2}{(\rho S)_1} \frac{1}{s(s^2 - c_1^2\gamma_1^2)} \right\}$$

$$+ 2 \frac{\Lambda_3 r}{(1+\Lambda_3 r)} \sum e^{-\left[\beta_0 + j\beta_1 + k\beta_2 + l\beta_3 - \frac{x}{c_1}\right]s} \left\{ \frac{\Lambda_2 E_2 e^{-\gamma_2(b-a)}}{(\rho S)_2} \frac{c_2}{c_1} \frac{1}{s} + \frac{\Lambda_2 c_2^2 E_2 \gamma_2^2 e^{-\gamma_2(b-a)}}{(\rho S)_2} \frac{c_2}{c_1} \frac{1}{(s^2 - c_2^2\gamma_2^2)} \right\}$$

$$+ \frac{1}{(1+\Lambda_3 r)} \sum e^{-\left[\beta_0 + j\beta_1 + k\beta_2 + \left(\frac{2l-1}{2}\right)\beta_3 - \frac{x}{c_1}\right]s} \Bigg\langle \Bigg\{ \frac{\Lambda_1 c_1^2 E_1 \gamma_1^2 e^{-\gamma_1 a}}{(\rho S)_1} \frac{1}{s(s^2 - c_1^2\gamma_1^2)} + \frac{\Lambda_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1} \frac{1}{s}$$

$$- \frac{\Lambda_2 \Lambda_3 c_2^2 E_2 \gamma_2^2}{(\rho S)_2} \frac{1}{s(s^2 - c_2^2\gamma_2^2)} - \frac{\Lambda_4 E_2}{(\rho S)_2} \frac{1}{s} \Bigg\} + \Lambda_3 r \left\{ - \frac{\Lambda_1 c_1 E_1 \gamma_1 e^{-\gamma_1 a}}{(\rho S)_1} \frac{1}{(s^2 - c_1^2\gamma_1^2)} + \frac{\Lambda_2 c_2 E_2 \gamma_2}{(\rho S)_2} \frac{c_2}{c_1} \frac{1}{(s^2 - c_2^2\gamma_2^2)} \right\} \Bigg\rangle$$

$$+ \frac{1}{(1+\Lambda_3 r)} \sum e^{-\left[\beta_0 + j\beta_1 + k\beta_2 + \left(\frac{2l+1}{2}\right)\beta_3 - \frac{x}{c_1}\right]s} \Bigg\langle \Bigg\{ \frac{\Lambda_1 c_1^2 E_1 \gamma_1^2 e^{-\gamma_1 a}}{(\rho S)_1} \frac{1}{s(s^2 - c_1^2\gamma_1^2)} + \frac{\Lambda_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1} \frac{1}{s}$$

$$- \frac{\Lambda_2 \Lambda_3 c_2^2 E_2 \gamma_2^2}{(\rho S)_2} \frac{1}{s(s^2 - c_2^2\gamma_2^2)} - \frac{\Lambda_4 E_2}{(\rho S)_2} \frac{1}{s} \Bigg\} - \Lambda_3 r \left\{ - \frac{\Lambda_1 c_1 E_1 \gamma_1 e^{-\gamma_1 a}}{(\rho S)_1} \frac{1}{(s^2 - c_1^2\gamma_1^2)} + \frac{\Lambda_2 c_2 \gamma_2 E_2}{(\rho S)_2} \frac{c_2}{c_1} \frac{1}{(s^2 - c_2^2\gamma_2^2)} \right\} \Bigg\rangle$$

$$+ \sum e^{-\left[j\beta_1 + k\beta_2 + l\beta_3 + \frac{x}{c_1}\right]s} \left\{ \frac{\Lambda_1 E_1}{(\rho S)_1} \frac{1}{s} + \frac{\Lambda_1 c_1^2 E_1 \gamma_1^2}{(\rho S)_1} \frac{1}{s(s^2 - c_1^2\gamma_1^2)} \right\}$$

(continued on page 26)

$$+\frac{(1-\Lambda_3 r)}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+\left(\frac{2j-1}{2}\right)\beta_1+k\beta_2+\left(\frac{2l+1}{2}\right)\beta_3+\frac{x}{c_1}\right]s}\left\{\frac{\Lambda_1 E_1}{(\rho S)_1}\frac{1}{s}+\frac{\Lambda_1 c_1^2 E_1\gamma_1^2}{(\rho S)_1}\frac{1}{s(s^2-c_1^2\gamma_1^2)}\right\}$$

$$-2\frac{\Lambda_3 r}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3+\frac{x}{c_1}\right]s}\left\{\frac{\Lambda_2 E_2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\frac{c_2}{c_1}\frac{1}{s}+\frac{\Lambda_2 c_2^3 E_2\gamma_2^2 e^{-\gamma_2(b-a)}}{(\rho S)_2 c_1}\frac{1}{(s^2-c_2^2\gamma_2^2)}\right\}$$

$$-\frac{1}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+\left(\frac{2l-1}{2}\right)\beta_3+\frac{x}{c_1}\right]s}\left\langle\left\{\frac{\Lambda_1 c_1^2 E_1\gamma_1^2 e^{-\gamma_1 a}}{(\rho S)_1}\frac{1}{s(s^2-c_1^2\gamma_1^2)}+\frac{\Lambda_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{1}{s}\right.\right.$$

$$\left.-\frac{\Lambda_2\Lambda_3 c_2^2 E_2\gamma_2^2}{(\rho S)_2}\frac{1}{s(s^2-c_2^2\gamma_2^2)}-\frac{\Lambda_4 E_2}{(\rho S)_2}\frac{1}{s}\right\}+\Lambda_3 r\left\{-\frac{\Lambda_1 c_1 E_1\gamma_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{1}{(s^2-c_1^2\gamma_1^2)}+\frac{\Lambda_2 c_2\gamma_2 E_2}{(\rho S)_2}\frac{c_2}{c_1}\frac{1}{(s^2-c_2^2\gamma_2^2)}\right\}\Big\rangle$$

$$-\frac{1}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+\left(\frac{2l+1}{2}\right)\beta_3+\frac{x}{c_1}\right]s}\left\langle\left\{\frac{\Lambda_1 c_1^2 E_1\gamma_1^2 e^{-\gamma_1 a}}{(\rho S)_1}\frac{1}{s(s^2-c_1^2\gamma_1^2)}+\frac{\Lambda_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{1}{s}\right.\right.$$

$$\left.-\frac{\Lambda_2\Lambda_3 c_2^2 E_2\gamma_2^2}{(\rho S)_2}\frac{1}{s(s^2-c_2^2\gamma_2^2)}-\frac{\Lambda_4 E_2}{(\rho S)_2}\frac{1}{s}\right\}-\Lambda_3 r\left\{-\frac{\Lambda_1 c_1 E_1\gamma_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{1}{(s^2-c_1^2\gamma_1^2)}+\frac{\Lambda_2 c_2\gamma_2 E_2}{(\rho S)_2}\frac{c_2}{c_1}\frac{1}{(s^2-c_2^2\gamma_2^2)}\right\}\Big\rangle$$

$$\left.-\frac{\Lambda_1 c_1^2 E_1\gamma_1^2 e^{-\gamma_1 x}}{(\rho S)_1}\frac{1}{s(s^2-c_1^2\gamma_1^2)}\right]-(3\lambda+2\mu)_1\alpha_1\frac{E_1}{(\rho S)_1}e^{-\gamma_1 x}\frac{1}{s} \qquad (3.57)$$

$$\sigma^{*}_{x2}(x,s) = (\lambda+2\mu)_2\Bigg[\Bigg[-2\frac{1}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3-\frac{b}{c_2}-\frac{x}{c_2}\right]s}\left\{\frac{\Lambda_1 E_1}{(\rho S)_1}\frac{c_1}{c_2}\frac{1}{s}+\frac{\Lambda_1 c_1^2 E_1\gamma_1^2}{(\rho S)_1}\frac{c_1}{c_2}\frac{1}{s(s^2-c_1^2\gamma_1^2)}\right\}$$

$$+\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3-\frac{a}{c_1}+\frac{a}{c_2}-\frac{x}{c_2}\right]s}\left\{\frac{\Lambda_2 E_2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\frac{1}{s}+\frac{\Lambda_2 c_2^2 E_2\gamma_2^2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\frac{1}{s(s^2-c_2^2\gamma_2^2)}\right\}$$

$$-\frac{(1-\Lambda_3 r)}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3+\frac{a}{c_1}+\frac{a}{c_2}-\frac{x}{c_2}\right]s}\left\{\frac{\Lambda_2 E_2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\frac{1}{s}+\frac{\Lambda_2 c_2^2\gamma_2^2 E_2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\frac{1}{s(s^2-c_2^2\gamma_2^2)}\right\}$$

$$+\frac{1}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3+\frac{a}{c_1}+\frac{b}{c_2}-\frac{x}{c_2}\right]s}\left\{\frac{\Lambda_1 E_1 c_1^2\gamma_1^2 e^{-\gamma_1 a}}{(\rho S)_1}\frac{c_1}{c_2}\frac{1}{s(s^2-c_1^2\gamma_1^2)}+\frac{\Lambda_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{c_1}{c_2}\frac{1}{s}\right.$$

$$\left.-\frac{\Lambda_2\Lambda_3 E_2\gamma_2^2 c_2^2}{(\rho S)_2}\frac{c_1}{c_2}\frac{1}{s(s^2-c_2^2\gamma_2^2)}-\frac{\Lambda_4 E_2}{(\rho S)_2}\frac{c_1}{c_2}\frac{1}{s}-\frac{\Lambda_1 c_1\gamma_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{c_1}{c_2}\frac{1}{(s^2-c_1^2\gamma_1^2)}+\frac{\Lambda_2 c_2 E_2\gamma_2}{(\rho S)_2}\frac{1}{(s^2-c_2^2\gamma_2^2)}\right\}$$

$$+\frac{1}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3-\frac{a}{c_1}+\frac{b}{c_2}-\frac{x}{c_2}\right]s}\left\{\frac{\Lambda_1 E_1 c_1^2\gamma_1^2 e^{-\gamma_1 a}}{(\rho S)_1}\frac{c_1}{c_2}\frac{1}{s(s^2-c_1^2\gamma_1^2)}+\frac{\Lambda_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{c_1}{c_2}\frac{1}{s}\right.$$

$$\left.-\frac{\Lambda_2\Lambda_3 E_2\gamma_2^2 c_2^2}{(\rho S)_2}\frac{c_1}{c_2}\frac{1}{s(s^2-c_2^2\gamma_2^2)}-\frac{\Lambda_4 E_2}{(\rho S)_2}\frac{c_1}{c_2}\frac{1}{s}+\frac{\Lambda_1 c_1\gamma_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{c_1}{c_2}\frac{1}{(s^2-c_1^2\gamma_1^2)}-\frac{\Lambda_2 c_2\gamma_2 E_2}{(\rho S)_2}\frac{1}{(s^2-c_2^2\gamma_2^2)}\right\}$$

$$+2\frac{1}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3-\frac{b}{c_2}+\frac{x}{c_2}\right]s}\left\{\frac{\Lambda_1 E_1}{(\rho S)_1}\frac{c_1}{c_2}\frac{1}{s}+\frac{\Lambda_1 c_1^2 E_1\gamma_1^2}{(\rho S)_1}\frac{c_1}{c_2}\frac{1}{s(s^2-c_1^2\gamma_1^2)}\right\}$$

(continued on page 28)

$$
-\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3+\frac{a}{C_1}-\frac{a}{C_2}+\frac{x}{C_2}\right]s}\left\{\frac{\Lambda_2 E_2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\frac{1}{s}+\frac{\Lambda_2 C_2^2\gamma_2^2 E_2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\frac{1}{s(s^2-C_2^2\gamma_2^2)}\right\}
$$

$$
+\frac{(1-\Lambda_3 r)}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3-\frac{a}{C_1}-\frac{a}{C_2}+\frac{x}{C_2}\right]s}\left\{\frac{\Lambda_2 E_2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\frac{1}{s}+\frac{\Lambda_2 C_2^2\gamma_2^2 E_2 e^{-\gamma_2(b-a)}}{(\rho S)_2}\frac{1}{s(s^2-C_2^2\gamma_2^2)}\right\}
$$

$$
-\frac{1}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3+\frac{a}{C_1}-\frac{b}{C_2}+\frac{x}{C_2}\right]s}\left\{\frac{\Lambda_1 C_1^2\gamma_1^2 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{C_1}{C_2}\frac{1}{s(s^2-C_1^2\gamma_1^2)}+\frac{\Lambda_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{C_1}{C_2}\frac{1}{s}\right.
$$

$$
\left.-\frac{\Lambda_2\Lambda_3 C_2^2\gamma_2^2 E_2}{(\rho S)_2}\frac{C_1}{C_2}\frac{1}{s(s^2-C_2^2\gamma_2^2)}-\frac{\Lambda_4 E_2}{(\rho S)_2}\frac{C_1}{C_2}\frac{1}{s}-\frac{\Lambda_1 C_1\gamma_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{C_1}{C_2}\frac{1}{(s^2-C_1^2\gamma_1^2)}+\frac{\Lambda_2 C_2\gamma_2 E_2}{(\rho S)_2}\frac{1}{(s^2-C_2^2\gamma_2^2)}\right\}
$$

$$
-\frac{1}{(1+\Lambda_3 r)}\sum e^{-\left[\beta_0+j\beta_1+k\beta_2+l\beta_3-\frac{a}{C_1}-\frac{b}{C_2}+\frac{x}{C_1}\right]s}\left\{\frac{\Lambda_1 C_1^2\gamma_1^2 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{C_1}{C_2}\frac{1}{s(s^2-C_1^2\gamma_1^2)}+\frac{\Lambda_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{C_1}{C_2}\frac{1}{s}\right.
$$

$$
\left.-\frac{\Lambda_2\Lambda_3 C_2^2\gamma_2^2 E_2}{(\rho S)_2}\frac{C_1}{C_2}\frac{1}{s(s^2-C_2^2\gamma_2^2)}-\frac{\Lambda_4 E_2}{(\rho S)_2}\frac{C_1}{C_2}\frac{1}{s}+\frac{\Lambda_1 C_1\gamma_1 E_1 e^{-\gamma_1 a}}{(\rho S)_1}\frac{C_1}{C_2}\frac{1}{(s^2-C_1^2\gamma_1^2)}-\frac{\Lambda_2 C_2\gamma_2 E_2}{(\rho S)_2}\frac{1}{(s^2-C_2^2\gamma_2^2)}\right\}
$$

$$
\left.-\frac{\Lambda_2 C_2^2\gamma_2^2 E_2 e^{-\gamma_2(x-a)}}{(\rho S)_2}\frac{1}{s(s^2-C_2^2\gamma_2^2)}\right]-(3\lambda+2\mu)_2\alpha_2\frac{E_2 e^{-\gamma_2(x-a)}}{(\rho S)_2}\frac{1}{s} \qquad (3.58)
$$

C. The Inverse Transform

The inverse transforms of Eqns.(3.57) and (3.58) are obtained by using the shift property of the Laplace transform, namely

$$L^{-1}\{f(s)e^{-\phi s}\} = F(t-\phi)H(t-\phi) \tag{3.59}$$

where ϕ is a positive constant, $H(t-\phi)$ is the shifted step function defined by

$$H(t-\phi) = \begin{cases} 0, & t<\phi \\ 1, & t>\phi \end{cases} \tag{3.60}$$

and $F(t)$ is the inverse transform of $f(s)$. The functions for which inverse transforms are needed in evaluating Eqn.(3.59) are of the form $\frac{1}{s}$, $\frac{1}{(s^2-a^2)}$ and $\frac{1}{s(s^2-b^2)}$. They are listed in Appendix B.

It is convenient to introduce the nondimensional quantities

$$\begin{aligned} \xi &= \frac{x}{a} \\ \tau &= \frac{c_1}{a}t \\ \Gamma_1 &= \gamma_1 a, \quad \Gamma_2 = \gamma_2 a \\ \sigma_{\xi 1}(\xi,\tau) &= \frac{\sigma_{x1}(x,t)}{(3\lambda+2\mu)_1\alpha_1\frac{E_1}{(\rho S)_1}} \\ \sigma_{\xi 2}(\xi,\tau) &= \frac{\sigma_{x2}(x,t)}{(3\lambda+2\mu)_2\alpha_2\frac{E_2}{(\rho S)_2}} \end{aligned} \tag{3.61}$$

with

$$\begin{aligned} \beta &= \frac{b}{a} \\ K &= \frac{(3\lambda+2\mu)_2\alpha_2}{(3\lambda+2\mu)_1\alpha_1}\,\frac{E_2/(\rho S)_2}{E_1/(\rho S)_1} \end{aligned} \tag{3.62}$$

$${}_p\Phi_q = {}_p\Phi_q(j,k,l,r,\beta,\xi), \quad (p=1,2;\ q=1,\cdots 10)$$

where ${}_p\Phi_q$ is listed in Appendix B. With this the inverse transforms of Eqns. (3.57) and (3.58) are

$$
\begin{aligned}
\sigma_{\xi 1}(\xi,\tau) = &- \sum \cosh \Gamma_1 [\tau - {}_1\Phi_1]\, H(\tau - {}_1\Phi_1) \\
&- \frac{(1-\gamma_{21})}{(1+\gamma_{21})} \sum \cosh \Gamma_1 [\tau - {}_1\Phi_2]\, H(\tau - {}_1\Phi_2) \\
&+ \frac{2}{(1+\gamma_{21})} K \sum e^{-\Gamma_2(\beta-1)} \left\{ 1 + \sinh \frac{\Gamma_2}{r} [\tau - {}_1\Phi_3] \right\} H(\tau - {}_1\Phi_3) \\
&+ \frac{1}{(1+\gamma_{21})} \sum \Big(e^{-\Gamma_1} \left\{ \cosh \Gamma_1 [\tau - {}_1\Phi_4] - \gamma_{21} \sinh \Gamma_1 [\tau - {}_1\Phi_4] \right\} \\
&\qquad + K \left\{ \sinh \frac{\Gamma_2}{r} [\tau - {}_1\Phi_4] - \cosh \frac{\Gamma_2}{r} [\tau - {}_1\Phi_4] \right\} \Big) H(\tau - {}_1\Phi_4) \\
&+ \frac{1}{(1+\gamma_{21})} \sum \Big(e^{-\Gamma_1} \left\{ \cosh \Gamma_1 [\tau - {}_1\Phi_5] + \gamma_{21} \sinh \Gamma_1 [\tau - {}_1\Phi_5] \right\} \\
&\qquad - K \left\{ \sinh \frac{\Gamma_2}{r} [\tau - {}_1\Phi_5] + \cosh \frac{\Gamma_2}{r} [\tau - {}_1\Phi_5] \right\} \Big) H(\tau - {}_1\Phi_5) \\
&+ \sum \cosh \Gamma_1 [\tau - {}_1\Phi_6]\, H(\tau - {}_1\Phi_6) \\
&+ \frac{(1-\gamma_{21})}{(1+\gamma_{21})} \sum \cosh \Gamma_1 [\tau - {}_1\Phi_7]\, H(\tau - {}_1\Phi_7) \\
&- \frac{2}{(1+\gamma_{21})} K \sum e^{-\Gamma_2(\beta-1)} \left\{ 1 + \sinh \frac{\Gamma_2}{r} [\tau - {}_1\Phi_8] \right\} H(\tau - {}_1\Phi_8) \\
&- \frac{1}{(1+\gamma_{21})} \sum \Big(e^{-\Gamma_1} \left\{ \cosh \Gamma_1 [\tau - {}_1\Phi_9] - \gamma_{21} \sinh \Gamma_1 [\tau - {}_1\Phi_9] \right\} \\
&\qquad + K \left\{ \sinh \frac{\Gamma_2}{r} [\tau - {}_1\Phi_9] - \cosh \frac{\Gamma_2}{r} [\tau - {}_1\Phi_9] \right\} \Big) H(\tau - {}_1\Phi_9) \\
&- \frac{1}{(1+\gamma_{21})} \sum \Big(e^{-\Gamma_1} \left\{ \cosh \Gamma_1 [\tau - {}_1\Phi_{10}] + \gamma_{21} \sinh \Gamma_1 [\tau - {}_1\Phi_{10}] \right\} \\
&\qquad - K \left\{ \sinh \frac{\Gamma_2}{r} [\tau - {}_1\Phi_{10}] + \cosh \frac{\Gamma_2}{r} [\tau - {}_1\Phi_{10}] \right\} \Big) H(\tau - {}_1\Phi_{10}) \\
&- e^{-\Gamma_1 \xi} \cosh \Gamma_1 \tau
\end{aligned}
\tag{3.63}
$$

$$\sigma_{\xi 2}(\xi,\tau) = -\frac{2\gamma_{21}}{(1+\gamma_{21})} \sum \cosh \Gamma_1 [\tau - {}_2\Phi_1] H(\tau - {}_2\Phi_1)$$

$$+ K \sum e^{-\Gamma_2(\beta-1)} \cosh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_2] H(\tau - {}_2\Phi_2)$$

$$-\frac{(1-\gamma_{21})}{(1+\gamma_{21})} K \sum e^{-\Gamma_2(\beta-1)} \cosh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_3] H(\tau - {}_2\Phi_3)$$

$$+\frac{\gamma_{21}}{(1+\gamma_{21})} \sum \Big(e^{-\Gamma_1} \{ \cosh \Gamma_1 [\tau - {}_2\Phi_4] - \sinh \Gamma_1 [\tau - {}_2\Phi_4] \}$$

$$+ K \{ \gamma_{21}^{-1} \sinh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_4] - \cosh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_4] \} \Big) H(\tau - {}_2\Phi_4)$$

$$+\frac{\gamma_{21}}{(1+\gamma_{21})} \sum \Big(e^{-\Gamma_1} \{ \cosh \Gamma_1 [\tau - {}_2\Phi_5] + \sinh \Gamma_1 [\tau - {}_2\Phi_5] \}$$

$$- K \{ \gamma_{21}^{-1} \sinh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_5] + \cosh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_5] \} \Big) H(\tau - {}_2\Phi_5)$$

$$+\frac{2\gamma_{21}}{(1+\gamma_{21})} \sum \cosh \Gamma_1 [\tau - {}_2\Phi_6] H(\tau - {}_2\Phi_6)$$

$$- K \sum e^{-\Gamma_2(\beta-1)} \cosh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_7] H(\tau - {}_2\Phi_7)$$

$$+\frac{(1-\gamma_{21})}{(1+\gamma_{21})} K \sum e^{-\Gamma_2(\beta-1)} \cosh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_8] H(\tau - {}_2\Phi_8)$$

$$-\frac{\gamma_{21}}{(1+\gamma_{21})} \sum \Big(e^{-\Gamma_1} \{ \cosh \Gamma_1 [\tau - {}_2\Phi_9] - \sinh \Gamma_1 [\tau - {}_2\Phi_9] \}$$

$$+ K \{ \gamma_{21}^{-1} \sinh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_9] - \cosh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_9] \} \Big) H(\tau - {}_2\Phi_9)$$

$$-\frac{\gamma_{21}}{(1+\gamma_{21})} \sum \Big(e^{-\Gamma_1} \{ \cosh \Gamma_1 [\tau - {}_2\Phi_{10}] + \sinh \Gamma_1 [\tau - {}_2\Phi_{10}] \}$$

$$- K \{ \gamma_{21}^{-1} \sinh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_{10}] + \cosh \frac{\Gamma_2}{r} [\tau - {}_2\Phi_{10}] \} \Big) H(\tau - {}_2\Phi_{10})$$

$$- K e^{-\Gamma_2(\xi-1)} \cosh \frac{\Gamma_2}{r} \tau \qquad (3.64)$$

The solution for the stress as given by Eqns.(3.63) and (3.64) has a simple and convenient interpretation. There are in each of these equations eleven terms. Ten of these contain the unit step function and consequently make no contribution unless their argument is positive. The arguments of these step functions differ by integral multiples of the time required for a wave to traverse a given layer. Thus, as time proceeds, the various terms can be identified in a systemic manner as reflected waves. If the values of $_p\Phi_q$ in Eqns.(3.63) and (3.64) are displayed in a $\xi_j - \tau$ solution space, a typical result is shown in Fig.2. In this diagram particular values of $\frac{b}{a} = 1.33$ and $\frac{c_1}{c_2} = 4$ have been used.

The method of determining the stresses $\sigma_{\xi 1}$ and $\sigma_{\xi 2}$ as a function of τ for a particular value of ξ_j is shown best by an example. A typical value ξ_j is indicated by the dashed line $a, b, c, \cdots$ in Fig.2. At point "a" the only non-zero term of Eqn.(3.63) is the last one when $\tau = 0^+$. Because of the delta function time dependence for the heat generation rate, the stress jumps to a finite value at $\tau = 0^+$. At point "b", a value of τ slightly smaller than $_1\Phi_4(0,0,0)$, $\sigma_{\xi 1}$ is still given by the last term of Eqn.(3.63). At $\tau =$ "c", a jump in stress is given by the fourth term of Eqn.(3.63) by virtue of the positive argument of the step function $H(\tau - {}_1\Phi_4)$. From points "c" to "d" there are then, two operative terms, the tenth and the fourth. At point "e" the sixth term creates a second jump because of the positive argument of the step function $H(\tau - {}_1\Phi_6)$. Between points "e" and

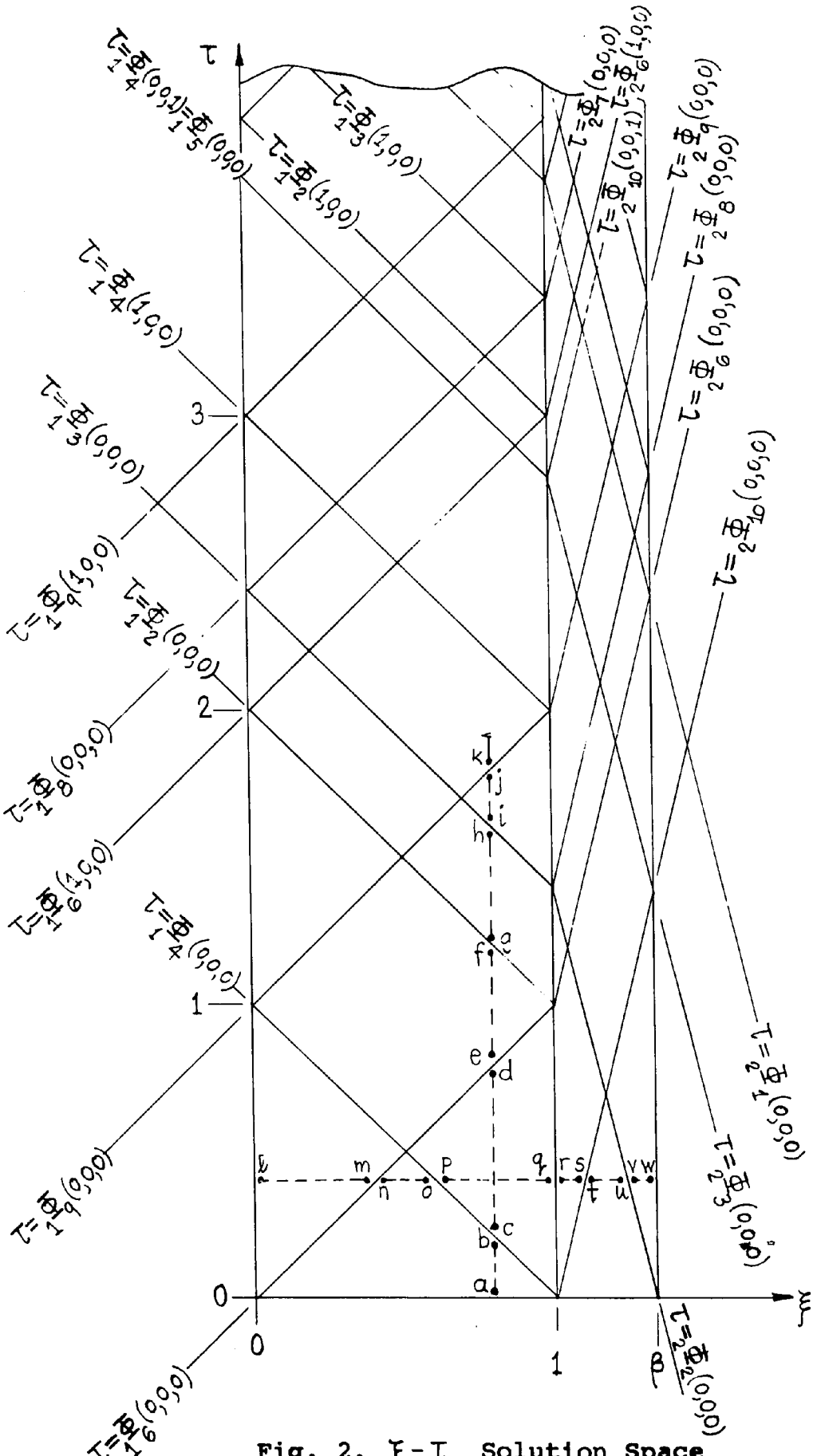


Fig. 2. ξ-τ Solution Space

"f" there are three terms operative, the tenth, the fourth and the sixth. As the time τ increases, successive step functions become non-zero and the corresponding terms multiplied by them become operative. In each term the series is evaluated for the values of j, k and l associated with the particular Φ in question. For example, the value of the series $\sum$ for the term corresponding to ${}_1\Phi_4(0,0,0)$ is 1 ; for the term corresponding ${}_1\Phi_4(1,0,0)$ it is $\frac{(1-\gamma_{21})}{(1+\gamma_{21})}$ and for the term corresponding to ${}_1\Phi_4(0,0,1)$ it is $-\frac{(1-\gamma_{21})}{(1+\gamma_{21})}$.

Calculating the stress as a function of the spatial coordinate ξ for a fixed value of τ is accomplished in a similar way. A typical time τ is represented by the dashed line $l, m, n, \cdots w$ in Fig.2. Choosing to begin at $\xi = 0^+$, there are two terms operative in Eqn.(3.63), the sixth and the eleventh. At $\xi = 0$ these two terms cancel and $\sigma_{\xi 1} = 0$ at $\xi = 0$ as it should by boundary condition Eqn.(2.19). Across "m"-"n" there is a jump due to the negative argument of the step function in the sixth term. From "n" to "o" then, only the eleventh term is non-zero. Across "o"-"p" the fourth term contributes a value since the argument of H is positive. Hence from "p" to "q" the eleventh and fourth terms are non-zero. The σ_ξ stress at $\xi = 1$ (bond stress) may be calculated from either Eqn.(3.63) or (3.64) because of the continuity of σ_ξ at $\xi = 1$. Proceeding into media 2 Eqn.(3.63) must be used. The operative terms in the second layer are determined in an analogous way.

CHAPTER IV

NUMERICAL EXAMPLES

A. The Half-Space

As a simple illustration of the analytical results of Chapter III, the homogeneous half-space shown in Fig.3 is considered. Zaker [15] also considered this case. Since this case is obtained by letting the interface boundary in Fig.1 approach infinity, a different interpretation is necessary for the nondimensionalization used in Eqn.(3.63). The physical distance X is measured from the surface of the half-space. The value of X where the energy deposition is e^{-2} times the value at the surface is taken as the length for nondimensionalization. Thus the nondimensional length $\xi = \gamma x/2$ where γ is the absorption factor. The only meaningful terms in Eqn.(3.63) are the sixth (with the value of $\sum$ being unity) and the last.

Fig.3 shows the nondimensional stress σ_ξ as a function of the nondimensional time τ at several values of ξ. Fig.4 gives σ_ξ as a function of ξ with the time τ as parameter. The stress discontinuity has the value unity, with the maximum tensile stress being nearly 0.5 for ξ greater than approximately 1.0. It is apparent from Fig.4 that as the time τ increases, a stress wave maintaining the stress free boundary condition at the surface propagates into the half-space with a jump discontinuity $\Delta\sigma_\xi = +1$. If the absorption factor γ is large, there exists a correspondingly small value of the physical distance X for ξ relatively

small, say $\xi = 1$. Thus for large γ , it is possible to generate relatively large tensile stresses near the surface, creating the possibility of surface spallation.

B. The Single Layer

The next case treated is a single layer sustaining energy deposition, as shown in Fig.5. The surface $\xi = 1$ may be considered in contact with a media of zero specific acoustic impedance and zero energy deposition. Examination of Eqn.(3.63) shows that the solution space shown in Fig.6 is divided into diamond shaped regions by lines on which $(\tau - \Phi)$ is constant. Only the values Φ_2 , Φ_4, Φ_6 and Φ_9 are involved. The maximum tensile stress occurs at $\xi = 0.5$. Fig.7 shows the nondimensional stress at this point as a function of nondimensional time for several values of Γ . For a fixed value of ξ near 0.5 , say $\xi = 0.5^-$, the stress begins at $\tau = 0$ as compression until a tensile jump of unit magnitude occurs across the line $\tau = \Phi_6(0,0,0)$. For increasing τ , the change in stress is not significant until $\tau = \Phi_4(0,0,0)$, which has a value slightly larger than 0.5 . At this time another tensile jump of magnitude $e^{-\Gamma}$ occurs. When $\xi = 0.5$, these two tensile discontinuities occur simultaneously at $\tau = 0.5$ and hence the total tensile stress increase at $\tau = 0.5$ is $1 + e^{-\Gamma}$. Note that for large Γ , say 4 or greater, the maximum tensile or compressive stress is 0.5. The stress is periodic with period 2.

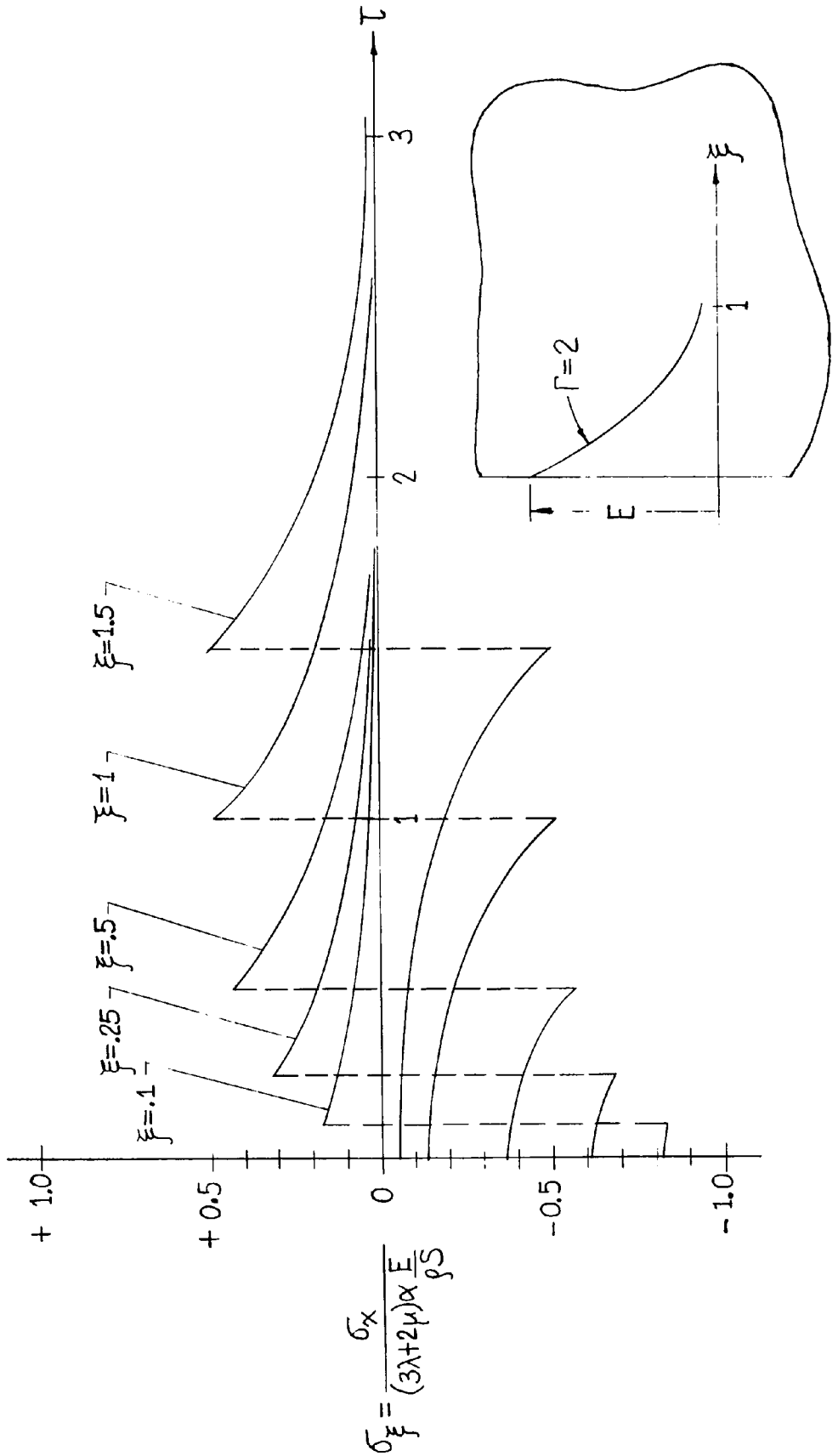


Fig. 3. Nondimensional Stress vs. Nondimensional Time for Half-Space.

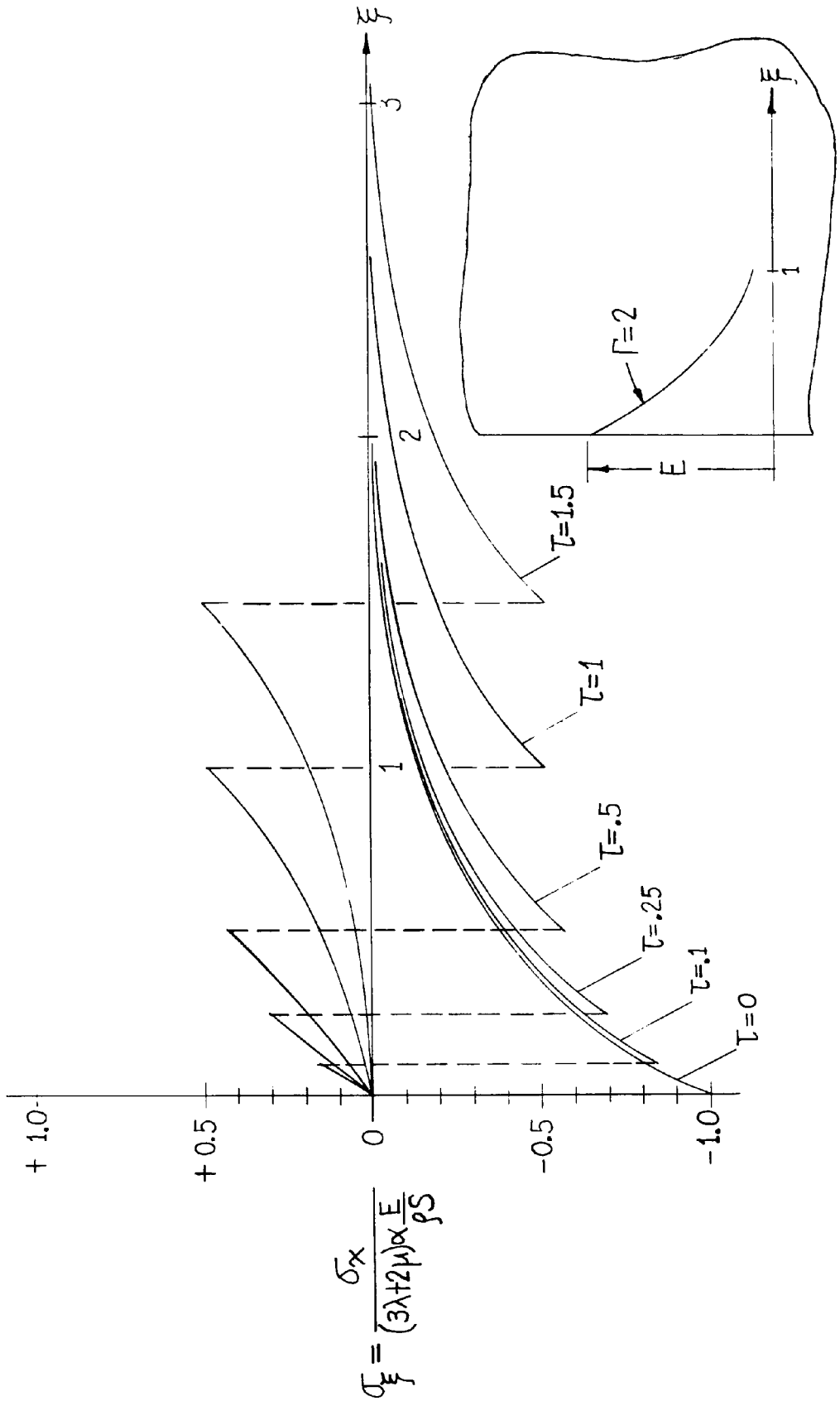


Fig. 4. Nondimensional Stress vs. Nondimensional Distance From Surface for Half-Space.

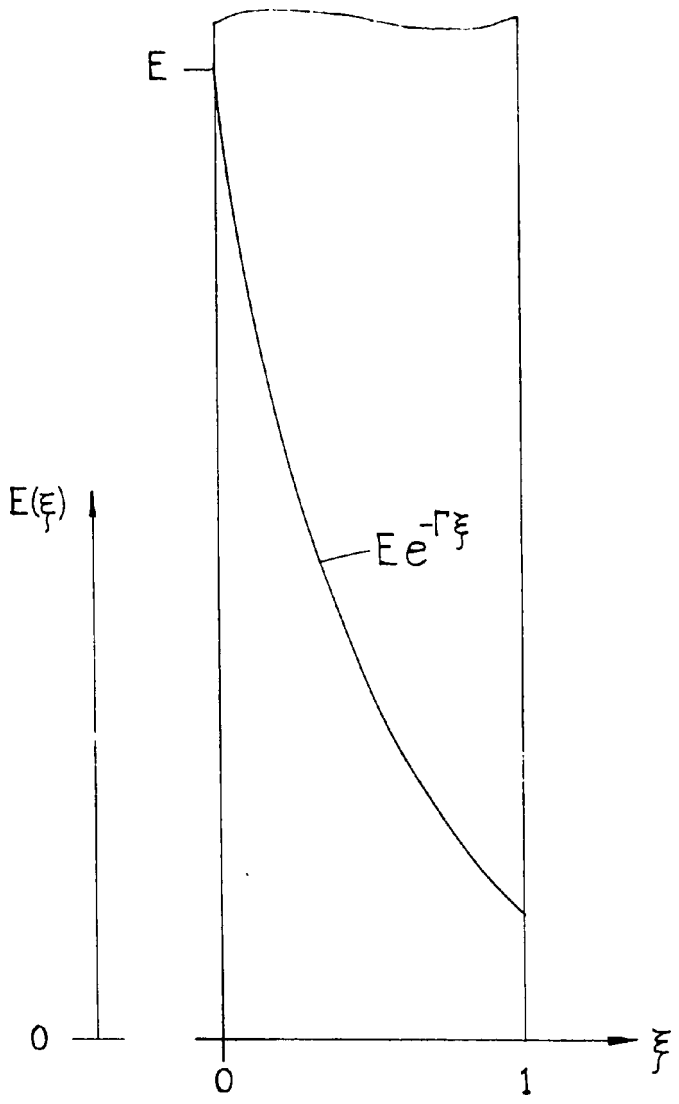


Fig. 5. Energy Deposition Diagram for Single Layer.

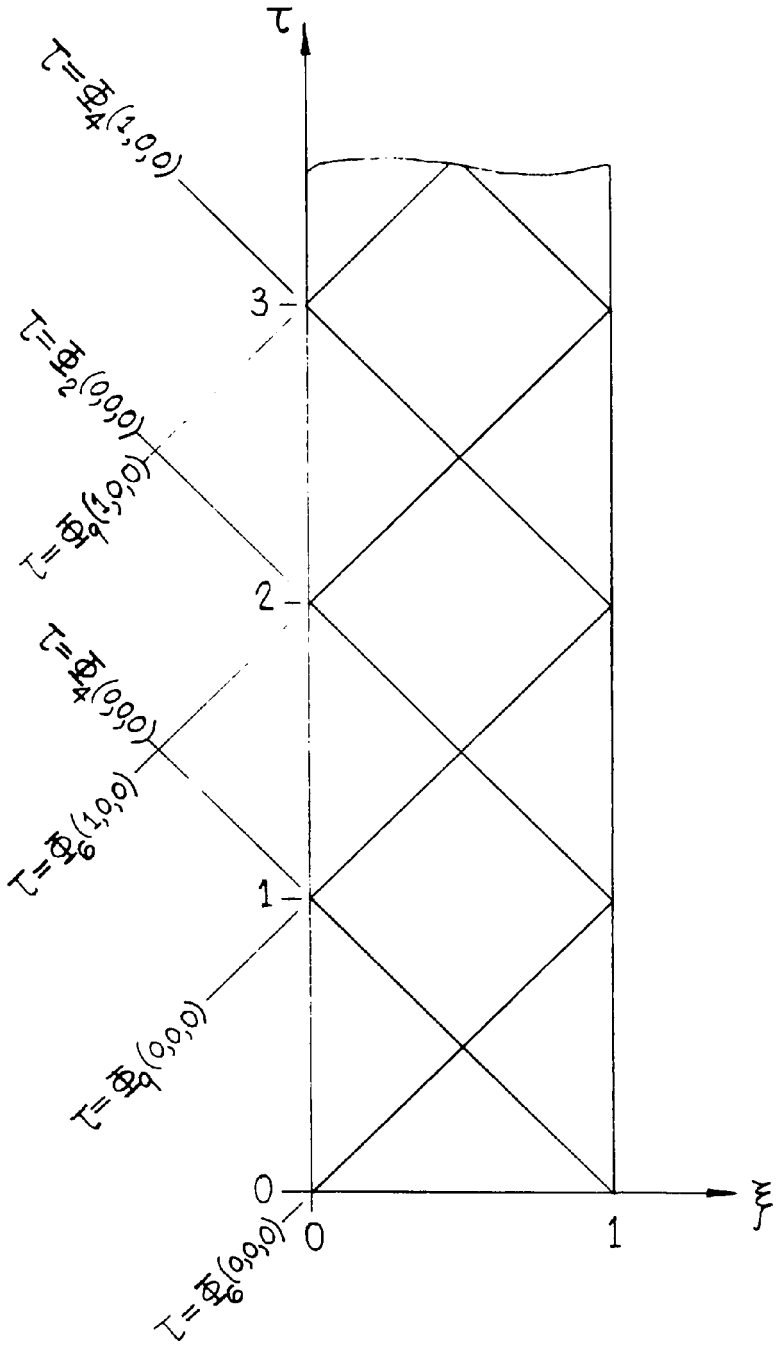


Fig. 6. - ξ-τ Solution Space for Single Layer.

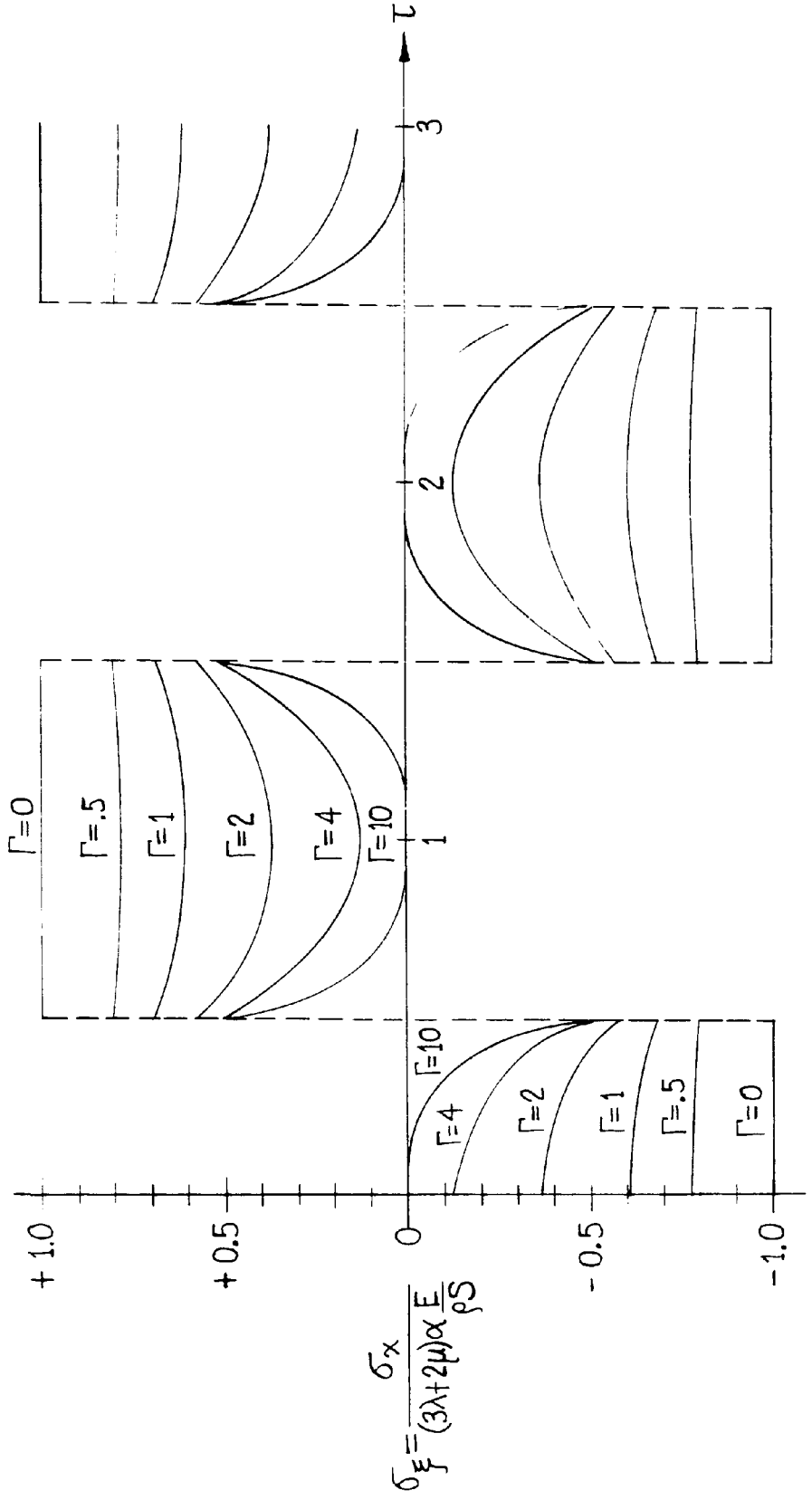


Fig. 7. Nondimensional Stress vs. Nondimensional Time at $\xi = 0.5$ for Single Layer.

C. The Two-Layer Media

1. Front Layer Strong Absorber--Back Layer Weak Absorber

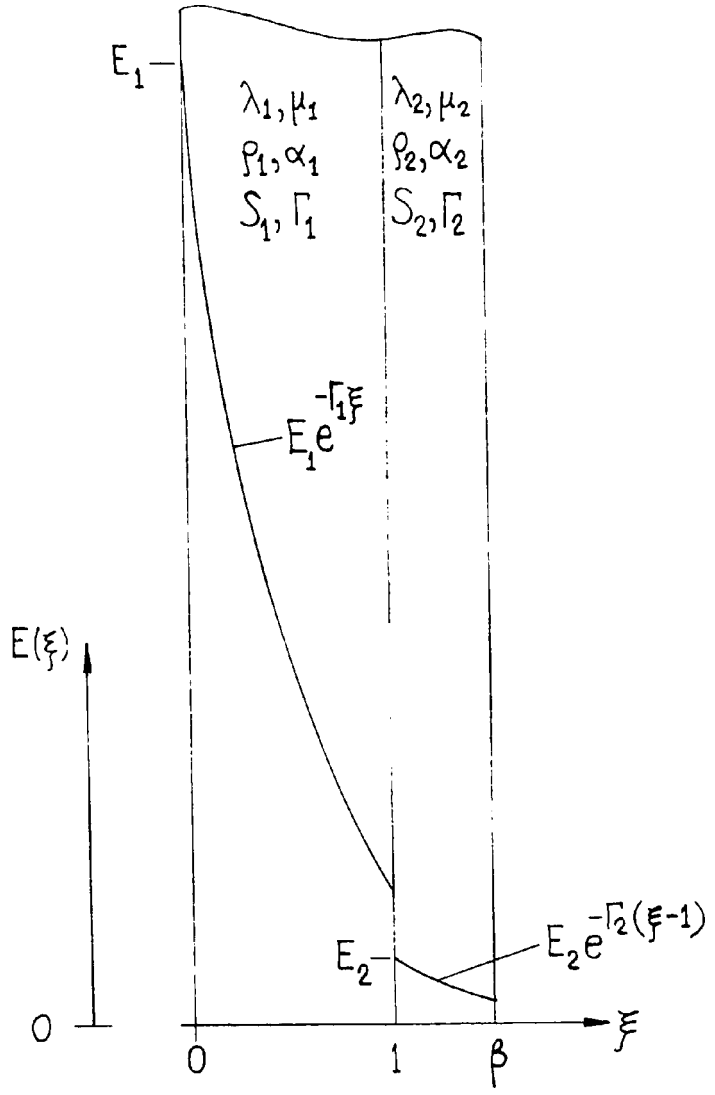
The parameters for this example are described in Fig.8. The value of Γ is the same as in the single layer example. The results show the effect of adding a layer of material to the rear surface. The ratio of energy absorption at $\xi=1$ expressed by $\left.\frac{E_1}{E_2}\right|_{\xi=1}$ is taken to be 2 with the other parameters chosen to give the value of $K=0.037$. The $\xi-\tau$ solution space is shown in Fig.9. Fig.10 shows the nondimensional stress at $\xi=0.5$ as a function of the nondimensional time. The maximum tensile stress is at $\xi=0.5$ and occurs at $\tau=0.5$ combining the tensile jump of unit magnitude across $\tau=\Phi_6(0,0,0)$ and the tensile jump across $\tau=\Phi_4(0,0,0)$ of magnitude $(1+\gamma_{21})^{-1}(e^{-\Gamma_1}-K)<e^{-\Gamma_1}$. Hence addition of the rear layer has reduced the maximum tensile stress from $\sigma_\xi=0.57$ to $\sigma_\xi=0.49$. As time increases and multiple reflections occur, the stresses are attenuated by stress wave reflection and refraction at the plane $\xi=1$.

The bond stress at $\xi=1$ may be of particular importance in practical application. It may be determined from either Eqn.(3.63) or (3.64). Fig.10 shows the bond stress. The maximum tensile value is about 0.38 at $\tau=1.90$. Because of the relatively small value of the tensile jump across $\tau=\Phi_3(0,0,0)$, this factor was ignored in calculating the bond stress.

2. Front and Back Layers Absorbing

The parameters considered in this example are shown in Fig.11. The wave speed ratio and the specific acoustic impedance ratio are sufficiently different to show a wide range of effects. The ratio of energy absorption at $\xi = 1$ is taken as 0.4 with the other parameters chosen to give the value of $K = 0.57$. The ξ-τ solution space for a value of $\beta = 1.5$ is shown in Fig.12. Because of the different parameters, the maximum tensile stress of 0.43 occurs at $\xi = 0.5^-$ due to the tensile jump of unit magnitude across $\tau = \Phi_6(0,0,0)$. In this case the stress jump across $\tau = \Phi_4(0,0,0)$ is compressive and hence combines to give a substantially reduced tensile stress at $\xi = 0.5$. Fig.13 shows the stress at $\xi = 0.1$ through 0.4, $\xi = 0.5^-$ and at $\xi = 0.5^+$. Fig.14 shows the stress at $\xi = 0.5$ (which here is <u>not</u> the location of maximum tensile stress) as well as the bond stress.

Since the stress jump across $\tau = \Phi_3(0,0,0)$ is of considerable relative magnitude, it is instructive to show the effect of changing the thickness ratio β. Decreasing β will create the possibility of a higher tensile stress in the front layer. The magnitude of the increase depends upon the value of β. Fig.15 shows the ξ-τ solution space for $\beta = 1.25$. In the front layer the jump across $\tau = \Phi_3(0,0,0)$ at $\xi = 0.5$ now occurs at $\tau = 2.33$. This creates a substantial difference in the maximum tensile stress there. Fig.16 shows the stress at $\xi = 0.5$ and the bond stress for this case.



$$K = \frac{(3\lambda+2\mu)_2\alpha_2}{(3\lambda+2\mu)_1\alpha_1}\,\frac{E_2/(\rho S)_2}{E_1/(\rho S)_1} = 0.037$$

$$\beta = 1.50\ ,\quad \Gamma_1 = \Gamma_2 = 2\ ,\quad r = 0.9\ ,\quad \gamma_{21} = \frac{(\rho c)_2}{(\rho c)_1}$$

Fig. 8. Energy Deposition Diagram for Two-Layer Media; Front Layer Strong Absorber.

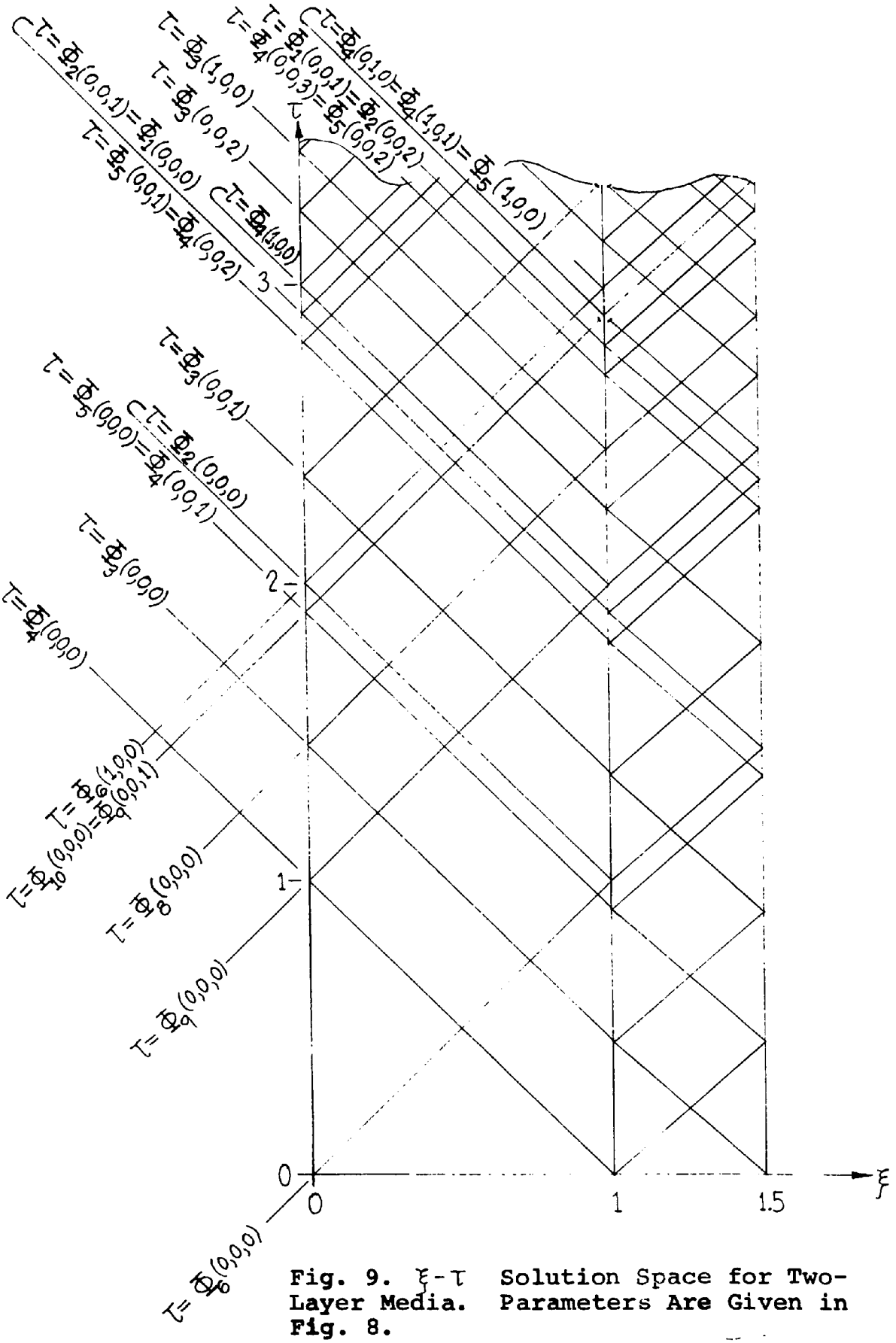


Fig. 9. ξ-τ Solution Space for Two-Layer Media. Parameters Are Given in Fig. 8.

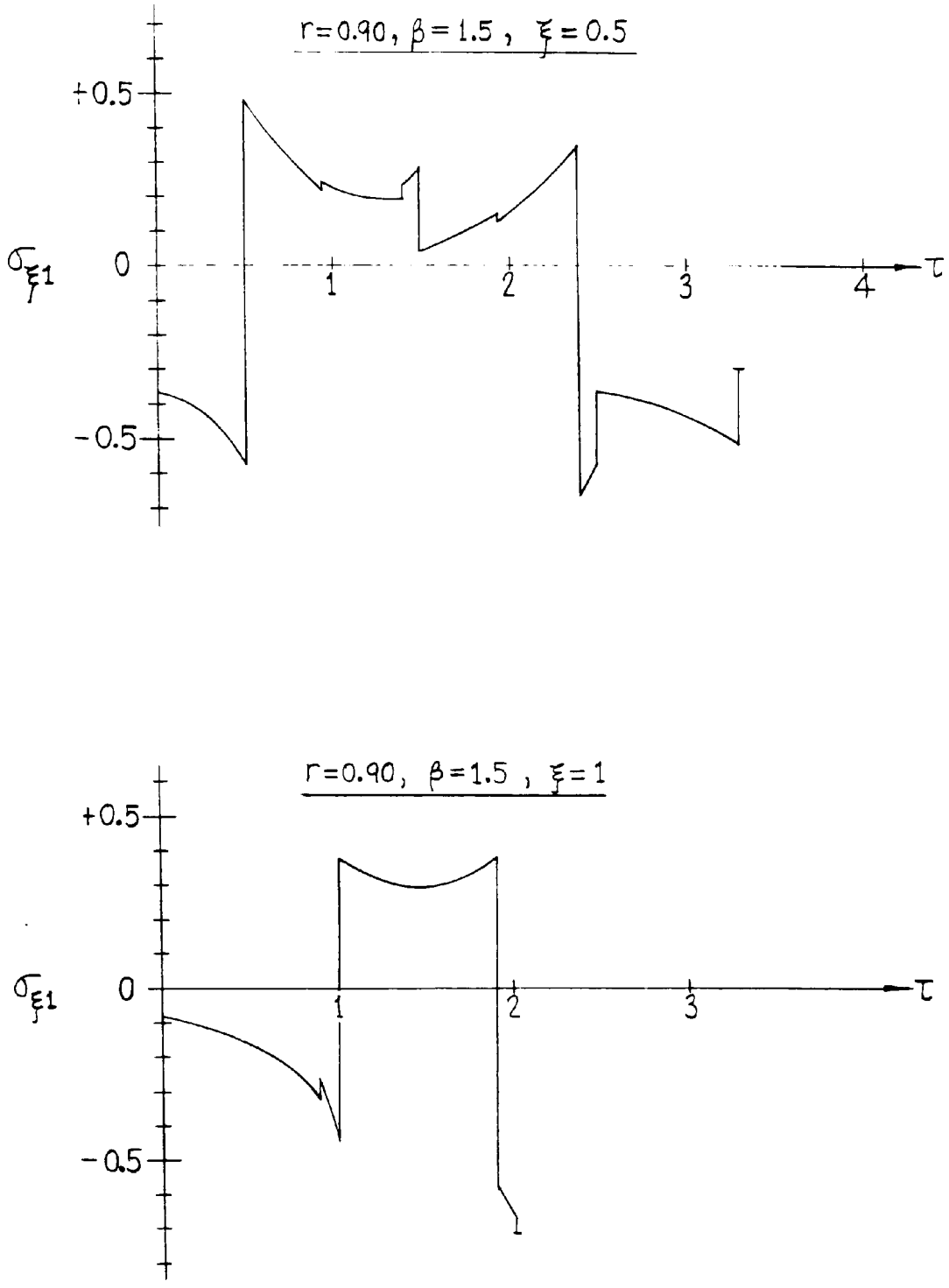
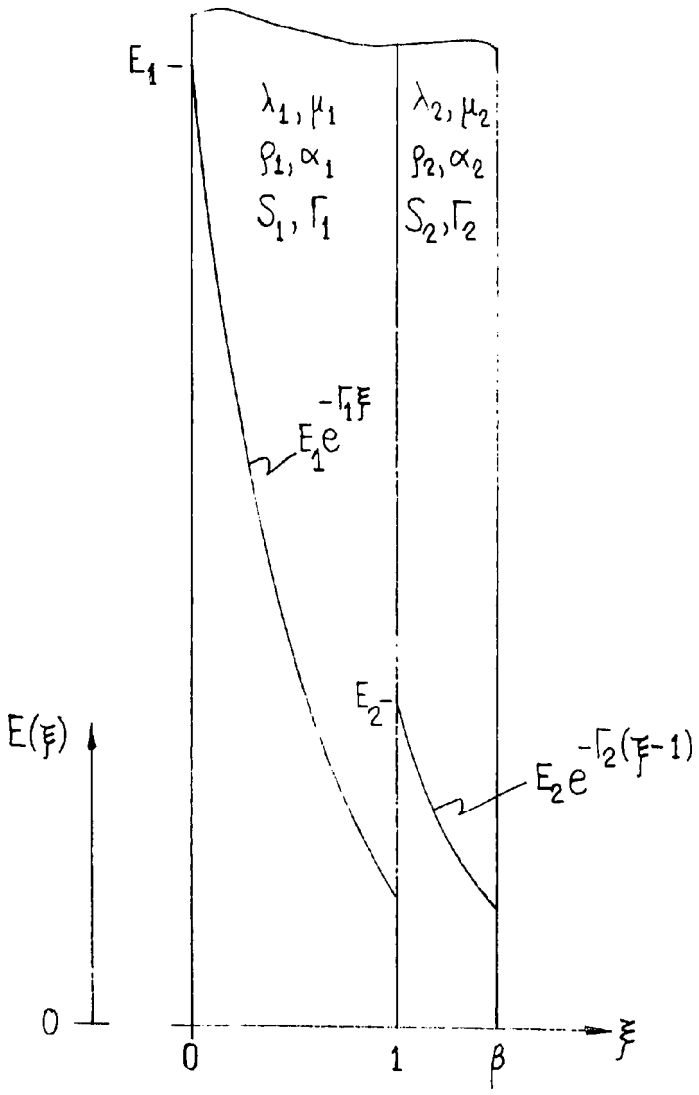


Fig. 10. Nondimensional Stress vs. Nondimensional Time for Example shown in Fig. 8. $\xi = 0.5$ is the Middle Plane of Front Layer (Top). $\xi = 1$ is the Bond Plane (Bottom).



$$K = \frac{(3\lambda+2\mu)_2 \alpha_2}{(3\lambda+2\mu)_1 \alpha_1} \frac{E_2/(\rho S)_2}{E_1/(\rho S)_1} = 0.57,$$

$\beta = 1.50$ and 1.25, $\Gamma_1 = \Gamma_2 = 2$, $r = 7.35$, $\gamma_{21} = \frac{(\rho c)_2}{(\rho c)_1} = 0.20$

Fig. 11. Energy Deposition Diagram for Two-Layer Media; Both Layers Absorbing.

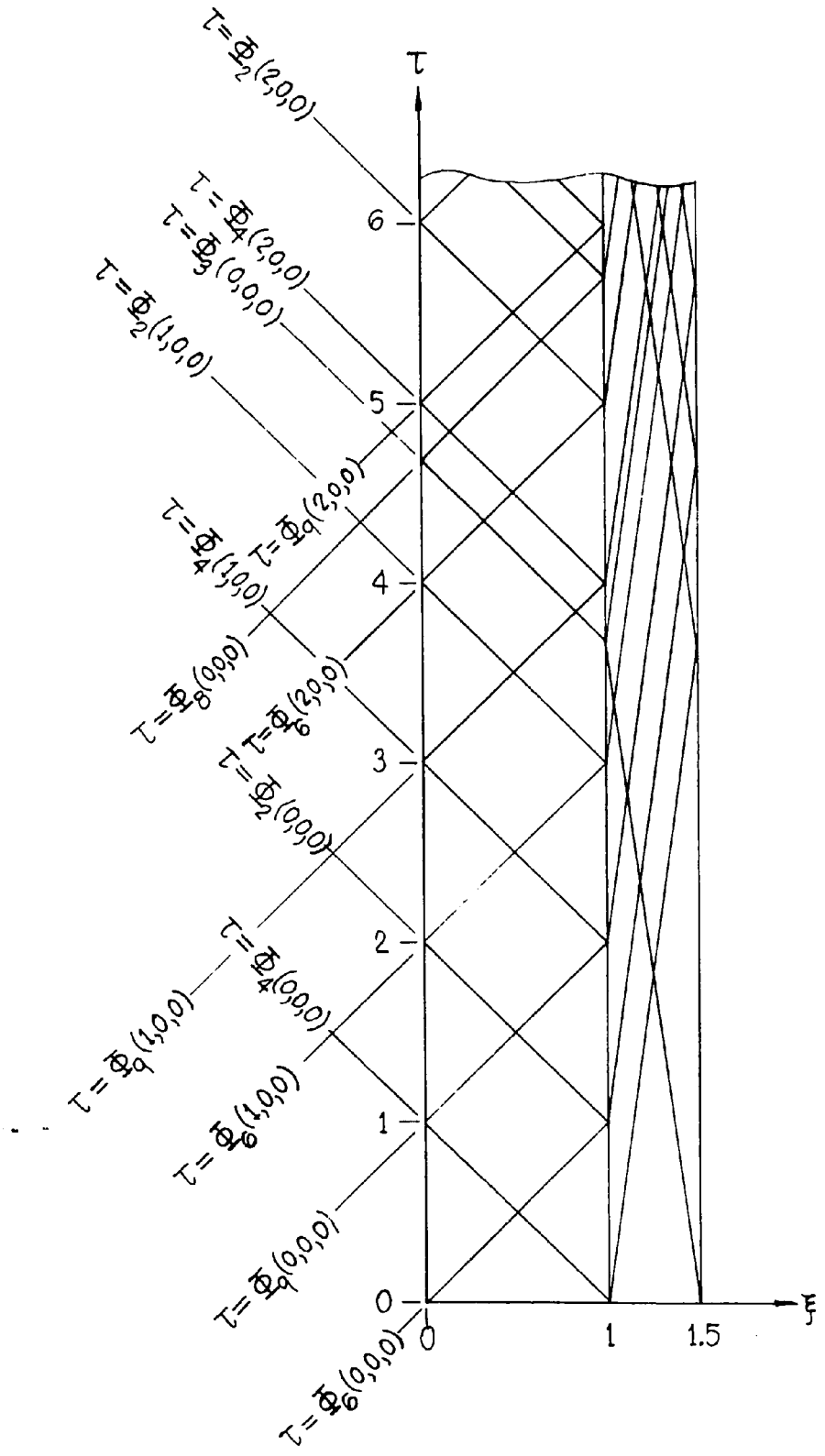


Fig. 12. $\xi-\tau$ Solution Space for Two-Layer Media. Parameters Are Given In Fig. 11. Thickness Ratio $\beta = 1.5$

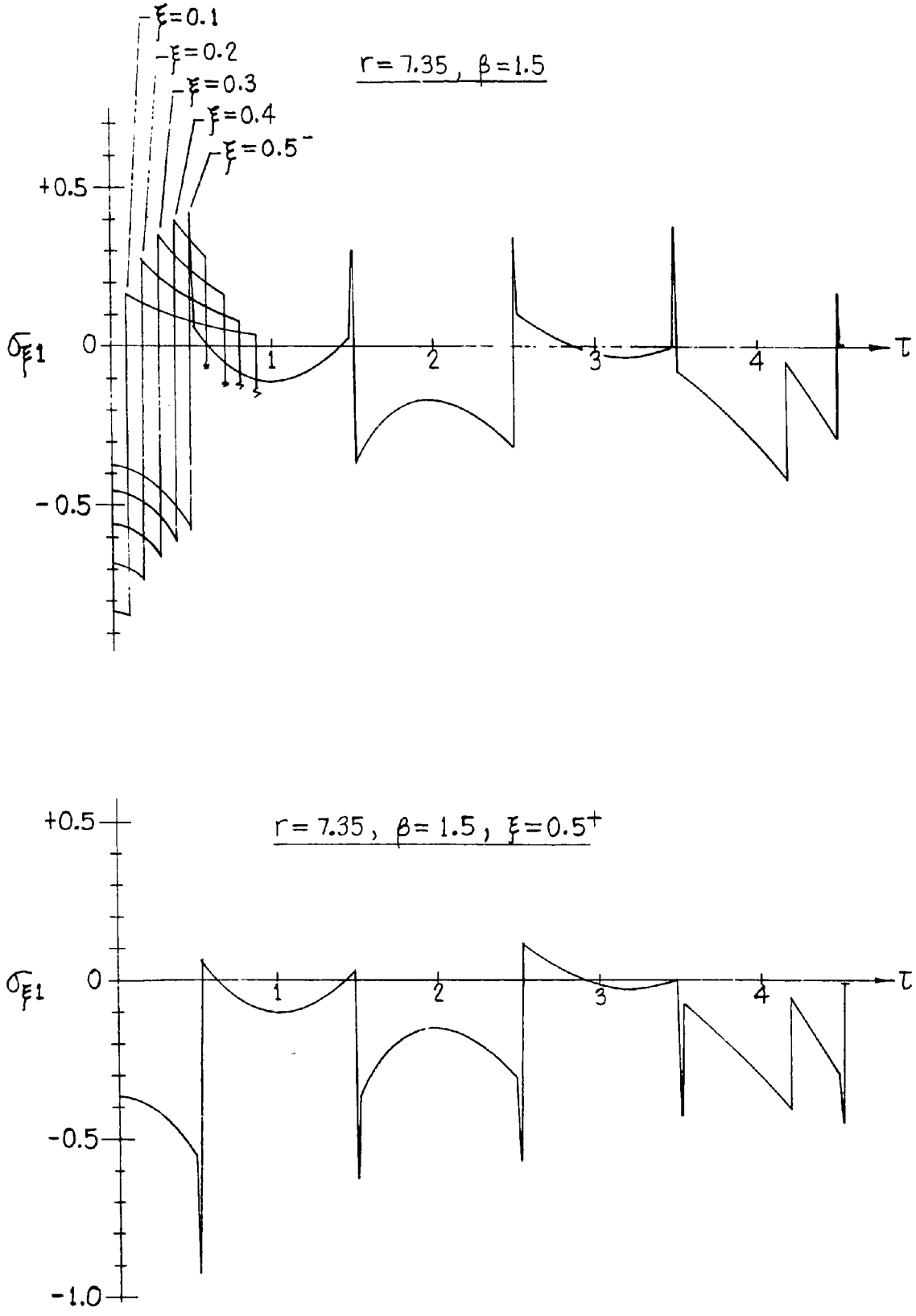


Fig. 13. Nondimensional Stress vs. Nondimensional Time for Example shown in Fig. 11. $\xi = 0.5$ is the Middle Plane of Front Layer.

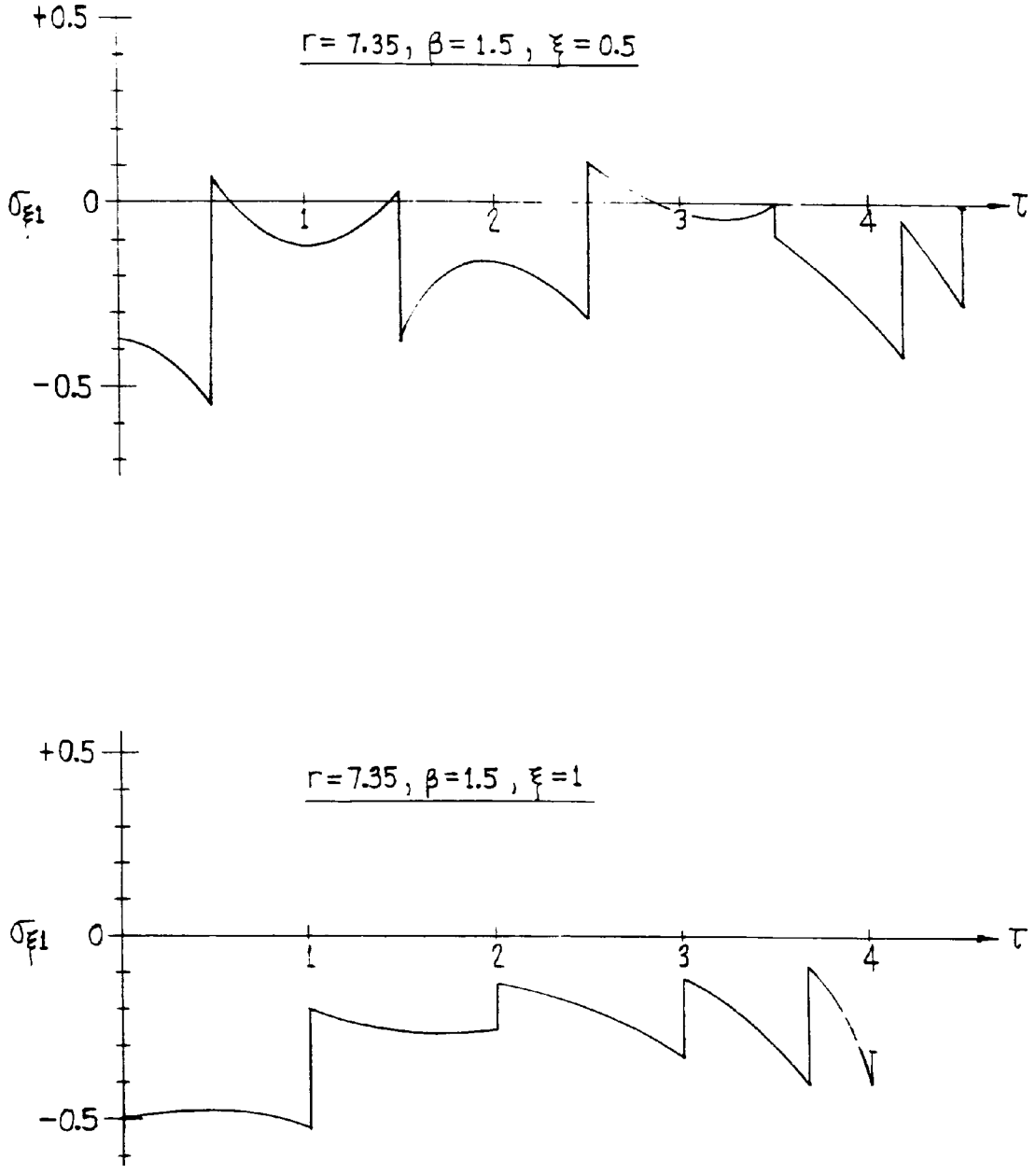


Fig. 14. Nondimensional Stress vs. Nondimensional Time for Example shown in Fig. 11. $\xi = 0.5$ is the Middle Plane of Front Layer (Top). $\xi = 1$ is the Bond Plane (Bottom).

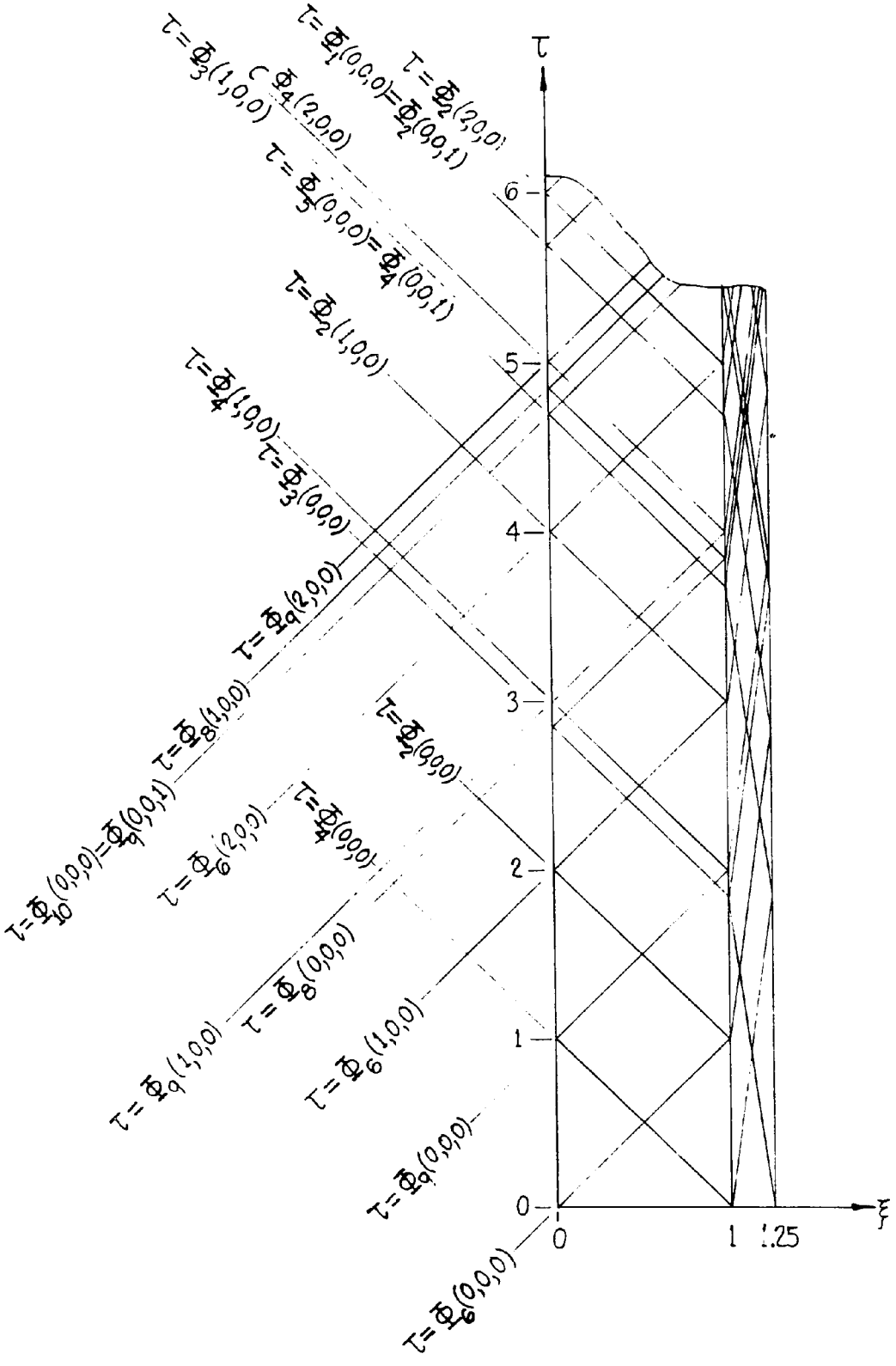


Fig. 15. ξ-τ Solution Space for Two-Layer Media. Parameters Are Given in Fig. 11. Thickness Ratio $\beta = 1.25$.

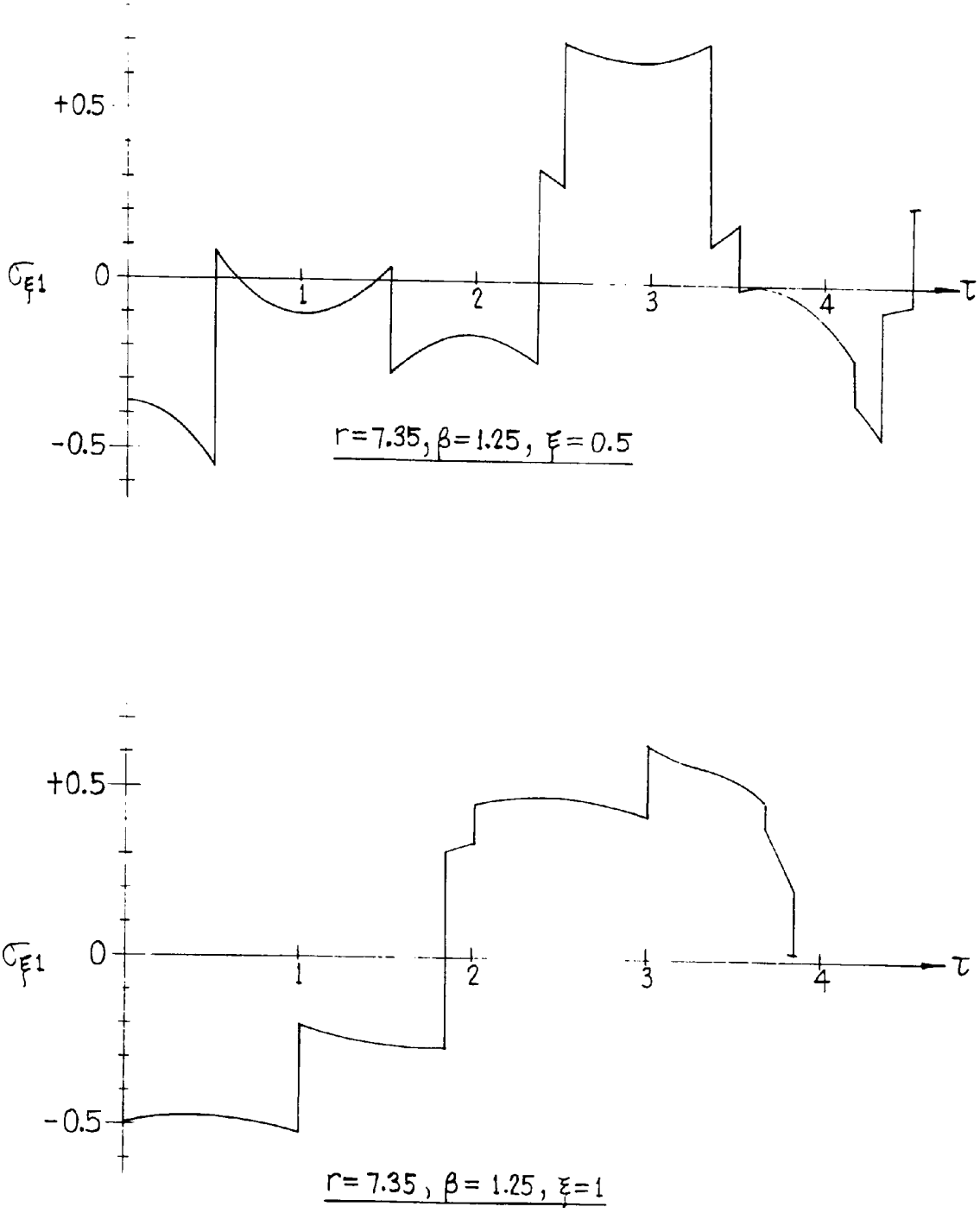


Fig. 16. Nondimensional Stress vs. Nondimensional Time for Example shown in Fig. 15. $\xi = 0.5$ is the Middle Plane of Front Layer (Top). $\xi = 1$ is the Bond Plane (Bottom).

CHAPTER V

CONCLUSIONS

The transient mechanical response of a layered media due to impulsive energy deposition has been analyzed. The media is formed of two elastic, homogeneous and isotropic plane layers joined by an interface molecular bond. A one-dimensional formulation has resulted from neglecting in-plane strain components. The equations of linear thermo-elasticity, assuming temperature independent properties, have been solved retaining inertia and neglecting thermo-mechanical coupling and heat conduction.

A solution for the stresses has been developed which is valid for all values of time. The present investigation provides a systematic method of accounting for multiple reflections in the media.

The magnitude of the maximum stress in the layer material and at the bond line depends on the thickness and wave speed ratios of the layered media for given energy deposition functions. The maximum layer stress (or bond stress) depends markedly upon the manner in which the generated stress waves combine to either reinforce or cancel. This suggests that a stress optimumization procedure may be developed in a specific application. This may be of particular importance where the material is subject to brittle fracture caused by excessive tensile stress.

Finally, from the solution of the problem of a thermal-

ly conducting homogeneous layer, it has been shown that the effect of heat conduction in determining the initial transient stresses is negligible.

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APPENDIX A

SINGLE LAYER INCLUDING HEAT CONDUCTION

1. Formulation of Problem

The governing equations including heat conduction and neglecting thermomechanical coupling are given by Eqns.(2.13) and (2.14). They are

$$(\lambda+2\mu)\frac{\partial^2 u(x,t)}{\partial x^2} - (3\lambda+2\mu)\alpha\frac{\partial\theta(x,t)}{\partial x} = \rho\frac{\partial^2 u(x,t)}{\partial t^2} \tag{A1.1}$$

and

$$k\frac{\partial^2\theta(x,t)}{\partial x^2} + Q(x,t) = \rho S\frac{\partial\theta(x,t)}{\partial t} \tag{A1.2}$$

where the symbols have the same meaning as given in Chapter II.

The strain-displacement and the stress-strain relationships are given by Eqns.(2.3) and (2.4). The region considered is shown in Fig.17. Assuming the external boundaries to be adiabatic, i.e. no heat flux through the surfaces at $x=0$ and $x=a$, the thermal and mechanical boundary conditions are

$$\frac{\partial\theta(0,t)}{\partial x} = 0, \quad t\geq 0 \tag{A1.3}$$

$$\frac{\partial\theta(a,t)}{\partial x} = 0, \quad t\geq 0 \tag{A1.4}$$

$$\sigma_x(0,t) = 0, \quad t\geq 0 \tag{A1.5}$$

and

$$\sigma_x(a,t) = 0, \quad t\geq 0 \tag{A1.6}$$

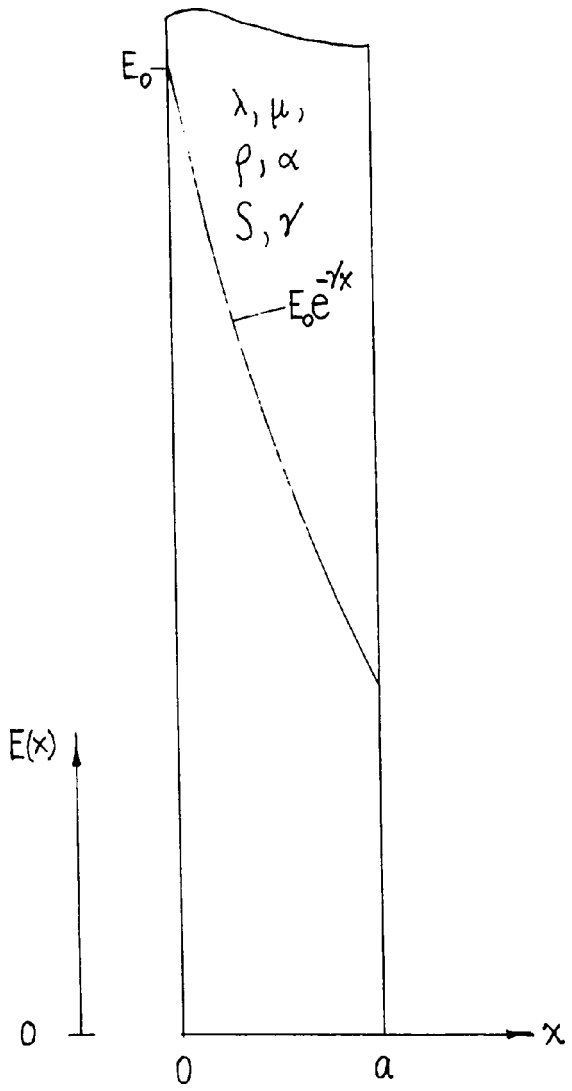


Fig. 17. Definition Diagram for Single Layer.

For an initially undisturbed media, the initial conditions are

$$u(x,0) = 0 \; ; \quad 0 \leq x \leq a \tag{A1.7}$$

$$\frac{\partial u(x,0)}{\partial t} = 0 \; , \quad 0 \leq x \leq a \tag{A1.8}$$

and

$$\theta(x,0) = 0 \; ; \quad 0 \leq x \leq a \tag{A1.9}$$

Applying the Laplace transform defined by Eqn.(3.1) to Eqns.(A1.1) and (A1.2) and using the initial conditions Eqns.(A1.7) through (A1.9) gives

$$\frac{c^2 \partial^2 u^*(x,s)}{\partial x^2} - \frac{s^2}{c^2} u^*(x,s) = \Lambda \frac{\partial \theta^*(x,s)}{\partial x} \; ; \quad 0 \leq x \leq a \tag{A1.10}$$

and

$$\frac{\partial^2 \theta^*(x,s)}{\partial x^2} - \frac{s}{\kappa} \theta^*(x,s) = -\frac{Q^*(x,s)}{k} \; ; \quad 0 \leq x \leq a \tag{A1.11}$$

where

$$\kappa = \frac{k}{\rho S}$$

$$c^2 = \frac{\lambda + 2\mu}{\rho} \tag{A1.12}$$

and

$$\Lambda = \frac{(3\lambda + 2\mu)\alpha}{(\lambda + 2\mu)}$$

Applying the transform to the boundary conditions Eqns. (A1.3) through (A1.6) gives

$$\frac{\partial \theta^*(0,s)}{\partial x} = 0 \tag{A1.13}$$

$$\frac{d\theta^*(a,s)}{dx} = 0 \qquad \text{(A1.14)}$$

and

$$\frac{du^*(0,s)}{dx} = \Lambda\theta^*(0,s) \qquad \text{(A1.15)}$$

$$\frac{du^*(a,s)}{dx} = \Lambda\theta^*(a,s) \qquad \text{(A1.16)}$$

Using the deposition function shown in Fig.17 assumed to have a delta function time dependence

$$Q^* = E_0 e^{-\gamma x} \qquad \text{(A1.17)}$$

2. Solution

The general solution of (A1.11) is

$$\theta^*(x,s) = \theta_h^*(x,s) + \theta_p^*(x,s)$$

where θ_h^* is the solution of the homogeneous equation and θ_p^* is a particular solution. A complete solution of (A1.11) is

$$\theta^*(x,s) = Ae^{\sqrt{s/\kappa}\,x} + Be^{-\sqrt{s/\kappa}\,x} + \frac{\kappa E_0 e^{-\gamma x}}{k}\frac{1}{(s-\kappa\gamma^2)} \qquad \text{(A2.1)}$$

A and B are determined from the boundary conditions Eqns.(A1.13) and A1.14). These give

$$\sqrt{s/\kappa}\,A - \sqrt{s/\kappa}\,B = \frac{\kappa\gamma E_0}{k}\frac{1}{(s-\kappa\gamma^2)}$$

$$\sqrt{s/\kappa}\,A - \sqrt{s/\kappa}\,B = \frac{\kappa\gamma E_0 e^{-\gamma a}}{k}\frac{1}{(s-\kappa\gamma^2)}$$

The determinant of the coefficients of A and B is

$$\Delta_1 = \begin{vmatrix} \sqrt{s/\kappa} & -\sqrt{s/\kappa} \\ \sqrt{s/\kappa}\, e^{\sqrt{s/\kappa}\, a} & \sqrt{s/\kappa}\, e^{-\sqrt{s/\kappa}\, a} \end{vmatrix}$$

or

$$\Delta_1 = \frac{s}{\kappa}\left\{ e^{\sqrt{s/\kappa}\, a} - e^{-\sqrt{s/\kappa}\, a} \right\} \tag{A2.2}$$

Solving for $\Delta_1 A$ and $\Delta_1 B$ gives

$$\Delta_1 A = \frac{\kappa\gamma E_0 e^{-\gamma a}}{k} \sqrt{s/\kappa}\, \frac{1}{(s-\kappa\gamma^2)} - \frac{\kappa\gamma E_0}{k} \sqrt{s/\kappa}\, \frac{1}{(s-\kappa\gamma^2)}\, e^{-\sqrt{s/\kappa}\, a} \tag{A2.3}$$

and

$$\Delta_1 B = \frac{\kappa\gamma E_0 e^{-\gamma a}}{k} \sqrt{s/\kappa}\, \frac{1}{(s-\kappa\gamma^2)} - \frac{\kappa\gamma E_0}{k} \sqrt{s/\kappa}\, \frac{1}{(s-\kappa\gamma^2)}\, e^{\sqrt{s/\kappa}\, a} \tag{A2.4}$$

Eqn. (A2.2) may be written as

$$\frac{1}{\Delta_1} = \frac{\kappa}{s} e^{-\sqrt{s/\kappa}\, a} \frac{1}{\left\{1 - e^{-2\sqrt{s/\kappa}\, a}\right\}} \tag{A2.5}$$

But

$$\frac{1}{\left\{1 - e^{-2\sqrt{s/\kappa}\, a}\right\}} = 1 + e^{-2\sqrt{s/\kappa}\, a} + e^{-4\sqrt{s/\kappa}\, a} + \cdots = \sum_{n=0}^{\infty} e^{-2n\sqrt{s/\kappa}\, a}$$

and is absolutely and uniformly convergent for $Re(s) > 0$.

Hence

$$\begin{aligned} \theta^*(x,s) = {} & \frac{\kappa^{3/2}\gamma E_0 e^{-\gamma a}}{k} \frac{1}{\sqrt{s}(s-\kappa\gamma^2)} \left[\left[\sum_{n=0}^{\infty} \exp\left\{ -\left[\frac{2na + (a-x)}{\sqrt{\kappa}} \right] \sqrt{s} \right\} \right. \right. \\ & \left. \left. + \sum_{n=0}^{\infty} \exp\left\{ -\left[\frac{2na + (a+x)}{\sqrt{\kappa}} \right] \sqrt{s} \right\} \right] \right] \\ & - \frac{\kappa^{3/2}\gamma E_0}{k} \frac{1}{\sqrt{s}(s-\kappa\gamma^2)} \left[\left[\sum_{n=0}^{\infty} \exp\left\{ -\left[\frac{2na + (2a-x)}{\sqrt{\kappa}} \right] \sqrt{s} \right\} \right. \right. \\ & \left. \left. + \sum_{n=0}^{\infty} \exp\left\{ -\left[\frac{2na - x}{\sqrt{\kappa}} \right] \sqrt{s} \right\} \right] \right] \\ & + \frac{\kappa E_0 e^{-\gamma x}}{k} \frac{1}{(s-\kappa\gamma^2)} \end{aligned} \tag{A2.6}$$

Eqn.(A1.10) can now be solved since $\theta^*(x,s)$ is determined. This is given by

$$u^*(x,s) = u_h^*(x,s) + u_p^*(x,s) \tag{A2.7}$$

with

$$u_h^*(x,s) = De^{\frac{s}{c}x} + Ee^{-\frac{s}{c}x} \tag{A2.8}$$

and

$$u_p^*(x,s) = Fe^{\sqrt{s/k}\,x} + Ge^{-\sqrt{s/k}\,x} + He^{-\gamma x} \tag{A2.9}$$

The constants F, G and H are obtained by equating coefficients in Eqn.(A1.10) after the substitution of Eqn.(A2.9). The result is

$$F = -\frac{Ac^2k\gamma E_0 e^{-\gamma a}}{k}\frac{1}{s(s-c^2/k)(s-k\gamma^2)}\sum_{n=0}^{\infty}\exp\left\{-\left[\frac{2n,a+a}{\sqrt{k}}\right]\sqrt{s}\right\}$$

$$+\frac{Ac^2k\gamma E_0}{k}\frac{1}{s(s-c^2/k)(s-k\gamma^2)}\sum_{n=0}^{\infty}\exp\left\{-\left[\frac{2na+2a}{\sqrt{k}}\right]\sqrt{s}\right\}$$

$$G = \frac{Ac^2k\gamma E_0 e^{-\gamma a}}{k}\frac{1}{s(s-c^2/k)(s-k\gamma^2)}\sum_{n=0}^{\infty}\exp\left\{-\left[\frac{2na+a}{\sqrt{k}}\right]\sqrt{s}\right\} \tag{A2.10}$$

$$-\frac{Ac^2k\gamma E_0}{k}\frac{1}{s(s-c^2/k)(s-k\gamma^2)}\sum_{n=0}^{\infty}\exp\left\{-\left[\frac{2na+a}{\sqrt{k}}\right]\sqrt{s}\right\}$$

and

$$H = \frac{Ac^2k\gamma E_0}{k}\frac{1}{(s^2-c^2\gamma^2)(s-k\gamma^2)}$$

The determination of D and E is made by satisfying the boundary conditions, Eqns.(A1.15) and (A1.16). This gives

$$\begin{matrix} D & -E & = R_1 \\ e^{\frac{s}{c}a} & -e^{-\frac{s}{c}a} & \\ e^{\frac{s}{c}a} D & -e^{-\frac{s}{c}a} E & = R_2 \end{matrix} \qquad \text{(A2.11)}$$

where

$$R_1 = -\frac{c}{\sqrt{k}}\frac{1}{\sqrt{s}} F + \frac{c}{\sqrt{k}}\frac{1}{\sqrt{s}} G + c\gamma\frac{1}{s} H + \Lambda c \frac{1}{s}\theta^*(0,s) \qquad \text{(A2.12)}$$

and

$$R_2 = -\frac{c}{\sqrt{k}}\frac{1}{\sqrt{s}} e^{\sqrt{s/k}\,a} F + \frac{c}{\sqrt{k}}\frac{1}{\sqrt{s}} e^{-\sqrt{s/k}\,a} G$$

$$+ c\gamma e^{-\gamma a}\frac{1}{s} H + \Lambda c \frac{1}{s}\theta^*(a,s) \qquad \text{(A2.13)}$$

The determinant of the coefficients of D and E is

$$\Delta_2 = \begin{vmatrix} 1 & -1 \\ e^{\frac{s}{c}a} & -e^{-\frac{s}{c}a} \end{vmatrix}$$

or

$$\Delta_2 = e^{\frac{s}{c}a} - e^{-\frac{s}{c}a} \qquad \text{(A2.14)}$$

Solving for $\Delta_2 D$ and $\Delta_2 E$ gives

$$\Delta_2 D = R_2 - e^{-\frac{s}{c}a} R_1 \qquad \text{(A2.15)}$$

and

$$\Delta_2 E = R_2 - e^{\frac{s}{c}a} R_1 \qquad \text{(A2.16)}$$

Eqn.(A2.14) may be written as

$$\frac{1}{\Delta_2} = e^{-\frac{s}{c}a}\frac{1}{\left\{1 - e^{-2\frac{s}{c}a}\right\}}$$

$$\frac{1}{\left\{1 - e^{-2\frac{s}{c}a}\right\}} = 1 + e^{-2\frac{s}{c}a} + e^{-4\frac{s}{c}a} + \cdots = \sum_{p=0}^{\infty} e^{-2p\frac{s}{c}a}$$

which is absolutely and uniformly convergent for $Re(s) > 0$.

Eqn.(A2.7) becomes

$$u^*(x,s) = R_2 \sum_{p=0}^{\infty} \exp\left\{-\left[\frac{2pa+(a-x)}{c}\right]s\right\} - R_1 \sum_{p=0}^{\infty} \exp\left\{-\left[\frac{2pa+(2a-x)}{c}\right]s\right\}$$
$$+ R_2 \sum_{p=0}^{\infty} \exp\left\{-\left[\frac{2pa+(a+x)}{c}\right]s\right\} - R_1 \sum_{p=0}^{\infty} \exp\left\{-\left[\frac{2pa+x}{c}\right]s\right\}$$
$$+ F e^{\sqrt{s/k}\,x} + G e^{-\sqrt{s/k}\,x} + H e^{-\gamma x} \qquad \text{(A2.17)}$$

The transformed stress is

$$\sigma_x^*(x,s) = (\lambda+2\mu)\frac{du^*(x,s)}{dx} - (3\lambda+2\mu)\alpha\,\theta^*(x,s) \qquad \text{(A2.18)}$$

Using (A2.17) and (A2.6) gives $\sigma_x^*(x,s)$ as

$$\sigma_x^*(x,s) = (\lambda+2\mu)\Bigg\langle \sum_{p=0}^{\infty} e^{-\left[\frac{2pa+x}{c}\right]s} \Bigg\{ 2\sqrt{s/k}\left[\frac{\Lambda c^2 k\gamma E_0 e^{-\gamma a}}{k} \frac{1}{s(s-c^2/k)(s-k\gamma^2)} \sum_{n=0}^{\infty} e^{-\left[\frac{2na+a}{\sqrt{k}}\right]\sqrt{s}}\right]$$
$$- \sqrt{s/k}\left[\frac{\Lambda c^2 k\gamma E_0}{k} \frac{1}{s(s-c^2/k)(s-k\gamma^2)} \sum_{n=0}^{\infty} e^{-\left[\frac{2na+2a}{\sqrt{k}}\right]\sqrt{s}}\right]$$
$$- \sqrt{s/k}\left[\frac{\Lambda c^2 k\gamma E_0}{k} \frac{1}{s(s-c^2/k)(s-k\gamma^2)} \sum_{n=0}^{\infty} e^{-\left[\frac{2na}{\sqrt{k}}\right]\sqrt{s}}\right]$$
$$+ \frac{\Lambda c^2 k\gamma^2 E_0}{k} \frac{1}{(s^2-c^2\gamma^2)(s^2-k\gamma^2)}$$
$$+ \frac{2\Lambda k^{3/2}\gamma E_0 e^{-\gamma a}}{k} \frac{1}{\sqrt{s}(s-k\gamma^2)} \sum_{n=0}^{\infty} e^{-\left[\frac{2na+a}{\sqrt{k}}\right]\sqrt{s}}$$
$$- \frac{\Lambda k^{3/2}\gamma E_0}{k} \frac{1}{\sqrt{s}(s-k\gamma^2)} \left\langle \sum_{n=0}^{\infty} e^{-\left[\frac{2na+2a}{\sqrt{k}}\right]\sqrt{s}} + \sum_{n=0}^{\infty} e^{-\left[\frac{2na}{\sqrt{k}}\right]\sqrt{s}} \right\rangle$$
$$+ \frac{\Lambda k E_0}{k} \frac{1}{(s-k\gamma^2)} \Bigg\}$$

$$+\sum_{p=0}^{\infty} e^{-\left[\frac{2pa+(a-x)}{c}\right]s}\left\{\sqrt{s/k}\left[\frac{\Lambda c^2 k\gamma/E_0 e^{-\gamma a}}{k}\frac{1}{s(s-c^2/k)(s-k\gamma^2)}\sum_{n=0}^{\infty} e^{-\left[\frac{2na}{\sqrt{k}}\right]\sqrt{s}}\right]\right.$$

$$-2\sqrt{s/k}\left[\frac{\Lambda c^2 k\gamma/E_0}{k}\frac{1}{s(s-c^2/k)(s-k\gamma^2)}\sum_{n=0}^{\infty} e^{-\left[\frac{2na+a}{\sqrt{k}}\right]\sqrt{s}}\right]$$

$$+\sqrt{s/k}\left[\frac{\Lambda c^2 k\gamma/E_0 e^{-\gamma a}}{k}\frac{1}{s(s-c^2/k)(s-k\gamma^2)}\sum_{n=0}^{\infty} e^{-\left[\frac{2na+2a}{\sqrt{k}}\right]\sqrt{s}}\right]$$

$$-\frac{2\Lambda k^{3/2}\gamma/E_0}{k}\frac{1}{\sqrt{s}(s-k\gamma^2)}\sum_{n=0}^{\infty} e^{-\left[\frac{2na+a}{\sqrt{k}}\right]\sqrt{s}}$$

$$+\frac{\Lambda c^2 k\gamma^2 E_0 e^{-\gamma a}}{k}\frac{1}{(s^2-c^2\gamma^2)(s-k\gamma^2)}$$

$$\left.+\frac{\Lambda k^{3/2}\gamma E_0 e^{-\gamma a}}{k}\frac{1}{\sqrt{s}(s-k\gamma^2)}\left\langle\sum_{n=0}^{\infty} e^{-\left[\frac{2na}{\sqrt{k}}\right]\sqrt{s}}+\sum_{n=0}^{\infty} e^{-\left[\frac{2na+2a}{\sqrt{k}}\right]\sqrt{s}}\right\rangle\right.$$

$$\left.+\frac{\Lambda k E_0 e^{-\gamma a}}{k}\frac{1}{(s-k\gamma^2)}\right\}$$

$$-\sum_{p=0}^{\infty} e^{-\left[\frac{2pa+(a+x)}{c}\right]s}\left\{\sqrt{s/k}\left[\frac{\Lambda c^2 k\gamma/E_0 e^{-\gamma a}}{k}\frac{1}{s(s-c^2/k)(s-k\gamma^2)}\sum_{n=0}^{\infty} e^{-\left[\frac{2na}{\sqrt{k}}\right]\sqrt{s}}\right]\right.$$

$$-2\sqrt{s/k}\left[\frac{\Lambda c^2 k\gamma/E_0}{k}\frac{1}{s(s-c^2/k)(s-k\gamma^2)}\sum_{n=0}^{\infty} e^{-\left[\frac{2na+a}{\sqrt{k}}\right]\sqrt{s}}\right]$$

$$+\sqrt{s/k}\left[\frac{\Lambda c^2 k\gamma/E_0 e^{-\gamma a}}{k}\frac{1}{s(s-c^2/k)(s-k\gamma^2)}\sum_{n=0}^{\infty} e^{-\left[\frac{2na+2a}{\sqrt{k}}\right]\sqrt{s}}\right]$$

$$-\frac{2\Lambda k^{3/2}\gamma/E_0}{k}\frac{1}{\sqrt{s}(s-k\gamma^2)}\sum_{n=0}^{\infty} e^{-\left[\frac{2na+a}{\sqrt{k}}\right]\sqrt{s}}$$

$$+\frac{\Lambda c^2 k\gamma^2 E_0 e^{-\gamma a}}{k}\frac{1}{(s^2-c^2\gamma^2)(s-k\gamma^2)}$$

$$+\frac{\Lambda K^{3/2}\gamma E_0 e^{-\gamma a}}{k}\frac{1}{\sqrt{s}(s-K\gamma^2)}\left\langle\sum_{n=0}^{\infty}e^{-\left[\frac{2na}{\sqrt{K}}\right]\sqrt{s}}+\sum_{n=0}^{\infty}e^{-\left[\frac{2na+2a}{\sqrt{K}}\right]\sqrt{s}}\right\rangle$$

$$+\frac{\Lambda K E_0 e^{-\gamma a}}{k}\frac{1}{(s-K\gamma^2)}\Bigg\}$$

$$-\sum_{p=0}^{\infty}e^{-\left[\frac{2pa+(2a-x)}{c}\right]s}\Bigg\{2\sqrt{s/K}\left[\frac{\Lambda c^2K\gamma E_0 e^{-\gamma a}}{k}\frac{1}{s(s-c^2/K)(s-K\gamma^2)}\sum_{n=0}^{\infty}e^{-\left[\frac{2na+2a}{\sqrt{K}}\right]\sqrt{s}}\right]$$

$$-\sqrt{s/K}\left[\frac{\Lambda c^2K\gamma E_0}{k}\frac{1}{s(s-c^2/K)(s-K\gamma^2)}\sum_{n=0}^{\infty}e^{-\left[\frac{2na+2a}{\sqrt{K}}\right]\sqrt{s}}\right]$$

$$-\sqrt{s/K}\left[\frac{\Lambda c^2K\gamma E_0}{k}\frac{1}{s(s-c^2/K)(s-K\gamma^2)}\sum_{n=0}^{\infty}e^{-\left[\frac{2na}{\sqrt{K}}\right]\sqrt{s}}\right]$$

$$+\frac{\Lambda c^2K\gamma^2E_0}{k}\frac{1}{(s^2-c^2\gamma^2)(s-K\gamma^2)}$$

$$+\frac{2\Lambda K^{3/2}\gamma E_0 e^{-\gamma a}}{k}\frac{1}{\sqrt{s}(s-K\gamma^2)}\sum_{n=0}^{\infty}e^{-\left[\frac{2na+a}{\sqrt{K}}\right]\sqrt{s}}$$

$$-\frac{\Lambda K^{3/2}\gamma E_0}{k}\frac{1}{\sqrt{s}(s-K\gamma^2)}\left\langle\sum_{n=0}^{\infty}e^{-\left[\frac{2na+2a}{\sqrt{K}}\right]\sqrt{s}}+\sum_{n=0}^{\infty}e^{-\left[\frac{2na}{\sqrt{K}}\right]\sqrt{s}}\right\rangle$$

$$+\frac{\Lambda K E_0}{k}\frac{1}{(s-K\gamma^2)}\Bigg\}$$

$$-\frac{\Lambda c^2\sqrt{K}\gamma E_0 e^{-\gamma a}}{k}\frac{1}{\sqrt{s}(s-c^2/K)(s-K\gamma^2)}\sum_{n=0}^{\infty}e^{-\left[\frac{2na+(a-x)}{\sqrt{K}}\right]\sqrt{s}}$$

$$+\frac{\Lambda c^2\sqrt{K}\gamma E_0}{k}\frac{1}{\sqrt{s}(s-c^2/K)(s-K\gamma^2)}\sum_{n=0}^{\infty}e^{-\left[\frac{2na+(2a-x)}{\sqrt{K}}\right]\sqrt{s}}$$

$$-\frac{\Lambda c^2\sqrt{K}\gamma E_0 e^{-\gamma a}}{k}\frac{1}{\sqrt{s}(s-c^2/K)(s-K\gamma^2)}\sum_{n=0}^{\infty}e^{-\left[\frac{2na+(a+x)}{\sqrt{K}}\right]\sqrt{s}}$$

$$+\frac{\Lambda c^2\sqrt{\kappa}\gamma E_0}{k}\frac{1}{\sqrt{s}(s-c^2/\kappa)(s-\kappa\gamma^2)}\sum_{n=0}^{\infty}e^{-\left[\frac{2na+x}{\sqrt{\kappa}}\right]\sqrt{s}}$$

$$-\frac{\Lambda c^2\kappa\gamma^2 E_0 e^{-\gamma x}}{k}\frac{1}{(s^2-c^2\gamma^2)(s-\kappa\gamma^2)}\Bigg\rangle\Bigg\rangle$$

$$-(3\lambda+2\mu)\alpha\Bigg\{\frac{\kappa^{3/2}\gamma E_0 e^{-\gamma a}}{k}\frac{1}{\sqrt{s}(s-\kappa\gamma^2)}\Bigg\langle\sum_{n=0}^{\infty}e^{-\left[\frac{2na+(a-x)}{\sqrt{\kappa}}\right]\sqrt{s}}+\sum_{n=0}^{\infty}e^{-\left[\frac{2na+(a+x)}{\sqrt{\kappa}}\right]\sqrt{s}}\Bigg\rangle$$

$$-\frac{\kappa^{3/2}\gamma E_0}{k}\frac{1}{\sqrt{s}(s-\kappa\gamma^2)}\Bigg\langle\sum_{n=0}^{\infty}e^{-\left[\frac{2na+(2a-x)}{\sqrt{\kappa}}\right]\sqrt{s}}+\sum_{n=0}^{\infty}e^{-\left[\frac{2na+x}{\sqrt{\kappa}}\right]\sqrt{s}}\Bigg\rangle$$

$$+\frac{\kappa E_0 e^{-\gamma x}}{k}\frac{1}{(s-\kappa\gamma^2)}\Bigg\} \qquad \text{(A2.19)}$$

Eqn.(A2.19) may be written as

$$\sigma^*(x,s)=\sum_{p=0}^{\infty}e^{-\delta_{1p}s}\psi_1(s)+\sum_{p=0}^{\infty}e^{-\delta_{2p}s}\psi_2(s)$$

$$+\sum_{p=0}^{\infty}e^{-\delta_{3p}s}\psi_3(s)+\sum_{p=0}^{\infty}e^{-\delta_{4p}s}\psi_4(s)$$

$$+\psi_5(s)+\psi_6(s) \qquad \text{(A2.20)}$$

with

$$\delta_{1p}=\frac{2pa+x}{c},\quad \delta_{2p}=\frac{2pa+(a-x)}{c},$$
$$\delta_{3p}=\frac{2pa+(a+x)}{c},\quad \delta_{4p}=\frac{2pa+(2a-x)}{c} \qquad \text{(A2.21)}$$

and $\psi_1(s)$, $\psi_2(s)$, $\psi_3(s)$ and $\psi_4(s)$ are the corresponding multiplying factors in Eqn.(A2.19).

Using the shift property of the Laplace transform Eqn. (3.59), the inverse transforms determined in Appendix B, and the nondimensionalization defined by Eqn.(3.61) together with a dimensionless conduction parameter $\omega=\frac{\kappa}{ac}$, the in-

verse transform of Eqn.(A2.20) is

$$\sigma_{\xi}(\xi,\tau)=\sum_{P=0}^{\infty}H(\tau-\Delta_{1P})\left[\left[-\frac{e^{-\Gamma}}{(1-\omega^2\Gamma^2)}\left\{\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})-\Gamma(2n+1)}\,\mathrm{Erfc}\left(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\right)\right.\right.\right.$$

$$-\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})+\Gamma(2n+1)}\,\mathrm{Erfc}\left(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\right)$$

$$-\omega\Gamma\left[\sum_{n=0}^{\infty}e^{\omega^{-1}(\tau-\Delta_{1P})-\omega^{-1}(2n+1)}\,\mathrm{Erfc}\left(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}-\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{1P}}\right)\right.$$

$$\left.\left.-\sum_{n=0}^{\infty}e^{\omega^{-1}(\tau-\Delta_{1P})+\omega^{-1}(2n+1)}\,\mathrm{Erfc}\left(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}+\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{1P}}\right)\right]\right\}$$

$$+\frac{1}{2}\frac{1}{(1-\omega^2\Gamma^2)}\left\{\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})-2\Gamma(n+1)}\,\mathrm{Erfc}\left(\frac{n+1}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\right)\right.$$

$$-\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})+2\Gamma(n+1)}\,\mathrm{Erfc}\left(\frac{n+1}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\right)$$

$$-\omega\Gamma\left[\sum_{n=0}^{\infty}e^{\omega^{-1}(\tau-\Delta_{1P})-2\omega^{-1}(n+1)}\,\mathrm{Erfc}\left(\frac{n+1}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}-\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{1P}}\right)\right.$$

$$\left.\left.-\sum_{n=0}^{\infty}e^{\omega^{-1}(\tau-\Delta_{1P})+2\omega^{-1}(n+1)}\,\mathrm{Erfc}\left(\frac{n+1}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}+\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{1P}}\right)\right]\right\}$$

$$+\frac{1}{2}\frac{1}{(1-\omega^2\Gamma^2)}\left\{\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})-2n\Gamma}\,\mathrm{Erfc}\left(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\right)\right.$$

$$-\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})+2n\Gamma}\,\mathrm{Erfc}\left(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\right)$$

$$-\omega\Gamma\left[\sum_{n=0}^{\infty}e^{\omega^{-1}(\tau-\Delta_{1P})-2\omega^{-1}n}\,\mathrm{Erfc}\left(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}-\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{1P}}\right)\right.$$

$$\left.\left.-\sum_{n=0}^{\infty}e^{\omega^{-1}(\tau-\Delta_{1P})+2\omega^{-1}n}\,\mathrm{Erfc}\left(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}+\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{1P}}\right)\right]\right\}$$

$$+\frac{1}{(1-\omega^2\Gamma^2)}\Big\{\cosh\Gamma(\tau-\Delta_{1P})-e^{\omega\Gamma^2(\tau-\Delta_{1P})}+\omega\Gamma\sinh\Gamma(\tau-\Delta_{1P})\Big\}$$

$$+e^{-\Gamma}\Big\{\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})-\Gamma(2n+1)}\operatorname{Erfc}\Big(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\Big)$$

$$-\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})+\Gamma(2n+1)}\operatorname{Erfc}\Big(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\Big)\Big\}$$

$$-\frac{1}{2}\Big\{\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})-2\Gamma(n+1)}\operatorname{Erfc}\Big(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\Big)$$

$$-\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})+2\Gamma(n+1)}\operatorname{Erfc}\Big(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\Big)$$

$$+\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})-2\Gamma n}\operatorname{Erfc}\Big(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\Big)$$

$$-\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{1P})+2\Gamma n}\operatorname{Erfc}\Big(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{1P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{1P}}\Big)\Big\}$$

$$+e^{\omega\Gamma^2(\tau-\Delta_{1P})}\Big]\Big]$$

$$+\sum_{P=0}^{\infty}H(\tau-\Delta_{2P})\Big[\Big[\frac{1}{2}\frac{e^{-\Gamma}}{(1-\omega^2\Gamma^2)}\Big\{\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{2P})-2\Gamma n}\operatorname{Erfc}\Big(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\Big)$$

$$-\sum_{n=0}^{\infty}e^{\omega\Gamma^2(\tau-\Delta_{2P})+2\Gamma n}\operatorname{Erfc}\Big(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\Big)$$

$$-\omega\Gamma\Big[\sum_{n=0}^{\infty}e^{\omega^{-1}(\tau-\Delta_{2P})-2\omega^{-1}n}\operatorname{Erfc}\Big(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}-\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{2P}}\Big)$$

$$-\sum_{n=0}^{\infty}e^{\omega^{-1}(\tau-\Delta_{2P})+2\omega^{-1}n}\operatorname{Erfc}\Big(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}+\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{2P}}\Big)\Big]\Big\}$$

$$+\frac{1}{(1-\omega^2\Gamma^2)}\Bigg\{\sum_{n=0}^{\infty} e^{\omega\Gamma^2(\tau-\Delta_{2P})-\Gamma(2n+1)}\,\mathrm{Erfc}\left(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2(\tau-\Delta_{2P})+\Gamma(2n+1)}\,\mathrm{Erfc}\left(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\right)$$

$$-\omega\Gamma\Bigg[\sum_{n=0}^{\infty} e^{\omega^{-1}(\tau-\Delta_{2P})-\omega^{-1}(2n+1)}\,\mathrm{Erfc}\left(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}-\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{2P}}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega^{-1}(\tau-\Delta_{2P})+\omega^{-1}(2n+1)}\,\mathrm{Erfc}\left(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}+\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{2P}}\right)\Bigg]\Bigg\}$$

$$-\frac{1}{2}\frac{1}{(1-\omega^2\Gamma^2)}\Bigg\{\sum_{n=0}^{\infty} e^{\omega\Gamma^2(\tau-\Delta_{2P})-2\Gamma(n+1)}\,\mathrm{Erfc}\left(\frac{n+1}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2(\tau-\Delta_{2P})+2\Gamma(n+1)}\,\mathrm{Erfc}\left(\frac{n+1}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\right)$$

$$-\omega\Gamma\Bigg[\sum_{n=0}^{\infty} e^{\omega^{-1}(\tau-\Delta_{2P})-2\omega^{-1}(n+1)}\,\mathrm{Erfc}\left(\frac{n+1}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}-\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{2P}}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega^{-1}(\tau-\Delta_{2P})+2\omega^{-1}(n+1)}\,\mathrm{Erfc}\left(\frac{n+1}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}+\frac{1}{\sqrt{\omega}}\sqrt{\tau-\Delta_{2P}}\right)\Bigg]\Bigg\}$$

$$+\frac{e^{-\Gamma}}{(1-\omega^2\Gamma^2)}\Big\{\cosh\Gamma(\tau-\Delta_{2P})-e^{\omega\Gamma^2(\tau-\Delta_{2P})+\omega\Gamma\sinh\Gamma(\tau-\Delta_{2P})}\Big\}$$

$$+\frac{e^{-\Gamma}}{2}\Bigg\{\sum_{n=0}^{\infty} e^{\omega\Gamma^2(\tau-\Delta_{2P})-2\Gamma n}\,\mathrm{Erfc}\left(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2(\tau-\Delta_{2P})+2\Gamma n}\,\mathrm{Erfc}\left(\frac{n}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\right)$$

$$+\sum_{n=0}^{\infty} e^{\omega\Gamma^2(\tau-\Delta_{2P})-2\Gamma(n+1)}\,\mathrm{Erfc}\left(\frac{n+1}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2(\tau-\Delta_{2P})+2\Gamma(n+1)}\,\mathrm{Erfc}\left(\frac{n+1}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\right)\Bigg\}$$

$$-\Big\{\sum_{n=0}^{\infty} e^{\omega\Gamma^2(\tau-\Delta_{2P})-\Gamma(2n+1)}\,\mathrm{Erfc}\Big(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}-\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\Big)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2(\tau-\Delta_{2P})+\Gamma(2n+1)}\,\mathrm{Erfc}\Big(\frac{n+1/2}{\sqrt{\omega}\sqrt{\tau-\Delta_{2P}}}+\sqrt{\omega}\Gamma\sqrt{\tau-\Delta_{2P}}\Big)\Big\}$$

$$+e^{-\Gamma+\omega\Gamma^2(\tau-\Delta_{2P})}\Big]\Big]$$

$$-\sum_{P=0}^{\infty} H(\tau-\Delta_{3P})\Big[\Big[\text{same as second term with } \Delta_{2P} \text{ replaced by } \Delta_{3P}\Big]\Big]$$

$$-\sum_{P=0}^{\infty} H(\tau-\Delta_{4P})\Big[\Big[\text{same as first term with } \Delta_{1P} \text{ replaced by } \Delta_{4P}\Big]\Big]$$

$$+\Big[\Big[\frac{1}{2}\frac{1}{(1-\omega^2\Gamma^2)}\Big\langle e^{-\Gamma}\Big\{\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau-[2n+(1-\xi)]\Gamma}\,\mathrm{Erfc}\Big(\frac{n+(1-\xi)/2}{\sqrt{\omega}\sqrt{\tau}}-\sqrt{\omega}\Gamma\sqrt{\tau}\Big)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau+[2n+(1-\xi)]\Gamma}\,\mathrm{Erfc}\Big(\frac{n+(1-\xi)/2}{\sqrt{\omega}\sqrt{\tau}}+\sqrt{\omega}\Gamma\sqrt{\tau}\Big)$$

$$-\omega\Gamma\Big[\sum_{n=0}^{\infty} e^{\omega^{-1}\tau-\omega^{-1}[2n+(1-\xi)]}\,\mathrm{Erfc}\Big(\frac{n+(1-\xi)/2}{\sqrt{\omega}\sqrt{\tau}}-\frac{1}{\sqrt{\omega}}\sqrt{\tau}\Big)$$

$$-\sum_{n=0}^{\infty} e^{\omega^{-1}\tau+\omega^{-1}[2n+(1-\xi)]}\,\mathrm{Erfc}\Big(\frac{n+(1-\xi)/2}{\sqrt{\omega}\sqrt{\tau}}+\frac{1}{\sqrt{\omega}}\sqrt{\tau}\Big)\Big]\Big\}$$

$$-\Big\{\sum_{n=0}^{\infty} e^{\omega\Gamma^2-[2n+(2-\xi)]\Gamma}\,\mathrm{Erfc}\Big(\frac{n+(1-\xi/2)}{\sqrt{\omega}\sqrt{\tau}}-\sqrt{\omega}\Gamma\sqrt{\tau}\Big)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2+[2n+(2-\xi)]\Gamma}\,\mathrm{Erfc}\Big(\frac{n+(1-\xi/2)}{\sqrt{\omega}\sqrt{\tau}}+\sqrt{\omega}\Gamma\sqrt{\tau}\Big)$$

$$-\omega\Gamma\Big[\sum_{n=0}^{\infty} e^{\omega^{-1}\tau-\omega^{-1}[2n+(2-\xi)]}\,\mathrm{Erfc}\Big(\frac{n+(1-\xi/2)}{\sqrt{\omega}\sqrt{\tau}}-\frac{1}{\sqrt{\omega}}\sqrt{\tau}\Big)$$

$$-\sum_{n=0}^{\infty} e^{\omega^{-1}\tau+\omega^{-1}[2n+(2-\xi)]}\,\mathrm{Erfc}\left(\frac{n+(1-\xi/2)}{\sqrt{\omega}\sqrt{\tau}}+\frac{1}{\sqrt{\omega}}\sqrt{\tau}\right)\Bigg]\Bigg\}$$

$$+e^{-\Gamma}\Bigg\{\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau-[2n+(1+\xi)]\Gamma}\,\mathrm{Erfc}\left(\frac{n+(1+\xi)/2}{\sqrt{\omega}\sqrt{\tau}}-\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau+[2n+(1+\xi)]\Gamma}\,\mathrm{Erfc}\left(\frac{n+(1+\xi)/2}{\sqrt{\omega}\sqrt{\tau}}+\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)$$

$$-\omega\Gamma\Bigg[\sum_{n=0}^{\infty} e^{\omega^{-1}\tau-\omega^{-1}[2n+(1+\xi)]}\,\mathrm{Erfc}\left(\frac{n+(1+\xi)/2}{\sqrt{\omega}\sqrt{\tau}}-\frac{1}{\sqrt{\omega}}\sqrt{\tau}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega^{-1}\tau+\omega^{-1}[2n+(1+\xi)]}\,\mathrm{Erfc}\left(\frac{n+(1+\xi)/2}{\sqrt{\omega}\sqrt{\tau}}+\frac{1}{\sqrt{\omega}}\sqrt{\tau}\right)\Bigg]\Bigg\}$$

$$-\Bigg\{\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau-(2n+\xi)\Gamma}\,\mathrm{Erfc}\left(\frac{n+\xi/2}{\sqrt{\omega}\sqrt{\tau}}-\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau+(2n+\xi)\Gamma}\,\mathrm{Erfc}\left(\frac{n+\xi/2}{\sqrt{\omega}\sqrt{\tau}}+\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)$$

$$-\omega\Gamma\Bigg[\sum_{n=0}^{\infty} e^{\omega^{-1}\tau-\omega^{-1}(2n+\xi)}\,\mathrm{Erfc}\left(\frac{n+\xi/2}{\sqrt{\omega}\sqrt{\tau}}-\frac{1}{\sqrt{\omega}}\sqrt{\tau}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega^{-1}\tau+\omega^{-1}(2n+\xi)}\,\mathrm{Erfc}\left(\frac{n+\xi/2}{\sqrt{\omega}\sqrt{\tau}}+\frac{1}{\sqrt{\omega}}\sqrt{\tau}\right)\Bigg]\Bigg\}\Bigg\rangle$$

$$-\frac{e^{-\Gamma\xi}}{(1-\omega^2\Gamma^2)}\left\{\cosh\Gamma\tau-e^{\omega\Gamma^2\tau}+\omega\Gamma\sinh\Gamma\tau\right\}\Bigg]\Bigg]$$

$$-\Bigg[\Bigg[\frac{e^{-\Gamma}}{2}\Bigg\{\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau-[2n+(1-\xi)]\Gamma}\,\mathrm{Erfc}\left(\frac{n+(1-\xi)/2}{\sqrt{\omega}\sqrt{\tau}}-\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau+[2n+(1-\xi)]\Gamma}\,\mathrm{Erfc}\left(\frac{n+(1-\xi)/2}{\sqrt{\omega}\sqrt{\tau}}+\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)$$

$$+\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau-[2n+(1+\xi)]\Gamma}\,\mathrm{Erfc}\left(\frac{n+(1+\xi)/2}{\sqrt{\omega}\sqrt{\tau}}-\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau+[2n+(1+\xi)]\Gamma}\,\mathrm{Erfc}\left(\frac{n+(1+\xi)/2}{\sqrt{\omega}\sqrt{\tau}}+\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)\Bigg\}$$

$$-\frac{1}{2}\Bigg\{\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau-[2n+(2-\xi)]\Gamma}\,\mathrm{Erfc}\left(\frac{n+(2-\xi)/2}{\sqrt{\omega}\sqrt{\tau}}-\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau+[2n+(2-\xi)]\Gamma}\,\mathrm{Erfc}\left(\frac{n+(2-\xi)/2}{\sqrt{\omega}\sqrt{\tau}}+\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)$$

$$+\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau-[2n+\xi]\Gamma}\,\mathrm{Erfc}\left(\frac{n+\xi/2}{\sqrt{\omega}\sqrt{\tau}}-\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)$$

$$-\sum_{n=0}^{\infty} e^{\omega\Gamma^2\tau+[2n+\xi]\Gamma}\,\mathrm{Erfc}\left(\frac{n+\xi/2}{\sqrt{\omega}\sqrt{\tau}}+\sqrt{\omega}\,\Gamma\sqrt{\tau}\right)\Bigg\}$$

$$+e^{-\Gamma\xi+\omega\Gamma^2\tau}\Bigg]\Bigg] \qquad \text{(A2.22)}$$

In Eqn. (A2.22)

$$\Delta_{1P} = 2P+\xi$$

$$\Delta_{2P} = 2P+(1-\xi) \qquad \text{(A2.23)}$$

$$\Delta_{3P} = 2P+(1+\xi)$$

and

$$\Delta_{4P} = 2P+(2-\xi)$$

3. Remarks on Solution

The solution now contains the conductivity parameter ω which is proportional to the thermal conductivity (or ther-

mal conductivity (or thermal diffusivity). For most engineering materials and for a physical thickness of 1 cm, ω is on the order of 10^{-5} to 10^{-6}. To determine the effect of ω, the example of the single layer slab in Chapter IV was recalculated from Eqn. (A2.22) for $\Gamma=2$ and $\omega=10^{-3}$. No significant different with the previous results for $\omega=0$ was found for values of the nondimensional time up to $\tau=4$. For $\omega=10^{-2}$, a difference of approximately one percent was found in the maximum positive stress up to $\tau=4$. Because the interest of the problem is in the range of $\omega\sim10^{-5}$ to 10^{-6} and τ in the approximate range of $0<\tau<10$, neglecting conduction is an extremely good assumption.

APPENDIX B

MISCELLANEOUS CALCULATIONS

1. Equivalence of Series Used in Chapter III.

In Eqn.(3.50) it was asserted that

$$\sum_{i=0}^{\infty} Z^i = \sum_{i=0}^{\infty} \sum_{\substack{j,k,l=0 \\ (j+k+l=i)}}^{i} \frac{i!}{j!k!l!} Z_1^j Z_2^k (-Z_3)^l e^{-(j\beta_1+k\beta_2+l\beta_3)s}$$

where $Z = Z_1 e^{-\beta_1 s} + Z_2 e^{-\beta_2 s} - Z_3 e^{-\beta_3 s}$ by Eqn.(3.43) and β_1, β_2 and β_3 are functions of a, b, C_1 and C_2 as given by Eqn.(3.50). The second summation indicates all possible combinations of j, k and l where their sum is equal to i. Expanding the left hand side of Eqn.(3.50) after introducing Z gives

$$\begin{aligned}
\sum_{i=0}^{\infty} Z^i = {} & 1 && (i=0)\\
& + \left[Z_1 e^{-\beta_1 s} + Z_2 e^{-\beta_2 s} - Z_3 e^{-\beta_3 s} \right] && (i=1)\\
& + \Big[Z_1^2 e^{-2\beta_1 s} + Z_2^2 e^{-2\beta_2 s} + Z_3 e^{-2\beta_3 s} \\
& \qquad + 2 Z_1 Z_2 e^{-(\beta_1+\beta_2)s} - 2 Z_2 Z_3 e^{-(\beta_2+\beta_3)s} - 2 Z_1 Z_3 e^{-(\beta_1+\beta_3)s} \Big] && (i=2)\\
& + \Big[Z_1^3 e^{-3\beta_1 s} + Z_2^3 e^{-3\beta_2 s} - Z_3 e^{-3\beta_3 s} \\
& \quad + 3 Z_1 Z_3^2 e^{-(\beta_1+2\beta_3)s} + 3 Z_2 Z_3^2 e^{-(\beta_2+2\beta_3)s} + 3 Z_1 Z_3^2 e^{-(\beta_1+2\beta_3)s} \\
& \quad + 3 Z_1^2 Z_2 e^{-(2\beta_1+\beta_2)s} - 3 Z_1^2 Z_3 e^{-(2\beta_1+\beta_3)s} \\
& \quad - 3 Z_2^2 Z_3 e^{-(2\beta_2+\beta_3)s} - 6 Z_1 Z_2 Z_3 e^{-(\beta_1+\beta_2+\beta_3)s} \Big] && (i=3)\\
& + \left[\cdots\cdots \right.
\end{aligned}$$

The right hand side of Eqn. (3.48) gives for

$i=0$:

$i=0 \Longrightarrow j=k=\ell=0$ and the only term is

$$\frac{0!}{0!0!0!} = 1 \qquad \text{by definition.}$$

$i=1$:

There are three combinations:

(i) $j=1, k=0, \ell=0 \longrightarrow \frac{1!}{1!0!0!} Z_1^1 e^{-\beta_1 s} = Z_1 e^{-\beta_1 s}$

(ii) $j=0, k=1, \ell=0 \longrightarrow \frac{1!}{0!1!0!} Z_2^1 e^{-\beta_2 s} = Z_2 e^{-\beta_2 s}$

(iii) $j=0, k=0, \ell=1 \longrightarrow \frac{1!}{0!0!1!} (-Z_3)^1 e^{-\beta_3 s} = -Z_3 e^{-\beta_3 s}$

$i=2$:

There are six combinations:

(i) $j=1, k=2, \ell=0 \longrightarrow \frac{2!}{1!1!0!} Z_1^1 Z_2^1 e^{-(\beta_1+\beta_2)s} = 2Z_1 Z_2 e^{-(\beta_1+\beta_2)s}$

(ii) $j=0, k=1, \ell=1 \longrightarrow \frac{2!}{0!1!1!} Z_2^1 (-Z_3)^1 e^{-(\beta_2+\beta_3)s} = -2Z_2 Z_3 e^{-(\beta_2+\beta_3)s}$

(iii) $j=1, k=0, \ell=1 \longrightarrow \frac{2!}{1!0!1!} Z_1^1 (-Z_3)^1 e^{-(\beta_1+\beta_3)s} = -2Z_1 Z_3 e^{-(\beta_1+\beta_3)s}$

(iv) $j=2, k=0, \ell=0 \longrightarrow \frac{2!}{2!0!0!} Z_1^2 e^{-2\beta_1 s} = Z_1^2 e^{-2\beta_1 s}$

(v) $j=0, k=2, \ell=0 \longrightarrow \frac{2!}{0!2!0!} Z_2^2 e^{-2\beta_2 s} = Z_2^2 e^{-2\beta_2 s}$

(vi) $j=0, k=0, \ell=2 \longrightarrow \frac{2!}{0!0!2!} (-Z_3)^2 e^{-2\beta_3 s} = Z_3^2 e^{-2\beta_3 s}$

$i=3$:

There are ten combinations:

(i) $j=1, k=1, l=1 \longrightarrow \frac{3!}{1!1!1!} Z_1^1 Z_2^1 (-Z_3)^1 e^{-(\beta_1+\beta_2+\beta_3)s} = -6 Z_1 Z_2 Z_3 e^{-(\beta_1+\beta_2+\beta_3)s}$

(ii) $j=0, k=1, l=2 \longrightarrow \frac{3!}{0!1!2!} Z_2^1 (-Z_3)^2 e^{-(\beta_2+2\beta_3)s} = 3 Z_2 Z_3^2 e^{-(\beta_2+2\beta_3)s}$

(iii) $j=0, k=0, l=3 \longrightarrow \frac{3!}{0!0!3!} (-Z_3)^3 e^{-3\beta_3 s} = -Z_3^3 e^{-3\beta_3 s}$

(iv) $j=1, k=2, l=0 \longrightarrow \frac{3!}{1!2!0!} Z_1^1 Z_2^2 e^{-(\beta_1+2\beta_2)s} = 3 Z_1 Z_2^2 e^{-(\beta_1+2\beta_2)s}$

(v) $j=1, k=0, l=2 \longrightarrow \frac{3!}{1!0!2!} Z_1^1 (-Z_3)^2 e^{-(\beta_1+2\beta_3)s} = 3 Z_1 Z_3^2 e^{-(\beta_1+2\beta_3)s}$

(vi) $j=0, k=3, l=0 \longrightarrow \frac{3!}{0!3!0!} Z_2^3 e^{-3\beta_2 s} = Z_2^3 e^{-3\beta_2 s}$

(vii) $j=3, k=0, l=0 \longrightarrow \frac{3!}{3!0!0!} Z_1^3 e^{-3\beta_1 s} = Z_1^3 e^{-3\beta_1 s}$

(viii) $j=0, k=2, l=1 \longrightarrow \frac{3!}{0!2!1!} Z_2^2 (-Z_3)^1 e^{-(2\beta_2+\beta_3)s} = -3 Z_2^2 Z_3 e^{-(2\beta_2+\beta_3)s}$

(ix) $j=2, k=0, l=1 \longrightarrow \frac{3!}{2!0!1!} Z_1^2 (-Z_3)^1 e^{-(2\beta_1+\beta_3)s} = -3 Z_1^2 Z_3 e^{-(2\beta_1+\beta_3)s}$

(x) $j=2, k=1, l=0 \longrightarrow \frac{3!}{2!1!0!} Z_1^2 Z_2^1 e^{-(2\beta_1+\beta_2)s} = 3 Z_1^2 Z_2 e^{-(2\beta_1+\beta_2)s}$

By comparing these two expansions term by term up to and including $i=3$, it is seen that they are identical. By induction, the two expansions are identical for all values of the index i.

2. Derivation of Untabulated Inverse Laplace Transforms Used in Chapter III and Appendix A.

In Eqns.(3.62), (3.63) and (A2.19) the inverse Laplace transforms of the following types of terms are needed:

$$L^{-1}\left\{\frac{1}{\sqrt{s}(s-a)}\,e^{-c\sqrt{s}}\right\} \tag{B2.1}$$

$$L^{-1}\left\{\frac{1}{\sqrt{s}(s-a)}\,\frac{1}{(s-b)}\,e^{-c\sqrt{s}}\right\} \tag{B2.2}$$

$$L^{-1}\left\{\frac{1}{(s^2-a^2)}\,\frac{1}{(s-b)}\right\} \tag{B2.3}$$

Inverse Transform of Eqn.(B2.1)

From the convolution property of the Laplace transform for suitably restricted functions $f(s)$ and $g(s)$ (see reference [21] or [22])

$$L^{-1}\{f(s)g(s)\} = F(t)\circledast G(t) \tag{B2.4}$$

where $\circledast$ is the convolution operator defined by

$$F(t)\circledast G(t) = \int_0^t F(\tau)G(t-\tau)\,d\tau \tag{B2.5}$$

In Eqn.(B2.5) $f(s)$ is the transform of $F(t)$ and $g(s)$ is the transform of $G(t)$. Choosing

$$f(s)=\frac{1}{\sqrt{s}}\,e^{-c\sqrt{s}} \quad \text{and} \quad g(s)=\frac{1}{(s-a)}$$

and using formulas 5.6-(1) and 5.2-(2) from [23] gives

$$F(t) = \frac{e^{-c^2/4t}}{\sqrt{\pi t}} \quad \text{and} \quad G(t) = e^{at}$$

Using these and applying Eqn.(B2.5) to Eqn.(B2.1) gives

$$L^{-1}\left\{\frac{1}{\sqrt{s}}\, e^{-c\sqrt{s}}\, \frac{1}{(s-a)}\right\} = \frac{e^{at}}{\sqrt{\pi}} \int_0^t \frac{e^{-\frac{c^2}{4\tau} - a\tau}}{\sqrt{\tau}}\, d\tau \qquad \text{(B2.6)}$$

A change of variable $x = \sqrt{\tau}$ in Eqn.(B2.6) gives

$$L^{-1}\left\{\frac{1}{\sqrt{s}}\, e^{-c\sqrt{s}}\, \frac{1}{(s-a)}\right\} = \frac{2}{\sqrt{\pi}}\, e^{at} \int_0^{\sqrt{t}} e^{-ax^2 - \frac{c^2}{4x^2}}\, dx \qquad \text{(B2.7)}$$

Eqn.(7.4.33) of [24] gives the integration formula

$$\int e^{-a^2x^2 - \frac{b^2}{x^2}}\, dx = \frac{\sqrt{\pi}}{4a}\left[e^{2ab}\, \mathrm{erf}\left(ax + \frac{b}{x}\right) + e^{-2ab}\, \mathrm{erf}\left(ax - \frac{b}{x}\right)\right] + \text{const.} \qquad \text{(B2.8)}$$

when $a \neq 0$

where the error function $\mathrm{erf}(x)$ is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

The complementary error function $\mathrm{Erfc}(x)$ is defined as

$$1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

The error function is odd and approaches unity for large values of its argument.

After some manipulation

$$L^{-1}\left\{\frac{1}{\sqrt{s}}\, e^{-c\sqrt{s}}\, \frac{1}{(s-a)}\right\} = \frac{1}{2\sqrt{a}}\left[e^{at - c\sqrt{a}}\, \mathrm{Erfc}\left(\frac{c}{2\sqrt{t}} - \sqrt{at}\right)\right.$$

$$\left. - e^{at + c\sqrt{a}}\, \mathrm{Erfc}\left(\frac{c}{2\sqrt{t}} + \sqrt{at}\right)\right] \qquad \text{(B2.9)}$$

Inverse Transform of Eqn.(B2.2)

The inverse transform of $\frac{1}{\sqrt{s}} e^{-c\sqrt{s}}$ is known from [25] and a convolution of this result with the inverse transform of $\frac{1}{(s-a)} \cdot \frac{1}{(s-b)}$ is needed. Applying Eqn.(B2.5) with

$F(t) = e^{bt}$ and $G(t) = e^{at}$ gives

$$L^{-1}\left\{\frac{1}{(s-a)}\frac{1}{(s-b)}\right\} = e^{at}\int_0^t e^{(b-a)\tau}\, d\tau$$

or

$$L^{-1}\left\{\frac{1}{(s-a)}\frac{1}{(s-b)}\right\} = \frac{e^{at}}{(a-b)}\left\{1 - e^{-(a-b)t}\right\} \qquad \text{(B2.10)}$$

From [25],

$$L^{-1}\left\{\frac{1}{\sqrt{s}} e^{-c\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}} e^{-\frac{c^2}{4t}} \qquad \text{(B2.11)}$$

Convolving Eqns.(B2.10) and (B2.11) gives

$$L^{-1}\left\{\frac{1}{(s-a)}\frac{1}{(s-b)}\frac{1}{\sqrt{s}} e^{-c\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{a(t-\tau)}}{(a-b)}\left\{1 - e^{-(a-b)(t-\tau)}\right\}\frac{e^{-\frac{c^2}{4\tau}}}{\sqrt{\tau}}\, d\tau$$

After a change of variable $x = \sqrt{\tau}$ and the use of Eqn.(7.433) of [24]

$$L^{-1}\left\{\frac{1}{(s-a)}\frac{1}{(s-b)}\frac{1}{\sqrt{s}} e^{-c\sqrt{s}}\right\} = \frac{2e^{at}}{\sqrt{\pi}(a-b)}\left[\frac{\sqrt{\pi}}{4\sqrt{a}}\left\{e^{c\sqrt{a}}\,\mathrm{erf}\left(\sqrt{a}\,x + \frac{c}{2x}\right) + e^{-c\sqrt{a}}\,\mathrm{erf}\left(\sqrt{a}\,x - \frac{c}{2x}\right)\right\}_{x=0}^{x=\sqrt{\tau}}\right.$$
$$\left. - e^{-(a-b)t}\frac{\sqrt{\pi}}{4\sqrt{b}}\left\{e^{c\sqrt{b}}\,\mathrm{erf}\left(\sqrt{b}\,x + \frac{c}{2x}\right) + e^{-c\sqrt{b}}\,\mathrm{erf}\left(\sqrt{b}\,x - \frac{c}{2x}\right)\right\}_{x=0}^{x=\sqrt{t}}\right]$$

for $\sqrt{a}, \sqrt{b} \neq 0$

Considerable algebraic simplification yields

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)}\frac{1}{(s-b)}\frac{1}{\sqrt{s}}e^{-c\sqrt{s}}\right\} = \frac{1}{2(a-b)}\left\{\frac{1}{\sqrt{a}}\left[e^{at-c\sqrt{a}}\,\mathrm{Erfc}\left(\frac{c}{\sqrt{4t}}-\sqrt{at}\right)\right.\right.$$

$$\left. - e^{at+c\sqrt{a}}\,\mathrm{Erfc}\left(\frac{c}{\sqrt{4t}}+\sqrt{at}\right)\right]$$

$$-\frac{1}{\sqrt{b}}\left[e^{bt-c\sqrt{a}}\,\mathrm{Erfc}\left(\frac{c}{\sqrt{4t}}-\sqrt{bt}\right)\right.$$

$$\left.\left. - e^{bt+c\sqrt{a}}\,\mathrm{Erfc}\left(\frac{c}{\sqrt{4t}}+\sqrt{bt}\right)\right]\right\} \qquad \text{(B2.12)}$$

for

$$\sqrt{a}, \sqrt{b} \neq 0$$

Inverse Transform of Eqn. (B2.3)

From Formula 17 in Appendix 3 of [21]

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2-a^2)}\right\} = \frac{1}{a}\sinh at \qquad \text{(B2.13)}$$

Using Eqn. (B2.5) one has

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2-a^2)}\frac{1}{(s-b)}\right\} = \frac{1}{a}\int_0^t e^{b(t-\tau)}\sinh a\tau\, d\tau$$

Two integrations by parts

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2-a^2)}\frac{1}{(s-b)}\right\} = \frac{1}{(a^2-b^2)}\left[\cosh at - e^{bt} + \frac{b}{a}\sinh at\right], \quad a^2 \neq b^2 \qquad \text{(B2.14)}$$

3. Values of ${}_p\Phi_q$ defined by Eqn. (3.62)

The values of ${}_p\Phi_q(j,k,\ell,r,\beta,\xi) \equiv {}_p\Phi_q$ given by Eqn. (3.62) are

$${}_1\Phi_1 = 2[1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)-\xi$$

$${}_1\Phi_2 = [1+r(\beta-1)]+(2j+1)+2k[1+r(\beta-1)]+(2\ell-1)r(\beta-1)-\xi$$

$${}_1\Phi_3 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)-\xi$$

$$ {}_1\Phi_4 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+(2\ell-1)r(\beta-1)-\xi $$

$$ {}_1\Phi_5 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+(2\ell+1)r(\beta-1)-\xi $$

$$ {}_1\Phi_6 = 2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)+\xi $$

$$ {}_1\Phi_7 = [1+r(\beta-1)]+(2j-1)+2k[1+r(\beta-1)]+(2\ell+1)r(\beta-1)+\xi $$

$$ {}_1\Phi_8 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)+\xi $$

$$ {}_1\Phi_9 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+(2\ell-1)r(\beta-1)+\xi $$

$$ {}_1\Phi_{10} = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+(2\ell+1)r(\beta-1)+\xi $$

and

$$ {}_2\Phi_1 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)+r\beta-r\xi $$

$$ {}_2\Phi_2 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)-1+r-r\xi $$

$$ {}_2\Phi_3 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)+1+r-r\xi $$

$$ {}_2\Phi_4 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)+1+r\beta-r\xi $$

$$ {}_2\Phi_5 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)-1+r\beta-r\xi $$

$$ {}_2\Phi_6 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)-r\beta+r\xi $$

$$ {}_2\Phi_7 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)+1-r+r\xi $$

$$ {}_2\Phi_8 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)-1-r+r\xi $$

$$ {}_2\Phi_9 = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)+1-r\beta+r\xi $$

$$ {}_2\Phi_{10} = [1+r(\beta-1)]+2j+2k[1+r(\beta-1)]+2\ell r(\beta-1)-1-r\beta+r\xi $$