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The University of Michigan, Ph.D., 1965
Engineering, nuclear

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RADIATIVE TRANSFER PROBLEMS IN PLANE GEOMETRY

by
Charles Edward Siewert

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1965

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ACKNOWLEDGEMENTS

The author wishes to express sincere appreciation for the guidance and supervision given by the Doctoral Committee Chairman, Professor Paul F. Zweifel. The direction and encouragement given by Professor Zweifel on the University of Michigan campus and in the year we spent abroad far exceeded the aspirations of any student. In addition to the normal role of providing professional leadership, both Dr. and Mrs. Zweifel contributed enormous amounts of their time and energy toward making my stay abroad a pleasant and very enriching experience.

The discussion time and the advice given by Professor R. K. Osborn at the onset of this dissertation were of paramount importance and are appreciated. Professors K. M. Case, F. C. Shure, and G. C. Summerfield were most helpful, and their aid through the many technical discussions that were held is recognized with gratitude. In particular, much of Appendix D of this volume is due to Professors G. C. Summerfield and P. F. Zweifel. Drs. J. C. Stewart and K. D. Pyatt, Jr., originally suggested the topic of Radiative Transfer, and their enlightening discourses served as a preface to this work.

I wish to express my thanks to the Çekmece Nuclear Center in Istanbul, Turkey, for its hospitality and the use of its facilities during several visits there. In particular, the consultations with Drs. K. İmre and E. Özizmir were most helpful and their inspiration was responsible for much of the

progress made in this dissertation. During my stay in the Middle East Technical University in Ankara, Turkey, Professors Feza Gürsey and Erdal Inönü were very generous with their time, and their advice on many topics is thankfully acknowledged. Also, I wish to thank the Middle East Technical University for providing every opportunity for the productive use of its facilities. The hospitality of the University of Wisconsin is also noted.

In conclusion, I would like to acknowledge the financial support of the U.S. Atomic Energy Commission through its Special Fellowship Program and the Ford Foundation through its Faculty Development Program. These two agencies relaxed the pressure of financial burdens and thus provided the opportunity for the research done here.

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1. INTRODUCTION

The problem of describing the transfer of radiation through an atmosphere was first introduced by Lord Rayleigh in 1871. This field of study was further developed by the classical works of Schuster (1) and Schwarzschild (2) in 1905 and 1906, respectively. Since that time, the problem has been studied extensively by astrophysicists who are trying to interpret better the frequency spectrum of radiation emitting from the stars. Various physical models have been used to describe the processes involved. Milne (3), in another classic paper, presents a phenomenological derivation and discusses the concept of Local Thermodynamic Equilibrium. Although such recent work as that by Osborn and Klevans (4), using the techniques of Quantum Mechanics, has given sophistication to the development of the equations governing the transfer of radiant energy through plasmas, the Local Thermodynamic Equilibrium model presented by Milne still forms the basis for much of the work now being done.

The field of radiative transfer remained in a rather underdeveloped state of mathematical advancement for many years. The technique for solving even the simplest of physical models, i.e., making the assumption that the absorption coefficient is a constant independent of the frequency of the radiation (gray case), was not developed until the early 1930's. At that time, the Wiener-Hopf method (5, 6) was introduced and thus enabled one to obtain exact solutions for at least some of the important quantities involved.

Chandrasekhar (7), in his monumental work on the theory of radiative transfer, solved many of the problems for the gray case. Case (8), while considering the theory of one-speed neutron transport (identical to the gray case), developed the method that bears his name for constructing the complete solution.

As usual, the choice between an accurate physical model and a mathematically solvable one is present here. One finds that the equations become particularly difficult if the absorption coefficient for the radiation has any dependence on frequency other than constant. Because an attempt to solve any of the realistic physical models of a stellar atmosphere forces one to seek a desirable way to obtain an approximate solution, the need for an exact solution for some test model is basic. Until recently, the gray case was the only test model for which an exact solution was known, and indeed this solution was used as standard against which approximating mathematical and numerical schemes could be checked.

The next step toward arriving at a realistic formulation of the problem is to allow the absorption coefficient to have a dependence on the frequency of the radiation (nongray problem). In general, once the absorption coefficient is taken to be frequency dependent, the task of obtaining exact solutions becomes quite formidable, if not impossible. Many techniques for approximating the nongray atmosphere as well as for constructing the solution thereof have been advanced in the past few years (cf., for example, Refs. 9-13). In addition, Gingerich (14) has made an extensive study of the theory of

numerical solutions. Even by using modern high-speed computers, numerical solutions are not very satisfactory. Since the equations involved here are highly nonlinear, the convergence rate of the solutions is very slow (cf. Appendix F).

As the attempt to solve the nongray case in radiative transfer progressed, the Uniform Picket Fence model was introduced (15). The solution for this model, using the Eddington approximation, is given in Ref. (15) and in a higher discrete ordinate approximation by King (16). Stewart (17) has constructed an exact solution, using a Wick-Chandrasekhar discrete ordinates procedure (7) and going to the limit as the number of ordinates becomes infinite. In the present work, we have extended Case's method, which was developed for the one-speed neutron transport problem (gray case), and thus have been able to construct the complete solution for the Uniform Picket Fence model.

As a way of introduction to Case's method of normal modes, Chapter III presents his solution to the gray problem, in Chapter IV the method for solving the Uniform Picket Fence equations is developed, and in Chapter V we solve the Generalized Picket Fence model. Here the absorption coefficient is allowed to take on N values rather than two, as in the previous chapter. This solution is very important because now we have a considerably more realistic model to use as a test case. No longer is the gray problem the final stage as far as mathematically exact solutions are concerned. By taking the value of N to be sufficiently large, the representation of the structure of many absorption coefficients can be made reasonably accurate.

In the majority of the work presented herein, the effect of radiation scattering is neglected. The inclusion of a scattering term in the models considered here would make the system of equations completely analogous to the general multi-group formulation of Neutron Transport Theory. Although the inclusion of scattering is quite easily incorporated into the model, it makes a fundamental difference when one proceeds with the solution.

The fact that the determinant of the transfer matrix in the scattering-free model vanishes is very important. Without this condition the solutions for half-space problems appear not to be expressible in closed form. Conversely, when, as in our model, this determinant vanishes we are led to closed form solutions which, in many cases, are simple extensions of the gray problem results.

The results presented here represent one of the first explicit results that have been obtained from Case's method for other than the one-speed case. Thus, in addition to their application in astrophysics, they would appear to have the merit of contributing to the mathematical understanding of the singular eigenfunction approach.

2. RADIATIVE TRANSFER IN STELLAR ATMOSPHERES

2.1 Introduction

By a "stellar atmosphere" we shall mean the layers of a star that are directly accessible to observation, in the sense that a detectable amount of radiation characteristic of these layers reaches the observer without being scattered or absorbed (18). In the present work, we attempt to predict the spectrum of radiation that reaches the observer. We assume that the atmosphere can be represented by an infinite half-space with the source of radiation being deep within the interior of the medium. We will further assume that the atmosphere is stratified into parallel planes so that only one space variable is necessary to describe a given position in the medium (we thus have symmetry in the other two dimensions).

To describe the radiation field in the star, we use the specific intensity, $\psi_\nu(z,\mu)$, which is defined as follows (7). If we consider an element of area $d\sigma$ with normal $\underline{\hat{n}}$ located at position $\underline{r}$ in the medium, then the amount of radiant energy, dE, in a specified frequency interval, $\nu,\nu+d\nu$, that crosses this area in a direction $d\underline{\Omega}$ about $\underline{\Omega}$ in a time dt about t is defined as

$$dE_\nu \triangleq \psi_\nu(\underline{r},\underline{\Omega},t)\,d\nu\,d\sigma\,|\cos\theta|\,d\Omega\,dt, \qquad (2.1)$$

where

$$\cos\theta = \underline{\hat{\Omega}}\cdot\underline{\hat{n}}.$$

If the intensity, ψ_ν, is independent of direction at a given point, the radiation field is said to be isotropic at

that point. If the intensity is the same at every point in space and in all directions, the radiation field is said to be homogeneous and isotropic. We will neglect polarization effects in the radiation field.

The interaction of the matter in the medium with the radiation field will be described by $K_\nu(\underline{r})$, the coefficient of absorption per unit mass (cm^2/g) and the coefficient of spontaneous emission, $\epsilon_\nu(\underline{r})$ (with units erg/g ster rad and assumed to be isotropic).

2.2 Basic Equations of Radiative Transfer

If we allow the element of area, $d\sigma$, to have a thickness dS, then the balance of radiant energy along the direction $\underline{\Omega}$ is given by the equation of transfer; i.e.,

$$\frac{d\psi_\nu}{ds}(\underline{r},\underline{\Omega}) = -\rho(\underline{r})K_\nu(\underline{r})\psi_\nu(\underline{r},\underline{\Omega}) + \rho(\underline{r})\epsilon(\underline{r}), \qquad (2.2)$$

where $\rho(\underline{r})$ is the density of the medium at position $\underline{r}$. In the case of plane symmetry (to be used throughout this work), Eq. (2.2) reduces to

$$\mu \frac{\partial\psi_\nu}{\partial z}(z,\mu) = -\rho(z)K_\nu(z)\psi_\nu(z,\mu) + \rho(z)\epsilon_\nu(z), \qquad (2.3)$$

where the axially symmetric field characterized by ψ_ν is a function of z, the distance into the atmosphere, and the angle, $\cos^{-1}\mu$, between the direction considered and the inward normal. (In order to agree with much of the literature in astrophysics, we would have to change μ to $-\mu$. We use the current notation in order to agree with the new work that has been done recently in the theory of neutron transport.)

Assuming that the system obeys the equation of radiative equilibrium (19), we have in addition to Eq. (2.3) the condition,

$$\int_0^\infty d\nu \epsilon_\nu(z) = \frac{1}{2}\int_0^\infty d\nu K_\nu(z) \int_{-1}^{1} \psi_\nu(z,\mu')d\mu'. \qquad (2.4)$$

This means that there are no mechanisms other than the radiation for transporting energy in the atmosphere; i.e., at a given point in the atmosphere, an equal amount of energy is absorbed as is emitted.

2.3 The Hypothesis of Local Thermodynamic Equilibrium

Let us consider an isolated element of volume $d^3\underline{r}$ in which there are many particles and photons in equilibrium. If this system is in thermodynamic equilibrium at a temperature, T, then the amount of radiation of a given frequency and direction that is absorbed must be exactly equal to the amount of radiant energy emitted in this direction and frequency; i.e.,

$$\rho(\underline{r})K_\nu(\underline{r})\psi'_\nu(\underline{r},\underline{\Omega}) = \rho(\underline{r})\epsilon(\underline{r}),$$

where ψ'_ν is the specific intensity of radiation in $d^3\underline{r}$. However, it has been shown (20) that the specific intensity for a system in thermodynamic equilibrium at temperature T is given by the Planck black body spectrum, $B_\nu(T)$;

$$B_\nu(T) = (2h\nu^3/c^2)[e^{h\nu/kT} - 1]^{-1}. \qquad (2.5)$$

Thus we obtain the relation known as Kirchhoff's law (7),

$$\epsilon_\nu(\underline{r}) = K_\nu(\underline{r})B_\nu[T(\underline{r})]. \qquad (2.6)$$

In reality, the atmosphere is not in thermodynamic equilibrium because of the "external" field of radiation that is produced from a source deep within the star. We say that the atmosphere is in Local Thermodynamic Equilibrium when it is possible to define at each point in space a temperature $T(\underline{r})$ such that Kirchhoff's law is valid (7). Thus we see that Eq. (2.6) may be used to define a "temperature" at every point in the star (18).

Local Thermodynamic Equilibrium (L.T.E.) can be thought of as the high-density limit in the following sense. Consider again the element of volume, $d^3\underline{r}$, in which the density of particles is very high. Without the "external" radiation field, this volume would be in equilibrium at temperature T and thus emitting according to the black body law. Now let the external field be "turned on." The particles in $d^3\underline{r}$ will thus absorb radiation, become excited to higher energy levels, and increase their kinetic energy. If the density of particles is high, the newly absorbed energy will be distributed among the totality of particles present via particle-particle collisions. The system thus reaches equilibrium at a new, higher "local" temperature. The system now re-emits energy, again in the black body spectrum but at a higher temperature. In this limit, the effect of the external field is simply to increase the temperature at every position in the star.

The opposite of L.T.E. is the low-density limit. Here we would expect the radiation from the external field to be absorbed and simply re-emitted in a different direction because the absorbing particles would not share their energy with the atoms present, since the density is low. This idea forms what is known as Monochromatic Equilibrium (3). The radiative transfer equations that are generally presented in the study of stellar atmospheres are, thus,

$$\mu \frac{\partial \psi_\nu}{\partial z}(z,\mu) + \rho(z)K_\nu(z)\psi_\nu(z,\mu) = \rho(z)K_\nu(z)B_\nu[T(z)], \qquad (2.7)$$

for the L.T.E. limit, and

$$\mu \frac{\partial \psi_\nu}{\partial z}(z,\mu) + \rho(z)K_\nu(z)\psi_\nu(z,\mu) = \frac{1}{2}\rho(z)K_\nu(z)\int_{-1}^{1}\psi_\nu(z,\mu')d\mu', \qquad (2.8)$$

for the Monochromatic Equilibrium limit. We can quite easily incorporate the ideas of the two limits by writing Eq. (2.7) and Eq. (2.8) in the form,

$$\begin{aligned}\mu \frac{\partial \psi_\nu}{\partial z}(z,\mu) &+ \rho(z)[K_\nu(z) + S_\nu(z)]\psi_\nu(z,\mu) \\ &= \rho(z)K_\nu(z)B_\nu[T(z)] + \frac{1}{2}S_\nu(z)\rho\int_{-1}^{1}\psi_\nu(z,\mu')d\mu'.\end{aligned} \qquad (2.9)$$

Here S_ν denotes a coefficient for pure scattering. We note that once the gray approximation is made the two models, Eq. (2.7) and Eq. (2.8), reduce to equations of the same form for the integrated density. The solution of the gray problem is considered in the next chapter.

3. THE SOLUTION OF THE EQUATIONS OF RADIATIVE TRANSFER FOR THE CASE OF A GRAY ATMOSPHERE

3.1 Introduction

In this chapter we develop the solution to the equation of radiative transfer under the assumption that the atmosphere is gray. The atmosphere is said to be gray when the absorption coefficient is independent of the frequency of radiation. The assumption of a gray atmosphere is the usual first approximation made in this problem, and is analogous to the case of constant cross section or one-speed theory in the realm of neutron transport theory (21). In addition, we will consider only the equation obtained from the L.T.E. approximation (it turns out, however, that for the gray case we are led to the same equation that is obtained from the low-density or Monochromatic Equilibrium assumption). Finally, we consider an atmosphere that is stratified in parallel planes so that we need only to use one space variable, z.

3.2 Reduction of the Frequency Dependent Equations

For the case of a gray atmosphere, the continuity equation takes the form,

$$\mu \frac{\partial \psi_\nu}{\partial x}(x,\mu) + \psi_\nu(x,\mu) = B_\nu[T(z)], \qquad (3.1)$$

where the variable, x, is the optical thickness defined by

$$x = \int^{z} K\rho(z')dz'.$$

The constant absorption coefficient is denoted by K. If we integrate Eq. (3.1) over $d\nu$ from zero to infinity, we obtain

$$\mu \frac{\partial \psi}{\partial x}(x,\mu) + \psi(x,\mu) = (\sigma/\pi)T^4(x), \qquad (3.2)$$

where

$$\psi(x,\mu) \triangleq \int_0^\infty d\nu \psi_\nu(x,\mu).$$

The term involving $T^4(x)$ in Eq. (3.2) comes simply from integrating the black body formula over $d\nu$. Stefan's radiation constant is denoted by σ.

In addition to Eq. (3.1), we have the conservation condition, which for the gray problem becomes

$$\int_0^\infty d\nu B_\nu[T(x)] = \frac{1}{2}\int_0^\infty d\nu \int_{-1}^{1} \psi_\nu(x,\mu')d\mu' \qquad (3.3)$$

or

$$(\sigma/\pi)T^4(x) = \frac{1}{2}\int_{-1}^{1} \psi(x,\mu')d\mu'. \qquad (3.4)$$

Substituting this result into Eq. (3.2), we find

$$\mu \frac{\partial \psi}{\partial x}(x,\mu) + \psi(x,\mu) = \frac{1}{2}\int_{-1}^{1} \psi(x,\mu')d\mu'. \qquad (3.5)$$

Exactly the same equation is obtained from the Monochromatic Equilibrium limit when K_ν is independent of frequency.

Integrating Eq. (3.5) over $d\mu$ from -1 to 1 produces the result,

$$\frac{\partial}{\partial x}\int_{-1}^{1} \mu'\psi(x,\mu')d\mu' = 0;$$

i.e., the total energy flux (or current in the neutron transport theory vernacular) is a constant. Following the general convention (7), we define

$$\int_{-1}^{1} \mu' \psi(x,\mu') d\mu' = -(F/2), \tag{3.6}$$

where F is called the flux constant. In astrophysics, it is customary to normalize the problem in the following manner:

$$\sigma T_e^4 = \pi F$$

so that Eq. (3.6) takes the form,

$$\int_{-1}^{1} \mu' \psi(x,\mu') d\mu' = -(\sigma T_e^4/2\pi). \tag{3.7}$$

The effective temperature, denoted by T_e, is now the only unspecified parameter in the analysis (this is analogous to the power normalization in neutron transport theory).

3.3 Solution by Expansion in Normal Modes

We now develop the solution to the equation,

$$\mu \frac{\partial \psi}{\partial x}(x,\mu) + \psi(x,\mu) = \frac{1}{2} \int_{-1}^{1} \psi(x,\mu') d\mu', \tag{3.8}$$

following Case's method (8). The fact that the equation processes translational invariance suggests that we look for eigenfunctions of the form,

$$\psi_\eta(x,\mu) = e^{-x/\eta} \phi_\eta(\mu).$$

Substituting this <u>ansatz</u> into Eq. (3.8) gives

$$[1 - (\mu/\eta)] \phi_\eta(\mu) = \frac{1}{2} \int_{-1}^{1} \phi_\eta(\mu') d\mu'. \tag{3.9}$$

Since Eq. (3.9) is homogeneous, we arbitrarily select the

following normalization.[1]

$$\int_{-1}^{1} \phi_\eta(\mu')d\mu' = 1. \tag{3.10}$$

Thus Eq. (3.9) takes the form,

$$(\eta-\mu)\phi_\eta(\mu) = \eta/2. \tag{3.11}$$

Considering first that η is not on the real interval $[-1,1]$, we obtain the two discrete solutions [after applying the normalization condition, Eq. (3.10)].

$$\phi_{0\pm}(\mu) = \tfrac{1}{2}[\eta_0/(\eta_0 \mp \mu)], \tag{3.12}$$

where η_0 (as well as $-\eta_0$) is a zero of the dispersion function, $\Lambda(z)$,

$$\Lambda(z) = 1 - z \tanh^{-1}(1/z). \tag{3.13}$$

Case has shown that there are only two zeros of $\Lambda(z)$ (for $z \notin [-1,1]$), η_0 and $-\eta_0$, and in this conservative case (c=1 in Case's notation) the two zeros coalesce at infinity. For this situation (c=1), the two discrete eigenfunctions are degenerate and are given by[2]

$$\psi_1(x,\mu) = \tfrac{1}{2} \tag{3.14}$$

and

$$\psi_2(x,\mu) = \tfrac{1}{2}(x-\mu). \tag{3.15}$$

---

[1]This is permissible except for the trivial solution, $\phi_\eta(\mu) = 0$.

[2]Here, $\psi_1(x,\mu)$ is a solution to Eq. (3.11) corresponding to $\eta = \infty$, whereas $\psi_1(x,\mu)$ and $\psi_2(x,\mu)$ are both solutions to Eq. (3.8).

In addition to these two discrete modes, we have a set of continuum modes with eigenvalues on the real interval, [-1,1];

$$\phi_\eta(\mu) = (\eta/2)[P/(\eta-\mu)] + \lambda(\eta)\delta(\eta-\mu), \qquad (3.16)$$

where $\lambda(\eta)$, obtained by applying the normalization condition, is given by

$$\lambda(\eta) = 1 - \eta \tanh^{-1} \eta. \qquad (3.17)$$

The symbol, P, in Eq. (3.16) denotes that the Cauchy principal value of integrals involving $\phi_\eta(\mu)$ is to be taken.

As is usually the case in boundary value problems, we represent the solution to Eq. (3.8) as a linear superposition of the eigensolutions. We, therefore, expand the solution in the form,

$$\psi(x,\mu) = A_1\psi_1(x,\mu) + A_2\psi_2(x,\mu) + \int_{-1}^{1} A(\eta)e^{-x/\eta}\phi_\eta(\mu)d\eta, \qquad (3.18)$$

where A_1, A_2, and $A(\eta)$ are arbitrary expansion coefficients that will be used to make the solution meet the boundary conditions and the normalization.

Mutiplying Eq. (3.18) by $\mu d\mu$ and integrating from -1 to 1 and using the relation (8),

$$\int_{-1}^{1} \mu\phi_\eta(\mu)d\mu = 0,$$

determines A_2 according to the normalization given by Eq. (3.7); i.e.,

$$A_2 = 3\sigma T_e^4/2\pi. \qquad (3.19)$$

It is possible to write Eq. (3.8) in the following integral form.

$$\psi(x,\mu) = \int_0^x (dt/\mu)[\tfrac{1}{2}\int_{-1}^{1}\psi(t,\mu')d\mu']e^{-(x-t)/\mu},\ \mu > 0 \qquad (3.20)$$

and

$$\psi(x,\mu) = -\int_x^\infty (dt/\mu)[\tfrac{1}{2}\int_{-1}^{1}\psi(t,\mu')d\mu']e^{-(x-t)/\mu},\ \mu < 0. \qquad (3.21)$$

Quite generally (7), the convergence of the above integrals requires that

$$\lim_{x\to\infty} \tfrac{1}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' e^{-x} = 0,$$

and since for the continuum modes

$$\psi(x,\mu) \sim e^{-x/\eta},$$

our first boundary condition is that $\eta \notin [-1,0]$. The second boundary condition is that there is no radiation incident on the medium; i.e.,

$$\psi(0,\mu) = 0,\ \mu > 0.$$

Applying these boundary conditions to Eq. (3.18) produces the result,

$$A_2\mu/2 = (A_1/2) + \int_0^1 A(\eta)\phi_\eta(\mu)d\eta,\ \mu > 0. \qquad (3.22)$$

The fact that an arbitrary function (in this case, $A_2\mu/2$) can be expanded in terms of the eigenfunctions of the transport equation is discussed in detail in Appendix D. There these functions are shown to obey a very general partial

range completeness theorem. At this point we can determine the expansion coefficients from use of the orthogonality relations developed by Kuščer et al. (22). Applying the orthogonality conditions, we find

$$A(\eta) = -\frac{1}{6}X(-\eta)g(1,\eta)A_2, \tag{3.23}$$

where $X(-\eta)$ is Case's X-function for the half-range and $g(1,\eta)$ is a tabulated function (23); i.e.,

$$g(1,\eta) = 1/[\lambda^2(\eta) + (\pi^2\eta^2/4)]. \tag{3.24}$$

The constant, A_1, is given by

$$A_1 = \frac{3}{2}\int_0^1 [\mu^2/X(-\mu)]d\mu A_2. \tag{3.25}$$

Since $X(-\mu)$ for $\mu \in [0,1]$ is also well tabulated (24), the extrapolated endpoint defined by

$$x_0 = \frac{3}{2}\int_0^1 [\mu^2/X(-\mu)]d\mu$$

can be easily evaluated. A computer program was written to make this calculation. The result,

$$x_0 = 0.71044,$$

is in excellent agreement with other exact methods of solution (25).

The solution to Eq. (3.8), subject to the proper boundary and normalization conditions, takes the form,

$$\psi(x,\mu) = (3\sigma T_e^4/4\pi)[x_0 + x - \mu - \frac{1}{3}\int_0^1 X(-\eta)g(1,\eta)e^{-x/\eta}\phi_\eta(\mu)d\eta]. \tag{3.26}$$

To obtain the mean intensity, defined by

$$J(x) \triangleq \frac{1}{2}\int_{-1}^{1}\psi(x,\mu')d\mu',$$

we simply integrate Eq. (3.26). The result is

$$J(x) = (3\sigma T_e^4/4\pi)[x_0 + x - \frac{1}{6}\int_0^1 X(-\eta)g(1,\eta)e^{-x/\eta}d\eta], \quad (3.27)$$

and thus from Eq. (3.4) we obtain the temperature distribution in the medium; i.e.,

$$T^4(x)/T_e^4 = \frac{3}{4}[x_0 + x - \frac{1}{6}\int_0^1 X(-\eta)g(1,\eta)e^{-x/\eta}d\eta]. \quad (3.28)$$

The temperature distribution, as given by Eq. (3.28), has been calculated using the IBM 7090 computer. The results are shown in Table 3.1. Having found the temperature distribution in the atmosphere, we are now able to determine the general solution for the frequency dependent intensity. Thus we write the solution to Eq. (3.1).

$$\psi_\nu(x,\mu) = \int_0^x (dt/\mu)B_\nu[T(t)]e^{-(x-t)/\mu}, \quad \mu > 0 \quad (3.29a)$$

and

$$\psi_\nu(x,\mu) = -\int_x^\infty (dt/\mu)B_\nu[T(t)]e^{-(x-t)/\mu}, \quad \mu < 0. \quad (3.29b)$$

Taking the jth moment (with respect to μ) of the intensity,

$$\int_{-1}^{1} d\mu\mu^j\psi_\nu(x,\mu) = \int_0^x dtB_\nu[T(t)]E_{j+1}(x-t) + (-1)^j\int_x^\infty dtB_\nu[T(t)]E_{j+1}(t-x), \quad (3.30)$$

Table 3.1 The temperature distribution for the gray atmosphere

x	$T(x)/T_e$	x	$T(x)/T_e$	x	$T(x)/T_e$	x	$T(x)/T_e$
0.000	0.8112	0.250	0.9082	2.300	1.2256	4.800	1.4258
0.010	0.8184	0.260	0.9110	2.400	1.2357	4.900	1.4322
0.020	0.8242	0.270	0.9138	2.500	1.2456	5.000	1.4386
0.030	0.8295	0.280	0.9166	2.600	1.2551	5.100	1.4448
0.040	0.8344	0.290	0.9193	2.700	1.2646	5.200	1.4510
0.050	0.8390	0.300	0.9220	2.800	1.2737	5.300	1.4571
0.060	0.8434	0.400	0.9472	2.900	1.2827	5.400	1.4631
0.070	0.8477	0.500	0.9700	3.000	1.2915	5.500	1.4691
0.080	0.8518	0.600	0.9910	3.100	1.3002	5.600	1.4750
0.090	0.8557	0.700	1.0106	3.200	1.3086	5.700	1.4807
0.100	0.8596	0.800	1.0288	3.300	1.3169	5.800	1.4865
0.110	0.8633	0.900	1.0460	3.400	1.3250	5.900	1.4922
0.120	0.8670	1.000	1.0624	3.500	1.3330	6.000	1.4978
0.130	0.8705	1.100	1.0780	3.600	1.3409	6.100	1.5033
0.140	0.8740	1.200	1.0929	3.700	1.3486	6.200	1.5088
0.150	0.8774	1.300	1.1071	3.800	1.3562	6.300	1.5143
0.160	0.8807	1.400	1.1208	3.900	1.3636	6.400	1.5196
0.170	0.8840	1.500	1.1340	4.000	1.3710	6.500	1.5250
0.180	0.8872	1.600	1.1468	4.100	1.3782	6.600	1.5302
0.190	0.8903	1.700	1.1591	4.200	1.3853	6.700	1.5354
0.200	0.8934	1.800	1.1710	4.300	1.3923	6.800	1.5406
0.210	0.8965	1.900	1.1826	4.400	1.3992	6.900	1.5457
0.220	0.8996	2.000	1.1938	4.500	1.4060	7.000	1.5507
0.230	0.9024	2.100	1.2047	4.600	1.4127	7.100	1.5557
0.240	0.9053	2.200	1.2153	4.700	1.4193	7.200	1.5607

where $E_i(x)$ is the standard exponential function (23). Thus, for the frequency dependent flux,

$$F_\nu(x) \triangleq -2\int_{-1}^{1}\mu'\Psi_\nu(x,\mu')d\mu' \tag{3.31}$$

or

$$F_\nu(x) = 2\int_x^\infty dtB_\nu[T(t)]E_2(t-x) - 2\int_0^x dtB_\nu[T(t)]E_2(x-t). \tag{3.32}$$

Writing Eq. (3.32) in terms of the dimensionless variable,

$$\alpha = h\nu/kT_e, \tag{3.33}$$

we find

$$\frac{F_\alpha(x)}{F} = \frac{\alpha^3}{0.30797}\left[\int_x^\infty dt\,\frac{E_2(t-x)}{e^{[\alpha T_e/T(x)]}-1} - \int_0^x dt\,\frac{E_2(x-t)}{e^{[\alpha T_e/T(x)]}-1}\right]. \tag{3.34}$$

Here F is the flux constant. Using the results of Eq. (3.28), Eq. (3.34) was calculated as a function of α and x. The results are shown in Table 3.2.

Another quantity of interest in this astrophysical problem is the law of darkening (7), i.e., the angular distribution of radiant energy (integrated over frequency) that is leaving the surface of the star. This is obtained by setting $x = 0$ in Eq. (3.25) and considering only $\mu < 0$. Using the technique of Shure and Natelson (26), we find

$$\Psi(0,\mu) = (3\sigma T_e^4/8\pi)[1/X(-\mu)],\ \mu < 0. \tag{3.35}$$

Table 3.2 The Monochromatic fluxes, $F_\alpha(x)$, in units of F

x	α							
	1	2	3	4	6	8	10	12
0.000	-0.08644	-0.18437	-0.20834	-0.17818	-0.08616	-0.03163	-0.010189	-0.003072
0.100	-0.07853	-0.17355	-0.20325	-0.17964	-0.09149	-0.03473	-0.011405	-0.003471
0.200	-0.07194	-0.16400	-0.19813	-0.18024	-0.09622	-0.03774	-0.012655	-0.003899
0.300	-0.06627	-0.15538	-0.19306	-0.18025	-0.10047	-0.04067	-0.013935	-0.004354
0.400	-0.06133	-0.14756	-0.18808	-0.17982	-0.10432	-0.04351	-0.015237	-0.004833
0.500	-0.05697	-0.14042	-0.18324	-0.17905	-0.10778	-0.04627	-0.016554	-0.005334
0.600	-0.05310	-0.13387	-0.17854	-0.17801	-0.11092	-0.04893	-0.017881	-0.005854
0.700	-0.04965	-0.12785	-0.17400	-0.17675	-0.11374	-0.05149	-0.019211	-0.006392
0.800	-0.04657	-0.12231	-0.16963	-0.17533	-0.11628	-0.05396	-0.020539	-0.006946
0.900	-0.04389	-0.11710	-0.16542	-0.17378	-0.11856	-0.05632	-0.021863	-0.007513
1.000	-0.04129	-0.11247	-0.16138	-0.17214	-0.12061	-0.05859	-0.023177	-0.008091
1.200	-0.03697	-0.10402	-0.15380	-0.16864	-0.12408	-0.06285	-0.025768	-0.009276
1.400	-0.03341	-0.09673	-0.14685	-0.16499	-0.12682	-0.06674	-0.028293	-0.010488
1.600	-0.33044	-0.09040	-0.14048	-0.16130	-0.12897	-0.07028	-0.030739	-0.011716
1.800	-0.02794	-0.08486	-0.13464	-0.15762	-0.13061	-0.07351	-0.033098	-0.012952
2.000	-0.02581	-0.08000	-0.12928	-0.15401	-0.13184	-0.07643	-0.035365	-0.014189

The gray case is now completely solved. We have the temperature distribution, Eq. (3.28); the law of darkening, Eq. (3.35); the monochromatic fluxes, Eq. (3.34); and the expressions for the integrated intensity and mean intensity, Eqs. (3.26) and (3.27), respectively. We thus formulate in the next chapter the exact solution to our first approach to the nongray problem, i.e., the Uniform Picket Fence model.

4. UNIFORM PICKET FENCE MODEL

4.1 Introduction

This chapter presents a solution of the equation of radiative transfer in the Picket Fence model (15) that is a generalization of the gray case described in Chapter 3. In the model, the absorption coefficient for the radiation is assumed to be representable as a set of two different constant values over the frequency spectrum.[1] This represents, to some approximation, absorption by resonance lines. In Section 4.2 we show how in this approximation, under the assumption of Local Thermodynamic Equilibrium (7), the equation of radiative transfer reduces to a set of two coupled transport equations similar to the two-group neutron transport equations considered by Żelazny and Kuszell (27).[2] The radiative case, however, is simpler because the "transfer matrix" takes a particularly simple (degenerate) form and has a vanishing determinant. Our method of solution is given in some detail, because it will serve as a guide to the more general case mentioned in footnote 1, below.

The method used is similar to that of Chapter 3, i.e., Case's method of singular eigensolutions (8, 21, 29, and 30).

A number of solutions to the Picket Fence model exist. Chandrasekhar (15) solves the euqations using the Eddington

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[1]The generalization to N different values of the absorption coefficient is discussed in Chapter 5.

[2]These authors used a highly inconvenient form of the (degenerate) eigensolutions. A similar inconvenient form has been used by Ferziger and Leonard (28), who considered a related problem.

approximation. Actually, Chandrasekhar's model is somewhat more general than ours in that he also considered scattering. Ours, in fact, corresponds to his case $c=1$. The case $c\neq 1$ involves a transfer matrix whose determinant does not vanish; this case is discussed in Section 4.7. Various other approximate and numerical solutions have been developed and have been summarized by Gingerich (14), who considered numerical solutions. We will not attempt to review any of this work, except to note that numerical solutions are extremely difficult. As noted in Chapter 1, Stewart (17) has obtained an exact solution. However, we believe our method is simpler and, in addition, possesses the merit that it can be readily generalized in the manner already mentioned.

A number of recent papers have considered the solution of the radiative transfer equations for various models in the nongray case, but these are not, strictly speaking, Picket Fence models (cf. Refs. 9-13).

As we have stated, Section 4.2 is devoted to a brief derivation of the Picket Fence equations. Thus, in Section 4.3, we begin the solution by introducing the normal modes. In Section 4.4, a completeness proof is sketched and orthogonality relations are deduced in Section 4.5. In Section 4.6, explicit solutions for the extrapolated endpoint, the temperature distribution, and the law of darkening are developed for the Milne problem (31). Actually, we show how to solve any of the standard half-range problems, e.g.,

the Green's function and the albedo problem [the slab albedo problem can be solved by extending the method of McCormick and Mendelson (32), but the Milne problem is of primary interest here].[3] In Section 4.7, we make a few remarks about the problem in which the transfer matrix is completely arbitrary [this case is of considerable interest in neutron physics and also corresponds to Chandrasekhar's $\varepsilon \neq 1$ (15)], but we have been able to obtain explicit solutions only for full-space problems. More work is needed on the half-space solutions.

Our notation is that of neutron physics, because Case's paper (8), which we draw so heavily upon, uses this notation.[4]

4.2 The Equations in the Picket Fence Model[5]

The equation of radiative transfer under the assumption of Local Thermodynamic Equilibrium has been written in Chapter 2 as

$$\mu \frac{\partial \psi_\nu}{\partial z}(z,\mu) + \rho(z) K_\nu \psi_\nu(z,\mu) = \rho(z) K_\nu B_\nu[T(z)]. \qquad (4.1)$$

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[3]The solution of the half-space Green's function requires a special technique. This is discussed in Appendix B.

[4]In order to compare our equations with those used in astrophysics (e.g., Ref. 7), one must change μ to $-\mu$, $\psi_\nu \longrightarrow I_\nu$.

[5]The derivation of the Picket Fence equations is given in a number of places, e.g., Ref. (15). A brief sketch is presented here for the sake of completeness.

We recall that in this equation, $B_\nu[T(z)]$ is the Planck black body function, given again for convenience:

$$B_\nu[T(z)] = (2h\nu^3/c^2)\{\exp[h\nu/kT(z)] - 1\}^{-1}. \qquad (4.2)$$

To solve Eq. (4.1), the Schwarzschild condition, which states local energy conservation, is used (cf. Eq. 2.5):

$$\int_0^\infty d\nu K_\nu B_\nu[T(z)] = \frac{1}{2}\int_0^\infty d\nu K_\nu \int_{-1}^{1} \psi_\nu(z,\mu)d\mu. \qquad (4.3)$$

Eq. (4.1) and Eq. (4.3) are two equations for the two unknowns, $\psi_\nu(z,\mu)$ and $T(z)$. Obviously they are nonlinear. However, if the absorption coefficient takes a certain form, the problem becomes much simpler. The so-called gray case refers to K_ν=constant and was solved in Chapter 3. For this situation, Eq. (4.1) reduced to the form,

$$\mu\frac{\partial\psi}{\partial x}(x,\mu) + \psi(x,\mu) = \frac{1}{2}\int_{-1}^{1}\psi(x,\mu')d\mu', \qquad (4.4)$$

where x is the optical thickness and $\psi(x,\mu)$ is the integrated energy density; i.e.,

$$\psi(x,\mu) \triangleq \int_0^\infty d\nu\,\psi_\nu(x,\mu). \qquad (4.5)$$

Eq. (4.4) is precisely the equation considered by Case (8), and refers to the gray atmosphere in L.T.E. or to the case of pure isotropic scattering.

In the Picket Fence Model, we have two constant values, K_1 and K_2, for the representation of K_ν. Let $\Delta\nu_i$ represent

the frequency region over which K_ν has the value K_i, and integrate Eqs. (4.1) and (4.3) to obtain

$$\mu \frac{\partial \psi_i}{\partial z}(z,\mu) + \rho(z)\psi_i(z,\mu)K_i = [\rho(z)K_i w_i/2\sum_{j=1}^{2} K_j w_j]\sum_{j=1}^{2} K_j \int_{-1}^{1} \psi_j(z,\mu')d\mu', \quad i=1,2. \tag{4.6}$$

Here $\psi_i(z,\mu)$ is defined by

$$\psi_i(z,\mu,) \triangleq \int_{\Delta\nu_i} d\nu \psi_\nu(z,\mu), \tag{4.7}$$

and w_i is given by

$$w_i \triangleq [\pi/\sigma T^4(z)]\int_{\Delta\nu_i} d\nu B_\nu[T(z)], \tag{4.8}$$

where σ is the Stefan-Boltzmann constant. To obtain Eq. (4.6), the Schwarzschild condition has been used. In the present model, it takes the form,

$$\sum_{j=1}^{2} K_j w_j[\sigma T^4(z)/\pi] = \frac{1}{2}\sum_{j=1}^{2} K_j \int_{-1}^{1} \psi_j(z,\mu')d\mu'. \tag{4.9}$$

We note that in general the w_i are functions of z. In order to solve the transport equation, Eq. (4.6), we shall have to assume the w_i are constant. This implies either the Uniform Picket Fence model (cf. Ref. 14) or a "Random" model (17). In the Uniform model, it is assumed that the frequency spectrum can be divided into ranges, $\Delta\nu_i$, so small that B [T(z)] may be considered constant in each range. In each

range it is assumed that the fractional frequency width covered by each K_i is constant. From these two assumptions, one readily verifies from Eq. (4.8) that the w_i are independent of z. In the Random model, one assumes that there are so many lines of random width randomly distributed throughout the spectrum that the distribution may be considered uniform on the average. Chandrasekhar, in Ref. (15), implicitly assumed one of these two models.

Eq. (4.6) can be put in more convenient notation. Let K_2 be the smaller of the K_i. Then define an optical variable x in terms of K_2; i.e.,

$$x \triangleq K_2 \int^z \rho(z')dz'. \tag{4.10}$$

Further denote K_i/K_2 by σ_i ($\sigma_2 \equiv 1$). Eq. (4.6) can thus be written in the form,

$$\mu \frac{\partial}{\partial x} \underline{\Psi}(x,\mu) + \underline{\Sigma}\, \underline{\Psi}(x,\mu) = \underline{C} \int_{-1}^{1} \underline{\Psi}(x,\mu')d\mu'. \tag{4.11}$$

Here $\underline{\Psi}(x,\mu)$ is a two-component vector with elements $\psi_i(x,\mu)$, while $\underline{\Sigma}$ and $\underline{C}$ are the following matrices.

$$(\underline{\Sigma})_{ij} = \sigma_i \delta_{ij} \tag{4.12a}$$

$$(\underline{C})_{ij} = \sigma_i w_i \sigma_j / 2 \sum_{i=1}^{2} \sigma_i w_i. \tag{4.12b}$$

$\underline{C}$ is the transfer matrix; we note that det $\underline{C}$ = 0.

We turn now to the solution of Eq. (4.11).

4.3 Eigenvalues and Eigensolutions

We proceed as Case did (8) by noting the translational invariance of Eq. (4.11). This suggests that the eigensolutions should transform according to the irreducible representations of the translation group (21). We first try the one-dimensional representation, $e^{-x/\eta}$; i.e., we assume a solution to Eq. (4.11) of the form,

$$\underline{\Psi}(x,\mu) = e^{-x/\eta}\underline{F}(\eta,\mu). \tag{4.13}$$

When this ansatz is substituted into Eq. (4.11), we obtain an eigenvalue equation for $\underline{F}(\eta,\mu)$, where η is the eigenvalue.

$$\frac{1}{\eta}\begin{bmatrix}\sigma_1\eta-\mu & 0\\ 0 & \eta-\mu\end{bmatrix}\underline{F}(\eta,\mu) = \underline{C}\int_{-1}^{1}\underline{F}(\eta,\mu')d\mu'. \tag{4.14}$$

The eigenvalue spectrum of Eq. (4.14) must be considered as three separate regions. There are two continuum regions and a discrete spectrum. Consider then the first of the continuous spectra.

Region 1: $\qquad \eta \in \left[-\frac{1}{\sigma_1}, \frac{1}{\sigma_1}\right].$

Here we write

$$\underline{F}^{(1)}(\eta,\mu) \tag{4.15}$$

$$= \begin{bmatrix}\frac{\eta P}{\sigma_1\eta-\mu} + \lambda_{11}(\eta)\delta(\sigma_1\eta-\mu) & 0\\ 0 & \frac{\eta P}{\eta-\mu} + \lambda_{12}(\eta)\delta(\eta-\mu)\end{bmatrix}\underline{C}\int_{-1}^{1}\underline{F}^{(1)}(\eta,\mu)d\mu.$$

The symbol P indicates that the Cauchy principal value is to be taken when integrals over these functions are performed.

The functions $\lambda_{1i}(\eta)$ are chosen such that Eq. (4.15) is satisfied. Since Eq. (4.15) is homogeneous, the normalization is arbitrary. We select

$$\int_{-1}^{1} \underline{F}^{(1)}(\eta,\mu')d\mu' = \begin{bmatrix} a_1(\eta) \\ a_2(\eta) \end{bmatrix}, \tag{4.16}$$

where, as yet, the $a_i(\eta)$ are unspecified functions of η. Integrating Eq. (4.15), we find

$$\begin{bmatrix} a_1(\eta) \\ a_2(\eta) \end{bmatrix} = \begin{bmatrix} 2\eta\tau(\sigma_1\eta)+\lambda_{11}(\eta) & 0 \\ 0 & 2\eta\tau(\eta)+\lambda_{12}(\eta) \end{bmatrix} \underline{C} \begin{bmatrix} a_1(\eta) \\ a_2(\eta) \end{bmatrix}, \tag{4.17}$$

where we have used the abbreviation, $\tau(x)$, for $\tanh^{-1}(x)$.

Eq. (4.17) is really two equations for the four unknowns, $\lambda_{1i}(\eta)$ and $a_i(\eta)$. Thus we note that the eigensolutions in this region are twofold degenerate. (At this point we deviate from the treatment of Ref. (27) in which the eigensolutions were left arbitrary. We explicitly take the degeneracy into account.) Clearly the two linearly independent choices for $\underline{F}^{(1)}(\eta,\mu)$ correspond to $a_1(\eta)=1$, $a_2(\eta)=0$ and $a_1(\eta)=0$, $a_2(\eta)=1$. Therefore, for region 1 we can write [solving for $\lambda_{1i}(\eta)$ and multiplying out Eq. (15)],

$$\underline{F}_1^{(1)}(\eta,\mu) = \begin{bmatrix} \dfrac{c_{11}\eta P}{\sigma_1\eta-\mu} + \delta(\sigma_1\eta-\mu)[1-2\eta c_{11}\tau(\sigma_1\eta)] \\ \dfrac{c_{21}\eta P}{\eta-\mu} + \delta(\eta-\mu)[-2\eta c_{21}\tau(\eta)] \end{bmatrix} \tag{4.18a}$$

and

$$\underline{F}_2^{(1)}(\eta,\mu) = \begin{bmatrix} \dfrac{c_{12}\eta P}{\sigma_1\eta-\mu} + \delta(\sigma_1\eta-\mu)[-2\eta c_{12}^{\tau}(\sigma_1\eta)] \\ \dfrac{c_{22}\eta P}{\eta-\mu} + \delta(\eta-\mu)[1-2\eta c_{22}^{\tau}(\eta)] \end{bmatrix}. \qquad (4.18b)$$

We now consider the second continuum.

Region 2: $\eta \in \left[-1, -\frac{1}{\sigma_1}\right]$ and $\left[\frac{1}{\sigma_1}, 1\right]$.

Solving Eq. (4.14) for region 2, the factor, $\sigma_1\eta-\mu$, we note, is not singular. Thus only one coefficient, $\lambda_{22}(\eta)$, enters; and there is no degeneracy.

We easily find

$$\underline{F}^{(2)}(\eta,\mu) = \begin{bmatrix} \dfrac{c_{12}\eta}{\sigma_1\eta-\mu} \\ \dfrac{c_{22}\eta P}{\eta-\mu} + \delta(\eta-\mu)[1-2\eta c_{11}^{\tau}(1/\sigma_1\eta)-2\eta c_{22}^{\tau}(\eta)] \end{bmatrix}. \qquad (4.19)$$

Finally we have the discrete spectrum.

Region 3: $\eta \notin [-1,1]$.

Solving Eq. (4.14) for $\underline{F}(\eta,\mu)$ and integrating over μ, we find the dispersion relation,

$$\Omega(z) = 1 - 2zc_{11}^{\tau}(1/\sigma_1 z) - 2zc_{22}^{\tau}(1/z). \qquad (4.20)$$

That is, the discrete eigenvalues are the zeros of $\Omega(z)$. It can be shown readily (see Appendix C) that $\Omega(z)$ has only two zeros, $\eta_0 = \pm\infty$. Thus, we find that the discrete eigensolutions are also twofold degenerate. In this case, however,

unlike the degeneracy of region 1, we can no longer use the one-dimensional representation of the translation group, Eq. (4.13). Since we are dealing with twofold degeneracy, the two-dimensional representation is appropriate. It can be generated by the basis set (21),

$$Y_1(x,\eta) = e^{-x/\eta} \longrightarrow 1 \tag{4.21}$$

$$Y_2(x,\eta) = xe^{-x/\eta} \longrightarrow x, \tag{4.22}$$

where the arrows indicate the limit for $\eta_0 \longrightarrow \infty$. The appropriate linear combinations that satisfy Eq. (4.11) we find to be

$$\underline{\Psi}_+(x,\mu) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \triangleq \underline{\phi}_+ \tag{4.23}$$

and

$$\underline{\Psi}_-(x,\mu) = \begin{bmatrix} w_1[x-(\mu/\sigma_1)] \\ w_2(x-\mu) \end{bmatrix}. \tag{4.24}$$

Instead of the continuum eigenfunctions, $\underline{F}^{(i)}(\eta,\mu)$, which we have just calculated, Eqs. (4.18) and (4.19), it is more convenient to use certain linear combinations. The use of these linear combinations simplifies both the completeness proof (Section 4.4) and the calculation of the normalization integrals (Section 4.5). We define, therefore,

$$\underline{\phi}_1(\eta,\mu) = \frac{1}{c_{11}}\underline{F}_1^{(1)}(\eta,\mu) - \frac{1}{c_{12}}\underline{F}_2^{(1)}(\eta,\mu) \tag{4.25a}$$

$$= \begin{bmatrix} \frac{1}{c_{11}}\,\delta(\sigma_1\eta-\mu) \\ -\frac{1}{c_{12}}\,\delta(\eta-\mu) \end{bmatrix}, \tag{4.25b}$$

$$\underline{\phi}_2^{(1)}(\eta,\mu) = \underline{F}_2^{(1)}(\eta,\mu), \tag{4.25c}$$

and

$$\underline{\phi}_2^{(2)}(\eta,\mu) = \underline{F}^{(2)}(\eta,\mu). \tag{4.25d}$$

We note that, if we wished, we could consider that we have only two continuum eigensolutions; namely, $\underline{\phi}_1(\eta,\mu)$ and

$$\underline{\phi}_2(\eta,\mu) \triangleq \underline{\phi}_2^{(1)}(\eta,\mu)\ \theta_1(\eta) + \underline{\phi}_2^{(2)}(\eta,\mu)\ \theta_2(\eta). \tag{4.26}$$

Here

$$\theta_i(\eta) = 1,\ \eta \in \text{region } i \tag{4.27}$$
$$= 0,\ \text{otherwise}.$$

Having determined the eigenvalue spectrum and the corresponding eigensolutions, we consider next the completeness theorem.

4.4 Completeness

Theorem I: The functions, $\phi_1(\eta,\mu)$, $\underline{\phi}_2^{(i)}(\eta,\mu)$, $\eta \in [0,1]$, and $\underline{\phi}_+$ form a complete set for functions defined on the "half-range," $0 \le \mu \le 1$, in the sense that an arbitrary function,[6] $\underline{\Psi}(\mu)$, defined for $0 \le \mu \le 1$ can be expanded in the form,

$$\underline{\Psi}(\mu) = A_+\underline{\phi}_+ + \int_{(1')} \alpha_1(\eta)\underline{\phi}_1(\eta,\mu)d\eta + \int_{(1')} \alpha_2(\eta)\underline{\phi}_2^{(1)}(\eta,\mu)d\eta + \int_{(2')} \alpha_2(\eta)\underline{\phi}_2^{(2)}(\eta,\mu)d\eta,\ 0 \le \mu \le 1. \tag{4.28}$$

[6]Actually, the solution to Eq. (4.28) incorporates results that are valid only if $\underline{\Psi}(\mu)$ obeys a Hölder condition on the interval [0,1] (33). However, in Ref. (21) it is shown how to interpret the formalism when $\underline{\Psi}(\mu)$ is a distribution.

Here the ranges of integration, 1' and 2', refer to those portions of regions 1 and 2 for which $\eta \geq 0$.

This is the half-range completeness theorem.[7] Before proving this theorem, we note that the expansion coefficients, $\alpha_1(\eta)$ and $\alpha_2(\eta)$ are taken to be scalars (i.e., multiples of the unit matrix).

We turn now to the proof of theorem I. Eq. (4.28) can be considered a singular integral equation for the expansion coefficients.[8] To prove the theorem, it is sufficient to prove that a solution exists. This, in turn, is done by solving the equation using the methods of Ref. (33). This yields expressions for the expansion coefficients; however, they are more conveniently obtained from the orthogonality relations developed in Section 4.5.

We begin by attempting an expansion in terms of the continuum modes alone; i.e.,

$$\underline{\psi}(\mu) = \int_{(1')} \alpha_1(\eta)\underline{\phi}_1(\eta,\mu)d\eta + \int_{(1')} \alpha_2(\eta)\underline{\phi}_2^{(1)}(\eta,\mu)d\eta + \int_{(2')} \alpha_2(\eta)\underline{\phi}_2^{(2)}(\eta,\mu)d\eta. \tag{4.29}$$

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[7] A full-range completeness theorem can be proved in a very similar manner and is used in Appendix B. Also, there is an analogous theorem for the other half-range, $-1 \leq \mu \leq 0$.

[8] Although we are proving the theorem for the given form of $\underline{C}$, i.e., Eq. (4.12b), it also can be proved assuming only that $\det \underline{C} = 0$.

Putting in the explicit forms of the eigensolutions, we obtain the two equations,

$$\phi_1(\mu) = (1/c_{11}\sigma_1)\alpha_1(\mu/\sigma_1) + c_{12}\int_{(1')} \alpha_2(\eta)\{[\eta P/(\sigma_1\eta-\mu)] + \delta(\sigma_1\eta-\mu)[-2\eta\tau(\sigma_1\eta)]\}d\eta + c_{12}\int_{(2')} \alpha_2(\eta)[\eta/(\sigma_1\eta-\mu)]d\eta \tag{4.30a}$$

and

$$\phi_2(\mu) = -[\alpha_1(\mu)\,\theta_1(\mu)/c_{12}] + \int_{(1')} \alpha_2(\eta)\{[c_{22}\eta P/(\eta-\mu)] + \delta(\eta-\mu)[1 - 2\eta c_{22}\tau(\eta)]\}d\eta + \int_{(2')} \alpha_2(\eta)\{[c_{22}\eta P/(\eta-\mu)] + \delta(\eta-\mu)[1 - 2\eta c_{22}\tau(\eta) - 2\eta c_{11}\tau(1/\sigma_1\eta)]\}d\eta. \tag{4.30b}$$

Here $\phi_1(\mu)$ and $\phi_2(\mu)$ are the two components of $\underline{\Psi}(\mu)$. Making the charge of variable $\mu \rightarrow \sigma_1\mu$, solving for $\alpha_1(\mu)$ from Eq. (4.30a) and substituting into Eq. (4.30b), we obtain

$$\phi_2(\mu) + \sigma_1(c_{11}/c_{12})\phi_1(\sigma_1\mu)\,\theta_1(\mu) = \omega(\mu)\alpha_2(\mu) + [c_{22} + c_{11}\,\theta_1(\mu)]P\int_0^1[\alpha_2(\eta)\eta/(\eta-\mu)]d\eta, \tag{4.31}$$

where we have defined

$$\omega(\mu) \triangleq 1 - 2\mu c_{22}\tau(\mu) - 2\mu c_{11}\tau(\sigma_1\mu)\,\theta_1(\mu) - 2\mu c_{11}\tau(1/\sigma_1\mu)\,\theta_2(\mu). \tag{4.32}$$

Taking the boundary values on the branch cut of the dispersion function, Eq. (4.20), we see that

$$\Omega_1^{\pm}(\mu) = 1 - 2\mu c_{11}\tau(\sigma_1\mu) - 2\mu c_{22}\tau(\mu) \pm \pi i\mu(c_{11}+c_{22}) \qquad (4.33a)$$

and

$$\Omega_2^{\pm}(\mu) = 1 - 2\mu c_{11}\tau(1/\sigma_1\mu) - 2\mu c_{22}\tau(\mu) \pm \pi i\mu c_{22}. \qquad (4.33b)$$

Here the subscripts on $\Omega^{\pm}(\mu)$ refer to the region in which the boundary value is evaluated. Thus, we find

$$\omega(\mu) = \tfrac{1}{2}[\Omega^{+}(\mu) + \Omega^{-}(\mu)] \qquad (4.34a)$$

and

$$c_{22} + c_{11}\,\theta_1(\mu) = (1/2\pi i\mu)[\Omega^{+}(\mu) - \Omega^{-}(\mu)]. \qquad (4.34b)$$

Proceeding in standard fashion (8, 33), we introduce the auxiliary function, N(z), defined by

$$N(z) \triangleq (1/2\pi i)\int_0^1 [\alpha_2(\eta)\eta d\eta/(\eta-z)], \qquad (4.35)$$

with boundary values,

$$N^{\pm}(\mu) = (P/2\pi i)\int_0^1 [\alpha_2(\eta)\eta d\eta/(\eta-\mu)] \pm \tfrac{1}{2}\mu\alpha_2(\mu). \qquad (4.36)$$

From Eq. (4.35), N(z) should be analytic in the cut plane and vanish as 1/z for large z. Substituting Eqs. (4.34) and (4.36) into Eq. (4.31), we find

$$\begin{aligned} &\Psi_2(\mu) + \sigma_1(c_{11}/c_{12})\Psi_1(\sigma_1\mu)\,\theta_1(\mu) \\ &= \Omega^{+}(\mu)N^{+}(\mu) - \Omega^{-}(\mu)N^{-}(\mu). \end{aligned} \qquad (4.37)$$

The solution to Eq. (4.37) is well known. It is expressed in terms of a function, X(z), which has the properties,

$$X^{+}(\mu)/X^{-}(\mu) = \Omega^{+}(\mu)/\Omega^{-}(\mu), \quad \mu > 0, \qquad (4.38)$$

but with $X^+(\mu) = X^-(\mu)$ for $\mu < 0$. That is, $X(z)$ is analytic in the complex plane cut, on the real line, from zero to 1; $X(z)$ must also obey other restrictions, as discussed in Ref. (8). The appropriate X-function for this case is

$$X(z) = [1/(1-z)]\exp[(1/\pi)\int_0^1 \mathrm{Arg}\ \Omega^+(\mu)[d\mu/(\mu-z)]. \qquad (4.39)$$

The solution to Eq. (4.37) is then

$$N(z) = [1/2\pi i X(z)]\int_0^1 \gamma(\mu)\{\psi_2(\mu) + (\sigma_1 c_{11}/c_{12})\psi_1(\sigma_1\mu)\ \theta_1\ (\mu)\}[d\mu/(\mu-z)], \qquad (4.40)$$

where

$$\gamma(\mu) \triangleq \mu[X^+(\mu)/\Omega^+(\mu)]. \qquad (4.41)$$

We see from Eq. (4.40) that $N(z)$ does not vanish at infinity as $1/z$ unless

$$\int_0^1 \gamma(\mu)[\psi_2(\mu) + (\sigma_1 c_{11}/c_{12})\psi_1(\sigma_1\mu)\ \theta_1\ (\mu)]d\mu = 0. \qquad (4.42)$$

In general, this condition will not be satisfied. However, we recall that we are actually expanding not $\underline{\Psi}(\mu)$ but $\underline{\Psi}(\mu) - A_+\underline{\phi}_+$ [comparing Eqs. (4.28) and (4.29)]. Replacing the $\psi_i(\mu)$ in Eq. (4.42) by $\psi_i(\mu) - A_+ w_i$, we see that Eq. (4.42) will be satisfied if A_+ is defined by

$$A_+ = \frac{\int_0^1 \underline{\tilde{\phi}}_+^\dagger \underline{W}(\mu)\underline{\Psi}(\mu)d\mu}{\int_0^1 \underline{\tilde{\phi}}_+^\dagger \underline{W}(\mu)\underline{\phi}_+ d\mu}, \qquad (4.43)$$

where

$$\underline{W}(\mu) = \begin{bmatrix} \sigma_1 \gamma(\mu/\sigma_1) & 0 \\ 0 & \gamma(\mu) \end{bmatrix} \tag{4.44}$$

and

$$\underline{\phi}_+^\dagger = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{4.45}$$

The coefficients are now determined. A_+ is given by Eq. (4.43), while the continuum coefficients may be determined from N(z), Eq. (4.40), using Eq. (4.36) to find $\alpha_2(\eta)$ and then obtaining $\alpha_1(\eta)$ from Eq. (4.30). In principle, the expansion coefficients could be found in this manner. However, the orthogonality relations derived in the next section provide a more convenient method. The existence of such relations is already strongly suggested by the form of Eq. (4.43), particularly because one verifies immediately that $\underline{\phi}_+^\dagger$ is a solution of the "adjoint" equation; i.e., Eq. (4.14) with $\underline{C}$ replaced by $\underline{\tilde{C}}$ (the superscript tilde means transpose) corresponding to eigenvalue $\eta_0 = \pm \infty$. In this case, the scalar product must be defined to include a "weight function," $\underline{W}(\mu)$.

4.5 Orthogonality and Normalization

Theorem II: The functions, $\underline{\phi}_1(\eta,\mu)$, $\underline{\phi}_2^{(i)}(\eta,\mu)$, and $\underline{\phi}_+$ are orthogonal to the corresponding solutions of the adjoint equation on the range [0,1] with weight function, $\underline{W}(\mu)$. That is,

$$\int_0^1 \underline{\tilde{\phi}}^\dagger(\eta',\mu)\underline{W}(\mu)\underline{\phi}(\eta,\mu)d\mu = 0, \quad \eta \neq \eta'. \tag{4.46}$$

We shall prove this theorem using the original continuum eigensolutions, $\underline{F}(\eta,\mu)$, rather than the $\underline{\phi}(\eta,\mu)$. This is clearly immaterial to the proof; the choice is made for convenience.

We rewrite Eq. (4.14) and the adjoint equation below in symbolic form.

$$[(\underline{\Sigma}/\mu) - (\underline{C}/\mu)\int_{-1}^{1}\cdot d\mu']\underline{F}(\eta,\mu) = (1/\eta)\underline{F}(\eta,\mu) \qquad (4.47a)$$

$$(\underline{\Sigma}/\mu) - (\underline{\tilde{C}}/\mu)\int_{-1}^{1}\cdot d\mu'\ \underline{F}^{\dagger}(\eta',\mu) = (1/\eta')\underline{F}^{\dagger}(\eta',\mu). \qquad (4.47b)$$

It is easily verified that Eqs. (4.47) have identical eigenvalue spectra. Furthermore, we see that

$$\underline{F}^{\dagger}(\eta,\mu,\underline{C}) = \underline{F}(\eta,\mu,\underline{\tilde{C}}); \qquad (4.48)$$

i.e., the adjoint functions are obtained from $\underline{F}(\eta,\mu)$ by interchanging in the latter c_{ij} and c_{ji}.[9] The discrete adjoint function is simply

$$\underline{\phi}_{+}^{\dagger} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \qquad (4.49)$$

To prove the theorem, multiply Eq. (4.47a) from the left by $\underline{F}^{\dagger}(\eta',\mu)\underline{W}(\mu)$. Then multiply the transpose of Eq. (4.47b) from the right with $\underline{W}(\mu)\underline{F}(\eta,\mu)$. Integrate both equations over $d\mu$ from zero to 1 and subtract. The terms involving $\underline{\Sigma}$ are identical since $\underline{\Sigma}$ and $\underline{W}(\mu)$ commute, and so we obtain

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[9] In making this replacement, $c_{ij} \Longleftrightarrow c_{ji}$, one must be careful not to express the components of the $\underline{\Sigma}$ matrix in terms of the c_{ij} until after the replacement has been made.

$$\int_0^1 \frac{d\mu}{\mu}\ \tilde{\underline{F}}^{\dagger}(\eta',\mu)\underline{W}(\mu)\underline{C}\int_{-1}^{1}\underline{F}(\eta,\mu')d\mu'$$

$$-\int_{-1}^{1}\tilde{\underline{F}}^{\dagger}(\eta',\mu')d\mu'\underline{C}\int_0^1 \frac{d\mu}{\mu}\ \underline{W}(\mu)\underline{F}(\eta,\mu) \qquad (4.50)$$

$$= [(1/\eta') - (1/\eta)]\int_0^1 \tilde{\underline{F}}^{\dagger}(\eta',\mu)\underline{W}(\mu)\underline{F}(\eta,\mu)d\mu.$$

Thus, to prove theorem II, it is necessary to show that the left-hand side of Eq. (4.50) vanishes if $\eta \neq \eta'$. Denote

$$\int_{-1}^{1}\underline{F}(\eta,\mu')d\mu' \triangleq \underline{A}(\eta) \qquad (4.51a)$$

and

$$\int_{-1}^{1}\underline{F}^{\dagger}(\eta',\mu')d\mu' \triangleq \underline{A}^{\dagger}(\eta'). \qquad (4.51b)$$

(With the proviso of footnote 9 on the preceding page, the elements $a_i^{\dagger}(\eta)$ of $\underline{A}^{\dagger}(\eta)$ are obtained from the elements $a_i(\eta)$ of $\underline{A}(\eta)$, under the transformation, $c_{ij} \longrightarrow c_{ji}$). Furthermore, let

$$\int_0^1 \frac{d\mu}{\mu}\ \underline{W}(\mu)\underline{F}(\eta,\mu) \triangleq \underline{B}(\eta). \qquad (4.52a)$$

$$\int_0^1 \frac{d\mu}{\mu}\ \underline{W}(\mu)\underline{F}^{\dagger}(\eta',\mu) = \underline{B}^{\dagger}(\eta'). \qquad (4.52b)$$

In this notation, the left-hand side of Eq. (4.50) becomes

$$\text{L.H.S.} = \tilde{\underline{B}}^{\dagger}(\eta')\underline{C}\ \underline{A}(\eta) - \tilde{\underline{A}}^{\dagger}(\eta')\underline{C}\ \underline{B}(\eta), \qquad (4.53)$$

or in summation form we have

$$\text{L.H.S.} = \sum_{i,j} b_i^\dagger(\eta')c_{ij}a_j(\eta) - \sum_{i,j} a_i^\dagger(\eta')c_{ij}b_j(\eta), \qquad (4.54)$$

where $b_i(\eta)$ and $b_i^\dagger(\eta')$ are the elements of $\underline{B}(\eta)$ and $\underline{B}^\dagger(\eta')$, respectively. Now, from the definition of c_{ij}, Eq. (4.12b), we note

$$c_{ij} = c_{2j}c_{i2}/c_{22}. \qquad (4.55)$$

When this expression is substituted into Eq. (4.54), the sums separate. Thus,

$$\begin{aligned} c_{22}(\text{L.H.S.}) = &\sum_i b_i^\dagger(\eta')c_{i2}\sum_j c_{2j}a_j(\eta) \\ &- \sum_i a_i^\dagger(\eta')c_{i2}\sum_j c_{2j}b_j(\eta). \end{aligned} \qquad (4.56)$$

Now we divide Eq. (4.56) by $\sum_j c_{2j}a_j(\eta)\sum_i a_i^\dagger(\eta')c_{i2}$ and call c_{22}(L.H.S.) divided by this product J. Thus we find that

$$J = \frac{\sum_i b_i^\dagger(\eta')c_{i2}}{\sum_i a_i^\dagger(\eta')c_{i2}} - \frac{\sum_j c_{2j}b_j(\eta)}{\sum_j c_{2j}a_j(\eta)}. \qquad (4.57)$$

The two terms on the right-hand side of Eq. (4.57) are related by $\eta' \Longleftrightarrow \eta$ and $c_{ij} \Longleftrightarrow c_{ji}$. A sufficient condition for the theorem to be true (J=0) is that one of these terms, the second for example, be a constant independent of η and of c_{ij}. We shall show that this is indeed true. We denote the second term of Eq. (4.57) by J_2. Substituting for $b_j(\eta)$, Eq. (4.52), we obtain

$$J_2 = \sum_j c_{2j} \int_0^1 \frac{d\mu}{\mu} \sigma_j \gamma(\mu/\sigma_j) f_j(\eta,\mu) / \sum_j c_{2j} a_j(\eta). \quad (4.58)$$

Here $f_j(\eta,\mu)$ are the components of any of the $\underline{F}(\eta,\mu)$ and can be written in the form [cf. Eq. (4.14)],

$$f_j(\eta,\mu) = \{[\eta P/(\sigma_j\eta-\mu)] + \delta(\sigma_j\eta-\mu)\lambda_j(\eta)\} \sum_{i=1}^{2} c_{ji} \int_{-1}^{1} f_i(\eta,\mu')d\mu', \quad (4.59a)$$

or

$$f_j(\eta,\mu) = \{[\eta P/(\sigma_j\eta-\mu)] + \delta(\sigma_j\eta-\mu)\lambda_j(\eta)\} \sum_{i=1}^{2} c_{ji} a_i(\eta). \quad (4.59b)$$

(This equation holds for all continuum modes as well as the discrete mode, although in various regions the delta function may vanish and the symbol, P, may be superflous. This makes no difference to the general form of $f_j(\eta,\mu)$. The function, $\lambda_j(\eta)$, of course must be chosen so that Eq. (4.59a) is consistent.) If Eq. (4.59b) is substituted into Eq. (4.58), and if we replace the product $c_{2j}c_{ji}$ by the equivalent, $c_{jj}c_{2i}$, The factor, $\sum_j c_{2j} a_j(\eta)$, cancels out of the numerator and denominator and we find

$$J_2 = \sum_j c_{jj} \int_0^1 \frac{d\mu}{\mu} \sigma_j \gamma(\mu/\sigma_j) \{[\eta P/(\sigma_j\eta-\mu)] + \lambda_j(\eta)\delta(\sigma_j\eta-\mu)\}. \quad (4.60)$$

The integration over the delta function can be carried out.

$$J_2 = \sum_j c_{jj}\lambda_j(\eta)[\gamma(\eta)/\eta] + \sum_j c_{jj}\int_0^{1/\sigma_j} d\mu[(1/\mu) - [P/(\mu-\eta)]]\gamma(\mu). \tag{4.61}$$

The integrals in Eq. (4.61) can be performed with the help of Identity I given in Appendix A. We find

$$J_2 = [\gamma(\eta)/\eta]\sum_j c_{jj}\lambda_j(\eta) + 1 - \omega(\eta)[\gamma(\eta)/\eta] \tag{4.62}$$

[$\omega(\eta)$ was defined in Eq. (4.32); we have also noted that $X(0) = 1$]. We should clarify the meaning of Eq. (4.62). First, we note that $\omega(\eta)$ is a different function in the two continuum regions and in the "discrete region." Similarly, $\lambda_j(\eta)$ takes a different value for each of the three continuum $\underline{F}(\eta,\mu)$ and for the discrete mode. That J_2 = constant must be verified for each of the four possibilities in order that the theorem be proved. For example, by comparing Eqs. (4.59) and (4.18a) and recalling that (4.18a) was obtained with the normalization,

$$\underline{A}(\eta) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{4.63}$$

we see that for the mode $\underline{F}_1^{(1)}(\eta,\mu)$,

$$\lambda_1(\eta) = (1/c_{11})[1 - 2\eta c_{11}\tau(\sigma_1\eta)] \tag{4.64}$$

and

$$\lambda_2(\eta) = (1/c_{21})[-2\eta c_{21}\tau(\eta)]. \tag{4.65}$$

Also for $\eta \in$ region 1 [cf. Eq. (4.32)],

$$\omega_1(\eta) = 1 - 2\eta c_{11}\tau(\sigma_1\eta) - 2\eta c_{22}\tau(\eta). \tag{4.66}$$

Inserting these values into Eq. (4.62), we find

$$J_2 = 1. \tag{4.67}$$

In the same way, we verify that $J_2=1$ when the eigensolutions, $\underline{F}_2^{(1)}(\eta,\mu)$ and $\underline{F}^{(2)}(\eta,\mu)$, are used [in the latter case, of course,

$$\omega_2(\eta) = 1 - 2\eta c_{11}\tau(1/\sigma_1\eta) - 2\eta c_{22}\tau(\eta) \tag{4.68}$$

must be used in Eq. (4.62)]. For the discrete mode, $\lambda=0$. But $\Omega^+(\eta_0) = \Omega^-(\eta_0) = \Omega(\eta_0) = 0$, by definition, and we see immediately that $J_2 = 1$. Thus we obtain

$$J_2 \equiv 1 \tag{4.69}$$

and the theorem is proved; i.e.,

$$[(1/\eta) - (1/\eta')]\int_0^1 \underline{\tilde{\phi}}^\dagger(\eta',\mu)\underline{W}(\mu)\underline{\phi}(\eta,\mu)d\mu = 0. \tag{4.70}$$

Our method of proving theorem II has several advantages. We mention two:

1. If the form of $\underline{W}(\mu)$ had not already been suggested, a similar procedure to the above could be used to find $\underline{W}(\mu)$.

2. The method can be generalized to the N-group problem, in which the above procedure serves as a guide.

For the normalization integrals, we return to the $\underline{\phi}(\eta,\mu)$ rather than the $\underline{F}(\eta,\mu)$, since we will expand in terms of the $\underline{\phi}(\eta,\mu)$.

The degenerate eigensolutions can be orthogonalized using a Schmidt type procedure. First we present the normalization integrals. Defining

$$(\underline{E},\underline{G}) \triangleq \int_0^1 \tilde{\underline{E}}^{\dagger}(\eta',\mu)\underline{\underline{W}}(\mu)\underline{G}(\eta,\mu)d\mu, \tag{4.71}$$

we have

$$(\underline{\phi}_+,\phi_+) = N_+ \tag{4.72a}$$

$$(\underline{\phi}_1,\underline{\phi}_1) = N_1(\eta)\delta(\eta-\eta') \tag{4.72b}$$

$$(\underline{\phi}_2^{(1)},\underline{\phi}_2^{(1)}) = N_2^{(1)}(\eta)\delta(\eta-\eta') \tag{4.72c}$$

$$(\underline{\phi}_2^{(2)},\underline{\phi}_2^{(2)}) = N_2^{(2)}(\eta)\delta(\eta-\eta') \tag{4.72d}$$

$$(\underline{\phi}_1,\underline{\phi}_2^{(1)}) = M_{12}(\eta)\delta(\eta-\eta') \tag{4.72e}$$

$$(\underline{\phi}_2^{(1)},\underline{\phi}_1) = M_{21}(\eta)\delta(\eta-\eta'). \tag{4.72f}$$

Here

$$N_+ = 2(\sigma_1 w_1+w_2) \tag{4.73a}$$

$$N_1(\eta) = \gamma(\eta)[(c_{11}+c_{22})/c_{11}^2c_{22}] \tag{4.73b}$$

$$N_2^{(1)}(\eta) = \gamma(\eta)\{\pi^2\eta^2(c_{11}+c_{22})c_{22} + [1 - 2\eta c_{22}\tau(\eta)]^2 + 4\eta^2 c_{11}c_{22}\tau^2(\sigma_1\eta)\} \tag{4.73c}$$

$$N_2^{(2)}(\eta) = \gamma(\eta)\Omega_2^+(\eta)\Omega_2^-(\eta) \tag{4.73d}$$

$$M_{ij}(\eta) = -(c_{ij}/c_{11}c_{22})\gamma(\eta)[1 - 2\eta c_{22}\tau(\eta) + 2\eta c_{22}\tau(\sigma_1\eta)] \tag{4.73e}$$

Explicitly, we write [cf. Eqs. (4.33)]

$$\Omega_1^+(\eta)\Omega_1^-(\eta) = [1 - 2\eta c_{11}\tau(\sigma_1\eta) - 2\eta c_{22}\tau(\eta)]^2 + \pi^2\eta^2(c_{11}+c_{22})^2 \tag{4.74a}$$

$$\Omega_2^+(\eta)\Omega_2^-(\eta) = [1 - 2\eta c_{11}\tau(1/\sigma_1\eta) - 2\eta c_{22}\tau(\eta)]^2 + \pi^2\eta^2 c_{22}^2. \tag{4.74b}$$

Since $\underline{\phi}_1$ and $\underline{\phi}_2^{(1)}$ are not orthogonal for $\eta=\eta'$, we introduce two new functions, $\underline{\chi}_1^{(1)}$ and $\underline{\chi}_2^{(1)}$ defined such that $\underline{\chi}_1^{(1)}$ is orthogonal to $\underline{\phi}_2^{(1)}$ and $\underline{\chi}_2^{(1)}$ is orthogonal to $\underline{\phi}_1$ [clearly, both $\underline{\chi}_j^{(1)}$ are orthogonal to $\underline{\phi}_2^{(2)}$]. Therefore,

$$\underline{\chi}_1^{(1)}(\eta,\mu) \triangleq [c_{11}^2/\gamma(\eta)][N_2^{(1)}(\eta)\underline{\phi}_1(\eta,\mu) - M_{12}(\eta)\underline{\phi}_2^{(1)}(\eta,\mu)] \tag{4.75a}$$

and

$$\underline{\chi}_2^{(1)}(\eta,\mu) \triangleq [c_{11}^2/\gamma(\eta)][N_1(\eta)\underline{\phi}_2^{(1)}(\eta,\mu) - M_{21}(\eta)\underline{\phi}_1(\eta,\mu)]. \tag{4.75b}$$

We find

$$(\underline{\chi}_1^{(1)},\underline{\phi}_1) = \gamma(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta)\delta(\eta-\eta') \tag{4.76a}$$

$$(\underline{\chi}_1^{(1)},\underline{\phi}_2^{(1)}) = 0 \tag{4.76b}$$

$$(\underline{\chi}_2^{(1)},\underline{\phi}_2^{(1)}) = \gamma(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta)\delta(\eta-\eta') \tag{4.76c}$$

and

$$(\underline{\chi}_2^{(1)},\underline{\phi}_1) = 0. \tag{4.76d}$$

With the formalism developed in this section, typical half-space problems can be solved immediately. We consider, in the next section, the Milne problem.

4.6 The Milne Problem

We seek the angular energy density, $\underline{\Psi}_M(x,\mu)$ in the source-free half-space subject to the boundary conditions (21),

1. $\underline{\Psi}_M(0,\mu) = \underline{0}$, $\mu \geq 0$ (zero re-entrant radiation)

2. $\underline{\Psi}_M(x,\mu) \sim \underline{\Psi}_-(x,\mu)$ for large x.

The second condition specifies that $\underline{\Psi}_M(x,\mu)$ diverges no more rapidly than the slowest diverging mode, $\underline{\Psi}_-(x,\mu)$.

The solution can be constructed from the normal modes of the transport equation. Condition (2) requires that no $\underline{\Psi}(\eta,x,\mu)$ be included for $\eta \in [-1,0]$. Thus we write

$$\underline{\Psi}_M(x,\mu) = A_+\underline{\Psi}_+(x,\mu) + A_-\underline{\Psi}_-(x,\mu) + \int_{(1')} \alpha_1(\eta)e^{-x/\eta}\underline{\phi}_1(\eta,\mu)d\eta + \int_{(1')} \alpha_2(\eta)e^{-x/\eta}\underline{\phi}_2^{(1)}(\eta,\mu)d\eta + \int_{(1')} \alpha_2(\eta)e^{-x/\eta}\underline{\phi}_2^{(2)}(\eta,\mu)d\eta. \tag{4.77}$$

The coefficient, A_-, we leave arbitrary (it depends upon the normalization). The other coefficients are obtained from condition (1). Setting x = 0 in Eq. (4.77), we have

$$-A_-\underline{\Psi}_-(0,\mu) = A_+\underline{\phi}_+ + \int_{(1')} \alpha_1(\eta)\underline{\phi}_1(\eta,\mu)d\eta + \int_{(1')} \alpha_2(\eta)\underline{\phi}_2^{(1)}(\eta,\mu)d\eta + \int_{(2')} \alpha_2(\eta)\underline{\phi}_2^{(2)}(\eta,\mu)d\eta, \quad \mu > 0. \tag{4.78}$$

Thus the coefficients are merely the half-range expansion coefficients for the function,

$$-A_-\underline{\Psi}_-(0,\mu) = A_-\begin{bmatrix} w_1/\sigma_1 \\ w_2 \end{bmatrix}\mu. \tag{4.79}$$

They are found immediately from the orthogonality relations,

$$A_+/A_- = (1/N_+)\int_0^1 \underline{\tilde{\phi}}_1^+(\mu)\underline{w}(\mu)\begin{bmatrix} w_1/\sigma_1 \\ w_2 \end{bmatrix}\mu d\mu, \qquad (4.80a)$$

or

$$A_+/A_- = (1/N_+)[w_2\int_0^1 \gamma(\mu)\mu d\mu + \sigma_1^2 w_1\int_0^{1/\sigma_1}\gamma(\mu)\mu d\mu]. \qquad (4.80b)$$

This expression can be put in terms of the X-function by the use of Identity IV, Appendix A. We find

$$\frac{A_+}{A_-} = \frac{3\sigma_1}{2(\sigma_1 w_2 + w_1)}\left\{w_2\int_0^1 \frac{\mu^2 d\mu}{X(-\mu)} + \sigma_1^2 w_1\int_0^{1/\sigma_1}\frac{\mu^2 d\mu}{X(-\mu)}\right\}, \qquad (4.81)$$

where the explicit form of N_+, Eq. (4.73a), has been used. Similarly,

$$\alpha_1(\eta)/A_- = \int_0^1 \underline{\tilde{X}}_1^{(1)}(\eta,\mu)\underline{w}(\mu)\begin{bmatrix} w_1/\sigma_1 \\ w_2 \end{bmatrix}\mu d\mu/\gamma(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta). \qquad (4.82)$$

This integration is somewhat lengthy, but completely straightforward. We find

$$\frac{\alpha_1(\eta)}{A_-} = \frac{-c_{11}c_{12}\eta w_2}{c_{22}}\,\frac{1 - 2\eta c_{22}\tau(\eta) + 2\eta c_{22}\tau(\sigma_1\eta)}{\gamma(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta)}. \qquad (4.83)$$

Similarly,

$$\frac{\alpha_2(\eta)}{A_-} = \frac{-\eta[w_2 + \sigma_1^2 w_1\,\theta_1(\eta)]}{\gamma(\eta)\Omega_2^+(\eta)\Omega_2^-(\eta)}. \qquad (4.84)$$

The expansion coefficients are now determined, so the problem is solved. Note that all results are expressed in terms of the two functions, $\tau(z)$ and $X(-\mu)[\Omega^{\pm}(\eta)$ are given in terms

of τ-functions, while $\gamma(\eta)$ is expressed in terms of $X(-\mu)$, $0 \leq \mu \leq 1$, in Identity IV, Appendix A].

The customary normalization (7) is to set

$$-2\pi \int_{-1}^{1} d\mu\mu \int_{0}^{\infty} \psi_\nu(x,\mu)d\nu = \sigma T_e^4, \tag{4.85}$$

where T_e is the "effective temperature." This leads to

$$A_- = 3\sigma_1 \sigma T_e/4\pi(\sigma_1 w_2 + w_1). \tag{4.86}$$

The energy density,

$$F(x) \triangleq 2\pi \int_{-1}^{1} d\mu \int_{0}^{\infty} \psi_\nu(x,\mu)d\nu, \tag{4.87}$$

is given by

$$F(x) = 2\pi \begin{bmatrix} \widetilde{1} \\ 1 \end{bmatrix} \int_{-1}^{1} \underline{Y}_M(x,\mu)d\mu, \tag{4.88}$$

or finally,

$$\begin{aligned} F(x)/4\pi = {} & (A_+ + xA_-)(w_1 + w_2) \\ & + \frac{1}{2} \int_{(1')} \{[\alpha_1(\eta)/c_{11}] - [\alpha_2(\eta)/c_{12}]\}e^{-x/\eta}d\eta \\ & + \frac{1}{2} \int_{(2')} \alpha_2(\eta)[1 + 2(c_{12}-c_{11})\eta\tau(1/\sigma_1\eta)]e^{-x/\eta}d\eta. \end{aligned} \tag{4.89}$$

The extrapolated endpoint, x_0, is defined as the distance from the boundary at which the asymptotic density extrapolates to zero. It is

$$x_0 = A_+/A_-, \tag{4.90}$$

which is given already by Eq. (4.81).

The temperature distribution is found from Eq. (4.9); i.e.,

$$\frac{\sigma T^4(x)}{\pi} = \frac{1}{2(\sigma_1 w_1 + w_2)} \begin{bmatrix} \tilde{\sigma}_1 \\ 1 \end{bmatrix} \int_{-1}^{1} \underline{\Psi}_M(x,\mu)\,d\mu. \tag{4.91}$$

We find

$$\frac{T^4(x)}{T_e^4} = \frac{3\sigma_1}{4(\sigma_1 w_2 + w_1)} \Big[x + x_0 - (c_{11}+c_{22}) \int_0^{1/\sigma_1} \frac{\eta e^{-x/\eta}}{\gamma(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta)}\,d\eta - c_{22} \int_{1/\sigma_1}^{1} \frac{\eta e^{-x/\eta}}{\gamma(\eta)\Omega_2^+(\eta)\Omega_2^-(\eta)}\,d\eta \Big]. \tag{4.92}$$

Writing this in terms of $X(-\mu)$, we have

$$\frac{T^4(x)}{T_e^4} = \frac{3\sigma_1}{4(\sigma_1 w_2 + w_1)} \Big[x + x_0 - \frac{(c_{11}+c_{22})(\sigma_1 w_2 + w_1)}{3\sigma_1(\sigma_1 w_1 + w_2)} \int_0^{1/\sigma_1} \frac{X(-\eta)e^{-x/\eta}}{\Omega_1^+(\eta)\Omega_1^-(\eta)}\,d\eta - \frac{c_{22}(\sigma_1 w_2 + w_1)}{3\sigma_1(\sigma_1 w_1 + w_2)} \int_{1/\sigma_1}^{1} \frac{X(-\eta)e^{-x/\eta}}{\Omega_2^+(\eta)\Omega_2^-(\eta)}\,d\eta \Big]. \tag{4.93}$$

Now that the temperature distribution is known, one can obtain the frequency dependent angular density, $\psi_\nu(z,\mu)$, simply by integrating Eq. (4.1) (after transforming it to the optical variable, x). The frequency dependent law of darkening is then given by $\psi_\nu(0,\mu)$.

The law of darkening for the integrated quantities is found from $\underline{\Psi}_M(0,\mu)$. Just as in the gray problem, if we consider $\underline{\Psi}_M(0,-\mu)$, $\mu > 0$, the integrals over $d\eta$ in Eq. (4.77) can be performed (for x=0) by extending the technique used in

Ref. (26). In this way we are able to obtain an exact solution for the law of darkening involving the integrated quantities. To this end, we begin with

$$\underline{Y}_M(0,-\mu) = A_+\underline{\phi}_+ + A_-\underline{Y}_-(0,-\mu) + \int_0^1 \alpha_2(\eta)\begin{bmatrix} c_{12}\eta/(\sigma_1\eta+\mu) \\ c_{22}\eta/(\eta+\mu) \end{bmatrix} d\eta, \quad \mu > 0. \tag{4.94}$$

We note that here the continuum part of the expansion becomes particularly simple because by restricting μ to be negative the eigensolutions ($\eta > 0$) are no longer singular.

From the definition of N(z), Eq. (4.35), we note that $\underline{Y}_M(0,-\mu)$ can be written in the form,

$$\underline{Y}_M(0,-\mu) = A_+\underline{\phi}_+ + A_-\underline{Y}_-(0,-\mu) + 2\pi i \begin{bmatrix} (c_{12}/\sigma_1)N(-\mu/\sigma_1) \\ c_{22}N(-\mu) \end{bmatrix}, \quad \mu > 0. \tag{4.95}$$

Recalling from the completeness proof developed in Section 4.4 that, in general, N(z) is given by

$$N(z) = [1/2\pi i X(z)]\int_0^1 \gamma(\eta)\{\psi_2(\eta) - A_+w_2 + (\sigma_1 c_{11}/c_{12})[\psi_1(\sigma_1\eta) - A_+w_1]\,\theta_1(\eta)\}d\eta/(\eta-z), \tag{4.96}$$

we find that for the Milne problem, $\psi_1(\mu) = A_-w_1\mu/\sigma_1$, $\psi_2(\mu) = A_-w_2\mu$,

$$N(-\mu) = [1/2\pi i X(-\mu)]\{\int_0^1 \gamma(\eta)(A_-w_2\eta - A_+w_2)[d\eta/(\eta+\mu)] + (\sigma_1 c_{11}/c_{12})\int_0^{1/\sigma_1} \gamma(\eta)(A_-w_1\eta - A_+w_1)[d\eta/(\eta+\mu)]\}. \tag{4.97}$$

Employing the identities in Appendix A, the above reduces to

$$N(-\mu) = [(\sigma_1 w_1 + w_2)/2\pi i][-A_+ - \mu A_- + [A_-/X(-\mu)]]. \quad (4.98)$$

Substituting this expression into Eq. (4.95) and noting Eq. (4.86), we obtain the desired result; namely,

$$\underline{\Psi}_M(0,-\mu) = [3\sigma_1 \sigma T_e/4\pi(\sigma_1 w_2 + w_1)] \begin{bmatrix} w_1/X(-\mu/\sigma_1) \\ w_2/X(-\mu) \end{bmatrix}. \quad (4.99)$$

It is clear how other half-space problems could be solved. For example, consider the albedo problem. Here we have a source-free half-space with incident distribution.[10]

$$\underline{\Psi}_{INC}(\mu) = \begin{bmatrix} \delta(\mu-\mu_1) \\ \delta(\mu-\mu_2) \end{bmatrix}, \quad \mu_i \geq 0. \quad (4.100)$$

Here the solution must not diverge at infinity, so we set

$$\begin{aligned} \underline{\Psi}_a(x,\mu) = A_+\underline{\phi}_+ &+ \int_{(1')} \alpha_1(\eta) e^{-x/\eta} \underline{\phi}_1(\eta,\mu)\, d\eta \\ &+ \int_{(1')} \alpha_2(\eta) e^{-x/\eta} \underline{\phi}_2^{(1)}(\eta,\mu)\, d\eta \\ &+ \int_{(2')} \alpha_2(\eta) e^{-x/\eta} \underline{\phi}_2^{(2)}(\eta,\mu)\, d\eta. \end{aligned} \quad (4.101)$$

Since

$$\underline{\Psi}_a(0,\mu) = \underline{\Psi}_{INC}(\mu), \quad \mu \geq 0, \quad (4.102)$$

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[10] If the albedo solution is to be used to generate the solution to a problem involving arbitrary incident distribution, then two albedo problems must be defined, just as two Green's functions are defined (cf. Appendix B).

the expansion coefficients are found as integrals of the adjoint functions times delta functions. We omit the details of this or other possible half-space problems except for the half-space Green's function discussed briefly in Appendix B.

4.7 The Case of General $\underline{C}$

As has been discussed in the introduction, the case of general $\underline{C}$ has interest both in neutron transport theory and in radiative transfer. In the latter, if we had included scattering in the resonance lines [$c \neq 1$, Ref. (15)] we would have been led to this case. We discuss this situation briefly in the present section. All results are presented without proof.

The eigenfunctions for region 1 have the same form as for $\det \underline{C} = 0$. They are given by Eqs. (4.18). For region 2, however, we find

$$\underline{F}^{(2)}(\eta,\mu) = \begin{bmatrix} c_{12}\eta/(\sigma_1\eta-\mu) \\ [\eta Pf(\eta)/(\eta-\mu)] + \delta(\eta-\mu)\lambda(\eta) \end{bmatrix}, \quad (4.103)$$

where

$$\lambda(\eta) = 1 - 2\eta c_{22}\tau(\eta) - 2\eta c_{11}\tau(1/\sigma_1\eta) + 4\eta^2 \det \underline{C}\ \tau(\eta)\tau(1/\sigma_1\eta) \quad (4.104)$$

and

$$f(\eta) = c_{22} - 2\eta \det \underline{C}\ \tau(1/\sigma_1\eta). \quad (4.105)$$

The discrete solutions are

$$\underline{F}_{i\pm}(\mu) = \begin{bmatrix} c_{12}\eta_i/(\sigma_1\eta_i \mp \mu) \\ \eta_i[c_{22} - 2 \det \underline{C}\ \eta_i\tau(1/\sigma_1\eta_i]/(\eta_i \mp \mu) \end{bmatrix}, \quad (4.106)$$

where η_i are positive roots of the dispersion function,

$$\Omega(z) = 1 - 2zc_{11}\tau(1/\sigma_1 z) - 2zc_{22}\tau(1/z) + 4z^2 \det \underline{C}\, \tau(1/z)\tau(1/\sigma_1 z). \tag{4.107}$$

It can be shown (34) that for $\det \underline{C} \leq 0$, $\Omega(z)$ has only two zeros while for $\det \underline{C} > 0$, it may have two or four, depending upon the relative magnitudes of the c_{ij} and σ_1. Note that $\Omega(z) = \Omega(-z)$; hence the roots occur in $\pm$ pairs, explaining the notation of Eq. (4.106).

Theorem III. The functions, $\underline{F}_i^{(1)}(\eta,\mu)$, $\underline{F}^{(2)}(\eta,\mu)$, and $\underline{F}_{i\pm}(\mu)$ form a complete set in the sense that an arbitrary function, $\underline{Y}(\mu)$, defined on the interval $[-1,1]$, can be expanded in the form,

$$\underline{Y}(\mu) = \sum_i A_{i+}\underline{F}_{i+}(\mu) + \sum_i A_{i-}\underline{F}_{i-}(\mu) + \int_{(1)} \alpha_1(\eta)\underline{F}_1^{(1)}(\eta,\mu)d\eta + \int_{(1)} \alpha_2(\eta)\underline{F}_2^{(1)}(\eta,\mu)d\eta + \int_{(2)} \alpha_3(\eta)\underline{F}^{(2)}(\eta,\mu)d\eta. \tag{4.108}$$

The proof of this theorem follows in analogous fashion to that of theorem I except that here it is unnecessary to introduce the X-function since $N(z)$ and $\Omega(z)$ have the same branch cut.

Theorem IV. The functions, $\underline{F}(\eta,u)$, are orthogonal to the adjoint functions, $\underline{F}^{\dagger}(\eta,\mu)$, on the interval $[-1,1]$ with weight

function μ. This applies to both continuum and discrete modes.[11]

This theorem is proved, in the usual manner, directly from the equations obeyed by $\underline{F}(\eta,\mu)$ and $\underline{F}^{\dagger}(\eta,\mu)$.[12]

In attempting to prove the half-range theorem, analogous to theorem I but for $\det \underline{C} \neq 0$, one proceeds in exactly the same manner as for theorem I. However, instead of arriving at Eq. (4.37), which is solved by Wiener-Hopf factorization, one finds an equation that cannot be factored; namely,

$$\mu h(\mu) = N^{+}(\mu)\Omega^{+}(\mu) - N^{-}(\mu)\Omega^{-}(\mu) + \det \underline{C}\ \mu a(\mu) P\int_0^1 [\eta/(\eta-\mu)]\mathcal{K}(\eta)d\eta + \det \underline{C}\ \mu P\int_0^1 [\eta/(\eta-\mu)]a(\eta)\mathcal{K}(\eta)d\eta. \qquad (4.109)$$

Here $h(\mu)$ is a known function of the components of the expansion function, $\underline{\Psi}(\mu)$, $N(z)$, and $\Omega(z)$ have the usual definitions; $\mathcal{K}(\eta)$ is the unkown expansion coefficient; and

$$a(\mu) = \int_{-1/\sigma}^{0} [d\eta/(\mu-\eta)]. \qquad (4.110)$$

Actually, this problem has been treated in Ref. (27), but not in a form that makes it possible to obtain the expansion coefficients in any simple way. While we have remedied this for the full-range case through theorems III and IV, we have thus far been unable to discover a similar scheme for the half-range problem.

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[11] It is possible even for the case $\det \underline{C} \neq 0$ that there is a degenerate double root at infinity (34). In this situation, $\underline{\Psi}(0,\mu)$ is not included in the orthogonal set.

[12] Both theorems III and IV apply also for the case $\det \underline{C} = 0$.

5. GENERALIZED PICKET FENCE MODEL

5.1 Introduction

In the preceding chapter, hereinafter referred to as IV, we presented exact solutions for the equations of radiative transfer with Local Thermodynamic Equilibrium for a particular model of the absorption coefficient, K_ν (Uniform Picket Fence model). In this model, K_ν was assumed to take only two different values, K_1 and K_2. In the present chapter, we generalize the results of IV to the case that K_ν can take on N values. The derivation of the basic (matrix) transport equation follows similar lines to that in IV; however, we briefly give the cogent points in the derivation below with a slight generalization of the case treated in IV. Namely, we include a scattering term in the transport equation which, however, we shall have to neglect before we can obtain an explicit solution.

We begin, then, with the equation (21),

$$\mu \frac{\partial \psi_\nu}{\partial z}(z,\mu) + \rho(z)(K_\nu+S_\nu)\psi_\nu(z,\mu) = \rho(z)K_\nu B_\nu[T(z)] + \rho(z)S_\nu \int_{\underline{\Omega}'} \psi_\nu(z,\mu')f(\underline{\Omega}'\cdot\underline{\Omega})d\underline{\Omega}'. \tag{5.1}$$

This equation is identical to Eq. (4.1), where the notation was defined, except that a monochromatic scattering term is included; S_ν is the scattering coefficient; and

$$\int_{\underline{\Omega}} f(\underline{\Omega}'\cdot\underline{\Omega})d\underline{\Omega} = 1. \tag{5.2}$$

We now assume that the frequency spectrum can be divided into

ranges $\Delta\nu_i$ in each of which $K\nu$ and $S\nu$ take on the same N different values $(K_1, K_2, \cdots, K_N;\ S_1, S_2, \cdots S_N)$ and that the fractional width, $w_n(i)$, of $\Delta\nu_i$ over which K_ν and S_ν have the same values is the same for all $\Delta\nu_i$.[1]

Following the procedure discussed in IV, we find the matrix transport equation [$dx = \rho(x)K_N dz$, $K_N > K_i$],

$$\mu \frac{\partial}{\partial x} \underline{\Psi}(x,\mu) + \underline{\Sigma}'\underline{\Psi}(x,\mu) = \underline{C} \int_{-1}^{1} \underline{\Psi}(x,\mu')d\mu' + \underline{C}' \int_{\underline{\Omega}'} \underline{\Psi}(x,\mu') f(\underline{\Omega}'\cdot\underline{\Omega})d\underline{\Omega}', \tag{5.3}$$

where

$$(\underline{\Sigma}')_{ij} = [\sigma_i + (S_i/K_N)]\delta_{ij}, \tag{5.4}$$

$$\sigma_i \triangleq K_i/K_N,\ \sigma_N = 1, \tag{5.5}$$

$$(\underline{C})_{ij} \quad \sigma_i\sigma_j w_i / 2\sum_{\alpha=1}^{N} \sigma_\alpha w_\alpha, \tag{5.6}$$

and

$$(\underline{C}')_{ij} = (S_i/K_N)\delta_{ij}. \tag{5.7}$$

Even for isotropic scattering [$f(\Omega'\cdot\Omega) = 1/4\pi$], we do not know how to obtain explicit solutions to this set of equations [$\det(\underline{C}+\underline{C}') \neq 0$]. However, if the scattering term may be neglected, we obtain a transport equation of the form.

$$\mu \frac{\partial}{\partial x} \underline{\Psi}(x,\mu) + \underline{\Sigma}\,\underline{\Psi}(x,\mu) = \underline{C} \int_{-1}^{1} \underline{\Psi}(x,\mu')d\mu', \tag{5.8}$$

where

$$(\underline{\Sigma})_{ij} = \sigma_i\delta_{ij}. \tag{5.9}$$

We note

$$\det \underline{C} = 0 \tag{5.10}$$

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[1]Note that we have assumed that the steps for K_ν and S_ν always occur at the same value of ν.

and, in fact,

$$\det \underline{\mathbf{M}} = 0,$$

where $\underline{\mathbf{M}}$ is any minor of any rank of $\underline{\mathbf{C}}$. Eq. (5.8) can be solved explicitly, by an extension of the technique used in IV. The procedure is described in the subsequent sections.

5.2 The Eigenvalues and Eigensolutions

As in IV, translational invariance suggests we seek solutions of Eq. (5.8) of the form,

$$\underline{\Psi}(\eta,x,\mu) = e^{-x/\eta}\underline{\mathbf{F}}(\eta,\mu). \tag{5.11}$$

Substituting this ansatz into Eq. (5.8), we obtain an equation for $\underline{\mathbf{F}}(\eta,\mu)$,

$$[\underline{\Sigma} - (\mu/\eta)\underline{\mathbf{E}}]\underline{\mathbf{F}}(\eta,\mu) = \underline{\mathbf{C}}\int_{-1}^{1}\underline{\mathbf{F}}(\eta,\mu')d\mu', \tag{5.12}$$

where $\underline{\mathbf{E}}$ is the unit matrix. First we consider the continuum solutions; i.e., $\eta \in [-1,1]$. In IV, it was necessary to divide this range into two subranges. Here, there are N such subranges:

$$\text{Region 1,}\quad \eta \in [-(1/\sigma_1),(1/\sigma_1)] \tag{5.13}$$

$$\text{Region 2,}\quad \eta \in [-(1/\sigma_2),-(1/\sigma_1)] \text{ and } [(1/\sigma_1),(1/\sigma_2)], \tag{5.14}$$

and thus, in general,

$$\text{Region i,}\quad \eta \in [-(1/\sigma_i),-(1/\sigma_{i-1})] \text{ and } [(1/\sigma_{i-1}),(1/\sigma_i)]. \tag{5.15}$$

The eigensolutions for the ith region take the form,

$$\underline{F}^{(i)}(\eta,\mu) = (\underline{P}_i+\underline{\Lambda}_i)\underline{C}\int_{-1}^{1}\underline{F}^{(i)}(\eta,\mu')d\mu', \qquad (5.16)$$

where

$$(\underline{P}_i)_{jk} = \eta[P/(\sigma_j\eta-\mu)]\delta_{jk} \qquad (5.17)$$

and

$$(\underline{\Lambda}_i)_{jk} = \lambda_j^{(i)}(\eta)\delta(\sigma_j\eta-\mu)\delta_{jk}. \qquad (5.18)$$

Here the $\lambda_j^{(i)}(\eta)$ are unspecified functions that must be selected such that Eq. (5.16) be consistent. We note that for $j < i$ the symbol P, denoting the Cauchy principal value, in Eq. (5.17) is superflous because the denominator can never vanish. Similarly, $\lambda_j^{(i)}(\eta)$ may be taken to be zero for $j < i$, since the argument of the delta function never vanishes. Thus we see that $\underline{F}^{(i)}(\eta,\mu)$ contains $(N+1-i)$ unknown functions $\lambda_j^{(i)}(\eta)$. In addition there are N unknown functions $a_j^{(i)}(\eta)$ in Eq. (5.16), defined by

$$\int_{-1}^{1}\underline{F}^{(i)}(\eta,\mu')d\mu' = \underline{A}^{(i)}(\eta) = \begin{bmatrix} a_1^{(i)}(\eta) \\ \vdots \\ a_N^{(i)}(\eta) \end{bmatrix}. \qquad (5.19)$$

Thus the solutions $\underline{F}^{(i)}(\eta,\mu)$ are $(N+1-i)$-fold degenerate. There are $(N+1-i)$ linearly independent eigensolutions in region i. We denote these by

$$\underline{F}_{\alpha}^{(i)}(\eta,\mu),\ \alpha = i,i+1,\cdots,N. \tag{5.20}$$

(For convenience in notation, we take α to run from i to N rather than from 1 to N+1-i.)

It is a straightforward matter to obtain the explicit form of the $\underline{F}_{\alpha}^{(i)}(\eta,\mu)$. We find [$\tau(x) \triangleq \tanh^{-1} x$]

$$\underline{F}_{\alpha}^{(i)}(\eta,\mu) = \begin{bmatrix} \frac{1}{\sigma_1\eta-\mu}\,\eta c_{1\alpha} \\ \vdots \\ \frac{1}{\sigma_{i-1}\eta-\mu}\,\eta c_{i-1,\alpha} \\ \frac{P}{\sigma_i\eta-\mu}\,\eta c_{i\alpha} - 2\eta c_{i\alpha}\tau(\sigma_i\eta)\delta(\sigma_i\eta-\mu) \\ \vdots \\ \frac{P}{\sigma_\alpha\eta-\mu}\,\eta c_{\alpha\alpha} + [1 - 2\eta c_{\alpha\alpha}\tau(\sigma_\alpha\eta) \\ \qquad - 2\eta\sum_{\beta=1}^{i-1} c_{\beta\beta}\tau(1/\sigma_\beta\eta)]\delta(\sigma_\alpha\eta-\mu) \\ \frac{P}{\sigma_{\alpha+1}\eta-\mu}\,\eta c_{\alpha+1,\alpha} - 2\eta c_{\alpha+1,\alpha}\tau(\sigma_{\alpha+1}\eta) \\ \vdots \qquad\qquad \times\,\delta(\sigma_{\alpha+1}\eta-\mu) \end{bmatrix}. \tag{5.21}$$

Although the derivation of Eq. (5.21) is tedious, it is easy to verify that it is a solution. In doing so, the relation,

$$c_{j\alpha}c_{\alpha k} = c_{jk}c_{\alpha\alpha}, \tag{5.22}$$

must be kept in mind.

Next we consider the discrete spectrum; i.e., $\eta \notin [-1,1]$. We obtain from Eq. (5.12)

$$\underline{F}(\eta,\mu) = \underline{D}\,\underline{C}\int_{-1}^{1}\underline{F}(\eta,\mu')d\mu', \tag{5.23}$$

where

$$(\underline{D})_{ij} = [1/(\sigma_i\eta-\mu)]\eta\delta_{ij}. \tag{5.24}$$

The eigenvalues are obtained by integrating Eq. (5.23) and noting that nontrivial solutions exist only if

$$\Omega(z) \triangleq \det\,(\underline{E}-\underline{T}\,\underline{C}) = 0, \tag{5.25}$$

where

$$(\underline{T})_{ij} = 2z\tau(1/\sigma_i z)\delta_{ij}. \tag{5.26}$$

To evaluate Eq. (5.25), we write (35)

$$\det(\lambda\underline{E}-\underline{T}\,\underline{C}) = \lambda^N - \mathrm{Tr}(\underline{T}\,\underline{C})\lambda^{N-1} + a_2\lambda^{N-2} - a_3\lambda^{N-3} + \cdots + a_N. \tag{5.27}$$

Here the coefficients of λ^{N-K}, $K = 2,3,\cdots,N$, are defined as the sum of all of the K by K minor determinants that can be formed using K of the diagonal elements of $\underline{T}\,\underline{C}$ (there are $N!/K!(N-K)!$ such determinants). One easily verifies that all such minor determinants of $\underline{T}\,\underline{C}$ vanish. Thus, setting $\lambda=1$ in Eq. (5.27) we obtain

$$\Omega(z) = 1 - 2z\sum_{\beta=1}^{N}c_{\beta\beta}\tau(1/\sigma_\beta z). \tag{5.28}$$

In Appendix C we show that $\Omega(z)$ has only two zeros, which are $\eta_0 = \pm\infty$. Thus the discrete eigenvalues are identical with those obtained in IV. The discrete eigensolutions are similar;

we find

$$\underline{\Psi}_+(x,\mu) = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_N \end{bmatrix} \triangleq \underline{\phi}_+ \tag{5.29}$$

and

$$\underline{\Psi}_-(x,\mu) = \begin{bmatrix} w_1[x - (\mu/\sigma_1)] \\ w_2[x - (\mu/\sigma_2)] \\ . \\ . \\ . \\ w_N(x-\mu) \end{bmatrix}. \tag{5.30}$$

As in IV, we shall choose to work mostly with certain linear combinations of the $\underline{F}_\alpha^{(i)}(\eta,\mu)$, which we shall call $\underline{\phi}_\alpha^{(i)}(\eta,\mu)$. These are defined by

$$[1 - 2\eta\sum_{\beta=1}^{i-1} c_{\beta\beta}\tau(1/\sigma_\beta\eta)]\underline{\phi}_\alpha^{(i)}(\eta,\mu) = (1/c_{1\alpha})\underline{F}_\alpha^{(i)}(\eta,\mu) - (1/c_{1,\alpha+1})F_{\alpha+1}^{(i)}(\eta,\mu), \tag{5.31a}$$

where $\alpha=i,i+1,\cdots,N-1$. Thus

$$\underline{\phi}_\alpha^{(i)}(\eta,\mu) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ (1/c_{1\alpha})\delta(\sigma_\alpha\eta-\mu) \\ -(1/c_{1,\alpha+1})\delta(\sigma_{\alpha+1}\eta-\mu) \\ 0 \\ 0 \\ . \\ . \\ . \end{bmatrix}, \quad \alpha=i,\cdots,N-1. \tag{5.31b}$$

Also we take

$$\underline{\phi}_N^{(i)}(\eta,\mu) = \underline{F}_N^{(i)}(\eta,\mu). \tag{5.32}$$

Thus we see that in each region there are (N-i) eigensolutions of the form (5.31b), containing only two nonzero elements, both involving only delta functions, and one eigensolution of the form Eq. (5.32).

Aside from the simple form taken by the $\underline{\phi}_\alpha^{(i)}(\eta,\mu)$, we see that these solutions take the same form in different regions; thus we can recombine the $\underline{\phi}_\alpha^{(i)}(\eta,\mu)$ in the following way. First, we note for K < N,

$$\underline{\phi}_K^{(i)}(\eta,\mu) = \underline{\phi}_K^{(j)}(\eta,\mu), \quad j=i,i+1,\cdots,K. \tag{5.33}$$

This suggests defining[2]

$$\underline{\phi}_K(\eta,\mu) = \underline{\phi}_K^{(j)}(\eta,\mu), \quad j \leq K < N \tag{5.34a}$$

$$= 0, \quad K < j. \tag{5.34b}$$

There are (N-1) eigensolutions of this form; since there are (N) of the $\underline{\phi}_N^{(i)}(\eta,\mu)$, we are now dealing with only (2N-1) different eigensolutions instead of $[\frac{1}{2}N(N+1)]$. If we, in fact, also define

$$\underline{\phi}_N(\eta,\mu) = \sum_{i=1}^{N} \underline{\phi}_N^{(i)}(\eta,\mu)\theta_i(\eta), \tag{5.35}$$

where

$$\theta_i(\eta) = 1, \ \eta \in \text{region } i$$
$$= 0, \text{ otherwise}, \tag{5.36}$$

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[2]Dr. J. Mika, Institute of Nuclear Research, Polish Academy of Science, Warsaw, in a private conversation, suggested this method of attack.

we need only consider N eigensolutions, $\underline{\phi}_1(\eta,\mu),\cdots,\underline{\phi}_N(\eta,\mu)$. To recapitulate, we have (N-1) eigensolutions of the form

$$\underline{\phi}_1(\eta,\mu) = \begin{bmatrix} (1/c_{11})\delta(\sigma_1\eta-\mu) \\ -(1/c_{12})\delta(\sigma_2\eta-\mu) \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \theta_1(\eta), \qquad (5.37a)$$

$$\underline{\phi}_2(\eta,\mu) = \begin{bmatrix} 0 \\ (1/c_{12})\delta(\sigma_2\eta-\mu) \\ -(1/c_{13})\delta(\sigma_3\eta-\mu) \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} [\theta_1(\eta)+\theta_2(\eta)], \qquad (5.37b)$$

or, in general,

$$\underline{\phi}_i(\eta,\mu) = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ (1/c_{1i})\delta(\sigma_i\eta-\mu) \\ -(1/c_{1,i+1})\delta(\sigma_{i+1}\eta-\mu) \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \sum_{\alpha=1}^{i} \theta_\alpha(\eta). \qquad (5.37c)$$

In addition, we have one eigensolution given by Eq. (5.35).

5.3 Completeness

Theorem V. The functions, $\underline{\phi}_i(\eta,\mu)$, $i=1,\cdots,N$, $\eta\geq 0$, and $\underline{\phi}_+$, are complete for arbitrary N-vector functions, $\underline{\psi}(\mu)$, defined on the "half-range," $0 \leq \mu \leq 1$.[3]

This theorem means that an N-component function, $\underline{\psi}(\mu)$, can be expanded in the form,

$$\underline{\psi}(\mu) = A_+\underline{\phi}_+ + \sum_{\beta=1}^{N} \int_0^{1/\sigma_\beta} \alpha_\beta(\eta)\underline{\phi}_\beta(\eta,\mu)d\eta, \quad \mu \in [0,1], \tag{5.38}$$

where A_+ and $\alpha_\beta(\eta)$ are "scalar" functions.

To prove this theorem, we proceed as in IV; i.e., we attempt an expansion in terms of the continuum modes alone.

$$\underline{\psi}(\mu) = \sum_{\beta=1}^{N-1} \int_0^{1/\sigma_\beta} \alpha_\beta(\eta)\underline{\phi}_\beta(\eta,\mu)d\eta + \int_0^1 \alpha_N(\eta)\underline{\phi}_N(\eta,\mu)d\eta. \tag{5.39}$$

Here the last term has been split off from the sum because the first (N-1) terms have the simple form given in Eq. (5.37c). These integrals can all be carried out explicitly.

$$\int_0^{1/\sigma_\beta} \alpha_\beta(\eta)\underline{\phi}_\beta(\eta,\mu)d\eta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{1}{\sigma_\beta c_{1\beta}}\,\alpha_\beta\left(\frac{\mu}{\sigma_\beta}\right) \\ -\frac{1}{\sigma_{\beta+1}c_{1,\beta+1}}\,\alpha_\beta\left(\frac{\mu}{\sigma_{\beta+1}}\right) S\left(\frac{\sigma_{\beta+1}}{\sigma_\beta}\,-\mu\right) \\ 0 \\ \vdots \end{bmatrix}, \tag{5.40}$$

[3] A discussion of the permissible degree of arbitrariness is given in IV; see also Ref. (21).

where S(x) is the unit step function,

$$S(x) = 1, \quad x \geq 0$$
$$= 0, \text{ otherwise.} \tag{5.41}$$

If we substitute the explicit form of $\phi_N(\eta,\mu)$ into Eq. (5.39), the integration over the delta functions can be carried out. When these results are substituted into Eq. (5.39), we obtain

$$\begin{aligned}\Psi_i(\mu) &= -(1/c_{1i}\sigma_i)\alpha_{i-1}(\mu/\sigma_i)S[(\sigma_i/\sigma_{i-1}) - \mu] \\ &+ (1/c_{1i}\sigma_i)\alpha_i(\mu/\sigma_i) - [2\mu c_{iN}\alpha_N(\mu/\sigma_i)\tau(\mu)/\sigma_i^2] \qquad (5.42a) \\ &+ c_{iN}P\int_0^1[\alpha_N(\eta)\eta d\eta/(\sigma_i\eta-\mu)], \quad i=1,2,\cdots,N-1,\end{aligned}$$

and

$$\begin{aligned}\Psi_N(\mu) &= -(1/c_{1N})\alpha_{N-1}(\mu)S[(1/\sigma_{N-1}) - \mu] + \alpha_N(\mu)\{1 - 2\mu c_{NN}\tau(\mu) \\ &- 2\mu\sum_{\beta=1}^{N-1}c_{\beta\beta}\tau(1/\sigma_\beta\mu)S[\mu - (1/\sigma_\beta)]\} \qquad (5.42b) \\ &+ c_{NN}P\int_0^1[\alpha_N(\eta)\eta d\eta/(\eta-\mu)].\end{aligned}$$

Here, $\Psi_i(\mu)$ is the ith component of $\underline{\Psi}(\mu)$. [We recall in obtaining Eq. (5.42b) that $\sigma_N \equiv 1$.]

We note from Eqs. (5.42) that the unknowns, $\alpha_i(\eta)$, $i=1,\cdots,N-1$, can be eliminated successively, starting with $i=1$ [$\alpha_0(\eta) \equiv 0$]. In this way, Eqs. (5.42) can be converted into a singular integral equation for $\alpha_N(\eta)$. To carry this out, we make the change of variable in Eq. (5.42a).

$$\mu/\sigma_i \longrightarrow \mu, \tag{5.43}$$

and multiply the equation by σ_i^2. Then we add all N equations,

including (5.42b), to obtain the simple result,

$$\sum_{i=1}^{N} \sigma_i^2 \psi_i(\sigma_i\mu) S[(1/\sigma_i) - \mu]$$

$$= \sum_{i=1}^{N} c_{ii} S[(1/\sigma_i) - \mu] P\int_0^1 [\alpha_N(\eta)\eta d\eta/(\eta-\mu)]$$

$$+ \alpha_N(\mu)\left\{1 - 2\mu\sum_{i=1}^{N} c_{ii}\{\tau(1/\sigma_i\mu)S[\mu - (1/\sigma_i)]\right.$$

$$\left. + \tau(\sigma_i\mu)S[(1/\sigma_i) - \mu]\}\right\}. \qquad (5.44)$$

The various step functions were introduced by the variable change since we must require the argument of $\psi_i(\mu)$, for example, to be less than or equal to unity. Thus, under (5.43),

$$\psi_i(\mu) \longrightarrow \psi_i(\sigma_i\mu)[\theta_1(\mu) + \theta_2(\mu) + \cdots + \theta_i(\mu)]. \qquad (5.45)$$

We note

$$\sum_{\alpha=1}^{i} \theta_\alpha(\mu) = S[(1/\sigma_i) - \mu]. \qquad (5.46)$$

Eq. (5.44) is now in canonical form, since we note, from the definition of $\Omega(z)$, Eq. (5.28), that

$$\Omega^{\pm}(\mu) = 1 - 2\mu\sum_{\beta=1}^{N} c_{\beta\beta}\tau(1/\sigma_\beta\mu)S[\mu - (1/\sigma_\beta)]$$

$$- 2\mu\sum_{\beta=1}^{N} c_{\beta\beta}\tau(\sigma_\beta\mu)S[(1/\sigma_\beta) - \mu] \qquad (5.47)$$

$$\pm \pi i\mu\sum_{\beta=1}^{N} c_{\beta\beta}S[(1/\sigma_\beta) - \mu].$$

Thus, Eq. (5.44) can be written

$$\sum_{i=1}^{N} \sigma_i^2 \phi_i(\sigma_i\mu) S[(1/\sigma_i) - \mu]$$

$$= \{[\Omega^+(\mu) - \Omega^-(\mu)]/2\pi i\mu\} P\int_0^1 [\alpha_N(\eta)\eta d\eta/(\eta-\mu)] \qquad (5.48)$$

$$+ \{[\Omega^+(\mu) + \Omega^-(\mu)]/2\}\alpha_N(\mu).$$

This equation is in standard form (cf. IV) and its solution is well known in terms of the X-function,

$$X(z) = [1/(1-z)]\exp\{\frac{1}{\pi}\int_0^1 \text{Arg } \Omega^+(\mu)[d\mu/(\mu-z)]\}. \qquad (5.49)$$

Once $\alpha_N(\eta)$ is known, the other $\alpha_i(\eta)$ may all be found from Eqs. (5.42). Also, the discrete mode is introduced, as in IV, by the condition at infinity on the auxiliary function,

$$N(z) = (1/2\pi i)\int_0^1 [\alpha_N(\eta)\eta d\eta/(\eta-z)]. \qquad (5.50)$$

Since the details are identical to those in IV, we go no further, except to note that the completeness theorem is hereby proved. The coefficients in the expansion, Eq. (5.38), can be found in principle from the above solution; but it is simpler to use the orthogonality relations derived in the next section.

5.4 Orthogonality

Theorem VI. The continuum functions, $\underline{\phi}(\eta,\mu)$, $\eta \in [0,1]$, and the discrete mode, $\underline{\phi}_+$, are orthogonal to the adjoint eigensolutions, $\underline{\phi}^\dagger(\eta,\mu)$, $\eta \in [0,1]$ and $\underline{\phi}_+^\dagger$, on the range $0 \leq \mu \leq 1$ with weight function $\underline{W}(\mu)$ where

$$[\underline{W}(\mu)]_{ij} = \sigma_i \nu(\mu/\sigma_i)\delta_{ij}. \qquad (5.51)$$

Here $\gamma(\mu)$ is defined as

$$\gamma(\mu) = \mu[X^+(\mu)/\Omega^+(\mu)]. \tag{5.52}$$

The proof of this theorem is a generalization of the one given in IV for N=2. As in IV, we prove orthogonality of the $\underline{F}_{\alpha}^{(i)}(\eta,\mu)$ rather than the $\underline{\phi}_K(\eta,\mu)$. This makes no difference since the statement of the theorem is that[4]

$$\int_0^1 \underline{\tilde{\phi}}^{\dagger}(\eta',\mu)\underline{W}(\mu)\underline{\phi}(\eta,\mu) = 0, \quad \eta \neq \eta', \tag{5.53a}$$

and we shall prove that

$$\int_0^1 \underline{\tilde{F}}^{\dagger}(\eta,\mu)\underline{W}(\mu)\underline{F}(\eta,\mu) = 0, \quad \eta \neq \eta'. \tag{5.53b}$$

Clearly, (5.53b) implies (5.53a), and vice versa. $\underline{F}^{\dagger}(\eta,\mu)$ obeys the adjoint equation given below, and may be obtained from $\underline{F}(\eta,\mu)$ under the interchange, $c_{ij} \Longleftrightarrow c_{ji}$; $\underline{\phi}^{\dagger}(\eta,\mu)$ and $\underline{\phi}(\eta,\mu)$ are related in the same way. As in IV, we note that care must be taken when this interchange is made. For example, $\underline{\phi}_+$ is given by Eq. (5.29). However, this form is a reduction of the form that is obtained if Eq. (5.12), with $\eta_0 = \pm \infty$, is solved directly, which is to say c_{ij} no longer appear. One finds easily, however,

$$\underline{\phi}_+^{\dagger} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \end{bmatrix}. \tag{5.54}$$

---

[4]The function, $\underline{\psi}_-(x,\mu)$, is not included in the orthogonal set because it is not a solution of Eqs. (5.12)--it satisfies Eq. (5.8).

Eqs. (5.21) for $\underline{F}_{\alpha}^{(i)}(\eta,\mu)$ and (5.31b) and (5.32) for $\underline{\phi}_{\alpha}(\eta,\mu)$ are in the form that the interchange, $c_{ij} \Longleftrightarrow c_{ji}$, gives the adjoint solutions.

We turn now to the proof of theorem VI; i.e., Eq. (5.53b). We begin, as in IV, with the equation obeyed by $\underline{F}(\eta,\mu)$ and the adjoint equation obeyed by $\underline{F}^{\dagger}(\eta,\mu)$. [To simplify notation, we allow $\underline{F}(\eta,\mu)$ and $\underline{F}^{\dagger}(\eta,\mu)$ to denote either discrete or continuum modes.] Thus,

$$[(\underline{\Sigma}/\mu) - (\underline{C}/\mu)\int_{-1}^{1}\cdot d\mu']\underline{F}(\eta,\mu) = (1/\eta)\underline{F}(\eta,\mu) \qquad (5.55a)$$

and

$$[(\underline{\Sigma}/\mu) - (\underline{\tilde{C}}/\mu)\int_{-1}^{1}\cdot d\mu']\underline{F}^{\dagger}(\eta',\mu) = (1/\eta')\underline{F}^{\dagger}(\eta',\mu) \qquad (5.55b)$$

The proof of the theorem proceeds, up to a point, precisely as in IV, where we considered N=2, except that the sums there should, of course, be taken from 1 to N. We are led, then, to an equation analogous to Eq. (4.62).

$$J_2 = 1 - [\omega(\eta)\gamma(\eta)/\eta] + \sum_{\alpha=1}^{N} c_{\alpha\alpha}\lambda^{\alpha}(\eta)[\gamma(\eta)/\eta], \qquad (5.56)$$

where

$$\omega(\eta) = \tfrac{1}{2}[\Omega^{+}(\eta) + \Omega^{-}(\eta)] \qquad (5.57)$$

The proof of the theorem hinges on proving that J_2 is independent of η and the c_{ij}. In fact, we shall show J_2 equal to unity. We showed this in IV by direct substitution of the $\lambda^{\alpha}(\eta)$. We note from Eq. (5.16) that any of the components $f_i(\eta,\mu)$ of $\underline{F}(\eta,\mu)$ can be written in the form,

$$f_\alpha(\eta,\mu) = \{[P/(\sigma_\alpha\eta-\mu)]\eta + \lambda^\alpha(\eta)\delta(\sigma_\alpha\eta-\mu)\}\sum_{\beta=1}^{N} c_{\alpha\beta}a_\beta(\eta), \qquad \sigma_\alpha\eta \le 1 \tag{5.58a}$$

or

$$f_j(\eta\ \mu) = [\eta/(\sigma_j\eta-\mu)]\sum_{\beta=1}^{N} c_{j\beta}a_\beta(\eta),\ \sigma_j\eta > 1, \tag{5.58b}$$

where

$$a_\beta(\eta) = \int_{-1}^{1} f_\beta(\eta,\mu')d\mu'. \tag{5.59}$$

[Thus the $\lambda^\alpha(\eta)$ are the unknown coefficients in Eq. (5.18) that one must find for explicit evaluation of the eigensolutions; they should be denoted by $\lambda^\alpha_{iK}(\eta)$ in order to distinguish the region and the degeneracy. However, we shall not have to use their explicit form. The symbol, $\lambda^\alpha(\eta)$, thus denotes any of the $\lambda^\alpha_{iK}(\eta)$.] Multiply Eq. (5.58a) by $c_{N\alpha}$ and sum over all α for which $\sigma_\alpha\eta \le 1$, then multiply Eq. (5.58b) by c_{Nj} and sum over all j for which $\sigma_j\eta > 1$. We integrate the resulting two equations over μ from -1 to 1 and add them to obtain

$$\begin{aligned}&\sum_\alpha^{S} c_{N\alpha}a_\alpha(\eta) + \sum_j^{NS} c_{Nj}a_j(\eta)\\ &= \sum_\alpha^{S} c_{N\alpha}[2\eta\tau(\sigma_\alpha\eta) + \lambda^\alpha(\eta)]\sum_{\beta=1}^{N} c_{\alpha\beta}a_\beta(\eta)\\ &+ \sum_j^{NS} c_{Nj}2\eta\tau(1/\sigma_j\eta)\sum_{\beta=1}^{N} c_{j\beta}a_\beta(\eta).\end{aligned} \tag{5.60}$$

Here the superscripts, S and NS, indicate that the sums are to be taken only over the singular [Eq. (5.58a)] and the

nonsingular [Eq. (5.58b)] sets, respectively. Using $c_{\alpha\beta}c_{N\alpha} = c_{\alpha\alpha}c_{N\beta}$ on the right-hand side of Eq. (5.60), the sums separate and a factor, $\sum_{\alpha=1}^{N} c_{N\alpha}a_{\alpha}(\eta)$, can therefore be canceled to give

$$1 - \sum_{\alpha}^{S} 2c_{\alpha\alpha}\eta\tau(\sigma_{\alpha}\eta) - \sum_{j}^{NS} 2c_{jj}\eta\tau(1/\sigma_{j}\eta) = \sum_{\alpha}^{S} c_{\alpha\alpha}\lambda^{\alpha}(\eta). \tag{5.61}$$

The left-hand side of Eq. (5.61) is exactly $\omega(\eta)$. Thus

$$\sum_{\alpha}^{S} c_{\alpha\alpha}\lambda^{\alpha}(\eta) = \omega(\eta). \tag{5.62}$$

Substituting this result into Eq. (5.56) [remembering that $\lambda^{\alpha}(\eta)$ only appears when $f_{\alpha}(\eta,\mu)$ is singular], we find

$$J_2 \equiv 1, \tag{5.63}$$

thus proving the theorem. We turn now to the calculation of the necessary normalization integrals.

5.5 Normalization

The results of the previous two sections can be used to expand functions $\underline{Y}(\mu)$ for $\mu \in [0,1]$ and to obtain the expansion coefficients once

(i) the normalization integrals are known, and

(ii) the degenerate eigenfunctions are orthogonalized.

As in IV, we introduce a new set of functions, $\underline{\chi}_K(\eta,\mu)$, constructed so as to be orthogonal to all of the $\underline{\phi}_K(\eta,\mu)$. We define our scalar product as

$$(\underline{U},\underline{V}) \triangleq \int_0^1 \underline{U}^\dagger(\eta',\mu)\underline{W}(\mu)\underline{V}(\eta,\mu)d\mu. \tag{5.64}$$

Abbreviating,

$$(\underline{\phi}_i,\underline{\phi}_j) = (i,j)\delta(\eta-\eta'), \tag{5.65}$$

we easily calculate

$$\begin{aligned}(i,j) = -\gamma(\eta)\{&(1/c_{1,i+1}c_{i+1,1})\delta^j_{i+1} + (1/c_{1i}c_{i1})\delta^j_{i-1} \\ &- [(1/c_{i1}c_{1i}) + (1/c_{1,i+1}c_{i+1,1})]\delta^j_i\},\end{aligned} \tag{5.66}$$

$$i,j < N.$$

Also, keeping theorem VI in mind, we have

$$(\underline{\phi}_i,\underline{\phi}_N) = \sum_{\alpha=1}^{i} (\underline{\phi}_i,\underline{\phi}_N^{(\alpha)}). \tag{5.67}$$

The product, $(\underline{\phi}_i,\underline{\phi}_N^{(\alpha)})$, is also easily calculated, and is seen to be independent of α. Thus the sum in Eq. (5.67) merely introduces a factor, i, and we find

$$(i,N) = i(c_{iN}/c_{i1})2\eta\gamma(\eta)[\tau(\sigma_{i+1}\eta) - \tau(\sigma_i\eta)], \tag{5.68a}$$

$$i < N-1$$

$$(N,i) = (i,N)(c_{Ni}/c_{1i})(c_{i1}/c_{iN}),\ i < N-1 \tag{5.68b}$$

$$= (w_N/w_i)(i,N),\ i < N-1. \tag{5.68c}$$

Also,

$$\begin{aligned}(N-1,N) = -\gamma(\eta)\{&(N-1)(c_{N-1,N}/c_{N-1,1})2\eta\tau(\sigma_{N-1}\eta) \\ &+ (1/c_{N1})\sum_{\alpha=1}^{N-1}[1 - 2\eta\sum_{\beta=1}^{\alpha-1}c_{\beta\beta}\tau(1/\sigma_\beta\eta) - 2\eta c_{NN}\tau(\eta)]\}.\end{aligned} \tag{5.69}$$

(N,N-1) may be found from (N-1,N) under the interchange of $c_{ij} \Longleftrightarrow c_{ji}$. Finally, we need

$$(\underline{\phi}_N, \underline{\phi}_N) = \sum_{i=1}^{N} (\underline{\phi}_N^{(i)}, \underline{\phi}_N^{(i)}), \tag{5.70}$$

where $(\underline{\phi}_N^{(i)}, \underline{\phi}_N^{(i)})$ may be calculated from the explicit form, Eq. (5.21), making use of the X-function identities of Appendix A. We quote the result.

$$\begin{aligned}(N,N) = \sum_{i=1}^{N} \Bigg\{ & \sum_{\alpha=1}^{N-1} c_{NN} 4\eta^2 c_{\alpha\alpha} \tau^2(\sigma_\alpha \eta) \\ & + [1 - 2\eta \sum_{\beta=1}^{i-1} c_{\beta\beta} \tau(1/\sigma_\beta \eta) - 2\eta c_{NN} \tau(\eta)]^2 \\ & + \sum_{\alpha=i}^{N} c_{NN} c_{\alpha\alpha} \pi^2 \eta^2 \Bigg\} \gamma(\eta).\end{aligned} \tag{5.71}$$

Eqs. (5.66), (5.68), (5.69), and (5.71) give all the normalization integrals that are necessary for the construction of the $\underline{\chi}_K(\eta,\mu)$. We write, in general,

$$\underline{\chi}_i(\eta,\mu) = \sum_{\alpha=1}^{N} \mathcal{K}_\alpha^{(i)} \underline{\phi}_\alpha(\eta,\mu), \tag{5.72}$$

where the $\mathcal{K}_\alpha^{(i)}$ are to be chosen such that

$$(\underline{\chi}_i, \underline{\phi}_j) = \delta_{ij} \delta(\eta - \eta'). \tag{5.73}$$

The $\mathcal{K}_\alpha^{(i)}$ are readily found from Eqs. (5.72) and (5.73). We take the scalar product of Eq. (5.72) from the right successively with the $\underline{\phi}_\beta(\eta,\mu)$, $\beta=1,\cdots,N$. This yields the following equations for the $\mathcal{K}_\alpha^{(i)}$.

$$
\begin{aligned}
0 &= (1,1)\mathcal{K}_1^{(i)} + (2,1)\mathcal{K}_2^{(i)} + 0 + \cdots + (N,1)\,\mathcal{K}_N^{(i)} \\
0 &= (1,2)\mathcal{K}_1^{(i)} + (2,2)\mathcal{K}_2^{(i)} + (3,2)\mathcal{K}_3^{(i)} + 0 + \cdots + (N,2)\,\mathcal{K}_N^{(i)} \\
0 &= 0 + (2,3)\mathcal{K}_2^{(i)} + (3,3)\mathcal{K}_3^{(i)} + (4,3)\mathcal{K}_4^{(i)} + 0 + \cdots + (N,3)\,\mathcal{K}_N^{(i)} \\
&\vdots \\
1 &= 0 + \cdots + (i-1,i)\mathcal{K}_{i-1}^{(i)} + (i,i)\mathcal{K}_i^{(i)} + (i+1,i)\mathcal{K}_{i+1}^{(i)} + 0 + \cdots + (N,i)\,\mathcal{K}_N^{(i)} \\
&\vdots \\
0 &= (1,N)\mathcal{K}_1^{(i)} + (2,N)\mathcal{K}_2^{(i)} + \cdots + (N-1,N)\mathcal{K}_{N-1}^{(i)} + (N,N)\,\mathcal{K}_N^{(i)}
\end{aligned}
\tag{5.74}
$$

We see that the first and $(N-1)^{th}$ equations have only three nonvanishing coefficients; all the rest have four such coefficients except the Nth, which has N. The set of equations (5.74) is easily solved for the $\mathcal{K}_\alpha^{(i)}$.

We note that we have set $(\underline{\chi}_K, \underline{\phi}_K) = \delta(\eta-\eta')$. Also, $(\underline{\chi}_K, \underline{\phi}_+) = 0$. The discrete (asymptotic) coefficient is found from the relation,

$$(\underline{\phi}_+, \underline{\phi}_+) = \sum_{i=1}^{N} \int_0^1 w_i \sigma_i \gamma(\mu/\sigma_i) d\mu. \tag{5.75}$$

Changing variables, this can be written

$$\begin{aligned}(\underline{\phi}_+, \underline{\phi}_+) &= \sum_{i=1}^{N} w_i \sigma_i^2 \int_0^{1/\sigma_i} \gamma(\mu) d\mu \\ &= 2 \sum_{K=1}^{N} w_K \sigma_K \sum_{i=1}^{N} c_{ii} \int_0^{1/\sigma_i} \gamma(\mu) d\mu.\end{aligned} \tag{5.76}$$

This integral can be evaluated from Eq. (A.12) in the limit $z \longrightarrow \infty$,

$$- \lim_{z \to \infty} zX(z) = \sum_{i=1}^{N} c_{ii} \int_0^{1/\sigma_i} \gamma(d\mu) d\mu. \tag{5.77}$$

But from Eq. (5.49) we see that the limit is -1. Thus

$$(\underline{\phi}_+, \underline{\phi}_+) = 2 \sum_{i=1}^{N} \sigma_i w_i. \tag{5.78}$$

In applying this result to obtain expansion coefficients, one might have an expansion of the form,

$$\underline{\Psi}(\mu) = A_+ \underline{\phi}_+ + \sum_{i=1}^{N} \int_0^{1/\sigma_i} \alpha_i(\eta) \underline{\phi}_i(\eta, \mu) d\eta. \tag{5.79}$$

Then from Eq. (5.76) we find

$$A_+ = (\underline{\phi}_+, \underline{\Psi}(\mu))/2\sum_{i=1}^{N}\sigma_i w_i \tag{5.80}$$

while

$$\alpha_i(\eta) = (\underline{\chi}_i, \underline{\Psi}(\mu)) . \tag{5.81}$$

5.6 The Milne Problem

The solution of the Milne problem is a simple generalization of that obtained in IV. We have

$$\underline{\Psi}_M(x,\mu) = A_+\underline{\phi}_+ + A_-\underline{\Psi}_-(x,\mu) + \sum_{i=1}^{N}\int_0^{1/\sigma_i}\alpha_i(\eta)e^{-x/\eta}\underline{\phi}_i(\eta,\mu)d\eta. \tag{5.82}$$

Here the coefficients are determined from the condition at $x=0$,

$$-\underline{\Psi}_-(0,\mu) = (A_+/A_-)\underline{\phi}_+ + \sum_{i=1}^{N}\int_0^{1/\sigma_i}\frac{\alpha_i(\eta)}{A^-}\underline{\phi}_i(\eta,\mu)d\eta, \qquad \mu \in [0,1], \tag{5.83}$$

and are immediately obtained from the orthogonality relations once the $\underline{\chi}_i(\eta,\mu)$ have been constructed. For the asymptotic solution, i.e., the part of $\underline{\Psi}_M(x,\mu)$ involving only $\underline{\Psi}_-(x,\mu)$ and $\underline{\phi}_+$, we have

$$A_+/A_- = -\int_0^1 \underline{\tilde{\phi}}_+^{\dagger}\underline{W}(\mu)\underline{\Psi}_-(0,\mu)d\mu/2\sum_{i=1}^{N}\sigma_i w_i . \tag{5.84}$$

Since

$$[\underline{\Psi}_-(0,\mu)]_i = -w_i\mu/\sigma_i , \tag{5.85}$$

we find

$$A_+/A_- = \sum_{i=1}^{N} w_i \int_0^1 \gamma(\mu/\sigma_i)\mu d\mu / 2\sum_{i=1}^{N} \sigma_i w_i . \qquad (5.86)$$

Changing variables and noting Eq. (5.6), this becomes

$$A_+/A_- = \sum_{i=1}^{N} c_{ii} \int_0^{1/\sigma_i} \gamma(\mu)\mu d\mu \qquad (5.87a)$$

or

$$A_+/A_- = \frac{3}{2}\left\{ \sum_{i=1}^{N} c_{ii} \int_0^{1/\sigma_i} [\mu^2/X(-\mu)]d\mu / \sum_{i=1}^{N} (c_{ii}/\sigma_i^3) \right\} . \qquad (5.87b)$$

To obtain the form Eq. (5.87b), Identity VIII in Appendix A has been used. The extrapolated endpoint is defined in terms of the quantity,

$$F_{asy}(x) \triangleq 2\pi \begin{bmatrix} \tilde{1} \\ 1 \\ . \\ . \\ . \end{bmatrix} \int_{-1}^{1} \underline{\Psi}_{M_{asy}}(x,\mu')d\mu' . \qquad (5.88)$$

This becomes, in our model,

$$F_{asy}(x) = 4\pi \sum_{i=1}^{N} w_i (A_+ + xA_-) . \qquad (5.89)$$

Thus the extrapolated endpoint, x_0, is

$$x_0 = A_+/A_- \qquad (5.90)$$

and is already given by Eqs. (5.87). In a similar manner the entire solution can be found and the temperature distribution thus deduced.

Just as in IV, the law of darkening for the integrated quantities, $\underline{\Psi}_M(0,\mu)$, $\mu < 0$, can easily be obtained. The fact that we restrict μ to be negative allows us to determine $\underline{\Psi}_M$

(for x=0) explicitly without actually knowing any of the expansion coefficients except $\alpha_N(\eta)$. The coefficient, $\alpha_N(\eta)$, is expressed in terms of the N-function, which then permits evaluation of the integrals involved. The procedure follows exactly as in IV. We simply quote the result.

$$\underline{\mathbf{F}}_{\mathbf{M}}(0,-\mu) = [3\sigma T_e^4/4\pi \sum_{i=1}^{N} (w_i/\sigma_i)] \begin{bmatrix} w_1/X(-\mu/\sigma_1) \\ w_2/X(-\mu/\sigma_2) \\ \vdots \\ w_N/X(-\mu) \end{bmatrix}, \quad \mu > 0. \qquad (5.91)$$

6. SUMMARY

Case's method of normal modes, which was developed to solve one-speed or gray transport problems, was extended here to construct solutions to the Uniform Picket Fence model and to the Generalized Picket Fence model of radiative transfer. Following Case's work, the solution to the given matrix transport equation was expanded in terms of the normal modes (both the discrete modes and the singular or distribution modes). The expansion coefficients then were determined from the boundary conditions.

The vector eigensolutions developed were found to satisfy full and half-range completeness theorems for the case where the transfer matrix, $\underline{C}$, had the form particular to the two above-mentioned models (Uniform Picket Fence model and Generalized Picket Fence model). In addition, these eigensolutions were found to obey rather remarkable full and half-range orthogonality theorems which thus greatly simplified the task of determining the unknown expansion coefficients.

The orthogonality relation developed utilizes the adjoint eigensolutions and has the interesting feature that the scalar product is defined, in the Generalized Picket Fence model, with a NxN matrix weight function.

Explicitly, we developed in this work the exact solutions for the law of darkening, the temperature distribution, and the extrapolated endpoint for the Milne problem in both the Uniform Picket Fence model and the Generalized Picket Fence model.

Some work was done for the situation where the transfer matrix, $\underline{C}$, was completely arbitrary (corresponding, quite generally, to problems in neutron transport theory or in the theory of radiative transfer). The eigensolutions for the case N=2 are presented and full-range completeness and orthogonality theorems are formulated. The half-range case appears not to have a closed form solution, however, more work is certainly indicated here.

A P P E N D I C E S

APPENDIX A

X-FUNCTION IDENTITIES

A.1 The Case N=2

Identity I:

$$X(z) = c_{22}\int_0^1 [\gamma(\mu)/(\mu-z)]d\mu + c_{11}\int_0^{1/\sigma_1} [\gamma(\mu)/(\mu-z)]d\mu. \quad (A.1)$$

The proof follows by writing Cauchy's theorem for X(z); i.e.,

$$X(z) = (1/2\pi i)\oint [X(z')/(z'-z)]dz', \quad (A.2)$$

where the contour can be shrunk down to include only the branch cut (the integrand vanishes at infinity). Thus we write

$$X(z) = (1/2\pi i)\int_0^1 [X^+(\mu) - X^-(\mu)][d\mu/(\mu-z)]. \quad (A.3)$$

Using Eqs. (4.34b) and (4.38), we obtain

$$X^+(\mu) - X^-(\mu) = 2\pi i\gamma(\mu)[c_{22} + c_{11}\theta_1(\mu)] \quad (A.4)$$

Entering Eq. (A.3) with this result gives Identity I.

Identity II:

$$X(z)X(-z) = [3\sigma_1(\sigma_1 w_1+w_2)/(\sigma_1 w_2+w_1)]\Omega(z). \quad (A.5)$$

For the proof, we note that the function,

$$F(z) \triangleq X(z)X(-z)(\sigma_1 w_2+w_1)/3\sigma_1(\sigma_1 w_1+w_2)\Omega(z), \quad (A.6)$$

is an entire function because it is analytic in the cut plane and its discontinuity across the cut vanishes. It is thus a constant.

Letting z approach infinity, we find

$$\lim_{z \to \infty} F(z) = 1 \tag{A.7}$$

because in the same limit,

$$X(z)X(-z) \longrightarrow -(1/z^2) \tag{A.8}$$

and

$$\Omega(z) \longrightarrow -(1/z^2)[(\sigma_1 w_2 + w_1)/3\sigma_1(\sigma_1 w_1 + w_2)]. \tag{A.9}$$

This proves Identity II.

Identity III:

$$X(z) = [3\sigma_1/2(\sigma_1 w_2 + w_1)]\{w_2 \int_0^1 [\mu/X(-\mu)(\mu - z)]d\mu + \sigma_1^2 w_1 \int_0^{1/\sigma_1} [\mu/X(-\mu)(\mu - z)]d\mu\}. \tag{A.10}$$

This is a nonlinear integral equation for X(z) that can be used by the ambitious to evaluate X(z) numerically. It is obtained by combining Identities I and II and noting the trivial,

Identity IV:

$$Y(\mu) = 3\mu\sigma_1(\sigma_1 w_1 + w_2)/(\sigma_1 w_2 + w_1)X(-\mu) \tag{A.11}$$

obtained from Eq. (A.5) by taking boundary values of both sides.

A.2 The Case of General N

The derivation of the X-function identities here is a trivial generalization of the corresponding derivation for N=2, so we present them without proof.

Identity V:

$$X(z) = \sum_{i=1}^{N} c_{ii} \int_0^{1/\sigma_i} [\Upsilon(\mu)/(\mu - z)]d\mu. \qquad (A.12)$$

Identity VI:

$$X(z)X(-z) = 3\Omega(z)/2 \sum_{i=1}^{N} (c_{ii}/\sigma_i^3). \qquad (A.13)$$

By combining Identities V and VI we get a nonlinear, nonsingular integral equation for the numerical evaluation of X(z). Thus we find

Identity VII:

$$X(z) = [3/2 \sum_{i=1}^{N} (c_{ii}/\sigma_i^3)] \sum_{\alpha=1}^{N} c_{\alpha\alpha} \int_0^{1/\sigma_\alpha} [\mu/X(-\mu)(\mu - z)]d\mu. \qquad (A.14)$$

Furthermore, Identity VIII is the trivial result obtained by taking boundary values of Eq. (A.13).

Identity VIII:

$$\Upsilon(\mu) = [3/2 \sum_{i=1}^{N} (c_{ii}/\sigma_i^3)][\mu/X(-\mu)]. \qquad (A.15)$$

APPENDIX B

THE HALF-SPACE GREEN'S FUNCTION

For the case, det $\underline{C}=0$, there is no solution to the infinite medium Green's function which is finite everywhere. However, we can define a "pseudo Green's function" for the infinite medium that goes to a constant at plus infinity and diverges at minus infinity, which can then be used to construct the half-space Green's function in the following way.

We wish to construct a solution to the homogeneous transport equation, Eq. (4.11), which approaches a constant bounded value at plus infinity, obeys the "jump conditions" at the source position, and has zero incident distribution. We note that there are two Green's functions, $\underline{G}_1$ and $\underline{G}_2$, corresponding to sources

$$\underline{q}_1 = \delta(x-x_0)\delta(\mu-\mu_0)\begin{bmatrix}1\\0\end{bmatrix} \qquad \text{(B.1a)}$$

$$\underline{q}_2 = \delta(x-x_0)\delta(\mu-\mu_0)\begin{bmatrix}0\\1\end{bmatrix}, \qquad \text{(B.1b)}$$

respectively. Each is found in a similar way, so we simply discuss $\underline{G}_1$.

Consider an artificial infinite medium Green's function of the form

$$\begin{aligned}\underline{G}_{1\infty}(x_0,\mu_0\longrightarrow x,\mu) = {} & A_{1+}\underline{\phi}_{+} + \int_0^{1/\sigma_1} \alpha_{11}(\eta)e^{-(x-x_0)/\eta}\underline{\phi}_1(\eta,\mu)d\eta \\ & + \int_0^{1/\sigma_1} \alpha_{12}(\eta)e^{-(x-x_0)/\eta}\underline{\phi}_2^{(1)}(\eta,\mu)d\eta \qquad \text{(B.2a)} \\ & + \int_{1/\sigma_1}^{1} \alpha_{12}(\eta)e^{-(x-x_0)/\eta}\underline{\phi}_2^{(2)}(\eta,\mu)d\eta, \quad x>x_0.\end{aligned}$$

$$\underline{G}_{1\infty}(x_0,\mu_0 \rightarrow x,\mu) = -A_{1-}\underline{\Psi}_-(x-x_0,\mu)$$
$$- \int_{-1/\sigma_1}^{0} \alpha_{11}(\eta) e^{(x_0-x)/\eta} \underline{\phi}_1(\eta,\mu)\,d\eta$$
$$- \int_{-1/\sigma_1}^{0} \alpha_{12}(\eta) e^{(x_0-x)/\eta} \underline{\phi}_2^{(1)}(\eta,\mu)\,d\eta \qquad (B.2b)$$
$$- \int_{-1}^{-1/\sigma_1} \alpha_{12}(\eta) e^{(x_0-x)/\eta} \underline{\phi}_2^{(2)}(\eta,\mu)\,d\eta, \quad x<x_0.$$

Here the coefficients are found in the usual manner; i.e., we set

$$\underline{G}_{1\infty}(x_0,\mu_0 \rightarrow x_0^+,\mu) - \underline{G}_{1\infty}(x_0,\mu_0 \rightarrow x_0^-,\mu) = 2\pi\mu\delta(\mu-\mu_0)\begin{bmatrix}1\\0\end{bmatrix}, \qquad (B.3)$$

which leads to

$$2\pi\mu\delta(\mu-\mu_0)\begin{bmatrix}1\\0\end{bmatrix} = A_{1+}\underline{\phi}_+ + A_{1-}\underline{\Psi}_-(0,\mu) + \int_{(1)} \alpha_{11}(\eta)\underline{\phi}_1(\eta,\mu)\,d\eta$$
$$+ \int_{(1)} \alpha_{12}(\eta)\underline{\phi}_2^{(1)}(\eta,\mu)\,d\eta + \int_{(2)} \alpha_{12}(\eta)\underline{\phi}_2^{(2)}(\eta,\mu)\,d\eta. \qquad (B.4)$$

Theorems III and IV apply to the above and thus we can find A_{1+}, A_{1-}, $\alpha_{11}(\eta)$, and $\alpha_{12}(\eta)$. Note that $\underline{G}_{1\infty}$ obeys the first two boundary conditions; i.e., it goes to a constant at $+\infty$ and it obeys the "jump condition," Eq. (B.3). That it diverges at $-\infty$ is not important here. We now must fix the zero incident distribution condition. To this end, we write

$$\underline{G}_{1_{hs.}}(x_0,\mu_0 \rightarrow x,\mu) = \underline{G}_{1_\infty}(x_0,\mu_0 \rightarrow x,\mu) - A'_{1+}\underline{\phi}_+ - \int_0^{1/\sigma_1} \alpha'_{11}(\eta)e^{-x/\eta}\underline{\phi}_1(\eta,\mu)d\eta - \int_0^{1/\sigma_1} \alpha'_{12}(\eta)e^{-x/\eta}\underline{\phi}_2^{(1)}(\eta,\mu)d\eta - \int_{1/\sigma_1}^{1} \alpha'_{12}(\eta)e^{-x/\eta}\underline{\phi}_2^{(2)}(\eta,\mu)d\eta, \quad x>0. \tag{B.5}$$

Setting x=0, we obtain the condition for zero incident distribution; namely,

$$\underline{G}_{1_\infty}(x_0,\mu_0 \rightarrow x,\mu) = A'_{1+}\underline{\phi}_+ + \int_0^{1/\sigma_1} \alpha'_{11}(\eta)\underline{\phi}_1(\eta,\mu)d\eta + \int_0^{1/\sigma_1} \alpha'_{12}(\eta)\underline{\phi}_2^{(1)}(\eta,\mu)d\eta + \int_{1/\sigma_1}^{1} \alpha'_{12}(\eta)\underline{\phi}_2^{(2)}(\eta,\mu)d\eta, \quad \mu \geq 0. \tag{B.6}$$

Since the left-hand side of Eq. (B.6) is known, we can determine the unknown coefficients, A'_{1+}, $\alpha'_{11}(\eta)$, and $\alpha'_{12}(\eta)$, with the aid of theorems I and II. The half-space Green's function corresponding to source $\underline{q}_1$ is thus known and given by Eq. (B.5). In exactly the same way, $\underline{G}_{2_{hs.}}$ can be found.

APPENDIX C

THE NUMBER OF DISCRETE EIGENVALUES

The discrete eigenvalues, $\eta_{i\pm}$, are defined as the zeros of the dispersion function, $\Omega(z)$, Eq. (5.28). Here we verify that there are only two zeros for any N.

Since $\Omega(z)$ is analytic in the cut plane and vanishes at infinity, the number of zeros is $1/2\pi$ times the change in the argument of $\Omega(z)$ as a contour encircling the cut is traversed (36). Because $\Omega^+(\mu) = [\Omega^-(\mu)]^*$ and $\Omega(z) = \Omega(-z)$, the change in argument is four times the change in going from $0+i\epsilon$ to $1+i\epsilon$. Call this change $\Delta_+(0,1)$. We have

$$\Delta_+(0,1) = \sum_{i=0}^{N-1} \Delta_+[(1/\sigma_i),(1/\sigma_{i+1})], \tag{C.1}$$

where we define $1/\sigma_0 = 0$.

From Eq. (5.47), we can write for the boundary value in region i

$$\Omega_i^+(\mu) = 1 - 2\mu \sum_{\beta=1}^{i-1} c_{\beta\beta} \tau(1/\sigma_\beta \mu) - 2\mu \sum_{\beta=i}^{N} c_{\beta\beta}[\tau(\sigma_\beta \mu) + (\pi i/2)]. \tag{C.2}$$

From Eq. (C.2) it is easily verified that

$$\Delta_+[0,(1/\sigma_1)] = \pi \tag{C.3}$$

$$\Delta_+[(1/\sigma_i),(1/\sigma_{i+1})] = 0, \quad i=1,2,\cdots,N-1. \tag{C.4}$$

Thus

$$\Delta_+(0,1) = \pi \tag{C.5}$$

and the total change as the cut is encircled is 4π. Thus $\Omega(z)$ has two zeros. This result, of course, applies for any N and requires only that the transfer matrix have the degenerate form given by Eq. (5.6).

APPENDIX D

PARTIAL RANGE COMPLETENESS FOR THE EIGENFUNCTIONS OF THE ONE-SPEED TRANSPORT EQUATION

D.1 Introduction

Case (8) proved that the (singular) eigenfunctions of the one-speed, one-dimensional neutron transport equation with isotropic scattering obey a partial range completeness theorem, which can be stated as follows.[1]

Theorem. A function, $\psi(\mu)$, defined on the range $a \leq \mu \leq b$, with $-1 < a < b < 1$, may be expanded in terms of the continuum eigenfunctions of the transport equation, $\phi_\nu(\mu)$, with $a \leq \nu \leq b$. If $a = -1$ or $b = 1$, one of the two discrete modes, $\phi_{0\pm}(\mu)$, must be included in the expansion; if $a = -1$ and $b = 1$, both discrete modes must be included. However, for expansions of this type the continuation of the expansion outside of the partial range is not always finite.[2] If we insist that the continuation always be finite we are led to the following theorem.

Modified Theorem. A function, $\psi(\mu)$, defined on the range $a \leq \mu \leq b$ may be expanded in terms of the continuum eigenfunctions, $\phi_\nu(\mu)$, with $a \leq \nu \leq b$ plus: (1) both discrete modes if $a < 0 < b$, and (2) one discrete mode if $a \geq 0$ or $b \leq 0$.

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[1] We are drawing heavily on the notation of Refs. (8), (29), and (30).

[2] Case pointed out in Ref. (8) that the expansion coefficients were not unique if a and b were both on the same side of the origin. Actually, as we shall see, they are not unique at all according to Case's statement of the theorem, except specifically for the full and half-range cases.

In Section D.2, we review the proof of Case's theorem. Then, in Section D.3, we describe our approach and show how it leads to the modified theorem as stated above. In Section D.4, we discuss briefly the finite discontinuity that appears in the case α or $\beta = 0$ and attempt to discuss some of the results of this appendix in at least quasi-physical terms. Finally, in Section D.5, we make some remarks on the time-dependent case in which it appears that no general partial range completeness theorem exists.

D.2 The Partial Range Completeness Theorems

For convenience in what follows, we sketch briefly the proof of the partial range completeness theorem, as presented by Case. We begin by attempting to expand $\psi(\mu)$ in the form,

$$\psi(\mu) = \int_{\alpha}^{\beta} A(\nu)\phi_{\nu}(\mu)\,d\nu, \quad \alpha \leq \mu \leq \beta, \tag{D.1}$$

where $A(\nu)$ is the coefficient to be determined. The completeness proof consists of proving that Eq. (D.1), viewed as a singular integral equation for $A(\nu)$, possesses a solution. This is done by actually constructing the solution according to the methods described by Muskhelishvili (33).

If we insert the explicit expression for $\phi_{\nu}(\mu)$ into Eq. (D.1), we obtain

$$\psi(\mu) = A(\mu)\lambda(\mu) + P\int_{\alpha}^{\beta} (c\nu/2)[1/(\nu-\mu)]A(\nu)\,d\nu. \tag{D.2}$$

The solution of this equation is effected by introducing an auxiliary function,

$$N(z) = (1/2\pi i)\int_{\alpha}^{\beta}(c\nu/2)[A(\nu)/(\nu-z)]d\nu. \qquad (D.3)$$

N(z) has the following properties, assuming that a sufficiently well-behaved $A(\nu)$ exists.

1. N(z) is analytic in the complex plane cut from α to β along the real axis.
2. $N(z) \sim 1/z$ as $z \longrightarrow \pm \infty$.
3. $N^{+}(\mu) + N^{-}(\mu) = (P/i\pi)\int_{\alpha}^{\beta}(c\nu/2)A(\nu)[d\nu/(\nu-\mu)]$.
4. $N^{+}(\mu) - N^{-}(\mu) = (c\mu/2)A(\mu)$.

In terms of N(z), we can write Eq. (D.2) as

$$(c\mu/2)\psi(\mu) = N^{+}(\mu)\Lambda^{+}(\mu) - N^{-}(\mu)\Lambda^{-}(\mu). \qquad (D.4)$$

We introduce a new function, X(z), cut from α to β, and satisfying

$$X^{+}(\mu)/X^{-}(\mu) = \Lambda^{+}(\mu)/\Lambda^{-}(\mu), \quad \alpha \leq \mu \leq \beta. \qquad (D.5)$$

In terms of X(z), Eq. (D.4) takes the form,

$$\gamma(\mu)\psi(\mu) = X^{+}(\mu)N^{+}(\mu) - X^{-}(\mu)N^{-}(\mu), \qquad (D.6)$$

where

$$\gamma(\mu) \equiv (c\mu/2)[X^{+}(\mu)/\Lambda^{+}(\mu)]. \qquad (D.7)$$

If X(z) exists, then the solution of Eq. (D.6) is

$$N(z) = [1/X(z)](1/2\pi i)\int_{\alpha}^{\beta}[\psi(\mu)\gamma(\mu)/(\mu-z)]d\mu. \qquad (D.8)$$

A function that obeys the "ratio condition" is

$$X_0(z) = \exp\{(1/2\pi i)\int_\alpha^\beta d\mu[\ln(\Lambda^+(\mu)/\Lambda^-(\mu))/(\mu-z)]\}, \qquad (D.9)$$

or

$$X_0(z) = \exp[(1/\pi)\int_\alpha^\beta \theta(\mu)[d\mu/(\mu-z)]], \qquad (D.10a)$$

where

$$\theta(\mu) = \mathrm{Arg}\Lambda^+(\mu). \qquad (D.10b)$$

To verify conditions (1) and (2) on the function N(z), it is necessary to consider the behavior of $X_0(z)$ at the endpoints of the cut.

$$X_0(z) \longrightarrow (\alpha-z)^{-\theta(\alpha)/\pi}, \; z \longrightarrow \alpha; \qquad (D.11a)$$

$$X_0(z) \longrightarrow (\beta-z)^{\theta(\beta)/\pi}, \; z \longrightarrow \beta, \qquad (D.11b)$$

$\theta(\mu)$ is a monotonic increasing odd function of μ; and $\theta(1) = \pi$ (Figure D.1). The cases $\alpha = -1$, $\beta = +1$ (the two "half-range"

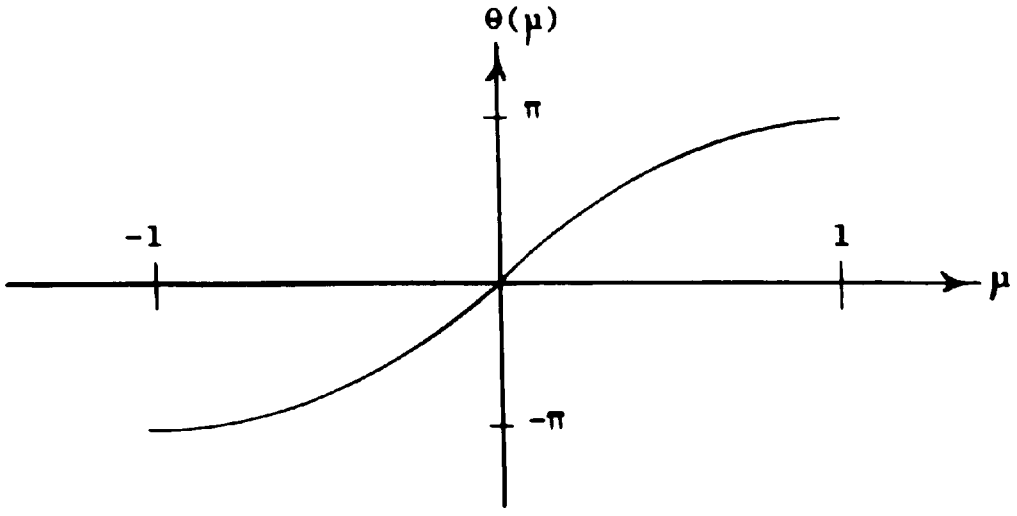

Figure D.1 The function $\theta(\mu)$ versus μ for a typical case

cases) are discussed in detail in Refs. (8), (21), (29), and (30). We consider the "partial range" case (i.e., all cases other than the full and half-range).[3] For the partial range, $X_0(z)$ leads to an acceptable behavior of N(z); thus the solution to Eq. (D.1) exists, and $A(\nu)$ can be determined from property (4) of N(z).

However, the following difficulty arises: the functions,

$$X_1(z) = X_0(z)/(\alpha-z) \text{ [or } X_0(z)/(\beta-z)], \qquad (D.12a)$$

and

$$X_2(z) = X_0(z)/(\alpha-z)(\beta-z) \qquad (D.12b)$$

also obey the "ratio condition." Furthermore, it is not difficult to see that $X_1(z)$ and $X_2(z)$ lead to the correct behavior of $\psi(\mu)$ as μ approaches the endpoints from within. They do lead to the wrong behavior for N(z) at infinity (condition 2) but this occurs also in the full and half-range cases and, like these, this behavior can be corrected by the introduction of discrete modes--one if the $X_1(z)$ function is used and both if the function $X_2(z)$ is introduced. This is the "nonuniqueness" referred to in Section D.1.

If we insist that the continuation of $\psi(\mu)$ be finite, we must use the function $X_2(z)$ if $\alpha < 0 < \beta$ and $X_1(z)$ if $\alpha \geq 0$ or $\beta \leq 0$. This means that at least one discrete mode

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[3]See, however, Section D.4 of this appendix for some remarks that are appplicable to the half-range case.

will always be necessary, while if the interval (α,β) encompasses the origin the two discrete modes are required.[4]

D.3 A Modified Partial Range Expansion

It is convenient to introduce the orthogonality relations of Kuščer et al. (22) and the closure property used by Kuščer and Shure (37) to demonstrate completeness of the normal modes. We first show that the expansion involving $X_0(z)$ has infinite discontinuities at the point $\mu = \alpha,\beta$. We denote by $\gamma_i(\mu)$ the following:

$$\gamma_i(\mu) = (c\mu/2)[X_i^+(\mu)/\Lambda^+(\mu)], \quad i=0,1,2. \tag{D.13}$$

Using the orthogonality relation (22),

$$\int_\alpha^\beta d\mu \gamma_0(\mu)\phi_\nu(\mu)\phi_{\nu'}(\mu)d\mu = \gamma_0(\nu)\Lambda^+(\nu)\Lambda^-(\nu)\delta(\nu-\nu'), \tag{D.14}$$

and Eq. (D.3), we find

$$A(\nu) = \int_\alpha^\beta \gamma_0(\mu')\phi_\nu(\mu')\psi(\mu')d\mu'/\gamma_0(\nu)\Lambda^+(\nu)\Lambda^-(\nu). \tag{D.15}$$

Thus we have

$$\psi(\mu) = \int_\alpha^\beta d\mu'\psi(\mu')\int_\alpha^\beta d\nu[\gamma_0(\mu')\phi_\nu(\mu')\phi_\nu(\mu)/\gamma_0(\nu)\Lambda^+(\nu)\Lambda^-(\nu)]. \tag{D.16}$$

---

[4]This statement of the partial range theorem is seen not to be in conflict with Case's original statement for the (physically important) full and half-range cases.

The quantity,

$$\int_{\alpha}^{\beta} d\nu[\gamma_0(\mu')\phi_\nu(\mu')\phi_\nu(\mu)/\gamma_0(\nu)\Lambda^+(\nu)\Lambda^-(\nu)] = \delta(\mu-\mu'), \qquad (D.17)$$

$$\mu',\mu \in [\alpha,\beta].$$

This is the closure relation. We denote by $\Delta_0(\mu',\mu)$ the value of the integral in Eq. (D.17) for $\mu' \in [\alpha,\beta]$, $\mu \notin [\alpha,\beta]$. Insertion of this quantity into Eq. (D.16) leads to $\psi_c(\mu)$, the continuation of $\psi(\mu)$; i.e.,

$$\psi_c(\mu) = \int_{\alpha}^{\beta} \psi(\mu')\Delta_0(\mu'\mu)d\mu'. \qquad (D.18)$$

It can be shown that

$$\psi_c(\mu) = \int_{\alpha}^{\beta} d\mu'[\gamma_0(\mu')\psi(\mu')/X_0(\mu)(\mu'-\mu)], \quad \mu \notin [\alpha,\beta]. \qquad (D.19)$$

It is a rather simple matter to see that $\psi_c(\alpha)$ and $\psi_c(\beta)$ diverge. Consider, for example, $\mu \sim \alpha$. We can write (D.19) as

$$\psi_c(\mu) = [\psi(\alpha)/X_0(\mu)]\int_{\alpha}^{\beta}\gamma_0(\mu')[d\mu'/(\mu'-\mu)] + [1/X_0(\mu)]\int_{\alpha}^{\beta}\gamma_0(\mu')[\psi(\mu') - \psi(\alpha)][d\mu'/(\mu'-\mu)]. \qquad (D.20)$$

However, we have (22)

$$\int_{\alpha}^{\beta}\gamma_0(\mu')[d\mu'/(\mu'-\mu)] = X_0(\mu) - 1. \qquad (D.21)$$

Thus,

$$\psi_c(\mu) = \psi(\alpha) - [\psi(\alpha)/X_0(\mu)] + [1/X_0(\mu)]\int_{\alpha}^{\beta}\gamma_0(\mu')\{[(\psi(\mu') - \psi(\alpha)]/(\mu'-\mu)\}d\mu'. \qquad (D.22)$$

The integral in Eq. (D.22) approaches a finite limit, generally not $\psi(\alpha)$, as $\mu \longrightarrow \alpha$. The factor, $1/X_0(\mu)$ diverges because $X_0(\mu)$ has a (fractional power) zero at $\mu = \alpha$ A similar argument shows that $\psi_c(\beta)$ also diverges. These arguments only apply for $\alpha < 0 < \beta$. For the case $\alpha > 0$, $\psi_c(\alpha) = \psi(\alpha)$, since $X_0(\mu)$ now approaches infinity. Then $\psi_c(\mu)$ is poorly behaved only at $\mu = \beta$. Similarly, for $\beta < 0$, $\psi_c(\mu)$ is poorly behaved only at $\mu = \alpha$. It should be noted that the difficulties at $\mu = \alpha$ do not occur if $\psi(\alpha) = 0$, and similarly at $\mu = \beta$, if $\psi(\beta) = 0$.

To remove this difficulty, it is necessary to introduce the function, $X_2(\mu)$, rather than $X_0(\mu)$, since $X_2(\mu)$ approaches infinity as $\mu \longrightarrow \alpha,\beta$. (For the case $\alpha > 0$ or $\beta < 0$, one of the two $X_1(z)$ functions must be used.) In this case, as we have discussed in Section D.2, it is necessary to introduce the two discrete modes, $\phi_{0\pm}(\mu)$, into the expansion in order for N(z) to have the proper behavior at infinity. (For $\alpha > 0$ or $\beta < 0$, only one discrete mode is needed.)

Thus, the partial range expansion takes the form,

$$\psi(\mu) = a_{0+}\phi_{0+}(\mu) + a_{0-}\phi_{0-}(\mu) + \int_{\alpha}^{\beta} A(\nu)\phi_{\nu}(\mu)d\nu. \qquad (D.23)$$

Again, the expansion coefficients, $A(\nu)$ and a_{0+}, are

$$A(\nu) = \int_{\alpha}^{\beta} w_2(\mu')\phi_{\nu}(\mu')\psi(\mu')d\mu'/w_2(\nu)\Lambda^{+}(\nu)\Lambda^{-}(\nu) \qquad (D.24a)$$

and

$$a_{0\pm} = \int_{\alpha}^{\beta} w_2(\mu')\phi_{0\pm}(\mu')\psi(\mu')d\mu'/N_{0\pm}, \qquad (D.24b)$$

where

$$w_2(\mu') = (\nu_0^2-\mu'^2)(c\mu'/2)[X^+(\mu')/\Lambda^+(\mu')], \qquad (D.25)$$

and

$$N_{0\pm} = \mp(c\nu_0/2)^2 2\nu_0 X_2(\pm\nu_0). \qquad (D.26)$$

Using these expressions, $\psi(\mu)$ can be written

$$\psi(\mu) = \int_\alpha^\beta d\mu' \psi(\mu') H(\mu',\mu), \qquad (D.27)$$

where

$$\begin{aligned} H(\mu',\mu) = \int_\alpha^\beta d\nu [w_2(\mu')\phi_\nu(\mu')\phi_\nu(\mu)/w_2(\nu)\Lambda^+(\nu)\Lambda^-(\nu)] \\ + w_2(\mu')\phi_{0+}(\mu')\phi_{0+}(\mu)/N_{0+} \qquad (D.28) \\ + w_2(\mu')\phi_{0-}(\mu')\phi_{0-}(\mu)/N_{0-}. \end{aligned}$$

If $\mu,\mu' \in [\alpha,\beta]$, (D.28) is simply the closure relation; i.e.,

$$H(\mu',\mu) = \delta(\mu'-\mu), \quad \mu,\mu' \in [\alpha,\beta]. \qquad (D.29)$$

If we continue $\psi(\mu)$ outside the interval, we have

$$\psi_c(\mu) = \int_\alpha^\beta d\mu' \psi(\mu') \Delta_2(\mu',\mu), \qquad (D.30)$$

where $\Delta_2(\mu',\mu)$ is given by $H(\mu',\mu)$, for $\mu \notin [\alpha,\beta]$. We thus find

$$\psi_c(\mu) = [1/(\nu_0^2-\mu^2)X_2(\mu)]\int_\alpha^\beta w_2(\mu')\phi(\mu')[d\mu'/(\mu'-\mu)]. \qquad (D.31)$$

In order to examine $\psi_c(\mu)$ as μ approaches α, we write

$$\psi_c(\mu) = [\psi(\alpha)/(\nu_0^2-\mu^2)X_2(\mu)]\int_\alpha^\beta w_2(\mu')[d\mu'/(\mu'-\mu)] + [1/(\nu_0^2-\mu^2)X_2(\mu)]\int_\alpha^\beta w_2(\mu')[\psi(\mu') - \psi(\alpha)][d\mu'/(\mu'-\mu)]. \tag{D.32}$$

It can be shown that

$$\int_\alpha^\beta w_2(\mu')[d\mu'/(\mu'-z)] = (\nu_0^2-z^2)X_2(z) - 1, \tag{D.33}$$

and thus Eq. (D.32) becomes

$$\psi_c(\mu) = \psi(\alpha) - [\psi(\alpha)/(\nu_0^2-\mu^2)X_2(\mu)] + [1/(\nu_0^2-\mu^2)X_2(\mu)]\int_\alpha^\beta w_2(\mu')[\psi(\mu')-\psi(\mu)][d\mu'/(\mu'-\mu)]. \tag{D.34}$$

The integral in Eq. (D.34) approaches a finite limit, generally not $\psi(\alpha)$, as $\mu \longrightarrow \alpha$. The function, X_2, diverges and we obtain

$$\psi_c(\alpha) = \psi(\alpha). \tag{D.35}$$

In exactly the same manner it follows that

$$\psi_c(\beta) = \psi(\beta). \tag{D.36}$$

Thus by using the appropriate X(z), we have an expansion for $\psi(\mu)$ that has a finite continuation outside the interval $[\alpha,\beta]$.

Note that if the interval $[\alpha,\beta]$ does not include the origin, then the infinite discontinuity occurs only at one of the endpoints. Then the $X_1(z)$ function is appropriate and only a single discrete mode is needed.

D.4 General Discussion

In the previous sections of this appendix, we have shown how an arbitrary function defined on the interval $[\alpha,\beta]$ may be expanded in terms of the normal modes of the transport equation in a number of different ways, depending upon the choice of the X(z) function. This does not mean, however, that the set of normal modes is "overcomplete" because we have seen that each of the possible expansions differs in its behavior at the endpoints α and β.

To understand why the infinite discontinuity encountered in some of the cases might be undesirable, it is helpful to formulate a typical problem that might involve a "partial range" expansion. In such a case, one would be dealing with a homogeneous equation in two variables, say x and μ, whose solution was uniquely specified by the behavior at $x = \infty$ along with the boundary values at another surface, say $x=0$, for some portion of the entire range of μ, say $\alpha \leq \mu \leq \beta$. Then the expansion coefficients would be obtained by expanding the boundary values for the range $[\alpha,\beta]$ in terms of the appropriate normal modes of the homogeneous equation, after which the complete solution would be found by continuation of the expansion to all values of μ. Thus an infinite discontinuity could not be tolerated.

Unfortunately, it is difficult to formulate a physical problem in transport theory where the "partial range," $[\alpha,\beta]$, is other than the "half-range," $[-1,0]$ or $[0,1]$. Thus, if the condition of analytic continuability is imposed, it must

be imposed by analogy with the half-range case.[5] That case is, unfortunately again, degenerate in the following sense. Consider the range [0,1]. Then

$$X_0(0) = 1$$
$$X_0(z) = (1-z),\ z \longrightarrow 1$$

so that the appropriate X(z) in this case is $X_1(z)$.

$$X_1(z) = X_0(z)/(1-z). \tag{D.37}$$

Neither $X_0(z)$ nor $X_2(z)$ will lead to an expansion of any sort, so that the ambiguity associated with the "partial range" does not arise.

It is interesting to note, furthermore, that the function $\psi(\mu)$ has a finite discontinuity in the half-range case at $\mu=0$. In fact, this also occurs in the partial range cases, $[\alpha,0]$ and $[0,\beta]$, whatever the value of α or β. To see this, we write, as previously (specializing to the case $[0,\beta]$),

$$\psi(\mu) = \int_0^\beta d\mu'\psi(\mu')\delta(\mu'-\mu),\ \mu \in [0,\beta] \tag{D.38a}$$

and

$$\psi_c(\mu) = \int_0^\beta d\mu'\psi(\mu')\Delta_1(\mu',\mu),\ \mu \notin [0,\beta], \tag{D.38b}$$

where

$$\Delta_1(\mu',\mu) = w_1(\mu')/(\mu'-\mu)(\nu_0-\mu)X_1(\mu), \tag{D.39}$$

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[5] In the half-range case, it is well known [see Ref. (38)] that the incident distribution and the behavior at infinity uniquely specify the solution.

and the weight function is

$$w_1(\mu) = (\nu_0-\mu)(c\mu/2)[X_1^+(\mu)/\Lambda^+(\mu)]. \qquad (D.40)$$

Substituting the expression for $\Delta_1(\mu',\mu)$ into equation (D.38b), we find that

$$\psi_c(\mu) = \psi(\mu) - [\psi(\mu)/(\nu_0-\mu)X_1(\mu)] + [1/(\nu_0-\mu)X_1(\mu)]\int_\alpha^\beta d\mu'[w_1(\mu')/(\mu'-\mu)][\psi(\mu') - \psi(\mu)]. \qquad (D.41)$$

Letting μ approach zero, we see that the integral in Eq. (D.41) approaches a finite limit, M.

$$\lim_{\mu\to 0}\psi_c(\mu) = \psi(0) - [\psi(0)/\nu_0] + (M/\nu_0). \qquad (D.42)$$

This result is not difficult to understand physically, at least for the half-range case. Consider a typical half-space problem, e.g., the Milne problem. There it is well known (21) that the solution to the neutron transport equation may have a discontinuity at $\mu = 0$, $x = 0$. Physically, this corresponds to the fact that there may be neutrons within the half-space flowing parallel to the interface (μ=0), but there are certainly no neutrons outside the medium moving in that direction.

D.5 The Time Dependent Case

A normal mode solution of the time dependent transport equation has been developed by Bowden (39) and is discussed also in Ref. (21). In this approach, the Laplace transform of the time dependent angular density, $\psi_s(x,\mu)$, is sought.

The definition of $\psi_s(x,\mu)$ is

$$\psi_s(x,\mu) = \int_0^\infty e^{(1-s)t}\psi(x,\mu,t)dt, \qquad (D.43)$$

so that the time independent case corresponds to s=1. The normal modes, $\phi_{\nu s}(\mu)$, are solutions of the equation,

$$(\nu-\mu)\phi_{\nu s}(\mu) = (c\nu/2s)\int_{-1}^{1}\phi_{\nu s}(\mu')d\mu'. \qquad (D.44)$$

The solutions to the various time independent problems may be taken over to the time dependent cases almost directly, if in the former we make the substitution, $c \rightarrow c/s$. We say "almost directly" because the two discrete roots, which are the zeros of the dispersion function,

$$\Lambda_s(\nu) = 1 - (c\nu/2s)\int_{-1}^{1}[d\mu/(\nu-\mu)], \qquad (D.45)$$

are present only for s contained in a certain region, S_i, of the complex s-plane [see Ref. (39) and Figure D.2]. That is,

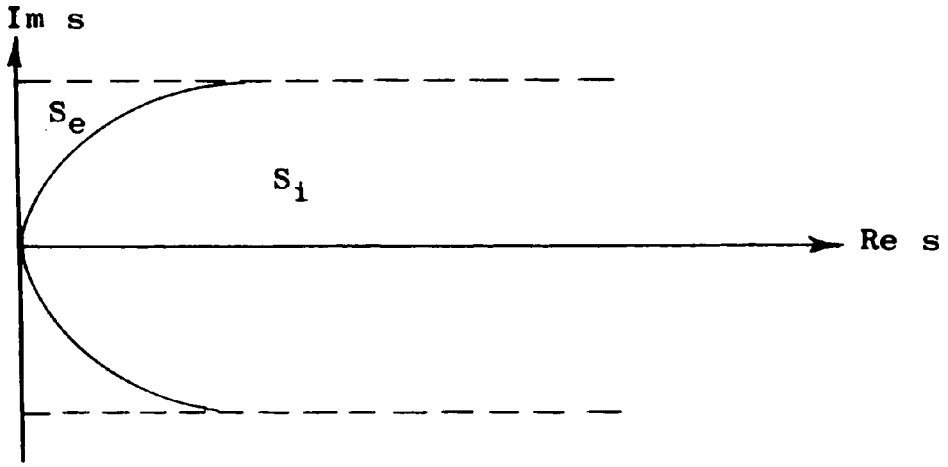

Figure D.2 The region S_i, in the complex s-plane, within which the dispersion function, $\Lambda_s(\nu)$, has two zeros; i.e., for $s \in S_i$, there are two discrete eigenfunctions; for $s \notin S_i$, there are none

for $s \in S_i$, there are two discrete roots and thus two discrete eigenfunctions; for $s \notin S_i$, there are only the continuum eigenfunctions.

The sort of completeness theorem that is useful in the solution of time dependent problems involves the possibility of expanding a function, $\psi(\mu)$, in terms of the $\phi_{\nu s}(\mu)$ for any value of s.

For $s \in S_i$, there is no problem. The function, $\theta_s(\mu)$, or rather $Re\theta_s(\mu)$ [since in the time dependent case $\theta_s(\mu)$ is complex], has essentially the same behavior as in the time independent case, and the various full and partial range completeness theorems, as well as the remarks on analytic continuation, go over directly.

However, for $s \notin S_i$, the situation is changed. There $Re\theta_s(\mu)$ has typically the behavior shown in Figure D.3 [see also Ref. (21)]. We see that the function, $X_{0s}(\mu)$ will be

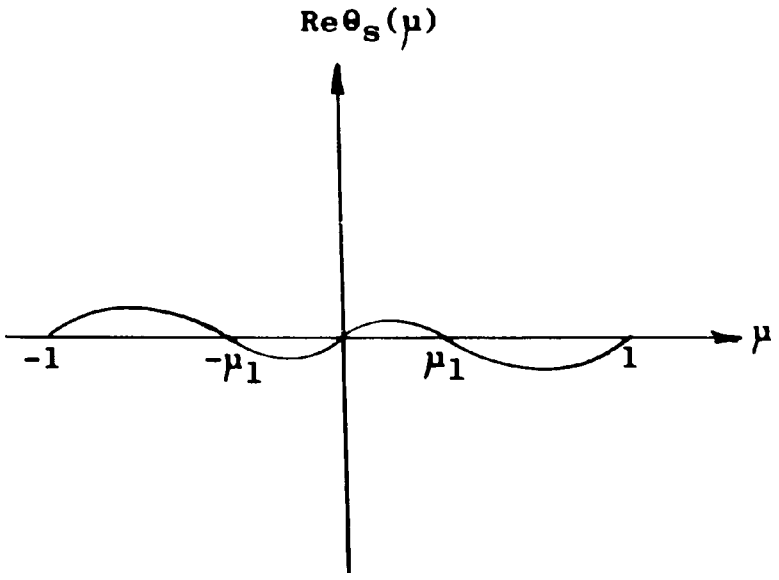

Figure D.3 $Re\theta_s(\mu)$ versus μ with $s \notin S_i$ for a typical case

constant in the vicinity of the endpoints -1, 0 or 1 [cf. Eqs. (D.13a) and (D.13b)]. However, for some values of α and/or β, the function, $X_{0s}(\mu)$, will have a fractional power zero. This behavior, as we have shown earlier, cannot be allowed without introducing an infinite discontinuity in $\psi_c(\mu)$. Thus the function, $X_{1s}(\mu)$ or $X_{2s}(\mu)$, must be used. But then $N_s(z)$ will have the wrong behavior at infinity and there are no discrete modes to correct this behavior. Thus a partial range theorem exists only for very special values of α and β for $s \notin S_i$; i.e., $-1 \leq \alpha \leq -\mu_1$, $\mu_1 \leq \beta \leq 1$, and α or $\beta = 0$.

APPENDIX E

MULTIGROUP TRANSPORT EQUATION WITH CONSTANT CROSS SECTIONS

It is well known (21) that the energy dependent neutron transport equation,

$$\underline{\Omega}\cdot\nabla\Psi(\underline{r},\underline{\Omega},E) + \Sigma_T(E)\Psi(\underline{r},\underline{\Omega},E) = \int\int\Psi(\underline{r},\underline{\Omega}',E')\Sigma(E'\rightarrow E,\underline{\Omega}'\rightarrow\underline{\Omega})d\underline{\Omega}'dE', \tag{E.1}$$

can be reduced to a one-speed transport equation for the integrated angular density,

$$\Psi(r,\underline{\Omega}) \triangleq \int\Psi(\underline{r},\underline{\Omega},E)dE,$$

in the case that the cross sections are independent of the initial neturon energy. [Note that the transfer cross section,

$$\Sigma(E' \longrightarrow E,\underline{\Omega}' \longrightarrow \underline{\Omega}),$$

includes the effects of elastic scattering, inelastic scattering, and fission.]

If the total cross section, $\Sigma_T(E)$, is independent of E and if $\Sigma(E' \longrightarrow E,\underline{\Omega}' \longrightarrow \underline{\Omega})$ can be approximated by a step function of E', then successive integrations of Eq. (E.1) over each of the ranges for which this quantity is independent of E' yields a set of coupled multigroup equations, each with the same total cross section (40). Even in the constant cross section case, when it is possible to obtain a one-group equation it may still be desirable to obtain the angular density as a function of energy.

In this appendix, we present a particularly simple method for solving the one-dimensional transport equation in the multigroup approximation, assuming the same total cross section in each group. This method reduces the N-group problem to the solution of N one-group equations.

The multigroup equations that we consider are written in the form,

$$\mu \frac{\partial}{\partial x} \underline{\Psi}(x,\mu) + \underline{\Psi}(x,\mu) = \underline{C} \int_{-1}^{1} \underline{\Psi}(x,\mu')d\mu' . \qquad (E.2)$$

Here $\underline{\Psi}(x,\mu)$ is an N-dimensional column vector whose i^{th} component represents the angular density in the ith group. $\underline{C}$ is a matrix whose elements c_{ij} give the transfer cross section from group j to group i.

In general, a similarity transformation exists that will reduce the matrix $\underline{C}$ to Jordan canonical form (41). Calling this matrix $\underline{S}$, the transformed problem becomes

$$\mu \frac{\partial}{\partial x} \underline{\Psi}'(x,\mu) + \underline{\Psi}'(x,\mu) = \underline{C}' \int_{-1}^{1} \underline{\Psi}'(x,\mu')d\mu' \qquad (E.3)$$

where

$$\underline{\Psi}'(x,\mu) = \underline{S}\, \underline{\Psi}(x,\mu) \qquad (E.4a)$$

and

$$\underline{C}' = \underline{S}\, \underline{C}\, \underline{S}^{-1} . \qquad (E.4b)$$

Since the canonical form of $\underline{C}'$ is triangular, the set of equations, Eq. (E.3), can be solved from the "bottom up." That is, the equation for $\psi_N(x,\mu)$ is a homogeneous one-group equation whose solution is well known (29). This result can

then be substituted into the equation for $\psi_{N-1}(x,\mu)$, which becomes an inhomogeneous one-group equation whose solution is equally well known (30). In this way, the entire set can be solved trivially; application of the inverse matrix, S^{-1}, to the calculated $\underline{\Psi}'(x,\mu)$ yields the desired result.

As an example, we consider the two-group albedo problem. Let us assume for simplicity that the eigenvalues of the matrix $\underline{C}$ are distinct. Then the matrix $\underline{C}$ can actually be diagonalized by a similarity transformation with

$$\underline{S} = 1/(\lambda_1-\lambda_2)\begin{bmatrix}(\lambda_1-c_{22})/c_{12} & 1\\ -(\lambda_2-c_{22})/c_{12} & -1\end{bmatrix}. \qquad \text{(E.5)}$$

We also find

$$\underline{S}^{-1} = \begin{bmatrix}c_{12} & c_{12}\\ \lambda_1 - c_{11} & \lambda_2 - c_{11}\end{bmatrix}. \qquad \text{(E.6)}$$

Here λ_i are the eigenvalues of the matrix $\underline{C}$; i.e.,

$$\lambda_1 = \tfrac{1}{2}[\mathrm{Tr}\,\underline{C} + \sqrt{(\mathrm{Tr}\,\underline{C})^2 - 4\ \det\,\underline{C}}] \qquad \text{(E.7a)}$$

and

$$\lambda_2 = \tfrac{1}{2}[\mathrm{Tr}\,\underline{C} - \sqrt{(\mathrm{Tr}\,\underline{C})^2 - 4\ \det\,\underline{C}}]. \qquad \text{(E.7b)}$$

The transformed equation now reduces to two one-group equations with the value of "c" = $2\lambda_1, 2\lambda_2$, respectively, in the notation of Ref. (29). Specifically we have

$$\mu \frac{\partial}{\partial x}\psi_1'(x,\mu) + \psi_1'(x,\mu) = \lambda_1 \int_{-1}^{1}\psi_1'(x,\mu')d\mu' \qquad \text{(E.8a)}$$

and

$$\mu \frac{\partial}{\partial x}\psi_2'(x,\mu) + \psi_2'(x,\mu) = \lambda_2 \int_{-1}^{1}\psi_2'(x,\mu')d\mu'. \qquad \text{(E.8b)}$$

The boundary conditions are (in analogy with the one-speed problem, we are assuming that the medium is neither multiplying nor conservative)

$$\lim_{x \to \infty} \underline{\Psi}(x,\mu) = \underline{0} \tag{E.9a}$$

and

$$\underline{\Psi}(0,\mu) = \begin{bmatrix} \delta(\mu-\mu_1) \\ \delta(\mu-\mu_2) \end{bmatrix}, \quad \mu > 0. \tag{E.9b}$$

The transformed boundary conditions thus are

$$\lim_{x \to \infty} \underline{\Psi}'(x,\mu) = \underline{0} \tag{E.10a}$$

and

$$\underline{\Psi}'(0,\mu) = \underline{S} \begin{bmatrix} \delta(\mu-\mu_1) \\ \delta(\mu-\mu_2) \end{bmatrix}, \quad \mu > 0. \tag{E.10b}$$

Explicitly, Eq. (E.10b) gives

$$\Psi_1'(0,\mu) = [1/(\lambda_1-\lambda_2)]\{[(\lambda_1-c_{22})/c_{12}]\delta(\mu-\mu_1) + \delta(\mu-\mu_2)\}, \tag{E.11a}$$

$$\mu > 0$$

and

$$\Psi_2'(0,\mu) = -[1/(\lambda_1-\lambda_2)]\{[(\lambda_2-c_{22})/c_{12}]\delta(\mu-\mu_1) + \delta(\mu-\mu_2)\}, \tag{E.11b}$$

$$\mu > 0.$$

Eqs. (E.8), subject to conditions (E.10), are easily solved, using the method of Ref. (29). The results are

$$\Psi_i'(x,\mu) = A_{i+}\phi_{i+}(\mu)e^{-x/\eta_{i0+}} + \int_0^1 A_i(\eta)\phi_{i\eta}(\mu)e^{-x/\eta}d\eta, \tag{E.12}$$

$$i=1,2,$$

$$A_{i+} = -[1/\lambda_i \eta_{i0} X_i(\eta_{i0})][S_{i1}\gamma_i(\mu_1) + S_{i2}\gamma_i(\mu_2)] \qquad \text{(E.13a)}$$

and

$$A_i(\eta) = [1/(\eta_{i0}-\eta)\gamma_i(\eta)\Lambda_i^+(\eta)\Lambda_i^-(\eta)][S_{i1}(\eta_{i0}-\mu_1)\gamma_i(\mu_1)\phi_{i\eta}(\mu_1) + S_{i2}(\eta_{i0}-\mu_2)\gamma_i(\mu_2)\phi_{i\eta}(\mu_2)]. \qquad \text{(E.13b)}$$

The desired result is obtained from

$$\underline{\psi}(x,\mu) = \underline{S}^{-1}\underline{\psi}'(x,\mu). \qquad \text{(E.14)}$$

Thus

$$\psi_i(x,\mu) = S_{i1}^{-1}[A_{1+}\phi_{1+}(\mu)e^{-x/\eta_{10}} + \int_0^1 A_1(\eta)\phi_{1\eta}(\mu)e^{-x/\eta}d\eta] + S_{i2}^{-1}[A_{2+}\phi_{2+}(\mu)e^{-x/\eta_{20}} + \int_0^1 A_2(\eta)\phi_{2\eta}(\mu)e^{-x/\eta}d\eta], \qquad \text{(E.15)}$$

$$i=1,2.$$

APPENDIX F

NUMERICAL ATTEMPTS

The purpose of this appendix is to illustrate, by means of a simple example, the difficulties (alluded to in Chapter 1) involved in attempting to solve numerically the nongray radiative transfer problem. We take as our model an absorption coefficient proportional to the frequency, ν.

The transfer equation and the Schwarzschild condition take the forms

$$\mu \frac{\partial \psi_\nu}{\partial x} + \nu \psi_\nu(x,\mu) = \nu B_\nu[T(x)] \tag{F.1}$$

and

$$\int_0^\infty d\nu \nu B_\nu[T(x)] = \frac{1}{2}\int_0^\infty d\nu \nu \int_{-1}^{1} \psi_\nu(x,\mu')d\mu', \tag{F.2}$$

respectively. Here we use the optical variable,

$$dx \triangleq \rho(z)a dz, \tag{F.3}$$

where

$$K_\nu \triangleq a\nu. \tag{F.4}$$

The left-hand side of Eq. (F.2) can be evaluated. We find

$$\int_0^\infty d\nu \nu B_\nu[T(x)] = \sigma_1 T^5(x), \tag{F.5}$$

where

$$\sigma_1 = (2h/c^2)(4!)(k/h)^4 \mathcal{S}_5. \tag{F.6}$$

Here

$$\mathcal{S}_N = \sum_{\alpha=1}^{\infty} (1/\alpha^N). \tag{F.7}$$

Eq. (F.2) now takes the form,

$$\sigma_1 T^5(x) = \frac{1}{2}\int_0^\infty d\nu\, \nu \int_{-1}^{1} \psi_\nu(x,\mu')d\mu'. \qquad (F.8)$$

Our iteration procedure is as follows. First, in Eq. (F.1) we assume some initial temperature distribution (the gray case provides a good starting point). Knowing then $B_\nu[T(x)]$, we solve Eq. (F.1) for $\psi_\nu(x,\mu)$; i.e.,

$$\psi_\nu(x,\mu) = \int_0^x (dt/\mu)B_\nu[T(t)]e^{-(x-t)/\mu},\ \mu > 0 \qquad (F.9a)$$

$$\psi_\nu(x,\mu) = -\int_x^\infty (dt/\mu)B_\nu[T(t)]e^{-(x-t)/\mu},\ \mu < 0. \qquad (F.9b)$$

Substituting Eqs. (F.9) into the right-hand side of Eq. (F.8) we find a new temperature distribution. We now use this temperature distribution in Eqs. (F.9) and thus iterate.

We put the above problem on the IBM 7090 computer and attempted to find the resultant solution for the temperature distribution. The results are shown schematically in Figure F.1. The solution, while at first appears to be quite good, does not converge at any reasonable rate that would make this procedure beneficial. It appears that the solution, after one given iteration, does not change very drastically and, in fact, the iteration reproduces very nearly the distribution that is originally taken.

From the work done here, we conclude that such a straightforward scheme cannot be very successful. This, of course, prompted us to look at the Picket Fence model from an analytical viewpoint.

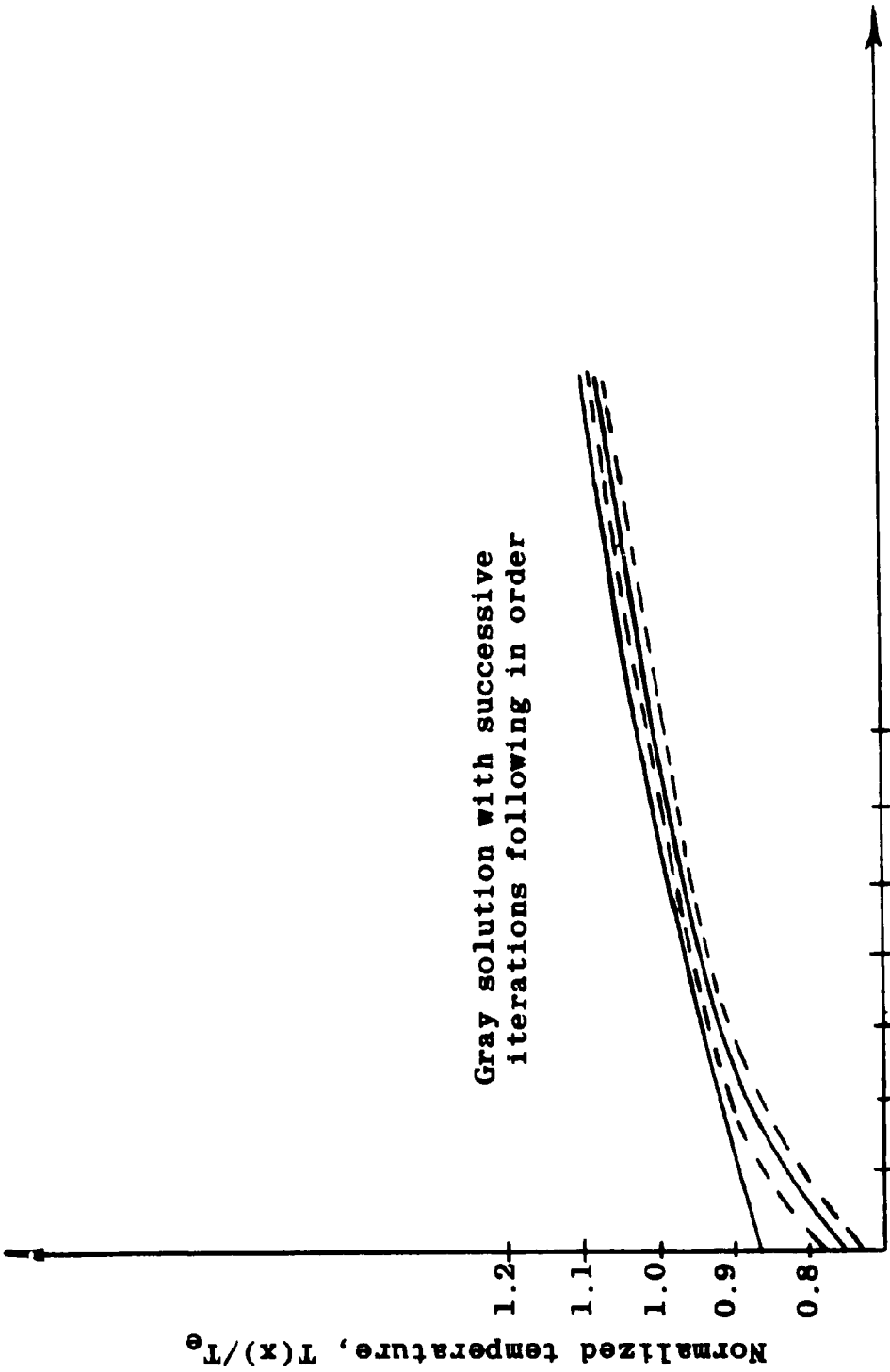

Figure F.1 Iteration results for nongray temperature distribution

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