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University of Michigan, Ph.D., 1961
Physics, general

University Microfilms, Inc., Ann Arbor, Michigan

THE KINETIC THEORY OF

FULLY IONIZED GASES

by
Ralph Lewis Guernsey

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in
The University of Michigan
1960

Doctoral Committee:

Professor George E. Uhlenbeck, Chairman
Professor Robert C. F. Bartels
Assistant Professor George W. Ford
Assistant Professor William C. Meecham
Professor Marcellus L. Wiedenbeck

ACKNOWLEDGEMENT

The author wishes especially to acknowledge the continued guidance of Professor G. E. Uhlenbeck, whose suggestions and criticisms inspired much of this work. Many thanks are also due to Dr. G. W. Ford for many helpful discussions. Finally sincere thanks are due to Mr. George Prosser and the UMRI Reports Office staff for the typing and reproduction, and to Mr. David Mage and Misses Kathleen Lathers, Elaine Kehrer and Priscilla Holt for writing in the equations.

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PART I

THE EQUILIBRIUM THEORY OF FULLY IONIZED GASES

CHAPTER I

INTRODUCTION

1. HISTORICAL BACKGROUND

The theory of ordinary gases in the equilibrium state may be said to have reached formal completeness with the development in 1937 of the Mayer theory of imperfect gases.[1,2] This theory gives the equation of state and all the thermodynamic functions as expansions in powers of the density, in which the coefficient (virial coefficients) are expressed in terms of the interatomic potential.

The analagous problem for ionized gases (or ionic solutions) presents considerably more difficulty because of the long-range character of the Coulomb interactions, and has only recently been treated with any degree of completeness. The first attack on this problem was made by Debye and Hückel,[3] whose method is discussed briefly in Section 2. This method gives the correct results in the limit of very low density, but cannot be generalized to obtain higher approximations, and is based on some rather dubious assumptions.

The first attempt to obtain corrections to the Debye-Hückel limiting law was by Kramers,[4] as later modified by Berlin and Montroll.[5] These authors use a "charge-averaging" process, in which the charge of each ion is treated as a continuous variable, and the distribution in these variables considered to be Gaussian. The Berlin and Montroll results come closer to experiment than Debye-Hückel (to which they reduce

at low density) but the improvement is not great, and the model does not appear to approximate physical reality under any conditions except extremely low density, when Debye-Hückel will do as well.

A theory offering more hope of success was presented by Bogolubov[6] in 1946, though it was not carried beyond Debye-Hückel by this author. This method involves the solution of differential equations for the distribution functions; a modification of the theory is presented in some detail in Chapter III.

In 1950 Mayer[7] obtained a complete formal solution of the problem by means of a rearrangement of terms in his original cluster integral expansion. Unlike earlier authors, he also included short range "ordinary gas" interactions. This work, together with some explicit calculations and the corresponding results of the distribution functions is presented in Chapter II.

2. THE DEBYE-HÜCKEL THEORY*

We consider a gas consisting of ν types of ions, in which there are N_σ ions of type σ, each having charge Z_σ. We suppose from the outset that the gas is neutral (which is certainly necessary for equilibrium in the limit of infinite volume), so that:

$$\sum_\sigma Z_\sigma N_\sigma = 0 \qquad (1.1)$$

Further if the deviations from the perfect gas are assumed to be slight, the average Coulomb energy must be much less than the average

*The presentation follows very closely that of Landau and Lifschitz, Ref. 8.

kinetic energy. This is equivalent to

$$e^2/r^{1/3} << kT \tag{1.2}$$

where r is the average specific volume. We will give another interpretation of (1.2) at the end of this section.

We wish to determine the contribution of the Coulomb interaction to the average internal energy of the gas (denoted by U_{Coul}). One can easily show that this is given by

$$U_{Coul} = \frac{e}{2} \sum_{\sigma} Z_{\sigma} N_{\sigma} \psi_{\sigma} \tag{1.3}$$

where ψ_{σ} is the average electric potential acting on a particle of type σ due to all the other ions in the gas.

In order to calculate ψ_{σ} we first consider the distribution of charge about a given ion of type σ. Let the density of ions of type σ' at a vector distance $\vec{r}$ from the given ion be* $n_{\sigma'}(r, \sigma)$, where $r = |\vec{r}|$. Then, following Debye, we make the following assumptions:

(a) $$n_{\sigma'}(r,\sigma) = \frac{N_{\sigma'}}{V} \exp[-Z_{\sigma'} e \phi(r,\sigma)/kT] \tag{1.4}$$

where $\phi(r,\sigma)$ is the average electric potential "felt" by a charge a distance r from a given charge of type σ, and

(b) $\phi(r,\sigma)$ satisfies Poisson's equation

$$\Delta \phi(r,\sigma) = -4\pi e \sum_{\sigma'} Z_{\sigma'} n_{\sigma'}(r,\sigma) - 4\pi e Z_{\sigma} \delta(r) \tag{1.5}$$

If (1.4) is substituted into (1.5), the equation becomes

*Except for normalization $n_{\sigma}(r,\sigma)$ is just the two particle distribution function; $\frac{n_{\sigma'}(r,\sigma)}{N_{\sigma'}}$ must be symmetric in σ, σ'.

$$\Delta\phi(r,\sigma) = \frac{-4\pi e}{V}\sum_{\sigma'} Z_{\sigma'} N_{\sigma'} \exp\left[-\frac{Z_{\sigma'} e\phi(r,\sigma)}{kT}\right] - 4\pi e Z_\sigma \delta(\vec{r}) \qquad (1.6)$$

This equation should give $\phi(r,\sigma)$, but is in general difficult to solve. If one makes the further assumption, which is of the same nature as (1.2)

$$\left| Z_\sigma e\phi(r,\sigma)/kT \right| << 1 \qquad (1.7)$$

and expands the exponential in (1.6) one finds

$$\Delta\phi(r,\sigma) = \frac{-4\pi e}{V}\sum_{\sigma'} Z_{\sigma'} N_{\sigma'} + \frac{4\pi e^2}{kTV}\sum_{\sigma'} Z_{\sigma'}^2 N_{\sigma'} \phi(r,\sigma) + \ldots . \qquad (1.8)$$
$$-4\pi Z_\sigma e\delta(r)$$

The first term vanishes by (1.1), and (1.8) becomes approximately

$$\Delta\phi(r,\sigma) = K^2\phi(r,\sigma) - 4\pi Z_\sigma e\,\delta(r) \qquad (1.9)$$

where $$K^2 = \frac{4\pi e^2}{kTV}\sum_\sigma Z_\sigma^2 N_\sigma \qquad (1.10)$$

The solution of (1.9 which vanishes at ∞ is

$$\phi(r,\sigma) = \frac{Z_\sigma e}{r}\exp[-Kr] \qquad (1.11)$$

According to (1.11), a charge a distance R from a given ion of type σ "feels," on the average, not the infinite range Coulomb potential $Z_\sigma e/R$ but a "screened" or "damped" Coulomb potential with an effective range of $\frac{1}{K}$, due to the shielding effect of the other ions in the gas. Thus $1/K$ is sometimes known as the Debye shielding radius, and $\exp[-KR]$ as the Debye damping factor. This "Debye damping," which removes the divergences due to the infinite range of the Coulomb field, must be a feature of any theory of fully ionized gases (or ionic solutions).

In the immediate neighborhood of the given charge, one may expand the exponential in (1.11)

$$\phi(R,\sigma) = \frac{Z_\sigma e}{R} - Z_\sigma e\kappa + \dots$$

The first term represents the potential due to the given ion. The second is clearly the potential due to all the other ions at the given charge, which is by definition ψ_σ .

Thus

$$\psi_\sigma = -Z_\sigma e\kappa \tag{1.12}$$

and substituting in (1.3)

$$U_{Coul} = -\frac{1}{2}\kappa e^2 \sum_\sigma Z_\sigma^2 N_\sigma = -\frac{\kappa^3}{8\pi} V kT \tag{1.13}$$

By integrating the thermodynamic relation

$U/T^2 = -\frac{\partial}{\partial T}\left(\frac{F}{T}\right)$, one finds for the correction to the Helmholtz free energy

$$F_{Coul} = -\frac{\kappa^3}{12\pi} V kT \tag{1.14}$$

and $$p = -\frac{\partial F}{\partial V} = kT\left(\sum_\sigma \frac{N_\sigma}{V} - \frac{\kappa^3}{24\pi}\right) \tag{1.15}$$

The second term in (1.15) will be small compared to the first, provided

$$(1/\kappa) >> v^{1/3} \tag{1.16}$$

with $$v = \frac{V}{\sum_\sigma N_\sigma} = \frac{V}{N} \tag{1.17}$$

From the definition (1.10 of κ , since the Z's are of order unity, this is equivalent to (1.2). Thus there are two length parameters

associated with the Coulomb field: e^2/kT , which is the approximate distance at which the absolute value of the Coulomb potential becomes equal to the thermal energy, and $1/\kappa$, the distance at which the effective potential is cut off. In order for our results to be approximately valid the first must be much less than the average distance between ions, and the second much greater.

We now make a few remarks about Debye's other assumptions. As noted before, $n_{\sigma'}(r,\sigma)/N_\sigma$, is the two particle distribution function, and must be symmetric under interchange of σ, σ'. Comparing with (1.4) this means that any $\phi(r,\sigma)$ which satisfies the assumption (a) must have the form

$$\phi(r,\sigma) = Z_\sigma e\, f(r) \tag{1.18}$$

where $f(r)$ is independent of σ. The approximate solution (1.11) has this form, as is already obvious from the approximate differential equation (1.9). However, the general solution of (1.6) does not have this form, so the assumptions (a) and (b) are really inconsistent, and one cannot logically go beyond the Debye result.

Note further that the assumption (1.7) is not equivalent to (1.2), and obviously cannot hold for all r . If we put in the result (1.11) for $\phi(r,\sigma)$, (1.17) can only hold for

$$r >> |Z_\sigma Z_{\sigma'} e^2/kT| \tag{1.19}$$

Thus it is really incorrect to use (1.11) in the neighborhood of the origina, as was done in calculating ψ_σ.

The nature of the difficulty is shown even more clearly if we use another method of calculating ψ_σ. By definition, ψ_σ is given by:

$$\psi_\sigma = \int d\vec{r} \sum_{\sigma'} \frac{Z_{\sigma'} e}{r} n_{\sigma'}(r, \sigma) \tag{1.20}$$

Using (1.4), (1.11) for $n_{\sigma'}(r, \sigma)$ this becomes

$$\psi_\sigma = e \int \frac{d\vec{r}}{r} \sum_{\sigma'} Z_{\sigma'} \frac{N_{\sigma'}}{V} \exp\left[- Z_\sigma Z_{\sigma'} e^2 \exp(-\kappa r)/r \right] \tag{1.21}$$

If one expands the exponential and keeps only the first two terms, the result (1.12) is obtained. However, the integrals over the remaining terms diverge! This difficulty is more apparent than real for the terms in the σ' sum for which $Z_\sigma Z_{\sigma'} > 0$, since the integrand then has the form $\frac{\exp(-\alpha r)}{r}$, $(\alpha > 0)$ in the neighborhood of the origin, and thus vanishes rapidly as $r \to 0$. On the other hand, the terms for which $Z_\sigma Z_{\sigma'} < 0$ (attractive potential) definitely diverge. This is not surprising, since it is well known that an equilibrium cannot be established for a system of point charges. These divergences at short range must be removed by introducing the usual short range repulsive interactions. For example, if we replace (1.11) by

$$\begin{aligned} \phi(r, \sigma) &= \frac{Z_\sigma e}{r} \exp(-\kappa r) && r > d_0 \\ &= \infty && r < d_0 \end{aligned} \tag{1.22}$$

with

$$\frac{e^2}{kT} \leq d_0 << 1/\kappa \tag{1.23}$$

then (1.21) gives the Debye results plus corrections of order $\left(\frac{e^2\kappa}{kT}\right)$, $\left(\frac{e^2\kappa}{kT}\right)\left(\frac{e^2\kappa}{kTd_0}\right)$, etc., times the Debye results.

The range of validity of the results (1.13)—(1.15) is thus seen

to depend on the "size" of the ions as well as the temperature and density. However, one may make a general statement that these equations give the correct behavior in the limit of infinite dilution (zero density).(1.15) is thus known as the Debye-Hückel limiting law.

3. SOME ORDERS OF MAGNITUDE

Throughout the dissertation we assume that we are dealing with a fully ionized gas. To obtain some idea of the validity of this assumption, we note that from the law of mass action or Saha Formula it follows that the degree of ionization is given by:

$$\alpha = \frac{1}{\sqrt{1+K}}$$

with:

$$K \cong 4\times10^{-13}\frac{N}{V}T^{-3/2}\exp\left(\frac{1.16\times10^{4}I}{T}\right)$$

where N/V is in $(cm)^{-3}$, T in °K and the (first) ionization energy I is in e.V. This shows that for temperatures of 10^{6}°K or higher and densities of less than 10^{20} all gases will at least be once ionized. Even at 10^{5}°K this is the case for densities less than 10^{16}. For a deuterium plasma under these circumstances it is therefore completely justified to consider the number of ions and electrons as fixed. For heavier gases this will not be true for temperature regions where kT becomes comparable with higher ionization energies. However, because of the exponential dependence these regions will be narrow, so that usually the assumption of fixed numbers of ions will be a good approximation.

As mentioned in the last section, there are two characteristic lengths associated with the Coulomb field, e^2/kT and $1/\kappa$. Taking

$1/\kappa \cong \left(\frac{kTV}{4\pi e^2 N}\right)^{1/2}$, one gets: $1/\kappa = 5.5\, T^{1/2}/(N/V)^{1/2}$

$$e^2/kT = 1.3 \times 10^{-13}\, T^{-1}$$

If short range forces are introduced another length (or lengths) appear, the collision diameter d_o. For the validity of all successive approximation methods discussed in this thesis, it is necessary that $e^2\kappa/kT$ and $2\pi d_o^3 N/V$ are small. In Table I, we give for a few temperatures and densities some representative numbers, using $d_o = 5 \times 10^{-8}$ cm.

TABLE I

T	N/V	e^2/kT	$1/\kappa$	$e^2\kappa/kT$	$\frac{2\pi d_o^3 N}{V}$
10^5	10^{14}	1.3×10^{-8}	1.7×10^{-4}	7.7×10^{-5}	7.8×10^{-8}
	10^{16}		1.7×10^{-5}	7.7×10^{-4}	7.8×10^{-6}
	10^{18}		1.7×10^{-6}	7.7×10^{-3}	7.8×10^{-4}
10^6	10^{14}	1.3×10^{-9}	5.5×10^{-4}	2.4×10^{-6}	
	10^{16}		5.5×10^{-5}	2.4×10^{-5}	
	10^{18}		5.5×10^{-6}	2.4×10^{-3}	

Note that for these temperatures e^2/kTd_o is of order one. For a deuterium plasma d_o is of order 10^{-13}, hence then $e^2/kTd_o \gg 1$. This leads to a peculiar difficulty, which will be discussed in Chapter II. Finally we remark that for an ionic solution both the ionization potential and the Coulomb force are reduced by the dielectric constant of the solvent.

Since for water this is around 85, even at room temperatures and for $N/V = 10^{18}$ there is complete ionization and $e^2\kappa/kT << 1$ so that the Debye theory is applicable.

4. PLAN OF THE PRESENT WORK

Part I of the dissertation is aimed at developing the equilibrium theory of a fully ionized gas in such a way that it may readily be generalized to non-equilibrium situations. In Chapter II the work of Mayer is reviewed and generalized to include expressions for the distribution functions. In addition some explicit calculations are carried out for the special case when the short range forces are of the hard sphere type. Chapter III is devoted to the development of a new method which uses the summation technique of Mayer in solving the differential equations of Bogolubov.

Part II of the dissertation deals with the non-equilibrium theory of a fully ionized gas, starting from the Liouville equation. In Chapter IV the general formalism is developed, and the zeroth and first approximations obtained and discussed. Chapter V treats the important special case of the uniform state. It is shown that the H theorem is satisfied for the resulting kinetic equation. Further this equation (of Fokker-Planck type) is related to a Boltzmann equation with a momentum-dependent pseudo-potential.

CHAPTER II

THE MAYER THEORY

1. SOME RESULTS FROM THE THEORY OF GAS MIXTURES

Mayer's theory of imperfect gases may be generalized to include mixtures of an arbitrary number of types. To express these results, we need some definitions from the theory of graphs.[2]

A linear graph is any collection of points and lines such that each pair of points is either joined by a line or not joined by a line. A doubly-labelled linear graph is a linear graph in which the points are divided into ν subsets called "types" and the points of each type are numbered. Each point is thus distinguished by a pair of numbers, denoting its type and its number within the type. A path between any two points is a continuous chain of lines (or a single line) joining them. A connected graph is one for which there is at least one path joining each pair of points. An articulation point is a point of a graph such that if all lines are disconnected from this point, the graph is divided into two or more disjoint connected parts. A star is a connected graph with no articulation points.

For the expression of the distribution functions one further labelling is necessary. Here a set of points are distinguished as roots. The points of the graph are thus divided into two classes (roots and non-roots), which are in turn divided into subsets (types), and each subset is numbered among itself. The labelling of the roots is always

considered as fixed. By the set (k_σ) of roots we shall always mean a particular set of k_1 roots of type 1,... k_ν roots of type ν . A simple $P_{(k_\sigma)}$ graph is a rooted graph with the set (k_σ) of roots, such that: (a) it contains no lines joining the root points; (b) it becomes a star if these lines are added and (c) each pair of non-root points is connected by at least one path containing no root points.

For a gas containing N_σ particles of type σ ($\sigma = 1, \ldots \nu$) the Helmholtz free energy is defined by

$$F = -kT \ln Z \tag{2.1}$$

with

$$Z = Q\left(\prod_\sigma \frac{\lambda_\sigma^{-3N_\sigma}}{N_\sigma !}\right) \qquad \lambda_\sigma = \frac{h}{(2\pi m_\sigma kT)^{1/2}}$$

$$Q = \int_V \cdots \int d_N(\vec{r}_N) \exp\left(-\phi(\vec{r}_N)/kT\right) \tag{2.2}$$

Here we have used the further abbreviations:

$$d_N(\vec{r}_N) = \prod_{\sigma=1}^{\nu} \prod_{i=1}^{N_\sigma} d\vec{r}_i^{\,\sigma}$$

$$\phi(\vec{r}_N) = \sum_{\sigma \le \tau \le \nu} \sum \sum_{i=1}^{N_\sigma} \sum_{j=1}^{N_\tau} \phi_{ij}^{\sigma\tau} \left[1 - \tfrac{1}{2}\delta_{\sigma\tau}(1+\delta_{ij})\right] \tag{2.3}$$

For the total volume element and total potential energy; $\phi_{ij}^{\sigma\tau} = \phi^{\sigma\tau}(|\vec{r}_i^{\,\sigma} - \vec{r}_j^{\,\tau}|)$ is the potential between the i^{th} particle of type σ and the j^{th} particle of type τ . Similarly, in the normalization we shall use, the space distribution function for s_1 particles of type 1,...

S_ν particles of type ν is defined by

$$n_{(s\sigma)}(\vec{r}_s) = \frac{V^s}{N^s} \prod_\sigma N_\sigma^{s_\sigma} \frac{\int \cdots \int d_{N-s}(\vec{r}_{N-s}) \exp\{-\phi(\vec{r}_N)/kT\}}{Q} \quad (2.4)$$

where $S = \sum_{\sigma=1}^{\nu} S_\sigma$ and $d_{N-s}(\vec{r}_{N-s}) = \prod_{\tau=1}^{\nu} \prod_{i=S_\tau+1}^{N_\tau} d\vec{r}_i^{\,\tau}$

Following Mayer, we define

$$f_{ij}^{\sigma\tau} = \exp[-\phi_{ij}^{\sigma\tau}/kT] - 1 \quad (2.5)$$

In terms of these, one can show

$$F = -kT\left\{\sum_\sigma N_\sigma(1 + v_\sigma/\lambda_\sigma^3) + V\sum_{k} \beta_{(k\sigma)} \prod_\sigma v_\sigma^{-k_\sigma}\right\} \quad (2.6)$$

where $v_\sigma = \frac{V}{N_\sigma}$ and the sum goes over all sets of values of $k_1, \ldots k_\nu$ such that $\sum_\sigma k_\sigma \geq 2$. Finally

$$\beta_{(k\sigma)} = \frac{1}{V \prod_\sigma k_\sigma!} \int \cdots \int d_k(\vec{r}_k) \sum_{(S_k)} \prod f_{ij}^{\lambda\mu} \quad (2.7)$$

Here the sum is over all different doubly labelled stars containing k_1 points of type 1..., k_ν points of type ν and the product is over all lines in a given star $S_{(k\sigma)}$, where a factor of $f_{jk}^{\lambda\mu}$ is included if there is a line joining j^{th} point of type λ and the k^{th} point of type μ in the given star. For the distribution functions one obtains:

$$m_{(s_\sigma)}(\vec{r}_s) = W_{(s_\sigma)} \exp\left\{\sum_{(k_\sigma)} C_{(s_\sigma)}^{h_{(k_\sigma)}(\vec{r}_k)}\right\} \qquad (2.8)$$

where

$$W_{(s_\sigma)} = \prod_\sigma \left(\frac{N_\sigma}{N}\right)^{s_\sigma} \exp\left\{-\Phi(\vec{r}_s)/kT\right\} \qquad (2.9)$$

The round sum in (2.8) is over all subsets $\left(\sum_\sigma k_\sigma \geq 2\right)$ of the given set (s_σ) and $h_{(k_\sigma)}$ is a function of the $\sum_\sigma k_\sigma$ variables $\vec{r}_1' \ldots \vec{r}_k'$; ... ...; $\vec{r}_1^{\,\nu} \ldots \vec{r}_{k_\nu}^{\,\nu}$ given by:

$$h_{(k_\sigma)} = \sum_{\sum_\sigma l_\sigma \geq 1} C_{(k_\sigma, l_\sigma)} \prod_\sigma \nu_\sigma^{-l_\sigma} \qquad (2.10)$$

$$C_{(k_\sigma l_\sigma)} = \int \cdots \int \frac{d_{\underline{l}}(\vec{r}_{\underline{l}})}{\prod_\sigma l_\sigma!} \sum_{(P(k_\sigma, l_\sigma))} \prod f_{ij}^{\lambda\mu} \qquad (2.11)$$

Here the sum is over all graphs $P(k_\sigma, l_\sigma)$ which are simple $P(k_\sigma)$ graphs with the (fixed) set (k_σ) of roots and l_1 non-roots of type 1,..., l_ν non-roots of type ν, and the product is over all lines in a given graph $P(k_\sigma, l_\sigma)$. The proof of (2.6) to (2.11) may be accomplished by a slight generalization of the methods of Ref. 2. Either (2.6) or (2.8) is sufficient in principle to obtain all the thermodynamic funtions of a gas mixture for which the interatomic potentials are known. However,

the case of long-range forces again presents special difficulties which require special methods. Mayer's treatment of the problem is presented in the next section.

2. THE MAYER SUMMATION TECHNIQUE: THE FREE ENERGY

If one considers $\phi_{ij}^{\sigma\tau}$ to be a Coulomb potential $\phi_{ij}^{\sigma\tau} = \frac{Z_\sigma Z_\tau e^2}{|\vec{R}_i^\sigma - \vec{r}_j^\tau|}$ the integrals $B_{(k\sigma)}, C_{(k\sigma, l\sigma')}$ do not converge. However, Mayer[7] has shown how it is possible to rearrange the sums* in (2.6) so as to obtain an expansion in which each coefficient is a convergent integral.

For this purpose Mayer introduces, instead of a pure Coulomb potential, the sum of a damped Coulomb potential and a short-range potential

$$\phi_{ij}^{\sigma\tau} = 4\pi e^2 Z_\sigma Z_\tau \phi(|\vec{R}_i^\sigma - \vec{r}_j^\tau|) + \psi_{\sigma\tau}(|\vec{R}_i^\sigma - \vec{r}_j^\tau|) \qquad (2.12)$$

where

$$\phi(R) = \frac{e^{-\alpha R}}{4\pi R} \qquad (\alpha > 0) \qquad (2.13)$$

and $\psi_{\sigma\tau}$ is a short-range potential satisfying

$$\psi_{\sigma\tau}(R) = 0 \qquad R > a_{\sigma\tau} \qquad (2.14)$$

If (2.12) is put in (2.5), the integrals all converge. However, in the limit of zero damping ($\alpha \to 0$) there is again a divergence. To remove

*Note that each $\beta_{(k\sigma)}$ is itself a sum of terms: Mayer's method consists in picking out certain terms from each β and summing these first.

this, we first expand the functions as follows:

$$f_{ij}^{\sigma\tau} = f^{\sigma\tau}(|\vec{r}_i^{\sigma} - \vec{r}_j^{\tau}|)$$
$$f^{\sigma\tau}(R) = g_{\sigma\tau}(R) + [1 + g_{\sigma\tau}(R)]\sum_1^{\infty} \frac{(-Z_\sigma Z_\tau \lambda \phi(R))^m}{m!} \tag{2.15}$$

with

$$g_{\sigma\tau} = \exp[-\psi_{\sigma\tau}/kT] - 1 \tag{2.16}$$

$$\lambda = 4\pi e^2/kT \tag{2.17}$$

After this expansion it is clear that we may reinterpret (2.7) in terms of a different kind of doubly labelled graph. We consider graphs which have two types of lines: (a) dotted lines, or g-bonds, which may occur either zero or one times between any pair of points and correspond to a factor $g_{ij}^{\sigma\tau}$; (b) solid lines or ϕ-bonds which may occur any number of times between any pair of points and correspond to a factor:

$$\phi_{ij}^{\sigma\tau} = -Z_\sigma Z_\tau \lambda \phi(|r_i^{\sigma} - r_j^{\tau}|) \tag{2.18}$$

The n! can be considered as the symmetry number of the n-tuple ϕ-bond.

$$\beta_{(k\sigma)} = \frac{1}{V \prod_{\sigma} k_\sigma!} \int \cdots \int d_k(\vec{r}_k) \sum_{(\{g^{\lambda\mu}_{ij}\})} \prod \frac{g_{ij}^{\lambda\mu} (\phi_{ij}^{\lambda\mu})^{m_{ij}^{\lambda\mu}}}{m_{ij}^{\lambda\mu}!} \tag{2.19}$$

where the sum is over all "generalized" doubly labelled stars of the type described above, with the set $(\vec{r}_\sigma)$ of points.

To remove the convergence difficulties as $\alpha \to 0$, it is convenient to define* "prototype graphs" in which each linear chain of ϕ-bonds is replaced by a single solid line. Here we mean by a linear chain of ϕ-bonds a set of points successively joined by just one ϕ-bond. There are of course an infinite number of generalized stars corresponding to each prototype with the exception of type 1 defined below, differing from each other by the number of points in each of the linear chains of ϕ-bonds. It is easy to show that the summed contribution of integrals corresponding to the same prototype is convergent in the limit $\alpha \to 0$, and is found by associating with each solid line in the prototype a "damped Coulomb potential" or Debye-bond of the form $\frac{-Z_\sigma Z_\tau \lambda \exp(-\kappa R)}{4\pi R}$ where κ is given by (1.10).

In order to prove this, we need the following theorem:

Let

$$\phi_m(r_{ij}) = \int \cdots \int \prod_{k=1}^{m} d\vec{r}_k \, \phi(r_{i1}) \phi(r_{12}) \cdots \phi(r_{m-1,m}) \phi(r_{mj}) \tag{2.20}$$

where ϕ is given by (2.13). Then

$$\lim_{\alpha \to 0} \sum_{n=0}^{\infty} (-\lambda^2)^n \phi_n(R) = \frac{e^{-\kappa R}}{4\pi R} \equiv \phi_D(R) \tag{2.21}$$

The proof is straightforward, and may be found in Ref. 7.

*In the usual graph language, a "prototype" is a homeomorphic type with respect to ϕ-bonds.

We now turn to the evaluation of the various prototype contributions. There are two types which must be considered separately: (1) type 1 consisting of two points joined by a single ϕ-bond, and (2) type c consisting of a single cycle of ϕ-bonds.

(1) Type 1.—From (2.19) if $k_\sigma = k_\tau = 1$ and all other k's are zero,

$$\beta_{(k_\sigma)} \to Z_\sigma Z_\tau \lambda \int_0^\infty R e^{-\alpha R} dR = \frac{Z_\sigma Z_\tau \lambda}{\alpha^2}$$

To find the contribution to the sum:

$$\sum \equiv \sum_k \beta_k \prod_\sigma \nu_\sigma^{-k_\sigma} \tag{2.22}$$

occurring in the free energy F (see eq. (2.6)) one must multiply the result by $1/(\nu_\sigma \nu_\tau)$ and sum over all different pairs σ, τ, which gives

$$\sum_\ell = -\frac{\lambda}{2\alpha^2}\left(\sum_\sigma Z_\sigma \nu_\sigma^{-1}\right)^2 = 0 \tag{2.23}$$

by the assumption of electrical neutrality (Eq. (1.1)). Thus: the contribution of type 1 is zero due to electrical neutrality.

(2) Type c.—According to (2.19) the contribution to the sum $\sum$ of a given cycle containing n_1 points of type 1..., m_ν points of type ν is

$$\prod_\sigma (-\lambda Z_\sigma^2)^{m_\sigma} \phi_{m-1}(0) \nu_\sigma^{-m_\sigma}/m_\sigma!$$

where ϕ_{n-1} is defined by (2.20) and $n \equiv \sum_\sigma n_\sigma$. Since there are $(n-1)!/2$ different ways of arranging distinguishable points in a cycle, the total contribution of the cycles is

$$\Sigma_c = \sum_{n=2}^{\infty} \frac{(n-1)!}{2} \phi_{n-1}(0) \sum_{(n_\sigma)} \prod_\sigma \left(\frac{-\lambda Z_\sigma^2}{\nu_\sigma}\right)^{n_\sigma} \Big/ n_\sigma! \qquad \sum_\sigma n_\sigma = n$$

$$= \sum_{n=2}^{\infty} \frac{\phi_{n-1}(0)}{2n} \left[\sum_\sigma \left(\frac{-\lambda Z_\sigma^2}{\nu_\sigma}\right)\right]^n = \sum_{n=2}^{\infty} \frac{1}{2n} (-\kappa^2)^n \phi_{n-1}(0) \qquad (2.24)$$

Differentiating with respect to $(-\kappa^2)$ one finds

$$\frac{d\Sigma_c}{d(-\kappa^2)} = \frac{1}{2} \sum_{n=1}^{\infty} (-\kappa^2)^n \phi_n(0) = \frac{1}{2} \lim_{R \to 0} \left(\phi_0(R) - \phi(R) \right) \qquad (2.25)$$

$$= \frac{1}{2} \lim_{\substack{R \to 0 \\ \alpha \to 0}} \frac{\left(e^{-\kappa R} - e^{-\alpha R}\right)}{4\pi R} = -\frac{\kappa}{8\pi}$$

Therefore $\Sigma_c = -\kappa^3/12\pi$ and (2.26)

$$F_{Coul} = -kT V \Sigma_c = \frac{kT V \kappa^3}{12\pi} \qquad (2.27)$$

which is the Debye result (1.14). Thus one may state <u>the contribution of the cycles gives the Debye limiting law</u>.

(3) <u>All other types</u>.—Having disposed of the special cases (1), (2) we may treat all other types of graphs in a general fashion. A given

prototype is identified by the set (m_σ) of junctions in the prototype, the numbers μ_σ of ϕ-bond chain endings on type σ , with a total of $\mu = \frac{1}{2}\sum_\sigma \mu_\sigma$ chains, and the number $A[(\mu_\sigma, m_\sigma)]$ of different doubly labelled prototypes corresponding to the same free* prototype.**

For a given doubly labelled graph corresponding to such a prototype, with $m_{\sigma i}$ intermediate*** points of type σ in the ith g-bond chain,

$\sum_\sigma m_{\sigma i} = m_i =$ total number of intermediate points of type σ ,

we have a contribution

$$(-\lambda)^\mu \prod_\sigma Z_\sigma^{\mu_\sigma} \prod_i (-\lambda Z_\sigma^2)^{m_{\sigma i}} \frac{v_\sigma^{-(n_\sigma + m_\sigma)}}{(n_\sigma + m_\sigma)!} I[(m_\sigma, \mu_\sigma); (\phi_{m_i})]$$

where

$$I[(m_\sigma, \mu_\sigma); (\phi_{m_i})]$$

is the integral in which the ith ϕ-bond chain (joining say the jth prototype junction of type σ with the kth junction of type τ) is replaced by a factor of $\phi_{m_i}(x_j^\sigma, x_k^\tau)$. In addition one must multiply by the number $\prod_\sigma \left[\frac{(n_\sigma + m_\sigma)!}{m_\sigma! \prod_i m_{\sigma i}!}\right]$ of ways of choosing the set (m_σ) of junctions and the set($m_{\sigma i}$) of intermediate points from the total set of ($n_\sigma + m_\sigma$) points; the number $m_i!$ of different ways

---

*By free we shall mean each point is identified as to type but not numbered.

**We should also specify the number and arrangement of k-bonds; this however remains fixed throughout the argument.

***By intermediate points we shall mean points which are not junctions in the prototype, i.e., not in the set (m_σ).

of arranging the n_i points in the ith chain; and the number $A[\mu_\sigma, m_\sigma]$ of different doubly labelled prototypes corresponding to the given free prototype. Summing over all values of the ($n_{\sigma i}$) such that $\sum_\sigma n_{\sigma i} = n_i$ and then summing over all (n_i), one finds, using (2.21)

$$\Sigma_{(\mu_\sigma m_\sigma)} = (-\lambda)^\mu \Pi \left[\frac{Z_\sigma^{\mu_\sigma}}{m_\sigma! \nu_\sigma^{m_\sigma}}\right] A[(\mu_\sigma, m_\sigma)] I[(\mu_\sigma, m_\sigma); \phi_D] \quad (2.28)$$

where

$$I[(\mu_\sigma, m_\sigma); \phi_0] = \frac{1}{V}\int \cdots \int d_m(\vec{r}_m) \Pi g_{ij}^{\sigma\tau} \frac{(\phi_{D,ij})^{m_{ij}^{\sigma\tau}}}{m_{ij}^{\sigma\tau}!} \quad (2.29)$$

In (2.29) the product is over all lines of the prototype* which is again a generalized star, but where now each solid line corresponds to a Debye bond ϕ_D while each dotted line still corresponds to a g-bond. The total sum Σ is therefore:

$$\Sigma = \Sigma_\ell + \Sigma_c + \sum_{(\mu_\sigma m_\sigma)} \sum (\mu_\sigma, m_\sigma) \quad (2.30)$$

*Here the prototype is considered to be labelled but the result is independent of the labelling.

and

$$F = -kT\left\{\sum_\sigma N_\sigma\left(1+\frac{\nu_\sigma}{\lambda_\sigma^3}\right)+V\cdot\Sigma\right\} \tag{2.31}$$

so that all the thermodynamic properties of the gas are known if $\psi_{\sigma\tau}$ is known. In Section 4 we give some explicit calculations for a two type gas with $\psi_{\sigma\tau}$ of hard sphere form.

3. THE DISTRIBUTION FUNCTIONS

One may carry out a similar rearrangment of terms in (2,8)-(2.11) to find the distribution functions. Although these give essentially the same information as the free energy, they are of some interest for comparison with the non-equilibrium case, where the Gibbs theory does not apply and one must use distribution functions.

We consider the contribution of rooted prototypes to the effective potential

$$U_{(S_\sigma)} = -\ln\left[n_{(S_\sigma)}\prod_\sigma\left(\frac{N}{N_\sigma}\right)^{S_\sigma}\right] = \frac{\Phi_{(S_\sigma)}}{kT} - \sum_{(k_\sigma)C(S_\sigma)} h_{(k_\sigma)} \tag{2.32}$$

Here even the Debye contribution need not be treated as a special case. A given rooted prototype is specified by the roots (k_σ), the non-root junctions (l_σ), the ϕ-bond chain endings on type $\sigma(\mu_\sigma)$ with $\frac{1}{2}\sum_\sigma \mu_\sigma = \mu$ the total number of such chains, and the number $A[(k_\sigma, l_\sigma, \mu_\sigma)]$ of doubly labelled rooted prototypes corresponding to the same free rooted prototype. A particular graph corresponding

to such a prototype requires the additional specification of the numbers ($p_{\sigma i}$) of intermediate points in the ith chain ($\sum_\sigma p_{\sigma i} = p_i$). Similarly to (2.28), one finds*

$$U_{(k_\sigma,l_\sigma,\mu_\sigma)} = -(-\lambda)^\mu \prod_\sigma \frac{Z_\sigma^{\mu_\sigma}}{l_\sigma!\,\nu_\sigma^{l_\sigma}} \sum_{(p_i)} \sum_{\substack{(p_{\sigma\lambda}) \\ \sum_\sigma p_{\sigma\lambda}=p_\lambda}} \prod_\lambda \frac{p_\lambda!}{p_{\sigma\lambda}!}\left(\frac{-Z_\sigma^2\lambda}{\nu_\sigma}\right)^{p_{\sigma\lambda}} \times$$

$$\times A\left[(k_\sigma,l_\sigma,\mu_\sigma)\right]\, I\left[(k_\sigma,l_\sigma,\mu_\sigma);\phi_{p_\lambda}\right]$$

$$= -\frac{(-\lambda)^\mu Z_\sigma^{\mu_\sigma}}{l_\sigma!\,\nu_\sigma^{l_\sigma}} A\left[(k_\sigma,l_\sigma,\mu_\sigma)\right]\, I\left[(k_\sigma,l_\sigma,\mu_\sigma);\phi_D\right] \qquad (2.33)$$

where the I's are defined analogously to (2.29) for simple $P_{(k_\sigma)}$ graphs.** The distribution functions are then

$$\eta_{(s_\sigma)} = \prod_\sigma \left(\frac{N_\sigma}{N}\right)^{s_\sigma} \exp\left[-\sum_{(k_\sigma)\subset(s_\sigma)} \sum_{(l_\sigma,\mu_\sigma)} U_{(k_\sigma,l_\sigma,\mu_\sigma)}\right] \qquad (2.34)$$

The results of Debye is given by taking $k_\sigma = k_\tau = \mu_\sigma = \mu_\tau = 1$ all l's zero, whence

$$U_{(k_\sigma,l_\sigma,\mu_\sigma)} \to U_{\sigma\tau}(|\vec{r}_{\sigma\tau}|) = Z_\sigma Z_\tau \lambda \phi_D(|\vec{r}_{\sigma\tau}|) \qquad (2.35)$$

---

*The sum over (p_i) is from 0 to ∞ for each p_i ; the direct bonds (no intermediate particles) between the set k_σ come from $\Phi_{(s_\sigma)}$ in (2.32).

**Note that the U's and the I's are functions of the position vectors of the root points.

which is readily seen to be equivelant to Debye's result.

4. EXPLICIT RESULTS FOR THE CASE WHEN THE SHORT RANGE FORCES ARE OF HARD SPHERE TYPE

If a particular model is assumed for the short range interactions, eqs. (2.31), (2.30), (2.28), (2.26), and (2.23) make it possible to calculate the thermodynamic properties of the gas as accurately as desired. We choose the simplest model for this type of calculation, namely hard-sphere interactions.

Accordingly we assume a potential of the form

$$\begin{aligned} \Psi_{\sigma\tau}(R) &= \infty \qquad R<a_{\sigma\tau} \\ &= 0 \qquad R>a_{\sigma\tau} \end{aligned} \tag{2.36}$$

and thus, by (2.16)

$$\begin{aligned} g_{\sigma\tau}(R) &= -1 \qquad R<a_{\sigma\tau} \\ &= 0 \qquad R>a_{\sigma\tau} \end{aligned} \tag{2.37}$$

With this assumption and the condition that $a_{\sigma\tau} \ll \nu_\mu^{1/3} \ll 1/K$ for all σ, τ, μ, the contribution of certain graphs may be found without any assumptions about the relative size of $\lambda/4\pi$ and $a_{\sigma\tau}$. The more important of these are displayed in Fig. 1.

From eqs. (2.28), (2.29), the contribution to the sum $\sum$ of graphs (a) to (e) of Figure 1 may readily be computed. One obtains for the contribution of these graphs

$$\sum_{1a} = \frac{-2\pi}{3} \sum_{\sigma} \sum_{\tau} \frac{a^3_{\sigma\tau}}{\nu_\sigma \nu_\tau} \tag{2.38}$$

$$\Sigma_{1b} = -\frac{2\pi^2}{9}\sum_{\sigma}\sum_{\tau}\sum_{\mu}\frac{1}{\nu_\sigma \nu_\tau \nu_\mu}\left[\frac{a_{\sigma\tau}^6}{24} - \frac{3}{8}a_{\sigma\tau}^4\left(a_{\sigma\tau}^2 + a_{\tau\mu}^2\right)\right.$$
$$+\frac{2}{3}a_{\sigma\tau}^3\left(a_{\sigma\mu}^3 + a_{\tau\mu}^3\right) - \frac{3}{8}a_{\sigma\tau}^2\left(a_{\sigma\mu}^4 - 2a_{\sigma\mu}^2 a_{\tau\mu}^2 + a_{\tau\mu}^4\right)$$
$$+\frac{1}{24}a_{\sigma\mu}^6 - \frac{3}{8}a_{\sigma\mu}^4 a_{\tau\mu}^2 + \frac{2}{3}a_{\sigma\mu}^3 a_{\tau\mu}^3$$
$$\left. -\frac{3}{8}a_{\sigma\mu}^2 a_{\tau\mu}^4 + \frac{1}{24}a_{\tau\mu}^6\right] \qquad (2.39)$$

This may also be written in the more compact but less symmetric form*

$$\Sigma_{1b} = \frac{-2\pi^2}{9}\sum_{\sigma}\sum_{\tau}\sum_{\mu}\frac{1}{\nu_\sigma \nu_\tau \nu_\mu}\left[\frac{a_{\sigma\tau}^6}{8} - \frac{9}{4}a_{\sigma\tau}^4 a_{\tau\mu}^2 + 2a_{\sigma\tau}^3 a_{\tau\mu}^3 + \frac{3}{4}a_{\sigma\tau}^2 a_{\tau\mu}^2 a_{\sigma\mu}^2\right] \qquad (2.40)$$

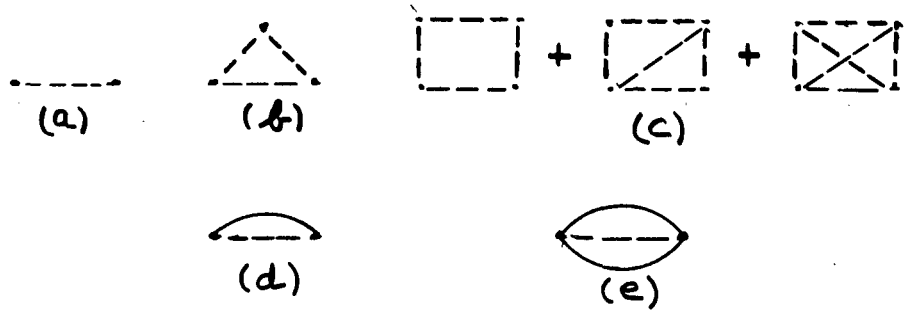


Fig. 1. Graphs whose contribution to the free energy does not depend critically on the ratios $\lambda/(4\pi a_{\sigma\tau})$.

By a purely dimensional argument, the graphs (c) of Fig. 1 give a contribution

$$\Sigma_{1c} = O\left[\frac{1}{\nu}\left(\frac{\pi a^3}{\nu}\right)^3\right] \qquad (2.41)$$

---

*If is clear that for only one type this reduces to the ordinary gas result $\left[-\frac{5}{12}\left(\frac{2\pi a^3}{3\nu}\right)^2\right]$.

where ν may be taken as the minimum ν_σ and a as the maximum $a_{\sigma\tau}$. Since we do not go beyond second order in $\pi a^3/\nu$, we ignore this contribution.

Then:

$$\Sigma_{1d} = \frac{\lambda}{2}\sum_\sigma \sum_\tau \frac{Z_\sigma Z_\tau}{\nu_\sigma \nu_\tau}\left[-\frac{a_{\sigma\tau}}{\kappa} e^{-\kappa a_{\sigma\tau}} + \frac{1}{\kappa^2}\left(1 - e^{-\kappa a_{\sigma\tau}}\right)\right]$$

$$\cong \frac{1}{2}\sum_\sigma \sum_\tau \frac{Z_\sigma Z_\tau}{\nu_\sigma \nu_\tau}\left[\frac{\lambda a_{\sigma\tau}^2}{2} - \frac{a_{\sigma\tau}^3}{3}(\lambda\kappa)\right] \qquad (2.42)$$

since $\kappa a_{\sigma\tau}$ is always small. Similarly

$$\Sigma_{1e} = -\frac{\lambda}{4(4\pi)}\sum_\sigma \sum_\tau \frac{Z_\sigma Z_\tau}{\nu_\sigma \nu_\tau}\frac{\left(1 - e^{-2\kappa a_{\sigma\tau}}\right)}{2\kappa}$$

$$\cong -\frac{1}{16\pi}\sum_\sigma \sum_\tau \frac{Z_\sigma Z_\tau}{\nu_\sigma \nu_\tau}\lambda^2 a_{\sigma\tau} \qquad (2.43)$$

We now turn to a class of graphs whose contribution depends critically on the ratio $\lambda/(4\pi a)$. Some of the simplest are shown in Fig. 2.



Fig. 2. Two point graphs whose contribution to the free energy depends strongly on the ratios $\lambda/(4\pi a_{\sigma\tau})$.

It is necessary to make certain assumptions about the ratios $\lambda/(4\pi a_{\sigma\tau})$. We list three possibilities.

(A) $\lambda/(4\pi a_{\sigma\tau}) \leq 1$ all σ, τ

This might be encountered in an ionic solution.

(B) The gas consists of small electrons (type 1) and large ions so that

$$\lambda/(4\pi a_{11}) \gg 1 \; ; \; \lambda/(4\pi a_{\sigma\tau}) \leq 1 \quad \sigma, \tau \text{ not both 1}$$

(C) The gas consists of electrons and stripped nuclei so that

$$\lambda/(4\pi a_{\sigma\tau}) \gg 1 \qquad \text{all } \sigma, \tau$$

Case (C) is the most difficult, and is of importance in thermonuclear reactions where one has ionized hydrogen, deuterium, etc.

Cases (A) and (B) may be treated together. The contribution of the graphs of Fig. 2 is easily seen to be

$$\Sigma_2 = \sum_\sigma \sum_\tau \frac{4\pi}{\nu_\sigma \nu_\tau} \int_{a_{\sigma\tau}}^\infty dR \, R^2 \times$$
$$\times \left[\exp\left(\frac{Z_\sigma Z_\tau \lambda e^{-\kappa R}}{4\pi R}\right) - 1 + \frac{Z_\sigma Z_\tau \lambda e^{-\kappa R}}{4\pi R} - \frac{Z_\sigma^2 Z_\tau^2 \lambda^2 e^{-2\kappa R}}{2(4\pi R)^2} \right] \tag{2.44}$$

If $a_{\sigma\tau} \gg \lambda/4\pi$, the exponential may be expanded* to give

$$\Sigma_2 = -\frac{1}{6} \sum_\sigma \sum_\tau \frac{Z_\sigma^3 Z_\tau^3 \lambda^3}{\nu_\sigma \nu_\tau (4\pi)^2} M_1(3\kappa a_{\sigma\tau}, \infty) + o\left[\frac{1}{\nu}\left(\frac{\lambda\kappa}{4\pi}\right)^2 \frac{\lambda}{4\pi a_{\sigma\tau}}\right]$$

$$\approx \frac{1}{6(4\pi)^2} \sum_\sigma \sum_\tau \frac{Z_\sigma^3 Z_\tau^3 \lambda^3}{\nu_\sigma \nu_\tau} (\gamma + \ln 3\kappa a_{\sigma\tau}) \qquad \left(\frac{\lambda}{4\pi a_{\sigma\tau}} \ll 1\right) \tag{2.45}$$

*This means that only the first two graphs in Fig. 2 need be considered, the others being negligible.

where $M_m(x,y) = \int_1^y \frac{e^{-xz}\,dz}{z^m}$ (2.46)

and we have used the expansion

$$M_1(x,\infty) = -(\gamma + \ln x) + x + \frac{x^2}{2\cdot 2!} + \cdots (x \ll 1) \quad (2.47)$$

If $a_{\sigma\tau}$ is not large with respect to $\lambda/4\pi$, we break the integral in (2.44) into two parts: from $a_{\sigma\tau}$ to $\sqrt{\frac{|Z_\sigma Z_\tau|\lambda}{4\pi K}}$ and from $\sqrt{\frac{|Z_\sigma Z_\tau|\lambda}{4\pi K}}$ to ∞. This is convenient since

$$\sqrt{\frac{|Z_\sigma Z_\tau|\lambda K}{4\pi}} \ll 1$$ and therefore

$$\frac{\lambda}{4\pi} \ll \sqrt{\frac{|Z_\sigma Z_\tau|\lambda}{4\pi K}} \ll 1/K \quad (2.48)$$

This means that the last part of the integral may be treated in the same manner as was used in obtaining (2.45). In the first part, since $KR \ll 1$, one may expand $e^{-KR} = 1 - KR + \cdots$. After some simplification, one obtains:

$$\begin{aligned}\sum_2 \cong 4\pi \sum_\sigma \sum_\tau \frac{1}{\nu_\sigma \nu_\tau} \Bigg\{ & a_{\sigma\tau}^3 \left[\frac{1}{3} - \frac{1}{2}\frac{Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}} + \frac{1}{2}\left(\frac{Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right)^2 + \frac{1}{3}\frac{Z_\sigma Z_\tau \lambda K}{4\pi}\right] - a_{\sigma\tau}^3 \exp\left(-\frac{Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right)\left[\frac{1}{3} - \frac{1}{6}\left(\frac{Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right)\right. \\ & \left. + \frac{1}{6}\left(\frac{Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right)^2 + \frac{1}{3}\frac{Z_\sigma Z_\tau \lambda K}{4\pi}\right] - \left(\frac{Z_\sigma Z_\tau \lambda}{4\pi}\right)^3 \left[\right. \\ & -\frac{1}{6}\left(\gamma + \ln\left(3\sqrt{\frac{|Z_\sigma Z_\tau|\lambda K}{4\pi}}\right)\right) \\ & \left. + \frac{1}{6} M_1\left(Z_\sigma Z_\tau \sqrt{\frac{\lambda K}{4\pi |Z_\sigma Z_\tau|}}, \sqrt{\frac{|Z_\sigma Z_\tau|\lambda}{4\pi K}}\Big/ a_{\sigma\tau}\right) + \frac{11}{36}\right] \Bigg\}. \end{aligned} \quad (2.49)$$

So far no assumptions have been made about the size of $\lambda/(4\pi a_{\sigma\tau})$. If all $a_{\sigma\tau} \gg \lambda/4\pi$, eq. (2.49) clearly reduces to (2.45). If $a_{\sigma\tau} \cong \lambda/4\pi$ for all σ, τ, it is easy to show that:

$$\Sigma_2 = O\left(\frac{1}{\nu}\left(\frac{\lambda K}{4\pi}\right)^2\right)$$

For the case (B) mentioned above, it is clear that*

$$\Sigma_2 = \frac{-\lambda^3}{(4\pi\nu_1)^2}\left[\frac{1}{6}M_1\left(3\sqrt{\frac{\lambda K}{4\pi}},\infty\right) + \frac{1}{6}M_1\left(\sqrt{\frac{\lambda K}{4\pi}},\infty\right) + \frac{11}{36}\right]$$

$$+ 4\pi \sum_{\substack{\sigma \\ \text{not both 1}}} \sum_{\tau} \frac{1}{\nu_\sigma \nu_\tau}\left\{a_{\sigma\tau}^3\left[\frac{1}{3} - \frac{1}{2}\frac{Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}} + \frac{1}{2}\left(\frac{Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right)^2 + \frac{1}{3}\frac{Z_\sigma Z_\tau \lambda K}{4\pi}\right]\right.$$

$$- a_{\sigma\tau}^3 \exp\left(\frac{-Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right)\left[\frac{1}{3} - \frac{1}{6}\frac{Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}} + \frac{1}{6}\left(\frac{Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right)^2 + \frac{1}{3}\frac{Z_\sigma Z_\tau \lambda K}{4\pi}\right]$$

$$- \left(\frac{Z_\sigma Z_\tau \lambda}{4\pi}\right)^3\left[\frac{1}{6}M_1\left(3\sqrt{\frac{|Z_\sigma Z_\tau|\lambda K}{4\pi}},\infty\right) + \frac{1}{6}M_1\left(Z_\sigma Z_\tau\sqrt{\frac{\lambda K}{4\pi|Z_\sigma Z_\tau|}}, \sqrt{\frac{|Z_\sigma Z_\tau|\lambda}{4\pi K}}\frac{1}{a_{\sigma\tau}}\right)\right.$$

$$\left.\left. + \frac{11}{36}\right]\right\} + O\left[\frac{1}{\nu}\left(\frac{\lambda K}{4\pi}\right)^2\right] \qquad (2.50)$$

$$\left(\frac{\lambda}{4\pi a_{11}} \gg 1 \;;\; \frac{\lambda}{4\pi a_{\sigma\tau}} \leq 1 \quad \sigma, \tau \text{ not both } 1\right)$$

*Of course $Z_1 = -1$, $e^{-\lambda/4\pi a_{11}} \ll\ll 1$.

For case (C) eq. (2.49) is still valid; however in the terms for which $Z_\sigma Z_\tau < 0$ (e.g. electron-proton forces) the quantities $\exp\left(\frac{-Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right)$ and $M_1\left(Z_\sigma Z_\tau \sqrt{\frac{\lambda K}{4\pi |Z_\sigma Z_\tau|}}, \sqrt{\frac{|Z_\sigma Z_\tau| \lambda}{4\pi K}} \frac{1}{a_{\sigma\tau}}\right)$ are very large. It is not difficult to show that the leading terms in (2.49) are then given by:

$$\sum_2 \cong -\frac{1}{6(4\pi)^2} \sum_{\sigma} \sum_{Z_\sigma Z_\tau < 0} \frac{(Z_\sigma Z_\tau \lambda)^3}{\nu_\sigma \nu_\tau} \left[\exp\left(\frac{-Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right) \ln\left(\frac{-Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right) - \int_0^{\frac{-Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}} e^z \ln z \, dz \right]$$

$$\frac{\lambda}{4\pi a_{\sigma\tau}} \gg 1 \qquad \text{all } \sigma, \tau \qquad (2.51)$$

However there is no reason to believe that the contribution of graphs with three or more points will be negligible compared to this; it appears that a different technique is necessary to evaluate the free energy with any accuracy for case (C). We merely remark that the factor $\exp\left(\frac{-Z_\sigma Z_\tau \lambda}{4\pi a_{\sigma\tau}}\right)$ ($\sim 10^{5000}$ for a hydrogen plasma at 10^6°K) makes this case fundamentally different from the others.

It is not difficult to show that the contribution of all three point graphs is negligible compared to the largest of $\left(\frac{\lambda K}{4\pi}\right)^2, \left(\frac{4\pi a^3}{\nu}\right)^2, (Ka)^3$ for cases (A) and (B). It is suspected (but not proved) that graphs of four or more points will also give a negligible contribution.

CHAPTER III

THE BOGOLUBOV METHOD

1. THE DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

As mentioned in Chapter I, Bogolubov[6] has devised a method of obtaining the distribution functions by the solution of differential equations. The present chapter is a modification of his method.

To simplify the notation, it is convenient to consider the type of ion as a kind of internal coordinate, which can assume the values $\sigma = 1, 2 \ldots \nu$. We write γ for the combined coordinates $(\vec{r}, \sigma)$. All latin indices will always refer to specific particles. The charge of an ion of type σ is again $Z_\sigma e$, and their number is:

$$N_\sigma = \sum_{i=1}^{N} \delta_{\sigma_i \sigma}$$

Charge neutrality is then expressed by:

$$\sum_{i=1}^{N} Z_{\sigma_i} = \sum_{\sigma=1}^{\nu} Z_\sigma N_\sigma = 0$$

With this notation eq.(2.4) for the s particle distribution function may be rewritten in the form:

$$n_s(\gamma_1 \cdots \gamma_s) = \frac{V^s}{N^s} \prod_{i=1}^{s} N_{\sigma_i} \frac{\int \cdots \int d\gamma_{s+1} \cdots d\gamma_N \exp\{-\lambda \Phi_N - \Psi_N\}}{Q_N} \tag{3.1}$$

where again $\lambda = 4\pi e^2/kT$ and:

$$\Phi_N(\gamma_1 \cdots \gamma_N) = \sum_{1 \le j < k \le N} \sum Z_{\sigma_j} Z_{\sigma_k} \phi(|\vec{r}_j - \vec{r}_k|) \tag{3.2}$$

$$\Psi_N(\gamma_1 \cdots \gamma_N) = \frac{1}{kT} \sum_{i<k \le N} \sum \psi_{\sigma_i \sigma_k}(|r_i - r_k|) \tag{3.3}$$

$$Q_N = \int \cdots \int \prod_{i=1}^{N} d\vec{r}_i \exp\{-\lambda \Phi_N - \Psi_N\} \tag{3.4}$$

with Φ, Ψ defined by (2.12)-(2.14).

(3.1) may be differentiated with respect to one of the arguments $\vec{r}_i$ to give*

$$\frac{\partial n_s}{\partial \vec{r}_i} + \lambda n_s \frac{\partial \Phi_s}{\partial \vec{r}_i} + n_s \frac{\partial \Psi_s}{\partial \vec{r}_i} + \sum_{j=s+1}^{N} \int d\vec{r}_j \frac{Z_{\sigma_i} Z_{\sigma_j}}{N_{\sigma_j} v} \lambda \frac{\partial \phi(|\vec{r}_i - \vec{r}_j|)}{\partial \vec{r}_i} n_{s+1}(\gamma_1 \cdots \gamma_s, \gamma_j)$$

$$+ \sum_{j=s+1}^{N} \frac{\int d\vec{r}_j}{kT N_{\sigma_j} v} \frac{\partial \psi_{\sigma_i \sigma_j}(|\vec{r}_i - \vec{r}_j|)}{\partial \vec{r}_i} n_{s+1}(\gamma_1 \cdots \gamma_s, \gamma_j) = 0 \tag{3.5}$$

Clearly all terms in the sum over j which refer to the same type of particle will give the same contribution; the number of terms of a given type σ is $N_\sigma - s_\sigma$ where $s_\sigma = \sum_{i=1}^{s} \delta_{\sigma_i \sigma}$. Thus (3.5) may be rewritten

$$\frac{\partial n_s}{\partial \vec{r}_i} + \lambda \frac{\partial \Phi_s}{\partial \vec{r}_i} + \frac{\partial \Psi_s}{\partial \vec{r}_i} + \frac{\lambda}{v} Z_{\sigma_i} \sum_\sigma \frac{(N_\sigma - s_\sigma) Z_\sigma}{N_\sigma} \int d\vec{r} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} n_{s+1}(\gamma_1 \cdots \gamma_s, \gamma)$$

$$+ \frac{1}{kTv} \sum_\sigma \frac{(N_\sigma - s_\sigma)}{N_\sigma} \int d\vec{r} \frac{\partial \psi_{\sigma_i \sigma}(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} n_{s+1}(\gamma_1 \cdots \gamma_s, \gamma) = 0$$

If we let V, $N_\sigma \to \infty$ and consider only distribution functions for

*We use $\partial/\partial \vec{r}_i$ to denote the gradient with respect to the ith position argument. Where not stated the arguments of n_s, Φ_s, Ψ_s are $\gamma_1 \cdots \gamma_s$.

which $S_\sigma = \sum_{i=1}^{s} \delta_{\sigma\sigma_i} << N_\sigma$, this becomes finally

$$\frac{\partial m_s}{\partial \vec{r}_i} + \lambda \frac{\partial \Phi_s}{\partial \vec{r}_i} + \frac{\partial \Psi_s}{\partial \vec{r}_i} + \frac{\lambda}{\nu} Z_{\sigma_i} \sum_\sigma \int d\vec{r}\, Z_\sigma \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} m_{s+1}(y_1 \cdots y_s, y)$$

$$+ \frac{1}{kT\nu} \int d\vec{r}\, \frac{\partial \Psi_{\sigma_i \sigma}(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} m_{s+1}(y_1 \ldots y_s, y) = 0 \tag{3.6}$$

A slight further modification of (3.6) is useful: if one puts

$$m_s = m_s e^{-\Psi_s} \tag{3.7}$$

the following equation is obtained for m_s

$$\frac{\partial m_s}{\partial \vec{r}_i} + \lambda m_s \frac{\partial \Phi_s}{\partial \vec{r}_i} + \frac{\lambda}{\nu} \int d\vec{r} \sum_\sigma Z_\sigma Z_{\sigma_i} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} (1 + f_s(y)) m_{s+1}(y)$$

$$- \frac{1}{\nu} \int d\vec{r} \sum_\sigma \frac{\partial f_s(y)}{\partial \vec{r}_i} m_{s+1}(y) = 0 \tag{3.8}$$

with:

$$m_{s+1}(y) \equiv m_{s+1}(y_1 \cdots y_s, y)$$

$$f_s(y) \equiv f_s(y_1 \cdots y_s; y) = \exp\left[-\frac{1}{kT} \sum_{i=1}^{s} \psi_{\sigma_i \sigma}(r_i, r)\right] - 1$$

$$= \prod_{i=1}^{s} (1 + f(y_i, y)) - 1 \tag{3.9}$$

where:

$$f(y_i, y) \equiv f_{\sigma_i \sigma}(|\vec{r}_i - \vec{r}|) = e^{-\frac{1}{kT} \psi_{\sigma_i \sigma}(r_i, r)}$$

denotes the Mayer f-function.

For the boundary conditions, we note that at large separations, the correlation between the particles should vanish. In our normalization this is equivalent to

$$\lim_{\text{all } |\vec{r}_{ij}| \to \infty} m_s(y_1 \cdots y_s) = \prod_{i=1}^{s} \left(\frac{N\sigma_i}{N}\right) \tag{3.10}$$

The rest of this chapter is devoted to solving (3.8), (3.10) approximately.

2. ORDINARY GAS MIXTURES

In order to give an idea of the method, we first solve (3.8) for the case $\lambda = 0$, i.e., with only short-range forces. In this case (3.8) becomes

$$\frac{\partial m_s}{\partial \vec{r}_i} = \frac{1}{v} \int d\vec{r} \sum_{\sigma} \frac{\partial f_s(y)}{\partial \vec{r}_i} m_{s+1}(y) \tag{3.8a}$$

Let the range of the short range potential $\psi_{\sigma\tau}(R)$ be $a_{\sigma\tau}$, then it follows from the definition (3.9) that

$$f_s(y) = 0 \text{ if } |\vec{r}_j - \vec{r}| > a_{\sigma_j,\sigma} \qquad j = 1, \ldots s \tag{3.11}$$

This means that if

$$a = \max (a_{\sigma\tau}) \tag{3.12}$$

the integral on the right is the gradient of a term of order $4\pi a^3/v$. This suggests that (3.9a) may be solved approximately by expanding m_s in powers of this parameter. This clearly corresponds to the virial expansion. To carry it out we expand m_s formally in powers of the

density $1/\nu$, and write:

$$m_s = \sum_{a=1}^{\infty} m_s^{(a)} (1/\nu)^a \tag{3.13}$$

Substituting (3.13) into (3.8a) and equating coefficients of like powers of $1/\nu$, one finds

$$\frac{\partial m_s^{(a)}}{\partial \vec{r}_i} = \int d\vec{r} \sum_{\sigma} \frac{\partial f_s(Y)}{\partial \vec{r}_i} m_{s+1}^{(a-1)}(Y) \tag{3.14}$$

The boundary condition (3.10) may be written in terms of the coefficients $m_s^{(a)}$ as

$$\lim_{all\ |\vec{r}_{i,j}| \to \infty} m_s^{(0)} = \prod_{i=1}^{s} \left(\frac{N\sigma_i}{N}\right) \; ; \; \lim_{all\ |\vec{r}_{i,j}| \to \infty} m_s^{(a)} = 0 \quad a > 0 \tag{3.15}$$

The first of equations (3.14) takes the form

$$\frac{\partial m_s^{(0)}}{\partial \vec{r}_i} = 0 \tag{3.16}$$

or, with (3.15),

$$m_s^{(0)} = \prod_{i=1}^{s} \left(\frac{N\sigma_i}{N}\right) = \text{const.} \tag{3.17}$$

For $m_s^{(1)}$ one has

$$\frac{\partial m_s^{(1)}}{\partial \vec{r}_i} = \int d\vec{r} \sum_{\sigma} \frac{N\sigma}{N} m_s^{(0)} \frac{\partial f_s(Y)}{\partial \vec{r}_i} \tag{3.18}$$

The solution of (3.18) has the form

$$m_s^{(1)} = m_s^{(0)} \sum_{\sigma} \frac{N\sigma}{N} \int d\vec{r}\, f_s(Y) + \text{const.} \tag{3.19}$$

The constant must be chosen, according to the boundary condition (3.15), to make $\lim_{|\vec{r}_{ij}|\to\infty} m_s^{(1)} = 0$.

Expanding the product expressing $f_s(Y_1 \cdots Y_s; Y)$ in terms of the Mayer functions $f(Y_i, Y)$ (see Eq. 3.9) one easily shows that for $|\vec{r}_{ij}| \to \infty$ all terms in the integral (3.19) containing products of two or more functions $f(Y_i, Y)$ will go to zero, so that

$$\lim_{|r_{ij}|\to\infty} \int d\vec{r}\, f_s(Y) = \int d\vec{r} \sum_{i=1}^{s} f(Y_i, Y) = \int d\vec{\rho} \sum_{i=1}^{s} f_{\sigma_i \sigma}(\rho) \tag{3.20}$$

This determines the constant in (3.19), so one has finally:

$$m_s^{(1)} = m_s^{(0)} \sum_{\sigma} \frac{N_\sigma}{N} \int d\vec{r} \left[f_s(Y) - \sum_{i=1}^{s} f(Y_i, Y) \right] \tag{3.21}$$

The equation for $m_s^{(2)}$ is

$$\frac{\partial m_s^{(2)}}{\partial \vec{r}_i} = \int d\vec{r} \sum_{\sigma} \frac{\partial f_s(Y)}{\partial \vec{r}_i}\, m_{s+1}^{(1)}(Y_1 \cdots Y_s, Y) \tag{3.22}$$

According to (3.21)

$$m_{s+1}^{(1)}(Y_1 \cdots Y_s, Y) = m_s^{(0)} \frac{N_\sigma}{N} \sum_{\sigma'} \int d\vec{r} \left[f_{s+1}(Y_1 \cdots Y_s, Y; Y') - \sum_{i=1}^{s} f(Y_i, Y) - f(Y, Y') \right] \tag{3.23}$$

It is easy to see that:

$$f_{s+1}(Y_1 \cdots Y_s, Y; Y') = f_s(Y_1 \cdots Y_s; Y') \left[1 + f(Y, Y') \right] + f(Y, Y') \tag{3.24}$$

Thus (3.22) becomes

$$\frac{\partial m_s^{(2)}}{\partial \vec{r}_i} = m_s^{(0)} \sum_\sigma \sum_{\sigma'} \left(\frac{N_\sigma}{N}\right)\left(\frac{N_{\sigma'}}{N}\right) \int d\vec{r} \int d\vec{r}\,' \frac{\partial f_s(Y)}{\partial \vec{r}_i} \left\{ \left[1 + f(Y,Y')\right] f_s(Y) - \sum_{i=1}^{s} f(Y_i, Y') \right\} \tag{3.25}$$

It is not difficult to show, as in the argument preceding (3.21), that the solution of this equation which satisfies the boundary condition (3.15) is

$$m_s^{(2)} = m_s^{(0)} \left\{ \left[\frac{1}{2} \left[\sum_\sigma \frac{N_\sigma}{N} \int d\vec{r} \left(f_s(Y) - \sum_{i=1}^{s} f(Y_i, Y) \right) \right] \right]^2 \right.$$

$$\left. + \frac{1}{2} \sum_\sigma \sum_{\sigma'} \left(\frac{N_\sigma}{N}\right)\left(\frac{N_{\sigma'}}{N}\right) \int d\vec{r} \int d\vec{r}\,' f(Y,Y') \left[f_s(Y)\, f_s(Y') - \sum_{i=1}^{s} f(Y_i, Y)\, f(Y_i, Y') \right] \right\} \tag{3.26}$$

The equations (3.7), (3.12), (3.21), and (3.26) are readily seen to be equivalent to the first two terms in $1/n$ of the expansion (2.8)-(2.11). One can go on in this fashion, but it seems difficult to obtain the complete expansion by this method.

3. PURE COULOMB FORCES

If all the Ψ's are taken equal to zero in (3.8) (and thus also $f_s(y) = 0$) one finds as the appropriate equation for point charges

$$\frac{\partial m_s}{\partial \vec{r}_i} + \lambda m_s \frac{\partial \Phi_s}{\partial \vec{r}_i} + K_0^2 \int d\vec{r} \sum_\sigma Z_\sigma Z_{\sigma_i} \frac{\partial \phi(|\vec{r}_i - \vec{r}_i|)}{\partial \vec{r}_i} m_{s+1}(Y) = 0 \tag{3.8b}$$

with

$$K_0^2 \equiv \lambda / n \tag{3.27}$$

One may not now assume that the integral term is small. In fact, if one assumes that the Debye results are approximately correct, this term will be of order unity. This points up again the basic difference between ordinary gases and ionized gases. In an ordinary gas the integral term, representing the effect of collisions with other particles in the gas is small, since they must be close to produce any effect. Thus a triple collision is less important than a binary collision by a factor of order a^3/v etc. For an ionized gas, on the other hand, long-range interactions are important, and one must consider the effect of interactions with all other particles in the gas.

As mentioned in Chapters I, II the appropriate small parameters of the problem is $\lambda K/4\pi$. If we restrict ourselves to not too small particle separations ($> \lambda/4\pi$) the second term in (3.8b) may be assumed to give a contribution proportional to this. Thus one may expand formally in λ and keep only the first few terms.*

The quantity K_0^2 may be used as an auxiliary expansion parameter; the terms proportional to $(K_0^2)^m$ will then represent the effect of interactions of the s-tuple with n additional particles. Since, as noted above, the cooperative long distance shielding effects due to a large number of ions may not be neglected, we must take all powers of K_0^2 for each power of λ .

*As mentioned earlier, for small $|\vec{r}_{ij}|$ one must introduce short-range forces; further unless a $>> \lambda/4\pi$ one must either take all terms in the λ expansion, or treat the Coulomb forces on the same basis as the short-range forces for small separations.

The above remarks suggest an expansion of the form

$$m_s = \sum_{b=0}^{\infty}\sum_{c=0}^{\infty} m_s^{(b,c)} \lambda^b (K_0^2)^c = \sum_{b=0}^{\infty} m_s^{(b)} \lambda^b \tag{3.28}$$

Putting (3.28) in (3.8b) and equating like powers of λ, K_0^2

$$\frac{\partial m_s^{(b,c)}}{\partial \vec{r}_i} + m_s^{(b-1,c)} \frac{\partial \Phi_s}{\partial \vec{r}_i} + \int d\vec{r} \sum_\sigma Z_\sigma Z_{\sigma i} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} m_{s+1}^{(b,c-1)}(Y) = 0 \tag{3.29}$$

For c = 0 the last term is not present, and the equation is

$$\frac{\partial m_s^{(b,0)}}{\partial \vec{r}_i} + m_s^{(b-1,0)} \frac{\partial \Phi_s}{\partial \vec{r}_i} = 0 \tag{3.30}$$

Clearly $m_s^{(0,0)} = m_s^{(0)}$ is given by (3.17), and one readily shows by induction from (3.30) that

$$m_s^{(b,0)} = \frac{m_s^{(0)} (-1)^b (\Phi_s)^b}{b!} \tag{3.31}$$

For $b = 0$, $c > 0$ (3.29) becomes

$$\frac{\partial m_s^{(0,c)}}{\partial \vec{r}_i} + \int d\vec{r} \sum_\sigma Z_\sigma Z_{\sigma i} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} m_{s+1}^{(0,c-1)}(Y) = 0 \tag{3.32}$$

For c = 1 This is

$$\frac{\partial m_s^{(0,1)}}{\partial \vec{r}_i} = -\int d\vec{r} \sum_\sigma Z_\sigma Z_{\sigma i} \frac{N_\sigma}{N} m_s^{(0)} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} = 0 \tag{3.33}$$

where the right side vanishes because

$$\int d\vec{r}\, \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} = \frac{\partial}{\partial \vec{r}_i} \int d\vec{r}\, \phi(|\vec{r}|) = 0 \qquad (3.34)$$

In view of the boundary condition (3.10) which implies

$$\lim_{\text{all } |\vec{r}_{i,j}| \to \infty} m_s^{(b,c)} = 0 \qquad b + c > 0 \qquad (3.35)$$

Eq. (3.34) gives

$$m_s^{(0,1)} = 0 \qquad (3.36)$$

and by a simple induction using (3.33),

$$m_s^{(0,c)} = 0 \qquad c > 0 \qquad (3.37)$$

For b = 1, $c > 0$, eq. (3.29) takes the form*

$$\frac{\partial m_s^{(1,c)}}{\partial \vec{r}_i} = -\int d\vec{r} \sum_\sigma z_\sigma z_{\sigma i} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} m_{s+1}^{(1,c-1)}(Y) \qquad (3.38)$$

If c = 1 this becomes, in view of (3.31)

$$\frac{\partial m_s^{(1,1)}}{\partial \vec{r}_i} = m_s^{(0)} \int d\vec{r} \sum_\sigma z_\sigma \frac{N_\sigma}{N} z_{\sigma i} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \Phi_{s+1}(Y_1 .. Y_s, Y) \qquad (3.39)$$

From the definition of Φ_s, [eq. (3.2)]

*The second term of eq. (3.29) vanishes in this case, because of eq. (3.37).

$$\Phi_{s+1}(\gamma_1 \cdots \gamma_s, \gamma) = \Phi_s(\gamma_1, \ldots \gamma_s) + \sum_{j=1}^{s} Z_\sigma Z_{\sigma_j} \phi(|\vec{r}_j - \vec{r}|) \tag{3.40}$$

Substituting from (3.40) in (3.39) and noting that the first term gives no contribution,

$$\frac{\partial m_s^{(1,1)}}{\partial \vec{r}_i} = m_s^{(0)} \left(\sum_\sigma Z_\sigma^2 \frac{N_\sigma}{N} \right) \int d\vec{r} \sum_{j=1}^{s} Z_{\sigma_i} Z_{\sigma_j} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \phi(|\vec{r} - \vec{r}_j|) \tag{3.41}$$

The solution of eq. (3.41) which satisfies eq. (3.35) is

$$m_s^{(1,1)} = m_s^{(0)} \left(\sum_\sigma Z_\sigma^2 \frac{N_\sigma}{N} \right) \int d\vec{r} \sum_{i<j=1}^{s} \sum Z_{\sigma_i} Z_{\sigma_j} \phi(|\vec{r}_i - \vec{r}|) \phi(|\vec{r} - \vec{r}_j|) \tag{3.42}$$

Introducing

$$A = \sum_\sigma Z_\sigma^2 \frac{N_\sigma}{N} \tag{3.43}$$

$$\Phi_s^{(m)} = \sum_{i<j=1}^{s} \sum Z_{\sigma_i} Z_{\sigma_j} \phi_m(|\vec{r}_i - \vec{r}_j|) \quad ; \quad \Phi_s^{(0)} = \Phi_s \tag{3.44}$$

where ϕ_m is given by (2.20),

Eq. (3.42) may be written

$$m_s^{(1,1)} = m_s^{(0)} A \Phi_s^{(1)} \tag{3.45}$$

and from eq. (3.31)

$$m_s^{(1,0)} = - m_s^{(0)} \Phi_s^{(0)} \tag{3.46}$$

We assert that in general

$$m_s^{(1,c)} = (-1)^{c+1} A^c m_s^{(0)} \Phi_s^{(c)} \tag{3.47}$$

Proof:

(a) Eq. (3.47) holds for c = 0 by eq. (3.46)

(b) If eq. (3.47) holds for c = n - 1, eq. (3.38) for c = n states

$$\frac{\partial m_s^{(1,m)}}{\partial \vec{r}_i} = -(-1)^m A^{m+1} m_{s+1}^{(0)} \int d\vec{r} \sum_\sigma Z_\sigma Z_{\sigma_i} \frac{\partial \phi(|r_i - r|)}{\partial \vec{r}_i} \Phi_{s+1}^{(m-1)}(\gamma_1 \ldots \gamma_s, \gamma)$$

Now:

$$\Phi_{s+1}^{(m-1)}(\gamma_1 \ldots \gamma_s, \gamma) = \Phi_s^{(m-1)}(\gamma_1 \ldots \gamma_s) + \sum_{j=1}^{s} Z_\sigma Z_{\sigma_j} \phi_{m-1}(r, r_j) \tag{3.48}$$

$$m_{s+1}^{(0)} = m_s^{(0)} \frac{N_\sigma}{N}$$

The contribution from $\Phi_s^{(m-1)}$ vanishes, and therefore one can write:

$$\frac{\partial m_s^{(1,m)}}{\partial \vec{r}_i} = (-1)^{m+1} A^m m_s^{(0)} \frac{\partial}{\partial \vec{r}_i} \int d\vec{r} \sum_{j=1}^{s} Z_{\sigma_i} Z_{\sigma_j} \phi(r_i - r)\, \phi_{m-1}(r, r_j)$$

$$= \frac{\partial}{\partial \vec{r}_i} \left[(-1)^{m+1} A^m m_s^{(0)} \sum_{i<j \le s} \sum Z_{\sigma_i} Z_{\sigma_j} \phi_m(|\vec{r}_i - \vec{r}_j|) \right] \tag{3.49}$$

Eq. (3.49), together with the boundary condition (3.35), shows that if eq. (3.47) holds for c = n - 1, it holds for c = n.

(a) and (b) constitute an inductive proof of eq. (3.47).

One may sum eq. (3.47) over c, using the Mayer summation theorem (2.21) and the fact (which follows from the definitions (1.10), (3.27), and (3.43) of κ^2, κ_0^2, A) that

$$\kappa^2 = \kappa_0^2 A \tag{3.50}$$

Thus

$$m_s^{(1)} = \sum_{c=0}^{\infty} (\kappa_0^2)^c m_s^{(1,c)} = -m_s^{(0)} \sum_{c=0}^{\infty} (-\kappa^2)^c \Phi_s^{(c)}$$

$$= -m_s^{(0)} \sum_{i \in 1} \sum_{j \notin s} Z_{\sigma_i} Z_{\sigma_j} \sum_{c=0}^{\infty} (-\kappa^2)^c \phi_c(|\vec{r}_i - \vec{r}_j|)$$

$$= -m_s^{(0)} \sum_{i \in 1} \sum_{j \notin s} Z_{\sigma_i} Z_{\sigma_j} \phi_D(|\vec{r}_i - \vec{r}_j|) \equiv -m_s^{(0)} \Phi_s^D \tag{3.51}$$

where ϕ_D is defined by (2.21) and the last equality constitutes a definition of Φ_s^D . This is clearly equivalent to the Debye result (2.35).

For b = 2, using (3.47), eq. (3.29) becomes

$$\frac{\partial m_s^{(2,c)}}{\partial \vec{r}_i} = (-1)^c m_s^{(0)} A^c \Phi_s^{(c)} \frac{\partial \Phi_s^{(0)}}{\partial \vec{r}_i} - \int d\vec{r} \, Z_{\sigma_i} \sum_{\sigma} Z_{\sigma} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} m_{s+1}^{(2,c-1)}(Y) \tag{3.52}$$

This may be simplified by the substitution

$$m_s^{(2,c)} = (-1)^c m_s^{(0)} \left[\frac{A^c}{2} \sum_{l_1 + l_2 = c} \Phi_s^{(l_1)} \Phi_s^{(l_2)} + f_s^{(c)} \right] \tag{3.53}$$

One obtains, often some simplifications, for the new unknown function $f_s^{(c)}$:

$$\frac{\partial \rho_s^{(c)}}{\partial \vec{r}_i} = \int d\vec{r}\, Z_{\sigma_i} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \Big[\frac{1}{2} A^{c-1} B \sum_{l_1+l_2=c-1} \sum_{j=1} \sum_{k=1} Z_{\sigma_j} Z_{\sigma_k} \cdot \phi_{l_1}(|\vec{r}_j - \vec{r}|) \phi_{l_2}(|\vec{r}_k - \vec{r}|) + \sum_\sigma Z_\sigma \frac{N_\sigma}{N} \rho_{s+1}^{(c-1)}(Y) \Big] \tag{3.54}$$

with

$$B = \sum_\sigma Z_\sigma^3 \frac{N_\sigma}{N} \tag{3.55}$$

The form of the first term of (3.54) suggests another substitution of the form

$$\rho_s^{(c)} = A^{c-1} B \sum_{l_1+l_2+l_3=c-1} \int d\vec{r} \Big[\sum_{i<j<k\le s} Z_{\sigma_i} Z_{\sigma_j} Z_{\sigma_k} \phi_{l_1}(|\vec{r}_i - \vec{r}|) \phi_{l_2}(|\vec{r}_j - \vec{r}|) \phi_{l_3}(|\vec{r}_k - \vec{r}|) + \frac{1}{2} \sum_{i\neq j} Z_{\sigma_i}^2 Z_{\sigma_j} \phi_{l_1}(|\vec{r}_i - \vec{r}|) \phi_{l_2}(|\vec{r}_i - \vec{r}|) \phi_{l_3}(|\vec{r}_j - \vec{r}|) \Big] + g_s^{(c)} \tag{3.56}$$

Putting (3.56) in (3.54) one finds

$$\frac{\partial}{\partial \vec{r}_i} g_s^{(c)} = \frac{A^{c-2} B^2}{2} \sum_{l_1+l_2+l_3=c-2} Z_{\sigma_i} \int d\vec{r} \int d\vec{r}' \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \phi_{l_1}(|\vec{r} - \vec{r}'|) \phi_{l_2}(|\vec{r} - \vec{r}'|) \sum_{j\neq i} \phi_{l_3}(|\vec{r}' - \vec{r}_j|) + Z_{\sigma_i} \int d\vec{r}_i \frac{\partial \phi(|\vec{r}_i' - \vec{r}|)}{\partial \vec{r}_i} \sum_\sigma Z_\sigma \frac{N_\sigma}{N} g_{s+1}^{(c-1)}(Y) \tag{3.57}$$

Again being guided by the first term, one may put

$$g_s^{(c)} = \frac{A^{c-2} B}{2} \sum_{l_1+l_2+l_3+l_4=c-2} \int d\vec{r} \int d\vec{r}' \, \phi_{l_1}(|\vec{r} - \vec{r}'|) \phi_{l_2}(|\vec{r} - \vec{r}'|) \cdot \sum_{j\le k\le s} \phi_{l_3}(|\vec{r}_j - \vec{r}|) \phi_{l_4}(|\vec{r}' - \vec{r}_k|) + \Lambda_s^{(c)} \tag{3.58}$$

which gives for $\Lambda_s^{(c)}$ the equation

$$\frac{\partial \Lambda_s^{(c)}}{\partial \vec{r}_i} = \int d\vec{r}\, Z_{\sigma_i} \frac{\partial \Phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \sum_\sigma Z_\sigma \frac{N_\sigma}{N} \Lambda_{s+1}^{(c-1)}(Y) \tag{3.59}$$

Since $\Lambda_s^{(0)} = 0$, one readily shows by induction that

$$\Lambda_s^{(c)} = 0 \tag{3.60}$$

From (3.53), (3.56), (3.58), and (3.60) one has finally

$$m_s^{(2,c)} = m_s^{(0)} (-1)^c \Big\{ \frac{A^c}{2} \sum_{l_1+l_2=c} \sum \bar{\Phi}_s^{(l_1)} \bar{\Phi}_s^{(l_2)}$$

$$+ A^{c-1} B \sum_{l_1+l_2+l_3=c-1} \sum \sum \int d\vec{r} \Big[\sum_{i<j<k\le s} \sum \sum Z_{\sigma_i} Z_{\sigma_j} Z_{\sigma_k} \phi_{l_1}(|\vec{r}_i - \vec{r}|)\, \phi_{l_2}(|\vec{r}_j - \vec{r}|)\, \phi_{l_3}(|\vec{r}_k - \vec{r}|)$$

$$+ \frac{1}{2} \sum_{i\neq j} \sum Z_{\sigma_i}^2 Z_{\sigma_j} \phi_{l_1}(|\vec{r}_i - \vec{r}|)\, \phi_{l_2}(|\vec{r}_i - \vec{r}|)\, \phi_{l_3}(|\vec{r}_j - \vec{r}|) \Big] +$$

$$+ \frac{A^{c-2} B^2}{2} \sum_{l_1+l_2+l_3+l_4=c-2} \sum \sum \sum \int d\vec{r} \int d\vec{r}'\, \phi_{l_1}(|\vec{r} - \vec{r}'|)\, \phi_{l_2}(|\vec{r} - \vec{r}'|) \sum_{i<j\le s} \sum \phi_{l_3}(|\vec{r}_i - \vec{r}|)\, \phi_{l_4}(|\vec{r}' - \vec{r}_j|)$$

(3.61)

Using eq. (2.21) one may sum eq. (3.61) over c without difficulty to obtain

$$m_s^{(2)} = \sum_{c=0}^{\infty} (K_0^2)^c m_s^{(2,c)}$$

$$= m_s^{(0)} \Big\{ \frac{1}{2} \left(\bar{\Phi}_s^{(0)}\right)^2 K_0^2 B \int d\vec{r} \Big[\sum_{i<j<k\le s} \sum \sum \phi_0(|\vec{r}_i - \vec{r}|)\, \phi_0(|\vec{r}_j - \vec{r}|)\, \phi_0(|\vec{r}_k - \vec{r}|) +$$

$$* Z_{\sigma_i} Z_{\sigma_j} Z_{\sigma_k} + \tfrac{1}{2} \sum_{i \neq j} \sum Z_{\sigma_i}^2 Z_{\sigma_j} \phi_0^2(|\vec{r}_i - \vec{r}|)\, \phi_0(|\vec{r} - \vec{r}_j|)\Big]$$

$$+ \frac{\kappa_0^2 B}{2} \int d\vec{r} \int d\vec{r}' \Big[\phi_0(|\vec{r} - \vec{r}'|)\Big]^2 \sum_{i<j\leq s} \sum \phi_0(|\vec{r}_i - \vec{r}|)\, \phi_0(|\vec{r}' - \vec{r}_j|)\Big\}. \tag{3.62}$$

Eqs. (3.50), (3.62) give the first two terms of the Mayer result (eqs. (2.33) and (2.34)) but, as for the ordinary gas, it would appear very difficult to obtain the complete general expansion from eqs. (3.28), (3.29).

4. THE GENERAL CASE

If both long and short range forces are present one would expect, from the results of the last two sections, that a triple expansion is necessary to solve the general equation (3.8). Here, however, a formal difficulty arises in that the three parameters used in sections 2 and 3 are not independent; in fact, we have the relation (3.27), i.e.,

$$\kappa_0^2 = \lambda/v$$

To resolve the difficulty, one must examine the terms in eq. (3.8) more closely. The remarks made in sections 2 and 3 about the terms proportional to $1/v$ and λ still apply. For the terms proportional to λ/v, one must establish a criterion for determining which terms will be small in $\lambda/(4\pi|r_{ij}|)$ and a^3/v and which terms will be of order one and thus should be associated with the auxiliary parameter κ_0^2. From the discussion of the previous sections, it is clear

that this will depend on the range of the integrand. The portions of the integrand which vanish for $|\vec{r}_i - \vec{r}| > a \quad i = 1 \ldots s \quad a << v^{1/3}$ will give a contribution of order $\lambda/a \; a^3/v \cong (\kappa a)^2$ and may thus be considered as proportional to both small parameters. Terms which have no such "cutoff" however, or have an effective cutoff of order $1/\kappa$ will give a contribution of order unity, and thus may be identified with the auxiliary long range interaction parameter κ_0^2 .

In view of these considerations, it is convenient to split the function $m_{s+1}(y_1 \ldots y_s, y)$ into two parts:

$$m_{s+1}(y) = \overline{m}_{s+1}(y) + \tilde{m}_{s+1}(y) \tag{3.63}$$

where $\overline{m}_{s+1}(y)$ consists of all terms* in $m_{s+1}(y)$ which vanish for $|\vec{r}_i - \vec{r}| > b \quad i = 1 \ldots s \quad b << v^{1/3}$ and $\tilde{m}_{s+1}(y)$ contains all the remaining terms. As mentioned in Section 1, the function $f_s(y) = f_s(y_1 \ldots y_s; y)$ has a cutoff, i.e., it vanishes for $|\vec{r}_i - \vec{r}| > 2a \quad i = 1 \ldots s$, where a is the maximum

*This may be stated more precisely in terms of the graphs of Chapter II. The function $m_{s+1}(y)$ is a sum of integrals corresponding to various rooted graphs in which the roots are subsets of the s+1 points associated with the arguments $(y_1 \ldots y_s, y)$. This is the division into "terms" referred to above.
The classification of a graph as belonging to $\overline{m}_{s+1}$ or $\tilde{m}_{s+1}$ may be visualized in terms of a mechanical analogue. One may imagine each Coulomb bond as being replaced by a rubber band of infinite elasicity and each k-bond by a stiff inelastic wire which will break if any tension is applied. One then grasps the figure at the point associated with the argument y and pulls. If the figure stretches indefinitely, the graph belongs to $\tilde{m}_{s+1}$, if some wire eventually breaks, it belongs to $\overline{m}_{s+1}$.

collision diameter.

In the terms in (3.8) where $m_{s+1}(Y)$ is multiplied with $f_s(Y)$ the splitting (3.63) of $m_{s+1}(Y)$ is therefore not necessary. Hence, we replace (3.8) by:

$$\frac{\partial m_s}{\partial \vec{r}_i} + \lambda m_s \frac{\partial \Phi_s}{\partial \vec{r}_i} + \kappa_0^2 \int d\vec{r}\, Z_{\sigma_i} \sum_{\sigma} Z_\sigma \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \tilde{m}_{s+1}(Y)$$

$$- \frac{1}{\nu} \int d\vec{r} \sum_{\sigma} \frac{\partial f_s(Y)}{\partial \vec{r}_i} m_{s+1}(Y) + \lambda \left(\frac{1}{\nu}\right) \int d\vec{r}\, Z_{\sigma_i} \cdot$$

$$\cdot \sum_{\sigma} Z_\sigma \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \left[\tilde{m}_{s+1}(Y) + f_s(Y)\, m_{s+1}(Y) \right] = 0 \tag{3.8c}$$

One may solve (3.8c) by treating λ, $1/\nu$, κ_0^2 as independent parameters and then let $\kappa_0^2 \longrightarrow \lambda/\nu$. Following the previous discussion we will attempt to solve (3.8c) approximately by the expansion

$$m_s = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} m_s^{(a,b,c)} \left(\frac{1}{\nu}\right)^a \lambda^b (\kappa_0^2)^c = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left(\frac{1}{\nu}\right)^a \lambda^b m_s^{(a,b)} \tag{3.64}$$

Substituting (3.64) into (3.8c) and equating like powers of $\frac{1}{\nu}$, λ, κ_0^2, one finds

$$\frac{\partial m_s^{(a,b,c)}}{\partial \vec{r}_i} + m_s^{(a,b-1,c)} \frac{\partial \Phi_s}{\partial \vec{r}_i} + \int d\vec{r}\, Z_{\sigma_i} \sum_{\sigma} Z_\sigma \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} m_{s+1}^{(a,b,c-1)}(Y)$$

$$- \int d\vec{r} \sum_{\sigma} \frac{\partial f_s(Y)}{\partial \vec{r}_i} m_{s+1}^{(a-1,b,c)}(Y) + \int d\vec{r}\, Z_{\sigma_i} \sum_{\sigma} Z_\sigma \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \cdot$$

$$\cdot \left[\tilde{m}_{s+1}^{(a-1,b-1,c)}(Y) + f_s(Y)\, m_{s+1}^{(a-1,b-1,c)}(Y) \right] = 0 \tag{3.65}$$

For b = c = 0, (3.65) reduces to (3.14), so $m_s^{(a,0,0)}$ is the same as $m_s^{(a)}$ of section 2 (cf. eqs, (3.17), (3.21), and (3.26).

For b = 0, c > 0, eq. (3.65) becomes

$$\frac{\partial m_s^{(a,0,c)}}{\partial \vec{r}_i} + Z_{\sigma_i} \int d\vec{r} \sum_\sigma Z_\sigma \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \tilde{m}_{s+1}^{(a,0,c-1)}(Y)$$

$$- \int d\vec{r} \sum_\sigma \frac{\partial f_s(Y)}{\partial \vec{r}_i} m_{s+1}^{(a-1,0,c)}(Y) = 0 \qquad (3.66)$$

We first observe that

$$\tilde{m}_{s+1}^{(a,0,0)} = \frac{N_\sigma}{N} m_s^{(a,0,0)} \qquad (3.67)$$

independent of Y . This is clearly true for a = 0, 1, 2 by the results of section 2. It is true in general because $m_{s+1}^{(a,0,0)}(Y)$ depends on Y only through f in combinations such as $f(Y_i; Y)$, $\int dy' f(y_i y') f(y'; y_i)$ etc. that vanish for $|\vec{r}_i - \vec{r}| > a, 2a$ etc. We then assert that

$$m_s^{(a,0,c)} = 0 \qquad c > 0 \qquad (3.68)$$

Proof:

(a) Eq. (3.68) holds for a = 0, c = 1, since eq. (3.68) than becomes (using (3.67))

$$\frac{\partial m_s^{(0,0,1)}}{\partial \vec{r}_i} = - \int d\vec{r} Z_\sigma \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \frac{N_\sigma}{N} m_s^0$$

(b) If Eq. (3.68) holds for $a+c<n$, it holds for a + c = n. This is obvious from eq. (3.67) and the boundary condition (3.10). (For c = 1 in eq. (3.66), eq. (3.67) must also be used). Eq. (3.68), together with the remarks of the preceding page shows that $m_s^{(a,0)} = \sum_{c=0}^{\infty} (K_0^2)^c m_s^{(a,0,c)} = m_s^{(a,0,0)}$ is the same as $m_s^{(a)}$ obtained in section 2 in the absence of long-range forces.

If a = 0, eq. (3.65) becomes

$$\frac{\partial m_s^{(0,b,c)}}{\partial \vec{r}_i} + m_s^{(0,b-1,c)} \frac{\partial \Phi_s}{\partial \vec{r}_i} Z_{\sigma_i} \int d\vec{r} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \sum_{\sigma} Z_\sigma m_s^{(0,b,c-1)}(y) = 0 \tag{3.69}$$

Since no short-range bonds are involved in the equations (3.69), it is clear that

$$\tilde{m}_{s+1}^{(0,b,c-1)}(y) = m_{s+1}^{(0,b,c-1)}(y) \tag{3.70}$$

Eqs. (3.69), and (3.70) are equivalent to eq. (3.29) so that $m_s^{(0,1,c)}$ and $m_s^{(0,2,c)}$ can be determined as in section 3, and are given by eqs. (3.47), and (3.61).

We have thus to consider only the "cross-terms" for which neither a nor b are zero. For a = 1, b = 1, c = 0, eq. (3.66) becomes

$$\frac{\partial m_s^{(1,1,0)}}{\partial \vec{r}_i} + m_s^{(0)} \int d\vec{r} \sum_{\sigma} \frac{N_\sigma}{N} \left[f_{s+1}(y) - \sum_{i=1}^{s} f_1(y_i, y) \right] \frac{\partial \Phi_s}{\partial \vec{r}_i} + m_s^{(0)} \int d\vec{r} \sum_{\sigma} \frac{\partial f_s(y)}{\partial \vec{r}_i} \frac{N_\sigma}{N} \cdot$$
$$\cdot \left[\Phi_s + \sum_{j=1}^{s} Z_\sigma Z_{\sigma_j} \phi(|\vec{r} - \vec{r}_j|) \right] + m_s^{0} \int d\vec{r} Z_{\sigma_i} \sum_{\sigma} Z_\sigma \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \frac{N_\sigma}{N} f_s(y) = 0 \tag{3.71}$$

where we have used eq. (3.21) for $m_s^{(1,0,0)}$, eqs. (3.31) and (3.40) for $m_{s+1}^{(0,1,0)}(y)$ and eq. (3.17) for $m_{s+1}^{(0,0,0)} = m_{s+1}^{(0)}$.

For a = b = 1, c > 0 one has

$$\frac{\partial m_s^{(1,1,c)}}{\partial \vec{r}_i} + z_{\sigma_i}\int d\vec{r}\,\frac{\partial \phi(|\vec{r}_i-\vec{r}|)}{\partial \vec{r}_i}\sum_{\sigma} z_{\sigma}\,\tilde{m}_{s+1}^{(1,1,c-1)}(y)$$

$$+ (-1)^c A^c m_s^{(0)}\int d\vec{r}\sum_{\sigma}\frac{N_\sigma}{N}\frac{\partial f_s(y)}{\partial \vec{r}_i}\left[\Phi_s^{(c)} + \sum_{j=1}^{s} z_\sigma z_{\sigma_j}\,\phi_c(|\vec{r}_j-\vec{r}|)\right] = 0 \tag{3.72}$$

The solution of eq. (3.71) which satisfies the boundary condition is

$$m_s^{(1,1,0)} = -m_s^{(0)}\left\{\Phi_s\int d\vec{r}\sum_{\sigma}\frac{N_\sigma}{N}\left[f_s(y) - \sum_{i=1}^{s} f(y_i;y)\right]\right.$$

$$\left.+\sum_{\sigma} z_\sigma \frac{N_\sigma}{N}\int d\vec{r}\left[\sum_{j=1}^{s} z_{\sigma_j}\,\phi(|\vec{r}_j-\vec{r}|)\,f_s(y) - \sum_{j=1}^{s} z_{\sigma_j}\,\phi(|\vec{r}_i-\vec{r}|)\,f(y_j;y)\right]\right\} \tag{3.73}$$

The form of eq. (3.73) suggests a substitution

$$m_s^{(1,1,c)} = (-1)^{c+1} m_s^{(0)}\left\{A^c\,\Phi_s^{(c)}\int d\vec{r}\sum_{\sigma}\frac{N_\sigma}{N}\left[f_s(y) - \sum_{i=1}^{s} f(y_i;y)\right]\right.$$

$$\left. A^c\sum_{\sigma} z_\sigma \frac{N_\sigma}{N}\int d\vec{r}\sum_{i=1}^{s} z_{\sigma_i}\,\phi_c(|\vec{r}_i-\vec{r}|)\left[f_s(y) - f(y_i;y)\right] + f_s^{(1,1,c)}\right\} \tag{3.74}$$

It is clear that the terms in $m_s^{(1,1,c)}(y)$ which contribute to $\tilde{m}_s^{(1,1,c)}(y)$ are: (a) terms independent of y, (b) terms in which the point y is connected to the fixed points y_i or additional integration variables y' etc. by means of one or more ϕ-bond chains (no f-bonds) and (c) the point y is connected (directly or indirectly) to some integration variables whose only direct connection to the remaining points (fixed or integration) is by means of ϕ-bond chains. It follows that ;

$$\tilde{m}_{s+1}^{(1,1,c)}(\gamma) = \frac{N_\sigma}{N} m_s^{(1,1,c)} + (-1)^c A^c m_s^{(0)} \frac{N_\sigma}{N} \left\{ \sum_{i=1}^{s} \phi_c(|\vec{r}_i - \vec{r}|) \int d\vec{r}' \sum_{\sigma'} \frac{N_{\sigma'}}{N} \cdot \right.$$

$$\cdot \left[f_s(\gamma') - \sum_{j=1}^{s} f(\gamma, \gamma') \right] + \sum_{\sigma'} Z_{\sigma'} \frac{N_{\sigma'}}{N} \int d\vec{r}' \left[Z_{\sigma'} \phi_c(|\vec{r}_i - \vec{r}|) f_s(\gamma') \right.$$

$$\left. \left. + \sum_{i=1}^{s} \phi_c(|\vec{r}_i - \vec{r}|) f(\gamma, \gamma') \right] \right\} + \tilde{\rho}_s^{(1,1,c)} \qquad (3.75)$$

Putting (3.74), and (3.75) in (3.72)

$$\frac{\partial \rho_s^{(1,1,c)}}{\partial \vec{r}_i} = A^{c-1} \sum_{\sigma} \sum_{\sigma'} \frac{Z_\sigma N_\sigma}{N} \frac{Z_{\sigma'} N_{\sigma'}}{N} \int d\vec{r} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} Z_{\sigma_i} \sum_{j=1}^{s} Z_{\sigma_j} \int d\vec{r}' \cdot$$

$$\cdot \phi_{c-1}(|\vec{r}_j - \vec{r}'|) f(\gamma; \gamma') + Z_{\sigma_i} \sum_{\sigma} Z_\sigma \frac{N_\sigma}{N} \int d\vec{r} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \tilde{\rho}_{s+1}^{(1,1,c-1)}(\gamma)$$

(3.76)

Using the fact that $\rho_s^{(1,1,0)} = 0$, one readily shows that the solution of the infinite set of equations (3.76) is

$$\rho_s^{(1,1,c)} = A^{c-1} \sum_{\sigma} \sum_{\sigma'} \frac{Z_\sigma N_\sigma}{N} \frac{Z_{\sigma'} N_{\sigma'}}{N} \sum_{\ell_1 + \ell_2 = c-1} \sum \int dr \int dr' \sum_{i \le j \le s} \sum Z_{\sigma_i} Z_{\sigma_j} \cdot$$

$$\cdot \phi_{\ell_1}(|\vec{r}_i - \vec{r}|) f(\gamma; \gamma') \phi_{\ell_2}(|\vec{r}' - \vec{r}_j|) \qquad (3.77)$$

Putting (3.77) in (3.76),

$$m_s^{(1,1,c)} = (-1)^{c+1} m_s^{(0)} \left\{ A^c \Phi_s^{(c)} \int d\vec{r} \sum_{\sigma} \frac{N_\sigma}{N} \left[f_s(\gamma) - \sum_{i=1}^{s} f(\gamma_i; \gamma) \right] \right.$$

$$\left. + A^c \int d\vec{r} \sum_{i=1}^{s} Z_{\sigma_i} \phi_c(|\vec{r}_i - \vec{r}|) \sum_{\sigma} \frac{Z_\sigma N_\sigma}{N} \left[f_s(\gamma) - f(\gamma_i; \gamma) \right] \right.$$

$$+A^{c-1}\sum_{l_1+l_2=c-1}\sum\int d\vec{r}\int d\vec{r}'\sum_{i\neq j\leq s}\sum Z_{\sigma_i}Z_{\sigma_j}\phi_{l_1}(|\vec{r}_i-\vec{r}|)\phi_{l_2}(|\vec{r}'-\vec{r}|)\sum_{\sigma}\sum_{\sigma'}\frac{Z_\sigma Z_{\sigma'}N_\sigma N_{\sigma'}}{N\,N}f(y_i;y') \tag{3.78}$$

Eq. (3.78) may be summed using eq. (2.21) to give

$$m_s^{(1,1)}=\sum_{c=0}^{\infty}(K_0^2)^c m_s^{(1,1,c)}=-m_s^{(0)}\left\{\Phi_s^D\int d\vec{r}\sum_{\sigma}\frac{N_\sigma}{N}\left[f_s(y)-\sum_{i=1}^{s}f(y_i;y)\right]\right.$$

$$\left.+\int d\vec{r}\sum_{i=1}^{s}Z_{\sigma_i}\phi_D(|\vec{r}_i-\vec{r}|)\sum_{\sigma}Z_\sigma\frac{N_\sigma}{N}\left[f_s(y)-f(y_i;y)\right]\right.$$

$$\left.+\int d\vec{r}\int d\vec{r}'\sum_{i\neq j\leq s}\sum Z_{\sigma_i}Z_{\sigma_j}\phi_D(|\vec{r}_i-\vec{r}|)\phi_D(|\vec{r}'-\vec{r}_j|)\sum_{\sigma}\sum_{\sigma'}\frac{Z_\sigma N_\sigma}{N}\frac{Z_{\sigma'}N_{\sigma'}}{N}f(y;y')\right. \tag{3.79}$$

Eqs. (3.17), (3.21), (3.26), (3.68), (3.51), (3.62), and (3.79) give all terms in the expansion (3.64) for which $a + b \leq 2$. Comparison with (2.33), and (2.34) shows that these are identical with the corresponding terms in the Mayer theory.

5. COMPARISON WITH OTHER METHODS

The method outlined in this chapter has certain advantages and disadvantages as compared to the Mayer treatment discussed in Chapter II.

As seen above, the principal disadvantages is that the complete expansion is difficult to obtain in general, and, in fact, has never

been found by the differential equation method as yet.

The method expounded here has at least two advantages over the Mayer theory:

(1) the physical meaning of the expansion is shown a little more clearly, and (2) the method may be generalized to cover non-equilibrium situations. Advantage (2) is, of course, more important, and forms the basis for Part II of this dissertation.

Compared to the original Bogolubov method for long-range forces,[6] the present approach has the advantage of showing how to obtain the distribution functions to an arbitrary degree of approximation.

PART II

THE NON-EQUILIBRIUM THEORY OF FULLY IONIZED GASES

CHAPTER IV

THE GENERAL THEORY

1. THE LIOUVILLE EQUATION; THE B-B-G-K-Y HIERARCHY

For the general case of a gas not in equilibrium, all information about the behavior of the distribution functions is contained in the Liouville equation. As in Part I we consider an arbitrary number of the types of ions, with N_σ of type σ ($\sigma = 1, 2 \cdots \mu$). The mass and charge of ions of type σ are denoted by m_σ and $Z_\sigma e$ respectively, where e is the charge of the electron. If the particles are numbered from 1 to $N = \sum_\sigma N_\sigma$, the k^{th} particle is specified by $y_k = (\vec{r}_k, \vec{p}_k, \sigma_k)$ where $\vec{r}_k$, $\vec{p}_k$ are its position and momentum vectors and σ_k (an integer between 1 and μ) gives its type. The numbers N_σ are related to the σ_j by

$$N_\sigma = \sum_{j=1}^{N} \delta_{\sigma\sigma_j} \tag{4.1}$$

and the condition for electrical neutrality is

$$\sum_{\sigma=1}^{N} Z_\sigma N_\sigma = \sum_{j=1}^{N} Z_{\sigma_j} = 0 \tag{4.2}$$

We define a distribution function $D(y_1 \text{ --------} y_N)$ such that $D(y_1 \text{----} y_N) \prod_{j=1}^{N} d\vec{r}_j\, d\vec{p}_j$ gives the probability of finding the system in a volume element $\prod_j d\vec{r}_j\, d\vec{p}_j$ about the point $(\vec{r}_1, \vec{p}_1, \text{----------} \vec{r}_N, \vec{p}_N)$ in the $6N$ dimensional phase space. D is clearly symmetric in the y's, i.e., independent of the numbering. In this notation the Liouville equation may be written

$$\frac{\partial D(y_1 \cdots y_N)}{\partial t} = [H(y_1 \cdots y_N), D(y_1 \cdots y_N)] \tag{4.3}$$

where $H(y_1, y_2 \cdots y_N)$ is the total Hamiltonian of the system and the Poisson bracket is defined by

$$[A, B] = \sum_{j=1}^{N} \left(\frac{\partial A}{\partial \vec{r}_j} \cdot \frac{\partial B}{\partial \vec{p}_j} - \frac{\partial A}{\partial \vec{p}_j} \cdot \frac{\partial B}{\partial \vec{r}_j} \right) \tag{4.4}$$

In the present case we will assume a Hamiltonian of the form

$$H(y_1 \cdots y_N) = \sum_{j=1}^{N} \left[\frac{1}{2m_{\sigma_j}} \left(\vec{p}_j - \frac{Z_{\sigma_j} e \vec{A}(\vec{r}_j)}{c} \right)^2 + Z_{\sigma_j} e\, U(\vec{r}_j) \right]$$

$$+ 4\pi e^2 \sum_{i<} \sum_{j \le N} Z_{\sigma_i} Z_{\sigma_j} \phi(|\vec{r}_i - \vec{r}_j|) \tag{4.5}$$

where $\vec{A}$, U , are the vector and scalar potentials of the external electromagnetic field and

$$\phi(R) = \frac{e^{-\alpha R}}{4\pi R} \tag{4.6}$$

(the damping α of the screened Coulomb field will eventually be allowed to approach zero). Magnetic interactions have been neglected, but it is not difficult to generalize the theory to include them, as will be indicated in section 3. eq. (4.3) now becomes

$$\frac{\partial D}{\partial t} = \left[-\mathcal{H}_N + 4\pi e^2 \Theta_N\right] D \tag{4.7}$$

where the operators $\mathcal{H}_N$ and Θ_N are defined by:

$$\mathcal{H}_N = \sum_{j=1}^{N} \left\{ \frac{(p_{j\alpha} - Z_{\sigma_j} e A_\alpha(\vec{r}_j))}{m_{\sigma_j}} \frac{\partial}{\partial r_{j\alpha}} \right.$$

$$\left. + \left[\frac{Z_{\sigma_j} e}{m_{\sigma_j} c} \left(p_{j\beta} - \frac{Z_{\sigma_j} e}{c} A_\beta(\vec{r}_j) \right) \frac{\partial A_\beta}{\partial r_{j\alpha}}(\vec{r}_j) - Z_{\sigma_j} e \frac{\partial U(\vec{r}_j)}{\partial r_{j\alpha}} \right] \frac{\partial}{\partial p_{j\alpha}} \right\} \tag{4.8}$$

$$\Theta_N = \sum_{i=1}^{N} \sum_{j \neq i}^{N} Z_{\sigma_i} Z_{\sigma_j} \frac{\partial \phi(|r_i - r_j|)}{\partial r_{i\alpha}} \left(\frac{\partial}{\partial p_{i\alpha}} - \frac{\partial}{\partial p_{j\alpha}} \right) \tag{4.9}$$

and where the usual summation convention for the scalar products is used with greek letters for dummy indices.

In principle eq. (4.7) determines the development in time of the gas from any initial distribution. For the application of approximation methods, however, it is more convenient to integrate over the coordinates and momenya of all but a small number of particles. Accordingly we define the partial distributions

$$F_s(y_1 \cdots y_s) = v^s \prod_{i=1}^{s} N_{\sigma_i} \int \cdots \int \prod_{j+s+1}^{N} d\vec{r}_j \, d\vec{p}_j \, D(y_1 \cdots y_N) \tag{4.10}$$

If (4.7) is now integrated over all but the first s particles, one finds

$$\frac{\partial F_s(y_1 \cdots y_s)}{\partial t} = -\mathcal{H}_s F_s(y_1 \cdots y_s) + 4\pi\varrho^2 O_s F_s(y_1 \cdots y_s)$$
$$+ 4\pi\varrho^2 \left(\frac{N - \sum_{i=1}^{s} \delta_{\sigma\sigma_i}}{V}\right) \int dy \, \Phi_s(y_1 \cdots y_s ; y) F_{s+1}(y_1 \cdots y_s ; y) \tag{4.11}$$

where

$$\Phi_s(y_1 \cdots y_s ; y) = Z_\sigma \sum_{i=1}^{s} Z_{\sigma_i} \frac{\partial \phi(|\vec{r}_i - \vec{r}|)}{\partial \vec{r}_i} \cdot \frac{\partial}{\partial \vec{p}_i} \tag{4.12}$$

and we have used the shorthand

$$\sum_{\sigma=1}^{\nu} \int d\vec{p} \int d\vec{r} \equiv \int dy \,. \tag{4.13}$$

If we now let $V \to \infty$, $N \to \infty$, $\frac{V}{N} \to v$ but restrict ourselves to finite s, (4.11) becomes

$$\frac{\partial F_s}{\partial t} + \mathcal{H}_s F_s = 4\pi\varrho^2 O_s F_s + \frac{4\pi\varrho^2}{v} \int dy \, \Phi_s(y) F_{s+1}(y) \tag{4.14}$$

where the arguments $y_1 \cdots y_s$ have been omitted. The eqs. (4.14) are a form of the B-B-G-K-Y) hierarchy (Bogolubov, Born, Green, Kirkwood,

Yvon). The remainder of this chapter is devoted to a scheme for solving these equations approximately.

2. THE KINETIC STAGE

Following Bogolubov, we now assume that after a certain time τ_0 the effect of an initial disturbance is smoothed out by "collisions" so that the s-particle distributions ($S > 1$) and the time derivative of the one-particle distributions depend on time only through F_1.

$$F_s(y_1 \cdots y_s, t) = F_s(y_1 \cdots y_s | F_1)$$

(4.15)

$$\frac{\partial F_1(y_1, t)}{\partial t} = A(y_1 | F_1)$$

(4.16)

For an ordinary gas one argues that τ_0 should be of the order of the "time of a collision". In an ionized gas, where collisions are not sharply defined, the treatment of Part I suggests that τ_0 should be taken as the time required for an average particle to travel a mean Debye radius. In putting in the expression for the Debye radius $mv^2 \approx kT$, one then gets as order of magnitude;

$$\tau_0 \sim \left(\frac{m}{4\pi e^2 n}\right)^{1/2}$$

(4.17)

For protons at a density of 10^{16} $(cm)^{-3}$ this is about 10^{-11} sec.; the electron distributions are smoothed out more quickly because of their

smaller mass.

Eqs. (4.14) resemble eqs. (3.8b), to which they reduce in equilibrium, very closely. From the discussion of Chapter III, it should be possible to treat $4\pi\ell^2$ as a small parameter (corresponding to λ in Chapter III) provided the particle separations are not too small. Similarly $4\pi\ell^2/\nu$ may be used as an auxiliary parameter (corresponding to κ_0^2). We thus propose to expand the quantities F_s and A of eqs. (4.15-16) as follows*:

$$F_s(y_1 \cdots y_s | F_1) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_s^{(m,n)}(y_1 \cdots y_N | F_1)(4\pi\ell^2)^m\left(\frac{4\pi\ell^2}{\nu}\right)^n = \sum_{m=0}^{\infty} F_s^{(m)}(1 F_1)(4\pi\ell^2)^m \tag{4.18}$$

$$A(y_1 | F_1) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(y_1 | F_1)(4\pi\ell^2)^m\left(\frac{4\pi\ell^2}{\nu}\right)^n = \sum_{m=0}^{\infty} A_m(y_1 | F_1)(4\pi\ell^2)^m \tag{4.19}$$

which define the quantities $F_s^{(m,n)}$, $F_s^{(m)}$, A_{mn} and A_m.

In addition the time derivative of any functional of F_1 must be expanded

$$\frac{\partial}{\partial t} F(1 F_1) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (4\pi\ell^2)^m\left(\frac{4\pi\ell^2}{\nu}\right)^n D_{mn} F(1 F_1) \tag{4.20}$$

where

$$D_{mn} F(1 F_1) = \left[A_{mn} \frac{\delta F}{\delta F_1}\right] \tag{4.21}$$

*Note that we expand $\frac{\partial F_1}{\partial t}$ and not F_1; in equilibrium $\frac{\partial F_1}{\partial t}$ vanishes but F_1 is Maxwellian and independent of the parameters.

Putting (4.16), (4.18), (4.19) in (4.14) for s = 1 and equating coefficients, one finds

$$A_{00}(y_1|F_1) = -\mathcal{H}_1(y_1)F_1(y_1) \tag{4.22}$$

$$A_{m0}(y_1|F_1) = 0 \qquad m>0 \tag{4.23}$$

$$A_{mn}(y_1|F_1) = \int dy\, \Phi_1(y_1;y) F_2^{(m,n-1)}(y_1,y) \qquad n>0 \tag{4.24}$$

Similarly, substituting (4.18), (4.20) in (4.14) for .

$$\sum_{p=0}^{m}\sum_{q=0}^{n} D_{pq} F_s^{(m-p,n-q)} + \mathcal{H}_s F_s^{(m,n)} = \Theta_s F_s^{(m-1,n)} + \int dy\, \Phi_s(y) F_{s+1}^{(m,n-1)}(y) \tag{4.25}$$

where $F_s^{(\tau,u)}$ is understood to be zero for t or u < 0 .

To solve eqs. (4.25) we introduce the kinetic streaming operator*

$$\mathcal{S}_\tau^{(s)} = e^{\tau \mathcal{H}_s} \tag{4.26}$$

*Note that $\mathcal{S}_\tau^{(s)}$ only contains the Hamilton operator $\mathcal{H}_s$ and not the interaction Θ_s . This is in contrast with the Bogolubov theory for neutral gases (short range forces). Also in Bogolubov's treatment of long range forces, the average field contribution is included in $\mathcal{H}_s$. The difficulty with this is that the $\mathcal{S}_\tau^{(s)}$ operator no longer has a simple exponential form. However the structure of the first approximation is formally simpler.

and require, following Bogolubov, that

$$\lim_{\tau\to\infty} \mathcal{S}_{-\tau}^{(s)}\left[F_s(y_1\cdots y_s \mid \mathcal{S}_\tau^{(1)} F_1) - \prod_{j=1}^{s} \mathcal{S}_\tau^{(1)}(y_j)\, F_1(y_j)\right] = 0 \tag{4.27}$$

In terms of the coefficients $F_s^{(m,n)}$ this implies

$$\lim_{\tau\to\infty} \mathcal{S}_{-\tau}^{(s)} F_s^{(0,0)}(y_1\cdots y_s \mid \mathcal{S}_\tau' F_1) = \mathcal{S}_{-\tau}^{(s)} \prod_{1}^{s} \mathcal{S}_\tau^{(1)} F_1(y_j) = \prod^{s} F_1(y_j) \tag{4.28}$$

where we have made use of the fact that due to the uncoupled nature of the Hamiltonian,

$$\mathcal{S}_\tau^{(s)}(y_1\cdots y_s) = \prod_{1}^{s} \mathcal{S}_\tau^{(1)}(y_i) \tag{4.28}$$

For the higher approximations

$$\lim_{\tau\to\infty} \mathcal{S}_{-\tau}^{(s)} F_s^{(m,n)}(\mid \mathcal{S}_\tau^{(1)} F_1) = 0 \tag{4.29}$$

The solution of eqs. (4.25) with the boundary conditions (4.28), (4.29) gives the higher distributions F_s as functionals of F_1. Substitution in (4.22) - (4.24) then gives the kinetic equations (analogues of the Boltzmann equation) for F_1 to any desired order. In the remainder of the dissertation the first two approximations (m = o,1) are derived and discussed.

3. THE ZEROTH APPROXIMATION; THE VLASOV EQUATIONS

For m = 0, n = 0 eq. (4.25) becomes

$$\left(D_{00} + \mathcal{H}_s\right) F_s^{(0,0)}(1 F_1) = 0 \tag{4.30}$$

To solve this equation we replace F_1 by $\mathcal{S}_\tau^{(1)} F_1$ and observe that

$$\left[D_{00} F(1 F_1)\right]_{F_1 \to \mathcal{S}_\tau^{(1)} F_1} = \left[-\mathcal{H}_1 \mathcal{S}_\tau^{(1)} F_1 \; \frac{\delta F(1 \mathcal{S}_\tau^{(1)} F_1)}{\delta(\mathcal{S}_\tau^{(1)} F_1)}\right]$$

$$= -\left[\frac{d \mathcal{S}_\tau^{(1)} F_1}{d\tau} \; \frac{\delta F(1 \mathcal{S}_\tau^{(1)} F_1)}{\delta(\mathcal{S}_\tau^{(1)} F_1)}\right] = -\frac{d}{d\tau} F(1 \mathcal{S}_\tau^{(1)} F_1) \tag{4.31}$$

Thus (4.30) may be written

$$\left(\frac{d}{d\tau} - \mathcal{H}_s\right) F_s^{(0,0)}(1 \mathcal{S}_\tau^{(1)} F_1) = 0 \tag{4.32}$$

which has the "solution"

$$F_s^{(0,0)}(1 \mathcal{S}_\tau^{(1)} F_1) = \mathcal{S}_{-\tau}^{(s)} F_s^{(0,0)}(1 F_1)$$

or

$$F_s^{(0,0)}(1 F_1) = \mathcal{S}_{-\tau}^{(s)} F_s^{(0,0)}(1 \mathcal{S}_\tau^{(1)} F_1) \tag{4.33}$$

Since the left side of (4.33) is independent of τ, we may take the limit as $\tau \to \infty$ and use the boundary condition (4.28) to obtain

$$F_s^{(0,0)}(y_1 \cdots y_s | F_1) = \lim_{\tau \to \infty} \mathcal{S}_{-\tau}^{(s)} F_s^{(0,0)}(y_1 \cdots y_s | \mathcal{S}_\tau^{(1)} F_1) = \prod_1^s F_1(y_i)$$

(4.34)

From (4.24) we have

$$A_{01}(y_1 | F_1) = \int dy \, \Phi_1(y_1; y) F_1(y_1) F_1(y)$$

(4.35)

It follows that

$$D_{01} F_s^{(0,0)} = \sum_{i=1}^{s} \prod_{j \neq i}^{s} F_1(y_j) \int dy \, \Phi_1(y_i; y) F_1(y) F_1(y_i)$$

$$= \int dy \, \Phi_s(y_1 \cdots y_s; y) F_1(y) F_s^{(0,0)}(y_1 \cdots y_s)$$

(4.36)

since $\Phi_s(y_1 \cdots y_s; y) = \sum_{j=1}^{s} \Phi_1(y_j, y)$

The equation for $F_s^{(0,1)}$ is now, using (4.36),

$$(D_{00} + \mathcal{N}_s) F_s^{(0,1)} = 0$$

(4.37)

This is precisely the same as (4.30); the only difference is that we must now use the boundary condition (4.29). Thus

$$F_s^{(0,1)}(|F_1) = \lim_{\tau \to \infty} \mathcal{S}_{-\tau}^{(s)} F_s^{(0,1)}(|\mathcal{S}_\tau^{(1)} F_1) = 0$$

(4.38)

The general equation for $F_s^{(0,n)}$ is

$$(D_{oo} + \mathcal{H}_s) F_s^{(o,n)} = \int dy\, \Phi_s(y) F_{s+1}^{(o,n-1)}(y) - \sum_{q=o}^{n} D_{oq} F_s^{(o,n-q)}$$

(4.39)

Using (4.38), (4.24), one readily shows by induction from (4.39) that

$$F_s^{(o,n)} = 0 \qquad n \geq 1$$

(4.40)

and hence

$$A_{om} = 0 \qquad m > 1$$

(4.41)

This completes the derivation of the zeroth approximation. The kinetic equation to this order is given by:

$$\frac{\partial F_1}{\partial t} = A_o(y_1 | F_1) = A_{oo}(y_1 | F_1) + \frac{4\pi e^2}{\nu} A_{o1}(y_1 | F_1)$$

$$= -\mathcal{H}_1 F_1 + \frac{4\pi e^2}{\nu} \int dy\, \Phi_1(y_1; y) F_1(y_1) F_1(y)$$

(4.42).

Introducing the explicit forms for the operators $\mathcal{H}_1$ and Φ_1, and using instead of the momentum the velocity:

$$\vec{w} = \frac{1}{m_\sigma} \left(\vec{p} - \frac{e Z_\sigma}{c} \vec{A}(r,t) \right)$$

(4.43)

as independent variable, it is easy to show that (4.42) can be written in the form:

$$\frac{\partial F_1}{\partial t} + w_{1\alpha} \frac{\partial F_1}{\partial r_{1\alpha}} + a_{1\alpha} \frac{\partial F_1}{\partial w_{1\alpha}} = 0$$

(4.44)

where the acceleration $\vec{a}$ is given by:

$$\vec{a}_1 = \frac{e Z_{\sigma_1}}{m_{\sigma_1}} \left\{ \vec{E}_0(\vec{r}_1) + \vec{E}_i(\vec{r}_1) + \frac{1}{c} [\vec{w}_1 \times \vec{H}_0] \right\}$$

Here $\vec{H}_0 = \text{CURL } \vec{A}$

$$\vec{E}_0 = \text{GRAD } U - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

(4.45)

are the outside magnetic and electric fields and:

$$\vec{E}_i = \frac{4\pi e}{v} \int dy \, Z_\sigma F_1(y) \frac{\partial \Phi(|\vec{r}_1 - \vec{r}|)}{\partial \vec{r}_1}$$

is clearly the average electric field at the point $\vec{r}_1$ due to all the other particles in the gas. Before discussing these results, we will indicate how (4.47) can be generalized to include magnetic interactions between the moving ions. Up to order of $(\bar{w}/c)^2$, where $\bar{w}$ is an average velocity, on obtains the <u>same</u> equation (4.44), but the acceleration $\vec{a}_1$ is now given by:

$$\vec{a}_1 = \frac{e Z_{\sigma_1}}{m_1} \left\{ \vec{E} + \frac{1}{c} [\vec{w}_1 \times \vec{H}] \right\}$$

with: $$\vec{E} = \vec{E}_0 + \vec{E}_i \,, \quad \vec{H} = \vec{H}_0 + \vec{H}_i$$

and where the average interaction fields $\vec{E}_i$, $\vec{H}_i$, satisfy the Maxwell equations:

$$\nabla \cdot \vec{E}_i = \frac{4\pi e}{v} \sum_\sigma Z_\sigma \int d\vec{w}\, F_1(\vec{r}, \vec{w}, \sigma, t) \tag{4.46}$$

$$\nabla \times \vec{E}_i = -\frac{1}{c} \frac{\partial \vec{H}_i}{\partial t} \tag{4.47}$$

$$\nabla \cdot \vec{H}_i = 0 \tag{4.48}$$

$$\nabla \times \vec{H}_i = \frac{4\pi e}{vc} \sum_\sigma Z_\sigma \int d\vec{w}\, \vec{w}\, F_1(\vec{r}, \vec{w}, \sigma, t) \tag{4.49}$$

We now return to eq. (4.45), neglecting from now on the magnetic interaction. If in addition there are no outside fields then, returning to the momentum variable, eq. (4.47) may be written as:

$$\frac{\partial F_1(y,t)}{\partial t} + \frac{\vec{p}}{m_\sigma} \cdot \frac{\partial F_1}{\partial \vec{r}} - \frac{4\pi e^2}{v} Z_\sigma \int dy' Z_{\sigma'} F_1(y') \frac{\partial \phi(|\vec{r} - \vec{r}'|)}{\partial \vec{r}} \cdot \frac{\partial F_1(y)}{\partial \vec{p}} = 0 \tag{4.50}$$

In this approximation, this is the basic kinetic equation which for the plasma plays the role similar to the Boltzmann equation for the case of a neutral gas. Just as the Boltzmann equation, it is non-linear and of the first order in the time derivative. However, in contrast to the Boltzmann equation, it is clear that <u>any</u> spatially uniform distribu-

tion $F_1 = F(\vec{p}, \sigma)$ is a solution of (4.50), since then there will be no electric field (because of charge neutrality), so that each term in (4.59) vanishes. The Vlasov equation can therefore not be expected to determine the velocity distribution of the ions.

Because (4.59) is of the first order in the time derivative, the basic problem will be the initial value problem: given F_1 for t = 0, find F_1 for any later time. The solution is quite difficult for the non-linear equation, but it can be solved for a small disturbance from a spatially uniform state. Put:

$$F_1(y,t) = F^{(0)}(\eta) + F^{(1)}(y,t) \tag{4.51}$$

Here η stands for $(\vec{p}, \sigma)$ and we will use the shorthand.

$$\int d\eta \equiv \sum_{\sigma} \int d\vec{p}$$

Neglecting second order terms in $F^{(1)}$ the linearized version of (4.59) becomes:

$$\frac{\partial F^{(1)}}{\partial t} + \frac{\vec{p}}{m_\sigma} \cdot \frac{\partial F^{(1)}}{\partial \vec{r}} - \frac{4\pi e^2}{\sigma} Z_\sigma \frac{\partial F^{(0)}}{\partial \vec{p}} \cdot \int dy' Z_{\sigma'} F^{(1)}(y') \frac{\partial \phi(\vec{r}-\vec{r}')}{\partial \vec{r}} = 0 \tag{4.52}$$

Landan[10] was the first to solve the initial value problem for the linearized Vlasov equation. He used the method of Laplace transforms. Later van Kampen[12] solved the same problem by using a development in normal modes. Case[15] has shown the precise connection between the two

methods. All these authors restrict themselves to the special case of a gas of electrons moving in a stationary positive background. However the generalization to the case of a mixture of an arbitrary number of ion types is quite straight forward. We will sketch here the normal mode treatment for the general case.

Taking Fourier transforms in $\vec{r}$ and defining

$$g(\eta,\vec{k},t)=\frac{1}{(2\pi)^3}\int d\vec{r}\, e^{-i\vec{k}\cdot\vec{r}}F^{(1)}(\eta,\vec{r},t) \tag{4.53}$$

one finds for g

$$\frac{\partial g(\eta,\vec{k},t)}{\partial t}+\frac{i\vec{k}\cdot\vec{p}}{m_\sigma}g(\eta,\vec{k},\tau)+\frac{iD_0(\eta,\vec{k})}{\pi k}\int d\eta' Z_{\sigma'}g(\eta,\vec{k},t)=0 \tag{4.54}$$

where*

$$D_0(\eta,\vec{k})=-\frac{4\pi e^2}{\nu}\frac{\pi}{k}Z_\sigma\vec{k}\cdot\frac{\partial}{\partial\vec{p}}F^{(0)}(\eta) \tag{4.55}$$

We now seek eigenfunctions of the form

$$g(\eta,\vec{k},t)=g_u(\eta,\vec{k})e^{-iku t} \tag{4.56}$$

*The definition of D_0 and the normalization of the eigenvalues are chosen to agree with Chapter V. We have used the fact that

$$\lim_{\alpha\to 0}\int d\vec{r}\,\frac{e^{-i\vec{k}\cdot\vec{r}}}{4\pi|\vec{r}|}e^{-\alpha|\vec{r}|}=\frac{1}{k^2}$$

Substituting this form in (4.63) and suppressing $\vec{k}$ in the functions one finds

$$\left(u - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) g_u(\eta, \vec{k}) = \frac{D_0(\eta)}{\pi k^2} \int \eta' Z_\sigma, g_u(\eta') \tag{4.57}$$

The normalization of the eigenfunctions is still arbitrary; we take

$$\int d\eta' Z_\sigma, g_u(\eta') = k^2 \tag{4.58}$$

Eq. (4.66) then becomes

$$\left(u - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) g_u(\eta) = \frac{D_0(\eta)}{\pi} \tag{4.59}$$

It is convenient to introduce "reduced" eigenfunctions; for any function $h(\eta)$, we define

$$\bar{h}(w) = \int d\eta Z_\sigma \delta\left(w - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) h(\eta) \tag{4.60}$$

Performing the "barring" operation on (4.59), one finds

$$(w - u)\,\bar{g}_u(w) = -\frac{\bar{D}_0(w)}{\pi} \tag{4.61}$$

By comparison with (4.59), g_u is related to $\bar{g}_u$ by

$$g_u(\eta) = \frac{D_0(\eta)}{\bar{D}_0\left(\frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right)} \bar{g}_u\left(\frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) \tag{4.62}$$

provided D_0 is zero when $\bar{D}_0$ is, which is certainly true for $F^{(0)}$ Maxwellian. The general solution of eq. (4.61) has the form

$$\bar{g}_u(w) = -\frac{\bar{D}_0(w)}{\pi} P\left(\frac{1}{w-u}\right) + \lambda(w)\delta(u-w) \tag{4.63}$$

where the symbol P (f) indicates that the Cauchy principal value is to be taken for any integral over the function. Since

$$\int dw\, \bar{g}_u(w) = \int d\eta\, Z_0 g_u(\eta) \tag{4.64}$$

the coefficient $\lambda(w)$ is determined from the normalization condition (4.67) to be

$$\lambda(w) = k^2 + H_u[\bar{D}_0] \tag{4.65}$$

where the Hilbert transform operator H_u is defined by

$$H_u[F] = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{dx\, F(x)}{x-u} \tag{4.66}$$

Eqs. (4.62), (4.63), and (4.64) determine the eigenfunctions* of (4.54).

To determine the solution of (4.54) for a given initial distribution $g(\eta, 0)$, one expands the initial distribution in terms of the eigenfunctions $g_u(\eta)$:

*As pointed out by Case[15], there is also in general a discrete spectrum. However, as shown by van Kampen, if $F^{(0)}$ is Maxwellian the eigenfunctions (4.63) (corresponding to real u) form a complete set.

$$g(\eta,0) = \int_{-\infty}^{\infty} du\, A(u,\vec{k})\, g_u(\eta)$$

(4.67)

Applying the "barring" operation (4.60), and using (4.63), the following integral equation is obtained for A(u)

$$\lambda(u)A(u) + \bar{D}_0(u)\, H_u[A] = \bar{g}(u,0)$$

(4.68)

This is a standard type of integral equation similar to those studied in some detail in Chapter V. We here merely state without proof that it has the unique solution

$$A(u) = \frac{\bar{g}_+(u,0)}{\varrho_0(u)} + \frac{\bar{g}_-(u,0)}{\varrho_0^*(u)}$$

(4.69)

where

$$\varrho_0(u) = \lambda(u) + i\,\bar{D}_0(u)$$

(4.70)

and

$$\bar{g}_\pm(u,0) = \int_{-\infty}^{\infty} dx\, \delta_\pm(u-x)\, \bar{g}(x,0)$$

(4.71)

$$\delta_\pm(y) = \lim_{\epsilon\to 0} \frac{1}{2\pi} \int_0^\infty e^{\pm ik(y\pm i\epsilon)}\, dk = \frac{1}{2}\delta(y) \pm \frac{i}{2\pi} P\left(\frac{1}{y}\right)$$

(4.72)

The solution of the initial value problem is then given by

$$g(\eta,t) = \int_{-\infty}^{\infty} d\omega\, A(\omega) e^{-i k \omega t} g_\omega(\eta) \tag{4.73}$$

and for the average potential one finds

$$\begin{aligned}\langle \Phi \rangle &= \frac{4\pi e}{v} \int dy' Z_{o1} \phi(|\vec{r} - \vec{r}'|) F^{(1)}(y',t) \\ &= \frac{4\pi e}{v} \int d\vec{r}' \phi(|\vec{r} - \vec{r}'|) \int d\vec{k} \int_{-\infty}^{\infty} d\omega\, e^{i[\vec{k}\cdot\vec{r} - k\omega t]} A(\omega) \cdot \\ &\quad \cdot \int d\eta' Z_{o1} g_\omega(\eta') \\ &= \frac{4\pi e}{v} \int d\vec{k} \int_{-\infty}^{\infty} d\omega\, e^{i[\vec{k}\cdot\vec{r} - k\omega t]} A(\omega)\end{aligned} \tag{4.74}$$

For $t > 0$, the first term of (4.69) does not contribute to (4.74), as can be seen by closing the path of the ω integration in the lower half plane. Thus the average potential is given by

$$\langle \Phi \rangle = \frac{4\pi e}{v} \int d\vec{k} \int_{-\infty}^{\infty} d\omega\, e^{i[\vec{k}\cdot\vec{r} - k\omega t]} \frac{\bar{g}_+(\omega,0)}{\rho_0(\omega)} \tag{4.75}$$

This is clearly a superposition of plane waves in which the amplitude is a product of two factors: $\bar{g}_+(\omega,0)$, which indicates how much of a given frequency was present in the initial distribution, and $1/\rho_0(\omega)$, a weight factor which determines the effectiveness of the given frequency in contributing to the average potential.

The factor $1/\varepsilon_0(\omega)$ seems to be characteristic of the collective interaction of charged particles; as we shall see in Chapter V, the average potential of (4.75) is intimately related to the effective potential of the "pseudo-collision" mechanism.

Van Kampen has shown how (4.75) may be evaluated approximately for various assumptions about the initial perturbation $g(\eta,k,0)$. A common feature is that for $t \to \infty$ the average potential always tends to zero, corresponding to a spatially uniform density distribution. In this sense the Vlasov equation shows with regard to the spatial distribution an irreversible behaviour.

4. THE FIRST APPROXIMATION

Returning to the general formalism of Section 2, we now wish to determine the functionals $F_2^{(1,n)}(y_1, y_2 | F_1)$, $A_{1n}(y_1 | F_1)$ for arbitrary n. Using eqs. (4.40), (4.41), eq. (4.25) may be written for $m = 1$, $n \geq 0$ as:

$$(D_{00} + \mathcal{H}_s) F_s^{(1,n)} = Q_s F_s^{(0,n)} + \int dy\, \Phi(y) F_s^{(1,n-1)} - D_{01} F_s^{(1,n-1)} - D_{1n} F_s^{(0,0)} \tag{4.76}$$

In order to solve this with the boundary condition (4.29), we state without proof* the following:

*The proof, which follows readily from (4.29), (4.31), and (4.26), may be found in Bogolubov[6] or Choh.[16]

Theorem I: Let $\Psi_s^{(m,n)}(1F_1)$ be a known functional; then the solution of the equation $(D_{00} + \mathcal{H}_s) F_s^{(m,n)}(1F_1) = \Psi_s^{(m,n)}(1F_1)$ which satisfies the boundary condition (4.29) is

$$F_s^{(m,n)}(1F_1) = \int_0^\infty d\tau \, \delta_{-\tau}^{(s)} \Psi_s^{(m,n)}(1 \, \delta_\tau^{(1)} F_1)$$

From the above theorem, it follows immediately* that the solution of eq. (4.76) for n = 0 is **

$$F_s^{(1,0)} = \int_0^\infty d\tau \, \delta_{-\tau}^{(s)} O_s \, \delta_\tau^{(s)} F_s^{(0,0)}$$

$$= \sum_{i<j\le s} \sum \prod_{k\neq i,j} F_1(y_k) \int_0^\infty d\tau \, \delta_{-\tau}^{(2)} O_2(y_i, y_j) \delta_\tau^{(2)} F_1(y_i) F_1(y_j)$$

(4.77)

The solution of (4.76) for n > 0 is simplified by the following theorem, which obviously holds for n = 0 and is proven by induction in Appendix II:

Theorem II: The solution of (4.76) which satisfies (4.29) has the form: $F_s^{(1,n)} = \sum_{i<j\le s} \sum F_2^{(1,n)}(y_i, y_j) \prod_{k\neq i,j} F_1(y_k)$.

Thus we may restrict ourselves to the case s = 2 for which (4.76) takes the form (using (4.23), (4.34), (4.12), (4.24), and Theorem II for s = 3):

*According to the convention used in writing (4.25)) of which (4.76) is a special case) the second and third terms on the right of (4.76) are not present for n = 0; the fourth vanishes by (4.23).

**We have used the explicit form (4.34) for $F_s^{(0,0)}$.

$$(D_{00}+\mathcal{H}_2)F_2^{(1,n)}(y_1,y_2)=\int dy\left\{F_2^{(1,n-1)}(y_1,y)\Phi(y_2;y)F_1(y_2)\right.$$
$$\left.+F_2^{(1,n-1)}(y_2,y)\Phi_1(y_1;y)F_1(y_1)+F_1(y)\Phi_2(y_1,y_2,y)F_2^{(1,n-1)}(y_1,y_2)\right\}$$
$$-D_{01}F_2^{(1,n-1)}(y_1,y_2) \tag{4.78}$$

Now note that in (4.78) the third term under the integral and also the last term (since it involves $A_{01}(y/F)$ contain the local charge density $\rho(r)$. Hence these terms will vanish in the spatially uniform case, where $\rho(r) = 0$ because of the electric neutrality. It is therefore advantageous to split $F_2^{(1,n)}$ into two parts:

$$F_2^{(1,n)}=C_n+D_n \tag{4.79}$$

where C_n satisfies

$$(D_{00}+\mathcal{H}_2)C_n(y_1,y_2)=\int dy\left[C_{n-1}(y_1,y)\Phi(y_2;y)F_1(y_2)\right.$$
$$\left.+C_{n-1}(y_2,y)\Phi_1(y_1;y)F_1(y_1)\right] \tag{4.80}$$

with

$$C_0=F_2^{(1,0)}=\int_0^\infty d\tau\, g_{-\tau}^{(2)}\,O_2\,g_\tau^{(2)}F_1(y_1)F_1(y_2) \tag{4.81}$$

and D_n accounts for the remaining terms

$$(D_{00} + \mathcal{H}_2) D_n(y_1, y_2) = \int dy \Big\{ D_{n-1}(y_1, y) \Phi_1(y_2; y) F_1(y_2) + D_{n-1}(y_2, y) \Phi_1(y_1; y) F_1(y_1) + F_1(y) \Phi_2(y_1, y_2; y) \cdot [C_{n-1}(y_1, y_2) + D_{n-1}(y_1, y_2)] \Big\} - D_{01} [C_{n-1}(y_1, y_2) + D_{n-1}(y_1, y_2)]$$

(4.82)

$$D_0 \equiv 0 \tag{4.83}$$

In the spatially uniform case $D_n \equiv 0$ as follows immediately by induction, and only the part C_n remains. C_n, as may be seen already from (4.80), involves the "linear chain" type interactions familiar from the study of the equilibrium state.

Using Theorem I, eq. (4.80) may be written as a recursion for the functionals C_n:

$$C_n(y_1, y_2) = \int dy \int_0^\infty d\tau \Big\{ O_2^\tau(y_2, y) C_{n-1}^\tau(y, y_1) F_1(y_2) + O_2^\tau(y_1, y) C_{n-1}(y, y_2) F_1(y_1) \Big\}$$

(4.84)

where

$$O_2^\tau(Y_1, Y_2) \equiv \delta_{-\tau}^{(2)}(Y_1, Y_2) O_2(Y_1, Y_2) \delta_\tau^{(2)}(Y_1, Y_2)$$

(4.85)

$$C_k^{\tau}(Y_1, Y_2) \equiv \mathcal{S}_{-\tau}^{(2)}(Y_1, Y_2) C_k(Y_1, Y_2 | \mathcal{S}_{\tau}^{(1)} F_1)$$

(4.86)

The particular form of (4.84) was obtained by observing that from the definitions (4.12), (4.9) of Φ_1, $\mathcal{O}_2$, and an integration by parts

$$\int dy\, \Phi_1(Y; y) F(y, Y) = \int dy\, \mathcal{O}_2(Y, y) F(y, Y)$$

(4.87)

A transformation of variables from y to $\mathcal{S}_{\tau}^{(1)} y$ was also used.

The solution of the recursion (4.84) may be written explicitly in the form

$$C_n(y_1, y_2) = \int_0^{\infty} d\tau_0 \int_{\tau_0}^{\infty} d\tau_1 \cdots \int_{\tau_{n-1}}^{\infty} d\tau_n \int \cdots \int dy^{(1)} \cdots dy^{(n)} \cdot$$

$$\cdot \sum_P \mathcal{O}_2^{\tau_0}(y_1, y^{(1)})\, \mathcal{O}_2^{\tau_1}(y^{(1)}, y^{(2)}) \cdots \mathcal{O}_2^{\tau_n}(y^{(n)}, y_2) \cdot$$

$$\cdot F(y_1) F(y_2) \prod_{i=1}^{n} F_1(y^i)$$

(4.88)

and $\sum P$ indicates a sum over all permutations of the argument pairs $(y_1, y^{(1)}) \cdots (y^{(n)}, y_2)$. The form (4.88), which is derived in Appendix II, shows clearly the chain nature of the interaction. A correspondence exists between the integrals and the rooted linear chains of Section 2.3, in which now a function F_1 is associated with each point

and an operator Θ_2 with each line in the graph. We will now show that in fact the integrals C_n reduce to the equilibrium Debye chains if F_1 is Maxwellian. The proof is based on the fact that if

$$F_1(y) = F_1(\eta) = \frac{N_\sigma}{N}(2\pi m_\sigma \theta)^{-3/2} e^{-\frac{p^2}{2m_\sigma\theta}} \tag{4.89}$$

then one easily shows that:

$$\Theta_2^\tau(y_1, y_2) F_1(y_1) F_1(y_2) = \frac{Z_{\sigma_1} Z_{\sigma_2}}{\theta} F_1(\eta_1) F_1(\eta_2) \cdot \cdot \frac{d}{d\tau}\phi\left(|r_{12} - \tau\left(\frac{\vec{p}_1}{m_{\sigma_1}} - \frac{\vec{p}_2}{m_{\sigma_2}}\right)|\right) \tag{4.90}$$

and

$$\int d\eta Z_\sigma F_1(\eta) = \sum_\sigma Z_\sigma^2 \frac{N_\sigma}{N} \equiv A \tag{4.91}$$

We now state the theorem:

Theorem III: If F_1 has the form (4.89), the solution of the recursion (4.84) is $C_n(y_1, y_2) = -Z_{\sigma_1} Z_{\sigma_2}\left(\frac{1}{\theta}\right)(-A/\theta)^n F_1(\eta_1) F_1(\eta_2)\phi_n(|\vec{r}_1 - \vec{r}_2|)$ where $\phi_n(|\vec{r}_1 - \vec{r}_2|) = \int \cdots \int d\vec{r}^{(1)} \cdots d\vec{r}^{(n)} \phi(|\vec{r}_1 - \vec{r}^{(1)}|) \prod_{k=1}^{n} \phi(|\vec{r}^{(k-1)} - \vec{r}^{(k)}|) \phi(|\vec{r}^{(n)} - \vec{r}_2|)$

Proof by induction:

(A) The theorem holds for n = 0. For using (4.90)

$$C_0 = \int_0^\infty d\tau\, \Theta_2^\tau(y_1, y_2) F_1(\eta_1) F_1(\eta_2) = \frac{Z_{\sigma_1} Z_{\sigma_2}}{\theta} F_1(\eta_1) F_1(\eta_2) \cdot \int_0^\infty d\tau \frac{d}{d\tau}\phi\left(|\vec{r}_1 - \vec{r}_2 - \tau\left(\frac{\vec{p}}{m_{\sigma_1}} - \frac{\vec{p}_2}{m_{\sigma_2}}\right)|\right)$$

$$= -\frac{Z_{\sigma_1} Z_{\sigma_2}}{\theta} F_1(\eta_1) F_1(\eta_2)\phi(|\vec{r}_1 - \vec{r}_2|) \tag{4.92}$$

(B) If the theorem holds for n = m - 1, it holds for n = m. For if F_1 has the form (4.89) and C_{m-1} satisfies the theorem, (4.84) becomes

$$C_m(y_1,y_2) = -Z_{\sigma_1} Z_{\sigma_2} F_1(\eta_1) F_1(\eta_2) \left(-\frac{A}{\theta}\right)^{m-1} \int d\eta\, Z_\sigma F_1(\eta) \int_0^\infty d\tau \cdot$$

$$\cdot \int d\vec{r} \left[\phi_{m-1}\left(|\vec{r} - \vec{r}_1 - \tau\left(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}_1}{m_{\sigma_1}}\right)|\right) \frac{d}{d\tau} \phi\left(|\vec{r} - \vec{r}_2 - \tau\left(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}_2}{m_{\sigma_2}}\right)|\right) \right.$$

$$\left. + \phi_{m-1}\left(|\vec{r} - \vec{r}_2 - \tau\left(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}_2}{m_{\sigma_2}}\right)|\right) \frac{d}{d\tau} \phi\left(|\vec{r}_1 - \vec{r} - \tau\left(\frac{\vec{p}_1}{m_{\sigma_1}} - \frac{\vec{p}}{m_\sigma}\right)|\right) \right]$$

$$= -\frac{Z_{\sigma_1} Z_{\sigma_2}}{\theta^2} F_1(\eta_1) F_1(\eta_2) \left(-\frac{A}{\theta}\right)^{m-1} \int d\eta\, Z_\sigma F_1(\eta) \int_0^\infty d\tau \cdot$$

$$\cdot \frac{d}{d\tau} \Phi_m\left(|\vec{r}_1 - \vec{r}_2 - \tau\left(\frac{\vec{p}_1}{m_{\sigma_1}} - \frac{\vec{p}_2}{m_{\sigma_2}}\right)|\right)$$

$$= -\frac{Z_{\sigma_1} Z_{\sigma_2}}{\theta} F_1(\eta_1) F_1(\eta_2) \left(-\frac{A}{\theta}\right)^m$$

which has the desired form.

The functional D_η is considerably more complicated, but much less important for states in which the charge density is nearly zero (it can be argued that such a state will be established in a very short time by electrostatic forces, cf. Spitzer.[17] D_n may also be written explicitely, but since we are really interested in $F_2^{(1)} = \sum \left(\frac{4\pi e^2}{\nu}\right)^n (C_n + D_n)$ it is more useful to obtain a formal expression for this.

The equations (4.78), (4.84) may be written formally as

$$F_2^{(1,n)} = (\mathcal{R}_0 + \mathcal{R}_1) F_2^{(1,n-1)} = (\mathcal{R}_0 + \mathcal{R}_1)^n F_2^{(1,0)} = (\mathcal{R}_0 + \mathcal{R}_1)^n C_0 \tag{4.93}$$

$$C_n = \mathcal{R}_0 C_{n-1} = \mathcal{R}_0^n C_0$$

(4.94)

where the operators $\mathcal{R}_0$, $\mathcal{R}_1$ are defined by

$$\mathcal{R}_0 F(y_1, y_2 | F_1) = \int dy \int_0^\infty d\tau \{ O_2^\tau(y_2, y) \mathcal{S}_{-\tau}^{(2)}(y, y_1) F(y, y_1 | \mathcal{S}_\tau^{(1)} F_1) F_1(y_2)$$
$$+ O_2^\tau(y_1, y) \mathcal{S}_{-\tau}^{(2)}(y, y_2) F(y, y_2 | \mathcal{S}_\tau^{(1)} F_1) F_1(y_1) \}$$

(4.95)

$$\mathcal{R}_1 F(y_1, y_2) | F_1) = \int_0^\infty d\tau \, \mathcal{S}_{-\tau}^{(2)}(y_1, y_2) \{ dy \, \mathcal{S}_\tau^{(1)}(y) F_1(y) \Phi_2(y_1, y_2; y) \cdot$$
$$F(y_1, y_2 | \mathcal{S}_\tau^{(1)} F_1) - [D_{01} F(y_1, y_2 | F_1)]_{F_1 \to \mathcal{S}_\tau^{(1)} F_1} \}$$

(4.96)

For $C \equiv \sum_0^\infty \left(\frac{4\pi e^2}{\nu}\right)^n C_n$ one then has

$$C = \sum_0^\infty \left(\frac{4\pi e^2}{\nu} \mathcal{R}_0\right)^n C_0 = \left(1 - \frac{4\pi e^2}{\nu} \mathcal{R}_0\right)^{-1} C_0$$

(4.97)

and similarly for $F_2^{(1)}$

$$F_2^{(1)} = \left[1 - \frac{4\pi e^2}{\nu} (\mathcal{R}_0 + \mathcal{R}_1')\right]^{-1} C_0 = \left[1 - \left(1 - \frac{4\pi e^2}{\nu} \mathcal{R}_0\right)^{-1} \left(\frac{4\pi e^2}{\nu}\right) \mathcal{R}_1\right]^{-1} \left(1 - \frac{4\pi e^2}{\nu} \mathcal{R}_0\right)^{-1} C_0$$
$$= \left[1 - \left(1 - \frac{4\pi e^2}{\nu} \mathcal{R}_0\right)^{-1} \frac{4\pi e^2}{\nu} \mathcal{R}_1\right]^{-1} C = C + \left[1 - \left(1 - \frac{4\pi e^2}{\nu} \mathcal{R}_0\right)^{-1} \frac{4\pi e^2}{\nu} \mathcal{R}_1\right]^{-1} \left(1 - \frac{4\pi e^2}{\nu} \mathcal{R}_0\right)^{-1} \cdot$$
$$\cdot \frac{4\pi e^2}{\nu} \mathcal{R}_1 C$$

(4.98)

so

$$D = F_2^{(1)} - C\cdot\left[1-\left(1-\frac{4\pi e^2}{\nu}\mathcal{K}_0\right)^{-1}\frac{4\pi e^2}{\nu}\mathcal{K}_1\right]^{-1}\left(1-\frac{4\pi e^2}{\nu}\mathcal{K}_0\right)^{-1}\frac{4\pi e^2}{\nu}\mathcal{K}_1 C$$
$$= \left(1-\frac{4\pi e^2}{\nu}\mathcal{K}_0\right)^{-1}\frac{4\pi e^2}{\nu}\mathcal{K}_1 C + \cdots$$

(4.99)

For convergence of these series' it is required that

$$\left|\frac{4\pi e^2}{\nu}(\mathcal{K}_0+\mathcal{K}_1)F\right| < F, \quad \left|\left(1-\frac{4\pi e^2}{\nu}\mathcal{K}_0\right)^{-1}\frac{4\pi e^2}{\nu}\mathcal{K}_1 F\right| < F$$

(4.100)

As in the equilibrium state, this may be insured* by taking the damping factor $\varkappa$ of the Coulomb field sufficiently large; the result is then analytically continued down to $\varkappa = 0$. The operator R_1 clearly represents the effect of the average electric field in producing correlation; in a nearly uniform state one may consider only the first term of the series (4.99).

We now return to the evaluation of C. The expression (4.88) or the recursion (4.84) is equivalent to the integral equation

$$C = C_0 + \frac{4\pi e^2}{\nu}\mathcal{K}_0 C = C_0 + \int_0^\infty d\tau \int dy \left\{ O_2^\tau(y_2, y)\, \mathcal{S}_\tau^{(2)}(y, y_1) C(y, y_1 | \mathcal{S}_\tau^{(1)} F_1) F_1(y_1) \right.$$
$$\left. + O_2^\tau(y_1, y)\, \mathcal{S}_{-\tau}^{(2)}(y, y_2) C(y, y_2 | \mathcal{S}_\tau^{(1)} F_1) F_1(y_2) \right\}$$

(4.101)

*Divergences are still present as $|\vec{r}_1 - \vec{r}_2| \longrightarrow 0$; these can only be removed by introducing short range forces.

It is difficult to solve this in general because of the functional dependence on $\delta_{\mathcal{L}}^{(1)} F_1$. However for the spatially uniform state considerable simplification occurs; when $F_1(y) \to F_1(\eta)$ independent of $\vec{r}$, then $\delta_{\mathcal{L}}^{(1)} F_1 = F_1$, $D = 0$ and $C(y_1, y_2) = C(\vec{r}_1 - \vec{r}_2, \eta_1, \eta_2)$. In this case (4.101) reduces to

$$C(\vec{r}_1 - \vec{r}_2, \eta_1, \eta_2) = \int_0^\infty d\tau \Big\{ Z_{\sigma_1} Z_{\sigma_2} \frac{\partial \phi(\vec{r}_1 - \vec{r}_2 - \tau(\frac{\vec{p}}{m_{\sigma_1}} - \frac{\vec{p}_2}{m_{\sigma_2}}))}{\partial \vec{r}_1} \Big(\frac{\partial}{\partial \vec{p}_1} - \frac{\partial}{\partial \vec{p}_2} \Big) F_1(\eta_1) F_1(\eta_2)$$

$$+ \frac{4\pi\ell^2}{\nu} \int dy \Big[Z_{\sigma_2} Z_\sigma \frac{\partial \phi(\vec{r}_2 - \vec{r} - \tau(\frac{\vec{p}_2}{m_{\sigma_2}} - \frac{\vec{p}}{m_\sigma}))}{\partial \vec{r}_2} \cdot \frac{\partial}{\partial \vec{p}_2} F_1(\eta_2) C\Big(\vec{r} - \vec{r}_1 - \tau(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}}{m_{\sigma_1}}); \eta, \eta$$

$$+ Z_{\sigma_1} Z_\sigma \frac{\partial \phi(\vec{r}_1 - \vec{r} - \tau(\frac{\vec{p}}{m_{\sigma_1}} - \frac{\vec{p}}{m_\sigma}))}{\partial \vec{r}_1} \cdot \frac{\partial}{\partial \vec{p}_1} F_1(\eta_1) C\Big(\vec{r} - \vec{r}_2 - \tau(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}_2}{m_{\sigma_2}}); \eta_1, \eta_2\Big)$$

(4.102)

Chapter V is devoted to the solution of this equation and the study of the resulting equation for F_1.

CHAPTER V

THE SPATIALLY UNIFORM CASE

1. REDUCTION OF THE INTEGRAL EQUATION*

The case when F_1 is independent of position, described by equations (4.113), (4.24), is of special interest because, (1) the kinetic equation can be obtained in a relatively simple form, and (2) the theory of the uniform state forms a starting point for various approximate theories such as the Chapman-Enskog development.

The integral equation (4.102) may be written in a somewhat neater form by introducing the new integration variables

$$\vec{R} = \begin{cases} \vec{r}_1 - \vec{r} - \tau\left(\dfrac{\vec{p}_1}{m_{\sigma_1}} - \dfrac{\vec{p}}{m_\sigma}\right) & \text{in the first term} \\ \vec{r}_2 - \vec{r} - \tau\left(\dfrac{\vec{p}_2}{m_{\sigma_2}} - \dfrac{\vec{p}}{m_\sigma}\right) & \text{in the second term} \end{cases}$$

Then (4.102) becomes

$$C(\vec{r}_{12}, \eta_1 \eta_2) = \int_0^\infty d\tau\, Z_{\sigma_1} Z_{\sigma_2} \frac{\partial \phi(|\vec{r}_{12} - \tau \vec{g}_{12}|)}{\partial \vec{r}_{12}} \cdot \left(\frac{\partial}{\partial \vec{p}_1} - \frac{\partial}{\partial \vec{p}_2}\right) F_1(\eta_1) F_1(\eta_2)$$

$$+ \frac{4\pi e^2}{v} \int_0^\infty d\tau \int d\eta \int d\vec{R} \left[- Z_\sigma Z_{\sigma_2} \frac{\partial \phi(|\vec{R} - \vec{r}_{12} + \tau \vec{g}_{12}|)}{\partial \vec{r}_{12}} \cdot \frac{\partial}{\partial \vec{p}_2} F_1(\eta_2) C(\vec{R}, \eta_1, \eta)\right.$$

*This section is a generalization and extension of the work of Balescu[18], who was the first to obtain the first approximation to the kinetic equation in the spatially uniform case for an electron gas in a positive background.

$$+Z_\sigma Z_{\sigma_1} \frac{\partial \phi(|\vec{R}+\vec{r}_{12}-\vec{r}_{g12}|)}{\partial \vec{r}_{12}} \cdot \frac{\partial}{\partial \vec{p}_1} F_1(\eta_1) C(\vec{R}, \eta_2, \eta) \Big] \qquad (5.1)$$

where:

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 \qquad \vec{g}_{12} = \frac{\vec{p}_1}{m_{\sigma_1}} - \frac{\vec{p}_2}{m_{\sigma_2}} \qquad (5.2)$$

From (5.1) follows that:

$$C(\vec{R}, \eta_1 \eta_2) = C(-\vec{R}, \eta_2 \eta_1)$$

which is consistent with the symmetry of the second distribution function. Furthermore one sees that the right hand side of (5.1) only involves the function: (i = 1,2)

$$\Gamma(\vec{r}_{12}, \eta_i) = \int d\eta Z_\sigma C(\vec{r}_{12}, \eta_i \eta) \qquad (5.3)$$

so that C can be found if Γ is known. By integrating (5.1) over η_2 (after multiplying with Z_{σ_2}) one then obtains an integral equation for Γ, which can be simplified by making a Fourier transform in $\vec{r}_{12}$. Calling:

$$L(\eta_1, \vec{k}) = \int d\vec{r}\ e^{i\vec{k}\cdot\vec{r}}\ \Gamma(\vec{r}, \eta_1) \qquad (5.4)$$

one obtains

$$\rho\left(\frac{\vec{k}\cdot\vec{p}}{k m_\tau}\right) L(\eta,\vec{k}) = q(\eta,\vec{k}) + 2i D(\eta,\vec{k}) \int d\eta' Z_{\sigma'} \delta_+\left(\frac{\vec{k}\cdot\vec{p}}{k m_\tau} - \frac{\vec{k}\cdot\vec{p}'}{k m_{\sigma'}}\right) L^*(\eta',\vec{k}) \tag{5.5}$$

where*

$$D(\eta,\vec{k}) = -\frac{4\pi e^2}{v}\left(\frac{\pi}{k}\right) Z_\sigma \vec{k}\cdot\frac{\partial}{\partial\vec{p}} F_1(\eta) \tag{5.6}$$

$$\rho\left(\frac{\vec{k}\cdot\vec{p}}{k m_\sigma}, k\right) = k^2 + 2i\int d\eta' Z_{\sigma'} \delta_+\left(\frac{\vec{k}\cdot\vec{p}}{k m_\sigma} - \frac{\vec{k}\cdot\vec{p}'}{k m_{\sigma'}}\right) D(\eta')$$

$$q(\eta,\vec{k}) = -\frac{2\pi i Z_\sigma}{k}\int d\eta' Z_{\sigma'}^* \delta_+\left(\frac{\vec{k}\cdot\left(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}'}{m_{\sigma'}}\right)}{k}\right) \vec{k}\cdot\left(\frac{\partial}{\partial\vec{p}} - \frac{\partial}{\partial\vec{p}'}\right) F_1(\eta) F_1(\eta') \tag{5.7}$$

(5.8)

Since the kernel of (5.5) depends on $\vec{p}'$ only in the combination $\frac{\vec{k}\cdot\vec{p}'}{k m_{\sigma'}}$, it is clear that L is completely determined by

$$\bar{L}(u) = \int d\eta Z_\sigma \delta\left(u - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) L(\eta) = \sum_\sigma Z_\sigma \int d\vec{p}_\perp L(m_\sigma u, \vec{p}_\perp) \tag{5.9}$$

---

*D and ρ reduce to D_0 and ρ_0 of Chapter IV if F_1 is replaced by $F^{(0)}$; δ_+ is defined by (4.72); the asterisk denotes the complex conjugate. The "undamped" form of the Coulomb potential ϕ has been used.

and in fact (5.5) may be written (suppressing the argument $\vec{k}$):

$$\rho\left(\frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) L(\eta) = q(\eta) + 2i D(\eta) \int_{-\infty}^{\infty} du' \, \delta_+\left(\frac{\vec{k}\cdot\vec{p}}{k m_\sigma} - u\right) \bar{L}^*(u') \tag{5.10}$$

Performing the barring operation on (5.10)

$$\rho(u)\, \bar{L}(u) = \bar{q}(u) + 2i \bar{D}(u) \int_{-\infty}^{\infty} du' \, \delta_+(u-u')\, \bar{L}^*(u') \tag{5.11}$$

We write ρ, q, L in terms of their real and imaginary parts

$$\rho = \rho_1 + i\rho_2 \qquad q = q_1 + i q_2 \qquad L = L_1 + i L_2 \tag{5.12}$$

and observe that by definition of the barring operation

$$\bar{q}_2 = -\pi \int d\eta\, Z_\sigma^2 \int d\eta'\, Z_{\sigma'}^2\, \delta\left(u - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) \delta\left(\frac{\vec{k}\cdot\vec{p}}{m_\sigma} - \frac{\vec{k}\cdot\vec{p}'}{m_{\sigma'}}\right) \vec{k}\cdot\left(\frac{\partial}{\partial\vec{p}} - \frac{\partial}{\partial\vec{p}'}\right) F_1(\eta) F_1(\eta')$$

$$= -\pi \int d\eta\, Z_\sigma^2 \int d\eta'\, Z_{\sigma'}^2\, \delta\left(u - \frac{\vec{k}\cdot\vec{p}'}{k m_{\sigma'}}\right) \delta\left(\frac{\vec{k}\cdot\vec{p}}{m_\sigma} - \frac{\vec{k}\cdot\vec{p}'}{m_{\sigma'}}\right) \vec{k}\cdot\left(\frac{\partial}{\partial\vec{p}} - \frac{\partial}{\partial\vec{p}'}\right) F_1(\eta) F_1(\eta')$$

$$= -\pi \int d\eta'\, Z_{\sigma'}^2 \int d\eta\, Z_\sigma^2\, \delta\left(u - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) \delta\left(\frac{\vec{k}\cdot\vec{p}'}{m_{\sigma'}} - \frac{\vec{k}\cdot\vec{p}}{m_\sigma}\right) \vec{k}\cdot\left(\frac{\partial}{\partial\vec{p}'} - \frac{\partial}{\partial\vec{p}}\right) F_1(\eta') F_1(\eta)$$

$$= -\bar{q}_2$$

(5.13)

so that $\bar{q}_2 = 0$ or $\bar{q}$ is real. Further

$$\rho_2(u) = \int d\eta\, Z_\sigma\, \delta\left(u - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) D(\eta) = \bar{D}(u) \tag{5.14}$$

Thus we have, equating real and imaginary parts of (5.11),

$$\begin{aligned} \rho_1 \bar{L}_1 - \rho_2 H[\bar{L}] &= \bar{q} + 2\rho_2 \bar{L}_2 \\ \rho_1 \bar{L}_2 + \rho_2 H[\bar{L}_2] &= 0 \end{aligned} \tag{5.15}$$

where H[f] again denotes the Hilbert transform of f.

These equations, which are of a standard type, are solved in the next section.

2. SOLUTION OF THE INTEGRAL EQUATIONS

Singular integral equations of the type (5.15), (5.16) have been studied in considerable detail by Russian mathematicians.[19] However, much of the formalism may be dispensed with in our case, due to the nature of the coefficients ρ_1 , ρ_2 . We will need the following result from the theory of Cauchy integrals:

Let f(x) be any function satisfying a Hölder condition on the real line. Then the complex function

$$\mathcal{F}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dx\, f(x)}{x - z} \tag{5.17}$$

is analytic except on the real line, which is a branch cut, and

$$\mathcal{H}^{-}(x) - \mathcal{H}^{-}(x) = f(x)$$

$$i\left(\mathcal{H}^{+}(x) + \mathcal{H}^{-}(x)\right) = H_x[f]$$

(5.18)

where

$$\mathcal{H}^{\pm}(x) = \lim_{\epsilon \to 0_+} \mathcal{H}(x \pm i\epsilon)$$

(5.19)

Equations (5.18) are known as the Plemelj formulae.

The following properties of $\mathcal{S}_1$, $\mathcal{S}_2$ are of interest.

(a) From the definition (5.7) of $\mathcal{S}$,

$$\mathcal{S}_1 = k^2 + H[\mathcal{S}_2]$$

(5.20)

(b) Since F_1 and its derivatives must vanish as $|\vec{p}| \to \infty$

$$\lim_{x \to \pm\infty} \mathcal{S}_1(x) = k^2 \qquad \lim_{x \to \pm\infty} \mathcal{S}_2(x) = 0$$

(5.21)

(c) It is assumed that $\mathcal{S}_1(x)$, $\mathcal{S}_2(x)$ are not simultaneously zero for any real x (this is certainly true if F_1 is Maxwellian).

We now turn to the solution of (5.17). Defining

$$\mathcal{L}_2(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dx\, \bar{L}_2(x)}{x - z}$$

(5.22)

and using (5.18), eq. (5.16) may be written

$$(\rho_1 + i\rho_2)\mathcal{L}_2^+(x) = (\rho_1 - i\rho_2)\mathcal{L}_2^-(x)$$

or dividing by $\rho_1 + i\rho_2$ (non-zero by (c) above)

$$\mathcal{L}_2^+(x) = e^{-2i\phi(x)}\mathcal{L}_2^-(x) \tag{5.23}$$

where*

$$\phi(x) = \arctan\left(\frac{\rho_2(x)}{\rho_1(x)}\right) \tag{5.24}$$

According to (5.22), ϕ satisfies

$$\lim_{x\to\pm\infty} \phi(x) = 0 \tag{5.25}$$

Consider now the complex function

$$\delta(z) = \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dy\,\phi(y)}{y-z}\right] \tag{5.26}$$

which is analytic in the cut plane, is nowhere zero, approaches unity as $|z| \to \infty$, and has the limiting values (cf. (5.18))

$$\delta^{\pm}(x) = e^{(H[\phi] \pm i\phi)} \tag{5.27}$$

*To make ϕ a single valued function, we take $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$.

It follows from (5.27) that

$$e^{-2i\phi(x)} = \mathcal{S}^-(x)/\mathcal{S}^+(x) \tag{5.28}$$

so that (5.23) may be written

$$\mathcal{L}_2^+(x) = \mathcal{S}^-(x)\,\mathcal{L}_2^-(x)/\mathcal{S}^+(x)$$

or multiplying by the non-zero bounded function $\mathcal{S}^+(x)$

$$\mathcal{L}_2^+\mathcal{S}^+ - \mathcal{L}_2^-\mathcal{S}^- = 0 \tag{5.29}$$

By definition of $\mathcal{L}_2(z)$, $\mathcal{S}(z)$ the function $\mathcal{L}_2\mathcal{S}$ is analytic in the cut plane and vanishes as $|z| \to \infty$; according to (5.29), it is continuous across the cut. Hence $\mathcal{L}_2\mathcal{S}$ is analytic <u>everywhere</u> and vanishes as $|z| \to \infty$; this can only be true if

$$\mathcal{L}_2\mathcal{S} = 0$$

or, since $\mathcal{S}$ is non-zero and bounded,

$$\mathcal{L}_2(z) = 0 \tag{5.30}$$

so finally

$$\bar{L}_2(x) = \mathcal{L}_2^+(x) - \mathcal{L}_2^-(x) = 0 \tag{5.31}$$

For the real part we have, putting $\bar{L}_2 = 0$

$$\rho_1 \bar{L}_1 - \rho_2 H[\bar{L}_1] = \bar{q} \tag{5.32}$$

Precisely as in proving (5.31), it may be shown that the homogeneous equation $\rho_1 \bar{L}_1 - \rho_2 \bar{L}_2 = 0$ has no non-trivial solution. It follows that the solution to (5.32), if it exists, is unique. In order to find a solution, we again make use of the Plemelj formula (5.18). Defining the complex functions

$$\mathcal{L}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dy \, \frac{\bar{L}_1(y)}{y - z} \tag{5.33}$$

$$\mathcal{R}(z) = \frac{k^2}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \, \frac{\rho_2(y)}{y - z} \tag{5.34}$$

Eqs. (5.18) give

$$\mathcal{L}^+ - \mathcal{L}^- = \bar{L}_1 \tag{5.35}$$

$$\mathcal{L}^+ + \mathcal{L}^- = -i H[\bar{L}_1] \tag{5.36}$$

$$\mathcal{R}^+ - \mathcal{R}^- = i \rho_2 \tag{5.37}$$

$$\mathcal{R}^+ + \mathcal{R}^- = k^2 + H[\rho_2] = \rho_1 \tag{5.38}$$

Substituting (5.35)-(5.38) in (5.32), one finds

$$2\left(\mathcal{R}^- \mathcal{L}^+ - \mathcal{R}^+ \mathcal{L}^-\right) = \bar{q} \tag{5.39}$$

or dividing by

$$2\mathcal{R}^+\mathcal{R}^- = \tfrac{1}{2}\left(\rho_1^2+\rho_2^2\right) = \tfrac{1}{2}(\rho)^2 \neq 0 \tag{5.40}$$

$$\left(\frac{\mathcal{L}}{\mathcal{R}}\right)^+ - \left(\frac{\mathcal{L}}{\mathcal{R}}\right)^- = \frac{2\bar{q}}{|\rho|^2} \tag{5.41}$$

Introducing the complex function:

$$\mathcal{T}(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{dy}{y-z}\,\frac{2\bar{q}(y)}{|\rho(y)|^2} \tag{5.42}$$

one concludes from (5.41) that:

$$\left(\frac{\mathcal{L}}{\mathcal{R}} - \mathcal{T}\right)^+ - \left(\frac{\mathcal{L}}{\mathcal{R}} - \mathcal{T}\right)^- = 0$$

and hence by the same argument as above $(\mathcal{L}/\mathcal{R}) - \mathcal{T}$ is identically zero, or:

$$\mathcal{L}(z) = \mathcal{R}(z)\,\mathcal{T}(z) \tag{5.43}$$

A solution of (5.42) is therefore given by:

$$\bar{L}_1(x) = \mathcal{L}^+(x) - \mathcal{L}^-(x) = \mathcal{R}^+\mathcal{T}^+ - \mathcal{R}^-\mathcal{T}^-$$

$$= \frac{1}{2}\left[(\mathcal{R}^+ + \mathcal{R}^-)(\mathcal{T}^+ - \mathcal{T}^-) + (\mathcal{R}^+ - \mathcal{R}^-)(\mathcal{T}^+ + \mathcal{T}^-)\right]$$

$$= \frac{\rho_1 \bar{q}}{|\rho|^2} + \rho_2 H\left[\frac{\bar{q}}{|\rho|^2}\right]$$

(5.44)

Eq. (5.44), together with (5.11), (5.4), (5.3) and (5.1) determines the two-particle distributions; however, as pointed out by Balescu,[18] the explicit form (5.44) is not necessary for the determination of the functional A_1, and thus the approximate kinetic equation. According to (4.18), (4.23)

$$A_1(y) = \sum_n^\infty \left(\frac{4\pi e^2}{\nu}\right)^n A_{1n}(y) = \frac{4\pi e^2}{\nu} \int dy' \, \Phi_1(y; y') F_2^{(1)}(y, y')$$

$$= \frac{4\pi e^2}{\nu} Z_\sigma \int d\vec{r}' \, \frac{\partial \phi(|\vec{r} - \vec{r}'|)}{\partial \vec{r}} \cdot \frac{\partial}{\partial \vec{p}} \int d\eta' Z_{\sigma'} F_2^{(1)}(y, y')$$

$$= \frac{4\pi e^2}{\nu} Z_\sigma \int d\vec{R} \, \frac{\partial \phi(|\vec{R}|)}{\partial \vec{R}} \cdot \frac{\partial}{\partial \vec{p}} \Gamma(\vec{R}, \eta)$$

$$= \frac{4\pi e^2}{\nu} Z_\sigma \int d\vec{R} \frac{1}{(2\pi)^3} \int d\vec{k}\ \frac{e^{i\vec{k}\cdot\vec{R}}}{k^2} (i\vec{k}) \cdot \frac{\partial}{\partial \vec{p}} \Gamma(\vec{R}, \eta)$$

$$= \frac{4\pi e^2}{\nu} \frac{i Z_\sigma}{(2\pi)^3} \int \frac{d\vec{k}}{k^2} \vec{k} \cdot \frac{\partial}{\partial \vec{p}} L(\eta, \vec{k})$$

(5.45)

Since A_1 is real, it is clear that only the imaginary part of L contributes to A_1 which is equivalent to the statement that only the part of $F_2^{(1)}$ which is odd in $\vec{R}$ contributes. Thus

$$A_1(y) = -\frac{4\pi e^2}{\nu} \frac{Z_\sigma}{(2\pi)^3} \int \frac{d\vec{k}}{k^2} \vec{k} \cdot \frac{\partial}{\partial \vec{p}} L_2(\eta, \vec{k})$$

(5.46)

But according to* (5.10), (5.15), (5.31)

$$L_2 = \frac{1}{|p|^2} \left[q_2 p_1 - q_1 p_2 + D(p_1 \bar{L} - p_2 H[\bar{L}]) \right]$$

$$= \frac{1}{|p|^2} \left[q_2 p_1 - q_1 p_2 + D\bar{q} \right]$$

(5.47)

* The same result may be obtained by eliminating the integral $\int_{-\infty}^{\infty} du' \delta_+(u-u') \bar{L}^*(u')$ between (5.10) and (5.11).

This can be put in a simpler form by using the expressions (5.7), (5.8) for ρ , q . Defining

$$f(\eta) = Z_\sigma F_1(\eta) \tag{5.48}$$

one may write

$$q_1 = \left(\frac{4\pi e^2}{\nu}\right)^{-1} \left[D H[\bar{f}] - f H[\bar{D}]\right] \tag{5.49}$$

$$q_2 = \left(\frac{4\pi e^2}{\nu}\right)^{-1} \left[D\bar{f} - f\bar{D}\right] \tag{5.50}$$

Using also (5.14), (5.20) in (5.47), this becomes

$$L_2 = \frac{k^2 q_2}{|\rho|^2} = k^2 \left(\frac{4\pi e^2}{\nu}\right)^{-1} |\rho|^{-2} (D\bar{f} - f\bar{D}) \tag{5.51}$$

Thus the kinetic equation becomes:

$$\frac{\partial F_1}{\partial t} = 4\pi e^2 A_1 = -\frac{4\pi e^2 Z_\sigma}{\nu} \int d\vec{k}\, \vec{k} \cdot \frac{\partial}{\partial \vec{p}} \left[\frac{D(\eta,k)\bar{f}(u) - \bar{D}(u,k) f(\eta)}{|\rho|^2}\right] \tag{5.52}$$

where $\vec{u} = k\cdot\vec{p}/k m_\sigma$. putting in the expressions for D, f and ρ:

$$\frac{\partial F_1}{\partial t} = \frac{2e^4 Z_\sigma^2}{\nu} \int d\vec{k}\, x$$

$$\times\left(\vec{k}\cdot\frac{\partial}{\partial\vec{p}}\right)\left[\frac{\int d\eta' Z_\sigma^2,\ \delta\left(\vec{k}\cdot(\vec{u}-\vec{u}')\right)\vec{k}\cdot\left(\frac{\partial}{\partial\vec{p}}-\frac{\partial}{\partial\vec{p}'}\right)F_1(\eta_1)F_1(\eta_2)}{\left|k^2-2\pi i\left(\frac{4\pi e^2}{\nu}\right)\int d\eta' Z_\sigma^2,\ \delta_+\left(\vec{k}\cdot(\vec{u}-\vec{u}')\right)\vec{k}\cdot\frac{\partial}{\partial\vec{p}'},F_1(\eta')\right|^2}\right]$$

(5.53)

The rest of the dissertation is devoted to the study of this equation.

3. THE APPROACH TO EQUILIBRIUM; THE H-THEOREM

If F_1 is the Maxwell distribution one easily sees that $\frac{\bar{D}}{D}=\frac{\bar{f}}{f}$ so that from (5.52) it follows that $\frac{\partial F_1}{\partial t}$ vanishes as expected. The question now arises whether this is a unique equilibrium, and whether the approach to equilibrium satisifes Boltzmann's H-theorem.

Following Boltzmann we introduce the quantity

$$H(t)=\int d\eta\, F_1(\eta)\ln F_1(\eta)$$

(5.54)

so that

$$\frac{dH}{dt}=\int d\eta\,\frac{\partial F_1(\eta)}{\partial t}\left(1+\ln F_1(\eta)\right)$$

(5.55)

If F_1 satisfies the Kinetic equation (5.52), we have

$$\frac{dH}{dt} = \int d\eta \left(1 + \ln F_1(\eta)\right) \left(\frac{-4\pi e^2 Z_\sigma}{(2\pi)^3}\right) \int d\vec{k}\, \vec{k} \cdot \frac{\partial}{\partial \vec{p}} \left[\frac{Df - f\bar{D}}{|p|^2}\right] \tag{5.56}$$

Integrating by parts with respect to η ,

$$\frac{dH}{dt} = \int d\vec{k} \int d\eta\, \frac{4\pi e^2 Z_\sigma}{(2\pi)^3} \left[\frac{D\bar{f} - f\bar{D}}{|p|^2}\right] \frac{\vec{k} \cdot \frac{\partial}{\partial \vec{p}} F_1(\eta)}{F_1(\eta)} \tag{5.57}$$

or by definition of D,

$$\frac{dH}{dt} = -\int d\vec{k} \int d\eta\, Z_\sigma h\left(\frac{\vec{k}\cdot\vec{p}}{k m_\sigma}, k\right) \frac{D(\eta)}{f(\eta)} \left(D\bar{f} - f\bar{D}\right) \tag{5.58}$$

where

$$h\left(\frac{\vec{k}\cdot\vec{p}}{k m_\sigma}, k\right) = \frac{\nu k}{\pi (2\pi)^3 \left| p\left(\frac{\vec{k}\cdot\vec{p}}{k m_\sigma}, k\right)\right|^2} \tag{5.59}$$

is a positive definite function, depending on η only through the combination $\frac{\vec{k}\cdot\vec{p}}{k m_\sigma}$. Now since for any function $g(\eta)$ one has identically

$$\int d\eta\, Z_\sigma g(\eta) = \int_{-\infty}^{\infty} du \int d\eta\, Z_\sigma\, \delta\left(u - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) g(\eta) \tag{5.60}$$

(5.58) may be written

$$\frac{dH}{dt} = -\int d\vec{k} \int du\, h(u,k) \int d\eta\, Z_\sigma\, \delta\left(u - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right)\left[\frac{D(\eta)}{f(\eta)} D(\eta)\bar{f}(u) - \bar{D}(u)D(\eta)\right]$$

(5.61)

or by the definition (4.60) of the barring operation

$$\frac{dH}{dt} = -\int d\vec{k} \int_{-\infty}^{\infty} du\, h(u)\left[\overline{\left(\frac{D^2}{f}\right)}\bar{f} - (\bar{D})^2\right]$$

(5.62)

But the quantity $\overline{ab}$ satisfies all the requirements for a scalar product, so we have for any two functions a, b the Schwarz inequality

$$\overline{a^2}\,\overline{b^2} \geq (\overline{ab})^2$$

(5.63)

Taking $a = \frac{D}{\sqrt{|f|}}$, $b = \sqrt{|f|}$

$$\overline{\left(\frac{D^2}{f}\right)}\bar{f} \geq (\bar{D})^2$$

(5.64)

Since h is always positive, it follows from (5.62) that

$$\frac{dH}{dt} \leq 0$$

(5.65)

The equality is satisfied only when the integrand of (5.62) is identically zero, which implies

$$\overline{\left(\frac{D^2}{f}\right)}\bar{f} - (\bar{D})^2 = 0$$

(5.66)

Using the definition of the barring operation, this may be written

$$\int d\eta Z_\sigma \int d\eta' Z_{\sigma'} \delta\left(u - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) \delta\left(u - \frac{\vec{k}\cdot\vec{p}'}{k m_{\sigma'}}\right)\left[\frac{f(\eta) D^2(\eta')}{f(\eta')} - D(\eta) D(\eta')\right] = 0 \tag{5.67}$$

Interchanging the integration variables η and η' and adding to the original expression, one finds

$$\int d\eta \int d\eta' \delta\left(u - \frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right) \delta\left(u - \frac{\vec{k}\cdot\vec{p}'}{k m_{\sigma'}}\right) Z_\sigma f(\eta) Z_{\sigma'} f(\eta') \left[\frac{D(\eta)}{f(\eta)} - \frac{D(\eta')}{f(\eta')}\right]^2 = 0 \tag{5.68}$$

Since the integrand is always ≥ 0 ($F_i(\eta)$ and hence $Z_\sigma f(\eta) = Z_\sigma^2 F_i(\eta)$ is required to be positive), it must vanish identically or

$$\left[\frac{D(\eta)}{f(\eta)} - \frac{D(\eta')}{f(\eta')}\right] = 0 \qquad \frac{\vec{k}\cdot\vec{p}}{k m_\sigma} = \frac{\vec{k}\cdot\vec{p}'}{k m_{\sigma'}} = u \tag{5.69}$$

where $\vec{k}$, $\vec{p}$, $\vec{p}'$, are arbitrary except for the condition $\vec{k}\cdot\left(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}'}{m_{\sigma'}}\right) = 0$. From the definition of D, (5.69) is equivalent to

$$\left[k \cdot \frac{\partial}{\partial \vec{p}} \ln f(\eta) - \vec{k}\cdot\frac{\partial}{\partial p'} \ln f(\eta')\right] = 0 \qquad \vec{k}\cdot\left(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}'}{m_{\sigma'}}\right) = 0 \tag{5.70}$$

Since the quantity in brackets is linear in $\vec{k}$, this is equivalent to

$$\vec{k}\cdot\left[\frac{\partial}{\partial\vec{p}}\ln f(\eta)-\frac{\partial}{\partial\vec{p}'}\ln f(\eta')\right]=-\beta\vec{k}\cdot\left(\frac{\vec{p}}{m_\sigma}-\frac{\vec{p}'}{m_{\sigma'}}\right) \tag{5.71}$$

where β is a constant.

If (5.71) is to hold for arbitrary $\vec{k}$, we must have

$$\frac{\partial}{\partial\vec{p}}\ln f(\eta)-\frac{\partial}{\partial\vec{p}'}\ln f(\eta')=-\beta\left(\frac{\vec{p}}{m_\sigma}-\frac{\vec{p}'}{m_{\sigma'}}\right)$$

or

$$\frac{\partial}{\partial\vec{p}}\ln f(\eta)=-\beta\left(\frac{\vec{p}}{m_\sigma}-\vec{V}\right) \tag{5.72}$$

where $\vec{V}$ is an arbitrary constant vector (independent of η). Eq.(5.72) may be integrated readily to give

$$\ln f(\eta)=-\frac{\beta m_\sigma}{2}\left(\frac{\vec{p}}{m_\sigma}-\vec{V}\right)^2+\ln A_\sigma$$

or

$$f(\eta)=A_\sigma e^{-\frac{\beta m_\sigma}{2}\left(\frac{\vec{p}}{m_\sigma}-\vec{V}\right)^2} \tag{5.73}$$

a Maxwell distribution around the average velocity V. Thus $\frac{dH}{dt}$ and $\frac{\partial F_1}{\partial t}$ are zero if and only if F_1 is Maxwellian.

4. INTERPRETATION OF THE KINETIC EQUATION IN TERMS OF A PSEUDO-BINARY INTERACTION

We have seen that the kinetic equation for an ionized gas comes out in a form which differs from the ordinary Boltzmann equation in that the collective interaction of large numbers of particles must be

taken into account. This is characterized in (5.52) by the factor $1/|\rho|^2$, which depends on F_1 (if only binary interactions had been considered, i.e., if $F_2^{(1)}$ had been replaced by $F_2^{(1,0)}$, this factor would have been replaced by $1/k^4$. However, as in the equilibrium theory, one might expect that the summed contribution of the chain interactions would lead to the replacement of the Coulomb potential by an effective damped potential. We will see that this is indeed the case, although the effective potential is of course more complicated than the Debye potential.

Choh[16] has shown* that for any binary interaction potential , a Kinetic equation of the form:

$$\frac{\partial F_1(y)}{\partial t} = \frac{1}{m}\int dy' \left[\frac{\partial \Phi(r-r';\eta,\eta')}{\partial \vec{r}}\cdot\left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'}\right) - \left(\frac{\partial \Phi}{\partial \vec{p}} - \frac{\partial \Phi}{\partial \vec{p}'}\right)\cdot\frac{\partial}{\partial \vec{r}}\right] \cdot \lim_{T\to\infty} S^{(2)}_{-T}(y,y')F_1(\eta)F_1(\eta')$$

(5.74)

where:

$$S^{(2)}_{-T} = \exp\left[-T\left\{\mathcal{H}_0 - \frac{\partial \Phi}{\partial \vec{r}}\cdot\left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'}\right) + \left(\frac{\partial \Phi}{\partial p} - \frac{\partial \Phi}{\partial p'}\right)\cdot\frac{\partial}{\partial r}\right\}\right] \quad (5.75)$$

*This is actually a slight generalization of Choh's result in so far as the interaction potential Φ in (5.74) is assumed to be dependent on the momenta of the particles. However, the method of proof is the same. (cf. Ref.16, p24 and ff.).

$$\mathcal{H}_0 = \frac{\vec{p}}{m_\sigma} \cdot \frac{\partial}{\partial \vec{r}} - \frac{\vec{p}'}{m_{\sigma'}} \cdot \frac{\partial}{\partial \vec{r}'} \tag{5.76}$$

can be reduced to the Boltzmann equation corresponding to the interaction potential Φ . Hence if one can show, that in some approximation and with an appropriately chosen Φ the kinetic equation (5.53) for a fully ionized gas can be cast into the form (5.74), then it follows that (5.53) can be transformed to the Boltzmann form.

For a weak field it is convenient to expand the operator $S^{(2)}_{-\tau}$ of (5.74) in powers of the field strength by means of the following identity (Yang and Lee[20])

$$e^{-\tau(\mathcal{H}_0+\lambda\mathcal{H}_1)} = e^{-\tau\mathcal{H}_0} - \lambda\int_0^\tau d\tau' e^{-\tau'\mathcal{H}_0}\mathcal{H}_1 e^{-(\tau-\tau')\mathcal{H}_0} + O(\lambda^2) \tag{5.77}$$

Taking

$$\lambda\mathcal{H}_1 = -\frac{\partial\Phi}{\partial\vec{r}}\cdot\left(\frac{\partial}{\partial\vec{p}} - \frac{\partial}{\partial\vec{p}'}\right) + \left(\frac{\partial\Phi}{\partial\vec{p}} - \frac{\partial\Phi}{\partial\vec{p}'}\right)\cdot\frac{\partial}{\partial\vec{r}} \tag{5.78}$$

and observing that for any function F of the indicated arguments .

$$\mathcal{H}_0 F(\eta,\eta') = 0 \tag{5.79}$$

$$e^{-\tau\mathcal{H}_0} F(\vec{r}-\vec{r}',\eta,\eta') = F(\vec{r}-\vec{r}'-\vec{g}\tau,\eta,\eta') \tag{5.80}$$

where

$$\vec{g} = \frac{\vec{p}}{m_\sigma} - \frac{\vec{p}'}{m_{\sigma'}} \tag{5.81}$$

Eq. (5.77) may be applied to (5.75) to give

$$\lim_{\tau\to\infty} S_{-\tau}^{(2)} F_1(\eta) F_1(\eta') \cong \left[1 + \int_0^\infty d\tau \; \frac{\partial \Phi(\vec{r}-\vec{r}'-\vec{g}\tau,\eta,\eta')}{\partial \vec{r}} \cdot \left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'}\right)\right] F_1(\eta) F_1(\eta') \tag{5.82}$$

Hence eq. (5.74) becomes approximately*

$$\frac{\partial F_1}{\partial t} \cong \frac{1}{\nu}\int d\eta' \left[\frac{\partial \Phi(\vec{r}-\vec{r}',\eta,\eta')}{\partial \vec{r}} \cdot \left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'}\right) - \left(\frac{\partial \Phi}{\partial \vec{p}} - \frac{\partial \Phi}{\partial \vec{p}'}\right) \cdot \frac{\partial}{\partial \vec{r}}\right]$$

$$\times \left[1 + \int_0^\infty d\tau \; \frac{\partial \Phi(\vec{r}-\vec{r}'-\vec{g}\tau,\eta,\eta')}{\partial \vec{r}} \cdot \left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'}\right)\right] F_1(\eta) F_1(\eta')$$

$$= \frac{1}{\nu}\int d\eta' \left[\frac{\partial \Phi}{\partial \vec{r}} \cdot \left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'}\right) - \left(\frac{\partial \Phi}{\partial \vec{p}} - \frac{\partial \Phi}{\partial \vec{p}'}\right) \cdot \frac{\partial}{\partial \vec{r}}\right] \int_0^\infty d\tau \; \frac{\partial \Phi(\vec{r}-\vec{r}'-\vec{g}\tau,\eta,\eta')}{\partial \vec{r}} \cdot \left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'}\right) F_1(\eta) F_1(\eta') \tag{5.83}$$

where the last equality follows by means of a partial integration.

Eq. (5.87) may be simplified somewhat by introducing the Fourier transform $\phi(\vec{k},\eta,\eta')$ of the interaction potential $\Phi(\vec{R},\eta,\eta')$

$$\Phi(\vec{R},\eta,\eta') = \frac{1}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{R}}\, \phi(\vec{k},\eta,\eta') \tag{5.84}$$

*The assumption is that the average potential is much less than the average kinetic energy; we assumed this at the outset for a Coulomb potential.

Putting this in (5.83), one easily shows

$$\frac{\partial F_1}{\partial t} = \frac{1}{(2\pi)^3 \nu} \int d\vec{k} \int d\eta' \left[\phi(\vec{k})\, \vec{k} \cdot \left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'} \right) + \vec{k} \cdot \left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'} \right) \phi \right] \cdot$$

$$\cdot \left[2\pi \delta_+ (\vec{k} \cdot \vec{g})\, \phi(-\vec{k})\, \vec{k} \left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'} \right) F_1(\eta) F_1(\eta') \right]$$

$$= \frac{1}{2(2\pi)^2 \nu} \int d\vec{k} \int d\eta'\, \vec{k} \cdot \frac{\partial}{\partial \vec{p}} \left[\phi(\vec{k}) \phi(-\vec{k})\, \delta(\vec{k} \cdot \vec{g})\, \vec{k} \cdot \left(\frac{\partial}{\partial \vec{p}} - \frac{\partial}{\partial \vec{p}'} \right) F_1(\eta) F_1(\eta') \right]$$

(5.85)

by a partial integration in $\vec{p}'$ and by using the fact that the integrand is symmetric in $\vec{k}$.

Comparing (5.85) with the kinetic equation (5.53) for a fully ionized gas, we see that the two equations are identical provided *

$$\phi(\vec{k}) = \frac{4\pi e^2 Z_\sigma Z_{\sigma'}}{\rho\left(\frac{\vec{k} \cdot \vec{p}}{k m_\sigma} \right)}$$

(5.86)

The choice of ϕ is not unique; what is really determined is $\phi(\vec{k})\phi(-\vec{k})$ $\delta\left(\vec{k} \cdot \left(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}''}{m_{\sigma'}} \right) \right)$. The presence of the δ function means for instances that the argument of ρ may be anything which reduces to $\frac{\vec{k} \cdot \vec{p}}{k m_\sigma}$ when $\frac{\vec{k} \cdot \vec{p}}{k m_\sigma} = \frac{\vec{k} \cdot \vec{p}'}{k m_{\sigma'}}$. The fact that ϕ occurs in the combination

*This "pseudo-potential" differs from ordinary potentials in that it is not Galilei invariant, i.e., it depends on the speed of the interacting pair relative to the background.

$\phi(\vec{k})\,\phi(-\vec{k})$ means that ρ^* or $|\rho|$ will do as well as ρ (the sign is determined by requiring $\phi(\vec{R})$ to reduce to the Coulomb potential in the limit of large k (small $|\vec{R}|$). The particular choice (5.86) was made because it seems to show clearest the connection with the Landau-Vlasov formalism (cf. eq. (4.75)).

At any rate, certain general properties of the potential may be observed regardless of what argument is chosen for ρ. If F_1 is replaced by a Maxwell distribution in ρ (which one should do when eq. (5.53) or eq. (5.89) is linearized) ρ becomes

$$\rho_0(u,k) = k^2+\kappa^2+\frac{4\pi e^2}{v}\sum Z_\sigma^2\frac{N_\sigma}{N}\left(\frac{m_\sigma}{2\pi\theta}\right)^{\frac{1}{2}} u e^{-m_\sigma u^2/2\theta}\cdot$$

$$\cdot\left[-2\sqrt{\pi}\int_0^{u(m_\sigma/2\theta)^{\frac{1}{2}}} dx\, e^{x^2}+i\pi\right] \tag{5.87}$$

where κ is the inverse Debye length defined by (1.10) and where one has used the identity:

$$P\int_{-X}^{+X}\frac{du'}{u'-u}e^{-\frac{m_\sigma u'^2}{2\theta}} = -2\sqrt{\pi}e^{-\frac{m_\sigma u^2}{2\theta}}\int_0^{u\sqrt{\frac{m_\sigma}{2\theta}}} dx\, e^{x^2} \tag{5.88}$$

For small u, therefore, corresponding to slow particles, $\rho_0(u)$ becomes $k^2+\kappa^2$, and $\Phi(\vec{R})$ becomes the Debye potential. For large u, $\rho(u)$ reduces to k^2; however,

$$\Phi(\vec{R}) = \frac{4\pi e^2 Z_\sigma Z_{\sigma'}}{(2\pi)^3} \int d\vec{k}\ \frac{e^{i\vec{k}\cdot\vec{R}}}{\rho\left(\frac{\vec{k}\cdot\vec{p}}{k m_\sigma}\right)} \tag{5.89}$$

does not reduce to the Coulomb potential for large $|\vec{p}|$, since an integral over the angle of $\vec{k}$, and hence over the argument of ρ, is involved. The motion produces a potential which is no longer spherically symmetric, but depends on the angle between $\vec{R}$ and $\vec{p}$. This is in accord with the results of Rosenbluth and Rostoker[21] who considered the problem of a single particle injected into a gas initially at equilibrium.

5. THE FOKKER-PLANCK FORM OF THE KINETIC EQUATION

It is well known that for a weak interaction potential, so that the average change of momentum in a collision is small, the Boltzmann equation can be approximated by the so-called Fokker-Planck diffusion equation:

$$\frac{\partial F_1}{\partial t} = \frac{\partial}{\partial p_\alpha}\left(a_\alpha F_1\right) + \frac{1}{2}\frac{\partial^2}{\partial p_\alpha \partial p_\beta}\left(b_{\alpha\beta} F_1\right) \tag{5.90}$$

Where $a_i = -\langle \Delta p_i \rangle$ is the average loss of momentum in a collision and the $b_{ij} = \langle \Delta p_i \Delta p_j \rangle$ are the quadratic averages of the momentum changes.

Since for the fully ionized plasma the pseudo-binary interaction Φ must be considered weak, one might expect that it must be possible to cast the basic kinetic equation (5.53) in the Fokker-Planck form. In fact, as can be verified quite easily and as already pointed out by Balescu, one can write (5.53) in the form (5.90) if one puts:

$$a_i(\eta) = \frac{-2Z_\sigma^2 e^4}{\nu}\int d\vec{k}\, k_i \left[\frac{k}{\pi\left(\frac{4\pi e^2}{\nu}\right)}\,\frac{\bar{D}}{|\rho|^2} + \vec{k}\cdot\frac{\partial}{\partial\vec{p}}\,\frac{\bar{f}}{|\rho|^2}\right]$$

$$= \frac{2e^4 Z_\sigma^2}{\nu}\int d\vec{k}\, k_i \int d\eta' Z_{\sigma'}^2 \left[\frac{1}{|\rho|^2}\,\delta\left(\vec{k}\cdot\left(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}'}{m_{\sigma'}}\right)\right)\vec{k}\cdot\frac{\partial}{\partial p'}F_1(\eta')\right.$$

$$\left. - F_1(\eta')k\cdot\frac{\partial}{\partial\vec{p}}\,\frac{\delta\left(\vec{k}\cdot\left(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}'}{m_{\sigma'}}\right)\right)}{|\rho|^2}\right] \tag{5.91}$$

and:

$$b_{ij}(\eta) = \frac{4e^4 Z_\sigma^2}{\nu}\int d\vec{k}\, k_i k_j\,\frac{\bar{f}}{|\rho|^2}$$

$$= \frac{4e^4 Z_\sigma^2}{\nu}\int d\vec{k}\,\frac{k_i k_j}{|\rho|^2}\int d\eta' Z_{\sigma'}^2\,\delta\left(k\cdot\left(\frac{\vec{p}}{m_\sigma} - \frac{\vec{p}'}{m_{\sigma'}}\right)\right)F_1(\eta') \tag{5.92}$$

Note that this is an exact transformation of the Kinetic equation (5.53). The a_i and b_{ij} are isotropic vector and tensor functions of the momentum

$\vec{p}$, which however still depend on the distribution function $F_1(\eta)$.

Although this transformation is quite suggestive, the formal character of the transformation should be emphasized. The kinetic equation (5.53) is <u>not</u> a linear diffusion equation. This is not even true if one linearizes the kinetic equation starting from the Maxwell distribution. Putting:

$$F_1(\eta) = F^{(0)}(\eta) + F^{(1)}(\eta)$$

Where $F^{(0)}$ is the Maxwell distribution and neglecting terms quadratic in $F^{(1)}$, one gets from (5.53) for $F^{(1)}$ a linear equation which can be written in the form:

$$\frac{\partial F^{(1)}(\eta)}{\partial t} = \frac{\partial}{\partial p_\alpha}\left(a_\alpha^{(0)} F^{(1)}\right) + \frac{1}{2}\frac{\partial^2}{\partial p_\alpha \partial p_\beta}\left(b_{\alpha\beta}^{(0)} F^{(1)}\right) +$$

$$+ \int d\eta' L(\eta, \eta') F^{(1)}(\eta') \qquad (5.93)$$

Where $a_i^{(0)}$, $b_{ij}^{(0)}$ are found from (5.91), (5.93) by replacing there $F_1(\eta)$ by $F^{(0)}(\eta)$, (and therefore ρ by the ρ_0 of (4.70)* and where:

*It can be seen that the $\vec{k}$-integrals occurring in $a_i^{(0)}$ and $b_{ij}^{(0)}$ diverge logarithmically at $k \to \infty$, which is due to the $r \to 0$ divergence of the Coulomb potential. One must use therefore a cut-off corresponding to a short-range repulsion.

$$L(\eta,\eta') = \frac{2e^4}{v} Z_\sigma^2 Z_{\sigma'}^2 \int d\vec{k}\, F^{(0)}(\eta)\, k_\alpha k_\beta \left\{ \delta\left(k\cdot\left(\frac{p}{m_\sigma} - \frac{p'}{m_{\sigma'}}\right)\right)\left[\frac{-p_\alpha}{m_\sigma kT}\frac{\partial}{\partial p_\beta}\left(\frac{1}{|p|^2}\right)\right]\right.$$

$$+ \frac{\partial}{\partial p_\alpha}\delta\left(k\cdot\left(\frac{p}{m_\sigma} - \frac{p'}{m_{\sigma'}}\right)\right)\left[\frac{-p_\alpha}{kT|p|^2}\left(\frac{1}{m_\sigma} + \frac{1}{m_{\sigma'}}\right) + \frac{m_\sigma}{m_{\sigma'}}\frac{\partial}{\partial p_\alpha}\left(\frac{1}{|p|^2}\right)\right]$$

$$\left.\frac{\partial^2}{\partial p_\alpha \partial p_\beta}\delta\left(k\cdot\left(\frac{p}{m_\sigma} - \frac{p'}{m_{\sigma'}}\right)\right)\frac{m_\sigma}{m_{\sigma'}|p|^2}\right\}$$

Because of the last term in (5.93), the linearized version of the kinetic equation is therefore not a "local" diffusion equation. It also seems difficult to interprete the a_i and the b_{ij} of eqs. (5.91), (5.92) as the first and second moments of the distribution of the momentum change in a binary collision with an effective interaction potential Φ. Only for the case* of one ion moving in a plasma which is in thermal equilibrium one obtains as Rosenbluth and Rostoker[26] have shown and as follows also from (5.53), for the momentum distribution of the one ion considered, a bona-fide Fokker-Planck diffusion equation.

*The relation of this case to the general case is clearly the same as the relation of the well known Rayleigh model to the Boltzmann equation.

APPENDIX I

THE LANDAU-VLASOV EQUATION WITH RADIATION FIELD

In Chapter IV the Landau-Vlasov equation was derived under the assumption of pure Coulomb interactions. Since the particles are in motion, there is also in general a magnetic interaction or radiation field. One way to handle this is to introduce the Fourier amplitudes of the interaction vector potential as generalized coordinates. One then has

$$\vec{A}\left(\vec{r}\right) = \vec{A}_0\left(\vec{r}\right) + \sum_{\lambda} q_{\lambda} \vec{A}_{\lambda}\left(\vec{r}\right) \tag{A.1}$$

where the $\vec{A}_{\lambda}$'s satisfy*

$$\left(\Delta + \frac{\omega_{\lambda}^2}{c^2}\right)\vec{A}_{\lambda} = \nabla \cdot \vec{A}_{\lambda} = 0 \tag{A.2}$$

Here the ω_{λ} are the characteristic frequencies of the volume; the sum in λ is over the polarization directions for a given frequency, as well as over the frequencies. The Hamiltonian for the system then becomes

$$H = \sum_{i=1}^{N} \left[\left(\vec{p}_i - Z_{\sigma_i} e \vec{A}\left(\vec{r}_i\right)/c\right)^2/\left(2 m_{\sigma_i}\right) + Z_{\sigma_i} e U\left(\vec{r}_i\right)\right]$$
$$+ \frac{1}{2}\sum_{\lambda}\left(p_{\lambda}^2 + \omega_{\lambda}^2 q_{\lambda}^2\right) + 4\pi e^2 \sum_{i<j\le N} \sum Z_{\sigma_i} Z_{\sigma_j} \phi\left(\left|\vec{r}_i - \vec{r}_j\right|\right) \tag{A.3}$$

*Since we are not concerned with relativistic invariance, we use the Coulomb gauge $\nabla \cdot \vec{A}_{\lambda} = 0$.

Introducing a distribution function $D(p_i, r_i, \sigma_i; p_\lambda q_\lambda)$ for the total system, one can derive in the usual way a Liouville equation for D. It is convenient to change from the momenta $\vec{p}_i$ of the particles to the velocities:

$$\vec{w}_i = \frac{1}{m\sigma_i}\left(\vec{p} - \frac{eZ\sigma_i}{c}\vec{A}(\vec{r}_i)\right)$$

(A.4)

After some transformations, the Liouville equation for the total system can then be written in the form:

$$\frac{\partial D(z_i, x_\lambda)}{\partial t} = \left\{ \sum_{i=1}^{N} \left[w_{i\alpha}\left(\frac{\partial}{\partial r_{i\alpha}} + \frac{eZ\sigma_i}{c} A_{\lambda\alpha}(\vec{r}_i)\frac{\partial}{\partial p_\lambda}\right) + \right.\right.$$

$$+ \frac{eZ\sigma_i}{m o_i}\left(\frac{w_{i\beta}}{c}\frac{\partial A_\alpha}{\partial r_{i\beta}} - \frac{w_{i\beta}}{c} + \sum_\lambda \frac{p_\lambda}{c} A_{\lambda\alpha}(\vec{r}_i) + \right.$$

$$\left.\left.\left. + \frac{1}{c}\frac{\partial A_\alpha(\vec{r}_i)}{\partial t} + \frac{\partial U(r_i)}{\partial r_{i\alpha}}\right)\frac{\partial}{\partial w_{i\alpha}}\right] + 4\pi e^2 \mathcal{O}_N \right\} D(z_i, x_\lambda)$$

(A.5)

Where $z_i \equiv (\vec{w}_i, \vec{r}_i, \sigma_i)$ and $x_\lambda \equiv (p_\lambda, q_\lambda)$. The operator $\mathcal{O}_n$ (see Ch. IV, eq. (49)) is now expressed as:

$$\mathcal{O}_N(z_1 .. z_N) = \sum_{i<j\le N}\sum Z_{\sigma_i} Z_{\sigma_j} \frac{\partial \phi(|r_i - r_j|)}{\partial \vec{r}_i}\left(\frac{1}{m\sigma_i}\frac{\partial}{\partial \vec{w}_i} - \frac{1}{m\sigma_j}\frac{\partial}{\partial \vec{w}_j}\right)$$

(A.6)

Partial distributions are now defined for s particles and the set (λ_K) of field coordinates as:

$$F_{s(\lambda k)}\left(z_1 \cdots z_s ; x_{(\lambda K)}\right) = \frac{V^s}{N^s} \prod_i N_{\sigma_i} \int \cdots \int (z_i ; x_\lambda)\, d_{N-s} z \prod_{\mu \neq (\lambda_K)} dq_\mu\, dp_\mu$$

(A.7)

Integrating (A.5) over all but s particles and the set (λ_K) of field coordinates, and taking the limit* as $N_\sigma, V \to \infty$ one finds for the modified B-B-G-K-Y hierarchy:

$$\frac{\partial F_s(\lambda K)}{\partial t} = \left(-\mathcal{H}_{s(\lambda K)} + 4\pi e^2 J_s\right) F_{s\,(\lambda_K)} + \frac{4\pi e^2}{v} \int \Phi_s(z)\, F_{s+1,(\lambda K)}(z_i, z)\, dz$$

$$+ \sum_{i=1}^{s} \frac{e Z \sigma_i}{m_{\sigma_i} c} \sum_{(\lambda K)} \left[w_{i\beta} q_\lambda \left(\frac{\partial A_{\lambda\alpha}}{\partial r_{i\beta}} - \frac{\partial A_{\lambda\beta}}{\partial r_{i\alpha}} \right) \frac{\partial}{\partial w_{i\alpha}} + p_\lambda A_{\lambda\alpha} \frac{\partial}{\partial w_{i\alpha}} - w_{i\alpha} A_{\lambda\alpha} \frac{\partial}{\partial p_\lambda} \right] F_{s(\lambda K)}$$

$$+ \sum_{i=1}^{s} \frac{e Z \sigma_i}{m_{\sigma_i} c} \sum_{\mu \neq (\lambda K)} \int dp_\mu\, dq_\mu \left[w_{i\beta} q_\mu \left(\frac{\partial A_{\mu\alpha}}{\partial r_{i\beta}} - \frac{\partial A_{\mu\beta}}{\partial r_{i\alpha}} \right) + p_\mu A_{\mu\alpha} \right] \frac{\partial}{\partial w_{i\alpha}} F_{s(\lambda_K)\,\mu}$$

$$- \frac{1}{v} \int dz\, \frac{e Z \sigma}{c} w_\alpha \sum_{(\lambda K)} A_{\lambda\alpha}(\vec{r}) \frac{\partial}{\partial p_\lambda} F_{s+1,(\lambda_K)}(z_i, z).$$

(A.8)

*In this limit the spectrum ω_λ becomes of course continuous. We continue to use the discrete notation since the modifications are obvious.

where

$$\mathcal{H}_{s(\lambda_K)} = \sum_{i=1}^{s} \left[w_{i\alpha} \frac{\partial}{\partial r_{i\alpha}} + \frac{Z\sigma_i e}{m\sigma_i} \left\{ E_{0\alpha}(\vec{r}_i) + \frac{1}{c} \left(\vec{w} \times \vec{H}_0(\vec{r}_i) \right)_\alpha \right\} \frac{\partial}{\partial w_{i\alpha}} \right]$$

$$+ \sum_{\lambda \in (\lambda_K)} \left(p_\lambda \frac{\partial}{\partial q_\lambda} - \omega_\lambda^2 q_\lambda \frac{\partial}{\partial p_\lambda} \right)$$

(A.9)

Here

$$\vec{E}_0(\vec{r}) = \nabla U(\vec{r}) - \frac{1}{c} \partial \vec{A}_0(\vec{r}, +)$$

and

$$\vec{E}(\vec{r}) = \nabla \times \vec{A}_0(\vec{r}, +)$$

(A.10)

are the applied fields. Φ_s is given by (4.12) with $\frac{\partial}{\partial \vec{p}_i}$ replaced by $\frac{1}{m\sigma_i} \frac{\partial}{\partial \vec{w}_i}$

One sees from (A.8), that the distribution function $F_{s(\lambda K)}$ is connected with distribution functions containing one more particle or one more field coordinate x_μ .

Generalizing the method of Bogolubov, we will seek solutions of the hierarchy (A.8), in which all distribution functions $F_{s(\lambda_K)}$ depend on the time only through the distribution functions of one particle (and no field coordinates) and of one field coordinate (and no particle coordinates). For these distribution functions $F_1(z_1)$ and $F_{0\lambda}(x_\lambda)$ we assume kinetic equations of the form:

$$\frac{\partial F_1}{\partial t} = A(z_1 | F_1, F_{0\lambda}) \; ; \; \frac{\partial F_{0\lambda}}{\partial t} = A(x_\lambda | F_1, F_{0\lambda}) \tag{A.11}$$

The last three terms in (A.8) vanish in the non-relativistic limit, and thus are small in the parameter $\overline{W}/c$ where $\overline{W}$ is an average speed. We will therefore expand all functionals, besides in $4\pi e^2$ and $4\pi e^2/v$ as in Ch.IV, in a third formal relativistic parameter δ assuming that the last three terms in (A.8) are proportional to δ. Accordingly we write:*

$$\frac{\partial F_1}{\partial t}(z_1) = \sum\sum\sum (4\pi e^2)^m \left(4\frac{\pi e^2}{v}\right)^n \delta^\ell A_{mn\ell}(z_1 | F)$$
$$= \sum\sum (4\pi e^2)^m \delta^i A_{mi}(z_1 | F) \tag{A.12}$$

$$\frac{\partial F_{0\lambda}(x_\lambda)}{\partial t} = \sum\sum\sum (4\pi e^2)^m \left(\frac{4\pi e^2}{v}\right)^n \delta^\ell A^\lambda_{mn\ell}(q_\lambda p_\lambda | F)$$
$$= \sum\sum A^\lambda_{m\ell}(q_\lambda p_\lambda | F) \tag{A.13}$$

*The indicated functional dependence on F is an abbreviation for the dependence on F_1 and $F_{0\lambda}$.

$$F_{s(\lambda_K)} = \sum\sum\sum (4\pi e^2)^m \left(\frac{4\pi e^2}{V}\right)^n \delta^{\ell} F_{s(\lambda_K)}^{(m,n,\ell)} \quad (1F)$$

$$= \sum\sum (4\pi e^2)^m \delta^{\ell} F_{s(\lambda_K)}^{(m,\ell)} \quad (1F)$$

(A.14)

Substituting the expansion (A.17) - (A.19) into (A.14), equating coefficients of the parameters, and letting $\delta \to 1$, one obtains:

$$A_{000} = -\mathcal{N}_1 F_1$$

$$A^{\lambda}_{00} = -\mathcal{N}_{\alpha\lambda} F_{0\lambda} \qquad (A.15)$$

$$A_{mn\ell}(\beta_1|F_1) = \int d\beta\, \Phi_1(\beta) F_2^{(m,n-1,\ell)}(\beta_1,\beta|F) \qquad (A.16)$$

$$+ \frac{z\sigma_1 e}{m\sigma_1 c} \sum_{\lambda} \int d p_\lambda dq_\lambda \left[w_{1\beta} q_\lambda \left(\frac{\partial A_{\lambda\alpha}(\vec{r}_1)}{\partial r_{1\beta}} - \frac{\partial A_{\lambda\beta}(\vec{r}_1)}{\partial r_{1\alpha}}\right) + q_\lambda A_{\lambda\alpha}(\vec{r}_1)\right]$$
$$\frac{\partial}{\partial w_{1\alpha}} F_{1\lambda}^{(m,n,\ell-1)}(\beta_1; q_\lambda p_\lambda|F)$$

(A.17)

$$A^{\lambda}_{mn\ell}(q_\lambda p_\lambda|F) = -\frac{1}{V}\int d\beta\, \frac{z\sigma e}{c} w_\alpha A_{\lambda\alpha}(\vec{r}) \frac{\partial}{\partial p_\lambda} F_{1\lambda}^{(m,n,\ell-1)}(\beta; q_\lambda p_\lambda|F)$$

(A.18)

and for the higher distributions

$$\sum_{p=0}^{m}\sum_{q=0}^{n}\sum_{r=0}^{\ell} D_{pqr} F_{s(\lambda_k)}^{(m-p,n-q,\ell-r)}(1F) = -\mathcal{N}_{s(\lambda_k)} F_{s(\lambda_k)}^{(m,n,\ell)} + \sigma_s F_{s(\lambda_k)}^{(m-1,n,\ell)}$$

$$+\int d\underline{z}\,\underline{\Phi}(\underline{z})F^{(m,n-1,\ell)}_{s+1,(\lambda_k)}+\sum_{i=1}^{s}\sum_{\lambda\in(\lambda_k)}\Big[w_{i\beta}\,e_\lambda\Big(\frac{\partial A_{\lambda\alpha}(\vec{r}_i)}{\partial r_{i\beta}}-\frac{\partial A_{\lambda\beta}(\vec{r}_i)}{\partial r_{i\alpha}}\Big)\frac{\partial}{\partial w_{i\alpha}}$$

$$+p_\lambda A_{\lambda\alpha}(\vec{r}_i)\frac{\partial}{\partial w_{i\alpha}}-w_{i\alpha}A_{\lambda\alpha}(\vec{r}_i)\frac{\partial}{\partial p_\lambda}\Big]F^{(m,n,\ell-1)}_{s(\lambda_k)}+\sum_{i=1}^{s}\sum_{\lambda\notin(\lambda_k)}\int dp_\lambda\,dq_\lambda \qquad \text{(A.19)}$$

$$\Big[w_{i\beta}\,e_\lambda\Big(\frac{\partial A_{\lambda\alpha}(\vec{r}_i)}{\partial r_{i\beta}}-\frac{\partial A_{\lambda\beta}(\vec{r}_i)}{\partial r_{i\alpha}}\Big)+p_\lambda A_{\lambda\alpha}(\vec{r}_i)\Big]\frac{\partial}{\partial w_{i\alpha}}F^{(m,n,\ell-1)}_{s(\lambda_k)\lambda}$$

$$-\frac{1}{\nu}\int d\underline{z}\,\frac{Z_\sigma e}{c}w_\alpha\sum_{\lambda\in(\lambda_k)}A_{\lambda\alpha}(\vec{r})\frac{\partial}{\partial p_\lambda}F^{(m,n,\ell-1)}_{s+1(\lambda_k)}(\underline{z})$$

We are here interested in only the Zeroth order in the field strength ($4\pi e^2$) and first order in the relativistic parameter δ , therefore we will calculate only A_{0m0} , A_{0m1} , A^{λ}_{0m0} , A^{λ}_{0m1} . For $\ell = 0$, the equations (A.15), (A17), (A.19) reduce to the corresponding equations for the non-relativistic limit; thus

$$A_{00}=A_{000}+\frac{4\pi e^2}{\nu}A_{010}$$

$$=\Big[-\mathcal{H}_1+\frac{4\pi e^2}{\nu}\int d\underline{z}\,F_1(\underline{z})\,\underline{\Phi}(\underline{z}_1;\underline{z})\Big]F_1(\underline{z}_1) \qquad \text{(A.20)}$$

while according to (A.16), (A.18)

$$A^{\lambda}_{000}=-\mathcal{H}_{0\lambda}F_{0\lambda} \qquad A^{\lambda}_{0n0}=0 \qquad n>0 \qquad \text{(A.21)}$$

or

$$A^{\lambda}_{00} = A^{\lambda}_{000} = -\mathcal{H}_{0\lambda} F_{0\lambda} \tag{A.22}$$

For the calculation of A_{0n1}, A^{λ}_{0n1}, we need to find $F^{(0,n,0)}_{s(\lambda_k)}$, $F^{(0,n,1)}_{s}$; for the former we have the equation

$$\left(D_{000} + \mathcal{H}_{s(\lambda_k)}\right) F^{(0,n,0)}_{s(\lambda_k)} = \int d_{\mathcal{Y}} \Phi_s(\mathcal{Y}) F^{(0,n-1,0)}_{s+1,(\lambda_k)} - D_{010} F^{(0,n-1,0)}_{s(\lambda_k)} \tag{A.23}$$

This is precisely the same equation as was found for $F^{(0,n)}_{s}$ in Chapter IV; the only difference is that we must solve it with the generalized boundary conditions

$$\lim_{\tau\to\infty} g^{s(\lambda_k)}_{-\tau} F^{(0,0,0)}_{s(\lambda_k)}(\,|g_\tau F) = \prod F_1(\mathcal{Y}_i) \prod_{\lambda\in(\lambda_k)} F_{0\lambda}(q_\lambda p_\lambda) \tag{A.24}$$

$$\lim_{\tau\to\infty} g^{s(\lambda_k)}_{-\tau} F^{(m,n,\ell)}_{s(\lambda_k)}(\,.\,|g_\tau F) = 0 \qquad m+n+\ell > 0 \tag{A.25}$$

It follows that

$$F^{(0,0,0)}_{s(\lambda_k)} = \prod_{i=1}^{s} F_1(\mathcal{Y}_i) \prod_{\lambda\in(\lambda_k)} F_{0\lambda}(q_\lambda, p_\lambda) \tag{A.26}$$

$$F^{(0,n,0)}_{s(\lambda_k)} = 0 \qquad n>0 \tag{A.27}$$

Using (A.27), (A.20), the equation for $F_s^{(0,m,1)}$ ((A.19) with m = 0, ℓ = 1, and the set (λ_k) empty) becomes

$$\left(D_{000} + \mathcal{H}_s\right) F_s^{(0,m,1)} = \int d\gamma\, \Phi_s(\gamma)\, F_{s+1}^{(0,m-1,1)}(\gamma) + \sum_{i=1}^{s} \sum_{\lambda} \frac{Z_{\sigma_i} e}{m_{\sigma_i} c} \int dq_\lambda\, dp_\lambda$$

$$\left[W_{i\beta} q_\lambda \left(\frac{\partial A_{\lambda\alpha}(\vec{r}_i)}{\partial r_{i\beta}} - \frac{\partial A_{\lambda\beta}(\vec{r}_i)}{\partial r_{i\alpha}} \right) + p_\lambda A_{\lambda\alpha}(\vec{r}_i) \right] \frac{\partial}{\partial W_{i\alpha}} F_{s\lambda}^{(0,m,0)}(q_\lambda p_\lambda)$$

$$- D_{010} F_s^{(0,m-1,1)} - D_{0m1} F_s^{(0,0,0)} \qquad \text{(A.28)}$$

For n = 0, this reduces to

$$\left(D_{000} + \mathcal{H}_s\right) F_s^{(0,0,1)} = \sum_{i=1}^{s} \sum_{\lambda} \frac{Z_{\sigma_i} e}{m_{\sigma_i} c} \int dq_\lambda\, dp_\lambda .$$

$$\left[W_{i\beta} q_\lambda \left(\frac{\partial A_{\lambda\alpha}(\vec{r}_i)}{\partial r_{i\beta}} - \frac{\partial A_{\lambda\beta}(\vec{r}_i)}{\partial r_{i\alpha}} \right) + p_\lambda A_{\lambda\alpha}(\vec{r}_i) \right] \frac{\partial}{\partial W_{i\alpha}} F_{s\lambda}^{(0,0,0)}(q_\lambda p_\lambda)$$

$$- D_{001} F_s^{(0,0,0)} = 0 \qquad \text{(A.29)}$$

where the last step follows from (A.26), (A.17). But according to Theorem I of Chapter IV, the solution of (A.29) with the boundary condition (A.25) is

$$F_s^{(0,0,1)} = 0 \qquad \text{(A.30)}$$

For $n > 0$, (A.28) becomes (using (A.27))

$$\left(D_{000} + \mathcal{H}_s\right) F_s^{(0,n,1)} = \int d_3 \Phi_s(3)\, F_{s+1}^{(0,n-1,1)}(3) - D_{010}\, F_s^{(0,n-1,1)} \tag{A.31}$$

or according to Theorem I,

$$F_s^{(0,n,1)}(1\cdots s \mid F_1) = \int_0^\infty d\tau\, g_{-\tau}^{(s)} \left\{ \int d_3 \Phi_s(1\cdots s; 3) F_{s+1}^{(0,n-1,1)}(1\cdots s\, 3 \mid g_\tau F) \right. \tag{A.32}$$

$$\left. - \left[D_{010}\, F_s^{(0,n-1,1)}(1\cdots s \mid F) \right]_{F \to g_\tau F} \right\}$$

It follows trivially by induction from (A.30), (A.32) that

$$F_s^{(0,n,1)} = 0 \qquad n > 0 \tag{A.33}$$

Using (A.33) in (A.17), (A.18) one finds

$$A_{001}(1 \mid F_1) = \frac{Z_\alpha e}{m_\alpha c} \sum_\lambda \int dp_\lambda\, dq_\lambda$$

$$\left[w_{1\beta} q_\lambda \left(\frac{\partial A_{\lambda\alpha}(\vec{r}_1)}{\partial r_{1\beta}} - \frac{\partial A_{\lambda\beta}(\vec{r}_1)}{\partial r_{1\alpha}} \right) + p_\lambda A_{\lambda\alpha}(\vec{r}_1) \right] \frac{\partial}{\partial w_{1\alpha}} F_{1\lambda}^{(0,0,0)}(3; q_\lambda p_\lambda \mid F) \tag{A.34}$$

$$A_{0n1}^\lambda = 0 \qquad n > 0 \tag{A.35}$$

$$A_{001}^\lambda(q_\lambda p_\lambda \mid F_1) = -\frac{1}{v} \int d_3 \frac{Z_\alpha e}{c} w_\alpha A_{\lambda\alpha}(\vec{r}) \frac{\partial}{\partial p_\lambda} F_{1\lambda}^{(0,0,0)}(3; q_\lambda p_\lambda \mid F) \tag{A.36}$$

$$A_{0n_1} = 0 \qquad n > 0 \tag{A.37}$$

where $F_{1\lambda}^{(0,0,0)}$ is given by (A.26).

From (A.20), (A.22), (A.24), (A.34) - (A.37), the kinetic equations become, to the desired order of approximation

$$\frac{\partial F_1}{\partial t}(z_1) = A_{00} + A_{01} = A_{000} + \frac{4\pi e^2}{\nu} A_{010} + A_{001} =$$

$$\left\{ -w_{1\alpha} \frac{\partial}{\partial r_{1\alpha}} - \frac{Z_\sigma e}{m_\sigma} \left[E_{0\alpha}(\vec{r}_1) + \frac{1}{c} (\vec{w} \times \vec{H}_0(\vec{r}_1))_\alpha \right.\right.$$

$$\left. - \frac{4\pi e}{\nu} \frac{\partial}{\partial r_{1\alpha}} \left(\int dz\, Z_\sigma F_1(z) \phi(|\vec{r}_1 - \vec{r}|) \right) + \frac{1}{c} \sum_\lambda \int dp_\lambda\, dq_\lambda\, F_{0\lambda}(q_\lambda p_\lambda) \right.$$

$$\left. \left(q_\lambda \left(\vec{w}_1 \times \left(\nabla_{\vec{r}_1} \times \vec{A}_\lambda(\vec{r}_1) \right) \right)_\alpha + p_\lambda A_{\lambda\alpha} \right) \right] \frac{\partial}{\partial w_{1\alpha}} \left. \right\} F_1(z_1) \tag{A.38}$$

$$\frac{\partial F_{0\lambda}}{\partial t} = A_{00}^\lambda + A_{01}^\lambda = A_{00}^\lambda + A_{001}^\lambda =$$

$$\left\{ -p_\lambda \frac{\partial}{\partial q_\lambda} + \left[\omega_\lambda^2 q_\lambda - \frac{1}{\nu} \int dz\, \frac{Z_\sigma e}{c} w_\alpha A_{\lambda\alpha}(\vec{r}) F_1(z) \right] \frac{\partial}{\partial p_\lambda} \right\} F_{0\lambda} \tag{A.39}$$

Eq. (A.38) may be expressed in terms of the average interaction scalar and vector potentials defined by

$$U_i(\vec{r}) = \frac{4\pi e}{\nu}\int d\gamma' Z_{\sigma_1} F_1(\gamma')\,\phi(|\vec{r}-\vec{r}'|)$$

(A.40)

$$\vec{A}_i(\vec{r}) = \sum_\lambda \int d q_\lambda\, dp_\lambda\, F_{0\lambda}(q_\lambda p_\lambda)\, q_\lambda \vec{A}_\lambda(\vec{r})$$

(A.41)

According to (A.39) we have

$$\frac{\partial \vec{A}_i}{\partial t} = \sum_\lambda \int dq_\lambda\, dp_\lambda\, q_\lambda \frac{\partial F_{0\lambda}}{\partial t}(q_\lambda p_\lambda)\,\vec{A}_\lambda(\vec{r})$$

$$= \sum_\lambda \int dq_\lambda\, dp_\lambda \left\{-p_\lambda \frac{\partial}{\partial q_\lambda} + \left[\omega_\lambda^2 q_\lambda - \frac{1}{\nu}\int d\gamma \frac{Z_\sigma e}{c} W_\alpha A_{\lambda\alpha}(\vec{r}) F_1(\gamma)\right]\frac{\partial}{\partial p_\lambda}\right\} F_{0\lambda}(q_\lambda, p_\lambda)$$

$$= \sum \int dq_\lambda\, dp_\lambda\, p_\lambda \vec{A}_\lambda(\vec{r})\, F_{0\lambda}(q_\lambda, p_\lambda)$$

(A.42)

where the last equality follows by partial integration. Therefore (A.38) may be written

$$\frac{\partial F_1(\gamma_1)}{\partial t} = -\left\{W_{1\alpha}\frac{\partial}{\partial r_{1\alpha}} + \frac{Z_{\sigma_1} e}{m_{\sigma_1}}\left[E_\alpha(\vec{r}_1) + \frac{1}{c}\left(W_1 \times H(\vec{r}_1)\right)_\alpha\right]\frac{\partial}{\partial W_{1\alpha}}\right\} F_1(\gamma_1)$$

where

$$\vec{E} = \vec{E}_0 + \vec{E}_i \qquad \vec{H} = \vec{H}_0 + H_i$$

(A.43)

with

$$\vec{E}_i(\vec{r}) = -\nabla_{\vec{r}} U_i(\vec{r}) - \frac{1}{c}\frac{\partial \vec{A}_i(\vec{r})}{\partial t} \qquad \vec{H}_i(\vec{r}) = \nabla_{\vec{r}} \times A_i(\vec{r}) \tag{A.44}$$

It remains to be shown that these fields satisfy the appropriate Maxwell equations. For the scalar potential, since $\Delta_{\vec{r}}\,\phi(|\vec{r}-\vec{r}'|) = -\delta(\vec{r}-\vec{r}')$, one has the Poisson equation

$$\Delta U_i(\vec{r}) = -\frac{4\pi e}{v}\sum_\sigma Z_\sigma \int d\vec{w}\, F_i(\vec{r},\vec{w},\sigma) = -4\pi\rho(\vec{r}) \tag{A.45}$$

where

$$\rho(\vec{r}) = e\sum_\sigma Z_\sigma \int d\vec{w}\, F_i(\vec{r},\vec{w},\sigma)$$

is clearly the average charge density. For the vector potential one has from (A.2), (A.41), (A.42), (A.39), and partial integration

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}_i(\vec{r}) =$$

$$-\frac{1}{v c^2}\sum_\lambda \int dq_\lambda dp_\lambda F_{0\lambda}(q_\lambda p_\lambda)\vec{A}_\lambda(\vec{r})\int dz' \frac{Z_\sigma e}{c} w'_\alpha A_{\lambda\alpha}(\vec{r}')F_i(z') \tag{A.46}$$

We wish to express the right hand side of (A.46) in terms of the current density and the potential U_i. For this purpose we define

$$\vec{j}(\vec{r}) = \frac{e}{v}\sum_\sigma Z_\sigma \int d\vec{w}\, \vec{w}\, F_i(\vec{r},\vec{w},\sigma) \tag{A.47}$$

According to (A.45), a relation exists between U_i and j for

$$\frac{\partial}{\partial t}\Delta U_i(\vec{r}) = -\frac{4\pi e}{v}\sum_\sigma Z_\sigma \int d\vec{w}\, \frac{\partial F_i(\vec{r},\vec{w},\sigma)}{\partial t}$$

$$= \frac{4\pi e}{V} \sum_{\alpha} Z_{\alpha} \int d\vec{w}\; w_{\alpha} \frac{\partial}{\partial r_{\alpha}} F_1(\vec{r}, \vec{w}, \sigma)$$

$$= 4\pi \nabla \cdot \vec{J}(\vec{r}) \tag{A.48}$$

where we have used (A.38) and an integration by parts. Accordingly we may write

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \tag{A.49}$$

where

$$\vec{J}_1(\vec{r}) = \frac{1}{4\pi} \frac{\partial}{\partial t} \nabla U_i(\vec{r}) \tag{A.50}$$

and

$$\nabla \cdot \vec{J}_2(\vec{r}) = 0 \tag{A.51}$$

Since the $\vec{A}_\lambda$ form a complete set for divergenceless functions, we may write

$$\vec{J}_2(\vec{r}) = \sum_{\lambda} a_\lambda(t)\, \vec{A}_\lambda(\vec{r}) \tag{A.52}$$

with*

$$a_\lambda(t) = \frac{1}{4\pi c^2} \int d\vec{r}\,' \vec{A}_\lambda(\vec{r}\,') \cdot \vec{J}_2(\vec{r}\,') \tag{A.53}$$

*The normalization of the $\vec{A}_\lambda$ is chosen to be
$\int d\vec{r}\, \vec{A}_\lambda(\vec{r}) \cdot A_{\lambda'}(\vec{r}) = 4\pi c^2 \delta_{\lambda\lambda'}$

From (A.52), (A.53),

$$\vec{J}_2(\vec{r}) = \frac{1}{4\pi c^2}\sum_\lambda \vec{A}_\lambda(\vec{r})\int d\vec{r}'\,\vec{A}_\lambda(\vec{r}')\cdot\vec{J}_2(\vec{r}')$$

$$= \frac{1}{4\pi c^2}\sum_\lambda \vec{A}_\lambda(\vec{r})\int d\vec{r}'\,\vec{A}_\lambda(\vec{r}')\cdot\left(\vec{J}(\vec{r}')-\vec{J}_1(\vec{r}')\right)$$

$$= \frac{1}{4\pi c^2}\sum_\lambda A_\lambda(\vec{r})\int d\vec{r}'\,\vec{A}_\lambda(\vec{r}')\cdot\vec{J}(\vec{r}') \tag{A.54}$$

where the last step follows from the fact that

$$\int d\vec{r}'\,\vec{A}_\lambda(\vec{r}')\cdot\vec{J}_1(\vec{r}') = \frac{1}{4\pi}\frac{\partial}{\partial t}\int d\vec{r}'\,\vec{A}_\lambda(\vec{r}')\,\nabla_{\vec{r}'}U_\lambda(\vec{r}')$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial t}\int d\vec{r}'\left[\nabla_{\vec{r}'}\cdot\left(U_\lambda(\vec{r}')\vec{A}_\lambda(\vec{r}')\right) - U_\lambda(\vec{r}')\nabla_{\vec{r}'}\cdot\vec{A}_\lambda(\vec{r}')\right]$$

$$= 0 \tag{A.55}$$

where we have used (A.2) and the divergence theorem. Putting the definition (A.47) of j in (A.54) one has

$$\vec{J}_2(\vec{r}) = \frac{1}{4\pi c^2 V}\sum_\lambda \vec{A}_\lambda(\vec{r})\int dz'\,\frac{Z_\alpha e}{c}\,w_\alpha' A_{\lambda\alpha}(\vec{r}')F_1(z') \tag{A.56}$$

Comparing (A.56) with (A.46) and noting that, from the definition of $F_{o\lambda}$ and the normalization of D,

$$\int dq_\lambda\, dp_\lambda\, F_{o\lambda}(q_\lambda, p_\lambda) = 1 \tag{A.57}$$

(A.44) may be written

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}_i = -\frac{4\pi}{c}\vec{j}_2 = -\frac{4\pi}{c}(\vec{j} - \vec{j}_1)$$

$$= -\frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\nabla U_i \qquad \text{(A.58)}$$

Eqs. (A.58) and (A.45) are equivalent to the Maxwell equations; one more equation should be added which determines the gauge. Rather than the usual Lorentz condition one has the restriction, which follows from (A.2) and (A.41), to the Coulomb gauge

$$\nabla \cdot \vec{A}_i = 0 \qquad \text{(A.59)}$$

APPENDIX II

SOLUTION OF THE INTEGRODIFFERENTIAL EQUATIONS FOR $F_s^{(1,n)}$

We wish first to prove Theorem II of Chapter IV which relates the functions $F_s^{(1,n)}$ to the functions $F_2^{(1,n)}$. The equation for $F_s^{(1,n)}$ is

$$(D_{00} + \mathcal{H}_s) F_s^{(1,n)} = O_s F_s^{(0,n)} \int dy\, \Phi_s(y)\, F_{s+1}^{(1,n-1)}(y)$$

$$- D_{01} F_s^{(1,n-1)} - D_{1n} F_s^{(0,0)} \tag{4.76}$$

We assert that the solutions of these equations with the boundary conditions

$$\lim_{\tau \to \infty} \Delta_{-\tau}^{(s)} F_s^{(1,n)} (1 \Delta_{\tau}^{(1)} F_1) = 0 \tag{4.29}$$

have the form

$$F_s^{(1,n)} = \sum_{i<} \sum_{j \le s} F_2^{(1,n)}(y_i, y_j) \prod_{k \neq i,j} F_1(y_k) \tag{A.60}$$

The proof is accomplished by mathematical induction. We first note that for n = 0, eq. (4.76) takes the form

$$(D_{00} + \mathcal{H}_s) F_s^{(1,0)} = O_s F_s^{(0,0)} \tag{A.61}$$

which has the solution, according to Theorem II of Chapter IV,

$$F_s^{(1,0)} = \int_0^\infty d\tau\, \mathcal{S}_{-\tau}^{(s)}\, O_s\, \mathcal{S}_{\tau}^{(s)}\, F_s^{(0,0)}$$

$$= \sum_{i<} \sum_{j\le s} \prod_{k\neq i,j} F_1(y_k) \int_0^\infty d\tau\, \mathcal{S}_{-\tau}^{(2)} O_2(y_i,y_j) \mathcal{S}_{\tau}^{(2)} F_1(y_i) F_1(y_j)$$

(A.62)

where we have used the explicit form (4.34) for $F_s^{(0,0)}$. Eq. (A.62) clearly has the form (A.58). If (A.60) holds for $n<m$, we observe that according to (4.21), (4.24) $D_{01} F_s^{(1,m-1)} = \sum_{i<}\sum_{j\le s}\Big[\prod_{k\neq i,j} F_1(y_k) D_\alpha F_2^{(1,m-1)}(y_i,y_j)$

$$+ \sum_{k\neq i,j} \prod_{l\neq i,j,k} F(y_l) \int dy\, F_1(y) \Phi_1(y_k;y) F_1(y_k)\Big]$$

(A.63)

$$D_{1m} F_s^{(0,0)} = \sum_{i=1}^{s} \prod_{j\neq i} F(y_j)\, dy\, \Phi_1(y_i;y) F_2^{(1,m-1)}(y_i,y)$$

(A.64)

Using the fact (which follows from the definition of Φ_s) that:

$$\Phi_s(y_1 \cdots y_s, y) = \sum_{i=1}^{s} \Phi_1(y_i,y)$$

(A.65)

one has

$$\int dy\, \Phi(y) F_{s+1}^{(1,m-1)} =$$

$$\sum_{i<}\sum_{j\le s}\Big[\prod_{k\neq i,j} F_1(y_k) \int dy\, F_1(y) \Phi_2(y_i,y_j;y) F_2^{(1,m-1)}(y_i,y_j)$$

$$+ F_2^{(1,m-1)}(y_i,y_j) \sum_{k\neq i,j} \prod_{l\neq i,j,k} F_1(y_l) \int dy\, F_1(y) \Phi_1(y_k;y) F_1(y_k)\Big]$$

$$+ \sum_{i=1}^{s} \Big[\prod_{j\neq i} F_1(y_j) \int dy\, \Phi_1(y_i;y) F_2^{(1,m-1)}(y_i,y)$$

$$+\sum_{j\neq i}\prod_{k\neq i,j} F_1(y_k)\int dy F_2^{(1,m-1)}(y_i,y)\Phi_1(y_j;y)F_1(y_j)\Big]$$

(A.66)

Thus eq. (4.76) may be written

$$(D_{00}+\mathcal{N}_s)F_s^{(1,m)} =$$
$$\sum_{i<j\leq s}\sum\prod_{k\neq i,j} F_1(y_k)\Big\{\int dy\Big[F_1(y)\Phi_2(y_i,y_j;y)F_2^{(1,m-1)}(y_i,y_j)+$$
$$+F_2^{(1,m-1)}(y_i,y)\Phi_1(y_j;y)F_1(y_j)$$
$$+F_2^{(1,m-1)}(y_j,y)\Phi_1(y_i;y)F_1(y_i)\Big]$$
$$-D_{01}F_2^{(1,m-1)}(y_i,y_j)\Big\}$$

(A.67)

But Theorem I of Chapter IV shows that this has the solution

$$F_s^{(1,m)}(y_1\cdots y_s|F_1) = \sum_{i<j\leq s}\sum\prod_{k\neq i,j} F_1(y_k)\int_0^\infty d\tau\, \mathcal{A}_\tau^{(2)}(y_i,y_j)\times$$
$$\times\Big\{\int dy\Big[\mathcal{A}_\tau^{(1)}(y)F_1(y)\Phi_2(y_i,y_j;y)F_2^{(1,m-1)}(y_i,y_j|\mathcal{A}_\tau^{(1)}F_1)$$
$$+F_2^{(1,m-1)}(y_i,y|\mathcal{A}_\tau^{(1)}F_1)\Phi_1(y_j;y)\mathcal{A}_\tau^{(1)}F_1(y_j)+(i\leftrightarrow j)\Big]$$
$$-\Big[D_{01}F_2^{(1,m-1)}(y_i,y_j|F_1)\Big]_{F_1\to\mathcal{A}_\tau^{(1)}F_1}\Big\}$$

(A.68)

Since (A.68) clearly has the form (A.60), this completes the inductive proof.

The next thing to be proved is that the expression

$$C_n(y_1,y_2) = \int_0^\infty d\tau_0 \int_{\tau_0}^\infty d\tau_1 \int \cdots \int_{\tau_{n-1}}^\infty d\tau_n \int \cdots \int dy^{(1)} \cdots dy^{(n)} \cdot$$
$$\cdot \sum P\, O_2^{\tau_0}(y_1,y^{(1)})\, O_2^{\tau_1}(y^{(1)},y^{(2)}) \cdots O_2^{\tau_n}(y^{(n)},y_2) F_1(y_1) F_1(y_2) \prod_{i=1}^{n} F_1(y^{(i)}) \tag{4.88}$$

with O_2^τ defined by (4.85), is a solution of the recursion,

$$C_n(y_1,y_2) = \int dy \int_0^\infty d\tau \left\{ O_2^\tau(y_2,y)\, C_{n-1}(y,y_1)\, F_1(y_2) + O_2^\tau(y_1,y)\, C_{n-1}^\tau(y,y_2)\, F_1(y_2) \right\} \tag{4.84}$$

where C_n is defined by (4.86). An inductive proof is again indicated. Eq. (4.81) shows that (4.88) is true for n = 0. If it is true for n = m-1, eq. (4.84) becomes*

$$C_m(y_1,y_2) = \int_0^\infty d\tau_0 \int_{\tau_0}^\infty d\tau_1 \cdots \int_{\tau_{n-1}}^\infty d\tau_n \int \cdots \int dy^{(1)} \cdots dy^{(n)} \cdot$$
$$\cdot \Big\{ O_2^{\tau_0}(y^{(m)},y_2) \sum P\, O_2^{\tau_1}(y_1,y^{(1)}) \cdots O_2^{\tau_m}(y^{(m-1)},y^{(m)}) + O_2^{\tau_0}(y_1,y^{(1)}) \sum P\, O_2^{\tau_1}(y^{(1)},y^{(2)}) \cdots O_2^{\tau_m}(y^{(m)},y_2) \Big\} \times$$
$$\times F_1(y_1) F_1(y_2) \prod_{k=1}^{m} F_1(y^{(k)}) \tag{A.69}$$

*The fact that O_2^τ is symmetric in its arguments has been used.

Eq. (A.69) contains all of the terms of (4.88) in which the first operator is either $O_2^{\tau_0}(y_1, y^{(1)})$ or $O_2^{\tau_0}(y^{(m)}, y_2)$. But the remaining terms can each be written in the form

$$\int dy^{(i)} \int dy^{(i+1)} O_2^{\tau_0}(y^{(i)}, y^{(i+1)}) F(y_1, y_2, y^{(i)}, y^{(i+1)})$$

and thus vanish by partial integration, using the definition of O_2.

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