DOCTORAL DISSERTATION SERIES

TITLE THE PHENOMENOLOGICAL NEUTRAL VECTOR MESON FIELD

AUTHOR WALTER WATARU WADA

UNIVERSITY OF MICH. DATE 1950

DEGREE Ph.D. PUBLICATION NO. 2471



UNIVERSITY MICROFILMS
ANN ARBOR · MICHIGAN

THE PHENOMENOLOGICAL NEUTRAL VECTOR MESON FIELD

by

Walter Wataru Wada

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1950

Committee in charge

Assistant Professor Edwin S. Lennox, Chairman
Professor Ernest F. Barker
Professor David M. Dennison
Associate Professor Wayne E. Hazen
Professor George Y. Rainich
Professor George E. Uhlenbeck

Acknowledgment

I wish to thank sincerely Professors Lennox and Uhlenbeck for their guidance and invaluable suggestions throughout the course of this investigation. My thanks are also due to Professor Laporte who so often gave me encouragement and advice during the initial phase of this investigation.

I also wish to express my gratitude to all other members of the staff and the graduate students of the Department of Physics for the congenial academic atmosphere that I always enjoyed.

This thesis is dedicated to my wife Helen S. Wada whose constant encouragement made possible to complete this work.

Lastly, I wish to acknowledge the Rackham School of Graduate Studies for granting me the Special Fellowship for 1946-1949.

Table of Contents

Acknowledgment 11

Chapter I. Introduction and Statement of Problem

Chapter II. Establishment of Macroscopic Proca Equations

1. Microscopic Vector Meson Field Equations 9

2. Lorentz-invariant Generalization of the Microscopic Proca Equations to the Macroscopic Field 13

3. Derivation of the Proposed Macroscopic Proca Equations 18

4. Continuation of the Derivation of Macroscopic Proca Equations (Tensor Coupling) 32

Chapter III. Dispersion Theory

1. Evaluation of the Magnitude of the Mesonic Electric Moment (Vector Coupling) 38

2. Application of Hoek's Method in the Derivation of a Wave Equation for the Neutral Proca Field Traversing Nuclear Matter (Vector Coupling) 43

3. Evaluation of the Magnitude of the Induced Spin Moment (Tensor Coupling) 48

4. Application of the Hoek's Method for the Case of the Tensor Coupling 53

5. Extinction Theorem of Oseen (Tensor Coupling) 56

6. Dispersion in Tensor Coupling 58

Chapter IV. Application of the Phenomenological Proca Equations

1. Derivation of the Field for the Čerenkov Radiation 63 (61)

2. Energy Radiated per Unit Length of the Path of the Nucleon 71

Appendix I. Lagrangian Formalism for the Phenomenological Field 76

Appendix II. Equations in Hoek's Method 80

Appendix III. Treatment of the Phenomenological Gordon-Klein Wave Equation without Dispersion under the Čerenkov Condition 82

Bibliography 92

Figures:

1. Physically Small Volume "dv" 24
2. The Position of a Nucleon with Respect to the Point of Observation and the Origin of the Coordinates 45
3. Geometrical Representation of the Equation (3.22) 51
4. Refraction of the Meson Waves 56
5. Čerenkov Radiation 62
6. Formation of Čerenkov Cone 65
7. Infinitesimal Displacement of s(r) 80
8. Contour Integrations to Solve the Wave Equation Under the Čerenkov Condition 84
9. Surfaces of Constant Phase 86
10. Field Represented by Complex Integration 87

Chapter I

Introduction and Statement of Problem

The meson theory of nuclear forces has been developed in analogy to the theory of the Coulomb force, which arises from the interaction of charged particles with the electromagnetic field. It was first pointed out by Yukawa[1] that the characteristic properties of the nuclear forces, such as the short range, the saturation effect, the β-decay, etc., can be understood simply on the basis of the existence of an intermediary field between that of the heavy particles and of the light particles when one lets the quanta of this new field corresponding to the photons of the electromagnetic case have a finite rest mass and positive and negative charges.

The range of forces arising from the rest mass μ of the field is $\frac{\hbar}{\mu c}$, which, when equated to the experimental value of 2×10^{-13}cm, yields a value of 200 electron masses for μ. The charges of the field not only suggest the way to understand the anomalies of the magnetic moments of the proton and the neutron because of the existence of the charged meson cloud in the vicinity of the nucleon, but also give rise to the Heisenberg's type exchange interaction between the proton and the neutron, leading to the saturation effect in the nuclear forces. Charged particles of this rest mass were later discovered in cosmic-rays.

---

[1] H. Yukawa, Proc. Phys.-Math. Soc. Japan, 17, 48 (1935)

Although the charged meson field was first proposed to account for the interactions between proton and neutron, the approximate equality of the interaction involving charge exchange (proton-neutron) and those involving no charge exchange (neutron-neutron and proton-proton) has made it necessary to introduce a neutral meson field as a part of the interaction field of the nuclear particles.

In spite of some qualitative agreements between experience and the predictions of the theory, however, the very extensive theoretical investigations that have been carried out in the past years have failed to devise a theory which account at all reasonably for the quantitative aspects of nuclear forces. These difficulties are perhaps connected with the defects of the quantum theory of fields, and the decisive judgment on the validity of the meson theory of the nuclear interactions may have to be postponed until the defects of the quantum theory of fields are completely cleared.

Assuming mesons are responsible for nuclear forces in this thesis we wish to exploit the analogy between the electromagnetic field and the meson field and establish a phenomenological meson field for nuclear matter corresponding to the phenomenological Maxwell field in the electromagnetic theory.

The essence of the phenomenological electromagnetic theory may be briefly stated in the following manner. When light goes through a transparent medium, because of the field vectors $\vec{E}$ and $\vec{H}$ associated with the light waves, the atoms and molecules within the medium are set into vibration. These resonators will in turn emit elementary waves. If the wave length of

the incoming radiation is long compared with the inter-resonator distances, these elementary waves will interfere constructively, thus forming a new wave front.

The parameters characterizing the new wave front in general depend upon the material constants and the density of the resonators which constitute the medium. Since the new wave front propagates through homogeneous medium in the same manner as the waves propagate in vacuum, in the phenomenological treatment one may suppress the microscopic interactions of light with matter, and consider an average field in the space occupied by matter, to which now phenomenological vectors $\vec{D}$ and $\vec{B}$ are ascribed along with $\vec{E}$ and $\vec{H}$. It is assumed here that the phenomenological field vectors are related to the vacuum vectors by material constants.

The phenomenological equations for the electromagnetic field in matter were derived by Lorentz from the microscopic field equations by means of averaging the physical quantities which characterize the process of scattering of the electromagnetic field by the scattering centers under the assumption that there exist a large number of scattering centers in the volume of the cube of a wave length.

In our attempt to extend the same considerations to the meson field we shall first consider the possibility that nuclei, especially heavy nuclei, are individually endowed with a meson wave dispersive property. It is well-known that nuclei in many respects exhibit properties similar to those possessed by solid or liquid matter. For instance, the binding energies of nuclei and the volume of a nucleus are directly

proportional to the number of the constituent particles, which presumably are homogeneously distributed throughout the nuclear matter. Since the average velocity of these particles is of the order of one-tenth of the velocity of light, they may be considered to be stationary with respect to the propagation of meson waves whose velocity is of the order of the velocity of light.

Let us consider the situation in which the meson wave is traveling through a homogeneous nuclear matter of a large extent. Since we are assuming that mesons are quanta of the field which represents the forces of the nucleons, the meson waves must interact with these particles with constitute the nuclear matter. Or, the meson waves must be absorbed and reemitted by nuclear particles, as photons traveling through matter are absorbed and reemitted by the constituent electrically charged particles. It is obvious that the greater the number of particles per unit volume, the greater the frequency of such events occurring in that volume.

Such a scattering effect of the meson waves by nucleons within nuclear matter may be looked at from a macroscopic point of view, as in the case of the phenomenológical electromagnetic theory, provided the mesons are of low energies and their de Broglie wave lengths are large enough to contain many scattering centers (nucleons) so that scattering occurs coherently. The way to establish the macroscopic description of coherent scattering is to generalize the meson field equations by introducing macroscopic material constants of the nuclear matter analogous to ϵ and μ in the Maxwell theory.

If one wishes to derive the macroscopic meson field equations from the microscopic equations by means of averaging, as in the derivation of the macroscopic electromagnetic theory, question arises at this point as to how perfect the coherent scattering of the meson waves would be in nuclear matter. In the low energy regions of the electromagnetic waves traveling through matter the number of scattering centers (atoms and molecules) within the volume of (wave length)^3 is of the order of 10^9. Therefore, in the electromagnetic case the coherent scattering can be considered to be perfect, and the resulting constructive intereference is well-defined.

Assuming a nuclear matter of very large extent, in the region of the energy of the meson field slightly beyond the rest energy (140 Mev) the de Broglie wave length is large enough to contain a large number of nucleons in the volume of the wave length. However, as the energy increases the wave length decreases rapidly, and at 200 Mev there are of the order of 500 nucleons in the volume of the wave length. The number goes down to the order of 25 at the meson energy of 300 Mev.

The fact that in the case of heavy nuclei the number of nucleons are of the order of 200 suggests that even in the region of low energies the coherent scattering that we look for cannot be as perfect as it is in the electromagnetic case. Consequently, the constructive interference which gives rise to the phenomenological field may not be as well defined for the case of the actual heavy nuclei. However, we shall tacitly assume here that the coherent scattering is not small

and the constructive interference of the meson waves does occur in non-negligible proportion.

Under such circumstances the phenomenological meson field equations can be obtained in a manner analogous to the phenomenological electromagnetic equations, and the wave equation arising from them becomes of the form

$$\left(\Delta - \frac{n_1^2}{c^2}\frac{\partial^2}{\partial t^2} - n_2^2 \kappa_0^2\right) A = 0,$$

where c is the velocity of light, k_0 the reciprocal Compton wave length of the quantum of the field, n_1 and n_2 the material constants, and A a function representing the field.

The fact that the Gordon-Klein equation has three terms indicates that, no matter what kind of meson field is considered, there will occur at most two indices of refraction, one corresponding to that in the Maxwell theory, viz. n_1, and the other to be associated with the Compton wave length of the meson, viz. n_2. Assuming a plane wave solution to this equation, one obtains for the phase velocity of the propagation of meson waves within the nuclear system an expression:

$$\frac{\omega}{\kappa} = \frac{c}{n_1}\sqrt{1 + n_2^2 \frac{\kappa_0^2}{\kappa^2}},$$

where ω and k are the frequency and the wave number of the meson wave, respectively.

In the following chapters we shall first establish the phenomenological field equations for the vector neutral mesons (Proca field), assuming the existence of appropriate phenomenological nuclear material constants. We shall later derive these equations from microscopic equations by means of averaging processes.

Next, the classical theory of the dispersion of meson waves by nucleons will be developed. This will make possible the proof of the existence of the phenomenological material constants, and, consequently, the indices of refraction from the microscopic point of view analogous to the corresponding electromagnetic theory.

Lastly, as an application of the phenomenological field, calculations of the mesonic Čerenkov radiation effect will be performed in a manner analogous to that of Tamm[1] for the electromagnetic Čerenkov radiation.

All these considerations that have been mentioned in this chapter will be developed in this thesis entirely within the domain of the classical field theory. Even if one grants our fundamental assumption concerning the coherent scattering of meson waves by nucleons constituting nuclear matter to be valid within certain limits, question still arises at this point as to whether or not an entirely classical treatment of the meson field will be able to give correct account of such nuclear phenomena as those under consideration. As a tentative answer to this question a discussion by Bhabha[2], whose classical treatment of meson dynamics is extensively made use of in this thesis, is quoted in the following.

"The classical theory of mesons, like that of photons, is of practical importance. The treatment of a field classically instead of quantum mechanically is equivalent to neglecting

---

[1] Ig. Tamm, Jour. Phys. U.S.S.R., 1, 439 (1939)
[2] H.J. Bhabha, Proc. Roy. Soc. A172, 384 (1939)

momentum properties of individual field quanta. We may therefore expect the classical theory to give correct results for processes in which momenta of individual mesons is small compared to the mass of nucleon. This covers the whole field of nuclear phenomena and an important part of cosmic-ray phenomena. The classical theory will give correct results for meson of a few hundred million electron-volts and will even give the correct order of magnitude for mesons of 10^9 electron-volts."

Chapter II

Establishment of Macroscopic Proca Equations

1. Microscopic vector meson field equations

The vector meson field equations, or Proca equations[1], may be stated in 4-dimensional notation[2] in the following manner:

$$\partial^{\alpha} \chi_{\alpha\beta} = \kappa_0 \phi_{\beta} - \frac{g}{\kappa_0} s_{\beta}$$
$$\partial_{\alpha} \phi_{\beta} - \partial_{\beta} \phi_{\alpha} = \kappa_0 \chi_{\alpha\beta} + f m_{\alpha\beta} \qquad (2.1)$$

where k_0 is the reciprocal Compton wave length of the quantum of the field, ϕ_{α} and $\chi_{\alpha\beta}$ are a 4-vector and an anti-symmetric tensor of the second rank for the field, respectively, and s_{α} and $m_{\alpha\beta}$ are corresponding quantities for the sources of the field. g and f are constants expressing vector and tensor coupling strength, respectively, between the Proca field and the source.

All Greek indices go as (1,2,3,0) and Latin indices as (1,2,3). Whenever two same indices occur together in the same term, they are summed over all components. Using co- and contravariant notations, we have $x^{\ell} = x_{\ell}$, $x^{0} = -x_0 = ct$, $\partial^{\ell} = \partial_{\ell} = \frac{\partial}{\partial x_{\ell}}$ and $\partial^{0} = -\partial_0 = -\frac{1}{c}\frac{\partial}{\partial t}$. When we want to pass from covariant a_{μ} form to contravariant a^{μ} form of a vector, we shall use the fundamental tensor $g^{\mu\nu}$ appropriate

---

[1] A. Proca, Comptes Rendus, 202, 1490 (1936)
[2] N. Kemmer, Proc. Roy. Soc., A166, 127 (1938)

for flat space-time, with components $g^{00} = -1$, $g^{11} = g^{22} = g^{33} = 1$, the other components vanishing.

The immediate consequences from (2.1) are:

$$\partial_\alpha \phi^\alpha = \frac{g}{\lambda_0^2} \partial_\alpha S^\alpha \tag{2.2}$$

$$\partial_\alpha \chi_{\alpha\beta} + \partial_\beta \chi_{\gamma\alpha} + \partial_\gamma \chi_{\alpha\beta} = -\frac{g}{\lambda_0^2} \left(\partial_\alpha m_{\beta\gamma} + \partial_\beta m_{\gamma\alpha} + \partial_\gamma m_{\alpha\beta} \right).$$

The source functions for the meson field are the probabilities per unit volume that a nucleon makes a transition from one state to another. The transition of a nucleon from a proton state to a neutron state and the converse constitute the sources for the positively and negatively charged mesons, respectively. Transitions involving no change in the charges of the nucleons provide the sources for the neutral meson field.

For the sake of simplicity we will concern ourselves only with the neutral Proca field for our discussion, in which case all components of the field functions are to be considered real and there should be the following continuity equation:

$$\partial_\alpha S^\alpha = 0. \tag{2.3}$$

The physical interpretation of this equation is that a proton stays as a proton and a neutron a neutron when a quantum of the neutral meson field is absorbed or emitted in the vector coupling of the field to a nucleon. From (2.2) we then obtain:

$$\partial_\alpha \phi^\alpha = 0. \tag{2.4}$$

The wave equations for the components of the 4-vector ϕ_α

are:

$$(\partial_\alpha \partial^\alpha - \kappa_0^2)\,\phi_\beta = -g\,s_\beta + f\,\partial^\alpha m_{\alpha\beta}. \tag{2.5}$$

It is instructive to rewrite the above equations (2.1)-(2.5) inclusive in the ordinary three dimensional vector notation so that we may compare them with the familiar electromagnetic equations in vector form. In doing so, we write the two anti-symmetric tensors in the following way:

$$\kappa_0 \chi_{\alpha\beta} = \begin{pmatrix} 0 & h_z & -h_y & e_x \\ -h_z & 0 & h_x & e_y \\ h_y & -h_x & 0 & e_z \\ -e_x & -e_y & -e_z & 0 \end{pmatrix}; \quad \frac{\kappa_0}{4\pi} m_{\alpha\beta} = \begin{pmatrix} 0 & m_z & -m_y & n_x \\ -m_z & 0 & m_x & n_y \\ m_y & -m_x & 0 & n_z \\ -n_x & -n_y & -n_z & 0 \end{pmatrix} \tag{2.6}$$

and the two 4-vectors as:

$$\frac{1}{4\pi} s_\alpha = \left(\frac{j_x}{c}, \frac{j_y}{c}, \frac{j_z}{c}; \rho\right). \tag{2.7}$$

The vectors $\vec{e}$ and $\vec{h}$ may be called mesonic electric and magnetic field strengths and $\vec{a}$ and b mesonic vector and scalar potentials of the Proca field. Variables for the microscopic field are written in small letters to distinguish them from macroscopic field variables which will be written in capital letters. This convention is employed in this chapter only.

The field equations (2.1) would now become:

$$\begin{aligned} &\operatorname{curl} \vec{h} - \frac{1}{c}\frac{\partial \vec{e}}{\partial t} + \kappa_0^2 \vec{a} = 4\pi g \frac{\vec{j}}{c}, \qquad \operatorname{div} \vec{e} + \kappa_0^2 b = 4\pi g \rho, \\ &\vec{e} + \vec{\nabla} b + \frac{1}{c}\frac{\partial \vec{a}}{\partial t} = -4\pi \frac{f}{\kappa_0} \vec{n}, \qquad \vec{h} - \operatorname{curl} \vec{a} = -4\pi \frac{f}{\kappa_0} \vec{m}. \end{aligned} \tag{2.8}$$

Equations (2.2) and (2.4) may be written:

$$\operatorname{curl} \vec{e} + \frac{1}{c}\frac{\partial \vec{h}}{\partial t} = -4\pi \frac{f}{\kappa_0}\left(\operatorname{curl} \vec{n} + \frac{1}{c}\frac{\partial}{\partial t}\vec{m}\right) \tag{2.9}$$

$$\operatorname{div} \vec{h} = -4\pi \frac{f}{\kappa_0} \operatorname{div} \vec{m}, \qquad \operatorname{div} \vec{a} + \frac{1}{c}\frac{\partial b}{\partial t} = 0.$$

Continuity equation (2.3) is:

$$\operatorname{div} \vec{j} + \frac{\partial \rho}{\partial t} = 0. \tag{2.10}$$

Finally, the wave equations (2.5) will be written as:

$$\begin{aligned} (\Delta - \tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2} - \kappa_0^2)\,\vec{a} &= -4\pi g \tfrac{\vec{j}}{c} - 4\pi \tfrac{1}{\kappa_0} \operatorname{curl} \vec{m} - \tfrac{1}{c}\tfrac{\partial \vec{n}}{\partial t} \\ (\Delta - \tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2} - \kappa_0^2)\,b &= -4\pi g \rho - 4\pi \tfrac{1}{\kappa_0} \operatorname{div} \vec{n} \end{aligned} \tag{2.11}$$

Considering the nucleons as point particle source functions the charge and current in the vector coupling will have the following expressions in the classical theory:

$$\rho = \delta(\vec{r} - \vec{r}'), \qquad \frac{\vec{j}}{c} = \rho \frac{\vec{u}}{c} = \frac{\vec{u}}{c}\,\delta(\vec{r} - \vec{r}').$$

Similarly, we assume the following expressions for the source functions in the tensor coupling in our classical theory:

$$\vec{m} = \vec{\sigma}\,\delta(\vec{r} - \vec{r}'), \qquad \vec{n} = \frac{\vec{u}}{c} m' = \frac{\vec{u}}{c} \vec{\sigma}\,\delta(\vec{r} - \vec{r}'), \tag{2.12}$$

where $\vec{u}$ is the velocity of a nucleon, $\vec{\sigma}$ a unit vector describing spin direction. Obviously, $\frac{\vec{j}}{c}$ is the relativistic counterpart to the non-relativistic source function ρ, and similarly $\vec{n}$, which may be called spin velocity, is the relativistic counterpart to the non-relativistic spin vector $\vec{m}$.

Note that the last equation of (2.9) is the Lorentz condition for the potential. In the electromagnetic theory this is introduced as an auxilliary condition so that one obtains wave equations for the vector and the scalar potentials. This is characteristic of a field in which potentials do not occur explicitly in the field equations, and consequently the potentials are not observable quantities (e.g. electromagnetic field and gravitational field).

If the quantum of the field possesses a finite rest mass, however, the potentials are observable quantities occurring explicitly in the field equations, as in the case of the Proca equations. Such a field is not "gauge invariant", and the Lorentz condition for the potentials should be a natural consequence from the field equations.

One obtains the electromagnetic equations by letting k_o and f go to zero and replacing the second equation in (2.1) with the second equation in (2.2). The "gauge invariance" that characterizes the electromagnetic field necessitates such rearrangement.

2. Lorentz-invariant generalization of the microscopic Proca equations to the macroscopic field

In our attempt to write down the macroscopic Proca equations we shall be guided as much as possible by the established way of generalizing the microscopic electromagnetic (Lorentz) equations to the macroscopic field (Maxwell). As was already mentioned in the previous section, the potential $(\vec{a}, b)$ in the vector meson field are observable field quantities along with the field strengths $\vec{e}$ and $\vec{h}$. Therefore, we must anticipate that there exist macroscopic counterparts to the potentials in addition to the macroscopic vectors $\vec{D}$ and $\vec{B}$.

To generalize the microscopic equations to the macroscopic field so that they may be put into a Lorentz-invariant form we first introduce two anti-symmetric tensors of the second rank constructed in the following manner, analogous to the case of

the Maxwell field:

$$\kappa_0 \Omega_{\alpha\beta} = \begin{pmatrix} 0 & H_z & -H_y & D_x \\ -H_z & 0 & H_x & D_y \\ H_y & -H_x & 0 & D_z \\ -D_x & -D_y & -D_z & 0 \end{pmatrix}; \quad \kappa_0 \Sigma_{\alpha\beta} = \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix}. \tag{2.13}$$

The next step is to construct two 4-vector potentials, one consisting of a vacuum spatial vector and the macroscopic scalar and the other the macroscopic spatial vector and a vacuum scalar:

$$\bar{\Psi}^\alpha = (\vec{A}, \psi); \quad \bar{\Phi}^\alpha = (\vec{F}, \phi) \tag{2.14}$$

where $\vec{A}$ and ϕ are vector and scalar potentials for the vacuum and $\vec{F}$ and ψ the corresponding quantities for the phenomenological field.

Then, corresponding to the microscopic Proca equations (2.1), we may propose the following set of equations for the macroscopic field:

$$\partial^\alpha \Omega_{\alpha\beta} = \kappa_0 \bar{\Psi}_\beta - \frac{1}{\kappa_0} S_\beta \tag{2.15}$$

$$\partial_\alpha \bar{\Phi}_\beta - \partial_\beta \bar{\Phi}_\alpha = \kappa_0 \Sigma_{\alpha\beta} + M_{\alpha\beta}.$$

As the immediate consequences from these equations we obtain, corresponding to (2.2):

$$\partial_\alpha \bar{\Psi}^\alpha = 0 \tag{2.16}$$

$$\partial_\alpha \Sigma_{\beta\gamma} + \partial_\beta \Sigma_{\gamma\alpha} + \partial_\gamma \Sigma_{\alpha\beta} = -\frac{1}{\kappa_0}\left(\partial_\alpha M_{\beta\gamma} + \partial_\beta M_{\gamma\alpha} + \partial_\gamma M_{\alpha\beta}\right),$$

where we assumed the macroscopic charge conservation law in the derivation of the top equation:

$$\partial_\alpha S^\alpha = 0 \tag{2.17}$$

Corresponding to s_α and $m_{\alpha\beta}$ of the microscopic field, we assume the following a 4-vector and an anti-symmetric tensor for the sources of the macroscopic field:

$$S_\alpha = 4\pi g \left(\frac{J_x}{c}, \frac{J_y}{c}, \frac{J_z}{c}, \Lambda \right)$$

$$M_{\alpha\beta} = 4\pi \frac{1}{\kappa_0} \begin{pmatrix} 0 & M_z & -M_y & N_x \\ -M_z & 0 & M_x & N_y \\ M_y & -M_x & 0 & N_z \\ -N_x & -N_y & -N_z & 0 \end{pmatrix} \tag{2.18}$$

Let us now rewrite equations (2.15) and (2.16) in three dimensional vector notation with which we are more familiar. Equations (2.15) then turn out to be:

$$\begin{aligned} &\operatorname{curl} \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} + \kappa_0^2 \vec{A} = 4\pi \frac{\vec{J}}{c} \\ &\operatorname{div} \vec{D} + \kappa_0^2 \chi = 4\pi \Lambda \\ &\vec{E} + \vec{\nabla}\phi + \frac{1}{c} \frac{\partial \vec{F}}{\partial t} = -\frac{4\pi}{\kappa_0} \vec{N} \\ &\vec{B} - \operatorname{curl} \vec{F} = -\frac{4\pi}{\kappa_0} \vec{M} \end{aligned} \tag{2.19}$$

The identities (2.16) arising from (2.15) are:

$$\begin{aligned} &\operatorname{curl} \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = -4\pi \frac{1}{\kappa_0} \left(\operatorname{curl} \vec{N} + \frac{1}{c} \frac{\partial \vec{M}}{\partial t} \right) \\ &\operatorname{div} \vec{A} = -4\pi \frac{1}{\kappa_0} \operatorname{div} \vec{M} \\ &\operatorname{div} \vec{A} + \frac{1}{c} \frac{\partial \chi}{\partial t} = 0 \end{aligned} \tag{2.20}$$

The charge conservation law (2.17) reads:

$$\mathrm{div}\,\vec{J} + \frac{\partial \Lambda}{\partial t} = 0.$$

Analogous to the case of the Maxwell equations, at this point we assume that in the rest frame of reference of matter each macroscopic field variable is related to its vacuum counterpart by a constant characteristic of nuclear matter in the following manner:

$$\vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H}, \quad \vec{F} = \nu \vec{A}, \quad \psi = \sigma \phi. \tag{2.21}$$

The energy conservation law may be derived from the equations (2.19), (2.20) and (2.21) in the usual way. We multiply scalarly the first equation in (2.19) by $\vec{E}$, the third by $k_o^2 \vec{A}$, the first equation in (2.20) by $\vec{H}$ and the third by $k_o^2 \phi$, and add up the results properly.[1]

$$\frac{c}{\partial t}\frac{1}{8\pi}\left[(\vec{E}\cdot\vec{D}) + (\vec{H}\cdot\vec{B}) + \kappa_o^2\left\{(\vec{A}\cdot\vec{F}) + \phi\psi\right\}\right] + \mathrm{div}\,\frac{c}{4\pi}\left[\left[\vec{E}\times\vec{H}\right] + \kappa_o^2 \vec{A}\phi\right]$$
$$= -(\vec{E}\cdot\vec{J}) - \kappa_o^2 c\,(\vec{A}\cdot\vec{N}) - \frac{c}{\kappa_o}\left(\vec{H}\cdot\left(\mathrm{curl}\,\vec{N} + \frac{1}{c}\frac{\partial \vec{M}}{\partial t}\right)\right).$$

Thus, we identify the energy density and current of our macroscopic field as follows:

$$W = \frac{1}{8\pi}\left[(\vec{E}\cdot\vec{D}) + (\vec{H}\cdot\vec{B}) + \kappa_o^2\left\{(\vec{A}\cdot\vec{F}) + \phi\psi\right\}\right]$$
$$\vec{S} = \frac{c}{4\pi}\left[\left[\vec{E}\times\vec{H}\right] + \kappa_o^2 \vec{A}\phi\right] \tag{2.22}$$

The right hand side of the energy continuity equation is the time rate of energy dissipation by the field due to its

---

[1] For the derivation of the same equation from the stress energy tensor of the macroscopic field see Appendix 1.

interaction with the sources. Note that W and $\vec{S}$ become the energy density and the Poynting vector of the phenomenological electromagnetic field in the limit $k_0 \to 0$. It is interesting to note that $\vec{S}$ is independent of the phenomenological material constants, as it is in the electromagnetic case.

By substituting (2.21) into (2.19) and (2.20) and eliminating we obtain the following set of second order equations for each of the field components:

$$\left(\Delta - \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\mu}{\nu}\kappa_0^2\right)\vec{A} - \left(1 - \frac{\epsilon\mu}{\sigma\nu}\right)\vec{\nabla}\,\mathrm{div}\,\vec{A} = -4\pi\left\{\frac{\mu}{\nu}\frac{\vec{J}}{c} + \frac{1}{\nu\kappa_0}\,\mathrm{curl}\,\vec{M} - \frac{\epsilon\mu}{\nu}\frac{1}{\epsilon}\frac{1}{c}\frac{\partial}{\partial t}\vec{N}\right\}$$

$$\left(\Delta - \frac{\sigma\rho}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\sigma}{\epsilon}\kappa_0^2\right)\phi = -4\pi\left\{\frac{\Lambda}{\epsilon} + \frac{1}{\kappa_0}\,\mathrm{div}\,\vec{N}\right\}$$

$$\left(\Delta - \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\mu}{\nu}\kappa_0^2\right)\vec{H} = -4\pi\frac{\kappa_0}{\nu}\vec{M} \qquad (2.23)$$

$$\left(\Delta - \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\mu}{\nu}\kappa_0^2\right)\vec{E} + \kappa_0^2\left(\frac{\sigma}{\epsilon} - \frac{\mu}{\nu}\right)\vec{\nabla}\phi = -4\pi\left\{\frac{1}{\epsilon}\vec{\nabla}\Lambda + \frac{\mu}{\nu}\kappa_0\vec{N}\right\}$$

Before we begin our task of deriving the proposed macroscopic Proca equations we introduce the following definitions for the macroscopic field components:

$$\vec{D} = \vec{E} + 4\pi\vec{P}, \qquad \vec{F} = \vec{A} + 4\pi\vec{Y},$$
$$\vec{B} = \vec{H} + 4\pi\vec{Q}, \qquad \vec{U} = \vec{A} + 4\pi\vec{Z}, \qquad (2.24)$$

where $\vec{P}$, $\vec{Q}$, $\vec{Y}$ and Z are moments per unit volume of the nuclear matter, corresponding to field components $\vec{E}$, $\vec{H}$, $\vec{A}$ and ϕ, respectively. It is obvious that each of the fields has two components, one due to g-coupling and the other to f-coupling of the field with sources.

$$\vec{P} = \vec{P}_g + \vec{P}_f, \qquad \vec{Y} = \vec{Y}_g + \vec{Y}_f, \qquad \vec{\Lambda} = \vec{\Lambda}_g + \vec{\Lambda}_f$$
$$\vec{Q} = \vec{Q}_g + \vec{Q}_f, \qquad Z = Z_g + Z_f, \qquad \Psi = \Psi_g + \Psi_f \tag{2.25}$$

By means of elimination from equations (2.19), (2.20), (2.24) we obtain the following wave equations for the potentials:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa_0^2\right)\psi = -4\pi\left\{\Lambda - \operatorname{div}\vec{P} - \Delta Z + \frac{1}{c}\frac{\partial}{\partial t}\operatorname{div}\vec{Y} - \operatorname{div}\vec{N}\right\}$$
$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa_0^2\right)\vec{A} = -4\pi\left\{\frac{1}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{P}}{\partial t} + \operatorname{curl}\vec{Q} - \operatorname{curl}\operatorname{curl}\vec{Y}\right. \tag{2.26}$$
$$\left. - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{Y} + \vec{V}\frac{1}{c}\frac{\partial}{\partial t}\Psi + \frac{1}{\kappa_0}\operatorname{curl}\vec{M} - \kappa_0 c\frac{\partial}{\partial t}\vec{N}\right\}$$

3. Derivation of the proposed macroscopic Proca equations

In this section a derivation of the proposed macroscopic Proca equations is given in a manner similar to what Lorentz gave in the derivation of the Maxwell equations from his microscopic electromagnetic equations.[1]

The procedure involves the method of obtaining average vector and scalar potentials from scattering centers (in our case nucleons) contained in a "physically" small volume within nuclear matter. By "physically" small volume it is meant a a volume large enough to contain a large number of scattering

---

[1] H.A. Lorentz, The Theory of Electrons (Leipzig, 1916); Van Vleck, The Theory of Electric and Magnetic Susceptibilities, Oxford University Press, 1 (1932)

centers but smaller in dimension than the wave length (λ) of the meson field, so that physical quantities characterizing the process of scattering of fields by the scattering centers do not differ appreciably from one scattering center to another within this volume $\Delta\nu$.

This condition enables one to obtain potentials arising from the volume $\Delta\nu$ by means of averaging over the scattering centers contained in that volume element. We then identify the average microscopic vector and scalar potentials arising from the averaged scattering of the field by individual nucleons contained in the small volume $\Delta\nu$ and integrated over the entire volume with the macroscopic vector and scalar potentials, respectively:

$$\overbrace{\vec{a}} = \vec{A}, \quad \overbrace{a} = \psi. \tag{2.27}$$

Here $\overbrace{\quad}$ indicates the process of averaging.

The scattering processes of the field by nucleons are characterized by the moments of the scattering centers which are induced by the external fields. The vibrations of these moments in turn send out waves of field, thus giving rise to scattering.

In the existence of the external meson field the mesonic charge g of a nucleon plays the role analogous to the electric charge in the existence of the external electromagnetic field, giving rise to $\vec{p}$ and $\vec{q}$ which are the analog of the electric and magnetic moments of a charged particle. In the case of the tensor coupling, however, the nucleon already possesses a permanent moment $\vec{m}$, the spin moment (see (2.12)), which

tends to be in the direction of the external magnetic field. There would also be a spin-velocity moment $\vec{n}$ corresponding to $\vec{q}$ in the g-coupling.

These moments $\vec{p}$, $\vec{q}$, $\vec{m}$ and $\vec{n}$ are the fundamental moments associated with the nucleon in the meson field, and they give rise to $\vec{P}_g$, $\vec{Q}_g$, $\vec{P}_f$ and $\vec{Q}_f$ vectors for the phenomenological field, as will be found in the discussions which follow this section. The question which is still unanswered is how the moments associated with the vector and scalar potentials arise.

Since $\vec{p}$ and $\vec{q}$ for g-coupling and $\vec{m}$ and $\vec{n}$ for f-coupling are the only moments associated with the nucleon, one may expect that the moments for the vector and the scalar potentials are the combinations of these fundamental moments with some other fundamental length, such as perhaps the average distance between nuclear particles. However, as will be shown, for isotropic nuclear matter the moments for the vector and the scalar potentials vanish in "dipole" approximation.

The first step in our procedure is to obtain solutions to the microscopic equations (2.11). Before we do so, it is more convenient to replace in the existence of the external field the charge density and current by the following expressions which automatically satisfy the continuity equation[1] (top equation in (2.10)):

$$\rho = \operatorname{div} \vec{p}, \quad \rho \frac{\vec{u}}{c} = -\frac{1}{c}\frac{\partial \vec{p}}{\partial t} + \operatorname{curl} \vec{q} \tag{2.28}$$

---

[1] D.I. Iwanenko and A. Sokolow, Jour. Phys. U.S.S.R., 3, 57 (1940)

where the vectors $\vec{p}$ and $\vec{q}$ are mesonic electric and magnetic polarization moments of a nucleon forming an anti-symmetric tensor of the second rank, $L_{\alpha\beta}$.

$$L_{\alpha\beta} = \begin{pmatrix} 0 & g_z & -g_y & -P_x \\ -g_z & 0 & g_x & -P_y \\ g_y & -g_x & 0 & -P_z \\ P_x & P_y & P_z & 0 \end{pmatrix}$$

Equation (2.28) can then be written as:

$$j_\beta = \partial^\alpha L_{\beta\alpha}.$$

In this form it is easy to prove that the charge conservation law is automatically satisfied.

For the later use it is desired at this point to introduce a tensor duel to $L_{\alpha\beta}$. Let us express the divergence of this tensor $L^*_{\alpha\beta}$ by γ_β.

$$\gamma_\beta = \partial^\alpha L^*_{\beta\alpha}$$

where, $\gamma_\beta = (\vec{l}_g, S_g)$.

Then, we obtain:

$$\operatorname{div}\vec{g} = S_g, \quad \operatorname{curl}\vec{f} + \frac{1}{c}\frac{\partial \vec{g}}{\partial t} = \vec{l}_g. \tag{2.29}$$

Proceeding similarly in the case of the tensor coupling, we have:

$$\operatorname{div}\vec{m} = S_f, \quad \operatorname{curl}\vec{n} + \frac{1}{c}\frac{\partial \vec{m}}{\partial t} = \vec{l}_f. \tag{2.30}$$

Note that when the equations (2.28) are substituted into the equations (2.11) the source functions in the g-coupling are of the form identical with those in the f-coupling. That is to say, the vectors $\vec{p}$ and $\vec{q}$ behave in manners similar to the vectors $\vec{n}$ and $\vec{m}$, respectively, with regard to the scattering of a neutral meson field. Because of this similarity of the source functions in the g and f-coupling and also because of the fact that they occur linearly, it is sufficient to investigate the g-coupling alone. Hence, we rewrite the equations (2.11) for the g-coupling only:

$$\begin{aligned} \left(\Delta - \tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2} - \kappa_0^2\right)\vec{a}_g &= -4\pi g\left(-\tfrac{1}{c}\tfrac{\partial \vec{p}}{\partial t} + c\,\mathrm{curl}\,\vec{q}\right) \\ \left(\Delta - \tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2} - \kappa_0^2\right) b_g &= -4\pi g\, d\dot{w}\vec{p}. \end{aligned} \tag{2.31}$$

The general solution of Gordon-Klein equation with a source function is obtained from Courant and Hilbert.[1]

$$\left(\Delta - \tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2} - \kappa_0^2\right)\phi(x,y,z,t) = -4\pi\lambda(x,y,z,t)$$

$$\phi(x,y,z,t) = \iiint_{-\infty}^{+\infty} dx'dy'dz' \frac{\lambda\left(x';y';z';t-\frac{R}{c}\right)}{R} + \kappa_0 \iiint\!\!\iint_{-\infty}^{+\infty} dx'dy'dz'dt' \, W(x,y,z,t;x';y';z';t')\,\lambda(x';y';z';t') \tag{2.32}$$

In this expression we have:

$$W(x,y,z,t;x';y';z';t') = -\frac{1}{2}\frac{J_1(\kappa_0\sqrt{\Gamma})}{\sqrt{\Gamma}}$$

$$\Gamma = c^2(t-t')^2 - R^2, \qquad R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

J_1 = Bessel function of the first order.

---

[1] Courant and Hilbert, Mathematische Physik II, p. 443, Berlin, Julius Springer (1937)

Note that when $k_0 \to 0$, which corresponds to the electromagnetic case, the function ϕ reduces to an ordinary retarded potential.

Solutions to equations (2.31) are now obtained as follows:

$$\vec{a}_g = g \iiint_{-\infty}^{+\infty} dx'dy'dz' \frac{1}{R} \left\langle -\frac{1}{c}\frac{\partial \vec{P}}{\partial t} + \mathrm{curl}_{r'} \vec{f} \right\rangle + k_0 g \iiint_{-\infty}^{+\infty} dx'dy'dz'dt' \, W \left\{ -\frac{1}{c}\frac{\partial \vec{P}}{\partial t'} + \mathrm{curl} \vec{q} \right\}$$

$$b_g = g \iiint_{-\infty}^{+\infty} dx'dy'dz' \frac{1}{R} \left\langle \mathrm{div}_{r'} \vec{P} \right\rangle + k_0 g \iiint_{-\infty}^{+\infty} dx'dy'dz'dt' \, W \left\{ \mathrm{div}_{r'} \vec{P} \right\}. \tag{2.33}$$

Here, $\langle \quad \rangle$ signifies "retardation" and r' the point (x', y', z'). Because of the point particle consideration for nucleons, it is permissible to assume the following expressions for $\vec{p}$ and $\vec{q}$:

$$\vec{P} = \vec{P}_0(\vec{r}, t)\, \delta(\vec{r} - \vec{r}\,'); \quad \vec{f} = \vec{f}_0(\vec{r}, t)\, \delta(\vec{r} - \vec{r}\,') \tag{2.34}$$

Performing spatial integrations with these δ-functions involved in the source functions, one obtains from (2.33) for one nucleon:

$$\vec{a}_{g(in)} = g \frac{1}{R} \left\langle -\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t} + \mathrm{curl} \vec{f}_0 \right\rangle + k_0 g \int_{-\infty}^{+\infty} dt' \, W(R) \left\{ -\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t'} + \mathrm{curl} \vec{f}_0 \right\}$$

$$b_{g(in)} = g \frac{1}{R} \left\langle \mathrm{div} \vec{P}_0 \right\rangle + k_0 g \int_{-\infty}^{+\infty} dt' \, W(R) \left\{ \mathrm{div} \vec{P}_0 \right\}, \tag{2.35}$$

where (in) signifies that these are potentials of the field scattered by one nucleon.

Let $\mathcal{N}$ be the nucleon density in the nuclear matter and dv a "physically small" volume defined in the previous

section. Since the scattering by $n\,dv$ nucleons in this "physically small" volume is <u>coherent</u>, the physical quantities $\vec{p}$ and $\vec{q}$ which characterize the process of scattering would not differ appreciably from one nucleon to another, and the potentials due to individual nucleons must be additive. Hence,

$$\widetilde{\vec{a}}_{g(dv)} = \mathcal{N} g \frac{1}{R} \widetilde{\left\langle -\frac{1}{c}\frac{\partial \vec{p}_0}{\partial t} + \operatorname{curl} \vec{q}_0 \right\rangle} dv + \mathcal{N} K_0 g \int_{-\infty}^{+\infty} dt' \, W(R) \widetilde{\left\{ -\frac{1}{c}\frac{\partial p_0}{\partial t} + \operatorname{curl} \vec{q}_0 \right\}} dv$$

$$\widetilde{b}_{g(dv)} = \mathcal{N} g \frac{1}{R} \widetilde{\left\langle \operatorname{div} \vec{p}_0 \right\rangle} dv + \mathcal{N} K_0 g \int_{-\infty}^{+\infty} dt' \, W(R) \widetilde{\left\{ \operatorname{div} p_0 \right\}} dv, \tag{2.36}$$

where $\widetilde{\quad}$ signifies "average" of quantities characterizing the scattering process over the volume dv. Obviously $\langle \quad \rangle$ and $\widetilde{\quad}$ commute with each other.

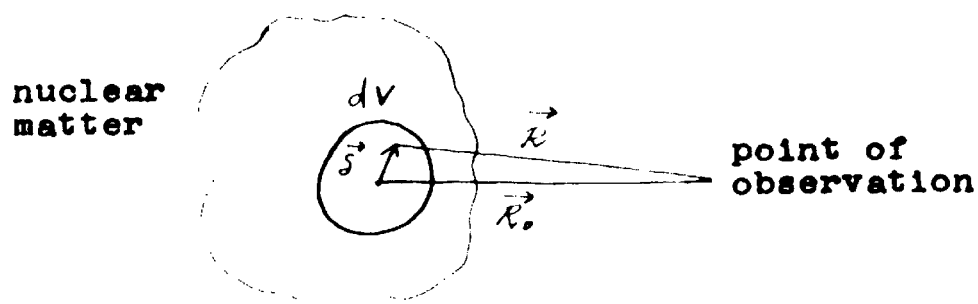


Fig. 1

Physically small volume dv

Let $\vec{R}$ refer to the distance from the point of observation to the position of each nucleon in dv-volume. If we now introduce $\vec{R}_0$ which is the distance from the point of observation to the center of dv-volume, any function of $\vec{R}$ can be expanded in Taylor series in power of a small distance $\vec{\delta}$

which refers to the nucleon in question from the center of the volume dV. Because of the smallness of $|\vec{s}|$ in comparison to R we may retain only the first two terms in the expansion.

Thus, we obtain:

$$\frac{1}{R} = \frac{1}{R_0} + (\vec{s}\cdot\vec{\nabla}\frac{1}{R_0}), \quad W(R) = W(R_0) + (\vec{s}\cdot\vec{\nabla}W(R_0)) \tag{2.37}$$

Hence, $$\frac{div\vec{p}_0}{R} = \frac{div\vec{p}_0}{R_0} + div\vec{p}_0(\vec{s}\cdot\vec{\nabla}\frac{1}{R_0}).$$

It is possible at this point to observe that, in view of the equations (2.28), the first term corresponds to the static potential and the second the dynamic potential due to the vibration of the moment $\vec{p}_0$, as it will be shown later. Thus, we write:

$$\frac{div\vec{p}_0}{R} = \frac{\rho}{R_0} + div\vec{p}_0(\vec{s}\cdot\vec{\nabla}\frac{1}{R_0}) \tag{2.38}$$

Similarly, $$div\vec{p}_0\cdot W(R) = \rho W(R_0) + div\vec{p}_0(\vec{s}\cdot\vec{\nabla}W(R_0)).$$

Since $$(\vec{s}\cdot\vec{\nabla}\frac{1}{R_0}) = div\frac{\vec{s}}{R_0} - \frac{div\vec{s}}{R_0},$$

we obtain:

$$div\vec{p}_0(\vec{s}\cdot\vec{\nabla}\frac{1}{R_0}) - div\vec{p}_0\,div\frac{\vec{s}}{R_0} - \frac{div\vec{p}_0\,div\vec{s}}{R_0}$$

$$= div\left(\frac{\vec{s}}{R_0}div\vec{p}_0\right) - \left(\frac{\vec{s}}{R}\cdot\vec{\nabla}div\vec{p}_0\right) - \frac{div\vec{p}_0\,div\vec{s}}{R_0}$$

$$= div\left(\frac{\vec{s}}{R_0}div\vec{p}_0\right) - \frac{1}{R_0}\{div(\vec{s}\,div\vec{p}_0) - div\vec{s}\,div\vec{p}_0\} - \frac{1}{R_0}div\vec{p}_0\,div\vec{s}$$

$$= div\left(\frac{\vec{s}}{R_0}div\vec{p}_0\right) - \frac{1}{R_0}div(\vec{s}\,div\vec{p}_0).$$

We shall disregard the first term in anticipation of the volume integration under the assumption that polarization moments vanish on the surface of the nuclear matter.

Using well-known vector identities,

$$\vec{s}\, div \vec{\rho}_0 = curl [\vec{s} \times \vec{\rho}_0] + \vec{\rho}_0 div \vec{s} - (\vec{\rho}_0 \cdot \vec{\nabla}) \vec{s} + (\vec{s} \cdot \vec{\nabla}) \vec{\rho}_0$$

$$(\vec{s} \cdot \vec{\nabla}) \vec{\rho}_0 = grad (\vec{s} \cdot \vec{\rho}_0) - (\vec{\rho}_0 \cdot \vec{\nabla}) \vec{s} - [\vec{s} \times curl \vec{\rho}_0] - [\vec{\rho}_0 \times curl \vec{s}].$$

However, $curl \vec{s} = 0$, $div \vec{s} = 3$ and $(\vec{\rho}_0 \cdot \vec{\nabla}) \vec{s} = \vec{\rho}_0$.

Hence, $div (\vec{s}\, div \vec{\rho}_0) = \Delta (\vec{s} \cdot \vec{\rho}_0) + div \vec{\rho}_0 - div [\vec{s} \times curl \vec{\rho}_0]$.

From (2.29), $curl \vec{\rho} + \frac{1}{c} \frac{\partial \vec{j}}{\partial t} = \vec{\ell}_g$, from which we obtain by spatial integration

$$curl \vec{\rho}_0 + \frac{1}{c} \frac{\partial \vec{j}_0}{\partial t} = \vec{\ell}_{g0}. \tag{2.39}$$

Hence,

$$div (\vec{s}\, div \vec{\rho}_0) = \Delta (\vec{s} \cdot \vec{\rho}_0) + div \vec{\rho}_0 + div \left[\vec{s} \times \frac{1}{c} \frac{\partial \vec{j}_0}{\partial t}\right] - div [\vec{s} \times \vec{\ell}_{g0}] \tag{2.40}$$

$$= \Delta (\vec{s} \cdot \vec{\rho}_0) + div \vec{\rho}_0 + div \frac{1}{c} \frac{\partial}{\partial t} [\vec{s} \times \vec{j}_0]$$

$$- div \left[\frac{1}{c} \frac{\partial \vec{s}}{\partial t} \times \vec{j}_0\right] - div [\vec{s} \times \vec{\ell}_{g0}].$$

However, $\frac{\partial \vec{s}}{\partial t}$ vanishes because in g-coupling $\vec{p}$ and $\vec{q}$ arise from the displacement of a nucleon under the influence of the external field and $\vec{s}$ is just a vector distance from the center of dv-volume to the equilibrium position of the nucleon in question. Therefore, $\vec{p}$ and $\vec{q}$ are time dependent, whereas $\vec{s}$ is time independent. This is not so in f-coupling on account of the spin being an intrinsic moment of the nucleon.

Referring to (2.38) we obtain from (2.40):

$$\frac{\langle \operatorname{div} \vec{P}_0 \rangle}{R} = \frac{1}{R_0} \left\langle \rho - \Delta(\vec{S}\cdot\vec{P}_0) - \operatorname{div} \vec{P}_0 + \operatorname{div} \frac{1}{c}\frac{\partial}{\partial t}[\vec{j}_0 \times \vec{S}] + \operatorname{div}[\vec{S} \times \vec{l}_{g_0}] \right\rangle .$$

Making use of the second equation in (2.37), we can write down the corresponding expression involving $W(R)$.

Thus, we obtain from (2.36), together with the fact that the process of averaging ~~~ commutes with the differentiation with respect to space and time (this is proven at the end of this chapter):

$$\begin{aligned} \underset{\sim}{l_{g(dv)}} = \frac{1}{R_0} \Big\langle & \varkappa_c g + \varkappa g - \operatorname{div} \varkappa g (\underset{\sim}{\vec{P}_0} - [\underset{\sim}{\vec{S} \times \vec{l}_{g_0}}]) \\ & + \operatorname{div} \frac{1}{c}\frac{\partial}{\partial t} \varkappa g [\underset{\sim}{\vec{j}_0 \times \vec{S}}] - \Delta \varkappa g (\underset{\sim}{\vec{S}\cdot\vec{P}_0}) \Big\rangle dv \qquad (2.41) \\ & + K_0 \int_{-\infty}^{+\infty} dt' W(R_0) \Big\{ \text{same except t' instead of t} \Big\} dv \end{aligned}$$

Here, $\underset{\sim}{l_{g(dv)}}$ is the average scalar potential arising from dv-volume whose center is at $\vec{R}_0$ with respect to the point of observation, $\varkappa$ the density of nucleons in the nuclear matter, and $\varkappa_c$ the density of the conduction current nucleons.

Since the total potential at the point of observation outside of the nuclear matter must be a sum of potentials contributed by individual dv's, we can write:

$$\underset{\sim}{l_g} = \underset{\sim}{l_{g(dv_1)}} + \underset{\sim}{l_{g(dv_2)}} + \cdots = \sum_n \underset{\sim}{l_{g(dv_n)}} .$$

Considering that our nuclear matter is large enough to contain a large number of dv's, we may replace the summation by an integration (subscript of R will be dropped now). Thus, we finally obtain:

$$\vec{b}_g = \iiint dV \frac{1}{R}\Big\langle \varkappa_c g + \varkappa g - \operatorname{div}\varkappa g\big(\vec{j}_0 - [\vec{s}\times\vec{j}_{g_0}]\big) + \operatorname{div}\frac{1}{c}\frac{\partial}{\partial t}\varkappa g\,[\vec{j}\times\vec{s}] \quad \Delta)\varkappa_1(\vec{s}\cdot\vec{F})\Big\rangle \qquad (2.42)$$
$$+ K_0\iiint dV\int_{-\infty}^{+\infty} dt'\,W(R)\Big\{\text{same except t' instead of t}\Big\}.$$

Let us proceed to the case of the vector potential. In order to avoid a great complexity we consider two parts of the first equation in (2.36) separately as follows:

$$\vec{A}^{(j_0)}_{g(dV)} = \varkappa g\frac{1}{R}\langle \operatorname{curl}\vec{j}_0\rangle dV + \varkappa g K_0\int_{-\infty}^{+\infty} dt'\,W(R)\Big\{\operatorname{curl}\vec{j}_0\Big\}dV \qquad (2.43)$$
$$\vec{A}^{(\rho_0)}_{g(dV)} = \varkappa g\frac{1}{R}\Big\langle -\frac{1}{c}\frac{\partial\vec{P}_0}{\partial t}\Big\rangle dV + \varkappa g K_0\int_{-\infty}^{+\infty} dt'\,W(R)\Big\{-\frac{1}{c}\frac{\partial}{\partial t'}\vec{P}_0\Big\}dV.$$

First we shall develop the top equation. Proceeding similarly to the previous case we begin with the following:

$$\frac{\operatorname{curl}\vec{j}_0}{R} = \frac{\operatorname{curl}\vec{j}_0}{R_0} + \operatorname{curl}\vec{j}_0\Big(\vec{s}\cdot\vec{\nabla}\frac{1}{R_0}\Big),$$

which, after a certain vector analytical manipulation together with the equation $\operatorname{div}\vec{j}_0 = s_g$. from (2.29), will reduce to the equation:

$$\frac{\operatorname{curl}\vec{j}_0}{R} = \frac{2\operatorname{curl}\vec{j}_0}{R_0} + \vec{s}\operatorname{div}\Big[\vec{j}_0\times\vec{\nabla}\frac{1}{R_0}\Big] + \frac{\operatorname{curl}\operatorname{curl}[\vec{s}\times\vec{j}_0]}{R_0} - \frac{\operatorname{curl}(\vec{s}\,s_{g_0})}{R_0}.$$

The second term on the right hand side of this equation further reduces in the volume integration to

$$\iiint dV\,\vec{s}\operatorname{div}\Big[\vec{j}_0\times\vec{\nabla}\frac{1}{R_0}\Big] = -\iiint dV\Big[\vec{j}_0\times\vec{\nabla}\frac{1}{R_0}\Big] = -\iiint dV\frac{\operatorname{curl}\vec{j}_0}{R_0}.$$

Substituting these equations into (2.43) and performing volume integration we obtain:

$$\vec{A}_g(j_0) = \iiint dV\frac{1}{R}\Big\langle \operatorname{curl}\varkappa g\big(\vec{j}_0 - \vec{s}\,s_{g_0} - \operatorname{curl}\operatorname{curl}[\vec{j}_0\times\vec{s}]\big)\Big\rangle \qquad (2.44)$$
$$+ K_0\iiint dV\int_{-\infty}^{+\infty} dt'\,W(R)\Big\{\text{same except t' instead of t}\Big\}.$$

Let us now proceed to the second equations in (2.43).

$$\frac{\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}}{R} = \frac{\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}}{R_0} + \frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}\left(\vec{\delta}\cdot\vec{\nabla}\frac{1}{R_0}\right)$$

$$= \frac{\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}}{R_0} + \vec{\nabla}\frac{1}{R_0}\left(\vec{\delta}\cdot\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}\right) + \frac{\left[\vec{\delta}\times\frac{1}{c}\frac{\partial}{\partial t}\,\mathrm{curl}\,\vec{P}_0\right]}{R_0} - \left[\vec{\delta}\times\mathrm{curl}\,\frac{\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}}{R_0}\right] \quad (2.45)$$

$$= \vec{\nabla}\frac{1}{R_0}\left(\vec{\delta}\cdot\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}\right) + \frac{\left[\vec{\delta}\times\frac{1}{c}\frac{\partial}{\partial t}\,\mathrm{curl}\,\vec{P}_0\right]}{R_0} + \vec{\delta}\,\mathrm{div}\,\frac{\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}}{R_0}$$

Under volume integration the first term becomes with proper regard to the boundary condition as follows:

$$\iiint dV\,\vec{\nabla}\frac{1}{R_0}\left(\vec{\delta}\cdot\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}\right) = -\iiint dv\,\frac{\vec{\nabla}\left(\vec{\delta}\cdot\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}\right)}{R_0}$$

$$= -\iiint dv\,\frac{\vec{\nabla}\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\delta}\cdot\vec{P}_0\right) - \vec{\nabla}\left(\frac{1}{c}\frac{\partial \vec{\delta}}{\partial t}\cdot\vec{P}_0\right)}{R_0} \quad (2.46)$$

With the aid of (2.39) the second term in (2.45) becomes:

$$\left[\vec{\delta}\times\frac{1}{c}\frac{\partial}{\partial t}\,\mathrm{curl}\,\vec{P}_0\right] = -\left[\vec{\delta}\times\frac{1}{c}\frac{\partial}{\partial t}\,\vec{q}_0\right] + \left[\vec{\delta}\times\frac{1}{c}\frac{\partial}{\partial t}\,\vec{\ell}g_0\right]$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left[\vec{q}_0\times\frac{1}{c}\frac{\partial \vec{\delta}}{\partial t}\right] + \frac{1}{c}\frac{\partial}{\partial t}\left[\vec{\delta}\times\vec{q}_0\right] - \left[\frac{1}{c}\frac{\partial \vec{q}_0}{\partial t}\times\frac{1}{c}\frac{\partial \vec{\delta}}{\partial t}\right] \quad (2.47)$$

$$+ \frac{1}{c}\frac{\partial}{\partial t}\left[\vec{\delta}\times\vec{\ell}g_0\right] - \left[\frac{1}{c}\frac{\partial \vec{\delta}}{\partial t}\times\vec{\ell}g_0\right]$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left[\vec{\delta}\times\vec{q}_0\right] + \frac{1}{c}\frac{\partial}{\partial t}\left[\vec{\delta}\times\vec{\ell}g_0\right].$$

The last term in (2.45) becomes under volume integration:

$$\iiint dv\,\vec{\delta}\,\mathrm{div}\,\frac{\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}}{R} = -\iiint dv\,\frac{\frac{1}{c}\frac{\partial \vec{P}_0}{\partial t}}{R} \quad (2.48)$$

Substituting (2.46), (2.47) and (2.48) into the second equation in (2.43), under volume integration we obtain finally:

$$\vec{A}_{g(P_0)} = \iiint dV \frac{1}{R} \Big\langle \frac{1}{c}\frac{\partial}{\partial t}\rho_g \vec{P}_0 - \frac{1}{c}\frac{\partial}{\partial t}[\vec{s}\times\vec{\ell}_{g_0}] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\rho_g[\vec{\xi}_0\times\vec{s}] + \vec{\nabla}\frac{1}{c}\frac{\partial}{\partial t}\rho_g(\vec{s}\cdot\vec{P}_0) \Big\rangle$$

$$+ K_0 \iiint dV \int_{-\infty}^{t-\infty} dt'\, W(R) \{\text{same except t' instead of t}\}.$$

When a conduction current of nucleons exist, because of (2.28), its density must be introduced in the form $\rho_c g \frac{\vec{u}}{c}(cond.)$ in conjunction with $\frac{1}{c}\frac{\partial}{\partial t}\rho_g \vec{P}_0$, the current induced by the external field.

Combining $\vec{A}_{g(g_0)}$ and $\vec{A}_{g(P_0)}$, the final expression for the vector potential turns out to be:

$$\vec{A}_g = \iiint dV \frac{1}{R} \Big\langle \rho_c \frac{\vec{u}}{c}(cond.) + \frac{1}{c}\frac{\partial}{\partial t}\rho_g(\vec{P}_0 - [\vec{s}\times\vec{\ell}_{g_0}]) + \vec{\nabla}\frac{1}{c}\frac{\partial}{\partial t}\rho_g(\vec{s}\cdot\vec{P}_0)$$

$$+ curl\,\rho_g(\vec{\gamma}_{g_0} - \vec{s}\,s_{g_0}) - curl\,curl\,\rho_g[\vec{\ell}_{g_0}\times\vec{s}] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\rho_g[\vec{\gamma}_{g_0}\times\vec{s}] \Big\rangle \qquad (2.49)$$

$$+ K_0 \iiint dV \int_{-\infty}^{t-\infty} dt'\, W(R) \{\text{same except t' instead of t}\}.$$

The parts of the scalar and vector wave equations in (2.26) arising from g-coupling alone are:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - K_0^2\right)\psi_g = -4\pi\left\{\Lambda - div\,P_g + \frac{1}{c}\frac{\partial}{\partial t}div\,\vec{\gamma}_g - \Delta Z_g\right\}$$

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - K_0^2\right)\vec{A}_g = -4\pi\left\{\frac{\vec{J}}{c} + \frac{1}{c}\frac{\partial}{\partial t}\vec{P}_g + curl\,\vec{\omega}_g - curl\,curl\,\vec{\gamma}_g - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\gamma}_g + \vec{\nabla}\frac{1}{c}\frac{\partial}{\partial t}Z\right\}.$$

According to (2.32) we write down solutions of these equations as follows:

$$\psi_g = \iiint dv \frac{1}{R} \left\langle \Lambda - div \vec{P}_g + \frac{1}{c}\frac{\partial}{\partial t} div \vec{\gamma}_g - \Lambda \overline{\overline{Z}}_g \right\rangle \qquad (2.50)$$
$$+ K_0 \iiint dv \int_{-\infty}^{t} dt' \, W(R) \left\{ \text{same except t' instead of t} \right\},$$

$$\vec{A}_g = \iiint dv \frac{1}{R} \left\langle \frac{\vec{J}}{c} + \frac{1}{c}\frac{\partial}{\partial t}\vec{P}_g + curl \vec{\omega}_g \; curl\, curl \vec{\gamma}_g \right.$$
$$\left. - \frac{1}{c}\frac{\partial}{\partial t} \vec{\gamma}_g + \vec{\nabla} \frac{1}{c}\frac{\partial}{\partial t} Z \right\rangle \qquad (2.51)$$
$$+ K_0 \iiint dv \int_{-\infty}^{t} dt' \, W(R) \left\{ \text{same except t' instead of t} \right\}.$$

Compare (2.42) with (2.50) and (2.49) with (2.51). Under the assumption that $\vec{A}_g = \overline{\overline{\vec{a}}}_g$, $\overline{\overline{\vec{\omega}}}_g = \overline{\overline{\vec{\omega}}}_g$ we arrive at the following identification of macroscopic variables in terms of the microscopic quantities:

$$\Lambda = \varkappa_{cg}(cond.), \quad \vec{J} = \varkappa_{cg} \overline{\overline{\vec{u}}}(cond.),$$
$$\vec{P}_g = \varkappa_g (\overline{\overline{\vec{p}_0}} - [\overline{\overline{\vec{s} \times \vec{l}_{g0}}}]), \quad \vec{\Omega} = \varkappa_g (\overline{\overline{\vec{q}_0}} - \vec{s}\, s_{g0}), \qquad (2.52)$$
$$\vec{\gamma}_g = \varkappa_g [\overline{\overline{\vec{q}_0 \times \vec{s}}}], \quad Z_g = \varkappa_g (\overline{\overline{\vec{s} \cdot \vec{p}_0}}).$$

However, since nucleons are isotropically distributed,

$$(\overline{\overline{\vec{s} \cdot \vec{p}_0}}) = \overline{\overline{\vec{s}}}\, s_{g0} = [\overline{\overline{\vec{s} \times \vec{l}_{g0}}}] = 0.$$

Moreover, when non-relativistic approximations are made to the motion of scattering centers $\vec{q}_0$ can be considered to vanish. Thus, in our approximation the only non-vanishing moment turns out to be $\vec{P}_g = \varkappa_g \overline{\overline{\vec{p}_0}}$. Going back to (2.21) and (2.24) we find $\mu_g = \nu_g = \sigma_g = 1$.

4. Continuation of the derivation of macroscopic Proca equations (tensor coupling)

The microscopic wave equation in tensor coupling are obtained from (2.11):

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa_0^2\right)\vec{a}_f = -4\pi \frac{f}{\kappa_0}\left(\operatorname{curl}\vec{m} - \frac{1}{c}\frac{\partial \vec{n}}{\partial t}\right)$$
$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa_0^2\right)\varphi_f = -4\pi \frac{f}{\kappa_0}\operatorname{div}\vec{n}. \qquad (2.53)$$

A comparison of these equations with (2.31) tells almost the entire story in tensor coupling—we may simply replace $g\vec{p}_0$ by $\frac{f}{\kappa_0}\vec{n}$ and $g\vec{q}$ by $\frac{f}{\kappa_0}\vec{n}$ in the development pursued in the previous section for vector coupling. However, there are a few places where modifications are necessary.

In the case of vector coupling moments $\vec{p}_0$ and $\vec{q}_0$ of a nucleon arose from the displacement from the equilibrium position caused by the external field. In contrast to this situation in the case of tensor coupling nucleons possess intrinsic spin moments already, and, hence, we may consider that the vector $\vec{\delta}$ refers to the distance from the center of the dV-volume to the instantaneous position of the nucleon, and not to its equilibrium position. Thus, $\vec{\delta}$ becomes time dependent, $\frac{\partial\vec{\delta}}{\partial t}$ being the induced velocity vector of the nucleon in question.

We recall that we disregarded the term $d\omega\left[\frac{1}{c}\frac{\partial\vec{p}_0}{\partial t}\times\vec{\delta}\right]$ in (2.40). In the tensor-coupling, however, $\left[\frac{1}{c}\frac{\partial\vec{\delta}}{\partial t}\times\vec{p}_0\right]$ will be replaced by $\frac{1}{\kappa_0}\left[\frac{\vec{u}}{c}\times\vec{m}_0\right]$, the relativistic counterpart to the non-relativistic spin vector having the same order of

magnitude as $\vec{n}$. Hence, this term must stay in the tensor coupling case. Moreover, in the existence of the conduction current of nucleons through nuclear matter the associated spin current density shall be introduced along with this spin current term induced by the external field.

Let us rewrite (2.47) for the tensor coupling case.

$$\left[\vec{\delta} \times \frac{1}{c}\frac{\partial}{\partial t}\,\mathrm{curl}\,\vec{n}_0\right] = -\frac{1}{c}\frac{\partial}{\partial t}\left[\vec{m}_0 \times \frac{1}{c}\frac{\partial \vec{\delta}}{\partial t}\right] + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\vec{\delta} \times \vec{m}_0\right]$$
$$- \left[\frac{1}{c}\frac{\partial \vec{m}_0}{\partial t} \times \frac{1}{c}\frac{\partial \vec{\delta}}{\partial t}\right] + \left[\vec{\ell}_{f_0} \times \frac{1}{c}\frac{\partial \vec{\delta}}{\partial t}\right]$$
$$+ \frac{1}{c}\frac{\partial}{\partial t}\left[\vec{\delta} \times \vec{\ell}_{f_0}\right]$$

Because of the analog of (2.29) in the tensor coupling, the third and fourth terms combined to give $\left[\mathrm{curl}\,\vec{n}_0 \times \frac{\vec{u}}{c}\right]$; which, however, is proportional to $\frac{u^2}{c^2}$ from (2.12). Hence, this term is disregarded.

With these remarks we can write down average microscopic scalar and vector potentials in tensor coupling corresponding to (2.42) and (2.49) as follows:

$$\widetilde{\ell_f} = \iiint dv\, \frac{1}{R}\Big\langle \mathrm{div}\, \mathcal{N}_c \frac{f}{K_0}\, \widetilde{\vec{n}_0}(\mathrm{cond.}) - \mathrm{div}\, \mathcal{N}\frac{f}{K_0}\left(\widetilde{\left[\vec{m}_0 \times \frac{u}{c}\right]} - \widetilde{\left[\delta \times \ell_{f_0}\right]}\right)$$
$$+ \frac{1}{c}\frac{\partial}{\partial t}\,\mathrm{div}\, \mathcal{N}\frac{f}{K_0}\widetilde{\left[\vec{m}_0 \times \vec{\delta}\right]} - \Delta\, \mathcal{N}\frac{f}{K_0}\left(\widetilde{\delta \cdot n_0}\right)\Big\rangle \tag{2.54}$$

$$+ K_0 \iiint dv \int_{-\infty}^{+\infty} dt'\, W(R)\left\{\text{same except } t' \text{ instead of } t\right\},$$

$$\vec{A}_f = \iiint dv \frac{1}{R} \Big\langle -\frac{1}{c}\frac{\partial}{\partial t}\varkappa_c \frac{1}{\kappa_0}\vec{n}_0(cond.) - \frac{1}{c}\frac{\partial}{\partial t}\varkappa\frac{1}{\kappa_0}\left([\vec{m}_0\times\frac{\vec{u}}{c}] - [\vec{S}\times\vec{l}_{f_0}]\right)$$
$$+ curl\,\varkappa\frac{1}{\kappa_0}\vec{m}_0(cond.) + curl\,\varkappa\frac{1}{\kappa_0}(\vec{m}_0 - \vec{S}s_{f_0}) \qquad (2.55)$$
$$- curl\,curl\,\varkappa\frac{1}{\kappa_0}[\vec{m}_0\times\vec{S}] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varkappa\frac{1}{\kappa_0}[\vec{m}_0\times\vec{S}] + \vec{\nabla}\frac{1}{c}\frac{\partial}{\partial t}\varkappa\frac{1}{\kappa_0}(\vec{S}\cdot\vec{n}_0)\Big\rangle$$
$$+ \kappa_0\iiint dv\int_{-\infty}^{+\infty}dt'\,W(R)\,\{\text{same except t' instead of t}\}.$$

Note $\vec{n}(cond.)$ and $\vec{m}(cond.)$ are introduced for the conduction current of nucleons through nuclear matter.

The part of wave equations in (2.26) arising from f-coupling are:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa_0^2\right)\varphi_f = -4\pi\left\{-div\,\vec{P}_f + div\,\vec{N} + \frac{1}{c}\frac{\partial}{\partial t}div\,\vec{\gamma}_f - \Delta Z_f\right\}$$

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa_0^2\right)\vec{A}_f = -4\pi\left\{-\frac{1}{\kappa_0}\frac{1}{c}\frac{\partial}{\partial t}\vec{N} + \frac{1}{c}\frac{\partial}{\partial t}\vec{P}_f + \frac{1}{\kappa_0}curl\,\vec{M}\right.$$
$$\left. + curl\,\vec{Q} - curl\,curl\,\vec{\gamma}_f - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\gamma}_f + \vec{\nabla}\frac{1}{c}\frac{\partial}{\partial t}Z_f\right\}.$$

Let us write down the solution of these equations in accordance to (2.32):

$$\varphi_f = \iiint dv\,\frac{1}{R}\Big\langle -div\,\vec{P}_f + div\,\vec{N} + \frac{1}{c}\frac{\partial}{\partial t}div\,\vec{\gamma}_f - \Delta Z_f\Big\rangle \qquad (2.56)$$
$$+ \kappa_0\iiint dv\int_{-\infty}^{+\infty}dt'\,W(R)\,\{\text{same except t' instead of t}$$

$$\vec{A}_f = \iiint dv\,\frac{1}{R}\Big\langle -\frac{1}{\kappa_0}\frac{1}{c}\frac{\partial}{\partial t}\vec{N} + \frac{1}{c}\frac{\partial}{\partial t}\vec{P}_f + \frac{1}{\kappa_0}curl\,\vec{M} + curl\,\vec{Q}_f$$
$$- curl\,curl\,\vec{\gamma}_f - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\gamma}_f + \vec{\nabla}\frac{1}{c}\frac{\partial}{\partial t}Z_f\Big\rangle \qquad (2.57)$$
$$+ \kappa_0\iiint dv\int_{-\infty}^{+\infty}dt'\,W(R)\,\{\text{same except t' instead of t}\}.$$

Then, under the assumption that $\psi_f = \overbrace{\ell_f}$, $\vec{A}_f = \overbrace{\vec{a}_f}$, we identify the macroscopic quantities appearing in the integrands of ψ_f and $\vec{A}_f$ as follows:

$$\begin{aligned}
&\vec{M} = \varkappa_c f \overbrace{\vec{m}_0}\,(\text{cond.}) \quad , \quad \vec{N} = \varkappa_c f \overbrace{\vec{n}_0}\,(\text{cond.}), \\
&\vec{P}_f = \varkappa \tfrac{f}{\kappa_0} \left(\overbrace{\left[\tfrac{\vec{u}}{c} \times \vec{m}_0\right]} - \overbrace{[\vec{s} \times \vec{\ell}_{f0}]} \right), \quad \vec{Q}_f = \varkappa \tfrac{f}{\kappa_0} \left(\overbrace{\vec{m}_0} - \overbrace{(\vec{s} S_{f0})} \right), \\
&\vec{Y}_f = \varkappa \tfrac{f}{\kappa_0} \overbrace{[\vec{m}_0 \times \vec{s}]}, \quad Z_f = \varkappa \tfrac{f}{\kappa_0} \overbrace{(\vec{s} \cdot \vec{n}_0)}.
\end{aligned} \tag{2.58}$$

However, because of isotropic distribution of nucleons is in all directions,

$$\overbrace{[\vec{s} \times \vec{m}_0]} = \overbrace{[\vec{s} \times \vec{\ell}_{f0}]} = \overbrace{(\vec{s} S_{f0})} = 0.$$

Moreover, we must put $\overbrace{\left[\tfrac{\vec{u}}{c} \times \vec{m}_0\right]} = 0$

when non-relativistic approximations are made to the motion of spin of a nucleon. Thus, in this approximation the only non-vanishing moment would be $\vec{Q}_f = \varkappa \tfrac{f}{\kappa_0} \overbrace{\vec{m}_0}$. Going back to (2.21) and (2.24) we find $\epsilon_f = \nu_f = \sigma_f = 1$.

The fact that we found $\nu = \sigma = 1$ for both couplings might suggest that the introduction of the phenomenological vector and scalar potentials in the formal argument presented in (2.14) was superfluous. This is probably not true in the exact theory, for the result we obtained is obviously a consequence of the dipole-type approximation that we introduced in (2.37), and one expects to obtain different results when higher order approximations beyond the dipole-type are also taken into the consideration.

The averaging and time differentiation are obviously interchangeable since the space and time coordinates are independent. The interchangeability of the averaging and space differentiation maybe proven in the following manner. Let $f(x,y,z,t)$ be any physical quantity characterizing the process of scattering of the field by a nucleon within a "physically small" volume whose center is at (x,y,z) at time t. The average of $f(x,y,z,t)$ over this volume θ can be defined as:

$$\overline{f}(x,y,z,t) = \frac{1}{\theta}\overset{\theta}{\iiint} dx'dy'dz' f(x+x', y+y', z+z', t).$$

Then,

$$\frac{\partial}{\partial x}\overline{f}(x,y,z,t) = \frac{1}{\theta}\overset{\theta}{\iiint} dx'dy'dz' \frac{\partial}{\partial x} f(x+x', y+y', z+z', t)$$

$$= \frac{1}{\theta}\overset{\theta}{\iiint} dx'dy'dz' \frac{\partial}{\partial x'} f(x+x', y+y', z+z', t)$$

$$= \overline{\frac{\partial}{\partial x} f(x,y,z,t)}.$$

As a consequence of this theorem we can deduce the following results immediately. Let us first average the equations (2.9):

$$\operatorname{div} \overline{\vec{h}} = -4\pi \frac{f}{\kappa_0} \operatorname{div} \overline{\vec{m}}.$$

Comparing this equation with (2.20), we find with the aid of the first expression in (2.58):

$$\overline{\vec{h}} = \vec{B}.$$

Averaging the first equation in (2.9) and substituting this relationship,

$$\operatorname{curl} \overline{\vec{e}} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -4\pi \frac{f}{\kappa_0}\left(\operatorname{curl} \overline{\vec{n}} + \frac{f}{c}\frac{\partial}{\partial t}\overline{\vec{m}}\right).$$

A comparison of this equation with (2.20) gives us, with the aid of the second expression in (2.58):

$$\vec{\vec{e}} = \vec{E}.$$

We summarize our results:

$$\vec{E} = \vec{\vec{e}}, \quad \vec{B} = \vec{\vec{h}}, \quad \vec{A} - \vec{F} = \vec{\vec{a}}, \quad \psi = \phi = \vec{\vec{b}},$$

$$\vec{D} = \vec{\vec{e}} + 4\pi\varkappa\left(g\vec{\vec{p}}_0 + \frac{f}{\kappa_0}\left[\frac{\vec{u}}{c}\times\vec{\vec{m}}_0\right]\right), \quad \vec{H} = \vec{\vec{h}} - 4\pi\varkappa\left(g\vec{\vec{g}}_0 + \frac{f}{\kappa_0}\vec{\vec{m}}_0\right), \tag{2.59}$$

$$\Lambda = \varkappa_e g, \quad \vec{J} = \varkappa_e g\vec{u}\,(\text{cond.}), \quad \vec{M} = \varkappa_e f\vec{m}_0\,(\text{cond.})$$

$$\vec{N} = \varkappa_e f\vec{\vec{n}}_0\,(\text{cond.}).$$

Chapter III

Dispersion Theory

1. Evaluation of the magnitude of the mesonic electric moment (vector coupling)

In this section we wish to evaluate the magnitude of the mesonic electric moment of a nucleon in an external meson field in terms of the fundamental constants of the nuclear matter. The classical relativistic equation of a nucleon in the neutral external meson field was given by Bhabha[1] in a manner similar to that of Dirac's derivation of the classical relativistic equation of motion of an electron[2]:

$$M_0 \dot{v}_\mu - \frac{2}{3}\frac{d^2}{c^2} v_\mu - \frac{2}{3}\frac{\vartheta^2}{c^5} v_\mu \dot{v}^2 - g^2 \frac{K_0^2}{c} v^\sigma \int_{-\infty}^{\tau} d\tau' \frac{S_\mu v_\sigma(\tau') - S_\sigma v_\mu(\tau')}{S^2} = -\frac{q}{c} G^{ext}_{\mu\sigma} v^\sigma, \tag{3.1}$$

where M_0 is the rest mass of a nucleon, v_μ the component of a 4-velocity vector referring to the nucleon, τ the proper time of the nucleon, and $S_\mu = x_\mu - z_\mu(\tau)$ where x_μ refers to the position of the nucleon and $z_\mu(\tau)$ its position along the world line from an arbitrary point on it. $G_{\mu\sigma}$ is a component of the external electric field and J_2 the Bessel function of the second order.

The first term in the equation is obviously the inertial force term. The second term is the familiar radiation damping force term. The third term is characteristic of Dirac's

---

[1] H.J. Bhabha, Proc. Roy. Soc. A172, 384 (1939) referred to as Bhabha I in this thesis.

[2] P.A.M. Dirac, Proc. Roy. Soc. A167, 148 (1938)

equation of motion, and does not occur in Lorentz's equation of motion of an electron. The term involving the integral arises from the fact that the meson field possesses a finite rest mass. When one lets $K_0 \to 0$, one obtains Dirac's equation of motion of an electron in the external electromagnetic field.

For the sake of simplicity we now consider the solution of (3.1) when the external force on the nucleon has the nature of a harmonic oscillation along the x-axis.

$$G_{10}^{ext.} = E_0 e^{i\omega t}, \quad G_{20}^{ext} = G_{30}^{ext.} = 0 \tag{3.2}$$

We apply the non-relativistic approximation to the motion of the source (nucleon), but retain relativistic treatment for the field. Further we assume that the binding energy of a nucleon in the nuclear matter can be expressed by a harmonic oscillator potential whose "spring constant" is λ . Thus, there should be a term in addition to the term due to the external field appearing on the right hand side of (3.1). Under these conditions we solve the equation (3.1) for the case where E_0 is small so that the velocity of the nucleon is assumed always small compared with the velocity of light.

Choosing the coordinate system so that the nucleon executes a harmonic oscillation about the origin along the x-axis, we may assume the solution of the equation (3.1) to be the form

$$z_1 = \beta_0 e^{i\omega t}, \quad z_2 \quad z_3 \quad , \tag{3.3}$$

where $\beta = \beta_0 e^{i\delta}$, δ the phase angle between β and E_0. It is further assumed that $\beta_0 \ll$, so that higher powers

of $\frac{\beta_0}{c}$ can be neglected.

$$v_1 = \frac{dz_1(\tau)}{d\tau} = \frac{dz_1}{dt}\frac{dt}{d\tau} = \frac{dz_1}{dt}\left\{1 - \frac{1}{c^2}\left(\frac{dz_1}{dt}\right)^2\right\}^{-\frac{1}{2}} \cong \frac{dz_1}{dt} = i\beta c e^{i\omega t}$$

$$v_0 = \frac{dz_0(\tau)}{d\tau} = \frac{dz_0}{dt}\frac{dt}{d\tau} = c\left\{1 - \frac{1}{c^2}\left(\frac{dz_1}{dt}\right)^2\right\}^{-\frac{1}{2}} \cong c$$

Since $\frac{d\tau}{dt} = \left\{1 - \frac{1}{c^2}\left(\frac{dz_1}{dt}\right)^2\right\}^{-\frac{1}{2}} \cong 1$, we may replace $d\tau$ by dt in our approximation.

$$\dot{v} \cong -\beta\omega e^{i\omega t}, \quad \ddot{v} \cong -i\beta\omega^2 e^{i\omega t}$$

Moreover,

$$S_0 = z_0(\tau) - z_0(\tau') \cong c(t - t')$$

$$S_1 = z_1(\tau) - z_1(\tau') = \frac{\beta}{\omega}\left(e^{i\omega t} - e^{i\omega t'}\right)$$

Therefore, $$S = \sqrt{S_0^2 - S_1^2} \cong c(t - t') = S_0.$$

Introducing the above results into the equation (3.1), the term involving the integral becomes:

$$-g^2\frac{K_0^2}{c}v^\sigma\int_{-\infty}^{\tau} d\tau' \frac{J_2(k_0 S)}{S^2}\left\{S_1 v_0(\tau') - S_0 v_1(\tau')\right\}$$

$$= -g^2\frac{K_0^2}{c}\frac{\beta}{\omega}e^{i\omega t}\left[\int_0^\infty ds\,\frac{J_2(k_0 s)}{s^2} - \int_0^\infty ds\,\frac{J_2(k_0 s)}{s^2}e^{-i\frac{\omega}{c}s}\right.$$

$$\left. - i\omega\int_0^\infty ds\,\frac{J_2(k_0 s)}{s}e^{-i\frac{\omega}{c}s}\right].$$

Introduce notations A, B and C for the following integrals:

$$A = \int_0^\infty ds\,\frac{J_2(s)}{s^2} = \frac{1}{3}$$

$$B = \int_0^\infty ds\,\frac{J_2(s)}{s^2}e^{-i\nu s} = \frac{i}{3}\left[-(\nu^2-1)^{3/2} + \nu^3 - \frac{3}{2}\nu\right] \quad \text{for } \nu = \frac{\omega}{k_0 c} > 1$$

$$C = \int_0^\infty ds\,\frac{J_2(s)}{s}e^{-i\nu s} = \frac{1}{2} - \nu^2 + \nu\sqrt{\nu^2-1} \quad \text{for } \nu = \frac{\omega}{k_0 c} > 1.$$

Then, the integral in question becomes:

$$-g^2\frac{K_0^2}{c}\frac{\beta}{\omega}e^{i\omega t}\left[(A-B)K_0c-i\omega C\right]. \tag{3.4}$$

Note $\nu=\frac{\omega}{K_0c}=\frac{\hbar\omega}{\mu c^2}>1$ means that the quantum energy of the meson field $\hbar\omega$ is greater than the rest energy of the meson μc^2 (μ= rest mass of the meson), indicating the frequency of vibration of nucleon ω , which is the same as the frequency of the field in classical theory, is greater than the frequency of the field corresponding to the rest energy of the meson. This condition must be satisfied for there to be free radiation of the meson field, otherwise the meson field will stay in the neighborhood of the nucleon, giving rise to only the time dependent proper field of the nucleon.

Substituting (3.2), (3.3) and (3.4) into (3.1) with binding term, we obtain:

$$-M_0\beta\omega+i\frac{2}{3}\frac{g^2}{c^3}\beta\omega^2-g^2\frac{K_0^2}{c}\frac{\beta}{\omega}\left[(A-B)K_0c-i\omega C\right]$$
$$=gE_0-\lambda\frac{\beta}{\omega}.$$

Let $\omega_0=\sqrt{\frac{\lambda}{M_0}}$ be the <u>natural</u> frequency of vibration of a nucleon due to the binding in the nuclear system, and solve above equation for $\frac{\beta}{\omega}$.

$$\frac{\beta}{\omega}=\frac{E_0g}{M_0}\frac{1}{\omega_0^2-\omega^2+i\frac{4}{3}\frac{g^2}{c^3M_0}\omega^3-\frac{g^2K_0^3}{3M_0}\left[1+i\left\{(\nu^2-1)^{3/2}-3\nu^2\sqrt{\nu^2-1}+2\nu^3\right\}\right]} \tag{3.5}$$

Note that this equation reduces to the familiar dispersion formula in the electromagnetic case when K_0 vanishes.

It is necessary at this point to investigate the relative importance of various terms in the denominator. We take

$\hbar\omega = 2\mu c^2 \sim 300\,Mev$, the nuclear fine structure constant $\frac{g^2}{\hbar c} \sim 0.3$, and $\mu_0 \sim 280$ electron masses.

$$\left|\frac{\text{radiation reaction term}}{\text{nucleon inertial term}}\right| = \frac{2}{3}\frac{g^2}{M_0 c^3}\omega = \frac{2}{3}\frac{g^2}{\hbar c}\frac{\hbar\omega}{M_0 c^2} \cong 0.06$$

$$\left|\frac{\text{meson mass term}}{\text{nucleon inertial term}}\right| = \frac{1}{3}\frac{g^2}{\hbar c}\frac{M_0}{M_0}\left(\frac{M_0 c^2}{\hbar\omega}\right)^2 \left|1 + i\left\{(\nu^2-1)^{3/2} - 3\nu^2\sqrt{\nu^2-1} + 2\nu^3\right\}\right|_{(\nu=2)}$$

$$\cong 0.006$$

In order to compute λ we consider that nuclear particles are bound to the system by a harmonic potential whose depth is $V_0 = 8\,Mev$ at the width equal to twice the internucleon distance $r_0 = 1.4 \times 10^{-13}\,cm$. Then, $V_0 = \frac{1}{2}\lambda r_0^2$, which gives $\lambda = 4 \times 10^{26}\,Mev/cm^2$.

Hence, $$\omega_0^2 = \frac{\lambda}{M_0} \cong 3.8 \times 10^{44}\,sec^{-2}.$$

For $\nu = 2$, $\hbar\omega \cong 300\,Mev$, $\omega \cong 4.8 \times 10^{23}\,sec^{-1}$.

$$\left|\frac{\text{binding term}}{\text{nucleon inertial term}}\right| = \frac{\omega_0^2}{\omega^2} \cong 0.001.$$

Thus, we find that in the region of meson energies at above 300 Mev the inertial force term in the equation of motion (3.1) is already the predominant one, others being negligible. Hence,

$$\frac{\beta}{\omega} \cong -\frac{E_0 g}{M_0}\frac{1}{\omega^2} = \alpha E_0, \qquad (3.6)$$

where $\alpha = -\frac{g}{M_0\omega^2}$ is the polarizability. Going from the one dimensional picture to the three dimensional representation,

we write $$\frac{\vec{\beta}}{\omega} = \alpha \vec{E}_0 .$$

Identifying $g\frac{\vec{\beta}}{\omega}$ with the average mesonic electric polarization moment $\overline{\vec{p}_0}$ in the physically small volume dv according to (2.52), the average polarization moment per unit volume $\vec{P}_g$ becomes:

$$\vec{P}_g = \varkappa \overline{\vec{p}_0} = \varkappa \alpha g \vec{E}_0 = \varkappa \left(-\frac{g^2}{M_0 \omega^2}\right) \vec{E}_0 .$$

Since $\vec{D} = \vec{E} + 4\pi \vec{P}_g$, $\epsilon_g = 1 + 4\pi \varkappa \alpha$. Let us compute the value of $4\pi\varkappa\alpha$ for actual nuclear matter for $\hbar\omega \cong 300$ Mev. Using the formula for the nuclear radius $R = 1.4 \times 10^{-13} A^{1/3}$ cm, where A is the atomic weight, the nucleon density turns out to be:

$$\varkappa = \frac{A}{\frac{4\pi}{3} R^3} = 0.87 \times 10^{38} \text{ cm}^{-3} .$$

This gives the value $4\pi\varkappa\alpha = -0.0085$.

That $4\pi\varkappa\alpha$ is negative, or $\epsilon < 1$, is due to the fact that a binding energy of 8 Mev is smaller than the energy of the field since the rest energy is already of the order of 150 Mev. This situation corresponds to the x-ray region in the electromagnetic case.

2. Application of Hoek's method in the derivation of a wave equation for the neutral Proca field traversing nuclear matter (vector coupling)

Hoek derived the wave equation for the electromagnetic field propagating through matter in which the index of

refraction n obeys the Lorentz-Lorenz law[1]

$$\frac{n^2-1}{n^2+2} = \frac{4\pi}{3} N\alpha ,$$

where N is the density of the scattering centers and α the polarizability of the individual scattering center (molecule). This law was originally derived for static fields using the concept of the inner field.

The short range nature of nuclear forces prohibits us from giving considerations similar to the underlying concept of the Lorentz-Lorenz law; however, in the radiation field where the field gets detached from the sources and travels through nuclear matter, there must exist an analog of Hoek's derivation of the wave equation in the region where the wave length is large compared with the average inter-nucleon distance.

The vector and scalar potentials for the radiation field of a vibrating nucleon in the non-relativistic approximation are derived in Bhabha I, p. 390:

$$\vec{A}(\vec{r}) = g\omega \frac{\vec{\beta}(r')}{\omega} \frac{e^{-i\kappa\rho}}{\rho} , \quad \phi(\vec{r}) = g\kappa \left(\frac{\vec{\rho}}{\rho} \cdot \frac{\vec{\beta}(r')}{\omega}\right) \frac{e^{-i\kappa\rho}}{\rho} , \tag{3.7}$$

where $\kappa = \frac{\sqrt{\omega^2 - \kappa_0^2 c^2}}{c}$, and $\rho = |\vec{r} - \vec{r}'|$ in which $\vec{r}'$ is the position of a nucleon and $\vec{r}$ the position of point of observation. For our purpose it is more convenient to resort to the Hertz vector $\vec{Z}$ rather than working directly with the vector and the scalar potentials.

$$\vec{A} = \frac{1}{c} \frac{\partial \vec{Z}}{\partial t} , \quad \phi = -\operatorname{div} \vec{Z} \tag{3.8}$$

---

[1]H. Hoek, Dissertation, University of Leiden (1939); Lectures on advanced electricity and magnetism by Professor G.E. Uhlenbeck, University of Michigan (1948)

When (3.7) and (3.8) are combined, we obtain for the Hertz vector:

$$\vec{Z}(x) = \frac{1}{i} g \frac{\vec{\beta}(x')}{\omega} \frac{e^{i(\omega t - kr)}}{r} = R(x,x')\vec{p}(x'), \qquad (3.9)$$

where $g\frac{\vec{\beta}}{\omega}$ is replaced by $\vec{p}$.

In the region of space away from a source, $\vec{Z}$ satisfies the Gordon-Klein wave equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - k_0^2\right)\vec{Z} = 0.$$

Since from the Proca equations

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \text{grad div } \vec{Z} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{Z},$$

$$E = \vec{\nabla}\,\text{div}\,\vec{Z} - \Delta\vec{Z} + k_0^2\vec{Z}$$

$$= (\text{curl curl} + k_0^2)\vec{Z}.$$

Introducing (3.9),

$$\vec{E}(x) = \frac{1}{i} g (\text{curl curl}_x + k_0^2) \frac{e^{-ikr}}{r} \vec{p}(x') \qquad (3.10)$$

$$= \Omega(x,x')\vec{p}(x').$$

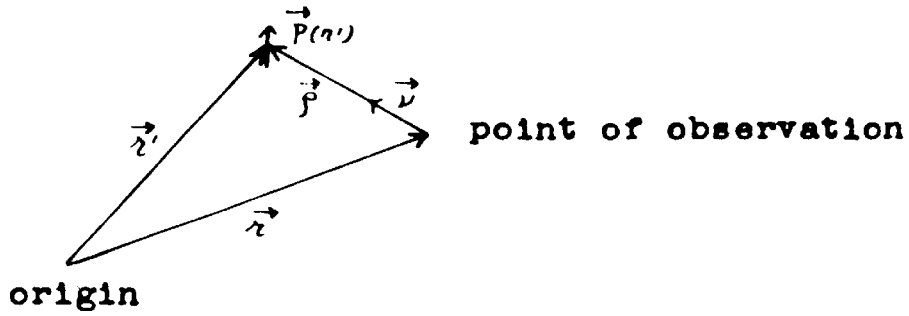


Fig. 2

The position of a nucleon with respect to the point of observation and the origin of the coordinates

If we let $\vec{E}^{(0)}$ be the external field at the position of the m-th nucleon, the total field at this nucleon would be:

$$\vec{E}(m) = \vec{E}^{(0)}(m) + \sum_{m'}{}' \Omega(m,m')\vec{p}(m').$$

Introducing the polarizability α, we may write:

$$\vec{p}(m) = \alpha \vec{E}(m) = \alpha\left[\vec{E}^{(0)} + \sum_{m'}{}' \Omega(m,m')\vec{p}(m')\right].$$

Assuming isotropic distribution of nucleons in the nuclear matter, we may replace the summation by the volume integral which excludes the neighborhood of the m-th nucleon by a small sphere whose radius is later reduced to zero. Thus we write:

$$\vec{p}(r) = \alpha\left[\vec{E}^{(0)}(r) + \mathcal{N}\int_{S(r)}^{V} dr'\,\Omega(r,r')\vec{p}(r')\right] \qquad (3.11)$$

where $\vec{p}(r)$ is the <u>average</u> polarization moment at $\vec{r}$, $\mathcal{N}$ the nucleon density, $S(r)$ a small sphere with its center at $\vec{r}$ and V the volume of the nuclear matter. This is an integral equation for $\vec{p}$.

For our development we need the following equations which are derived in Appendix 2:

$$\text{grad div}_r \int_{S(r)}^{V} dr'\,R(r,r')\vec{p}(r') = \int_{S(r)}^{V} dr'\,\text{grad div}_r R(r,r')\vec{p}(r') - \frac{4\pi}{3}\vec{p}(r) \qquad (3.12a)$$

$$\Delta_r \int_{S(r)}^{V} dr'\,R(r,r')\vec{p}(r') = \int_{S(r)}^{V} dr'\,\Delta_r R(r,r')\vec{p}(r') - 4\pi\vec{p}(r) \qquad (3.12b)$$

$$\text{curl curl}_r \int_{S(r)}^{V} dr'\,R(r,r')\vec{p}(r') = \int_{S(r)}^{V} dr'\,\text{curl curl}_r R(r,r')\vec{p}(r') + \frac{8\pi}{3}\vec{p}(r) \qquad (3.12c)$$

$$(\Delta_r + \kappa^2)\int_{S(r)}^{V} dr'\,R(r,r')\vec{p}(r') = -4\pi\vec{p}(r). \qquad (3.12d)$$

Making use of (3.12c),

$$(\text{curl curl}_r + \kappa_0^2)\int_{S(r)}^{V} dr'\,R(r,r')\vec{p}(r') = \int_{S(r)}^{V} dr'\,(\text{curl curl}_r + \kappa_0^2)R(r,r')\vec{p}(r') + \frac{8\pi}{3}\vec{p}(r), \qquad (3.13)$$

which, combined with (3.11) gives:

$$\left(1+\tfrac{8\pi}{3}\mathcal{N}\alpha\right)\vec{p}(r) = \alpha\left[\vec{E}^{(0)}(r) + \mathcal{N}\left(\mathrm{curl}\,\mathrm{curl}_r + K_0^2\right)\int_{S(r)}^{V} dr'\, R(r,r')\,\vec{p}(r')\right].$$

Let this equation now be operated on by $(\Delta_r + K^2)$.

$$\left(1+\tfrac{8\pi}{3}\mathcal{N}\alpha\right)\left(\Delta_r + K^2\right)\vec{p}(r) = \alpha\left[\left(\Delta_r + K^2\right)\vec{E}^{(0)}(r) + \mathcal{N}\left(\mathrm{curl}\,\mathrm{curl}_r + K_0^2\right)\left(\Delta_r + K^2\right)\int_{S(r)}^{V} dr'\, R(r,r')\,\vec{p}(r')\right].$$

Making use of (3.12b) and (3.12d) and noting $(\Delta_r + K^2)\,\vec{E}^{(0)}(r) = 0$, we obtain:

$$\left(1+\tfrac{8\pi}{3}\mathcal{N}\alpha\right)\left(\Delta_r + K^2\right)\vec{p}(r) = -4\pi\mathcal{N}\alpha\left(\mathrm{curl}\,\mathrm{curl}_r + K_0^2\right)\vec{p}(r),$$

or
$$\Delta_r\vec{p}(r) + \frac{1+\frac{8\pi}{3}\mathcal{N}\alpha}{1-\frac{4\pi}{3}\mathcal{N}\alpha}\frac{\omega^2}{c^2}\vec{p}(r) - K_0^2\vec{p}(r) = -\frac{4\pi\mathcal{N}\alpha}{1-\frac{4\pi}{3}\mathcal{N}\alpha}\vec{\nabla}\,\mathrm{div}_r\vec{p}(r).$$ [1]

Let $\dfrac{1+\frac{8\pi}{3}\mathcal{N}\alpha}{1-\frac{4\pi}{3}\mathcal{N}\alpha} = n^2$, then $\dfrac{n^2-1}{n^2+2} = \dfrac{4\pi}{3}\mathcal{N}\alpha$, $n^2-1 = \dfrac{4\pi\mathcal{N}\alpha}{1-\frac{4\pi}{3}\mathcal{N}\alpha}$.

$$\Delta_r\vec{p}(r) + n^2\frac{\omega^2}{c^2}\vec{p}(r) - K_0^2\vec{p}(r) = -(n^2-1)\,\vec{\nabla}\,\mathrm{div}_r\vec{p}(r) \qquad (3.14)$$

Identifying n^2 with ϵ and recalling $\vec{p} = \alpha\vec{E}$ and introducing one of the Proca equations $\epsilon\,\mathrm{div}\,\vec{E} + K_0^2\phi = 0$ from (2.19), the equation (3.14) becomes:

$$\left(\Delta + \epsilon\frac{\omega^2}{c^2} - K_0^2\right)\vec{E} + K_0^2\left(\frac{1}{\epsilon} - 1\right)\vec{\nabla}\phi = 0.$$

We have already concluded on page 31 that $\mu_g = \nu_g = \sigma_g = 1$ in the non-relativistic approximation. Hence, the present result of our non-relativistic theory agrees with the equation for $\vec{E}$ in (2.23), which was derived from the phenomenological

---

[1] One obtains Hoek's wave equation in the electromagnetic case by letting $K_0 \to 0$ and observing $\mathrm{div}_r\vec{p}(r) = 0$ from the equation which corresponds to (3.13).

Proca equations (2.19) and (2.20).

3. Evaluation of the magnitude of the induced spin moment (tensor coupling)

Following Dirac's method the classical relativistic equation of motion of a dipole in the external mesonic magnetic field was derived also by Bhabha.[1] When the non-relativistic approximation is made to the equation by considering a nucleon as infinitely heavy, and hence has no linear motion but only a gyration of the spin vector in the external mesonic magnetic field, the equation reduces to:

$$I\dot{\vec{M}} + K\left[\vec{M}\times\ddot{\vec{M}}\right] = \frac{f}{\kappa_0}\left[\vec{M}\times\vec{H}\right] + \frac{2}{3}\frac{f^2}{\kappa_0^2 c^3}\left[\vec{M}\times\dddot{\vec{M}}\right] + f^2\kappa_0\left[\vec{M}\times\vec{\vec{G}}\right] + \frac{f^2}{c}\left[\vec{M}\times\dot{\vec{M}}\right]. \tag{3.15}$$

In this equation $\vec{M}$ is a vector representing the spin of a nucleon which tends to be in the direction of the external field $\vec{H}$, I the angular momentum of the spin, which, when quantized, has the value of $\frac{\hbar}{2}$, K spin inertia and $\vec{\vec{G}}$ is the following expression

$$\vec{\vec{G}}(t) = 2\int_{-\infty}^{t} dt'\,\vec{M}(t')\,\frac{J_2(\kappa_0 u)}{u^2}$$

where $u = t - t'$. The first term in the equation (3.15) expresses obviously the mechanical torque, the second a torque arising from the inertia of spin, the third that due to the external field, the fourth that which is due to the

---

[1] H.J. Bhabha, Proc. Roy. Soc. A, 173, 314 (1941), referred to as Bhabha II in this thesis.

radiation reaction and the last two coming from the fact that the field possesses a finite rest mass. For a classical theory of a spinning particle in the electromagnetic field the last two terms do not appear.[1] Further, if the particle is an electron interacting with the electromagnetic field the spin inertia term should also go out since the electron does not possess such a mechanical constant. The reader is referred to other literature for the significance of a spin inertia[2] in the theory of nucleons.

Let us suppose that a weak periodic force $\vec{H}$ with frequency ω

$$\vec{H} = \vec{H}_0 e^{i\omega t} \tag{3.16}$$

acts on a nucleon and solve the equation (3.15) for a small gyrational amplitude of the spin. We may write for the total spin vector as follows:

$$\vec{M}(t) = \vec{M}_0 + \vec{M}_1 e^{i\omega t} + \vec{M}_2 e^{i(\omega t + \delta)}, \tag{3.17}$$

where $\vec{M}_0$ is the original, unperturbed spin vector, having the value $|\vec{M}_0|^2 = 1$, and $\vec{M}_1$ and $\vec{M}_2$ are the mutually perpendicular components of the induced spin moment. $\vec{M}_1$ and $\vec{M}_2$ are both perpendicular to $\vec{M}_0$. δ is a phase angle between the induced moments $\vec{M}_1$ and $\vec{M}_2$. The assumption that $\vec{H}_0$, $\vec{M}_1$ and $\vec{M}_2$ are small enables us to linearize the equation (3.15) whose solution is directly obtainable.

Since $\vec{M}_0$, $\vec{M}_1$ and $\vec{M}_2$ are mutually perpendicular and

---

[1] H.J. Bhabha and H.C. Corben, Proc. Roy. Soc. A173, 106 (1941)

[2] W. Heisenberg, Zeits. f. Physik 113, 61 (1939); W. Pauli, Meson Theory of Nucleon Forces, p.11, N.Y., Interscience Publishers, Inc. (1943)

$|\vec{M}_0|^2 = 1$, we have the following equations at our disposal:

$$[\vec{M}_0 \times \vec{M}_1] = \vec{M}_2 \left|\frac{M_1}{M_2}\right|$$
$$[\vec{M}_2 \times \vec{M}_0] = \vec{M}_1 \left|\frac{M_2}{M_1}\right|. \qquad (3.18)$$

Substituting (3.16), (3.17) and (3.18) into (3.15) and neglecting terms quadratic in $\vec{M}_1$, $\vec{M}_2$ and $\vec{H}$, we obtain:

$$i\omega I(\vec{M}_1 + \vec{M}_2 e^{i\delta}) + K\omega^2\left[\vec{M}_1\left|\frac{M_2}{M_1}\right| e^{i\delta} - \vec{M}_2\left|\frac{M_1}{M_2}\right|\right]$$
$$= \frac{f}{K_0}[\vec{M}_0 \times \vec{H}_0] + f^2\left\{i\frac{2}{3}\frac{\omega^3}{K_0^3c^3} + i\frac{2}{3}K_0 A + \frac{2}{3}K_0 - \frac{i\omega}{c}\right\}\left[\vec{M}_1\left|\frac{M_2}{M_1}\right| e^{i\delta} - \vec{M}_2\left|\frac{M_1}{M_2}\right|\right], \qquad (3.19)$$

where $A = \left\{\left(\frac{\omega}{K_0c}\right)^2 - 1\right\}^{3/2} - \left(\frac{\omega}{K_0c}\right)^3 + \frac{3}{2}\left(\frac{\omega}{K_0c}\right)$ arises from the term in (3.15) which involves the integral. This integral can be carried out in much the same manner as the integral occurring in the equation (3.1).

Let us introduce the following notations to simplify our equations:

$$\alpha = \frac{3IK_0^2}{2f^2}, \qquad \gamma = \frac{3}{2}\frac{KK_0^2}{2f^2},$$
$$\xi = \gamma\omega - \frac{K_0^3}{\omega}, \qquad \zeta = \frac{(\omega^2 - K_0^2c^2)^{3/2}}{c^3\omega}. \qquad (3.20)$$

Then, the equation (3.19) becomes:

$$\vec{M}_1\left[\alpha - (\zeta + i\xi)\left|\frac{M_2}{M_1}\right| e^{i\delta}\right] + \vec{M}_2\left[\alpha e^{i\delta} + (\zeta + i\xi)\left|\frac{M_1}{M_2}\right|\right] = \frac{3}{2}\frac{K_0}{f\omega i}[\vec{M}_0 \times \vec{H}_0]. \qquad (3.21)$$

Since $\vec{M}_0$ and $\vec{H}_0$ are both real and $\vec{M}_1$ and $\vec{M}_2$ are both pure imaginary and mutually perpendicular, the imaginary parts of the factors for $\vec{M}_1$ and $\vec{M}_2$ must individually vanish, giving the following equations:

$$\xi \cos\delta + \zeta \sin\delta = 0 , \quad \alpha \sin\delta + \xi \left|\frac{M_1}{M_2}\right| = 0.$$

Hence, $$\tan\delta = -\frac{\xi}{\zeta} , \quad \left|\frac{M_1}{M_2}\right| = \frac{\alpha}{\sqrt{\xi^2+\zeta^2}}.$$

Then, the equation (3.21) becomes:

$$\vec{M}_1\left[\alpha - \frac{\xi^2}{\alpha} - \frac{\zeta^2}{\alpha}\right] + \vec{M}_2 \left|\frac{M_1}{M_2}\right| 2\zeta = \frac{3K_0}{2f\omega i}\left[\vec{M}_0 \times \vec{H}_0\right]. \qquad (3.22)$$

This is equivalent to the equation derived in Bhabha II, page 335.

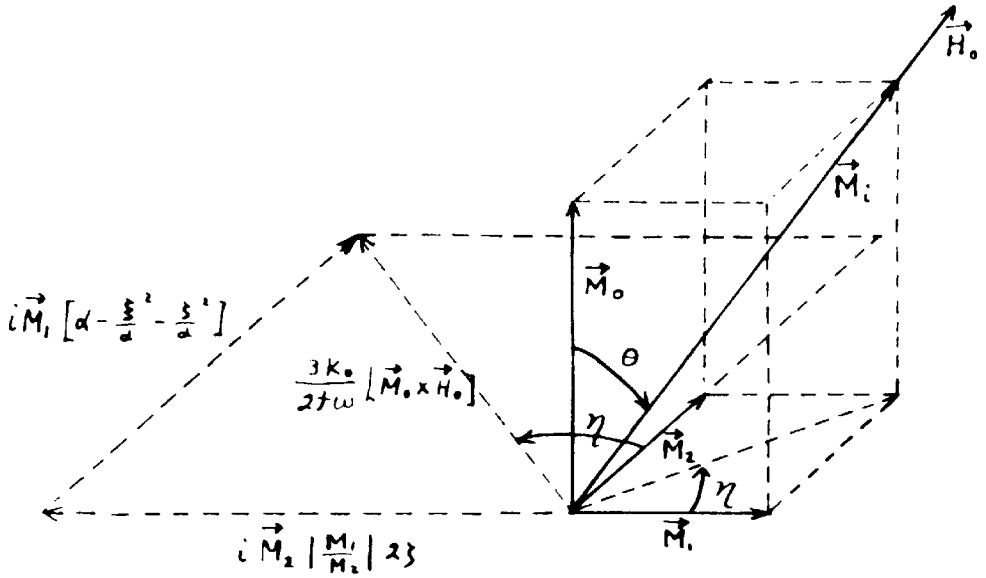


Fig. 3

Geometrical representation of the equation (3.22)

From the equation (3.22) and the above figure we obtain:

$$\tan\eta = \frac{2\alpha\zeta}{\alpha^2-\xi^2-\zeta^2} , \quad \sin\eta = \frac{2\alpha\zeta}{\sqrt{(\alpha^2-\xi^2-\zeta^2)+4\alpha^2\zeta^2}}.$$

Thus, we obtain for each component:

$$|\vec{M}_1| = \frac{3K_0H_0\sin\theta}{2f}\,\frac{\alpha}{\omega\sqrt{(\alpha^2-\xi^2-\zeta^2)^2+4\alpha^2\zeta^2}}$$

$$|\vec{M}_2| = \frac{3K_0H_0\sin\theta}{2f}\,\frac{\sqrt{\xi^2+\zeta^2}}{\omega\sqrt{(\alpha^2-\xi^2-\zeta^2)^2+4\alpha^2\zeta^2}}.$$

Hence,

$$\sqrt{|M_1|^2+|M_2|^2} = \frac{3K_0H_0\sin\theta}{2f}\,\frac{\sqrt{\alpha^2+\xi^2+\zeta^2}}{\omega\sqrt{(\alpha^2-\xi^2-\zeta^2)^2+4\alpha^2\zeta^2}},$$

where θ is the angle between the original spin vector $\vec{M}_0$ and the external magnetic field $\vec{H}_0$. Expressing the left hand side of the equation (3.22) by a complex function Z, we write:

$$Z = |\vec{M}_1|\left[\alpha-\frac{\xi^2}{\alpha}-\frac{\zeta^2}{\alpha}\right]+i|\vec{M}_2|\frac{2\zeta\alpha}{\sqrt{\xi^2+\zeta^2}}$$

$$= \sqrt{|M_1|^2+|M_2|^2}\,(\cos\Delta+i\sin\Delta).$$

Then, we obtain:

$$\cos\Delta = \frac{|\vec{M}_1|}{\sqrt{|\vec{M}_1|^2+|\vec{M}_2|^2}}\left[\alpha-\frac{\xi^2}{\alpha}-\frac{\zeta^2}{\alpha}\right] = \frac{|\alpha^2-\xi^2-\zeta^2|}{\sqrt{\alpha^2+\xi^2+\zeta^2}}$$

$$\sin\Delta = \frac{|\vec{M}_2|}{\sqrt{|\vec{M}_1|^2+|\vec{M}_2|^2}}\,\frac{2\zeta\alpha}{\sqrt{\xi^2+\zeta^2}} = \frac{2\alpha\zeta}{\sqrt{\alpha^2+\xi^2+\zeta^2}}.$$

Consequently,

$$Z = \sqrt{|\vec{M}_1|^2+|\vec{M}_2|^2}\,\frac{|\alpha^2-\xi^2-\zeta^2|+2i\alpha\zeta}{\sqrt{\alpha^2+\xi^2+\zeta^2}}.$$

This function represents a complex vector whose length is $\sqrt{|\vec{M}_1|^2+|\vec{M}_2|^2}$ and which lies in the direction of $[\vec{M}_0\times\vec{H}_0]$ on the plane defined by vectors $\vec{M}_1$ and $\vec{M}_2$. However, the vector $\vec{M}_1+\vec{M}_2$ is perpendicular to $[\vec{M}_0\times\vec{H}_0]$ on the complex plane, i.e. its phase is $\frac{\pi}{2}$ less than that of $[\vec{M}_0\times\vec{H}_0]$. Thus, $Ze^{-i\frac{\pi}{2}}$ represents the complex vector which is the projection of the induced spin vector $\vec{M}_1$ in the direction of the external field $\vec{H}_0$ onto the complex plane. Hence[1],

---

[1]It is of interest to consider the case in which the equation of motion is simply $I\dot{\vec{M}} = \frac{f}{K_0}[\vec{M}\times\vec{H}]$, corresponding to vanishing meson mass and the spin inertia. This is equivalent to letting γ, ξ and ζ in (3.20) equal to zero, and for this case one obtains $\frac{f}{K_0}\vec{M}_1 = \frac{3}{2}\frac{i}{\omega\alpha}\vec{H}_0$, i.e. the induced moment is a pure imaginary quantity. This is indeed expected from the fact the equation of motion is of the first order in time.

$$\frac{f}{K_0}\vec{M}_i = \frac{3}{2}\frac{i\sqrt{\alpha^2+\xi^2+\zeta^2}}{\omega\{|\alpha^2-\xi^2-\zeta^2|+2i\alpha\xi\}}\vec{H}_0 = \frac{3}{2}\frac{\sqrt{\alpha^2+\xi^2+\zeta^2}}{\omega}\frac{2\alpha\xi+i|\alpha^2-\xi^2-\zeta^2|}{(\alpha^2-\xi^2-\zeta^2)^2+4\alpha^2\xi^2}\vec{H}_0$$

Introducing β as the complex polarizability of the spin of a nucleon, we may write:

$$\frac{f}{K_0}\vec{M}_i = \beta\vec{H}_0 \qquad (3.23)$$

where

$$\beta = \frac{3}{2}\frac{\sqrt{\alpha^2+\xi^2+\zeta^2}}{\omega}\frac{2\alpha\xi+i|\alpha^2-\xi^2-\zeta^2|}{(\alpha^2-\xi^2-\zeta^2)^2+4\alpha^2\xi^2}.$$

In the expression for β, ξ may be regarded as expressing the effects of radiation reaction of the meson field, for it reduces to ω^2 when $K_0 \to 0$ (see (3.20)), the reaction term for a Maxwell field. In the absence of this radiation reaction term, $\vec{M}_1$ becomes infinite when $\xi = \alpha$, i.e. at frequency ω given by

$$\omega = \frac{I}{K - \frac{2}{3}\frac{f^2K_0}{\omega}}$$

This is a resonance phenomenon. The effect of radiation reaction term ξ is to make the amplitude of $\vec{M}_1$, consequently the scattering cross-section, <u>finite</u> even for this frequency.

4. Application of the Hoek's method for the case of the tensor coupling

The vector potential at $\vec{r}$ arising from the gyrating spin of a nucleon at $\vec{r}'$ is given in Bhabha II, page 338 as follows:

$$\vec{A}(x) = K\frac{[\vec{\rho}\times\vec{M}_i(x')]}{\rho}\frac{e^{-i\lambda\rho}}{\rho},$$

where $\vec{\rho} = \vec{r} - \vec{r}'$, $\rho = |\vec{r} - \vec{r}'|$.

However, for our purpose it is more convenient to write:

$$\vec{A}(r) = \mathrm{curl}_r \left[\vec{M}_i(r') \frac{e^{-ik\rho}}{\rho} \right] = \frac{f}{\kappa_0} \mathrm{curl}_r R(r,r') \vec{M}_i(r').$$

Since $\vec{H} = \mathrm{curl}\, \vec{A}$, we have:

$$\vec{H}(r) = \mathrm{curl}\, \mathrm{curl}_r \left[\vec{M}_i(r') R(r,r') \right].$$

The integral equation for $\vec{M}_1$ which corresponds to (3.11) is in our case:

$$\vec{M}_i(r) = \beta \left[\vec{H}^{(0)}(r) + \varkappa \int_{S(r)}^{V} dr' \, \mathrm{curl}\, \mathrm{curl}_r R(r,r') \vec{M}_i(r') \right], \quad (3.23)$$

where $H^{(0)}(r)$ is the external field. Making use of (3.12c),

$$\vec{M}_i(r) = \beta \vec{H}^{(0)}(r) + \varkappa\beta \left[\mathrm{curl}\, \mathrm{curl}_r \int_{S(r)}^{V} dr' R(r,r') \vec{M}_i(r') - \frac{8\pi}{3} \vec{M}_i(r) \right] \quad (3.24)$$

Let this equation be operated on by $(\Delta_r + \kappa^2)$ and make use of (3.12b).

$$(\Delta_r + \kappa^2) \vec{M}_i(r) = \beta (\Delta_r + \kappa^2) \vec{H}^{(0)}(r) - \frac{8\pi}{3} \varkappa\beta (\Delta_r + \kappa^2) \vec{M}_i(r)$$
$$+ \varkappa\beta \left\{ \mathrm{curl}\, \mathrm{curl}_r \int_{S(r)}^{V} dr' (\Delta_r + \kappa^2) R(r,r') \vec{M}_i(r') - 4\pi \, \mathrm{Curl}\, \mathrm{curl}_r \vec{M}_i(r) \right\}$$

However, $(\Delta_r + \kappa^2) \vec{H}^{(0)} = 0$, $(\Delta_r + \kappa^2) R(r,r') = 0$ and $\mathrm{div}\, \vec{M}_i(r) = 0$ from (3.23). Thus, we finally obtain:

$$\left[\Delta_r + \frac{1 + \frac{8\pi}{3} \varkappa\alpha}{1 - \frac{4\pi}{3} \varkappa\alpha} \left(\frac{\omega^2}{c^2} - \kappa_0^2 \right) \right] \vec{M}_i(r) = 0. \quad (3.25)$$

Hence,

$$n^2 = \frac{1 + \frac{8\pi}{3} \varkappa\alpha}{1 - \frac{4\pi}{3} \varkappa\alpha} \quad \text{or} \quad \frac{n^2 - 1}{n^2 + 2} = \frac{4\pi}{3} \varkappa\beta. \quad (3.26)$$

Identifying $n^2 = \mu_f$, the above equation becomes:

$$\left[\Delta - \frac{\mu_f}{c^2}\frac{\partial^2}{\partial t^2} - \mu_f \kappa_0^2\right]\vec{H} = 0.$$

Since it was already found on page 35 that in the non-relativistic approximation $\epsilon_f = \nu_f = \sigma_f = 1$, this equation arising from the non-relativistic approximations to the motion of the nucleon agrees with the equation for $\vec{H}$ in (2.20), which was derived from the phenomenological Proca equations (2.17).

As it is remarked by Bhabha and Iwanenko and Sokolow, for the case of the tensor coupling only a <u>transverse</u> field exists when non-relativistic approximations are made to the motion of the nucleon. Consequently the scalar potential vanishes identically (see Bhabha II, page 338) and we obtain the following wave equations from (2.20):

$$\left(\Delta - \frac{\mu_f}{c^2}\frac{\partial^2}{\partial t^2} - \mu_f \kappa_0^2\right)\vec{A} = -4\pi\left\{\frac{\mu_f}{\kappa_0}\frac{1}{c}\frac{\partial}{\partial t}\vec{N} - \frac{1}{\kappa_0}\operatorname{curl}\vec{M}\right\}$$

$$\left(\Delta - \frac{\mu_f}{c^2}\frac{\partial^2}{\partial t^2} - \mu_f \kappa_0^2\right)\vec{H} = -4\pi\kappa_0\vec{M} \qquad (3.27)$$

$$\left(\Delta - \frac{\mu_f}{c^2}\frac{\partial^2}{\partial t^2} - \mu_f \kappa_0^2\right)\vec{E} = 4\pi\kappa_0\mu_f\vec{N}.$$

Note that the modified Gordon-Klein operators in these equations have a unique form, indicating that these components of the field travel through nuclear matter with the same phase velocity, contrary to the case of the vector coupling discussed in the previous sections.

5. Extinction Theorem of Oseen (tensor coupling)

A careful observation of the equation (3.23) brings out a remarkable fact that while the external field $\vec{H}^{(o)}$, which is a part of $\vec{M}_1$ according to (3.23), satisfies the vacuum wave equation, $\vec{M}_1$ satisfies the modified wave equation (3.25). This fact suggests that the integral in (3.23) must split into two parts, one of which would cancel completely the incoming external field $\vec{H}^{(o)}$, leaving the other as the field which propagates through nuclear matter.

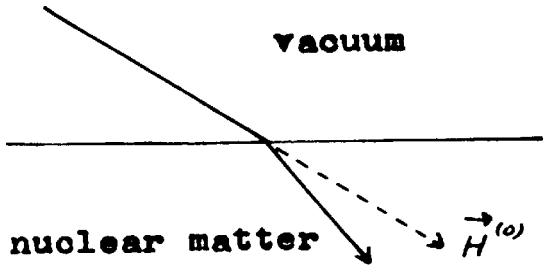


Fig. 4

Refraction of the meson waves

This is the analog of the Extinction Theorem of Oseen in the electromagnetic theory[1] in our case, and affords the explanation of the phenomenon of refraction.

From the equations

$$(\Delta_2 + K^2) R(2,2') = 0 , \quad (\Delta_2 + n^2 K^2) \vec{M}_i(2) = 0$$

we obtain:

$$R(2,2') = -\frac{1}{K^2} \Delta R(2,2'), \quad \vec{M}_i(2) = -\frac{1}{n^2 K^2} \Delta_2 \vec{M}_i(2).$$

Introducing a notation $\vec{I}(2) = \int_{S(2)}^{V} d2' \, R(2,2') \vec{M}_i(2')$,

---

[1] Lecture by Professor G.E. Uhlenbeck, loc. cit.

we have:

$$\kappa^2 \vec{I}(r) = -\int_{S(r)}^{V} dr' \; \vec{M}_i(r') \, \Delta_{r'} R(r,r')$$

$$n^2 \kappa^2 \vec{I}(r) = -\int_{S(r)}^{V} dr' \; R(r,r') \, \Delta_{r'} \vec{M}_i(r').$$

Substract the first equation from the second and use Green's theorem.

$$(n^2-1)\,\kappa^2 \vec{I}(r) = \int_{S(r)}^{V} dr' \left\{ \vec{M}_i(r') \, \Delta_{r'} R(r,r') - R(r,r') \, \Delta_{r'} \vec{M}_i(r') \right\}$$

$$= \iint_{\sigma} d\sigma \left\{ \vec{M}_i(r') \frac{\partial R(r,r')}{\partial n} - R(r,r') \frac{\partial \vec{M}_i(r')}{\partial n} \right\}$$

$$- \iint_{\sigma_0} d\sigma \left\{ \vec{M}_i(r') \frac{\partial R(r,r')}{\partial \eta} - R(r,r') \frac{\partial \vec{M}_i(r')}{\partial n} \right\}$$

The surface of the first integration is the entire outer surface of the nuclear matter, whereas that of the second integral is the surface of a small sphere of radius "a" exclouding the immediate neighborhood of the point $\vec{r}$. $\frac{\partial}{\partial \eta}$ signifies a normal derivative.

$$\lim_{a\to 0} \iint_{\sigma_0} d\sigma \; R(r,r') \frac{\partial \vec{M}_i(r')}{\partial \eta} = \lim_{a\to 0} [R]_{\rho=a} \iint_{\sigma} d\sigma \frac{\partial \vec{M}_i(r')}{\partial \rho} = 0$$

$$\lim_{a\to 0} -\iint_{\sigma_0} d\sigma \; \vec{M}_i(r') \frac{\partial R(r,r')}{\partial n} = \lim_{a\to 0} -\left[\frac{\partial R}{\partial \rho}\right]_{\rho=a} \iint_{\sigma} d\sigma \left\{ \vec{M}_i(r) + a\nu_\alpha \frac{\partial \vec{M}_i(r)}{\partial x_\alpha} + \right\}$$

$$= \frac{1}{a^2} 4\pi a^2 \vec{M}_i(r) = 4\pi \vec{M}_i(r).$$

Hence, we obtain

$$\vec{I}(r) = \frac{1}{(n^2-1)\kappa^2} \iint_{\sigma} d\sigma \left\{ \vec{M}_i(r') \frac{\partial R(r,r')}{\partial n} - R(r,r') \frac{\partial \vec{M}_i(r')}{\partial n} \right\} + \frac{4\pi}{(n^2-1)\kappa^2} \vec{M}_i(r).$$

Substitute this equation into (3.24).

$$\left(1+\frac{8\pi}{3}\mathcal{H}_1\beta\right)\vec{M}_i(r) = \beta\left[\vec{H}^{(0)}(r) + \frac{\mathcal{H}}{(n^2-1)K^2}\iint_{\circlearrowright} d\sigma\, \mathrm{curl}\,\mathrm{curl}_r\left\{\vec{M}_i(r')\frac{\partial R(r,r')}{\partial n} - R(r,r')\frac{\partial \vec{M}_i(r')}{\partial n}\right\} + \frac{4\pi\mathcal{H}}{(n^2-1)K^2}\,\mathrm{curl}\,\mathrm{curl}_r\,\vec{M}_i(r)\right]$$

However, one can easily verify

$$\left(1+\frac{8\pi}{3}\mathcal{H}_1\beta\right)\vec{M}_i = \frac{4\pi\mathcal{H}\beta}{(n^2-1)K^2}\,\mathrm{curl}\,\mathrm{curl}_r\,\vec{M}_i(r)$$

Thus, we finally obtain:

$$\vec{M}^{(0)}(r) + \frac{\mathcal{H}}{(n^2-1)K^2}\iint_{\circlearrowright} d\sigma\, \mathrm{curl}\,\mathrm{curl}_r\left\{\vec{M}_i(r')\frac{\partial R(r,r')}{\partial n} - R(r,r')\frac{\partial \vec{M}_i(r')}{\partial n}\right\} = 0$$

This is the proof of the anolog of the Extinction Theorem of Oseen in the tenser coupling case. The anolog does not exist in the vector coupling case because $\vec{p}$ does not satisfy a pure wave equation.

6. Dispersion in tensor coupling

Let us now introduce β obtained in (3.23) and separate the real and imaginary parts.

$$n^2 = \ell + im = \frac{1}{\ldots}\left[\ldots\right] \qquad (3.28)$$

We shall investigate in the following how the velocity of propagation and the absorption of the meson field by the nuclear

matter depend upon the real and the imaginary parts of n^2.

For the purpose of our discussion we write the wave equation in one dimensional form:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} - n^2 k_0^2\right) A_x = 0.$$

Assuming a plane wave solution $A_x = A_0 e^{(kx - \omega t)}$, we obtain

$$k^2 = n^2\left(\frac{\omega^2}{c^2} - k_0^2\right) = (\ldots)\left(\ldots\right)$$

$$= \left(\ldots\right)\sqrt{\ldots}\,\tan^{-1}\frac{m}{\ell}.$$

Hence,

$$k = \sqrt{\frac{\omega^2}{c^2} - k_0^2}\,(\ell^2 + m^2)^{1/4}\left\{\cos\left(\tfrac{1}{2}\tan^{-1}\frac{m}{\ell}\right) + \sin\left(\tfrac{1}{2}\tan^{-1}\frac{m}{\ell}\right)\right\},$$

$$\sqrt{\frac{\omega^2}{c^2} - k_0^2}\,(\ell^2 + m^2)^{1/4}\left\{\cos\left(\tfrac{1}{2}\tan^{-1}\frac{m}{\ell}\right) + \sin\left(\tfrac{1}{2}\tan^{-1}\frac{m}{\ell}\right)\right\}.$$

We take the upper branch for the complex function k, and write:

$$k = \frac{\omega}{c}(S_1 + iS_2),$$

where

$$S_1 = \frac{1}{\omega}\sqrt{\omega^2 - k_0^2 c^2}\,(\ell^2 + m^2)^{1/4}\cos\left(\tfrac{1}{2}\tan^{-1}\frac{m}{\ell}\right)$$

$$S_2 = \frac{1}{\omega}\sqrt{\omega^2 - k_0^2 c^2}\,(\ell^2 + m^2)^{1/4}\sin\left(\tfrac{1}{2}\tan^{-1}\frac{m}{\ell}\right).$$

Introducing k into $A_x = A_0 e^{(kx - \omega t)}$, we find

$$A_x = A_0 e^{\frac{\omega}{c}S_2 + \frac{\omega}{c}\left(\frac{S_1}{c} - t\right)}.$$

Thus, $\frac{c}{S_1}$ represents the phase velocity of propagation and $\frac{\omega S_2}{c}$ the attenuation in the amplitude of the field per cm propagating through the nuclear matter.

Several characteristic numbers for the meson energy range 180 Mev-340 Mev have been computed, using the following set of constants:

Fine structure constant $\frac{f^2}{\hbar c} = 0.3$, rest mass of π mean 280 electron masses, reciprocal Compton wave length of the

meson [illegible], spin angular momentum [illegible], nucleon density [illegible], and spin inertia [illegible].

Meson energy [illegible] (Mev)	180	220	260	300	340
de Broglie wave length (10^{-13} cm)	10.9	7.2	5.6	4.6	3.9
[illegible]	0.29 ± 0.79	0.83 ± 0.68	0.74 ± 0.39	0.49 ± 0.12	0.30 ± 0.19
n^2 = [illegible]	0.97 ± 0.89	1.60 ± 1.21	1.90 ± 0.64	1.60 ± 0.20	1.33 ± 0.22
S_1	0.67	1.03	1.19	1.12	1.04
Phase velocity c/S_1	1.5c	0.97c	0.84c	0.89c	0.96c
$\frac{S_1 \omega}{c}$	0.33k.	0.38k.	0.35k.	0.145k.	0.181k.

Taking [illegible] for the average inter-nucleon distance the number of nucleons in the volume of (de Broglie wave length)^3 vary from 500 to 20 as one goes from 180 Mev to 340 Mev. Assuming such small numbers of scattering centers are sufficient to give rise to phenomenological field, we do observe a region approximately between 200 Mev and 340 Mev where the phase velocity of the field is less than the velocity of light.

The fact that $\frac{K_0}{\lambda} > 1$ in the region less than 200 Mev of the meson field makes the phase velocity [illegible] always greater than c regardless of the values of n^2 in that region.

Chapter IV

Application of the Phenomenological Proca Equations

In this chapter we wish to calculate the rate of the radiation of energy in the form of the meson field per unit length of the path of a high energy nucleon as it penetrates through the nuclear matter. The method employed here is similar to that of Tamm[1] for the electromagnetic Čerenkov radiation.

The electromagnetic Čerenkov radiation arises when a source of the electromagnetic field, viz. an electron, moves through a refractive medium with velocity greater than the phase velocity of the propagation of the electromagnetic field itself. Microscopically speaking, an electron moving through matter, by virtue of its electromagnetic interaction with the constituent atoms and molecules, excites these resonators. The elementary electromagnetic waves emitted by the resonators will travel through the dispersive matter with the velocity c/n, where n is the index of refraction. If n is larger than unity, the velocity of the electron u may become greater than the phase velocity of the propagation of the electromagnetic waves. Under such circumstances a constructive interference of waves may occur along the surface of a cone emanating from the electron and moving with it, as in

---

[1]P.A. Čerenkov, Comptes Rendus de l'Acad. des Sciences, U.S.S.R. 14, 3, 101 (1937); Ig. Tamm, loc. cit.

the following diagram[1]:

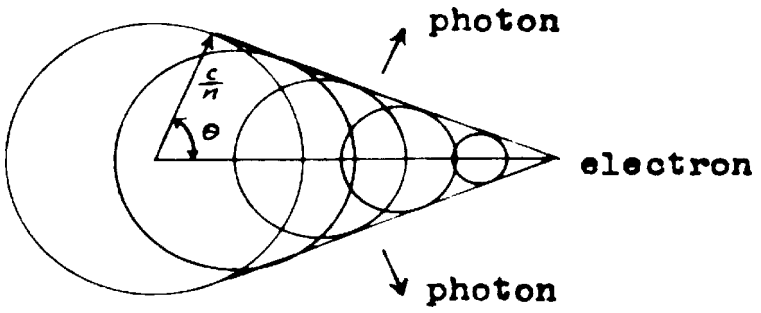


Fig. 5

Čerenkov Radiation

It is obvious that the cosine of the angle θ indicated in Fig. 5 should be equal to $\frac{c}{nu}$.

As a meson analog of the above-mentioned electromagnetic Čerenkov radiation we shall consider the collision between an extremely high energy nucleon with a heavy nucleus. Taking into consideration that the nuclear force range and the average inter-nucleonic distance are both same order of magnitude, 2×10^{-13} cm, there must be 20-40 nucleons along a diameter of a heavy nucleus, e.g. lead nucleus, which are within the range of interaction with the penetrating nucleons. These nucleons will interact strongly with the penetrating nucleon, creating mesons whose energies range from comparatively low values to extremely high. The mesons of extremely high energy

---

[1]One thing to be mentioned at this point is that the radiation which takes place here comes only indirectly from the electron moving through the dispersive medium, and is not connected with the deceleration of the charged particle. That is to say, an electron moving with constant velocity may emit radiation. The loss of energy due to the Čerenkov radiation will of coursedecelerate the electron, and there is a secondary radiation due to this deceleration, which, however, is much weaker than the Čerenkov radiation itself.

because of their wave lengths being smaller than the inter-nucleonic distances, will travel through the nuclear matter as if other nucleons were not present. The mesons of comparatively low energies on the other hand will be scattered by other nucleons in other parts of the nuclear matter. For those meson waves whose wave length are long compared with the average inter-nucleonic distance we may apply our phenomenological meson field equations.

It can easily be shown that a nucleon of 5 Bev has a velocity of 0.98c, and at 10 Bev its velocity is practically that of the velocity of light. We shall assume that such a high energy nucleon penetrates through nuclear matter of large extent at a constant velocity. Since it was found that the index of refraction in the vector coupling case was less than unity, for which the Čerenkov condition of phase velocity of the field less than the source velocity cannot be satisfied, we shall restrict ourselves to the tensor coupling where there exists a possibility of the phase velocity being less than the nucleon velocity for some range of frequencies in the low energy region.

1. Derivation of the radiation field

The wave equation for the vector potential is obtained from (3.25).

$$\left(\Delta - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} - n^2\lambda_0^2\right)\vec{A} = -4\pi\frac{f}{\lambda_0}\left\{\mathrm{curl}\,\vec{M} - \frac{n^2}{c}\frac{\partial}{\partial t}\vec{N}\right\}$$

$\vec{M}$ represents the spin of the moving nucleon and $\vec{N}$ its spin velocity.

For the sake of simplicity let us consider that a nucleon is moving through nuclear matter of an infinite extent with a constant velocity "u" along z-axis. Then, from (2.12) the source functions maybe written as

$$\vec{M} = \vec{\sigma}\,\delta(x)\,\delta(y)\,\delta(z-ut)$$
$$\vec{N} = \frac{u}{c}\vec{\sigma}\,\delta(x)\,\delta(y)\,\delta(z-ut). \qquad (4.1)$$

It will become clear later that the z-component of the vector potential will suffice for our calculation. The equation to be solved is, then

$$\left(\Delta - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} - \kappa_0^2\right) A_z = -4\pi \frac{f}{\kappa_0}\left\{curl_z \vec{M} - \frac{n^2}{c}\frac{\partial}{\partial t} N_z\right\}. \qquad (4.2)$$

Introducing (4.1) for the source functions we apply a Fourier transformation with respect to the time to this equation.

$$A_{z\omega}(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\; A_z(x,y,z,t)\, e^{-i\omega t}$$

$$(curl_z \vec{M})_\omega = \frac{1}{2\pi}\left\{\sigma_y \delta(y)\frac{\partial}{\partial x}\delta(x) - \sigma_x \delta(x)\frac{\partial}{\partial y}\delta(y)\right\}\frac{1}{u}\, e^{-i\frac{\omega z}{u}}$$

$$\left(\frac{\partial}{\partial t} N_z\right)_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \left(\frac{\partial}{\partial t} N_z\right) e^{i\omega t} = \frac{1}{2\pi}\frac{\sigma_z}{c}\, i\omega\, \delta(x)\delta(y)\, e^{-i\frac{\omega z}{u}}$$

In order to simplify our calculation we divide the vector potential into two parts, one with $-4\pi \frac{f}{\kappa_0} curl_z \vec{M}$ and the other $4\pi \frac{f}{\kappa_0}\frac{n^2}{c}\frac{\partial}{\partial t} N_z$ as the source functions:

$$(\Delta + \kappa^2) A^{(1)}_{z\omega} = 4\pi \frac{1}{2\pi}\frac{f}{\kappa_0}\frac{n^2}{c}\sigma_z\, i\omega\, \delta(x)\delta(y)\, e^{-i\frac{\omega z}{u}} \qquad (4.3)$$

$$(\Delta + \kappa^2) A^{(2)}_{z\omega} = -4\pi \frac{1}{2\pi}\frac{f}{\kappa_0}\frac{1}{u}\left\{\sigma_y \delta(y)\frac{\partial}{\partial x}\delta(x) - \sigma_x \delta(x)\frac{\partial}{\partial y}\delta(y)\right\} e^{-i\frac{\omega z}{u}} \qquad (4.4)$$

where $\kappa^2 = n^2\left(\frac{\omega^2}{c^2} - \kappa_0^2\right)$.

Our task now is to solve these equations. In order to do so we introduce $\frac{e^{-i\kappa R}}{R}$, the retarded solution of the homogeneous wave equation, as the Green's function for our inhomogeneous equation. $R^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$, where (x, y, z) is a fixed point at which the potential is computed and (ξ, η, ζ) refers to the source. Then, the general solution of (4.3) becomes

$$A_{z\omega}^{(1)} = -G^{(1)}(\omega) \iiint_{-\infty}^{+\infty} d\xi d\eta d\zeta\, \delta(\xi)\delta(\eta)\delta(\zeta)\, e^{-i\frac{\omega\zeta}{u}} \frac{e^{-i\kappa R}}{R}$$
$$= -G^{(1)}(\omega) \int_{-\infty}^{+\infty} d\zeta\, e^{-i\frac{\omega\zeta}{u}} \frac{e^{-i\kappa R}}{R}. \qquad (4.5)$$

Here, we introduced a notation $G^{(1)}(\omega) = \frac{1}{2\pi} \frac{n^2(\omega)}{\kappa_0} \frac{\sigma_z}{c^2} i\omega$.

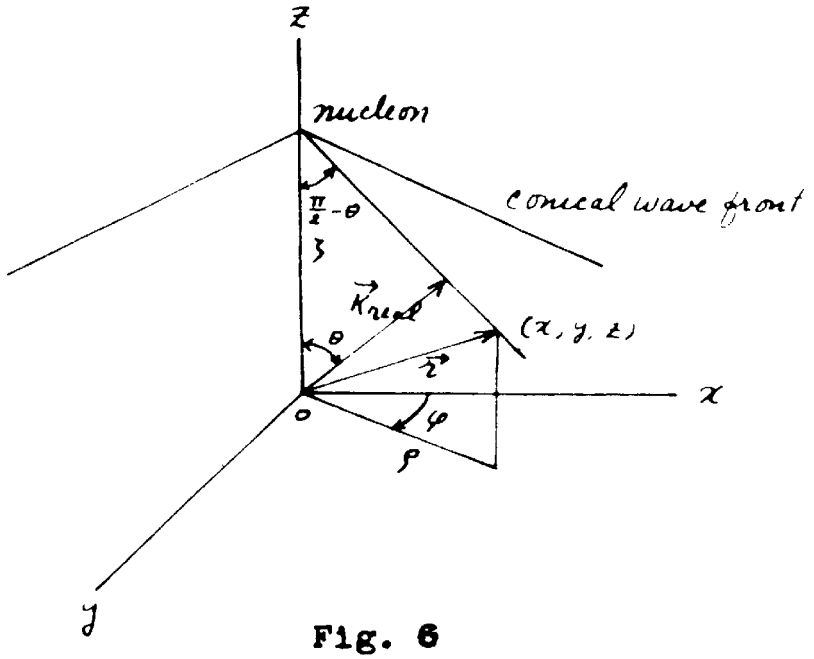


Fig. 6

Formation of the Čerenkov cone

We solve (4.4) similarly.

$$A^{(2)}_{z\omega} = \frac{1}{2\pi}\frac{1}{k_0 u}\iiint_{-\infty}^{+\infty} d\xi\, d\eta\, d\zeta \left[\sigma_y \delta(\eta)\frac{\partial}{\partial\xi}\delta(\xi) - \sigma_x \delta(\xi)\frac{\partial}{\partial\eta}\delta(\eta)\right] e^{-i\frac{\omega\zeta}{u}}\frac{e^{-ikR}}{R}$$

$$= \frac{1}{2\pi}\frac{1}{k_0 u}\int_{-\infty}^{+\infty} d\zeta\, e^{-i\frac{\omega\zeta}{u}}\left[\sigma_x \iint_{-\infty}^{+\infty} d\xi\, d\eta\, \delta(\xi)\delta(\eta)\frac{\partial}{\partial\eta}\frac{e^{-ikR}}{R}\right.$$

$$\left. - \sigma_y \iint_{-\infty}^{+\infty} d\xi\, d\eta\, \delta(\xi)\delta(\eta)\frac{\partial}{\partial\xi}\frac{e^{-ikR}}{R}\right]$$

However, $\frac{\partial}{\partial\xi}\frac{e^{-ikR}}{R} = ik\frac{x-\xi}{R}\frac{e^{-ikR}}{R}$, disregarding terms proportional to $\frac{1}{R^2}$. Performing the integration with respect to ξ and η, we obtain:

$$A^{(2)}_{z\omega} = \frac{1}{2\pi}\frac{1}{k_0}\frac{ik}{u}\int_{-\infty}^{+\infty} d\zeta\, e^{-i\frac{\omega\zeta}{u}}\frac{\sigma_x\,\zeta - \sigma_y\, x}{R}\frac{e^{-ikR}}{R}. \qquad (4.6)$$

We proceed to integrate (4.5) first.

Let $z-\zeta=\tau$, $R^2=\rho^2+(\zeta-z)^2$, where $\rho^2 = x^2+y^2$.

Then, $$A^{(1)}_{z\omega} = G^{(1)}(\omega)\, e^{-i\frac{\omega z}{u}}\int_{-\infty}^{+\infty} d\tau\, \frac{e^{i\frac{\omega\tau}{u} - ik\sqrt{\rho^2+\tau^2}}}{\sqrt{\rho^2+\tau^2}}.$$

(1) <u>For $\omega > 0$</u>

From (3.29) $$k = \frac{\omega}{c}(s_1 + is_2)$$

Hence, $$e^{-ik\sqrt{\rho^2+\tau^2}} = e^{-i\frac{\omega}{c}(s_1+is_2)\sqrt{\rho^2+\tau^2}}$$

$$= e^{-i\frac{\omega}{c}s_1\sqrt{\rho^2+\tau^2}}\, e^{\frac{\omega}{c}s_2\sqrt{\rho^2+\tau^2}}.$$

However, from the Fig. 6, $\sqrt{\rho^2+\tau^2} = \frac{\rho}{\cos\theta}$ = distance from the instantaneous position of the nucleon to the point of observation at (x, y, z). Since $\cos\theta$ equals the phase velocity of the field divided by the nucleon velocity, $\cos\theta = \frac{c}{u s_1}$. Thus,

$$\sqrt{\rho^2+\tau^2} = \rho \frac{u s_1}{c} = \rho \beta s_1 \quad (\beta = \frac{u}{c}).$$

Hence, $$A^{(1)}_{z\omega} = G'''(\omega)\, e^{-i\frac{\omega z}{u}}\, e^{\frac{\omega}{c} s_1 s_1 \beta \rho} \int_{-\infty}^{+\infty} d\tau \, \frac{e^{i\frac{\omega\tau}{u} - i\frac{\omega}{c} s_1 \sqrt{\rho^2+\tau^2}}}{\sqrt{\rho^2+\tau^2}}.$$

Here, $e^{-\frac{|\omega|}{c} s_1 s_2 \beta \rho}$ indicates the damping of the amplitude of the Čerenkov radiation field proper to a particular ω due to absorption. This term does not occur in Tamm's theory for the electromagnetic Čerenkov radiation because there he assumed absorption to be negligible.

Let $\tau = \rho \sinh\delta$, $\sqrt{\rho^2+\tau^2} = \rho\cosh\theta$, $d\tau = \rho\cosh\delta\, d\delta$.

$$A^{(1)}_{z\omega} = G'''(\omega)\, e^{-i\frac{\omega z}{u} + \frac{\omega}{c} s_1 s_2 \rho \beta} \int_{-\infty}^{+\infty} d\delta\, e^{i\frac{\omega}{u}\rho\sinh\delta - i\frac{\omega}{c} s_1 \rho\cosh\delta} \qquad (4.7)$$

Let $\frac{\omega}{u} = \rho\sinh\alpha$, $\frac{\omega}{c} s_1 = \rho\cosh\alpha$,

so that $\rho(\omega) = \frac{\omega}{c}\frac{s_1}{u}\sqrt{u^2 - \frac{c^2}{s_1^2}}$, $\frac{c}{s_1}$ and "u" are phase velocity of the field and velocity of the nucleon penetrating nucleon matter, respectively.

Let us suppose that there exists a range in ω for which the condition for the Čerenkov radiation is satisfied, i.e. for $\omega_0 \le \omega \le \omega_1$, $u > \frac{c}{s_1(\omega)}$. $\rho(\omega)$ is then <u>real</u>.

$$A''_{2\omega} = G''(\omega)\, e^{-i\frac{\omega z}{u}} + \frac{\omega}{c} S_1 S_2 \beta^2 \int_{-\infty}^{+\infty} ds\, e^{-i\beta\rho\cosh(s-\alpha)}$$

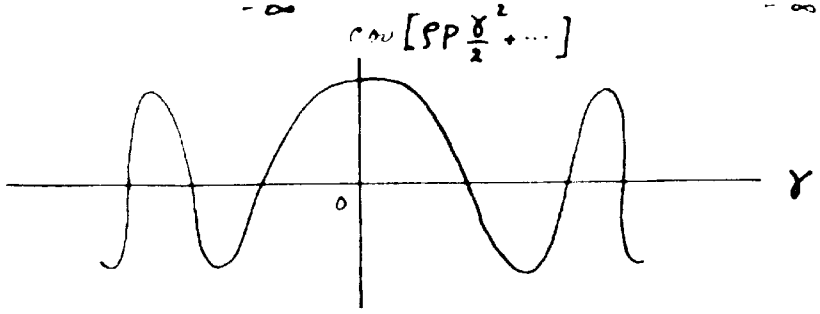
$$= G''(\omega)\, e^{-i\frac{\omega z}{u}} + \frac{\omega}{c} S_1 S_2 \beta^2 \int_{-\infty}^{+\infty} d\gamma\, e^{-i\beta\rho\cosh\gamma} \qquad (4.8)$$

$$= G''(\omega)\, e^{-i\frac{\omega z}{u}} + \frac{\omega}{c} S_1 S_2 \beta^2 \pi i\, H_0^{(2)}(\beta\rho)$$

$H_0^{(2)}(\beta\rho)$ is a Hankel function of the second kind.

By investigating the asymptotic behavior of the Fourier coefficient of the field for large ρ one can show that $A_{2\omega}$ represents a conical wave emanating from the instantaneous position of the nucleon, and moving with it.

$$\int_{-\infty}^{+\infty} d\gamma\, e^{-i\beta\rho\cosh\gamma} = \int_{-\infty}^{+\infty} d\gamma\, e^{-i\beta\rho(1+\frac{\gamma^2}{2}+\cdots)} = e^{-i\beta\rho}\int_{-\infty}^{+\infty} d\gamma\, e^{i\beta\rho(\frac{\gamma^2}{2}+\cdots)}$$



When $\cos\left[\beta\rho\frac{\gamma^2}{2}+\cdots\right]$ is integrated with respect to γ, the highly oscillating part cancels out, and only the region near the origin contributes to the integral. Hence, approximating the integrand $\cos\left[\beta\rho\frac{\gamma^2}{2}+\cdots\right]$ by $\cos\beta\rho\frac{\gamma^2}{2}$, we obtain:

$$\int_{-\infty}^{+\infty} d\gamma\, e^{-i\beta\rho\cosh\gamma} \cong e^{-i\beta\rho}\int_{-\infty}^{+\infty} d\gamma\cos\beta\rho\frac{\gamma^2}{2} = \sqrt{\frac{\pi}{\beta\rho}}\, e^{-i\beta\rho}$$

Here, we have made use of the integral $\int_{-\infty}^{+\infty} dx\cos x^2 = \sqrt{\frac{\pi}{2}}$.

From (4.5)

$$A^{(1)}_{z\omega}\, e^{i\omega t} \cong G^{(1)}(\omega)\sqrt{\frac{\pi}{\rho\rho}}\, e^{-\frac{\omega}{c}s_1 s_1 \rho\rho}\, e^{-i\frac{\omega z}{u} - i\rho\rho + i\omega t}$$

From $\cos\theta = \frac{c}{u s_1}$, $K_{real} = \frac{\omega}{c}s_1$, $\rho = \frac{\omega}{c}s_1\sqrt{1-\frac{c^2}{u^2 s_1^2}}$,

we obtain

$$\frac{\omega}{u} = K_{real}\cos\theta, \quad \rho = K_{real}\sin\theta.$$

Then, from Fig. 6, $\frac{|\omega| z}{u} + \rho\rho = K_{real}\cos\theta\, z + K_{real}\sin\theta\,\rho = (\vec{K}_{real}\cdot\vec{r})$.

Thus, we obtain:

$$A^{(1)}_{z\omega}\, e^{+i\omega t} \cong G^{(1)}(\omega)\sqrt{\frac{\pi}{\rho\rho}}\, e^{-\frac{|\omega|}{c}s_1 s_2 \rho\rho}\, e^{i(\omega t - \vec{K}_{real}\cdot\vec{r})},$$

where the vectors $\vec{K}_{real}$ and $\vec{r}$ are indicated in Fig. 6. Since this expression is independent of the angle around the z-axis, it represents a conical wave proper to a particular ω emanating from the nucleon and moving along with it.

(ii) <u>For $\omega < 0$</u>

We take $K = \frac{-|\omega|}{c}(s_1 + i s_2)$, from (3.29), and let $\frac{|\omega|}{u} = \rho\sin\alpha$.

$$A^{(1)}_{z\omega} = G^{(1)}(\omega)\, e^{-i\frac{\omega z}{u}}\, e^{-\frac{|\omega|}{c}s_1 s_2 \rho\rho} \int_{-\infty}^{+\infty} d\gamma\, e^{i\rho\rho\cosh\gamma} \tag{4.9}$$

$$= -G^{(1)}(\omega)\, e^{-i\frac{\omega z}{u}}\, e^{-\frac{|\omega|}{c}s_1 s_2 \rho\rho}\, \pi i\, H_0^{(1)}(\rho\rho)$$

$H_0^{(1)}$ is a Hankel function of the first kind.

(iii) <u>For regions $|\omega| > \omega_1$</u>, or <u>$\omega_0 > |\omega| \geq 0$</u>, $\rho(\omega)$ becomes pure imaginary. Letting $\rho(\omega) = -i\lambda(\omega)$, where $\lambda(\omega)$ is real,

$$\int_{-\infty}^{+\infty} d\gamma\, e^{-i\rho\rho\cosh\gamma} = \int_{-\infty}^{+\infty} d\gamma\, e^{-\lambda\rho\left(1+\frac{\gamma^2}{2}+\cdots\right)} \cong e^{-\lambda\rho}\int_{-\infty}^{+\infty} d\gamma\, e^{-\frac{\lambda\rho}{2}\gamma^2} = \sqrt{\frac{\pi}{\lambda\rho}}\, e^{-\lambda\rho}.$$

Therefore, for these ranges of frequency the Fourier coefficient of the field decays exponentially with ρ, so that there is no radiation at all.

Getting back to (4.5), from the previous discussions we may write:

$$\frac{y}{R} = \frac{\rho \sin\varphi}{R} = \sin\left(\frac{\pi}{2}-\theta\right)\sin\varphi = \cos\theta \sin\varphi = \frac{c}{u s_1}\sin\varphi$$

$$\frac{x}{R} = \frac{\rho\cos\varphi}{R} = \frac{c}{u s_1}\cos\varphi.$$

Hence, $$A^{(2)}_{z\omega} = \frac{1}{2\pi}\frac{1}{\kappa_0}\frac{ik}{u}\frac{c}{u s_1}(\sigma_x \sin\varphi - \sigma_y \cos\varphi)\int_{-\infty}^{+\infty} d\zeta\, e^{-i\frac{\omega}{u}\zeta}\frac{e^{-ikR}}{R}.$$

Proceeding similarly to (4.9), we have

$$A^{(2)}_{z\omega} = G^{(2)}(\omega)\, e^{-i\frac{\omega z}{u}} \begin{cases} e^{-\frac{\omega}{c}s_1 s_1 \beta\rho}\, \pi i\, H_0^{(2)}(\beta\rho) \\ \qquad \text{for } \omega_1 > \omega > \omega_0 \\ e^{-\frac{|\omega|}{c}s_1 s_1 \beta\rho}\, \pi i\, H_0^{(1)}(\beta\rho) \\ \qquad \text{for } -\omega_1 < \omega < -\omega_0, \end{cases} \tag{4.10}$$

where

$$G^{(2)}(\omega) = \frac{1}{2\pi}\frac{1}{\kappa_0}\frac{ik}{u}\frac{c}{u s_1}(\sigma_x \sin\varphi - \sigma_y \cos\varphi).$$

Combining (4.8), (4.9) and (4.10) we obtain:

$$A_{z\omega} = A^{(1)}_{z\omega} + A^{(2)}_{z\omega} = G(\omega)\, e^{-i\frac{\omega z}{u}}\, a(\rho(\omega),\beta), \tag{4.11}$$

where $$G(\omega) = G^{(1)}(\omega) + G^{(2)}(\omega)$$

$$a(\rho(\omega),\beta) = \begin{cases} 0, & \omega > \omega_1, \\ e^{-\frac{\omega}{c}s_1 s_2 \beta\rho}\, \pi i\, H_0^{(2)}(\beta\rho), & \omega_1 \geq \omega \geq \omega_0 \\ 0, & \omega_0 > \omega > -\omega_0, \\ e^{-\frac{|\omega|}{c}s_1 s_1 \beta\rho}\, \pi i\, H_0^{(1)}(\beta\rho), & -\omega_0 \geq \omega \geq -\omega_1 \\ 0, & \omega < -\omega_1. \end{cases} \tag{4.12}$$

The actual field is obtained by the inverse Fourier transformation:

$$A_z = \int_{-\infty}^{+\infty} d\omega\, A_{z\omega} e^{i\omega t} = \int_{-\infty}^{+\infty} d\omega\, G(\omega)\, e^{i\omega(t-\frac{z}{u})}\, a(\rho(\omega), \rho). \qquad (4.13)$$

2. Energy radiated per unit length of the path of the nucleon

The Poynting's vector for the phenomenological field is obtained from (2.22) with the scalar potential $\phi = 0$ (see page 55).

$$\vec{S} = \frac{c}{4\pi} [\vec{E} \times \vec{H}]$$

The energy emitted by the penetrating nucleon per unit length of its path through the nuclear matter is given by:

$$\frac{dW}{dl} = 2\pi\rho \int_{-\infty}^{+\infty} dt\, S_\rho = -\frac{\rho c}{2} \int_{-\infty}^{+\infty} dt\, E_z H_\varphi,$$

$$\text{or} \quad \frac{dW}{dl} = -\frac{\rho c}{4} \int_{-\infty}^{+\infty} dt\, [E_z H_\varphi^* + c.c.].$$

Away from the source E_z and H_φ are obtained from (2.17).

$$E_z = -\frac{1}{c}\frac{\partial}{\partial t} A_z, \quad H_\varphi = \frac{1}{\mu}\frac{\partial}{\partial \rho} A_z = \frac{1}{\eta^2}\frac{\partial}{\partial \rho} A_z$$

Introducing (4.11),

$$E_z = \frac{i}{c} \int_{-\infty}^{+\infty} d\omega\, \omega\, G(\omega)\, e^{i\omega(t-\frac{z}{u})}\, a(\rho(\omega), \rho)$$

$$H_\varphi^* = \int_{-\infty}^{+\infty} d\omega\, \frac{G^*(\omega)}{\eta^{2*}(\omega)}\, e^{-i\omega(t-\frac{z}{u})} \frac{\partial}{\partial \rho}\, a^*(\rho(\omega), \rho).$$

$$\frac{dW}{dl} = -\frac{i\beta}{4}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}d\omega\,\omega\,G(\omega)\,e^{i\omega(t-\frac{z}{u})}\,a(\rho(\omega),\beta)\int_{-\infty}^{+\infty}d\omega'\,\frac{G^*(\omega)}{n^{2*}(\omega')}\,e^{-i\omega'(t-\frac{z}{u})}\frac{\partial}{\partial\beta}a^*(\rho(\omega'),\beta) + c.c.$$

Integrating with respect to time first, using the formula

$$\int_{-\infty}^{+\infty} e^{i(\omega-\omega')t}\,dt = 2\pi\,\delta(\omega-\omega'),$$

$$\frac{dW}{dl} = -i\beta\frac{\pi}{2}\int_{-\infty}^{+\infty}d\omega\,\omega\,\frac{G(\omega)G^*(\omega)}{n^{2*}(\omega)}\,a(\rho(\omega),\beta)\frac{\partial}{\partial\beta}a^*(\rho(\omega),\beta) + c.c.$$

$$= -i\beta\frac{\pi}{2}\left[\int_0^{\infty}d\omega\cdot\omega\left\{\frac{G(\omega)G^*(\omega)}{n^{2*}(\omega)}\,a(\rho(\omega),\beta)\frac{\partial}{\partial\beta}a^*(\rho(\omega),\beta)\right\}\right.$$

$$\left. -\int_0^{\infty}d\omega\cdot\omega\left\{\frac{G(-\omega)G^*(-\omega)}{n^{2*}(-\omega)}\,a(\rho(-\omega),\beta)\frac{\partial}{\partial\beta}a^*(\rho(-\omega),\beta)\right\}\right]$$

$$+ c.c.$$

In view of the last equation in (3.20) for ζ, we find, from (3.28) that $n^{2*}(\omega) = n^2(-\omega)$.

$$G(\omega) = \frac{1}{2\pi}\frac{f}{k_0}\left\{n^2(\omega)\frac{\sigma_z}{c^2}i\omega + \frac{iK}{u}\frac{c}{uS_1}(\sigma_x\sin\varphi - \sigma_y\cos\varphi)\right\}$$

Dropping cross terms of the components of the spin vector $\vec{\sigma}$ in an anticipation of the averaging over the spin directions, we obtain:

$$G(\omega)G^*(\omega) = \frac{1}{4\pi^2}\frac{f^2}{k_0^2}\left\{n^2(\omega)n^{*2}(\omega)\frac{\overline{\sigma_z^2}}{c^4}\omega^2 + \frac{KK^*}{u^2}\frac{c^2}{u^2S_1^2}\left(\overline{\sigma_x^2}\sin^2\varphi + \overline{\sigma_y^2}\cos^2\varphi\right)\right\}$$

Thus, $G(\omega)G^*(\omega) = G(-\omega)G^*(-\omega)$.

$$\frac{dW}{d\ell} = i\rho\frac{\pi}{2}\int_0^\infty d\omega\,\omega\,G(\omega)G^*(\omega)\left[\frac{1}{n^2(\omega)}\left\{a^*(p(\omega),\rho)\frac{\partial}{\partial\rho}a(p(\omega),\rho) + \right.\right.$$

$$\left. + a(p(-\omega),\rho)\frac{\partial}{\partial\rho}a^*(p(-\omega),\rho)\right\} - \frac{1}{n^{2*}(\omega)}\left\{a^*(p(-\omega),\rho)\frac{\partial}{\partial\rho}a(p(-\omega),\rho) + \right.$$

$$\left.\left. + a(p(\omega),\rho)\frac{\partial}{\partial\rho}a^*(p(\omega),\rho)\right\}\right]$$

Obviously, from (4.12)

$$a(p(-\omega),\rho) = a^*(p(\omega),\rho)$$

Hence,

$$\frac{dW}{d\ell} = i\rho\frac{\pi}{2}\int_0^\infty d\omega\,\omega\,G(\omega)G^*(\omega)\left(\frac{1}{n^2(\omega)} + \frac{1}{n^{2*}(\omega)}\right)\times$$

$$\times\left[a^*(p(\omega),\rho)\frac{\partial}{\partial\rho}a(p(\omega),\rho) - a(p(\omega),\rho)\frac{\partial}{\partial\rho}a^*(p(\omega),\rho)\right]$$

Referring to (4.11),

$$\left[\quad\right] = \pi^2\left[e^{-\frac{\omega}{c}s_1s_2\beta\rho}H_0^{(1)}(\beta\rho)\frac{\partial}{\partial\rho}\left\{e^{-\frac{\omega}{c}s_1s_2\beta\rho}H_0^{(2)}(\beta\rho)\right\} - \right.$$

$$\left. - e^{-\frac{\omega}{c}s_1s_2\beta\rho}H_0^{(1)}(\beta\rho)\frac{\partial}{\partial\rho}\left\{e^{\frac{\omega}{c}s_1s_2\beta\rho}H_0^{(1)}(\beta\rho)\right\}\right]$$

$$= \pi^2 e^{-2\frac{\omega}{c}s_1s_2\beta\rho}\left\{H_0^{(1)}(\beta\rho)\frac{\partial}{\partial\rho}H_0^{(2)}(\beta\rho) - H_0^{(2)}(\beta\rho)\frac{\partial}{\partial\rho}H_0^{(1)}(\beta\rho)\right\}$$

Making use of the relations

$$H_0^{(1)} = J_0 + iN_0, \quad H_0^{(2)} = J_0 - iN_0,$$

$$\left\{ \quad \right\} = 2i \left\{ J_0(\rho s) \frac{\partial N_0(\rho s)}{\partial s} - N_0(\rho s) \frac{\partial J_0(\rho s)}{\partial s} \right\}$$

$$= -2i \left\{ J_0(\rho s) N_1(\rho s) - N_0(\rho s) J_1(\rho s) \right\}$$

$$= -\frac{4i}{\pi s}$$

Hence,

$$\frac{dW}{d\ell} = 2\pi^2 \int_{\omega_0}^{\omega_1} d\omega\, \omega\, \hat{f}(\omega) \hat{f}^*(\omega) \left(\frac{1}{n^2(\omega)} + \frac{1}{n^{2*}(\omega)} \right) e^{-2\frac{\omega}{c} s_1 s_2 \beta \rho}.$$

Introducing the values $\overline{\sigma_z^2} = \frac{1}{2}$, $\overline{\sigma_x^2 \sin^2\varphi} + \overline{\sigma_y^2 \cos^2\varphi} = \frac{1}{4}$,

$$\hat{f}(\omega)\hat{f}^*(\omega) = \frac{1}{4\pi^2} \frac{f^2}{\kappa_0^2} \frac{1}{2} \frac{\omega^2}{c^4} \left\{ n^2(\omega) n^{2*}(\omega) + \frac{1}{2\beta^4} \frac{s_1^2 + s_2^2}{s_1^2} \right\}.$$

Thus, we finally obtain:

$$\frac{dW}{d\ell} = \frac{1}{4} \frac{f^2}{\kappa_0^2} \frac{1}{c^4} \int_{\omega_0}^{\omega_1} d\omega\, \omega^3 \frac{n^2(\omega) + n^{2*}(\omega)}{n^2(\omega) n^{2*}(\omega)} \left\{ n^2(\omega) n^{2*}(\omega) + \frac{1}{2} \frac{1}{\beta^4} \frac{s_1^2 + s_2^2}{s_1^2} \right\} e^{-2\frac{\omega}{c} s_1 s_2 \beta \rho}. \tag{4.14}$$

For example, let us consider, from the results obtained on page 60, that the condition for the Čerenkov radiation is satisfied in the region of meson energy between $\hbar\omega_0 = 200$ Mev and $\hbar\omega_1 = 340$ Mev. We disregard $\frac{1}{2}\frac{1}{\beta^4}\frac{s_1^2 + s_2^2}{s_1^2}$ in comparison to $n^2(\omega) n^{2*}(\omega)$ since the latter is considerably larger than the former. We also assume that the index of refraction n does not change with the meson wave frequency in the energy range considered.

Thus, we have:

$$\frac{dW}{d\ell} = \frac{1}{4} \frac{f^2}{\kappa_0^2 c^4} (n^2 + n^{2*}) \int_{\omega_0}^{\omega_1} d\omega\, \omega^3 e^{-2\frac{\omega}{c} s_1 s_2 \beta \rho}.$$

---

[1] Jahnke and Emde, Tables of Functions, 4th Edition (1938)

Assuming that a nucleon penetrates through a heavy nucleus along its diameter with the velocity of light $(\beta = 1)$, ρ may-be taken as the average distance from the diameter to the surface. We take $\rho = \frac{1}{2}$ (radius of a heavy nucleus) $\simeq 5 \times 10^{-13}$ cm, $n^2 + n^{2k} \simeq 3.5$, $(S_1 S_2 \omega)_{av} \simeq 1.2 K_0$.

Then, the value of the exponential function is approximately:

$$e^{-2\omega S_1 S_2 \beta \rho} = \frac{1}{3}.$$

Multiplying $\frac{dW}{d\ell}$ by the diameter of a heavy nucleus $(\simeq 20 \times 10^{-13}$ cm$)$,

$$\frac{dW}{d\ell} \times 20 \times 10^{-13} \simeq \frac{f^2}{2\hbar^2} \frac{(\hbar\omega_1)^4 (\hbar\omega_2)^4}{(\mu_1 c^2)^3}$$

$$\simeq 5 \mu_0 c^2 \quad \text{for} \quad \frac{f^2}{\hbar^2} \sim 0.3.$$

This corresponds to the multiplicity of 2-3 at 250 Mev for each meson, which multiplied by the geometrical area of the heavy nucleus, gives 10^{-23} cm^2 as the order of magnitude of the cross-section of the meson production.

Appendix I

Lagrangian Formalism for the Phenomenological Field

The phenomenological Proca equations (2.15) and (2.16) are in a relativistically invariant representation. However, the equations in (2.19) hold only in a particular Lorentz frame of reference in which dispersive matter stays at rest. Consequently these equations are not independent of the frame of reference, and, therefore, not Lorentz-invariant.

In order to make them completely independent of the frame of reference we introduce a 4-vector velocity v_α referred to the matter. This 4-vector has the property that square of its length is -1, or $v_\alpha v^\alpha = -1$. Then, it can be proven that the following Lorentz-invariant relationships exist between the two tensors $\mathcal{F}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$[1]:

$$\mathcal{F}_{\alpha\beta} = \mu \mathcal{H}_{\alpha\beta} + (\epsilon\mu - 1)\{v_\alpha \sigma_\beta - v_\beta \sigma_\alpha - \epsilon_{\alpha\beta\gamma\delta} v^\gamma \sigma^\delta\}$$
$$- \epsilon_{\alpha\beta\gamma\delta} v^\gamma \sigma^\delta + (\epsilon\mu - 1)\{\mathcal{H}_{\beta\sigma} v^\sigma v_\alpha - \mathcal{H}_{\alpha\sigma} v^\sigma v_\beta\}. \tag{A1.1}$$

Also it can be proven that following covariant relationships exist between two 4-vectors $\mathcal{D}^\alpha$ and $\mathcal{I}^\alpha$:

$$\mathcal{D}^\alpha = \epsilon \mathcal{E}^\alpha + (\epsilon - 1) v^\alpha v_\beta \mathcal{E}^\beta$$
$$\mathcal{B}^\alpha = \mu \mathcal{H}^\alpha + (\mu - 1) v^\alpha v_\beta \mathcal{H}^\beta \tag{A1.2}$$

In all these equations the material constants ϵ, μ, and σ are assumed to be invariant constants. One may easily obtain

---

[1]This is the analog of the corresponding invariant relationships in the electromagnetic theory. See H. Minkowski, Göttinger Nachrichten, p. 53 (1908); Math. Ann. 68 (1910). Also, J.M. Jauch and K.M. Watson, Phys. Rev. 74, 950 (1948)

equation in (2.19) by going to the rest frame of reference for the matter. In order to do so we set $v_\ell = v^\ell = 0 \ (\ell = 1, 2, 3)$, $v_0 = -v^0 = 1$.

Let us postulate the following invariant Lagrangian density for the source free phenomenological Proca field:

$$\mathcal{L} = -\frac{\kappa_0^2}{2}\left\{\bar{\Phi}_\beta \bar{\Psi}^\beta + \frac{1}{2}\Omega_{\alpha\beta}\Sigma^{\alpha\beta}\right\} \qquad \text{(A1.3)}$$

This Lagrangian density reduces to that of the vacuum field[1] when the constants ϵ, μ, ν and σ are all set to unity.

The Lagrangian function is defined by

$$L = \int \mathcal{L}\, dx_0\, dx_1\, dx_2\, dx_3 .$$

Then, the variational principle $\delta \int L\, dx^0 = 0$ leads immediately to the Euler-Lagrangian equation

$$\frac{\partial \mathcal{L}}{\partial \bar{\Phi}_\beta} - \partial^\alpha \frac{\partial \mathcal{L}}{\partial(\partial^\alpha \bar{\Phi}_\beta)} = 0 .$$

Expressing the above Lagrangian density function in terms of $\bar{\Phi}_\alpha$ by means of the field equation (2.15) and (A1.1) and (A1.2) we obtain

$$\frac{\partial \mathcal{L}}{\partial \bar{\Phi}_\beta} = -\kappa_0^2 \bar{\Psi}^\beta , \qquad \frac{\partial \mathcal{L}}{\partial(\partial^\alpha \bar{\Phi}_\beta)} = -\kappa_0 \Omega_\alpha{}^\beta ,$$

which, when introduced to the Euler-Lagrangian equation, gives us

$$\partial^\alpha \Omega_{\alpha\beta} = \kappa_0^2 \bar{\Psi}_\beta .$$

This is our phenomenological Proca equations (2.15), thus verifying our Lagrangian density (A1.3).

The stress-energy density tensor is related to the

---

[1] N. Kemmer, loc. cit.

Lagrangian density function by the equation[1]

$$T_{\alpha\beta} = -(\partial_\beta \bar{\Phi}_\gamma)\frac{\partial \mathcal{L}}{\partial(\partial^\alpha \bar{\Phi}_\gamma)} + \delta_{\alpha\beta}\mathcal{L}. \qquad \text{(A1.4)}$$

The conservation laws for energy and momentum are given by the vanishing divergence of this stress-energy tensor.

$$\partial^\alpha T_{\alpha\beta} = -(\partial_\beta \bar{\Phi}_\delta)\partial^\alpha \frac{\partial \mathcal{L}}{\partial(\partial^\alpha \bar{\Phi}_\delta)} + \delta_{\alpha\beta}\frac{\partial \mathcal{L}}{\partial \bar{\Phi}_\sigma}\partial^\alpha \bar{\Phi}^\sigma$$

$$= -(\partial_\beta \Phi_\sigma)\left\{\partial^\alpha \frac{\partial \mathcal{L}}{\partial(\partial^\alpha \Phi_\sigma)} - \frac{\partial \mathcal{L}}{\partial \bar{\Phi}_\sigma}\right\} = 0$$

The equation (A1.4) gives us the following expression for the stress-energy tensor of the source-free field:

$$T_{\alpha\beta} = \kappa_0^2\left[\bar{\Psi}_\alpha \bar{\Phi}_\beta + \Omega_\alpha{}^\sigma \Sigma_{\beta\sigma} - \frac{1}{2}\delta_{\alpha\beta}\left(\Phi_\gamma \bar{\Psi}^\gamma + \frac{1}{2}\Omega_{\sigma\rho}\Sigma^{\sigma\rho}\right)\right] \qquad \text{(A1.5)}$$

In the presence of source we obtain the Lagrangian density function by replacing $\bar{\Psi}^\gamma$ by $\left(\bar{\Psi}^\gamma - \frac{1}{\kappa_0}S^\gamma\right)$ and $\Sigma^{\sigma\rho}$ by $\left(\Sigma^{\sigma\rho} + \frac{1}{\kappa_0}H^{\sigma\rho}\right)$ in the Lagrangian (A1.3), according to the equations (2.15). This new Lagrangian will be separated into three parts, first one which is independent of the source functions ($\mathcal{L}_0$), the second linearly dependent on them ($\mathcal{L}_1$) and the third quadratic in source functions. The last one will be disregarded.

$$\mathcal{L}_0 = -\frac{\kappa_0^2}{2}\left\{\Phi_\gamma \bar{\Psi}^\gamma + \frac{1}{2}\Omega_{\sigma\rho}\Sigma^{\sigma\rho}\right\}$$

$$\mathcal{L}_1 = g\,\bar{\Phi}_\gamma S^\gamma + \frac{1}{2}\Omega_{\sigma\rho}\Sigma^{\sigma\rho} \qquad \text{(A1.6)}$$

Substituting $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ into (A1.4) and taking the divergence,

---

[1] G. Wentzel, Einführung in die Quantentheorie der Wellen Felder, p. 9, Wien, Franz Denticke (1943)

we obtain:

$$\partial^\alpha T_{\alpha\beta} = \kappa_0^2 \left[S^\sigma \Sigma_{\beta\sigma} + \frac{1}{2\kappa_0} \Lambda^{\rho\sigma} \left\{ \partial_\rho M_{\sigma\rho} + \partial_\rho M_{\beta\sigma} + \partial_\sigma M_{\rho\beta} \right\} - \Psi_\alpha M_{\alpha\beta} \right].$$

For $\beta \neq o$, referring to (2.13) and (2.18), we obtain the general force law:

$$\vec{F} = 4\pi \left[\Lambda \vec{E} + \frac{1}{c} [\vec{J} \times \vec{B}] + \frac{1}{\kappa_0} \left\{ \vec{D} \times \left(\operatorname{curl} \vec{N} + \frac{1}{c} \frac{\partial \vec{M}}{\partial t} \right) - \vec{H} \operatorname{div} \vec{M} \right\} \right.$$

$$\left. + \kappa_0 \left\{ [\vec{A} \times \vec{M}] - 4\vec{N} \right\} \right].$$

For $\beta = o$, the general energy law will be obtained, which is already stated in (2.22). The work performed by the field per unit time is given by

$$\partial^\alpha T_{\alpha o} = -(\vec{E} \cdot \vec{J}) - \kappa_0 c \, (\vec{A} \cdot \vec{N}) - \frac{c}{\kappa_0} \left(\vec{H} \cdot \left(\operatorname{curl} \vec{N} + \frac{1}{c} \frac{\partial \vec{M}}{\partial t} \right) \right).$$

Appendix II

Equations in Hoek's Method

Refer to Chapter III, Section 2.

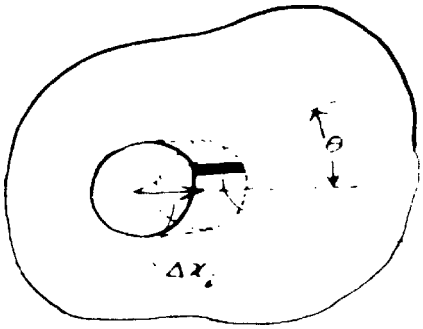
Let $I(r) = \int_{S(r)} K(r,r') F(r') dr'$, where $K(r,r') = \frac{e^{-\alpha \rho}}{\rho}$,

and F(r') is an analytic function of r'.

Then, $$\frac{\partial I(r)}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{I(x_1, \ldots, x_i + \Delta x_i, \ldots) - I(x_i)}{\Delta x_i}$$

$$= \int_{S(r)} dr \frac{\partial K(r,r')}{\partial x_i} f(r') - \int_{S(r)} d\sigma K(r,r') f(r') \nu_i ,$$

where ν_i is a unit vector along ρ from $\dot{r}$ to $\dot{r}'$ (see Fig. 2, page 45), and $\int_{S(r)} d\sigma$ signifies the surface integration over the small sphere s(r). The derivation of the above formula follows from the consideration that the center of the sphere is shifted by Δx_i, whereby a small volume is taken out on the right side of the sphere and added on the left side.



$dv = d\sigma \, \Delta x_i \cos\theta$

Fig. 7

Infinitesimal Displacement of s(r)

Since $|\vec{r}' - \dot{r}| = a$ is small when $\dot{r}'$ refers to a point on the surface of s(r), we may expand F(r) in the vicinity of $\dot{r}$ and retain the first two terms.

$$F(r') = F(r) + a \nu_i \frac{\partial F}{\partial x_i} + \cdots , \quad \text{where } \vec{\nu} = \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|}, \ \nu_i = \frac{x_i' - x_i}{\rho} .$$

Hence,

$$\int_{S(r)} d\sigma\, \nu_i\, R(r,r')\, F(r') = R(a) \int_{S(r)} d\sigma\, \nu_i \left[F(r) + a\, \nu_\alpha \frac{\partial F(r)}{\partial x_\alpha} \right].$$

Note the following equations:

$$\overline{\nu_i} = \frac{1}{4\pi a^2} \int_{S(r)} \nu_i\, d\sigma = 0\,, \qquad \overline{\nu_i\, \nu_j} = \frac{1}{4\pi a^2} \int_{S(r)} \nu_i\, \nu_j\, d\sigma = \frac{1}{3} \delta_{ij}$$

$$\overline{\nu_i\, \nu_j\, \nu_k} = 0\,, \qquad \overline{\nu_i\, \nu_j\, \nu_k\, \nu_\ell} = \frac{1}{15} (\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$$

We now obtain:

$$\lim_{a \to 0} \int_{S(r)} d\sigma\, \nu_i\, R(r,r')\, F(r') = \lim_{a \to 0} a R(a) \frac{\partial F(r)}{\partial x_\alpha} 4\pi a^2 \frac{1}{3} \delta_{i\alpha} \to 0$$

Hence,

$$\frac{\partial}{\partial x_i} \int_{S(r)}^{V} dr'\, R(r,r')\, F(r') = \int_{S(r)}^{V} dr'\, \frac{\partial R(r,r')}{\partial x_i} F(r').$$

Repeating the above processes,

$$\frac{\partial^2}{\partial x_i \partial x_j} \int_{S(r)}^{V} dr'\, R(r,r')\, F(r') = \frac{\partial}{\partial x_j} \int_{S(r)}^{V} dr'\, \frac{\partial R(r,r')}{\partial x_i} F(r')$$

$$= \int_{S(r)}^{V} dr'\, \frac{\partial^2 R(r,r')}{\partial x_i \partial x_j} F(r') - \int_{S(r)} d\sigma\, \nu_j\, \frac{\partial R(r,r')}{\partial x_i} F(r')$$

$$= \int_{S(r)}^{V} dr'\, \frac{\partial^2 R(r,r')}{\partial x_i \partial x_j} F(r') + \left[\frac{\partial R(\rho)}{\partial \rho} \right]_{\rho = a} \int_{S(r)} d\sigma\, \nu_i\, \nu_j \left[F(r) + a\, \nu_\alpha \frac{\partial F(r)}{\partial x_\alpha} + \cdots \right]$$

$$= \int_{S(r)}^{V} dr'\, \frac{\partial^2 R(r,r')}{\partial x_i \partial x_j} F(r') - \frac{4\pi}{3} F(r)\, \delta_{ij}.$$

Here, we made use of the equation:

$$\frac{\partial R(r,r')}{\partial x_i} = \frac{\partial R}{\partial \rho} \frac{\partial \rho}{\partial x_i} = \frac{\partial R}{\partial \rho} \frac{x_i - x_i'}{\rho} = -\nu_i \frac{\partial R}{\partial \rho}.$$

From this development we obtain the equations (3.12a)-(3.12d) inclusive.

Appendix III

Treatment of the Phenomenological Gordon-Klein Wave Equation without Dispersion under the Čerenkov Condition

The content of this appendix is a different treatment of the wave equation with a point source moving along z-axis under the Čerenkov condition when dispersion effect is disregarded. For the sake of simplicity we take a scalar wave equation of the following type:

$$\left(\Delta - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} - n^2 k_0^2\right)\psi = 4\pi\,\delta(x)\delta(y)\delta(z-ut), \qquad \text{(A3.1)}$$

where n is a constant. This equation differs from (4.1) in that the source function does not involve differential operators.

Let us now introduce notations $\frac{c}{n}t = \tau$, $nk_0 = \kappa$, and first solve the homogenous equation

$$(\Box - \kappa^2)\psi = 0, \qquad \text{(A3.2)}$$

where $\Box$ stands for the 4-dimensional Laplacian in x, y, z, and τ. Let r be the 4-dimensional distance between an arbitrary point (x', y', z', τ') and a fixed point (x, y, z, τ), and assume ψ (x, y, z, τ; x', y', z', τ') be symmetric in the 4-dimensional space.

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2 - c^2(\tau-\tau')^2}$$

Let $\psi = f(r)$. Then, $\psi_\mu = f' \frac{x_\mu}{r}$, $\psi_{\mu\nu} = f''\frac{x_\mu x_\nu}{r^2} + f'\left(\frac{\delta_{\mu\nu}}{r} - \frac{x_\mu x_\nu}{r^3}\right)$. Then the homogeneous equation (A3.2) becomes:

$$f'' + \frac{3}{r}f' - \kappa^2 f = 0 \qquad \text{(A3.3)}$$

We transform the dependent and independent variables as follows:

$$f(r) = \frac{\varphi}{r}, \quad \xi = i\kappa r$$

Then, the equation (A3.3) transforms to the Bessel differential equation of the first order:

$$\frac{d^2g}{d\xi^2} + \frac{1}{\xi}\frac{dg}{d\xi} + \left(1 - \frac{1}{\xi^2}\right) g = 0.$$

Hence, $g(\xi) = J_1(\xi)$, and we have:

$$\psi_1(r) = \frac{J_1(-i\kappa r)}{r}.$$

Using this function as the Green's function for the inhomogeneous equation (A3.1) we obtain its solution:

$$\psi(x,y,z,t) = \frac{\kappa}{2}\iiint_{-\infty}^{+\infty} dx'dy'dz'dt' S(x')S(y')S(z'+\beta n r')\frac{J_1(-i\kappa r)}{r}. \tag{A3.4}$$

Performing the integrations with respect to x', y' and z' first and noting that with $t' = i\lambda'$, $\int_{-\infty}^{+\infty} dt' = -i\int_{-i\infty}^{+i\infty} d\lambda'$, we have:

$$\psi(x,y,z,t) = -i\frac{\kappa}{2}\int_{-i\infty}^{+i\infty} d\lambda' \frac{J_1(-i\kappa r)}{r}\,^{1} \tag{A3.5}$$

where

$$r = \sqrt{x^2+y^2+(z-\beta n\lambda')^2-(t-\lambda')^2}.$$

Let two values of λ' which make r vanish be denoted by λ_1, and λ_2. Then,

$$r = \sqrt{\beta^2n^2-1}\,\sqrt{(\lambda'-\lambda_1)(\lambda'-\lambda_2)}$$

where

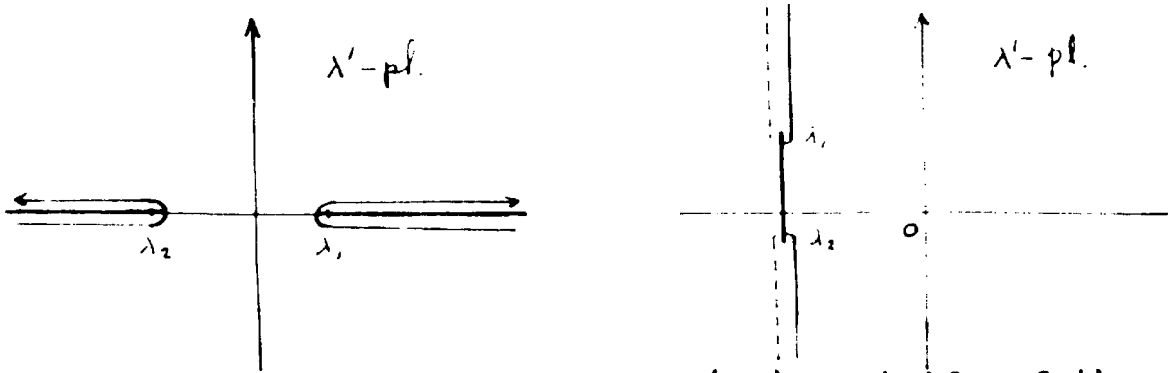
$$\lambda_{1,2} = \frac{1}{\beta^2n^2-1}\left\{z\beta n - t \pm \sqrt{(z-t\beta n)^2-\rho^2(\beta^2n^2-1)}\right\}.$$

Therefore, the integrand in (A3.5) has in general two branch points in the complex λ'-plane, and depending upon whether the

---

[1] The coefficient $\kappa/2$ is obtained from an analogous treatment of the equation (A3.1) in Courant-Hilbert II, p. 443

point of observation lies outside or inside the electromagnetic Čerenkov[1] cone the branch lines assume the following two different positions:



(i) inside of the electromagnetic Čerenkov cone $(ut-z)^2-\rho^2(\beta^2n^2-1)>0$

(ii) outside of the cone $(ut-z)^2-\rho^2(\beta^2n^2-1)<0$

Fig. 8

Contour integrations to solve the wave equation under the Čerenkov condition

(i) The complex integration along the entire imaginary axis can be replaced by one of the two integration paths along the real axis indicated, depending upon whether one wants advanced or retarded potential. For the retarded potential we take the path on the left. Since the integration along the entire path must be twice the integration along one branch of the path, e.g. from $-\infty$ to λ_2, we have:

$$\phi = -i\kappa \int_{-\infty}^{\lambda_2} d\lambda' \frac{J_1(-i\kappa r)}{r} . \qquad \text{(A3.6)}$$

Let us now perform the integration. First, we write r as:

$$r = \sqrt{\beta^2n^2-1}\,\sqrt{\zeta^2-\gamma^2}$$

where $$\zeta = \lambda' + \frac{\lambda - z\beta n}{\beta^2n^2-1} , \quad \gamma = \frac{\sqrt{(z-\lambda\beta n)^2-\rho^2(\beta^2n^2-1)}}{\beta^2n^2-1} .$$

---

[1] Ig. Tamm, loc. cit.

Then, (A3.6) transforms to:

$$\phi(\rho,z,\lambda) = -\frac{i\kappa}{\sqrt{\beta^2n^2-1}}\int_{-\infty}^{-\gamma} d\zeta\, \frac{J_1(-i\kappa\sqrt{\beta^2n^2-1}\sqrt{\zeta^2-\gamma^2})}{\sqrt{\zeta^2-\gamma^2}} = -\frac{e^{i\kappa\sqrt{\beta^2n^2-1}\,\gamma}-1}{(\beta^2n^2-1)\,\gamma}.$$

Hence, finally,

$$\phi(\rho,z,\lambda) = -\frac{e^{i\kappa\sqrt{\frac{(ut-z)^2}{\beta^2n^2-1}-\rho^2}}-1}{\sqrt{\beta^2n^2-1}\,\sqrt{\frac{(ut-z)^2}{\beta^2n^2-1}-\rho^2}}.$$

According to Tamm[2] the potential of the electromagnetic Čerenkov radiation is $\frac{1}{\sqrt{(ut-z)^2-\rho^2(\beta^2n^2-1)}}$. Thus, we find that ϕ consists of two parts, one purely electromagnetic type and the other non-electromagnetic type.

When we compare (A3.4) with the general solution of the same type of equation given by (2.32), we find that ϕ corresponds to the second part of the general solution, the first part being purely electromagnetic type. However, since in the limit $k_0 \to 0$, the function ϕ should reduce to that of a massless field, the part of ϕ which is purely electromagnetic type must cancel with the part which is electromagnetic type in the general solution for (A3.1). Thus, we obtain as the solution to our problem:

$$\phi(\rho,z,t) = -\frac{e^{i\kappa\sqrt{\frac{(ut-z)^2}{\beta^2n^2-1}-\rho^2}}}{\sqrt{(ut-z)^2-\rho^2(\beta^2n^2-1)}} \qquad \text{(A3.7)}$$

---

[1]For this integration see E. Schrödinger, Proc. Roy. Irish Acad. 47, 1 (1941)

[2]Ig. Tamm, loc. cit.

(ii) For this case, let the branch cut be the line $\overline{\lambda_1 \lambda_2}$. The integration along the entire imaginary axis may be replaced by one of the two paths indicated, where dotted lines are the paths on the lower Riemann surface. Taking the lower path for the retarded potential, we obtain:

$$\phi = -i\mathcal{K} \int_{-i\infty}^{\lambda_2} d\lambda' \frac{J_1(-i\mathcal{K}r)}{r} .$$

When integration is performed in the manner similar to that in (i), ϕ turns out to be the same as (A3.7) except the imaginary number i in the exponential function disappears. Thus, outside of the electromagnetic Čerenkov cone the field decays exponentially.

Let us summarize the behaviors of the field in three different cases:

(i) inside the electromagnetic Čerenkov cone $(ut - z > \rho\sqrt{\beta^2 n^2 - 1})$, ϕ is oscillatory, the surfaces of constant phase being represented by the family of curves in the diagram:

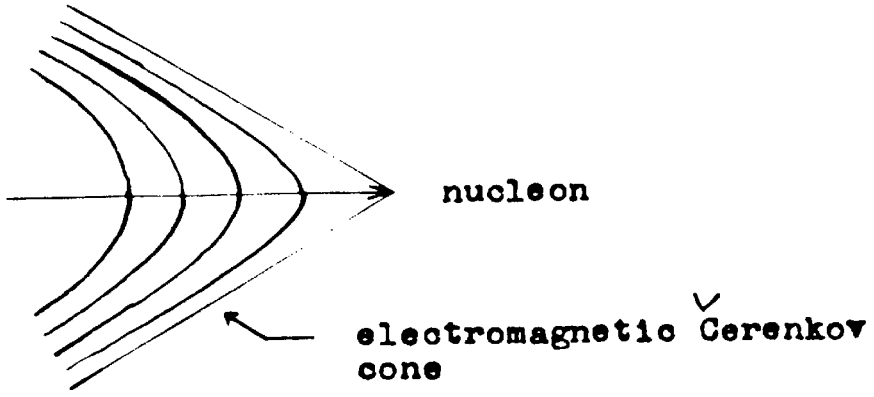


Fig. 9

Surfaces of constant phase

(ii) outside the cone ($ut - z < \rho\sqrt{\beta^2 n^2 - 1}$), ϕ decays exponentially and also is imaginary.

(iii) on the cone ($ut - z = \rho\sqrt{\beta^2 n^2 - 1}$), ϕ is infinite.

We shall now obtain a Fourier integral representation of this function. In order to do so we introduce notations

$$\frac{ut - z}{\sqrt{\beta^2 n^2 - 1}} = Z, \quad i\rho = r, \quad R^2 = Z^2 + r^2, \quad i\mathcal{K} = \mathcal{K}_0, \quad \frac{-n}{\sqrt{\beta^2 n^2 - 1}} = g.$$

Then, ϕ becomes $\phi = -g\,\frac{e^{\mathcal{K}_0 R}}{R}$, which according to Sommerfeld[1] has the following integral representation:

$$\phi = -g\int_0^\infty \lambda\, d\lambda\, \frac{J_0(\lambda r)}{\sqrt{\lambda^2 + \mathcal{K}_0^2}}\, e^{-\sqrt{\lambda^2 + \mathcal{K}_0^2}\, Z} \qquad \text{for } Z > 0.$$

Using relations $J_0(\lambda r) = \frac{1}{2}\left\{H_0^{(1)}(\lambda r) + H^{(2)}(\lambda r)\right\}$ and $H_0^{(1)}(\lambda r) = -H_0^{(2)}(\lambda r)$,

$$\phi = -\frac{g}{2}\int_{-\infty}^{+\infty} \lambda\, d\lambda\, \frac{H_0^{(1)}(\lambda r)}{\sqrt{\lambda^2 + \mathcal{K}_0^2}}\, e^{-\sqrt{\lambda^2 + \mathcal{K}_0^2}\, Z} \qquad \text{(A3.8)}$$

This integral has branch points at $\lambda = \pm i\mathcal{K}_0$. Hence, the path of integration along the entire real axis may be replaced by either one of the two paths indicated in the complex λ -plane.

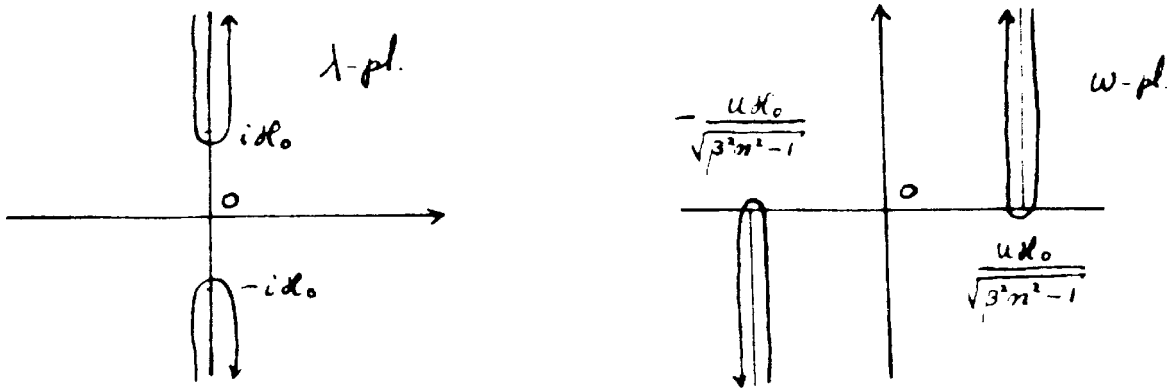


Fig. 10
Field represented by complex integration

---

[1] A. Sommerfeld, Ann. der Phys. 28, 683 (1908-09)

Let us make a substitution $\lambda^2 + \kappa_0^2 = \left(\frac{\beta^2 n^2 - 1}{u}\right)\omega^2$, from which follow $\lambda = \pm\sqrt{\left(\frac{\beta^2 n^2 - 1}{u^2}\right)^2 \omega^2 - \kappa_0^2}$, $\lambda\, d\lambda = \frac{\beta^2 n^2 - 1}{u^2}\omega\, d\omega$, and $\sqrt{\lambda^2 + \kappa_0^2} = -\sqrt{\frac{\beta^2 n^2 - 1}{u^2}}\,\omega$. Note that the branch points $\pm i\kappa_0$ in λ -plane both go over to the origins in ω -plane, one on the upper Riemann surface and the other on the lower surface. The two integration paths indicated in λ -plane for the integral (A3.8) individually go over to the integration path from $-i\infty$ to $+i\infty$ in ω -plane. Thus, we obtain:

$$\phi = \frac{\pi}{2u}\int_{-i\infty}^{+i\infty} d\omega\; H_0^{(1)}\left(\pm\sqrt{\frac{\beta^2 n^2 - 1}{u^2}\omega^2 - \kappa_0^2}\, r\right) e^{\frac{\sqrt{\beta^2 n^2 - 1}}{u}\omega Z}$$

The integrand has two branch points at $\pm\frac{u\kappa_0}{\sqrt{\beta^2 n^2 - 1}}$. Therefore, the integration along the entire imaginary axis maybe replaced by either one of the two paths indicated in ω -plane. Note that these two branch points go over to the origins in λ -plane, one on the upper Riemann surface and the other on the lower surface.

$$\phi_1 = \frac{\pi}{2u}\int_{\cap} d\omega\; H_0^{(1)}\left(\sqrt{\frac{\beta^2 n^2 - 1}{u^2} - \kappa_0^2}\, r\right) e^{\frac{\sqrt{\beta^2 n^2 - 1}}{u}\omega Z}$$

$$\phi_2 = \frac{\pi}{2u}\int_{\cup} d\omega\; H_0^{(1)}\left(-\sqrt{\frac{\beta^2 n^2 - 1}{u^2} - \kappa_0^2}\, r\right) e^{\frac{\sqrt{\beta^2 n^2 - 1}}{u}\omega Z}$$

The integration along the path $\cap$ for ϕ_1 must be twice the integration along one branch of the path, i.e. from $-i\infty$ to $\frac{-u\kappa_0}{\sqrt{\beta^2 n^2 - 1}}$. Similarly, the integration along $\cup$ for ϕ_2 must be twice the integration from $\frac{u\kappa_0}{\sqrt{\beta^2 n^2 - 1}}$ to $i\infty$.

Now, let $\omega = i\omega'$, and recall $\kappa_0 = i\kappa = ink_0$, $i\zeta = r$,

$$Z = \frac{ut - z}{\sqrt{\beta^2 n^2 - 1}}$$

$$\phi_1(\rho,z,t)=\frac{i\pi}{u}\int_{-\infty}^{\frac{-u\kappa}{\sqrt{\beta^2m^2-1}}}d\omega\ H_0^{(1)}\left(\sqrt{\frac{\beta^2m^2-1}{u^2}\omega^2-\kappa^2}\,\rho\right)e^{i\omega\left(t-\frac{z}{u}\right)},\quad \omega\le-\frac{u\kappa}{\sqrt{\beta^2m^2-1}}$$

$$\phi_2(\rho,z,t)=-\frac{i\pi}{u}\int_{\frac{u\kappa}{\sqrt{\beta^2m^2-1}}}^{\infty}d\omega\ H_0^{(2)}\left(\sqrt{\frac{\beta^2m^2}{u^2}\omega^2-\kappa^2}\,\rho\right)e^{i\omega\left(t-\frac{z}{u}\right)},\quad \omega\ge\frac{u\kappa}{\sqrt{\beta^2m^2-1}}$$

From the top of page 69,

$$\sqrt{\frac{\beta^2m^2-1}{u^2}\omega^2-\kappa^2}=p(\omega),\qquad \frac{u\kappa}{\sqrt{\beta^2m^2-1}}=\omega_0$$

Hence,

$$\phi(\rho,z,t)=\frac{1}{u}\int_{-\infty}^{+\infty}d\omega\ e^{i\omega\left(t-\frac{z}{u}\right)}a(\rho,p(\omega)),$$

where

$$a(\rho,S(\omega))=\begin{cases}-\pi i\,H_0^{(2)}(p\rho), & \omega\ge\omega_0\\ 0, & \omega_0>\omega>-\omega_0\\ \pi i\,H_0^{(1)}(p\rho), & \omega\le-\omega_0.\end{cases}$$

Agreement between this result and the result in (4.12) can easily be understood.

Lastly, we give an entirely different treatment of the equation (A3.1) in the following.

Let $\frac{c}{\eta}=c_1$, and introduce a notation $(x,y,z,ic_1t=x_4)$, $\alpha=1,2,3,4$. Then (A3.1) becomes:

$$\left(\frac{\partial^2}{\partial x_\alpha\,\partial x_\alpha}-\eta^2\kappa_0^2\right)\phi=-4\pi\,\delta(x)\delta(y)\delta(z-ut)\qquad\text{(A3.9)}$$

We would like to consider a certain rotational coordinate transformation, accoording to which in the new coordinate system the source lies at the origin. Let the transformation be written as

$$x_k'=a_{k\alpha}x_\alpha,$$

where $a_{\kappa j}$ is represented by the following orthogonal matrix

$$a_{\kappa j} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-i}{\sqrt{\beta^2 n^2 - 1}} & \frac{\beta n}{\sqrt{\beta^2 n^2 - 1}} \\ 0 & 0 & \frac{-\beta n}{\sqrt{\beta^2 n^2 - 1}} & \frac{-i}{\sqrt{\beta^2 n^2 - 1}} \end{pmatrix}.$$

One may easily verify the orthognality conditions:

$$a_{\alpha\kappa}\, a_{\alpha j} = a_{\kappa\alpha}\, a_{j\alpha} = \delta_{\kappa j}.$$

The Gordon-Klein operator remains invariant, because:

$$\frac{\partial^2}{\partial x_\alpha' \partial x_\alpha'} = a_{\alpha\ell}\, a_{\alpha j} \frac{\partial^2}{\partial x_\ell \partial x_j} = \delta_{\ell j} \frac{\partial^2}{\partial x_\ell \partial x_j} = \frac{\partial^2}{\partial x_\alpha \partial x_\alpha}.$$

According to this transformation we obtain:

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = \frac{-i}{\sqrt{\beta^2 n^2 - 1}} \left(x_3 + \frac{i u}{c} x_4 \right) = \frac{-i}{\sqrt{\beta^2 n^2 - 1}} (z - ut).$$

Thus, if we divide the equation (A3.1) $\frac{1}{\lambda_j + i\sqrt{\beta^2 n^2 - 1}}$ and replace $\delta(z - ut)$ by $\delta\left(\frac{z - ut}{i\sqrt{\beta^2 n^2 - 1}} \right)$, which must certainly be permissible, we obtain the following wave equation

$$\left(\frac{\partial^2}{\partial x_\alpha \partial x_\alpha} - n^2 k_0^2 \right) \phi' = -4\pi\, \delta(x_1')\, \delta(x_2')\, \delta(x_3'),$$

where

$$\phi' = \frac{\phi}{i\sqrt{\beta^2 n^2 - 1}}.$$

However, this is a wave equation for the case where a uniformly charged x_4'-axis constitutes the source. Hence, ϕ' is the same as the solution of the equation

$$\left(\underset{(x,y,z)}{\Delta} - n^2 k_0^2 \right) \psi = -4\pi\, \delta(x)\, \delta(y)\, \delta(z).$$

Thus, $\phi' = \frac{e^{ik_0 r}}{r}$, where $r = \sqrt{x_1'^2 + x_2'^2 + x_3'^2}$.

In terms of the original coordinates

$$r = \sqrt{x^2 + y^2 + \frac{(z-ut)^2}{1-\frac{u^2}{c_1^2}}} = i\sqrt{\frac{(z-ut)^2}{\beta^2 m^2 - 1} - \rho^2}.$$

Hence,

$$\phi = -\frac{e^{ik_0\sqrt{\frac{(ut-z)^2}{\beta^2 m^2 - 1} - \rho^2}}}{\sqrt{(ut-z)^2 - \rho^2(\beta^2 m^2 - 1)}}.$$

This agrees with (A3.7)

Bibliography

1. H.J. Bhabha, Proc. Roy. Soc. A172, 384 (1939); Proc. Roy. Soc. A173, 314 (1941)

2. H.J. Bhabha and H.C. Corben, Proc. Roy. A173, 106 (1941)

3. P.A. Čerenkov, Comptes Rendus de l'Acad. des Sciences, U.S.S.R., 14, 3, 101 (1937)

4. R. Courant and D. Hilbert, Mathematische Physik II, Berlin, Julius Springer (1937)

5. P.A.M. Dirac, Proc. Roy. Soc. A167, 148 (1938)

6. W. Heisenberg, Zeits. f. Physik 113, 61 (1939)

7. H. Hoek, Dissertation, University of Leiden (1939)

8. D.I. Iwanenko and A. Sokolow, Jour. Phys., U.S.S.R., 3, 57 (1940)

9. Jahnke and Emde, Tables of Functions, 4th Edition (1938)

10. J.M. Jauch and K.M. Watson, Phys. Rev. 74, 950 (1948)

11. N. Kemmer, Proc. Roy. Soc. A166, 127 (1938)

12. H.A. Lorentz, The Theory of Electrons, Leipzig (1916)

13. H. Minkowski, Göttinger Nachrichten 53 (1908); Math. Ann. 68 (1910)

14. W. Pauli, Meson Theory of Nuclear Forces, N.Y., Inter-Science Publishers, Inc. (1943)

15. A. Proca, Comptes Rendus, 202, 1490 (1936)

16. A. Sommerfeld, Ann. der Phys. 28, 683 (1908-09)

17. Ig. Tamm, Jour. Phys., U.S.S.R., 1, 439 (1939)

18. G.E. Uhlenbeck, Lectures on Advanced Electricity and Magnetism, University of Michigan (1948)

19. Van Vleck, The Theory of Electric and Magnetic Suceptibilities, Oxford University Press, (1932)

20. G. Wentzel, Einführung in die Quantentheorie der Wellen Felder, Wien, Franz Deuticke (1943)

21. H. Yukawa, Proc. Phys. Math. Soc. Japan, 17, 48 (1935)