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SUFFICIENT SETS FOR SOME SPACES OF ENTIRE FUNCTIONS

by

Dennis Murray Schneider

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of the requirements for the degree of
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INTRODUCTION

Let K be a family of continuous positive functions k on complex n-space $\mathbb{C}^n$ and let E be the vector space of all entire functions f on $\mathbb{C}^n$ such that $|f(z)| = O(k(z))$ as $|z| \to \infty$ for each $k \in K$. If $S \subset \mathbb{C}^n$, $f \in E$, and $k \in K$, let $||f||_{k,S} = \sup\{|f(z)|/k(z) : z \in S\}$. When $S = \mathbb{C}^n$ write $||f||_k$ for $||f||_{k,\mathbb{C}^n}$. Then $||\ ||_k$ (resp. $||\ ||_{k,S}$) is a seminorm on E, and the collection of seminorms $||\ ||_k$ (resp. $||\ ||_{k,S}$), $k \in K$, determine a locally convex topology on E which is denoted by (E,K) (resp. (E,K(S))). A subset S of $\mathbb{C}^n$ is called a sufficient set for E when the two families of seminorms $\{||\ ||_k : k \in K\}$ and $\{||\ ||_{k,S} : k \in K\}$ determine the same locally convex topology on E.

If there is a family $\Phi = \{\phi_1 \le \phi_2 \le \ldots\}$ of continuous positive functions on $\mathbb{C}^n$ such that K is the family of continuous positive functions on $\mathbb{C}^n$ which satisfy

$$\phi_n(z) = O(k(z)) \quad \text{as } |z| \to \infty,\ n = 1, 2, \ldots,$$

then the family K is called countably determined by the family Φ and we write $K = K(\Phi)$. For each $\phi_n \in \Phi$,

$$E(\phi_n) = \{f \text{ entire} : |f(z)| = O(\phi_n(z)) \text{ as } |z| \to \infty\}$$

is a Banach space with respect to the norm

$$||f||_n = \sup\{|f(z)|/\phi_n(z) : z \in \mathbb{C}^n\}$$

and $E = \bigcup_{n=1}^{\infty} E(\phi_n)$. (E,Φ) will denote the space E equipped with the inductive limit topology from the Banach spaces $E(\phi_n)$. If $S \subset \mathbb{C}^n$ is a set of uniqueness for the space E (i.e., any function $f \in E$ which vanishes on S vanishes identically), let

$$E(\phi_n,S) = \{f \in E : ||f||_{n,S} = \sup\{|f(z)|/\phi_n(z) : z \in S\} < \infty\}$$

Then $E(\phi_n,S)$ is a normed linear space and $E = \bigcup_{n=1}^{\infty} E(\phi_n,S)$. $(E,\Phi(S))$ will denote the space E equipped with the inductive limit topology from the $E(\phi_n,S)$. A subset S of $\mathbb{C}^n$ is called weakly sufficient when (E,Φ) is isomorphic to $(E,\Phi(S))$ as a topological vector space.

In Chapter 1 we derive some basic properties of sufficient and weakly sufficient sets. Using a result due to B.A. Taylor [14], we prove that a sufficient set is a weakly sufficient set. It follows from this that if S is a sufficient set for E, then $(E,K(S)) \simeq (E,K) \simeq (E,\Phi) \simeq (E,\Phi(S))$ where $\simeq$ denotes isomorphism as topological vector spaces. We also prove a Phragmén - Lindelöf type theorem which shows that if $K = K(\Phi)$ is countably determined, then the growth rate of any function $f \in E$ is determined by its growth rate on a sufficient set. Finally, in Section 1.3, we obtain a relationship between V.G. Iyer's [8] notion of an effective set and weakly sufficient sets. We prove that certain types of effective sets are weakly sufficient.

In Chapter 2 we shall develop a method for determining discrete sufficient sets for spaces (E,K) of entire functions on $\mathbb{C}^n$ when the functions $k \in K$ can be chosen as functions of $|z|$. Results in this direction were obtained by Taylor [13] who showed that the lattice points in the plane form a sufficient set for the space of entire functions on $\mathbb{C}^n$ of order less than two. Here we use methods similar to [13] and a technique due to G. Valiron [15] to obtain a generalization of this result to functions of several variables and to more general spaces of entire functions (see Theorem 2.2.1).

In Chapter 3 we shall determine sufficient sets for spaces (E,K) of entire functions on $\mathbb{C}^n$ when the growth rate in the real directions is different from the growth rate in the imaginary directions (see Theorem 3.2.8). Several special cases of the results obtained here were

given by L. Ehrenpreis [2;147;161] (see §3.3, Examples 3.4.1 and 3.4.3).

Sufficient sets were introduced by Ehrenpreis to study some interesting problems in analysis [2;3,4,13]. The duals of the spaces considered here are analytically uniform spaces [2;8]. Thus [2;12] if S is a sufficient set for a space (E,K), then every element in the dual of (E,K) has a representation as a Fourier integral

$$\int e^{izw}\, d\nu(w)$$

where ν is a bounded measure supported on the set S.

Also, we prove that if S is a sufficient set for (E,K) and if E is an algebra, then any $f \in E$ which is bounded on S is bounded and hence reduces to a constant (see Theorem 1.1.4). It follows from this and the fact that the lattice points in $\mathbb{C}^n$ form a sufficient set for the space of entire functions on $\mathbb{C}^n$ of order less than two that any entire function on $\mathbb{C}^n$ of order less than two which is bounded at the lattice points on $\mathbb{C}^n$ is a constant. In the case of one variable, this is a classical problem posed by Littlewood and solved by G. Polya, J.M. Whittaker, and others [16].

CHAPTER 1

THE SPACE (E,K)

§1.1 Sufficient Sets.

Let K be a family of continuous positive functions on $\mathbb{C}^n$ which tend to infinity with $|z|$, and let E be the vector space of entire functions f on $\mathbb{C}^n$ which satisfy $|f(z)| = O(k(z))$ for each $k \in K$. For any subset S of $\mathbb{C}^n$ define

$$||f||_{k,S} = \sup \{|f(z)| / k(z) : z \in S\}, \quad f \in E, \ k \in K.$$

When $S = \mathbb{C}^n$ write $||f||_k$ for $||f||_{k,\mathbb{C}^n}$. The seminorms $|| \ ||_k$ (resp. $|| \ ||_{k,S}$), $k \in K$, determine a locally convex topology on E, and E equipped with this topology is denoted by (E,K) (resp. (E,K(S))).

Definition 1.1.1. A subset S of $\mathbb{C}^n$ is called a sufficient set for E when the two families of seminorms $\{|| \ ||_k : k \in K\}$ and $\{|| \ ||_{k,S} : k \in K\}$ determine the same locally convex topology on E.

Thus S is a sufficient set for E if and only if (E,K) is isomorphic to (E,K(S))) as a topological vector space (denoted by $(E,K) \simeq (E,K(S))$). For every $k \in K$ and $f \in E$, $||f||_{k,S} \leq ||f||_k$ and thus the identity map $i : (E,K) \to (E,K(S))$ is always continuous. Consequently, a set S is a sufficient set for E if and only if for every $k \in K$ there is a $k' \in K$ and a constant $C > 0$ such that for each $f \in E$

$$||f||_k \leq C \ ||f||_{k',S}$$

or

$$|f(z)| \leq k'(z) \text{ on } S \Rightarrow |f(z)| \leq C\, k(z) \text{ on } \mathbb{C}^n.$$

This last condition is a type of Phragmén - Lindelöf principle. It shows that if S is a sufficient set for E, then the "global" growth rate of the functions in E is determined by their growth rate on S.

With some additional hypotheses on the family K we will obtain a sharper theorem of this type (see Theorem 1.2.8).

Definition 1.1.2. A subset S of $\mathbb{C}^n$ is called a set of uniqueness for a space E of entire functions if $f \in E$ and $f \equiv 0$ on $S \Rightarrow f \equiv 0$.

It is easy to see that if S is a sufficient set for (E,K), then S is a set of uniqueness for (E,K). For if $f \in E$, $f \equiv 0$ on S, and $k \in K$, then there is a $k' \in K$ and a constant $C > 0$ such that $||f||_k \leq C||f||_{k',S} = 0$. Consequently, $f \equiv 0$.

However, more is true. If S is a sufficient set for (E,K) and $f \in E$ with $|f(z)| \leq C$ on S, then $|f(z)| \leq C$ on $\mathbb{C}^n$ and hence f reduces to a constant. To prove this we need the following characterization of the bounded sets in (E,K) [13;437].

Proposition 1.1.3. A subset B of (E,K) is bounded if and only if there is a function $U(z) \geq 0$ such that $|f(z)| \leq U(z)$ for all $f \in B$ and $U(z) / k(z)$ is bounded for every $k \in K$.

Proof. If B is bounded, define

$$U(z) = \sup \{|f(z)| : f \in B\}.$$

Then clearly $|f(z)| \leq U(z)$ for all $f \in B$ and since B is bounded and

$$\frac{U(z)}{k(z)} = \sup \{\frac{|f(z)|}{k(z)} : f \in B\} \leq \sup \{||f||_k : f \in B\},$$

it follows that $U(z) / k(z)$ is bounded for all $k \in K$.

Conversely, if U(z) is such a function, then

$$\sup \{||f||_k : f \in B\} \leq ||U||_k$$

which is bounded for every $k \in K$. Thus B is bounded.

In the following theorem and in Theorem 1.2.8 we will need to assume that E is an algebra. In terms of the family K of growth conditions, this amounts to assuming that the product k_1k_2 belongs to K whenever k_1 and k_2 belong to K.

Theorem 1.1.4. Let S be a sufficient set for (E,K) and suppose that E is an algebra. Then any function f in E which is bounded on S is bounded and hence reduces to a constant.

Proof. It is sufficient to prove the theorem for $|f(z)| \leq 1$ for $z \in S$. For if $|f(z)| \leq C$ on S, replace f(z) by f(z) / C. Let

$$B = \{[f(z)]^j : j = 1, 2, \ldots\}.$$

Then B is a bounded set in (E,K(S)) and so B is a bounded set in (E,K), since S is sufficient. By Proposition 1.1.3, there is a function $U(z) \geq 0$ such that for all z,

$$|f(z)|^j \leq U(z), \quad j = 1, 2, \ldots$$

or

$$|f(z)| \leq [U(z)]^{1/j}, \quad j = 1, 2, \ldots .$$

Holding z fixed and letting $j \to \infty$ we obtain $|f(z)| \leq 1$. Since this holds for all z, the proof is complete.

Definition 1.1.5. The family K is said to be countably determined if there is a family $\Phi = \{\phi_1 \leq \phi_2 \leq \ldots\}$ of continuous positive functions on $\mathbb{C}^n$ which tend to infinity with $|z|$ such that K consists of the family of all continuous positive functions on $\mathbb{C}^n$ which satisfy

$$\phi_n(z) = O(k(z)), \quad n = 1, 2, \ldots .$$

We will write $K = K(\Phi)$ when the family K is countably determined by the family Φ.

It is easy to verify [12;444] that if $K = K(\Phi)$ is countably determined, then

$$E = \{f \text{ entire} : |f(z)| = O(\phi_n(z)) \text{ for some } n = 1, 2, \ldots\}.$$

Thus, $E = \bigcup_{n=1}^{\infty} E(\phi_n)$ where

$$E(\phi_n) = \{f \text{ entire} : |f(z)| = O(\phi_n(z))\}.$$

Each $E(\phi_n)$ is a Banach space with respect to the norm

$$||f||_n = \sup \{|f(z)| / \phi_n(z) : z \in \mathbb{C}^n\}.$$

Consequently, E may be given the inductive limit topology [10;78] from the sequence of Banach spaces $E(\phi_n)$. The space E equipped with this topology is denoted by (E,Φ).

In the following proposition, (E,τ_c) denotes E equipped with the topology of uniform convergence on compact subsets.

Proposition 1.1.6. (i) The seminorm topology on E is finer than the topology of uniform convergence on compact subsets. That is, the identity map $i : (E,K) \to (E,\tau_c)$ is continuous.

(ii) If $K = K(\Phi)$ is countably determined, then the inductive limit topology on E is finer than the seminorm topology. That is, the identity map $i : (E,\Phi) \to (E,K)$ is continuous.

Proof of (i). If D is any compact subset of $\mathbb{C}^n$ and if $k \in K$, then

$$\sup\{|f(z)| : z \in D\} \leq C_k ||f||_k$$

where $C_k = \sup \{k(z) : z \in D\}$. This proves (i).

Proof of (ii). The identity map $i : (E,\Phi) \to (E,K)$ is continuous if and only if each of the injections $j : E(\phi_n) \to (E,K)$ is continuous [10;79]. For every $k \in K$ and $\phi_n \in \Phi$,

$$||f||_k = \sup \{|f(z)| / k(z)\} \leq C_{n,k} ||f||_n$$

where $C_{n,k} = \sup \{\phi_n(z) / k(z)\}$. This proves that each injection is continuous.

Corollary 1.1.7. (E,K) is complete.

Proof. If $\{f_\alpha\}$ is a Cauchy net in (E,K), then by Proposition 1.1.6 (i) there is an entire function f such that $f_\alpha \to f$ uniformly on compact subsets. Let $k \in K$ and $\varepsilon > 0$ be given. There is an α_o such that $\alpha,\beta \geq \alpha_o$ implies

$$|f_\alpha(z) - f_\beta(z)| \leq \varepsilon k(z), \quad z \in \mathbb{C}^n.$$

Since $f_\beta \to f$ pointwise, this implies that if $\alpha \geq \alpha_o$,

$$|f_\alpha(z) - f(z)| \leq \varepsilon k(z), \quad z \in \mathbb{C}^n.$$

Thus $||f_\alpha - f||_k \leq \varepsilon$ for $\alpha \geq \alpha_o$ and this proves that $f_\alpha \to f$ in (E,K). Finally, since $f_{\alpha_o} - f \in E$, $f = f_{\alpha_o} - (f_{\alpha_o} - f) \in E$.

It is natural to ask when these two topologies on E coincide. Taylor [14] proved the following theorem which gives sufficient conditions for this to be the case.

Theorem 1.1.8. (Taylor). Let $K = K(\Phi)$ be countably determined and let $\psi_n = \log \phi_n$. Suppose that each ψ_n satisfies the following conditions:

(i) each ψ_n is plurisubharmonic;

(ii) $\psi_{n+1}(z) \geq \log (1 + |z|^2) + \tilde{\psi}_n(z)$ where $\tilde{\psi}_n(z) = \sup \{\psi_n(z + \xi) : |\xi| < 1\}$;

(iii) for every $n > 1$, $\psi_{n+1} - \psi_n$ is bounded above on every bounded set.

Then the inductive limit topology and the seminorm topology on E coincide. That is, $(E,K) \simeq (E,\Phi)$.

We will need the following characterization of the bounded sets in (E,K) and (E,Φ).

Proposition 1.1.9. Let $K = K(\Phi)$ be countably determined. The bounded sets in (E,K) and in (E,Φ) coincide, and each bounded set is contained in one of the bounded sets

$$B_n = \{f \in E : |f(z)| \leq n\phi_n(z)\}, \quad n = 1, 2, \ldots .$$

Proof. [14;380].

§1.2. Weakly Sufficient Sets.

Let $K = K(\Phi)$ be countably determined and let S be a set of uniqueness (see Definition 1.1.3) for (E,K). For each $\phi_n \in \Phi$,

$$E(\phi_n,S) = \{f \in E : |f(z)| = O(\phi_n(z)) \text{ for } z \in S\}$$

is a normed linear space with respect to the norm

$$||f||_{n,S} = \sup \{|f(z)| / \phi_n(z) : z \in S\},$$

and

$$E = \bigcup_{n=1}^{\infty} E(\phi_n,S).$$

Thus E may also be given the inductive limit topology determined by the spaces $E(\phi_n,S)$, and $(E,\Phi(S))$ will denote E equipped with this topology. The next proposition states that the "Φ topology" on E is finer than the "$\Phi(S)$ topology" on E, which is finer than the "K(S) topology".

Proposition 1.2.1. Let $K = K(\Phi)$ be countably determined. The identity mappings

$$i : (E,\Phi) \to (E,\Phi(S)), \qquad i : (E,\Phi(S)) \to (E,K(S))$$

are continuous.

Proof. A base of neighborhoods of zero in $(E,\Phi(S))$ consists of the set of absolutely convex envelopes of sets of the form $\bigcup_{n=1}^{\infty} V_n$ where V_n is a member of a base of absolutely convex neighborhoods of zero in $E(\phi_n,S)$ for each n [10;79]. It follows from this and the fact that the injections $E(\phi_n) \to E(\phi_n,S)$ are continuous that the injections $E(\phi_n) \to (E,\Phi(S))$ are continuous. Thus $i : (E,\Phi) \to (E,\Phi(S))$ is continuous.

The proof of the second assertion is the same as that of Proposition 1.1.6.

Definition 1.2.2. A subset S of $\mathcal{C}^n$ is called a weakly sufficient set for E if $(E,\Phi) \simeq (E,\Phi(S))$.

Theorem 1.2.3. Suppose that $K = K(\Phi)$ is countably determined and that Φ satisfies the conditions of Theorem 1.1.8. Then a sufficient set is a weakly sufficient set.

Proof. The theorem follows immediately from Definition 1.2.2, Theorem 1.1.8, and Proposition 1.2.1.

Corollary 1.2.4. Suppose that $K = K(\Phi)$ is countably determined and that Φ satisfies the conditions of Theorem 1.1.8. Then S sufficient implies

$$(E,K(S)) \simeq (E,K) \simeq (E,\Phi) \simeq (E,\Phi(S)).$$

In the following theorem we give two equivalent sufficient conditions for a set S to be weakly sufficient.

Theorem 1.2.5. A subset S of $\mathbb{C}^n$ is weakly sufficient for E if S satisfies either of the following equivalent conditions:

(i) $E(\phi_n,S)$ is complete for each $n = 1, 2, \ldots$.

(ii) For every n there is an m such that $E(\phi_n,S) \subset E(\phi_m)$ and the injection is continuous.

The proof that condition (ii) implies that S is weakly sufficient is the same as the proof of the first part of Proposition 1.2.1. Hence it only remains to show that the two conditions are equivalent. The proof will depend on Proposition 1.2.7 which is due to A. Grothendieck [3].

Proposition 1.2.6. Let E be an inductive limit of a sequence E_n of Fréchet spaces and let u be a sequentially closed linear map of a Fréchet space F into E. Then u is continuous.

Proof. [1;448].

Proposition 1.2.7. (Grothendieck). Let E be an inductive limit of a sequence E_n of Fréchet spaces and let u be a continuous linear mapping of a Fréchet space F into E. Then there is an m such that u is a continuous linear mapping of F into E_m.

Proof. [3;198].

Proof of Theorem 1.2.5. First suppose that $E(\phi_n,S)$ is complete for all n. The injection $E(\phi_n,S) \to (E,\Phi)$ is a sequentially closed linear map. For if $f_j \to f$ in $E(\phi_n,S)$ and $f_j \to g$ in (E,Φ), then $f_j \to f$ pointwise on $\mathbb{C}^n$ since the inductive limit topology is finer than the topology of uniform convergence on compact subsets (Proposition 1.1.6). Since

$f_j \to f$ pointwise on S and S is a set of uniqueness, $f \equiv g$. Thus, by Propositions 1.2.6 and 1.2.7, the injection $E(\phi_n,S) \to (E,\Phi)$ is continuous and in fact maps $E(\phi_n,S)$ into $E(\phi_m)$ for some m (continuously).

To prove the converse let $\{f_j\}$ be a Cauchy sequence in $E(\phi_n,S)$. By assumption there is an m such that the injection $E(\phi_n,S) \to E(\phi_m)$ is continuous. Thus $\{f_j\}$ is a Cauchy sequence in $E(\phi_m)$ and since $E(\phi_m)$ is complete there is an $f \in E(\phi_m)$ such that $f_j \to f$ in $E(\phi_m)$. It is routine (see the proof of Corollary 1.1.7) to show that $f \in E(\phi_n,S)$ and $f_j \to f$ in $E(\phi_n,S)$. Thus $E(\phi_n,S)$ is complete.

Condition (ii) in Theorem 1.2.5 is a Phragmén - Lindelöf type condition with the set S replacing the boundary of an angle. It insists that the set S should be "rich" enough to determine the growth rates of the functions in E. For, an equivalent statement is (essentially) that for every n there is an m such that

$$|f(z)| \le \phi_n(z) \text{ on } S \Rightarrow |f(z)| \le \phi_m(z) \text{ on } \mathbb{C}^n.$$

On the surface this is a finer condition on the growth rates of the functions in E than that imposed by the set S being a sufficient set. However in the following theorem we prove that if $K = K(\Phi)$ is countably determined, then sufficient implies condition (i), and hence condition (ii), of Theorem 1.2.5. We believe (but are unable to prove) that the converse of this theorem is also true.

Theorem 1.2.8. If $K = K(\Phi)$ is countably determined and S is a sufficient set for E, then for every n there is an m such that $E(\phi_n,S) \subset E(\phi_m)$ and the injection is continuous.

Proof. By Theorem 1.2.5 it is enough to show that $E(\phi_n,S)$ is complete for all n. Let $\{f_j\}$ be a Cauchy sequence in $E(\phi_n,S)$. The identity map

$$i : (E,\Phi(S)) \to (E,K(S))$$

is continuous (Proposition 1.2.1) and since S is a sufficient set, $(E,K) \simeq (E,K(S))$. Consequently, $\{f_j\}$ is a Cauchy sequence in (E,K) and since (E,K) is complete (Corollary 1.1.7) there is an $f \in E$ such that $f_j \to f$ in (E,K). It is routine (see the proof of Corollary 1.1.7) to show that $f \in E(\phi_n,S)$ and $f_j \to f$ in $E(\phi_n,S)$. Thus $E(\phi_n,S)$ is complete.

Corollary 1.2.9. If $K = K(\Phi)$ is countably determined, then a sufficient set is a weakly sufficient set.

Thus we see that a sufficient set is a weakly sufficient set even if the family Φ does not satisfy the conditions of Theorem 1.1.8. However, in this case it does not follow that all of the topologies on the space E coincide (see Corollary 1.2.4).

Theorem 1.2.10. If S is a weakly sufficient set for (E,Φ) and if E is an algebra, then any function in E which is bounded on S is bounded, and hence must reduce to a constant.

Proof. Let $f \in E$ and suppose that $|f(z)| \leq C$ on S. Let $g(z) = f(z) / C$ and let

$$B = \{[g(z)]^j : j = 1, 2, \ldots\}.$$

B is a bounded subset of $E(\phi_n,S)$ for each n and thus B is a bounded set in $(E,\Phi(S))$. Since S is weakly sufficient, B is bounded in (E,Φ). By Proposition 1.1.9, there is an n such that B is contained in the set

$$B_n = \{f \in E : |f(z)| \leq n\phi_n(z)\}.$$

Consequently, for all z,

$$|g(z)|^j \leq n\phi_n(z), \quad j = 1, 2, \ldots$$

or

$$|g(z)| \leq [n\phi_n(z)]^{1/j}, \quad j = 1, 2, \ldots .$$

Holding z fixed and letting $j \to \infty$ we obtain $|g(z)| \leq 1$. Since this holds for all z, the proof is complete.

§1.3 The Relationship Between Effective Sets and Weakly Sufficient Sets.

The notion of an effective set is due to Iyer [8]. In this section we shall prove that certain types of effective sets are weakly sufficient.

Let f be an entire function and let $M(r,f) = \max\{|f(z)| : |z| \le r\}$. The function f is said to be of order ρ if

$$\limsup_{r\to\infty} \frac{\log \log M(r,f)}{\log r} = \rho .$$

An entire function f of order ρ has type τ if

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} = \tau$$

Let $C(\rho,d)$ denote the set of all entire functions of order ρ and type less than d. A function of order less than ρ will be regarded as a function of order ρ and type 0 .

Let $[z_n]$ denote a distinct sequence of complex numbers such that $0 \le |z_1| \le |z_2| \le \ldots \le |z_n| \to \infty$. $[z_n]$ is said to have exponent of convergence ρ if ρ is the infimum of all positive numbers α for which $\sum_{n=1}^{\infty} |z_n|^{-\alpha}$ converges.

Definition 1.3.1. The set $[z_n]$ is called an effective set for $C(\rho,d)$ if the type of any function $f \in C(\rho,d)$ as determined on $[z_n]$ is equal to the type of f. That is, $[z_n]$ is an effective set for $C(\rho,d)$ if for every $f \in C(\rho,d)$,

$$\limsup_{n\to\infty} \frac{\log |f(z_n)|}{|z_n|^{\rho}} = \limsup_{r\to\infty} \frac{M(r,f)}{r^{\rho}} .$$

The following theorem [8; 360] gives a sufficient condition for a set $[z_n]$ to be an effective set for $C(\rho,d)$.

Theorem 1.3.2 (Iyer). Let $[z_n]$ have exponent of convergence ρ. Then $[z_n]$ is an effective set for $C(\rho,d)$ if there exists an entire function g with simple zeros at $z = z_n$ and an $h > \rho$ such that the following relations hold:

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{\log |g'(z_n)|}{|z_n|^{\rho}} = d \ ;$$

$$\text{(ii)} \qquad \lim \frac{\log |g(z)|}{|z|^{\rho}} = d$$

as $|z| \to \infty$ outside $A(h) = \bigcup_{n=1}^{\infty} \{ |z - z_n| \le |z_n|^{-h} \}$.

Let $\{d_n\}$ be a strictly increasing sequence of reals with $\lim_{n\to\infty} d_n = d$. For each n let $\phi_n(z) = \exp(d_n|z|^{\rho})$. Then $C(\rho,d) = \bigcup_{n=1}^{\infty} E(\phi_n)$.

Theorem 1.3.3. Let $[z_n]$ be an effective set for $C(\rho,d)$ which satisfies the conditions of Theorem 1.3.2. Then $[z_n]$ is weakly sufficient for $C(\rho,d)$.

Before we begin the proof of this theorem, we shall state in the form of lemmas several results used by Iyer in the proof of Theorem 1.3.2.

Let $[z_n]$ and g satisfy the conditions of Theorem 1.3.2.

Lemma 1.3.4. Let $f \in C(\rho,d)$ with $\limsup_{n\to\infty} \frac{\log |f(z_n)|}{|z_n|^{\rho}} = \beta$.

Choose η, $0 < \eta < d - \beta$, and λ so that $d - d\lambda^{\rho} \ge \beta + \eta$. Then

$$f(z) = \frac{g(z)}{g(\lambda z)} \sum_{n=1}^{\infty} \frac{f(z_n)\, g(\lambda z_n)}{g'(z_n)} \frac{1}{z - z_n}$$

Lemma 1.3.5. Let $a > 1$. There is a sequence $\{R_j\}$, $R_1 < \ldots < R_j \to \infty$ as $j \to \infty$, and $R_{j+1} \le aR_j$, such that no circle $|z| = R_j$ intersects any of the discs of either $A(h)$ or $A_{\lambda}(h)$ where

$$A_{\lambda}(h) = \bigcup_{n=1}^{\infty} \{ |z - z_n/\lambda| < |z_n/\lambda|^{-h} \} .$$

Proof of Theorem 1.3.3. First observe that if $S = [z_n]$ is an effective set for $C(\rho,d)$, then S is a set of uniqueness. Consequently, by Theorem 1.2.5, it is sufficient to show that for every n, $E(\phi_n,S)$ is continuously injected into $E(\phi_{n+2})$.

Let $f \in E(\phi_n,S)$ with $||f||_{n,S} \le 1$. Then

$$|f(z_j)| \le \exp\left[d_n |z_j|^\rho\right], \qquad j = 1, 2, \ldots . \tag{1.3.6}$$

If

$$\beta = \limsup_{n\to\infty} \frac{\log |f(z_n)|}{|z_n|^\rho},$$

then $\beta \le d_n$. Choose η, $0 < \eta < \frac{d_{n+1} - d_n}{2}$, and λ so that $d - d\lambda^\rho = d_n + \eta$. Finally choose $\varepsilon > 0$ so small that $\varepsilon < \eta$.

By Lemma 1.3.4,

$$f(z) = \frac{g(z)}{g(\lambda z)} \sum_{n=1}^{\infty} \frac{f(z_n)\, g(\lambda z_n)}{g'(z_n)} \frac{1}{z - z_n}. \tag{1.3.7}$$

Moreover, by Lemma 1.3.4, given $a = (d_{n+1}/(d_n + \eta + \varepsilon))^{-\rho} > 1$, there is a sequence $\{R_j\}$, $R_1 < R_2 < \ldots < R_j \to \infty$, $R_{j+1} \le aR_j$, such that none of the circles $|z| = R_j$ intersect any of the discs of either $A(h)$ or $A_\lambda(h)$. Using (i) and (ii) of Theorem 1.3.2, (1.3.6), and the choice of λ we obtain from (1.3.7) that

$$M(R_j,f) \le C \exp\left[(d - d\lambda^\rho + \varepsilon) R_j^\rho\right]$$

for $j \ge j_0 = j_0(\varepsilon)$ where C is a constant which is independent of f. Since $M(R,f)$ is an increasing function of R and $R_{j+1} \le aR_j$, it follows that $M(R,f) \le C \exp\left[(d - d\lambda^\rho + \varepsilon)\, a^\rho\, R^\rho\right]$ for $R \ge R_{j_0}$. Since $(d - d\lambda^\rho + \varepsilon)\, a^\rho = (d_n + \eta + \varepsilon)\, a^\rho = d_{n+1}$, we obtain

$$M(R,f) \le C \exp\left[d_{n+1} R^\rho\right], \qquad R \ge R_{j_0}. \tag{1.3.8}$$

This establishes that if $f \in E(\phi_n,S)$ with $||f||_{n,S} \le 1$, then $f \in E(\phi_{n+2})$. If $f \in E(\phi_n,S)$ and $||f||_{n,S} \ge 1$, we apply the preceding argument to $f / ||f||_{n,S}$ and obtain $f \in E(\phi_{n+2})$. Thus $E(\phi_n,S) \subset E(\phi_{n+2})$.

It remains to prove that the injection $E(\phi_n,S) \to E(\phi_{n+2})$ is continuous. Set

$$A = \sup\{\phi_{n+1}(R) / \phi_{n+2}(R) : R > 0\},$$

$$B = \sup\{\phi_{n+2}(R_0) / \phi_{n+2}(R) : R > R_0\},$$

where R_0 is the constant determined in (1.3.8). From (1.3.8) we obtain

$$\sup\{M(R,f) / \phi_{n+2}(R) : R > R_0\} \leq CA$$

and since

$$\sup_{R\leq R_0} \frac{M(R,f)}{\phi_{n+2}(R)} \leq \sup_{R\leq R_0}\left[\frac{M(R_0,f)}{\phi_{n+2}(R_0)} \quad \frac{\phi_{n+2}(R_0)}{\phi_{n+2}(R)}\right]$$

$$\leq ABC,$$

it follows that $||f||_{n+2} \leq ABC$. Thus we have proved that if $f \in E(\phi_n,S)$ with $||f||_{n,S} \leq 1$, then $||f||_{n+2} \leq ABC$. This establishes that the injection $E(\phi_n,S) \to E(\phi_{n+2})$ is continuous and completes the proof.

It is shown in [8;365] that the lattice points $z = m + in$, $m, n = 0, \pm 1, \pm 2, \ldots$, form an effective set for $C(2, \pi/2)$ which satisfies the conditions of Theorem 1.3.2. Consequently we have

Corollary 1.3.9. The lattice points in the plane form a weakly sufficient set for $C(2, \pi/2)$.

It will be shown in Chapter 2 that the lattice points in the plane form a sufficient set for the space of entire functions of order less than two. Since the proof that the lattice points form an effective set for $C(2,\pi/2)$ involves constructing a function of order 2 and type $\pi/2$ which vanishes at the lattice points, it follows that the lattice points do not form a weakly sufficient (nor sufficient) set for $C(2,d)$ when $d > \pi/2$.

CHAPTER 2

SUFFICIENT SETS FOR (E,K) WHEN THE CLASS K CONSISTS OF RADIAL FUNCTIONS

§2.1. Introduction.

In this chapter we shall develop a method for determining discrete sufficient sets for spaces (E,K) of entire functions on $\mathbb{C}^n$ when the functions $k \in K$ can be chosen as functions of $|z|$. Results in this direction were obtained by Taylor [13] who showed that the lattice points in the plane form a sufficient set for the space of entire functions on $\mathbb{C}$ of order less than two. Here we use methods similar to [13] and a technique due to G. Valiron [15] to obtain a generalization of this result to functions of several variables and to more general spaces of entire functions (see Theorem 2.2.1).

§2.2. The Main Theorem.

Let $V(r)$ be a positive increasing function of r such that $V(r) \to +\infty$ as $r \to \infty$. Let K be any class of continuous, positive, increasing functions $k(r)$ whose logarithms $\phi(r) = \log k(r)$ satisfy

(i) $\lim_{r\to\infty} \phi(r) = +\infty$,

(ii) $\exp(B_1^{-1}\phi(B_2^{-1}r)) \in K$ whenever $\exp(\phi) \in K$,

(iii) $\dfrac{[\phi(B_4 r)]^2}{\phi(r)} = o(V(r))$ as $r \to +\infty$,

where B_1, B_2 and B_4 are the constants appearing in (2.4.1).

Theorem 2.2.1. Let $S \subset \mathbb{C}^n$ such that $d(z,S) \leq \text{const } |z|/\sqrt{V(2|z|)}$ for all $z \in \mathbb{C}^n$. Then S is a sufficient set for (E,K).

The idea behind the proof of this theorem is to obtain an estimate for $M(r,f)$ in terms of the values of f on the set S. In particular, we shall prove (Theorem 2.3.16) that if f is any entire function (suitably normalized) then

$$\log M(r,f) \leq B_1 \log M(B_2 r,f,S) + B_3\left[\frac{\log M(B_4 r,f)}{\sqrt{V(r)}}\right]^2 \tag{2.2.2}$$

where B_1, B_2, B_3, B_4 depend on V. Roughly, it will follow from this inequality that

$$\log M(r,f) \leq \log M(B_2 r,f,S)$$

for $r \geq r_o$ where r_o is independent of f (see Lemma 2.3.14). Thus for $k \in K$,

$$\frac{M(r,f)}{k(r)} \leq \frac{M(B_2 r,f,S)}{k(r)} \leq \frac{M(B_2 r,f,S)}{k_1(B_2 r)}$$

where $k_1(r) = k(B_2^{-1}r)$. Consequently, if we restrict the norms to involve only the supremum for $r \geq r_o$, we have

$$||f||_k \leq ||f||_{k_1,S}.$$

Thus the set S is sufficient.

The details of the proof will be given in Section 2.4.

§2.3. A Growth Estimate for Entire Functions.

Throughout this chapter V will denote a pure $(n-1)$-dimensional analytic variety in $\mathbb{C}^n$. If $a \in \mathbb{C}^n$ and $\delta > 0$, then

$$V_\delta = \{z \in \mathbb{C}^n : d(z,V) < \delta\},$$

$$B_\delta = \{z \in \mathbb{C}^n : |z| < \delta\},$$

$$B_\delta(a) = \{z \in \mathbb{C}^n : |z-a| < \delta\}.$$

Finally, λ_m (resp. σ_m) will denote the m-dimensional Hausdorff measure (resp. the m-dimensional surface area measure) on $\mathbb{R}^m$, and τ_m will denote the volume of the unit ball in $\mathbb{R}^m$.

Proposition 2.3.1. Let V be a pure $(n-1)$-dimensional analytic variety

passing through the origin in $\mathbb{C}^n$. Then for any $r > 0$,

$$\tau_{2n-2} r^{2n-2} \leq \lambda_{2n-2}(V \cap B_r).$$

Proof. [10;16].

Proposition 2.3.2. Let V be a pure (n-1)-dimensional analytic variety in $\mathbb{C}^n$ and let r and δ be positive numbers. Then

$$\lambda_{2n}(V_\delta \cap B_r) \leq 3^{2n} \frac{\tau_{2n}}{\tau_{2n-2}} \delta^2 \lambda_{2n-2} (V \cap B_{r+2\delta}).$$

Proof. We begin by selecting a finite number of points $z_i \in V \cap \overline{B}_{r+\delta}$ which have the following properties:

(2.3.3) The balls $B_\delta(z_i)$ are pairwise disjoint.

(2.3.4) The balls $B_{2\delta}(z_i)$ cover $V \cap B_{r+\delta}$.

(2.3.5) The balls $B_{3\delta}(z_i)$ cover $V_\delta \cap B_r$.

If $V \cap \overline{B}_{r+\delta} = \emptyset$, then $V_\delta \cap B_r = \emptyset$ and the result follows. Hence we may suppose that $V \cap \overline{B}_{r+\delta} \neq \emptyset$. Let z_1 be an arbitrary point of $V \cap \overline{B}_{r+\delta}$. Now select

$$z_2 \in (V \cap \overline{B}_{r+\delta}) \setminus B_{2\delta}(z_1)$$

and inductively select

$$z_i \in (V \cap \overline{B}_{r+\delta}) \setminus \bigcup_{j=1}^{i-1} B_{2\delta}(z_{i-1}).$$

By the compactness of $V \cap \overline{B}_{r+\delta}$ there are at most a finite number k of points selected in this manner. It is clear from the construction that conditions (2.3.3) and (2.3.4) are satisfied. To verify condition (2.3.5) we note that if $\zeta \in V_\delta \cap B_r$ then $|\zeta| < r$ and there is a point $z \in V$ such that $|\zeta - z| < \delta$. Since $|z| < r + \delta$, condition (2.3.4) guarantees that $z \in B_{2\delta}(z_j)$ for some j and so $\zeta \in B_{3\delta}(z_j)$.

We now complete the proof of the proposition. Since V passes through the center of each of the balls $B_\delta(z_j)$, we may apply Proposition

2.3.1 to obtain the inequality

$$\tau_{2n-2}\delta^{2n-2} \le \lambda_{2n-2}(V\cap B_\delta(z_j))$$

for $1 \le j \le k$. It follows from (2.3.5) and this inequality that

$$\lambda_{2n}(V_\delta\cap B_r) \le \sum_{j=1}^{k} \tau_{2n}(3\delta)^{2n} = 3^{2n}\frac{\tau_{2n}}{\tau_{2n-2}}\delta^2 \sum_{j=1}^{k} \tau_{2n-2}\,\delta^{2n-2}$$

$$\le 3^{2n}\frac{\tau_{2n}}{\tau_{2n-2}}\delta^2 \sum_{j=1}^{k} \lambda_{2n-2}(V\cap B_\delta(z_j)).$$

Using (2.3.3) and the fact that $B_\delta(z_j)\subset B_{r+2\delta}$, $1 \le j \le k$, we obtain

$$\sum_{j=1}^{k} \lambda_{2n-2}(V\cap B_\delta(z_j)) = \lambda_{2n-2}\left(V \cap \bigcup_{j=1}^{k} B_\delta(z_j)\right) \le \lambda_{2n-2}(V\cap B_{r+2\delta}).$$

Combining the last two inequalities completes the proof.

Proposition 2.3.6. Let f be an entire function on $\mathbb{C}^n$ and let

$$V = \{z \in \mathbb{C}^n : f(z) = 0\}$$

If r and δ are positive numbers, then there is a constant $A > 0$ such that

$$\lambda_{2n}(V_\delta\cap B_r) \le A\, r^{2n-2}\,\delta^2(\log M(e(r+\delta), f) - \log|f(0)|).$$

Proof. By Proposition 2.3.2 we have

(2.3.7) $$\lambda_{2n}(V_\delta\cap B_r) \le \text{const}\ \delta^2\lambda_{2n-2}(V\cap B_{r+\delta})$$

where the constant depends only on the dimension of the space. Now Jensen's formula [9] states that

(2.3.8) $$N(r) + \log|f(0)| = \frac{1}{\sigma_{2n-1}}\int_{|\gamma|=1} \log|f(r\gamma)|\, d\sigma_{2n-1}(\gamma)$$

where

$$N(r) = \int_0^r \frac{n(t)\,dt}{t^{2n-1}}$$

and

$$n(t) = \begin{cases} \text{the number of zeros of } f \text{ in } |z| \le t, & n = 1, \\ \lambda_{2n-2}(V\cap B_t), & n > 1. \end{cases}$$

Now

$$N(er) \geq n(r) \int_r^{er} t^{-(2n-1)}\, dt = \begin{cases} n(r), & n = 1, \\ \dfrac{n(r)}{2n-2}\left(\dfrac{e^{2n-2} - 1}{(er)^{2n-2}}\right), & n > 1, \end{cases}$$

and so

$$n(r) \leq \text{const } r^{2n-2}\, N(er) \tag{2.3.9}$$

where the constant depends only on the dimension of the space.

Consequently,

$$\begin{aligned} \lambda_{2n}(V_\delta \cap B_r) &\leq \text{const } \delta^2\, \lambda_{2n-2}(V \cap B_{r+\delta}) && \text{by (2.3.7)} \\ &\leq \text{const } \delta^2\, r^{2n-2}\, N(e(r+\delta)) && \text{by (2.3.9)} \\ &= \text{const } \delta^2\, r^{2n-2}\left(\log M(e(r+\delta), f) - \log |f(0)|\right) && \text{by (2.3.8)} \end{aligned}$$

and this completes the proof.

Proposition 2.3.10. If f is analytic and satisfies $f(z) \neq 0, 1$ in $|z| < \delta$, then there exist absolute constants A and ℓ such that

$$|\log |f(z)| - \log |f(0)|| \leq A + \varepsilon\, |\log |f(0)||$$

for $|z| \leq \varepsilon \ell \delta$, $0 < \varepsilon < 1$.

Proof. It is sufficient to prove the result when $f(0) \geq 0$. For the general case replace f by $e^{-i\theta} f$ where $f(0) = re^{i\theta}$. By Schottky's theorem [5;49] there exist absolute constants A, B, C such that

$$|\log |f(z)|| \leq \left(A + B\, |\log |f(0)||\right)\left(\frac{\delta + z}{\delta - z}\right)^C$$

for $|z| < \delta$. For $x < \delta$ the function $x \to \dfrac{\delta + x}{\delta - x}$ is increasing and so we obtain

$$|\log |f(z)|| \leq A_1 + B_1\, |\log |f(0)|| \tag{2.3.11}$$

for $|z| \leq \delta/2$ where $A_1 = 3^C A$, $B_1 = 3^C B$. Since $f(z) \neq 0$ for $|z| < \delta$, $\log |f|$ is harmonic and (since $f(0)$ is real) we may apply the Poisson Integral Formula [6;31] to obtain

$$\log f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{re^{i\theta} + z}{re^{i\theta} - z} \log |f(re^{i\theta})|\, d\theta$$

for $|z| < r = \delta/2$. Using (2.3.11) to estimate the modulus of this integral we see that

$$(2.3.12) \qquad |\log f(z)| \le \frac{r + |z|}{r - |z|} \left(A_1 + B_1 \, |\log |f(0)||\right)$$

$$\le A_2 + B_2 \, |\log |f(0)||$$

for $|z| < r/2 = \delta/4$ where $A_2 = 3A_1$, $B_2 = 3B_1$. By the Cauchy Integral Formula,

$$\frac{d}{dz}(\log f(z)) = \frac{1}{2\pi i} \int_{|\zeta|=\delta/4} \frac{\log f(\zeta)}{(\zeta - z)^2} \, d\zeta$$

for $|z| < \delta/4$. Now using (2.3.12) we obtain the estimate

$$\left|\frac{d}{dz}\log f(z)\right| \le \frac{\delta}{4}\left(A_2 + B_2 \, |\log |f(0)||\right) (\delta/4 - \delta/8)^{-2}$$

$$= \frac{1}{\delta} (A_3 + B_3 \, |\log |f(0)||)$$

for $|z| < \delta/8$ where $A_3 = 16A_2$, $B_3 = 16B_2$. Thus

$$(2.3.13) \qquad |\log f(z) - \log f(0)| = \left|\int_0^z \frac{d}{d\zeta} \log f(\zeta) \, d\zeta\right|$$

$$\le \frac{|z|}{\delta}(A_3 + B_3 \, |\log |f(0)||)$$

for $|z| < \delta/8$.

Let $\ell = \min \{1/B_3, 1/8\}$ and let ε be given with $0 < \varepsilon < 1$. Then $\varepsilon\ell\delta < \delta/8$ and so we may apply (2.3.13) to obtain

$$|\log f(z) - \log f(0)| \le A_3 + \varepsilon \, |\log |f(0)||$$

for $|z| < \varepsilon\ell\delta$. Since

$$Re(\log f(z) - \log f(0)) = \log |f(z)| - \log |f(0)|$$

the assertion follows.

Lemma 2.3.14. Let u, ϕ, V be positive increasing functions of r tending to infinity for which there exist constants C_1 and $C_2 > 1$ such that the following conditions hold:

$$\text{(i)} \qquad \frac{\phi(C_2 r)^2}{\phi(r)} = o(V(r)),$$

$$\text{(ii)} \qquad u(r) = o(\phi(r)),$$

$$\text{(iii)} \qquad u(r) \le \phi(r) + C_1\left(\frac{u(C_2 r)}{\sqrt{V(r)}}\right)^2.$$

Then there is an r_o (which is independent of u) such that

$$u(r) \le 2\phi(r)$$

for $r \ge r_o$.

Proof. By (i) there is an r_o such that

$$\text{(2.3.15)} \qquad \frac{\phi(C_2 r)^2}{\phi(r)} \le \frac{1}{4C_1} V(r)$$

for $r \ge r_o$. If there is an $r > r_o$ with $2\phi(r) < u(r)$ then by (iii)

$$\phi(r) < C_1\left(\frac{u(C_2 r)}{\sqrt{V(r)}}\right)^2.$$

Using (2.3.15) and this inequality we obtain

$$\phi(C_2 r) \le \frac{V(r)\phi(r)}{4C_1\phi(C_2 r)} < \frac{u(C_2 r)^2}{4\phi(C_2 r)} .$$

Thus $2\phi(C_2 r) < u(C_2 r)$. Repeating this argument we obtain, for every $n = 1, 2, \ldots$, $2\phi(C_2^n r) < u(C_2^n r)$ which contradicts (ii).

Let β be the solution to the equation $\log|\beta| + \log|\beta - 1| = 1$ which is real and greater than 1. Fix ε, $0 < \varepsilon < 1$, let ℓ be the constant appearing in Proposition 2.3.10, and let $\psi(r)$ be a positive function defined for all $r > 0$. Let

$$\tilde{\psi}(r) = \sup\{\psi(t) : \frac{r}{2} \le t \le \frac{3r}{2}\}.$$

Theorem 2.3.16. Let $S \subset \mathbb{C}^n$ with $d(z,S) \leq \psi(|z|)$ for all z and let f be an entire function with $f(0) = \beta$. Then there exist constants $A_1 > 0$ depending on n and a constant A_2 depending on n and ε such that

$$\log M(r,f) \leq \frac{1}{1-\varepsilon}\left(A_1 + \log M(\frac{3r}{2} + \frac{\widetilde{\psi}(r)}{\varepsilon\ell}, f, S)\right)$$

$$+ A_2\left[\frac{\widetilde{\psi}(r)}{r} \log M(e(\frac{3r}{2} + \widetilde{\psi}(r)),f)\right]^2.$$

Proof. Let $a \in S$ and let

$$D = \{\zeta \in \mathbb{C}^n : |\zeta - a| \leq |a|/2\}.$$

Since f is entire, $\log|f|$ is subharmonic and hence satisfies the subaveraging property, i.e.,

$$\log|f(a)| \leq \left(\frac{2}{|a|}\right)^{2n} \tau_{2n}^{-1} \int_D \log|f(z)|\, d\lambda_{2n}(z).$$

If E is any measureable subset of D we may split this integral into the integral over E plus the integral over $D \setminus E$ and obtain the estimate

$$\log|f(a)| \leq \sup_E \log|f(\zeta)| + \frac{1}{\tau_{2n}}\left(\frac{2}{|a|}\right)^{2n} \lambda_{2n}(D\setminus E) \log M(\frac{3|a|}{2},f). \tag{2.3.17}$$

Now for $i = 0, 1$, set $V^i = \{z \in \mathbb{C}^n : f(z) = i\}$ and define

$$E = \{\zeta \in D : V^i \cap B_\delta(\zeta) = \emptyset \quad \text{for } i = 0, 1\}$$

where $\delta = \frac{\widetilde{\psi}(|a|)}{\varepsilon\ell}$. Then $D\setminus E \subset (V_\delta^{\,0} \cup V_\delta^{\,1}) \cap B_{\frac{3|a|}{2}}$

so that

$$\lambda_{2n}(D\setminus E) \leq \lambda_{2n}(V_\delta^{\,0} \cap B_{\frac{3|a|}{2}}) + \lambda_{2n}(V_\delta^{\,1} \cap B_{\frac{3|a|}{2}}).$$

Using Proposition 2.3.6 to estimate the right side of this inequality, we obtain

$$\lambda_{2n}(D\setminus E) \leq \text{const}\, |a|^{2n-2}\, \delta^2 \Big[\log M(e(\frac{3|a|}{2} + \delta), f) + \log M(e(\frac{3|a|}{2} + \delta), f - 1) - \log|f(0)| - \log|f(0) - 1|\Big] \tag{2.3.18}$$

However $\log (x + 1) \leq 1 + \log x$ for $x \geq 1$ and since $M(r,f-1) \leq M(r,f) + 1$ and $M(r,f) \geq |f(0)| = \beta > 1$,

$$\log M(e(\frac{3|a|}{2} + \delta), f - 1) \leq \log M(e(\frac{3|a|}{2} + \delta), f) + 1.$$

From the choice of β and this inequality we obtain from (2.3.18) that

$$\lambda_{2n} (D \setminus E) \leq \text{const } |a|^{2n-2} \delta^2 \log M(e(\frac{3|a|}{2} + \delta), f)$$

where the constant depends on the dimension of the space and the choice of β. Finally this inequality together with (2.3.17) gives

$$\log |f(a)| \leq \sup_E \log |f(\zeta)| + A \left(\frac{\delta}{r} \log M(e(\frac{3|a|}{2} + \delta), f)\right)^2. \tag{2.3.19}$$

We now proceed to obtain an estimate for $\sup \{\log |f(\zeta)| : \zeta \in E\}$. Let $E_1 = \{\zeta \in E : |f(\zeta)| < 1\}$. If $E = E_1$ the theorem follows from (2.3.19). Thus we may suppose that $E \setminus E_1 \neq \emptyset$. Now for any $\zeta \in E \setminus E_1$, $d(\zeta,S) \leq \psi(|\zeta|) \leq \tilde{\psi}(|a|) = \varepsilon\ell\delta$ and so $S \cap \overline{B}_{\varepsilon\ell\delta}(\zeta) \neq \emptyset$. Consider the function g of one complex variable defined for $|z| \leq \delta$ by

$$g(z) = f(\zeta + zs_1)$$

where $s_1 = \frac{s - \zeta}{|s - \zeta|}$ and $s \in S \cap \overline{B}_{\varepsilon\ell\delta}(\zeta)$. Applying Proposition 2.3.10 we obtain that there is a constant $A > 1$ such that

$$|\log |f(\zeta + zs_1)| - \log |f(\zeta)|| \leq A + \varepsilon |\log |f(\zeta)||$$

for $|z| \leq \varepsilon\ell\delta$. But since $\zeta \in E \setminus E_1$, $\log |f(\zeta)| \geq 0$ and so this inequality implies

$$\log |f(\zeta)| - \log |f(\zeta + zs_1)| \leq A + \varepsilon \log |f(\zeta)|$$

or

$$\log |f(\zeta)| \leq \frac{1}{1 - \varepsilon} (A + \log |f(\zeta + zs_1)|).$$

For $z = |s - \zeta|$, $\zeta + zs_1 = s \in S$ and $|z| \leq \varepsilon\ell\delta$ and so

$$\log |f(\zeta)| \leq \frac{1}{1 - \varepsilon} (A + \log M(\frac{3|a|}{2} + \varepsilon\ell\delta, f, S))$$

Consequently,

$$\sup_{E} |f(\zeta)| \le \sup_{E \setminus E_1} |f(\zeta)| \le \frac{1}{1-\varepsilon} \left(A + \log M\left(\frac{3|a|}{2} + \varepsilon\ell\delta,\ f,\ S\right)\right)$$

and combining this with (2.3.19) establishes the result.

§2.4. Proof of the Main Theorem.

Let $V(r)$ be a positive increasing function of r such that $V(r) \to +\infty$ as $r \to +\infty$. If $\psi(r) = \text{const } r/\sqrt{V(2r)}$, then $\tilde{\psi}(r) \le \text{const } r/\sqrt{V(r)}$. Consequently, if $S \subset \mathbb{C}^n$ with $d(z,S) \le \psi(|z|)$ for all z, Theorem 2.3.16 implies that there exists constants B_1, B_2, B_3, B_4 such that if f is any entire function with $f(0) = \beta$,

$$\log M(r,f) \le B_1 \log M(B_2 r,\ f,\ S) + B_3 \left(\frac{\log M(B_4 r, f)}{\sqrt{V(r)}}\right)^2. \tag{2.4.1}$$

Let K be any class of continuous positive increasing functions $k(r)$ whose logarithms $\phi(r) = \log k(r)$ satisfy

$$\text{(i)} \qquad \lim_{r\to\infty} \phi(r) = \infty\ ,$$

$$\text{(ii)} \qquad \exp(B_1^{-1}\phi(B_2^{-1}r)) \in K \quad \text{whenever } \exp\phi(r) \in K,$$

$$\text{(iii)} \qquad \frac{[\phi(B_4 r)]^2}{\phi(r)} = o(V(r)) \qquad \text{as } r\to\infty,$$

where B_1, B_2 and B_4 are the constants appearing in (2.4.1).

Theorem 2.2.1. Let S be a subset of $\mathbb{C}^n$ with

$$d(z,S) \le \text{const } r/\sqrt{V(2r)}$$

Then S is a sufficient set for (E,K).

Proof. For any subset S of $\mathbb{C}^n$ the injection $(E,K) \to (E, K(S))$ is continuous since $||f||_{k,S} \le ||f||_k$ for any $k \in K$. Consequently, it remains to show that the inverse is also continuous.

Following [13], given

$$k(r) = \exp(2\phi(r)) \in K$$

we define $k_1(z) = \exp(\phi_1(r))$ where $\phi_1(r) = B_1^{-1}\phi(B_2^{-1}r)$. Then $k_1 \in K$ and we will show that there is a constant $C > 0$ such that

$$||f||_{k_1,S} \leq 1 \Rightarrow ||f||_k \leq C$$

for all $f \in (E,K)$. Since this implies that $||f||_k \leq C\,||f||_{k_1,S}$ the proof will be complete.

Let $f \in (E,K)$ with $||f||_{k_1,S} \leq 1$. Then $\log |f(z)| \leq \phi_1(z)$ for $z \in S$.

Case 1. $f(0) = \beta$.

In this case we may apply Theorem 2.3.16 (see (2.4.1)) to obtain

$$\log M(r,f) \leq B_1\phi_1(B_2r) + B_3\left(\frac{\log M(B_4r,\ f)}{\sqrt{V(r)}}\right)^2$$

$$= \phi(r) + B_3\left(\frac{\log M(B_4r,\ f)}{\sqrt{V(r)}}\right)^2$$

Consequently, by Lemma 2.3.14, there is an $r_o > 0$ (independent of f) such that for $r \geq r_o$,

$$\log M(r,\ f) \leq 2\phi\ (r)$$

or

$$\sup_{|z| \geqslant r_o} \frac{|f(z)|}{k(z)} \leq 1.$$

Setting $C = \max\{k(r_o)/k(r) : r \leq r_o\}$ we obtain

$$\sup\{\frac{M(r,f)}{k(r)} : r \leq r_o\} \leq \sup\{\frac{M(r_o,r)}{k(r_o)}\,\frac{k(r_o)}{k(r)} : r \leq r_o\} \leq C.$$

Consequently, $||f||_k \leq C$.

Case 2. $f(0) \neq \beta$.

If $|f(0)| > \beta$, we apply Case 1 to the function $\beta f(z)/f(0)$ and deduce the same inequality.

If $|f(0)| < \beta$, we apply Case 1 to the function $f(z) - (f(0) - \beta)$ and deduce that $||f(z) - (f(0) - \beta)||_k \leq C$. Thus

$$||f||_k \le ||f(z) - (f(0) - \beta)||_k + ||f(0) - \beta||_k \le C + D$$

where $D = 2\beta \sup \{1/k(r) : r \ge 0\} \ge ||f(0) - \beta||_k$.

Corollary 2.4.2. If $S \subset C^n$ with $d(z,S) \le \text{const } |z|^{1-\rho/2}$, then S is a sufficient set for the space of entire functions of order less than ρ.

§2.5. A Related Theorem.

Iyer's notion of an effective set (see Section 1.3) dealt with the space $C(\rho,d)$ of entire functions of order ρ and type less than d. A set $S = [z_n]$ is effective for $C(\rho,d)$ if S is "rich enough" so that the type of any function as determined on S is equal to the type of the function. A sufficient set for the space E_ρ of entire functions of order less than ρ determines the type of a function in E_ρ in the following sense. There is a constant $C > 0$ such that if $f \in E_\rho$ and f has type d_1 on S, then f has type $\le Cd_1$. If K is countably determined, then a result of this type follows from the fact that sufficient implies that for every n there is an m such that $E(\phi_n,S)$ is continuously injected into $E(\phi_m)$ (see Theorem 1.2.8). However the following proposition gives an explicit upper bound on the type of f from its type on S.

Proposition 2.5.1. Let $S \subset \mathbb{C}^n$ with $d(z,S) \le |z|^{1-\rho/2}$ for all z. Let $f \in E_\rho$ have order $\rho_1 < \rho$. Then

$$\limsup_{\substack{|z|\to\infty \\ z\in S}} \frac{\log |f(z)|}{|z|^{\rho_1}} = d_1$$

implies

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_1}} \le Cd_1$$

where $C = d_1 B_1 B_2^{\rho_1}$ and B_1, B_2 are the constants determined in (2.4.1).

Proof. Let $f \in E_\rho$ with

$$\limsup_{\substack{|z|\to\infty \\ z\in S}} \frac{\log |f(z)|}{|z|^{\rho_1}} = d_1, \qquad \rho_1 < \rho.$$

Thus, given $\varepsilon > 0$ there is an r_o such that

$$\log |f(z)| \le (d_1 + \varepsilon)|z|^{\rho_1}$$

for $|z| \ge r_o$, $z \in S$. Letting $A = \max(M(r_o,f,S), 1)$ we obtain

$$M(r,f,S) \le A \exp((d_1 + \varepsilon)r^{\rho_1}), \qquad r > 0. \tag{2.5.2}$$

Case 1. $f(0) = \beta$.

By Theorem 2.3.16 and inequality (2.5.2) we obtain

$$\log M(r,f) \le B_1 \log M(B_2 r,f,S) + B_3\left(\frac{\log M(B_4 r,f)}{r^{\rho/2}}\right)^2$$

$$\le B_1 \log A + B_1(d_1+\varepsilon)(B_2 r)^{\rho_1} + B_3\left(\frac{\log M(B_4 r,f)}{r^{\rho/2}}\right)^2.$$

Since $\log M(B_4 r,f) = O(r^{\rho_1})$ and $\log M(B_4 r,f) = o(r^{\rho})$ we obtain

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_1}} \le (d_1 + \varepsilon)B_1 B_2^{\rho_1}.$$

Case 2. $f(0) \ne \beta$.

If $|f(z)| > \beta$ (resp. $|f(z)| < \beta$) we apply Case 1 to the function $\beta f(z)/f(0)$ (resp. $f(z) - (f(0) - \beta)$) and deduce the same inequality (see the proof of Theorem 2.2.1).

CHAPTER 3

SUFFICIENT SETS FOR (E,K) WHEN THE GROWTH IN THE REAL DIRECTIONS DIFFERS FROM THE GROWTH IN THE IMAGINARY DIRECTIONS

§3.1. Introduction.

In this chapter we shall determine sufficient sets for spaces (E,K) of entire functions on $\mathbb{C}^n$ when the growth rate in the real directions is different from the growth rate in the imaginary directions (see Theorem 3.2.8). Some special cases of the results obtained here were stated without proof by Ehrenpreis [2;147,161] (see §3.3, Examples 3.4.1 and 3.4.3).

§3.2. The Main Theorem.

If ρ is a non-negative real number, let P be a class of continuous positive functions $p(r)$ defined for $r > 0$ which satisfy

(3.2.0) $\qquad p(r) = o(r^{\rho}),$

(3.2.1) $\qquad p(\text{const } r) \geq \text{const } p(r),$

(3.2.2) $\qquad p(r_1 + r_2) \geq \text{const } (p(r_1) + p(r_2)),$

(3.2.3) $\qquad p(r_1 r_2) \geq \text{const } p(r_1)p(r_2),$

(3.2.4) $\qquad p(r_1^{\varepsilon}) + p(r_2^{\varepsilon}) \geq \text{const } p((r_1^2 + r_2^2)^{\varepsilon/2}),$

where the constant in (3.2.4) depends on ε and $0 < \varepsilon < 1$. Let K_a^{ρ} (resp. K_s^{ρ}) be a class of continuous positive functions of the form $k(z) = k_1(z)k_2(z)$ where $k_1(z)$ dominates $\exp(p(|z|))$ for every $p \in P$ (resp. $k_1(z) = \exp(p(|z|))$ for some $p \in P$) and $k_2(z) = k_2(re^{i\theta})$ dominates $\exp(Br^{\rho}|\sin\theta|)$ for all $B > 0$. It is further required that if $k \in K_a^{\rho}$ (resp. K_s^{ρ}) and $\phi_i = \log k_i$, $i = 1, 2$, then ϕ_1 and ϕ_2 should have minorants $\tilde{\phi}_1$ and $\tilde{\phi}_2$ such that

$$\tilde{k}(z) = \exp\,(\tilde{\phi}_1(z) + \tilde{\phi}_2(z)) \in K_a^\rho \text{ (resp. } K_s^\rho),$$

and for $i = 1, 2$,

$$\tilde{\phi}_i(r_1 + r_2) \le \text{const}\,(\tilde{\phi}_i(r_1) + \tilde{\phi}_i(r_2)), \tag{3.2.5}$$

$$\tilde{\phi}_i(r_1 r_2) \le \text{const}\,\tilde{\phi}_i(r_1)\,\tilde{\phi}_i(r_2), \tag{3.2.6}$$

$$\int_{-\infty}^{\infty} \frac{\tilde{\phi}_i(|x|^\varepsilon)}{1 + x^2}\,dx < \infty, \qquad 0 < \varepsilon < 1/\rho. \tag{3.2.7}$$

For the class of weight functions K_a^ρ (resp. K_s^ρ) the space (E,K_a^ρ) (resp. (E,K_s^ρ)) is the space of entire functions of order ρ which satisfy

$$|f(z)| \le \text{const}\; e^{p(|z|)}\; e^{B|z|^\rho |\sin\theta|}, \quad z = re^{i\theta},$$

for some $B > 0$ and some $p \in P$ (resp. for all $p \in P$).

Theorem 3.2.8. Let $S = \bigcup_{j=1}^{m} \{re^{i\alpha_j} : r \ge 0\}$ be a union of rays such that the positive and negative real axis are in S and the angular opening of each sector formed by these rays is less than π/ρ. Then S is a sufficient set for (E,K_a^ρ) and (E,K_s^ρ).

Proof. Let $k \in K_a^\rho$ (resp. K_s^ρ) be given. For each of the sectors Λ_j formed we shall show how to construct a $k_j \in K_a^\rho$ (resp. K_s^ρ) so that $f \in (E,K_a^\rho)$ (resp. (E,K_s^ρ)) and $|f(z)| \le k_j(z)$ for $z \in \text{bdry}\,\Lambda_j$, then $|f(z)| \le k(z)$ for $z \in \Lambda_j$. There are only two different types of sectors - a sector with one side on the real axis or a sector completely contained in the upper (or lower) half-plane. (Note that for $\rho < 2$ there may be no sectors of the second type.) We shall carry out the details of the proof for the first two sectors formed (see Figure 1). The other cases are similar.



Figure 1

The angular opening of both Λ_1 ($\beta = 0$) and Λ_2 ($\beta > 0$) is denoted by α. The condition $\beta = 0$ (resp. $\beta > 0$) will determine the sector Λ_1 (resp. Λ_2).

The idea of the proof is as follows. Map the sector Λ onto the upper half plane by means of the mapping $z \to (ze^{-i\beta})^{\pi/\alpha} = \lambda$. Let $\log k(z) = \phi_1(|z|) + \phi_2(z)$. Transfer these growth conditions to the upper half plane by means of

$$M_1(\lambda) = \phi_1(|z|), \qquad M_2(\lambda) = \phi_2(z).$$

Let M_1^* and M_2^* be the Poisson integrals in the upper half plane of M_1 and M_2, and let M_1^{**} and M_2^{**} be conjugate functions of M_1^* and M_2^*. It will follow that if

$$\tilde{\tilde{k}}(z) = \exp\,[M_1^*(\lambda) + M_2^*(\lambda) + i\,(M_1^{**}(\lambda) + M_2^{**}(\lambda))]$$

and if $|f(z)| \le |\tilde{\tilde{k}}(z)|$ on the boundary of Λ_1, then

$$|f(z)/\tilde{\tilde{k}}(z)| \to 0 \qquad \text{as } |z| \to \infty,\ z \in \Lambda.$$

This implies, by the maximum principle, that $|f(z)| \le |\tilde{\tilde{k}}(z)|$ for $z \in \Lambda$.

Let us now proceed with the details of the proof. Let $k \in K_a^\rho$ (resp. K_s^ρ) with $\log k = \phi_1 + \phi_2$. We may suppose that $\phi_2(re^{i\theta}) = \omega(r)r^\rho|\sin\theta|$ where $\omega(r) \to +\infty$ as $r \to +\infty$, $\phi_1(o) = \phi_2(o) = 0$, and ϕ_1 and ϕ_2 satisfy (3.2.5) - (3.2.7).

Map the sector Λ onto the upper half-plane by means of the mapping $z \to (ze^{-i\beta})^{\pi/\alpha} = \lambda$. Define

(3.2.9) $$M_1(\lambda) = \phi_1(|z|),$$

(3.2.10) $$M_2(\lambda) = \phi_2(z).$$

Observe that when M_1 is restricted to the real axis it satisfies (3.2.5) and (3.2.6). Also

(3.2.11) $$\int_{-\infty}^{\infty} \frac{M_i(x)}{1+x^2}dx < +\infty, \qquad i = 1, 2.$$

Let M_1^* and M_2^* denote the Poisson integrals in the upper half-plane

of M_1 and M_2 respectively.

Lemma 3.2.12. There is a constant c_1 such that $M_1^*(x,y) \le c_1 M_1(|\lambda|)$ where $\lambda = x + iy$.

Proof. Using the definition of the Poisson integral and the subadditivity of M_1 on the real axis we have

$$M_1^*(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yM_1(t)}{(x-t)^2 + y^2}dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yM_1(x+t)}{t^2 + y^2}\,dt$$

$$\le \frac{\text{const}}{\pi}\int_{-\infty}^{\infty} \frac{yM_1(x)}{t^2 + y^2}\,dt + \frac{\text{const}}{\pi}\int_{-\infty}^{\infty} \frac{yM_1(t)}{t^2 + y^2}\,dt.$$

The first term is const $M_1(x)$. Using (3.2.5) to estimate the second term we obtain

$$\int_{-\infty}^{\infty} \frac{yM_1(t)}{t^2 + y^2}\,dt = \int_{-\infty}^{\infty} \frac{M_1(t)}{1 + (t/y)^2}\,\frac{dt}{y} = \int_{-\infty}^{\infty} \frac{M_1(y \cdot t/y)}{1 + (t/y)^2}\,\frac{dt}{y}$$

$$\le \text{const } M_1(y)\int_{-\infty}^{\infty} \frac{M_1(t/y)}{1 + (t/y)^2}\,\frac{dt}{y} = \text{const } M_1(y).$$

Thus we obtain $M_1^*(x,y) \le \text{const } (M_1(x) + M_1(y)) \le \text{const } M_1(|\lambda|)$ where $\lambda = x + iy$. This completes the proof.

Lemma 3.2.13. If $p \in P$ and $\phi_1(|x|)$ dominates $p(|x|)$, then $M_1^*(x,y)$ dominates $p(|\lambda|)$ where $\lambda = x + iy$.

Proof. We have

$$M_1^*(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yM_1(t)}{(x-t)^2 + y^2}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yM_1(x+t)}{t^2 + y^2}\,dt.$$

Since $\phi_1(|x|)$ dominates $p(|x|)$ for $|x|$ large, $M_1(x)$ dominates $p(|x|^{\alpha/\pi})$ for $|x|$ large. From this fact, (3.2.2), and (3.2.1) we obtain

$$M_1(x + t) \ge \text{const} + p(|x + t|^{\alpha/\pi})$$

$$\ge \text{const} + p(c|x|^{\alpha/\pi}) + p(c|t|^{\alpha/\pi})$$

$$\ge \text{const} + \text{const}\,[p(|x|^{\alpha/\pi}) + p(|t|^{\alpha/\pi})].$$

Consequently,

$$M_1^*(x,y) \geq \text{const} \int_{-\infty}^{\infty} \frac{y}{t^2+y^2}\,dt + \text{const} \int_{-\infty}^{\infty} \frac{p(|x|^{\alpha/\pi})y}{t^2+y^2}\,dt +$$

$$\text{const} \int_{-\infty}^{\infty} \frac{p(|t|^{\alpha/\pi})y}{t^2+y^2}\,dt.$$

The first term is a constant; the second is const $p(|x|^{\alpha/\pi})$. Using (3.2.3) and (3.2.0) to estimate the third term we obtain

$$\text{const} \int_{-\infty}^{\infty} \frac{p(y^{\alpha/\pi}|t/y|^{\alpha/\pi})y}{t^2+y^2}\,dt \geq \text{const}\, p(|y|^{\alpha/\pi}) \int_{-\infty}^{\infty} \frac{p(|t/y|^{\alpha/\pi})y}{t^2+y^2}\,dt$$

$$= \text{const}\, p(|y|^{\alpha/\pi}).$$

Combining these inequalities we obtain

$$M_1^*(x,y) \geq \text{const}\,(p(|x|^{\alpha/\pi}) + p(|y|^{\alpha/\pi}))$$

$$\geq \text{const}\, p(|\lambda|^{\alpha/\pi}), \qquad \lambda = x + iy, \quad \text{by (3.2.4).}$$

Lemma 3.2.14. For all $x,y \in R$ with $y > 0$,

(i) $M_2(x+y) \leq \text{const}\,(M_2(x) + M_2(y))$,

(ii) $M_2(xy) \leq \text{const}\, M_2(x)\, M_2(-y)$ if $x < 0$.

Proof of (i). If $x + y \geq 0$, the result is clear from (3.2.8). If $x + y < 0$ and $x < 0$, $y > 0$, then $x < x + y < 0$ and so $M_2(x+y) \leq M_2(x) \leq M_2(x) + M_2(y)$. Finally if $x < 0$ and $y < 0$,

$$M_2(x+y) = \phi_2(|x+y|^{\alpha/\pi} e^{i(\alpha+\beta)})$$

$$\leq \phi_2(C|x|^{\alpha/\pi} e^{i(\alpha+\beta)} + C|y|^{\alpha/\pi} e^{i(\alpha+\beta)})$$

$$\leq \text{const}\,[\phi_2(C|x|^{\alpha/\pi} e^{i(\alpha+\beta)} + \phi_2(C|y|^{\alpha/\pi} e^{i(\alpha+\beta)})]$$

$$\leq \text{const}\,[\phi_2(|x|^{\alpha/\pi} e^{i(\alpha+\beta)}) + \phi_2(|y|^{\alpha/\pi} e^{i(\alpha+\beta)})]$$

$$= \text{const}\,[M_2(x) + M_2(y)].$$

Proof of (ii). From (3.2.8) we have

$$M_2(x) = \begin{cases} \phi_2(|x|^{\alpha/\pi} e^{i\beta}), & x > 0, \\ \phi_2(|x|^{\alpha/\pi} e^{i(\alpha+\beta)}), & x < 0, \end{cases}$$

Consequently,

$$\begin{aligned} M_2(xy) &= \phi_2(|xy|^{\alpha/\pi} e^{i(\alpha+\beta)}) \\ &= \phi_2(e^{-i(\alpha+\beta)}|x|^{\alpha/\pi} e^{i(\alpha+\beta)}|y|^{\alpha/\pi} e^{i(\alpha+\beta)}) \\ &\le \text{const } \phi_2(|x|^{\alpha/\pi} e^{i(\alpha+\beta)})\ \phi_2(|y|^{\alpha/\pi} e^{i(\alpha+\beta)}) \\ &= \text{const } M_2(x)M_2(-y). \end{aligned}$$

Lemma 3.2.15. For all $x,y \in R$ with $y > 0$,

$$\text{(i)} \qquad M_2^*(x,y) \le \text{const } (M_2(x) + M_2(-y)).$$

If in addition $x > 0$ and $\beta = 0$, then

$$\text{(ii)} \qquad M_2^*(x,y) \le \text{const } (y/x)M_2(-x).$$

Proof of (i). From Lemma 3.2.4 (i) we have

$$\begin{aligned} M_2^*(x,y) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yM_2(x+t)}{t^2+y^2}\,dt \\ &\le \text{const } M_2(x)\int_{-\infty}^{\infty} \frac{y}{t^2+y^2}\,dt + \text{const}\int_{-\infty}^{\infty} \frac{yM_2(t)}{t^2+y^2}\,dt. \end{aligned}$$

The first term is equal to const $M_2(x)$. We estimate the second using Lemma 3.2.4 (ii) to obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{yM_2(t)}{t^2+y^2}\,dt &\le \text{const } M_2(-y)\int_{-\infty}^{\infty} \frac{M_2(t/y)}{1+(t/y)^2}\,\frac{dt}{y} \\ &= \text{const } M_2(-y). \end{aligned}$$

Combining this estimate with the previous one completes the proof of the first assertion.

Proof of (ii). Since $x > 0$, $M_2(x) = 0$ and so $M_2(t + x) \le \text{const } M_2(t)$. Thus

$$\begin{aligned} M_2^*(x,y) &= \frac{1}{\pi}\int_{-\infty}^{0} \frac{yM_2(t)}{(x-t)^2+y^2}\,dt \\ &= \frac{1}{\pi}\int_{-\infty}^{-x} \frac{yM_2(x+t)}{t^2+y^2}\,dt \end{aligned}$$

$$\leq \text{const} \int_{-\infty}^{-x} \frac{yM_2(t)}{t^2 + y^2}\, dt$$

$$\leq \text{const} \int_{-\infty}^{-x} \frac{yM_2(t)}{t^2}\, dt$$

$$= \text{const}\ y \int_{-\infty}^{-x} \frac{M_2(xt/x)}{t^2}\, dt$$

$$\leq \text{const}\ (y/x)\ M_2(-x) \int_{-\infty}^{-x} \frac{M_2(t/x)}{(t/x)^2}\, \frac{dt}{x} \quad \text{(by Lemma 3.2.14 (ii))}$$

$$\leq \text{const}\ (y/x)\ M_2(-x).$$

Lemma 3.2.16. Let $z = re^{i\gamma}$ where $\alpha/4 \leq \gamma \leq \alpha$ if $\beta = 0$ and $\beta \leq \gamma \leq \alpha+\beta$ if $\beta > 0$, and let $\lambda = (ze^{-i\beta})^{\pi/\alpha} = x + iy$. Then there is a constant K which depends only on α and β such that

$$M_2(x) + M_2(-y) \leq 2K\phi_2(z).$$

Proof. We shall carry out the proof in the case where $\beta = 0$. The case where $\beta > 0$ is similar.

Since $\lambda = z^{\pi/\alpha} = x + iy$, we have ($r = |z|$)

$$x = r^{\pi/\alpha}\cos\ (\pi\gamma/\alpha), \qquad y = r^{\pi/\alpha}\sin\ (\pi\gamma/\alpha)$$

Thus

$$M_2(x) = \begin{cases} \phi_2(r|\cos\ (\pi\gamma/\alpha)|^{\alpha/\pi}\ \exp\ (i\alpha)), & \alpha/2 \leq \gamma \leq \alpha, \\ \phi_2(0), & 0 \leq \gamma \leq \alpha/2, \end{cases}$$

and

$$M_2(-y) = \phi_2(r|\sin\ (\pi\gamma/\alpha)|^{\alpha/\pi}\ \exp\ (i\alpha)).$$

Since $\gamma \geq \alpha/4$ there is a constant K depending on α such that $\sin\alpha \leq K \sin\gamma$. Thus

$$\phi_2(r|\sin\ (\pi\gamma/\alpha)|^{\alpha/\pi}\exp\ (i\alpha)) = r^\rho |\sin\ (\pi\gamma/\alpha)|^{\rho\alpha/\pi}|\sin\alpha|\ \omega(r\ |\sin(\pi\gamma/\alpha)|^{\alpha/\pi})$$

$$\leq Kr^\rho|\sin\gamma|\omega(r)$$

$$= K\phi_1(z)$$

since $|\sin(\pi\gamma/\alpha)|^{\rho\alpha/\pi} \le 1$ and $|\sin(\pi\gamma/\alpha)|^{\alpha/\pi} \le 1$. The same inequality holds for $M_2(x)$ and the proof is complete.

Lemma 3.2.17. If $\beta = 0$, $z = re^{i\gamma}$ with $0 \le \gamma \le \alpha/4$, and $\lambda = z^{\pi/\alpha} = x + iy$, then

$$M_2^*(x,y) \le \text{const } \phi_2(z)$$

Proof. Since $0 \le \gamma \le \alpha/4$, $x = r^{\pi/\alpha}\cos(\pi\gamma/\alpha) > 0$ and so

$$\begin{aligned} M_2(-x) &= \phi_2(|r|\ |\cos(\pi\gamma/\alpha)|^{\alpha/\pi}e^{i\alpha}) \\ &= r^\rho |\cos(\pi\gamma/\alpha)|^{\frac{\rho\alpha}{\pi}}(\sin\alpha)\omega(r|\cos(\pi\gamma/\alpha)|^{\alpha/\pi}) \\ &\le r^\rho\ \omega(r). \end{aligned}$$

Also $y/x = \tan(\pi\gamma/\alpha) \le \text{const} \sin(\pi\gamma/\alpha) \le \text{const} \sin\gamma$. Thus

$$\begin{aligned} M_2^*(x,y) &\le \text{const}\ (y/x)M_2(-x) \\ &\le \text{const}(\sin\gamma)r^\rho\ \omega(r) \\ &= \text{const}\ \phi_2(z). \end{aligned}$$

Lemma 3.2.18. Let $z = re^{i\gamma}$ and $\lambda = (ze^{i\beta})^{\pi/\alpha} = x + iy$.

(i) If $\beta = 0$ and $0 \le \gamma \le \alpha/4$, then $M_2^*(\lambda)$ dominates $A|z|^\rho \sin\gamma$ for all $A > 0$.

(ii) If $\beta = 0$ and $\gamma > \alpha/4$, or if $\beta > 0$, then $M_2^*(\lambda)$ dominates $A|z|^\rho$ for all $A > 0$.

Proof of (i). From the definition of the Poisson integral we obtain

$$M_2^*(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yM_2(x+t)}{t^2+y^2}\,dt \ge \frac{1}{\pi}\int_{-\infty}^{-x} \frac{yM_2(x+t)}{t^2+y^2}\,dt.$$

Since $\beta = 0$ and $0 \le \gamma \le \alpha/4$ it follows that $x \ge y \ge 0$. Thus $-x \le 0$ and so for $t \in (-\infty,-x)$, $x + t < 0$. It follows that for any $A > 0$ there is an A' (possibly negative) so that

$$M_2(x+y) \ge A' + A|x+t|^{\rho\alpha/\pi}.$$

Thus

$$\pi M_2^*(x,y) \ge A'\int_{-\infty}^{-x} \frac{y}{t^2+y^2}\,dt + A\int_{-\infty}^{-x} \frac{y|x+t|^{\rho\alpha/\pi}}{t^2+y^2}\,dt.$$

The first term on the right hand side is $A'(\frac{\pi}{2} - \tan^{-1}\frac{x}{y})$. To estimate the second we observe that there is a constant (depending on α and ρ) such that

$$|x + t|^{\rho\alpha/\pi} \geq \text{const}\left[|x|^{\rho\alpha/\pi} + |t|^{\rho\alpha/\pi}\right].$$

Thus

$$\int_{-\infty}^{-x} \frac{y|x+t|^{\rho\alpha/\pi}}{t^2 + y^2}\,dt \geq \text{const}\ |x|^{\rho\alpha/\pi} \int_{-\infty}^{-x} \frac{y}{t^2 + y^2}\,dt + \text{const} \int_{-\infty}^{-x} \frac{y|t|^{\rho\alpha/\pi}}{t^2+y^2}\,dt.$$

The first term is equal to const $|x|^{\rho\alpha/\pi}(\frac{\pi}{2} - \tan^{-1}\frac{x}{y})$. We estimate the second as follows. Since $t \leq -x \leq -y$ and $y > 0$, $|t/y| > 1$. Thus

$$\int_{-\infty}^{-x} \frac{y|t|^{\rho\alpha/\pi}}{t^2 + y^2}\,dt = y^{\rho\alpha/\pi} \int_{-\infty}^{-x} \frac{y|t/y|^{\rho\alpha/\pi}}{t^2 + y^2}\,dt \geq y^{\rho\alpha/\pi} \int_{-\infty}^{-x} \frac{y}{t^2 + y^2}\,dt$$

$$= y^{\rho\alpha/\pi}(\tfrac{\pi}{2} - \tan^{-1}\tfrac{x}{y}).$$

Combining these inequalities we obtain

$$\pi M_2^*(x,y) \geq (A' + \text{const}\ A|x|^{\rho\alpha/\pi} + \text{const}\ A|y|^{\rho\alpha/\pi})\ (\tfrac{\pi}{2} - \tan^{-1}\tfrac{x}{y}).$$

Now for $|\lambda| = |x + iy|$ large the right hand side is larger than const $A|\lambda|^{\rho\alpha/\pi}(\frac{\pi}{2} - \tan^{-1}\frac{x}{y})$.

For $|u| > 1$, $\tan^{-1}u < \frac{\pi}{2} - \frac{1}{2u}$ and since $x/y > 1$ we have

$$\frac{\pi}{2} - \tan^{-1}\frac{x}{y} \geq \frac{1}{2}\frac{y}{x} = \frac{1}{2}\tan\frac{\pi\gamma}{\alpha} \geq \frac{1}{2}\frac{\pi\gamma}{\alpha} \geq \text{const}\ \sin\gamma.$$

Thus for $|\lambda|$ large,

$$M_2^*(x,y) \geq \text{const}\ A|\lambda|^{\rho\alpha/\pi}(\sin\gamma)$$

$$= \text{const}\ A|z|^{\rho}\sin\gamma.$$

Proof of (ii). In this case, $\gamma \geq \gamma_o$ where $\gamma_o = \alpha/4$ if $\beta = 0$ and $\gamma_o = \beta$ otherwise. This implies $x \leq cy$ where c is a positive constant depending

on α and β. Thus

$$M_2^*(\lambda) \geq \frac{1}{\pi} \int_{-\infty}^{-cy} \frac{yM_2(x+t)}{t^2 + y^2}\, dt.$$

Estimating the right hand side of this inequality as in the proof of part (i) we obtain the result.

We now complete the proof of Theorem 3.2.8. First, consider the case $\Lambda = \Lambda_1$ ($\beta = 0$). Let c_1 be the constant appearing in Lemma 3.2.12, let c_2 be the larger of the constants appearing in Lemmas 3.2.15 (i) and 3.2.17, and let K be the constant appearing in Lemma 3.2.16. For $z \in \Lambda$ set

$$\tilde{k}(z) = \exp\left[\frac{M_1(\lambda)}{c_1} + \frac{M_2(\lambda)}{2Kc_2}\right],$$

$$\tilde{\tilde{k}}(z) = \exp\left[\frac{M_1^*(\lambda)}{c_1} + \frac{M_2^*(\lambda)}{2Kc_2} + i\left(\frac{M_1^{**}(\lambda)}{c_1} + \frac{M_2^{**}(\lambda)}{2Kc_2}\right)\right].$$

Thus $|\tilde{\tilde{k}}(z)| = \tilde{k}(z)$ for $z \in$ bdry Λ and $\tilde{k} \in K_a^\rho$ (resp. K_s^ρ). If $f \in (E,K_a^\rho)$ (resp. (E,K_s^ρ) and $|f(z)| \leq \tilde{k}(z)$ for $z \in$ bdry Λ, it follows that $|f(z)| \leq |\tilde{\tilde{k}}(z)|$ for $z \in$ bdry Λ. Moreover, by Lemmas 3.2.13 and 3.2.18, it follows that $|f(z) / \tilde{\tilde{k}}(z)| \to 0$ as $|z| \to \infty$, $z \in \Lambda$. Consequently, the maximum modulus principle implies that $|f(z)| \leq \tilde{\tilde{k}}(z)$ for $z \in \Lambda$. There are two cases to be considered; when $0 \leq \arg z \leq \alpha/4$, and when $\alpha/4 \leq \arg z \leq \alpha$.

Case 1. $0 \leq \arg z \leq \alpha/4$, $z \in \Lambda$.

It follows from (3.2.19) by the maximum principle that

$$\begin{aligned} |f(z)| \leq |\tilde{\tilde{k}}(z)| &\leq \exp\left[\frac{M_1^*(\lambda)}{c_1} + \frac{M_2^*(\lambda)}{2Kc_2}\right] \\ &\leq \exp\left[M_1(|\lambda|) + \phi_2(z)\right], \text{ by Lemmas 3.2.12, 3.2.17} \\ &\leq \exp\left[\phi_1(|z|) + \phi_2(z)\right] \\ &= k(z). \end{aligned}$$

Case 2. $\alpha/4 \le \arg z \le \alpha, \quad z \in \Lambda.$

Again, it follows from (3.2.19) by the maximum modulus theorem that in

$$|f(z)| \le \exp\left[\frac{M_1^*(\lambda)}{c_1} + \frac{M_2^*(\lambda)}{2Kc_2}\right]$$

$$\le \exp\left[\phi_1(|z|) + \frac{M_2(x) + M_2(-y)}{2K}\right], \quad \text{by Lemma 3.2.15(i)}$$

$$\le \exp\left[\phi_1(|z|) + \phi_2(z)\right], \quad \text{by Lemma 3.2.16}$$

$$= k(z).$$

Thus $|f(z)| \le \tilde{k}(z)$ for $z \in$ bdry Λ implies $|f(z)| \le k(z)$ for $z \in \Lambda$. This completes the proof when $\Lambda = \Lambda_1$. The proof when $\Lambda = \Lambda_2$ $(\beta > 0)$ is similar.

§3.3 Extensions to $\mathbb{C}^n$.

The extension of Theorem 3.2.8 to $\mathbb{C}^n$ is obtained as follows [2;135]. Let K_a^ρ (resp. K_s^ρ) be a family of continuous positive functions $k(z) = k^1(z_1) \ldots k^n(z_n)$ where $k^j(z_j)$, $j = 1, \ldots, n$ is a function satisfying the conditions in Section 3.2. (E,K_a^ρ) (resp. (E,K_s^ρ)) consists of all entire functions on $\mathbb{C}^n$ of order ρ which satisfy

$$|f(z)| \le \text{const} \exp\left[\sum_1^n p_i(|z_i|)\right] \exp\left[\sum_1^n B_i r_i^\rho \, |\sin \theta_i|\right]$$

where $z = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})$.

Theorem 3.3.1. For $j = 1, \ldots, n$ let S_j be a union of rays as described in Theorem 3.2.8 in z_j space. Then $S = S_1 \times \ldots \times S_n$ is a sufficient set for (E,K_a^ρ) and (E,K_s^ρ).

Proof. Let $k \in K_a^\rho$ (resp. K_s^ρ) be given. If $\Lambda = \prod_1^n \Lambda^j$ is a product of sectors Λ^j in the z_j plane, construct $\tilde{k}(z) = \tilde{k}^1(z_1) \ldots \tilde{k}^n(z_n)$ and

$\tilde{\tilde{k}}(z) = \tilde{\tilde{k}}'(z_1)\ldots k^n(z_n)$ where $\tilde{k}^j$ and $\tilde{\tilde{k}}^j$ are constructed for k^j and Λ^j as in §3.2. Then $|\tilde{\tilde{k}}(z)| = \tilde{k}(z)$ for z on the distinguished boundary of Λ and $\tilde{k} \in K_a^\rho$ (resp. K_s^ρ). Let $f \in (E,K_a^\rho)$ (resp. (E,K_s^ρ)) with $|f(z)| \leq \tilde{k}(z)$, z on the distinguished boundary of Λ. Since $|f(z)/\tilde{\tilde{k}}(z)| \to 0$ as $|z| \in \Lambda$, $z \in \Lambda$, the maximum modulus theorem implies that f assumes its maximum on the distinguished boundary of Λ. The remainder of the proof is the same as that of Theorem 3.2.8.

Theorem 3.3.2. For each j, let $\mathbb{R}_j$ be the real axis in the complex z_j plane. Let S be a sufficient set for the space of entire functions on $\mathbb{C}$ of order $\leq \rho$. Then $(S\cup\mathbb{R}_1)\times\ldots\times(S\cup\mathbb{R}_n)$ is a sufficient set for (E,K_a^ρ) and (E,K_s^ρ).

Proof. We shall carry out the details of the proof for the case n = 1. The extension to n > 1 is accomplished using the same technique as was used in the proof of Theorem 3.3.1.

Let S_ℓ be a sufficient set for (E,K_a^ρ) consisting of half lines through the origin as described in Theorem 3.3.1. Let $k = k_1k_2 \in K_a^\rho$ be given. There is a $k' = k_1'k_2' \in K_a^\rho$ such that $|f(z)| \leq k'(z)$ on S_ℓ implies $|f(z)| \leq k(z)$ on $\mathbb{C}$. Let $L_1,\ldots,L_n$ denote the half lines in S_ℓ which do not lie on the real axis, and for $j = 1,\ldots,n$, let α_j be the angle which L_j makes with the positive x-axis. Then, for each $j = 1,\ldots,n$, $k_2'(|z|e^{i\alpha_j}) \in K^\rho$ where K^ρ is a class of weight functions for the space of entire functions of order $\leq \rho$. Since S is sufficient for (E,K^ρ), there is a $k_{2,j}''$ in K^ρ such that

$$|f(z)| \leq k_{2,j}''(z) \text{ on } S \implies |f(z)| \leq k_2'(|z|e^{i\alpha_j}) \text{ on } \mathbb{C}.$$

Let $k_2''(z) = \inf\{k_{2,j}''(z) : j = 1,2,\ldots,n\}$. Then

$$|f(z)| \leq k_2''(z) \text{ on } S \implies |f(z)| \leq k_2'(|z|e^{i\alpha_j}) \text{ on } \mathbb{C}$$

for $j = 1,\ldots,n$. Choose k_1''', k_2''' such that

(i) $k_1'''(z) \le k_1'(z)$ on $\mathbb{C}$.

(ii) $k_1'''(z)k_2'''(z) \le k_2''(z)$ on $\mathbb{C}$.

(iii) $k_1'''k_2''' \in K_a^\rho$.

Let $k'''(z) = k_1'''(z)k_2'''(z)$ and suppose that

$$|f(z)| \le k'''(z) \text{ on } S \cup R.$$

Then

$$\begin{cases} |f(z)| \le k_2''(z) \text{ on } S \\ |f(x)| \le k_1'''(x) \text{ on } \mathbb{R} \end{cases}$$

$$\Rightarrow \begin{cases} |f(z)| \le k_2'(|z|e^{i\alpha_j}) \text{ on } \mathbb{C} \qquad j = 1, 2, \ldots, n \\ |f(x)| \le k_1'(x) \text{ on } \mathbb{R} \end{cases}$$

$$\Rightarrow |f(z)| \le k'(z) \text{ on } S_\ell$$

$$\Rightarrow |f(z)| \le k(z) \text{ on } \mathbb{C}.$$

This completes the proof.

§3.4. Some Examples.

Example 3.4.1. The Space $\widehat{\mathcal{E}'}$ of Fourier Transforms of Distributions with Compact Support.

Let $\mathcal{E}$ denote the space of infinitely differentiable functions on $\mathbb{R}^n$ with the topology of uniform convergence of functions and all their derivatives on compact sets. Then $\mathcal{E}'$ is the space of distributions with compact support and $\widehat{\mathcal{E}'}$ denotes the space of Fourier transforms of elements of $\mathcal{E}'$. According to the Paley-Wiener Theorem [7;21], the space $\widehat{\mathcal{E}'}$ can be identified with the space of entire functions of exponential type on $\mathbb{C}^n$ which have polynomial growth in the real directions. Thus $f \in \widehat{\mathcal{E}'}$ if and only if there are constants C, N, and A such that

$$|f(z)| \leq C(1 + |z|)^N e^{A|\mathrm{Im}z|}$$

for all $z \in \mathcal{C}^n$.

It is easy to see that if K_a^ρ consists of the family of all continuous positive functions $k(z) = k_1(|z|)k_2(|\mathrm{Im}z|)$ where k_1 dominates all polynomials and k_2 dominates all linear exponentials, then $(E,K_a^\rho) = \hat{\mathcal{E}}'$. Consequently, Theorem 3.3.1 yields the following result which is stated without proof in [2;161].

Theorem 3.4.2. For each j let $\mathbb{R}_j$ and A_j denote respectively the real axis and any other line through the origin in the z_j plane. Then $(\mathbb{R}_1 \cup A_1)\times\ldots\times(\mathbb{R}_n \cup A_n)$ is sufficient set for $\hat{\mathcal{E}}'$.

Example 3.4.3. The Space of Fourier Transforms of C^∞ Functions with Compact Support.

If $m > 0$ is an integer and $r > 0$, let $\mathcal{D}_r^m$ be the space of complex valued functions on $\mathbb{R}^n$ which are m times continuously differentiable and have their support in the ball $\{x : |x| \leq r\}$ (the support of a function is the closure of the set where it is different from zero). The topology is that of uniform convergence of functions and their first m derivatives. Let $\mathcal{D}^m = \cup \mathcal{D}_r^m$ and equip $\mathcal{D}^m$ with the inductive limit topology from the spaces $\mathcal{D}_r^m$. Finally, $\mathcal{D}_F = \cap \mathcal{D}^m$ and $\mathcal{D}_F$ is equipped with the projective limit topology from the spaces $\mathcal{D}^m$.

By the Paley-Wiener Theorem [7;21] the space $\hat{\mathcal{D}}_F$ of Fourier transforms of functions in $\mathcal{D}_F$ is the space of entire functions of exponential type which decay faster than the reciprocal of any polynomial in the real directions. Thus $f \in \hat{\mathcal{D}}_F$ if and only if there is a constant $A > 0$ such that for every positive integer N there is a constant C_N such that

$$|f(z)| \leq C_N(1 + |z|)^{-N} e^{A|\mathrm{Im}z|}.$$

Let K_s^ρ be the family of continuous positive functions $k(z) = k_1(|z|)k_2(|\mathrm{Im}z|)$ where $k_1(|z|) = (1 + |z|)^{-N}$, N a non-negative integer, and k_2 dominates all linear exponentials. Then [2;144] $(E,K_s^\rho) = \hat{\mathcal{D}}_F$ and Theorem 3.3.1 yields the following result [2;147].

Theorem 3.4.4. Let R_j and A_j, $j = 1, \ldots, n$, be as in Theorem 3.4.1. Then $(R_1 \cup A_1) \ldots (R_n \cup A_n)$ is a sufficient set for $\hat{\mathcal{D}}_F$.

Example 3.4.5. A Counterexample.

In this example we shall show that there is no analogue of Theorem 3.2.8 for the space E of entire functions of order 2 which satisfy

$$|f(z)| \leq Ce^{A_1|z|}\, e^{A_2|\mathrm{Im}z|^2}. \tag{3.4.6}$$

Let S be the union of the real axis with two lines L_1 and L_2 passing through the origin with the slope of L_1 (resp. L_2) bigger than 1 (resp. less than -1). We will show that S is not a sufficient set for E.

For suppose that S is a sufficient set for E. For each $\varepsilon > 0$ define

$$f_\varepsilon(z) = \exp(-(i + \varepsilon)z^2).$$

Thus

$$|f_\varepsilon(z)| = \exp(2xy - \varepsilon x^2 + \varepsilon y^2) \tag{3.4.7}$$

and since $2xy < \varepsilon x^2 + \varepsilon^{-1}y^2$ we obtain

$$|f_\varepsilon(z)| \leq \exp((\varepsilon + \varepsilon^{-1})y^2) \tag{3.4.8}$$

This proves that $f_\varepsilon \in E$ for all $\varepsilon > 0$.

Let us examine the behavior of f_ε on S. For $z = x + io \in R$, we obtain from (3.4.7)

$$|f_\varepsilon(x)| = \exp(-\varepsilon x^2) < 1. \tag{3.4.9}$$

If $y = ax$, $a > 1$, is the equation of L_1 and $z = (y/a) + iy$, then

(3.4.10) $$|f_\varepsilon(z)| = \exp((2a^{-1} - \varepsilon a^{-2} + \varepsilon)y^2).$$

Similarily, if $y = -bx$, $b > 1$, is the equation of L_2 and $z = (-y/b) + iy$, then

(3.4.11) $$|f_\varepsilon(z)| = \exp((-2b^{-1} - \varepsilon b^{-2} + \varepsilon)y^2).$$

Both $2a^{-1} - \varepsilon a^{-2} + \varepsilon$ and $-2b^{-1} - \varepsilon b^{-2} + \varepsilon$ decrease monotonically as ε decreases to zero. Thus if f_ε satisfies

(3.4.12) $$|f_\varepsilon(z)| \le Ce^{A_1|z|}e^{A_2|\mathrm{Im}z|^2}, \quad z \in S,$$

and $\varepsilon' < \varepsilon$, then $f_{\varepsilon'}$ satisfies the same inequality.

Since S is a sufficient set, Theorem 1.2.6 implies that given $\varepsilon_o > 0$ and constants C, A_1, A_2 such that f_{ε_o} satisfies (3.4.12), there exists constants C', A_1', A_2' such that for any $\varepsilon < \varepsilon_o$,

(3.4.13) $$|f_\varepsilon(z)| \le C' \exp[A_1'|z|] \exp[A_2'|\mathrm{Im}z|^2] \text{ on } \mathcal{C}.$$

Since $f_\varepsilon(z) \to \exp(-iz^2)$ pointwise as $\varepsilon \to 0^+$, (3.4.13) implies

(3.4.14) $$|\exp(-iz^2)| \le C' \exp[A_1'|z|] \exp[A_2'|\mathrm{Im}z|^2] \text{ on } \mathcal{C}.$$

We will now obtain a contradiction by showing that this last inequality cannot hold. For

$$|\exp(-iz^2)| = \exp(2xy)$$

and consequently inequality (3.4.14) is equivalent to

$$2xy \le A_1'(x^2 + y^2)^{1/2} + A_2'y^2 + \log C'.$$

If $x = A_2'y$, this last inequality is equivalent to

$$A_2'y^2 \le A_1'|y|[(A_2')^2 + 1]^{1/2} + \log C'$$

which is impossible for $A_2' > 0$ and y sufficiently large.

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