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ANALYSIS OF A PNEUMATIC TIRE UNDER LOAD

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NOMENCLATURE

English Letters

a, b, c, d	coefficients in the Fourier series expansion of deflections
D	carcass bending stiffness
E_{Tread}	tread modulus
E_{ϕ}, E_{θ}, F	orthotropic elastic constants
$\bar{E}_{\phi}, \bar{E}_{\theta}, \bar{F}$	orthotropic elastic constants
$F_1, \dots, F_{12}$	coefficients appearing in the linear algebraic equations for a, b, c
H	tire deflection
h	thickness of one ply of carcass
j	one-half the total number of carcass plies in a bias ply construction
k	spring constant
k_{ϕ}, k_{θ}	shell rotations
$k_{\phi\theta}$	shell twist
L, M, N	coefficients used in predicting cord load
$M_{\phi\phi}, M_{\theta\theta}$	shell moments
$N_{\phi\phi}, N_{\theta\theta}$	shell forces
n	end count
P	cord load
p	inflation pressure
$\bar{q}$	tread thickness
q_0	tread thickness at crown of tire

NOMENCLATURE (Continued)

r_0, r_1, r_2	shell radii of curvature
R_1	tread radius
t	ply thickness
U	strain energy
u, v, w	shell deflections
V	potential energy
W	work term in potential energy
w, z	coordinates
x, y, z	cartesian coordinates

Greek Letters

α	angle normal to the carcass midline, measured from tire midplane through the crown
$\bar{\alpha}$	angle normal to the tread, measured from the tire midplane through the crown
β	local cord angle
$\gamma_{\phi\theta}$	shear strain
$\epsilon_{\phi\phi}, \epsilon_{\theta\theta}$	strain
λ	green, or drum, cord angle
$\mathcal{U}$	strain-energy density
$\theta, \bar{\theta}$	angle of a point in the plane of the tire
δ	tread compression
Δ	total tire deflection
σ	stress

NOMENCLATURE (Concluded)

Subscripts

ϕ	meridional direction
θ	circumferential
f	final

I. FOREWORD

The problem of determining stresses and cord loads in a pneumatic tire carrying a load has not so far to the best of our knowledge, been solved in any rational way. It represents, however, a static shell problem of great technical interest since up to this time only experimental data have been available on the internal stress state of pneumatic tires. In general these data have been sketchy and not really reliable, due to the inadequacy of strain measuring techniques for elastomers and coated textiles.

Static solutions are a first step toward solution of the problem of a pneumatic tire under actual rolling or operating conditions. For the most part, it might be expected that such static solutions would not differ greatly from solutions for the rolling tire so long as the rotational speeds are relatively small, and so long as the centrifugal forces set up by the rotating tread are of a smaller order of magnitude than the internal pressure. In general, this would seem to imply that a static solution would be a valuable design tool in its own right and would further provide the first insight into internal conditions in such complex structures as aircraft tires and tires on earth moving equipment.

This problem can be solved by several methods. All methods are complicated by the fact that there are few, if any, solutions for a general non-axisymmetric loading on a shell. This means that a great deal of the necessary geometry in this problem has simply not been thoroughly worked out, thus hampering the development of techniques for solving the problem. It

is possible that direct integration of the differential equations governing this problem would provide a shorter and surer path toward reasonably accurate results. This possibility is now being investigated by this research group in a separate but parallel activity. At present, energy methods are used in this problem and provide a technique whereby a great deal of the information used in the problem of inflation of a pneumatic tire can be carried over to the problem of direct loading against a flat surface.

II. SUMMARY

The principle of potential energy is applied to the case of a toroidal shell pressed against a frictionless plane and carrying an internal pressure. The orthotropic nature of the tire carcass materials and the elasticity of the tire tread are included in the equations.

Previous work on the elastic characteristics of orthotropic laminates is brought together and correlated in such a way that the elastic response of a pneumatic tire to inflation and loading may be determined, provided one starts with the fundamentals which the designer must specify in building a tire: the cord properties, the rubber modulus, the geometry of the individual plies, the number of plies in the tire carcass, the geometry of the tire, the tread material, and the tread geometry. In this way, analysis of a tire under inflation and loading against a frictionless plane may proceed directly from the conceptual stage without lengthy intermediate calculations involving elastic characteristics or other information not directly related to the solution.

A digital computer program has been devised to accept the data listed in the preceding paragraph as input and give as output the elastic displacements of the tire, the cord loads and interply stresses, and the bead forces. These cord loads and interply stresses are calculated on the basis of both the membrane and bending effects in the shell, and hence may be considered reasonably accurate.

An example is done for a real automotive tire showing that the length of the contact patch and the general tire deflection characteristics are reasonably well represented for reflections of the order of those commonly used in automotive tire practice.

III. PRINCIPLE OF POTENTIAL ENERGY APPLIED TO AN ORTHOTROPIC TOROIDAL SHELL

Problems in elasticity are commonly attacked by many different methods. One of these is the construction and solution of the differential equations governing the loading of elastic structures. This is probably the most common approach to such problems and has many advantages. Somewhat less common is the energy or variational method which uses an energy principle to derive a set of conditions which the equilibrium state of an elastic body must fulfill under the action of applied loads. The advantage of the latter method is that it can often be expressed in terms of the displacements of a system, while the former method is most commonly thought of in terms of stresses in the system.

In view of the interest of the Tire and Suspension System Research Group in deformations and shape changes of a tire upon inflation, we felt it desirable to use some technique whereby displacements could be readily obtained. There was a more important consideration, however; because of the orthotropic nature of the structure, some difficulties were anticipated in constructing the proper form of the differential equations governing the displacements of a toroidal shell. Recent work in formulating strain-energy expressions for shells of revolution indicated that such strain-energy expressions could be reformulated for orthotropic bodies with relatively little difficulty. For these reasons, and because this problem seemed to lend itself to digital computer programming, solutions using energy techniques were attempted for the

axisymmetric, partial toroidal shell under internal pressure and vertical load by means of energy techniques.

A definition of the physical principle used here is a prerequisite to understanding the manipulations carried out in the remainder of this section. In physics it is well known that minimum energy principles occur in many types of problems. One of the commonest of these is the deflection of elastic bodies under the action of external forces. An energy principle for this general type of problem is properly formulated by considering the elastic body to be in a state of equilibrium under the action of these external, applied forces. While this state of equilibrium may be unknown in terms of its deformation from the unloaded state, it may be visualized that a small displacement about this equilibrium state would cause work to be done by the external forces acting on the elastic body as well as by the internal elastic forces. The potential energy of the system is then defined as the difference between the strain energy stored in the elastic body due to this small displacement, minus the work done by the external forces when undergoing this small displacement about the equilibrium position. Having defined this potential energy, the principle of potential energy may be stated in several ways, e.g.:

Principle of potential energy. Of all displacements satisfying given boundary conditions, those which satisfy the equilibrium conditions make the potential energy assume a stationary value. For stable equilibrium, the potential energy is a minimum.

Hence, if the potential energy functions for an elastic, orthotropic shell can be written, this principle may then be used to determine the deformation

characteristics of the shell under the action of external loads, and from these deformation characteristics, stresses of various kinds may be determined in the shell proper.

Knowles and Reissner¹ attempt to use a rigorous geometry of deformation to construct a strain-energy expression for thin, elastic shells on a rational basis. This strain-energy function includes certain terms not usually used in shell strain-energy functions and, as a consequence, should be considerably more accurate for cases in which displacements are highly sensitive to local factors. Equation (17) of Ref. 1 will be used as the fundamental strain-energy function for the tire treated as an elastic shell.

It should be noted that the particular problem encountered here is different from the one encountered in direct inflation of a pneumatic tire. In this particular case, one is faced with the situation in which a specified deflection is applied to the rim of the wheel. The process used in obtaining deflections of the tire may be seen most easily by reference to a very simple analogue. A sketch of this analogue is given in Fig. 1, in which two springs linked in series represent the carcass elasticity and tread elasticity, respectively, of a typical pneumatic tire.

In this figure the terminal point 0, representing the rim of the tire, is pressed downward a distance H . Terminal 1, representing the interface between the carcass and tread, moves with respect to 0 some intermediate distance w . The tread surface of the tire, represented by the terminal 2, moves a distance z relative to terminal 1. The problem is to solve for the dimensions w and z as a function of the spring constants k_1 , k_2 , and the total im-

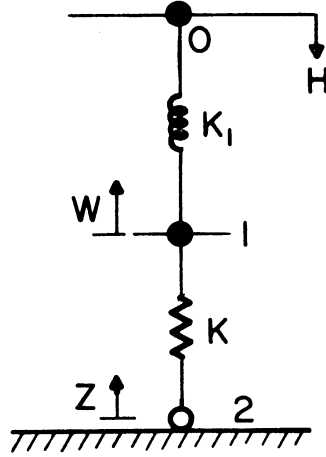


Fig. 1. Schematic drawing of series spring under fixed deflection.

posed deflection H . This may first be done by writing the total strain energy in the system, given here as:

$$U = 1/2 k_1(w)^2 + 1/2 k_2(z)^2 . \quad (1)$$

One may see from the geometry of the system that

$$H = w + z \quad z = H - w . \quad (2)$$

Substituting the value of z from Eq. (2) into Eq. (1), it is possible to express the total potential energy U in terms of the imposed deflection H and the intermediate deflection of reference point 1, namely w . This is expressed in Eq. (3):

$$U = 1/2 k_1 w^2 + 1/2 k_2 (H-w)^2 = 1/2 (k_1+k_2) w^2 - k_2 H w + 1/2 k_2 H^2 . \quad (3)$$

Applying the principle of minimum potential energy to this system implies that one should form the minimum of U with respect to possible deflections

w. Doing this, one obtains

$$\frac{\partial U}{\partial w} = 0 = (k_1 + k_2)w - k_2 H = 0 \quad (4)$$

from which one may obtain w as

$$w = \frac{k_2}{k_1 + k_2} H . \quad (5)$$

The very simple problem just described is almost exactly like the one which is faced when a pneumatic tire is pressed against a frictionless plane. In the case of the tire, one is faced with compression of the tread as well as deformation of the carcass, and these may be considered in the same way as the spring deflections were considered in the analogue problem of Fig. 1. There is one additional factor in the tire problem: the work done by the internal pressure must be taken into account. This may be handled exactly as it was in Ref. 2, in which the pure inflation of a pneumatic tire was considered. With this addition, the principle used in this report is the same as that described in the simpler problem of Fig. 1. This means that there will be a total of three possible terms entering into the formation of the total potential energy, $U-W$. The strain-energy term U will consist of two parts, analogous to those in the problem of Fig. 1, namely the strain energy stored in the carcass of the tire and the strain energy stored in the tire tread. The work done because of pressure forces moving through distances as displacement takes place will be represented by the third term.

IV. GEOMETRY OF TIRE SURFACES

In dealing with the tire in the problem of loading against a plane, the geometry used in Ref. 2 will be adapted for the carcass midline. This is illustrated in Fig. 2.

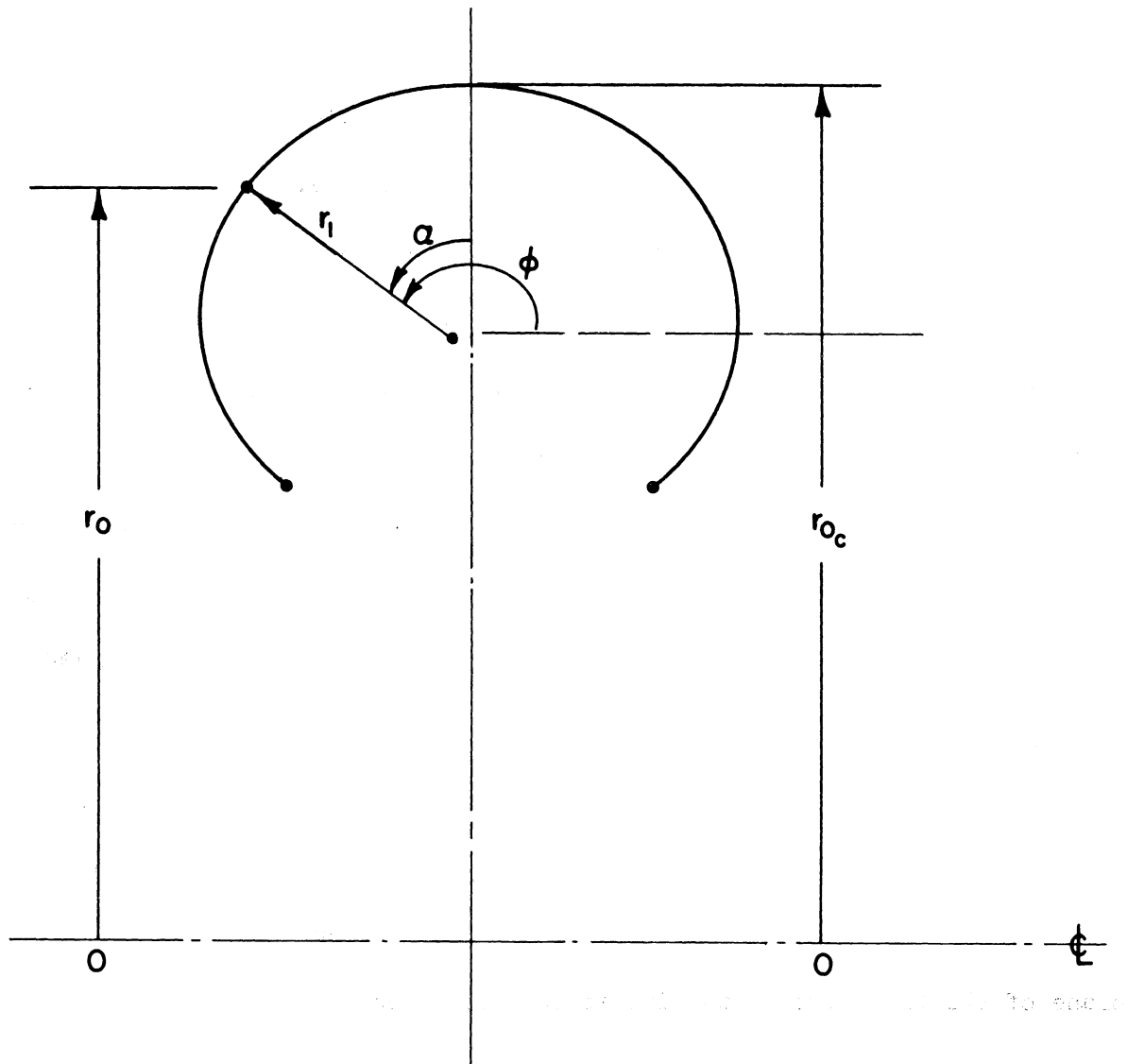


Fig. 2. Toroidal shell nomenclature.

Figure 2 shows that a meridional cross section of the tire may be described by a line whose curvature in its own plane is given by the quantity r_1 , while the distance of some point from the axle, or center of rotation, of the wheel is given by r_0 . The location of each point is defined by its angle α , where α is the angle between the vertical center line of the carcass midline and its radius of curvature. Again, a carcass midline is used as the defining line of the tire carcass geometry.

A rectangular cartesian coordinate system will be used to describe the basic location of points on the carcass midline, while displacements will be measured normal and tangential to this midline surface. This is illustrated in Fig. 3, again taken from Ref. 2. Here, w represents displacement normal to the carcass midline surface, u represents displacement tangential to it but directed in a meridional path, while v represents tangential displacement directed in a circumferential path.

It is next necessary to discuss the geometry of the tread attached to the carcass. Figure 4a shows a cross section of the carcass. The line CDE represents the carcass midline as previously described. Attached to this carcass midline is a tread section, defined by the tread radius of curvature R_1 , by the tread thickness at the crown q_0 , and by the angle α^* of the normal to the carcass midline lying directly above the shoulder of the tread, illustrated by points A and B. Figure 4b shows a view in a circumferential plane of the tire carcass and its associated tread.

In further analysis, we will assume that the contact pressures act vertically, that is, perpendicular to the flat roadway against which the

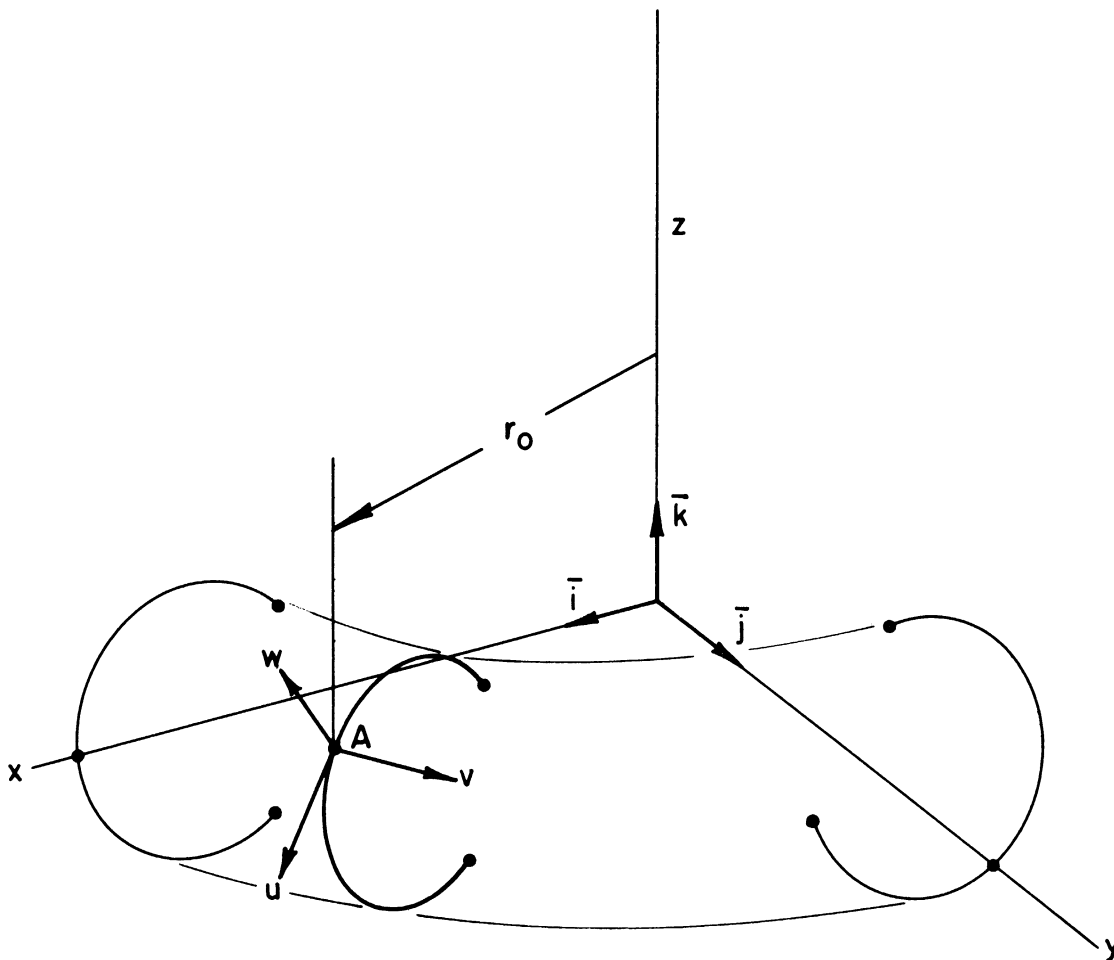


Fig. 3. Shell displacements u , v , and w .

tire might be pressed. This implies that the problem considered here is one in which the plane, or roadway, is frictionless.

It might logically be argued that one should assume contact loads normal to the surface A-B, of Fig. 4a, that is to say, perpendicular to the curving face of the tread. This assumption will be avoided for the present since (a) the resulting geometric relationships are considerably more complicated and, (b) for the relatively small deflections and motions visualized here, the differences between the two approaches should be small.

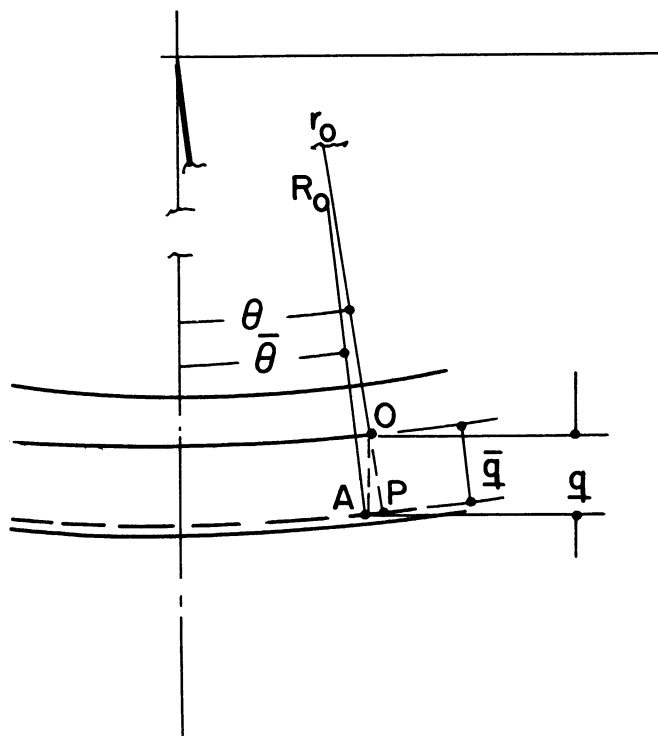
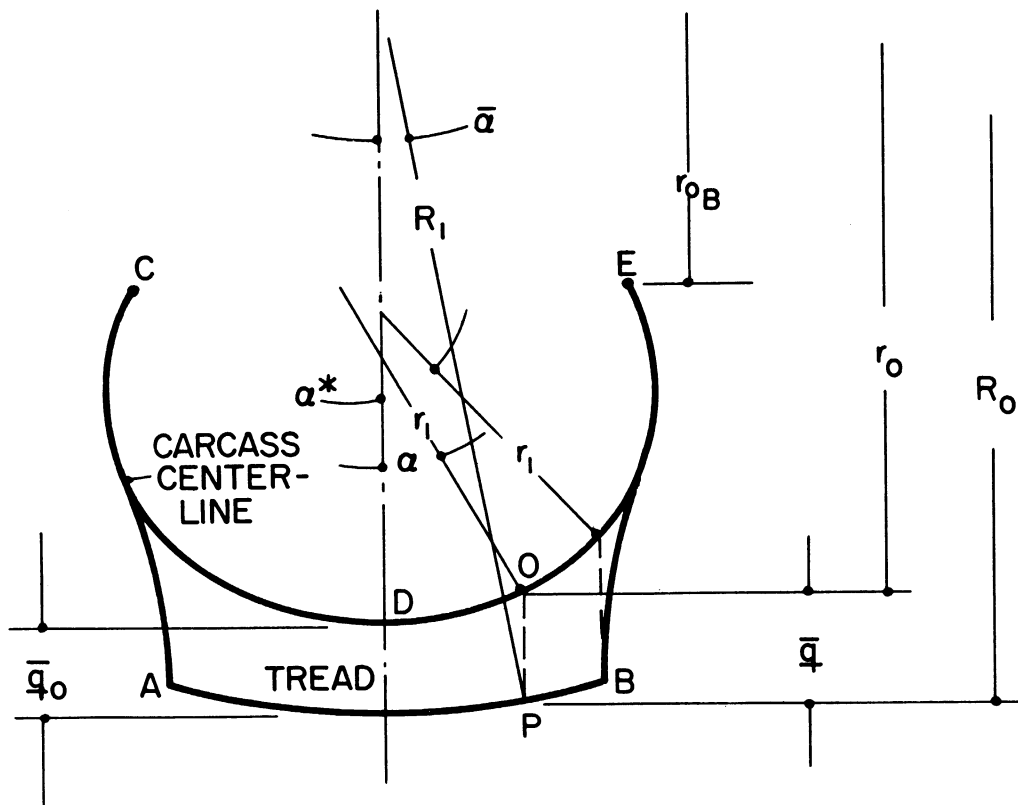


Fig. 4. Geometry of tire.

Certain relationships between the variables shown in Figs. 4a and 4b are needed for convenience in manipulations. These will now be worked out.

Again referring to Figs. 4a and 4b, radii r_1 and R_1 , defining points such as O and P, are related by the expression

$$\int_0^{\bar{\alpha}} R_1 \cos \bar{\alpha} d\bar{\alpha} = \int_0^{\alpha} r_1 \cos \alpha d\alpha \quad (6)$$

from which one angle may be found if the other is specified. It is also necessary to have some relation between the angles θ and $\bar{\theta}$. This is given by

$$R_0 \sin \bar{\theta} = r_0 \sin \theta . \quad (7)$$

Points O and A lie vertically above one another.

The thickness q_0 represents tread thickness from the carcass centerline measured at the crown. One must be able to compute the tread thickness at other points, such as O-P in Fig. 4a. This variable is given the symbol q and, if r_1 and R_1 are both specified, then one may write

$$\bar{q} = q_0 - \int_0^{\bar{\alpha}} R_1 \sin \bar{\alpha} d\bar{\alpha} + \int_0^{\alpha} r_1 \sin \alpha d\alpha , \quad (8)$$

where $\bar{\alpha}$ is obtained from Eq. (6). The vertical height q of the tread at this point is shown in Fig. 4b and is given by

$$q = \bar{q} \cos \theta = \cos \theta \left[q_0 - \int_0^{\bar{\alpha}} R_1 \sin \bar{\alpha} d\bar{\alpha} + \int_0^{\alpha} r_1 \sin \alpha d\alpha \right] . \quad (9)$$

The thickness q is necessary in order to be able to calculate the deflection due to compression of the tread itself arising from contact pres-

tures. In calculating deflection of this tread, it is convenient to consider that multiple cuts are generally used in the tread so that elements are essentially in uniaxial compression, that is to say, there is no extensive lateral support. In this case, one is justified in assuming that a tread element acts as an essentially uniaxial compression member, in which case

$$\delta(\theta, \alpha) = \frac{P_x(\theta, \alpha) q(\theta, \alpha)}{E_{\text{tread}}} . \quad (10)$$

From this last equation, one may express the contact pressure $P_x(\theta, \alpha)$ as a function of the tread compression, thickness, and elastic characteristics.

In Eq. (10) one must use the tread modulus given by the symbol E_{tread} . In calculating this value, one must remember to use the actual tread modulus and then correct it by an area reduction factor representing that portion of the tread area which actually is not cut out.

The geometry of the tread in contact with the frictionless plane is obviously of great interest here. Prior to this work, J. Sidles³ obtained a number of contact patch prints by inking a conventional automotive tire and pressing it against flat cardboard sheets. These prints show very clearly that the normal type of automotive tire construction utilizing heavy shoulders results in contact patches of constant width equal to the arc length of the tread, as shown in Fig. 4a. Now the problem is considerably less complex, since one is able to deal with a contact patch of fixed width but of unknown length.

In all subsequent calculations, we will assume that loads are large

enough to cause the full width of the contact patch to be developed. In addition, we will assume that loads are not large enough to cause the side-walls of the tire to bulge downward and contact the roadway, i.e., only the surface width A-B of Fig. 4a will be presumed to touch the road. The length of the contact patch will be assumed as an unknown and the contact patch treated as a rectangle. Again, observation of Sidles' experimental records seems to confirm this assumption.

Consider first the geometry of a contact surface which is perfectly flat and intrudes onto the undeflected shape of the tire by a maximum distance H_0 . The quantity H_0 thus is identical to the usually measured "tire deflection." Referring to Fig. 5, we see that the tire can deflect through two mechanisms: (a) deflection of the shell CDE proper, and (b) compression of the tread AQB.

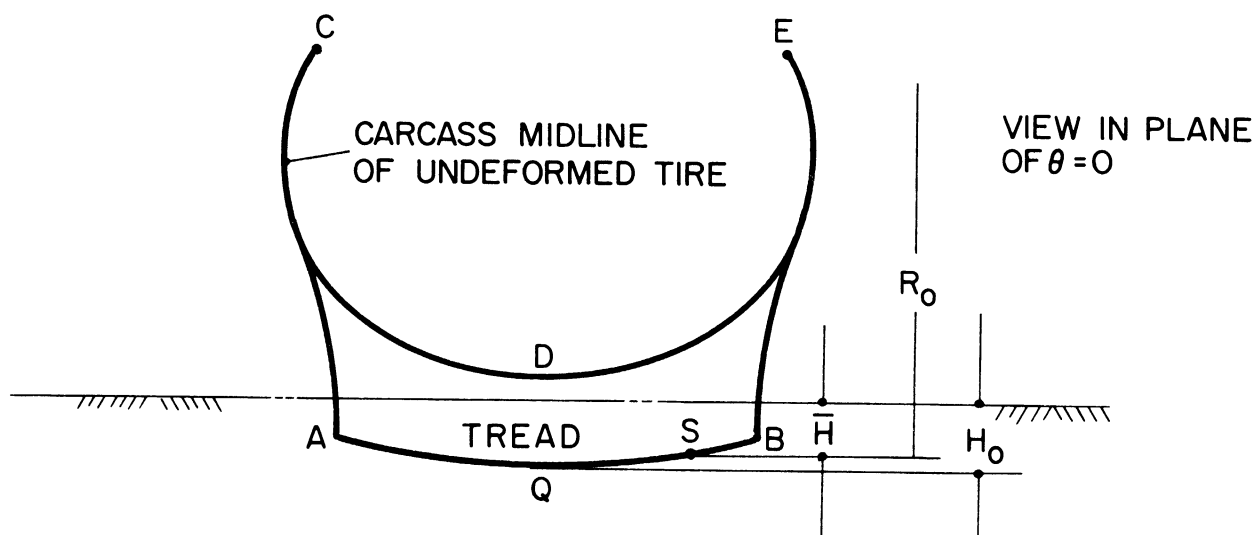


Fig. 5. Tire cross section and ground plane.

Each element in the contact patch must move through some vertical distance which will be denoted

$$H_x(\alpha, \theta) .$$

This quantity is obviously a function of the variables α and θ , and may be expressed as a function of $\bar{\alpha}$ and $\bar{\theta}$ if more convenient.

The vertical distance H_x can be determined from H_o and a knowledge of the tire tread geometry along the arc AQB. Using Fig. 5, H_x in the tread region AQB is given by the expressions

$$\bar{H} = H_o - \int_0^{\bar{\alpha}} R_1 \sin \bar{\alpha} d\bar{\alpha} \quad (11)$$

$$H_x = \bar{H} - R_o(1 - \cos \theta) \quad (12)$$

$$H_x = \bar{H} - R_o(1 - \cos \theta) = H_o - \int_0^{\bar{\alpha}} R_1 \sin \bar{\alpha} d\bar{\alpha} - R_o(1 - \cos \theta) . \quad (13)$$

The shell deflections u , v , and w have a vertical component. This may be worked out by reference to Fig. 4, in which the x component of membrane displacement is required. This becomes, expressed in terms of the usual shell displacements u , v , and w ,

$$\Delta_x = (u \cos \phi \cos \theta - v \sin \theta + w \sin \phi \cos \theta), \quad (14)$$

where Δ_x is positive downward, that is toward the ground.

We will further assume that the tire tread is in contact with the roadway at all points inside the contact patch. This precludes the possibility of

buckling motion upward, along with loss of contact pressure. Most available experimental evidence seems to indicate that this assumption is valid.

The total vertical motion of any point on the tire $H_X(\alpha, \theta)$ may be equated to the sum of the deflection of the tread $\delta_X(\alpha, \theta)$ and the deflection of the carcass itself $\Delta_X(\alpha, \theta)$. This may be expressed as

$$\begin{aligned} H_X(\alpha, \theta) &= -\Delta_X(\alpha, \theta) + \delta(\alpha, \theta) \\ &= -\Delta_X(\alpha, \theta) + \frac{p_X(\theta, \alpha) q(\theta, \alpha)}{E_{\text{tread}}} . \end{aligned} \quad (15)$$

From this, one may solve immediately for the value of the pressure distribution $p_X(\alpha, \theta)$ in the form

$$p_X(\theta, \alpha) = E_T \left[\frac{H_X(\alpha, \theta) + \Delta_X(\alpha, \theta)}{q(\theta, \alpha)} \right] . \quad (16)$$

In carrying out the sequence of calculations, the contact patch will be of constant width given by

$$\int_0^{\bar{\alpha}^*} R_1 \cos \bar{\alpha} d\bar{\alpha} . \quad (17)$$

The length of the contact patch will remain an unknown and will serve as a variable upon which the solution will iterate.

The entire problem may now be approached minimizing the energy with respect to the Fourier coefficients a_{mn} , b_{mn} , c_{mn} . From these, the pressure distribution may be computed. The new length of the contact patch may then be worked out, based on selecting points θ such that the mean pressure over

the line ($\theta = \text{constant}$) is equal to 0. This mean pressure may then be computed in any way that is convenient in a digital computer program.

Next, using the new length of contact patch length equal to that at which the pressure vanished, the problem is redone. We hope that a very small number of iterations will be necessary, since the tread region of the tire allows the width of the contact patch to be fixed from the beginning for almost all tires and loads of real interest. In general, experience has shown that two or three iterations are sufficient to achieve a reasonably accurate answer.

V. FOURIER SERIES REPRESENTATION OF TIRE CARCASS DEFLECTIONS

In applying energy methods to the solution of this elastic shell problem, it is convenient to assume series representations for the various deflections, and these series must satisfy identically the boundary conditions of the problem. Figures 4a and 4b show that the definitions of positive angle and positive displacement result in requiring the deflection functions to have certain symmetry characteristics. For example, deflections u and w must be even functions of the variable θ , such that

$$u(\theta, \alpha) = u(-\theta, \alpha) \quad (18)$$

and

$$w(\theta, \alpha) = w(-\theta, \alpha) . \quad (19)$$

Also, the tangential displacement v must be an odd function of θ , such that

$$v(\theta, \alpha) = -v(-\theta, \alpha) . \quad (20)$$

It may readily be shown that even functions of θ , when expressed in the form of a general Fourier series, retain only the cosine components. Furthermore, single-valued functions of angular arguments require integer coefficients of such angular arguments. It may similarly be shown that odd functions result in retention of sine terms in traditional Fourier series. Figures 4a and 4b also show that the deflections v and w must be even functions of the angle α , while u is an odd function of the angle α . Finally, under these con-

ditions, we may write the various displacements involved using doubly subscripted coefficients in the form

$$u = \sum_{n=1} a_n \sin \frac{n\pi\alpha}{\alpha_r} \sum_{k=0} e_k \cos k\theta = \sum_{k=0} \sum_{n=1} a_{kn} \cos k\theta \sin \frac{n\pi\alpha}{\alpha_r} \quad (21)$$

$$\begin{aligned} w &= w'(\theta)w''(\alpha) \\ &= \sum_{k=0} e_k \cos k\theta \sum_{n=1} b_n \left[1 - (-1)^n \cos \frac{n\pi\alpha}{\alpha_r} \right] \\ &= \sum_{k=0} \sum_{n=1} b_{kn} \cos k\theta \left[1 - (-1)^n \cos \frac{n\pi\alpha}{\alpha_r} \right] \end{aligned} \quad (22)$$

$$\begin{aligned} v &= \sum_{k=0} e_k \sin k\theta \sum_{m=1} g_m \left[1 - (-1)^m \cos \frac{m\pi\alpha}{\alpha_r} \right] \\ &= \sum_{k=0} \sum_{n=1} c_{kn} \sin k\theta \left[1 - (-1)^n \cos \frac{n\pi\alpha}{\alpha_r} \right] . \end{aligned} \quad (23)$$

The form of the series representation in α is chosen from Ref. 2 in such a way that the boundary conditions at the rim of the tire are completely satisfied, that is, all deflections vanish at those points.

The actual pressure distribution in the contact patch is dependent on the nature of the roadway surface, particularly its friction coefficient, as well as on the bending and shear rigidity of the tread elements. To determine the actual pressure distribution in a contact patch requires a knowledge

of tread design and is quite properly the subject of a separate study. For purposes of this report, we will assume a frictionless surface, in which the only component of pressure p is assumed to be colinear with w and to take on the same general form as w , although of course with different coefficients; one may express this pressure in the form

$$p_x(\theta, \alpha) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{mn} \cos m\alpha \cdot \cos n\theta . \quad (24)$$

No effort has been made to impose a condition that the pressure vanish at $\alpha = \alpha^*$, since, because of the shoulder or buttress-like structure there, the pressure does not vanish. Throughout this report the pressure distribution will be obtained from Eq. (16) without invoking the use of the Fourier coefficients d_{mn} .

VI. THE STRAIN-ENERGY INTERGRAL OVER THE COMPLETE TIRE

The strain energy over the entire tire carcass and tread must be calculated. To do this, consider the tire to consist of two parts: the first part is the carcass, which is symmetrically constructed but may not be symmetrically loaded; the second part may be thought of as that portion of the tread which is directly loaded by the pressure distribution in the contact patch. In view of the scalar nature of energy, these two parts may be added directly.

In considering first the strain energy of the tire carcass, one must refer to the work of Reissner and Knowles,¹ in which the strain-energy density is computed for a general isotropic shell in such a form that modification for orthotropic materials is possible. Working out the relations between membrane strains and rotations and shell forces and moments results in equations similar to those in Ref. 2, but with some modification. These are:

$$\begin{aligned}
N_{\phi\phi} &= \bar{E}_{\phi} \cdot h \cdot \epsilon_{\phi} + \bar{F} \cdot h \cdot \epsilon_{\theta} + \bar{E}_{\phi} \frac{h^3}{12} \left(\frac{1}{r_2} - \frac{1}{r_1} \right) k_{\phi}^* \\
N_{\theta\theta} &= \bar{E}_{\theta} \cdot h \cdot \epsilon_{\theta} + \bar{F} \cdot h \cdot \epsilon_{\phi} + \bar{E}_{\theta} \frac{h^3}{12} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) k_{\theta}^* \\
M_{\phi\phi} &= \bar{E}_{\phi} \frac{h^3}{12} k_{\phi}^* + \bar{F} \frac{h^3}{12} k_{\theta}^* + \bar{E}_{\phi} \frac{h^3}{12} \frac{\epsilon_{\phi}}{r_2} + \bar{F} \frac{h^3}{12} \frac{\epsilon_{\theta}}{r_2} \\
M_{\theta\theta} &= \bar{E}_{\theta} \frac{h^3}{12} k_{\theta}^* + \bar{F} \frac{h^3}{12} k_{\phi}^* + \bar{E}_{\theta} \frac{h^3}{12} \frac{\epsilon_{\theta}}{r_1} + \bar{F} \frac{h^3}{12} \frac{\epsilon_{\phi}}{r_1} \\
N_{\phi\theta} &= G_{\phi\theta} \cdot h \cdot \gamma_{\phi\theta} + G_{\phi\theta} \cdot \frac{h^3}{24} \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \left[\tau^* + \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \gamma_{\phi\theta} \right] \\
N_{\theta\phi} &= G_{\phi\theta} \cdot h \cdot \gamma_{\phi\theta} + G_{\phi\theta} \cdot \frac{h^3}{24} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \left[\tau^* + \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \gamma_{\phi\theta} \right] \\
M_{\phi\theta} &= G_{\phi\theta} \cdot \frac{h^3}{12} \left(\tau^* + \frac{\gamma_{\phi\theta}}{r_2} \right) \\
M_{\theta\phi} &= G_{\phi\theta} \cdot \frac{h^3}{12} \left(\tau^* + \frac{\gamma_{\phi\theta}}{r_1} \right)
\end{aligned} \tag{25}$$

where the usual notation for stress-strain relations has been adopted, in

which

$$\begin{aligned}
\bar{E}_{\phi} &= \frac{F}{\left(\frac{F}{E_{\theta}} - \frac{E_{\phi}}{F} \right)} = \bar{E}_{\phi} \text{ (cord angle)} \\
\bar{E}_{\theta} &= \frac{F}{\left(\frac{F}{E_{\phi}} - \frac{E_{\theta}}{F} \right)} = \bar{E}_{\theta} \text{ (cord angle)} \\
F &= \frac{E_{\phi}}{\left(\frac{F}{E_{\theta}} - \frac{E_{\phi}}{F} \right)} = \bar{F} \text{ (cord angle)} ,
\end{aligned} \tag{26}$$

where E_ϕ , E_θ and F must be determined at each point on the surface of the tire carcass as a function of the cord angle and of the elastic properties of the materials that make up the carcass. Thus, each of the elastic constants may be considered known functions of the original construction parameters of the tire.

Reference 1 contains derivations of the strain-displacement relations for thin shells. These will be required here and are summarized as follows:

$$\begin{aligned}
\epsilon_\phi &= \frac{1}{r_1} \left(\frac{\partial u}{\partial \phi} + w \right) \\
\epsilon_\theta &= \frac{1}{r_0} \left(u \cos \phi + w \sin \phi + \frac{\partial v}{\partial \theta} \right) \\
\gamma_{\phi\theta} &= \frac{1}{r_0} \frac{\partial u}{\partial \theta} - \frac{v \cos \phi}{r_0} + \frac{1}{r_1} \frac{\partial v}{\partial \phi} \\
k_\phi^* &= - \frac{1}{r_1^2} \left[\frac{u}{r_1} \frac{\partial r_1}{\partial \phi} + w - \frac{1}{r_1} \frac{\partial r_1}{\partial \phi} \frac{\partial w}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2} \right] \\
k_\theta^* &= - \left[- \frac{u}{r_0 r_1} \left(1 - \frac{r_1}{r_2} \right) \cos \phi + \frac{w}{r_2^2} + \frac{1}{r_0 r_1} \frac{\partial w}{\partial \phi} \cos \phi + \frac{1}{r_0^2} \frac{\partial^2 w}{\partial \theta^2} \right] \\
\tau^* &= \frac{1}{r_1 r_0} \left[\frac{\partial u}{\partial \theta} \left(1 - \frac{r_1}{r_2} \right) - \frac{v}{r_0} (r_1 \sin \phi \cos \phi - r_0 \cos \phi) \right. \\
&\quad \left. + r_0 \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \frac{\partial v}{\partial \phi} + \frac{2}{r_0} \frac{\partial w}{\partial \theta} r_1 \cos \phi - 2 \frac{\partial^2 w}{\partial \phi \partial \theta} \right] .
\end{aligned} \tag{27}$$

The general expression for strain-energy density may now be written:

$$\begin{aligned}
\mathcal{U} &= \frac{1}{2} \left[N_{\phi\phi} \epsilon_\phi + N_{\theta\theta} \epsilon_\theta + \frac{1}{2} (N_{\phi\theta} + N_{\theta\phi}) \gamma_{\phi\theta} + M_{\phi\phi} \left(k_\phi^* + \frac{\epsilon_\phi}{r_1} \right) \right. \\
&\quad \left. + M_{\theta\theta} \left(k_\theta^* + \frac{\epsilon_\theta}{r_2} \right) + \frac{1}{2} M_{\phi\theta} \left(\tau^* + \frac{\gamma_{\phi\theta}}{r_2} \right) + \frac{1}{2} M_{\theta\phi} \left(\tau^* + \frac{\gamma_{\phi\theta}}{r_1} \right) \right] .
\end{aligned} \tag{28}$$

It is now necessary to substitute Eqs. (27) into Eqs. (25), and the resulting equations into Eq. (28). Obviously this final equation will be of great length and will not be written out fully here. It may be summarized by noting the form which the substitution of Eqs. (25) into Eq. (28) gives. This is shown in Eq. (29).

$$\begin{aligned}
\mathcal{H} = & \frac{h^2}{2} \left[\left(1 + \frac{h^2}{12r_1r_2} \right) \left(\epsilon_{\phi}^2 \bar{E}_{\phi} + \epsilon_{\theta}^2 \bar{E}_{\theta} + 2\epsilon_{\phi}\epsilon_{\theta} \bar{F} \right) \right. \\
& + \gamma_{\phi\theta}^2 \cdot G_{\phi\theta} \left\{ 1 + \frac{h^2}{24} \left[\left(\frac{1}{r_1} - \frac{1}{r_2} \right)^2 + \left(\frac{1}{r_2^2} + \frac{1}{r_1^2} \right) \right] \right\} \\
& + \frac{h^2}{12} \left\{ k_{\phi}^{*2} \bar{E}_{\phi} + k_{\theta}^{*2} \bar{E}_{\theta} + \tau^{*2} G_{\phi\theta} + 2 \left(\frac{\epsilon_{\phi} k_{\phi}^* \bar{E}_{\phi}}{r_2} + \frac{\epsilon_{\theta} k_{\theta}^* \bar{E}_{\theta}}{r_1} \right) \right. \\
& \left. \left. + 2\bar{F} \left(k_{\phi}^* k_{\theta}^* + k_{\phi}^* \frac{\epsilon_{\theta}}{r_2} + k_{\theta}^* \frac{\epsilon_{\phi}}{r_1} \right) + \tau^* \gamma_{\phi\theta} G_{\phi\theta} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \right\} \right] .
\end{aligned} \tag{29}$$

To expand Eq. (29) into statements involving the displacement functions u , v , and w , one must work out the various strain and rotation quantities in terms of these displacements. This is simplified somewhat by adopting a notation in which

$$\frac{\partial}{\partial \phi} = \quad , \quad \frac{\partial}{\partial \theta} = \quad ^{\wedge} .$$

The strain and rotation quantities become, using this notation,

$$\epsilon_{\phi} = \frac{1}{r_1} (u' + w)$$

$$\epsilon_{\phi}^2 = \frac{1}{r_1^2} (u'^2 + 2u'w + w^2)$$

$$\epsilon_{\theta} = \frac{1}{r_0} (u \cos \phi + w \sin \phi + \hat{v})$$

$$\epsilon_{\theta}^2 = \frac{1}{r_0^2} (u^2 \cos^2 \phi + w^2 \sin^2 \phi + \hat{v}^2 + 2uw \sin \phi \cos \phi + 2u\hat{v} \cos \phi + 2w\hat{v} \sin \phi)$$

$$\begin{aligned} \epsilon_{\phi} \epsilon_{\theta} &= \frac{1}{r_1 r_0} (u' + w)(u \cos \phi + w \sin \phi + \hat{v}) \\ &= \frac{1}{r_1 r_0} (u \cdot u' \cos \phi + w \cdot u' \sin \phi + u' \cdot \hat{v} + u \cdot w \cos \phi + w^2 \sin \phi + w\hat{v}) \end{aligned}$$

$$\gamma_{\phi\theta} = \frac{1}{r_0} (\hat{u} - v \cos \phi) + \frac{1}{r_1} v'$$

$$\gamma_{\phi\theta}^2 = \frac{1}{r_0^2} (\hat{u}^2 - 2\hat{u}v \cos \phi + v^2 \cos^2 \phi) + \frac{1}{r_1^2} v'^2 + \frac{2}{r_0 r_1} (\hat{u} \cdot v' - v \cdot v' \cos \phi)$$

$$k_{\phi}^* = -\frac{1}{r_1^2} \left(u \frac{r_1'}{r_1} + w - w' \frac{r_1'}{r_1} + w'' \right)$$

$$\begin{aligned} k_{\phi}^{*2} &= \frac{1}{r_1^4} \left[u^2 \frac{r_1'^2}{r_1^2} + w^2 + \left(w' \frac{r_1'}{r_1} \right)^2 + (w'')^2 + 2u \frac{r_1' w}{r_1} - 2uw' \left(\frac{r_1'}{r_1} \right)^2 + 2uw'' \frac{r_1'}{r_1} \right. \\ &\quad \left. - 2w \cdot w' \frac{r_1'}{r_1} + 2w \cdot w'' - 2w' w'' \frac{r_1'}{r_1} \right] \end{aligned}$$

$$k_{\theta}^* = - \left[-\frac{u}{r_0 r_1} \left(1 - \frac{r_1}{r_2} \right) \cos \phi + \frac{w}{r_2^2} + \frac{1}{r_0 r_1} w' \cos \phi + \frac{\hat{w}}{r_0^2} \right]$$

$$\begin{aligned} k_{\theta}^{*2} &= \frac{u^2}{r_0^2 r_1^2} \left(1 - \frac{r_1}{r_2} \right)^2 \cos^2 \phi + \frac{w^2}{r_2^4} + \frac{1}{r_0^2 r_1^2} w'^2 \cos^2 \phi + \frac{\hat{w}^2}{r_0^4} \\ &\quad - \frac{2uw}{r_0 r_1 r_2^2} \left(1 - \frac{r_1}{r_2} \right) \cos \phi - \frac{2uw'}{r_0^2 r_1^2} \left(1 - \frac{r_1}{r_2} \right) \cos^2 \phi \\ &\quad - \frac{2u\hat{w}}{r_0^3 r_1} \left(1 - \frac{r_1}{r_2} \right) \cos \phi + \frac{2ww' \cos \phi}{r_0 r_1 r_2^2} + \frac{2w\hat{w}}{r_0^2 r_2^2} \\ &\quad + \frac{2w' \hat{w}}{r_0^3 r_1} \cos \phi \end{aligned}$$

$$\begin{aligned}
\tau^* &= \frac{1}{r_1 r_0} \left[\hat{u} \left(1 - \frac{r_1}{r_2} \right) + \frac{v \cos \phi}{r_0} (r_0 - r_1 \sin \phi) + v' r_0 \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \right. \\
&\quad \left. + \frac{2}{r_0} \hat{w} r_1 \cos \phi - 2 \hat{w}' \right] \\
\tau^{*2} &= \frac{1}{r_1^2 r_0^2} \left[\hat{u}^2 \left(1 - \frac{r_1}{r_2} \right)^2 + \frac{v^2 \cos^2 \phi}{r_0^2} (r_0 - r_1 \sin \phi)^2 + v'^2 r_0^2 \left(\frac{1}{r_2} - \frac{1}{r_1} \right)^2 \right. \\
&\quad + \frac{4}{r_0^2} \hat{w}^2 r_1^2 \cos^2 \phi + 4 \hat{w}'^2 + 2 \hat{u} v \cos \phi \frac{(1 - \frac{r_1}{r_2})}{r_0} (r_0 - r_1 \sin \phi) \\
&\quad + 2 \hat{u} v' r_0 \left(1 - \frac{r_1}{r_2} \right) \left(\frac{1}{r_2} - \frac{1}{r_1} \right) + 4 \hat{u} \hat{w} \frac{(1 - \frac{r_1}{r_2})}{r_0} r_1 \cos \phi - 4 \hat{u} \hat{w}' \left(1 - \frac{r_1}{r_2} \right) \\
&\quad + 2 v \cdot v' \cos \phi (r_0 - r_1 \sin \phi) \left(\frac{1}{r_2} - \frac{1}{r_1} \right) + 4 v \hat{w} r_1 \cos^2 \phi (r_0 - r_1 \sin \phi) \\
&\quad - 4 v \hat{w}' \frac{\cos \phi}{r_0} (r_0 - r_1 \sin \phi) + 4 \hat{w} v' r_1 \cos \phi \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \\
&\quad \left. - 4 \hat{w}' v' r_0 \left(\frac{1}{r_2} - \frac{1}{r_1} \right) - 8 \hat{w} \hat{w}' \frac{r_1 \cos \phi}{r_0} \right] \\
\epsilon_{\phi}^{k*} &= - \frac{1}{r_1^3} (u' + w) \left(u \frac{r_1'}{r_1} + w - w' \frac{r_1'}{r_1} + w'' \right) \\
&= - \frac{1}{r_1^3} \left[u \cdot u' \frac{r_1'}{r_1} + u' w - u' w' \frac{r_1'}{r_1} + u' w'' + u w \frac{r_1'}{r_1} + w^2 - w w' \frac{r_1'}{r_1} + w w'' \right] \\
\epsilon_{\theta}^{k*} &= - \frac{1}{r_0} (u \cos \phi + w \sin \phi + \hat{v}) \left[- \frac{u}{r_0 r_1} \left(1 - \frac{r_1}{r_2} \right) \cos \phi + \frac{w}{r_2^2} \right. \\
&\quad \left. + \frac{w'}{r_0 r_1} \cos \phi + \frac{\hat{w}}{r_0^2} \right] \\
&= \frac{1}{r_0^3} \left[u^2 \frac{r_0}{r_1} \left(1 - \frac{r_1}{r_2} \right) \cos^2 \phi - u w \sin^2 \phi \cos \phi - \frac{r_0}{r_1} u w' \cos^2 \phi \right. \\
&\quad - u \hat{w} \cos \phi + u w \frac{r_0}{r_1} \left(1 - \frac{r_1}{r_2} \right) \sin \phi \cos \phi - w^2 \sin^3 \phi \\
&\quad - w \cdot w' \frac{r_0}{r_1} \sin \phi \cos \phi - w \hat{w} \sin \phi + u \hat{v} \frac{r_0}{r_1} \left(1 - \frac{r_1}{r_2} \right) \cos \phi \\
&\quad \left. - w \hat{v} \sin^2 \phi - w' \hat{v} \frac{r_0}{r_1} \cos \phi - \frac{\hat{w}}{r_1} \right]
\end{aligned}$$

$$\begin{aligned}
k_{\phi}^* \cdot k_{\theta}^* &= \frac{1}{r_1^2} \left(u \frac{r_1'}{r_1} + w - w' \frac{r_1'}{r_1} + w'' \right) \left[- \frac{u}{r_0 r_1} \left(1 - \frac{r_1}{r_2} \right) \cos \phi + \frac{w}{r_2^2} \right. \\
&\quad \left. + \frac{w'}{r_0 r_1} \cos \phi + \frac{\hat{w}}{r_0^2} \right] \\
&= \frac{1}{r_1^2} \left[- \frac{u^2 r_1'}{r_0 r_1^2} \left(1 - \frac{r_1}{r_2} \right) \cos \phi + \frac{u w r_1'}{r_1 r_2^2} + \frac{u \cdot w' r_1' \cos \phi}{r_0 r_1^2} \right. \\
&\quad + \frac{u \hat{w} r_1'}{r_1 r_0^2} - \frac{u w}{r_0 r_1} \left(1 - \frac{r_1}{r_2} \right) \cos \phi + \frac{w^2}{r_2^2} + \frac{w \cdot w'}{r_0 r_1} \cos \phi \\
&\quad + \frac{w \hat{w}}{r_0^2} + \frac{u w' r_1'}{r_0 r_1^2} \left(1 - \frac{r_1}{r_2} \right) \cos \phi - \frac{w w' r_1'}{r_1 r_2^2} \\
&\quad - \frac{w'^2 r_1'}{r_0 r_1^2} \cos \phi - \frac{w' \hat{w} r_1'}{r_1 r_0^2} - \frac{u w''}{r_0 r_1} \left(1 - \frac{r_1}{r_2} \right) \cos \phi \\
&\quad \left. + \frac{w w''}{r_2^2} + \frac{w' w''}{r_0 r_1} \cos \phi + \frac{w'' \hat{w}}{r_0^2} \right]
\end{aligned}$$

$$\begin{aligned}
k_{\phi}^* \cdot \epsilon_{\theta} &= - \frac{1}{r_1^2 r_0} (u \cos \phi + w \sin \phi + \hat{v}) \left(u \frac{r_1'}{r_1} + w - w' \frac{r_1'}{r_1} + w'' \right) \\
&= - \frac{1}{r_1^2 r_0} \left[u^2 \frac{r_1'}{r_1} \cos \phi + u \cdot w \cos \phi - u w' \frac{r_1'}{r_1} \cos \phi \right. \\
&\quad + u w'' \cos \phi + u w \frac{r_1'}{r_1} \sin \phi + w^2 \sin \phi - w w' \frac{r_1'}{r_1} \sin \phi \\
&\quad \left. + w w'' \sin \phi + u \hat{v} \frac{r_1'}{r_1} + w \hat{v} - w' \hat{v} \frac{r_1'}{r_1} + \hat{v} w'' \right]
\end{aligned}$$

$$\begin{aligned}
k_{\theta}^* \cdot \epsilon_{\phi} &= - \frac{1}{r_1} (u' + w) \left[- \frac{u}{r_0 r_1} \left(1 - \frac{r_1}{r_2} \right) \cos \phi + \frac{w}{r_2^2} + \frac{w'}{r_0 r_1} \cos \phi + \frac{\hat{w}}{r_0^2} \right] \\
&= \frac{1}{r_1^2 r_0} \left[u \cdot u' \left(1 - \frac{r_1}{r_2} \right) \cos \phi - u' w \frac{r_1}{r_0} \sin^2 \phi - u' w' \cos \phi - \frac{r_1}{r_0} u' \hat{w} \right. \\
&\quad \left. + u w \left(1 - \frac{r_1}{r_2} \right) \cos \phi - w^2 \frac{r_1}{r_0} \sin^2 \phi - w w' \cos \phi - \frac{r_1}{r_0} w \hat{w} \right]
\end{aligned}$$

$$\begin{aligned}
\tau^* \cdot \gamma_{\phi\theta} &= \left[\frac{1}{r_0} (\hat{u} - v \cos \phi) + \frac{v'}{r_1} \right] \frac{1}{r_1 r_0} \left[\hat{u} \left(1 - \frac{r_1}{r_2} \right) + \frac{v \cos \phi}{r_0} (r_0 - r_1 \sin \phi) \right. \\
&\quad \left. + v' r_0 \left(\frac{1}{r_2} - \frac{1}{r_1} \right) + \frac{2}{r_0} \hat{w} r_1 \cos \phi - 2 \hat{w}' \right] \\
&= \frac{1}{r_1^2 r_0} \left[\hat{u}^2 \frac{r_1}{r_0} \left(1 - \frac{r_1}{r_2} \right) + \hat{u} v \cos \phi \frac{r_1}{r_0} \left(1 - \frac{r_1}{r_0} \sin \phi \right) \right. \\
&\quad + \hat{u} v' \left(\frac{r_1}{r_2} - 1 \right) + 2 \hat{u} \hat{w} \left(\frac{r_1}{r_0} \right)^2 \cos \phi - 2 \frac{r_1}{r_0} \hat{u} \hat{w}' \\
&\quad - \hat{u} v \left(1 - \frac{r_1}{r_2} \right) \left(\frac{r_1}{r_0} \right) \cos \phi - v^2 \cos^2 \phi \left(\frac{r_1}{r_0} \right) \left(1 - \frac{r_1}{r_0} \sin \phi \right) \\
&\quad - v v' \cos \phi \left(\frac{r_1}{r_2} - 1 \right) - 2 \hat{w} v \left(\frac{r_1}{r_0} \right)^2 \cos^2 \phi + 2 \hat{w}' v \left(\frac{r_1}{r_0} \right) \cos \phi \\
&\quad - \hat{u} v' \left(1 - \frac{r_1}{r_2} \right) + v v' \cos \phi \left(1 - \frac{r_1}{r_0} \sin \phi \right) + v'^2 \left(\frac{r_0}{r_2} - \frac{r_0}{r_1} \right) \\
&\quad \left. + 2 \hat{w} v' \frac{r_1}{r_0} \cos \phi - 2 \hat{w}' v' \right] .
\end{aligned}$$

The strain-energy density may now be determined by substitution of these quantities into Eq. (29); the integration of this density over the entire surface of the tire carcass must then be accomplished numerically. This will give the strain energy stored in the carcass itself.

Attention is next directed to the strain energy stored in the tire tread. This region is subjected to a compressive stress set up by the pressure distribution between the tire and the frictionless roadway. As previously discussed, this pressure distribution will be assumed to be perpendicular to the roadway and also perpendicular to the tire tread elements. We will draw on the fact that common American practice is to use multiple tread cuts for purposes of controlling friction coefficient, water and wet roadway characteristics, and noise. These tread cuts result in many small elements of tread

which act very much like compression members. In calculating the strain energy stored in the tread itself, one might consider this strain energy to be in the form of a density per unit area of contact patch, in which the strain-energy density is made up of the usual form that one would apply to a simple compression member, namely

$$\mathcal{H}'_x = \frac{1}{2} \frac{p_x^2(\theta, \alpha) q(\theta, \alpha)}{E_T}, \quad (30)$$

where the length or thickness of the tread is represented by the quantity q , while the local pressure is represented by p_x . Note also that in Eq. (30) it is necessary to use a modified tread modulus, which is obtained by taking the total elastic modulus of the tread material and multiplying it by the ratio formed by dividing the actual tread contact area by the total circumscribed area, that is to say, by reducing the tread modulus by a contact-area correction factor.

Recalling that from Eq. (16) it is possible to express the pressure distribution in the contact patch in terms of the tire deflection and the arbitrary or unknown carcass deflections, one may express Eq. (30) as

$$\mathcal{H}'_x = \frac{1}{2} E_T \frac{\{H_x(\theta, \alpha) + \Delta_x(\theta, \alpha)\}^2}{q(\theta, \alpha)}. \quad (31)$$

This represents strain-energy density in the tread and not total strain energy itself. One must integrate this strain-energy density over the tread area. As previously discussed, the width of this tread area is known because of the construction normally used in automotive tires; this width is limited by the shoulders of the tread. The length of the contact patch is still un-

known; one must guess such a length arbitrarily, perhaps from geometric considerations of the contact patch length, and examine the pressure distribution at the ends of this contact patch by integrating across the tread to determine the mean pressure intensity at this point. If this mean pressure intensity is indeed zero, then the guess as to the contact patch is a satisfactory one and conforms to reality. If this is the case, then no iteration is necessary. On the other hand, if the integration across the width of the tread results in a negative mean pressure intensity, it would indicate that the contact patch was too long in the original guess and should subsequently be reduced. This could be done by successive iteration. If the integration of the pressure intensity across the width of the tread resulted in a positive number, it would indicate that the contact patch was not long enough, and correction would have to be made in a subsequent iteration. Hopefully, the iteration process would not be lengthy and would be rapidly convergent.

VII. WORK DONE OVER THE COMPLETE TIRE

The work done over the complete tire is caused by the internal inflation pressure, and Ref. 2 covers this subject completely, since the inflation pressure remains axisymmetric independent of other loading characteristics. In this particular case, we will assume that the inflation pressure p is colinear with the normal deflection w of the tire carcass, and no correction will be made for the large deformations which cause deflections sufficient to interfere with this simple relationship. The work done by the internal pressure is given by

$$W = \int_{-\alpha_r}^{\alpha_r} 2\pi p w r_{0f} r_{1f} d\alpha \quad , \quad (32)$$

where p is the inflation pressure, and the deflection w will be such that $w = w(\theta, \alpha)$. Note that the radii of curvature r_0 and r_1 enter Eq. (32). These are given subscripts f since the variational nature of the intended equilibrium equation requires one to perform variations about the final position of the shell. In that case it is desirable to use final radii of curvature, as in Ref. 2. These radii of curvature are worked out in some detail in that reference and need not be repeated here. In general then, except for the fact that the deflection w now becomes a function of two variables rather than one, Eq. (32), and hence the work done by the internal pressure, is intended to be exactly the same as it was in the work on tire inflation only.

This simplified version of the work term has obvious shortcomings. The first of these is that in a non-axisymmetric loading process the radii of curvature no longer maintain the simple form predicted in Ref. 2. Now the radii of curvature in their final state are not functions of position alone around the meridional cross section of the tire, but are also functions of the circumferential angle θ . In fact, the radii of curvature are generally such that they now may no longer be described by the same simple axisymmetric geometry, and the radius r_{of} no longer intersects the axis of rotation of the tire in some cases. One should, to be strictly correct, include such effects in Eq. (32). Examination of the algebra necessary to include these effects reveals that it is quite lengthy. In view of the already excessive length of the program, we decided not to attempt to include these effects but rather to produce a simplified program which might have to be accepted as only approximately accurate. It is on this basis that the work term is computed.

VIII. EQUILIBRIUM EQUATIONS

One may construct the total potential energy of the tire as described by using Eqs. (29), (31), and (32). These give

$$\begin{aligned}
 V = & \int_0^{2\pi} \int_{-\alpha_r}^{\alpha_r} \pi \cdot r_o r_1 d\alpha \cdot d\theta + \iint_{\text{contact patch}} \pi_x'^* \cdot dA \\
 & - \int_0^{2\pi} \int_{-\alpha_r}^{\alpha_r} p \cdot w r_{of} \cdot r_{1f} d\alpha d\theta .
 \end{aligned} \tag{33}$$

One may now apply the principle of potential energy to Eq. (33) in order to derive equilibrium equations for the loaded, inflated shell. Doing this, one obtains symbolically

$$\frac{\partial V}{\partial a_{mn}} = 0 \tag{34}$$

$$\frac{\partial V}{\partial b_{mn}} = 0 \tag{35}$$

$$\frac{\partial V}{\partial c_{mn}} = 0 \tag{36}$$

where, if one were actually to expand equations (34), (35), or (36) in terms of the actual unknowns, namely the coefficients a , b , and c , one would obtain equations of great length. However, one must visualize exactly the types of equations involved, and for that reason one may, in Eqs. (37), produce a symbolic set of equations containing the essence of the problem. These are

given as

$$(F1)_{mn}a_{mn} + (F2)_{mn}c_{mn} + (F3)_{mn}b_{mn} = (F10)_{mn}$$

$$(F4)_{mn}a_{mn} (F5)_{mn}c_{mn} (F6)_{mn}b_{mn} = (F11)_{mn} \quad (37)$$

$$(F7)_{mn}a_{mn} (F8)_{mn}c_{mn} (F9)_{mn}b_{mn} = (F12)_{mn} ,$$

where there are $m+n$ of the first type of equations, $m+n$ of the second type, and $m+n$ of the third type. There are, thus, a number of equations equal to the number of unknowns in the problem. The order of coefficients a , c , b is not meaningful but arises because of previous practice in which the u coefficients appeared first in the matrix and the w coefficients last. Such a set of equations involving doubly subscripted coefficients can very easily exceed the capacity of hand solutions and is practical only in terms of digital computer techniques.

IX. CALCULATION OF CORD LOADS

Before beginning a stress analysis of the internal structure of the tire, one must set up a system of notation for identifying different plies in the tire lying at different crown cord angles. This may be done by reference to Fig. 6, in which it is assumed that the outermost layer of the carcass uses a ply whose crown cord angle is denoted as $+\beta$.

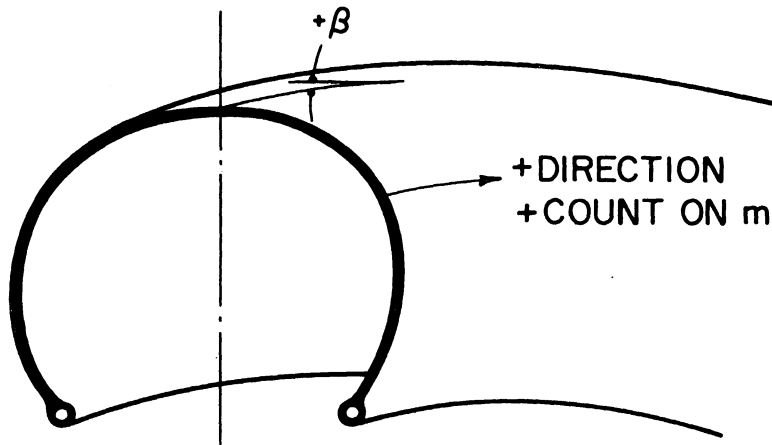


Fig. 6. Tire carcass notation.

Next, one must be able to identify various plies within the carcass. The system used to do this is illustrated in Fig. 7, in which the carcass midline is used as a reference point from which plies lying outside of the carcass midline are numbered positively beginning from the carcass midline, while plies lying inside the carcass midline are numbered negatively running from this same point. In an eight-ply tire, the outermost ply would thus be denoted as ply 4, while the innermost ply would be denoted as ply -4.

+Count		Angle	<u>m</u>	
		<u>+β</u>	4	
	+β	-β	3	
+β	-β	+β	2	
-β	+β	-β	1	
				℄
				-1
				-2
				-3
				-4

Fig. 7. Ply notation.

It is now necessary to work out separately the various components of cord loads. The first load is concerned with membrane effects, a subject treated in some detail by Clark.⁴ He showed that the cord load induced in a thin-walled structure due to some general stress state is given by the expression

$$P_A = \frac{t}{n} [L \cdot \sigma_\theta + M \cdot \sigma_\phi + N \cdot \sigma_{\phi\theta}] , \quad (38)$$

where t equals ply thickness, n equals end count, and L , M , and N are given as functions of cord angle.

In order to calculate the stresses σ_θ , σ_ϕ , and $\sigma_{\phi\theta}$, one must use the stress-strain relations and the strain-displacement relations of the shell, since shell deflection is the fundamental information given by the strain-energy method. To perform such a calculation, one must recall the following equations:

$$\begin{aligned}
\epsilon_{\phi} &= \frac{1}{r_1} \left(\frac{\partial u}{\partial \alpha} + w \right) \\
\epsilon_{\theta} &= \frac{1}{r_0} \left(-u \sin \alpha + w \cos \alpha + \frac{\partial v}{\partial \theta} \right) \\
\gamma_{\phi\theta} &= \frac{1}{r_0} \frac{\partial u}{\partial \theta} + \frac{v \sin \alpha}{r_0} + \frac{1}{r_1} \frac{\partial v}{\partial \alpha} \\
\sigma_{\phi} &= \bar{E}_{\phi} \epsilon_{\phi} + \bar{F} \epsilon_{\theta}; \quad \sigma_{\theta} = \bar{E}_{\theta} \epsilon_{\theta} + \bar{F} \epsilon_{\phi} \\
\sigma_{\phi\theta} &= G_{\phi\theta} \gamma_{\phi\theta} .
\end{aligned}$$

In this particular problem of membrane loads, the cord load induced in each of the piles is exactly the same, so that, for the j th ply,

$$\begin{aligned}
P_{Aj} &= P_A \\
P_{A-j} &= P_A .
\end{aligned} \tag{39}$$

It is next necessary to consider bending effects in the tire. This problem was not treated in previous reports covering inflation effects only; hence it must be worked out in considerably more detail here. First, let j be the number of the outer ply, that is, in a two-ply carcass $j = 1$, while in a four-ply carcass $j = 2$. The cord load in the outer ply, or j th ply, is given by

$$P_{Bj} = \frac{(M_{\theta\theta} \cos^2 \alpha + M_{\phi\phi} \sin^2 \alpha)}{n \cdot h} \left[\frac{(2j-1)}{(2j-1)^2 + (2j-3)^2 + \dots + 3^2 + 1^2} \right] , \tag{40}$$

where the total number of plies is $2j$ and where

$$\begin{aligned}
M_{\theta\theta} &= - \bar{D}_{\theta\theta} k_{\theta} - \bar{D} k_{\phi} \\
M_{\phi\phi} &= - \bar{D}_{\phi\phi} k_{\phi} - \bar{D} k_{\theta}
\end{aligned} \tag{41}$$

$$\begin{aligned}
\bar{D}_{\theta\theta} &= \bar{E}_{\theta\theta} \cdot \frac{2}{3} (jh)^3 \\
\bar{D}_{\phi\phi} &= \bar{E}_{\phi\phi} \cdot \frac{2}{3} (jh)^3 \\
\bar{D} &= \bar{F} \cdot \frac{2}{3} (jh)^3 ,
\end{aligned} \tag{42}$$

and where it is assumed that the ply thickness h is the same for each ply, that is, it is assumed here that plies are identical. To perform the calculations for $M_{\theta\theta}$ and $M_{\phi\phi}$, one must recall that the quantities k_{ϕ} and k_{θ} are related to displacements through the expressions

$$k_{\phi} = - \frac{1}{r_1^2} \left[\frac{u}{r_1} \cdot r_1' + w - \frac{r_1'}{r_1} \frac{\partial w}{\partial \alpha} + \frac{\partial^2 w}{\partial \alpha^2} \right] \tag{43}$$

$$k_{\theta} = - \left[\frac{u}{r_0 r_1} \left(1 - \frac{r_1}{r_2} \right) \sin \alpha + \frac{w \cos^2 \alpha}{r_0^2} - \frac{1}{r_0 r_1} \frac{\partial w}{\partial \alpha} \sin \alpha + \frac{1}{r_0^2} \frac{\partial^2 w}{\partial \theta^2} \right] . \tag{44}$$

The cord load induced by bending in one ply may be related to the cord loads induced by bending in other plies by the relation

$$P_{B_j} = \frac{(2m-1)}{(2j-1)} P_{B_j} \quad m \leq j \quad m \geq 1 \tag{45}$$

and

$$P_{B-m} = - (P_{B_m}) . \tag{46}$$

It is also true that twist is capable of inducing cord loads in such a structure since it is caused fundamentally by shearing stresses. The

shearing stresses on a given ply are related to the quantity $k_{\phi\theta}$ by the expression

$$\sigma_{\phi\theta_j} = -2G_{\phi\theta} k_{\phi\theta} \left[\left(j - \frac{1}{2} \right) h \right] , \quad (47)$$

where $\sigma_{\phi\theta}$ is evaluated at the midpoint of the outer ply of the laminate. This results in a cord load which is given by the expression

$$P_{C_j} = \frac{t}{n} \cdot N \cdot \left[-2G_{\phi\theta} k_{\phi\theta} \left(j - \frac{1}{2} \right) h \right] . \quad (48)$$

For other plies, one must write

$$P_{C_m} = \left(\frac{2m-1}{2j-1} \right) (-1)^{j-m} P_{C_j} \quad j \geq m \quad m \geq 1 . \quad (49)$$

For the inside half of the carcass, the dimension $(j - \frac{1}{2}) \cdot h$ or $(m - \frac{1}{2}) \cdot h$ becomes negative; the function N becomes negative so that the same sign is maintained, i.e.,

$$P_{C_{-m}} = P_{C_m} \quad m \geq 1 \quad (50)$$

where

$$\begin{aligned} k_{\phi\theta} = \tau^* = \frac{1}{r_1 r_0} \left[\frac{\partial u}{\partial \theta} \left(1 - \frac{r_1}{r_2} \right) + \frac{v}{r_0} (r_1 \sin \alpha \cos \alpha - r_0 \sin \alpha) \right. \\ \left. + r_0 \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \frac{\partial v}{\partial \alpha} - \frac{2}{r_0} \frac{\partial w}{\partial \theta} r_1 \sin \alpha - \frac{2 \partial^2 w}{\partial \alpha \partial \theta} \right] . \end{aligned} \quad (51)$$

A secondary cord load is introduced into the structure because bending itself causes interply stresses. These interply stresses load the cords and this factor must be taken into account in calculating total cord load. The expression for such a cord load is given by

$$P_{D_j} = \frac{h}{n} [N \cdot \sigma'_{\phi\theta_j}] , \quad (52)$$

where the induced shear stress $\sigma'_{\phi\theta}$ is given by

$$\sigma'_{\phi\theta_j} = \left[\frac{a_{13}(+\alpha)}{a_{33}(+\alpha)} (\bar{E}_{\theta k\theta} + \bar{F}_{k\phi}) + \frac{a_{23}(+\alpha)}{a_{33}(+\alpha)} (\bar{E}_{\phi k\phi} + \bar{F}_{k\theta}) \right] \left(j - \frac{1}{2} \right) h . \quad (53)$$

For the m th ply in the section of the carcass lying outside the carcass mid-line,

$$P_{D_m} = \left(\frac{2m-1}{2j-1} \right) P_{D_j} \quad m \geq 1 \quad j \geq m \quad (54)$$

while for the inner half of the carcass,

$$P_{D_{-m}} = - P_{D_m} . \quad (55)$$

It is now possible to construct the total cord load as the sum of four separate components. This is given by the expression

$$P_{tot_m} = P_{A_m} + P_{B_m} + P_{C_m} + P_{D_m} \quad (56)$$

for $m \geq 1$, while for the inner half of the tire carcass, one should use the expression

$$P_{tot_{-m}} = P_{A_m} - P_{B_m} + P_{C_m} - P_{D_m} \quad (57)$$

where $m \leq 1$. The cord load in each layer may now be computed as part of the standard digital computer program and printed as a portion of the output.

During the early stages of the construction of this program we felt that it might be beneficial to use some sort of sign test concerning the average overall cord load to determine whether or not the tension elastic properties

or compression elastic properties of the materials should be used in the digital computer program. Our general impression now is that the program should be evaluated temporarily on the basis of tension properties until some greater familiarity with it is obtained. For this reason the provision for changing the elastic properties in some restricted region because of cord compression in that region is not now being made available.

X. EXAMPLE ANALYSIS

A lengthy analysis such as this is always most meaningful if accompanied by an example of its application to an actual problem. It is also desirable to check the applicability of this computer program to a tire of dimensions commonly used in automotive applications. For the analysis, we purchased simultaneously two standard 8:00-14 four-ply automobile tires manufactured by the Cooper Tire Company. One of them was used and eventually dismembered in the inflation tests described in Technical Report No. 13 of this series. The second remained intact; we will assume that the elastic properties and carcass midline shape of this second tire are identical to those of the first tire, although, as pointed out by various sponsors' representatives, this assumption may not necessarily be true.

The elastic properties as generated from measurement of the Cooper tire given in Ref. 2, give almost all the necessary information for analysis of the tire loaded against a flat surface. However, one must determine the characteristics, not previously considered, of the tread itself. The tread radius may readily be obtained in the uninflated state by cutting and measuring a cardboard templet held against the tread. Tread modulus may be obtained by cutting out a small section of tread and subjecting it to compressive tests under stresses comparable to the internal pressure and the ground pressure. The effective tread modulus may be obtained by prorating this measured tread modulus by the effective area of the tread compared with its total area.

Finally, the thickness of the tread at the crown and the angle of extent of the tread α^* may be determined by measuring a section of the tire. This gives all necessary input data for the digital computer program except internal pressure and the deflection.

With this digital computer program, calculations were made for two cases of tire deflection using one inflation pressure. The inflation pressure was set at 20 psi, a nominal value, while the total tire deflections were 0.5 in. and 1.0 in. The second tire deflection value was felt to be consistent with actual tire operations. The first deflection value represents a situation closer to the usual small deflection theory, which this type of shell analysis is designed primarily to handle.

Costs are a major factor in operating a digital computer. It is relatively easy, where double series are involved, to obtain extremely large matrices, and even the generation of coefficients for such matrices becomes an expensive proposition. Costs limit calculation to a small total number of coefficients, so that one expresses the cross section of the tire with essentially three Fourier terms, while the circumferential shape is also expressed with three terms. This means that the accuracy is considerably less than that obtained in the previous analysis involving inflation effects, since in that case one could readily obtain six or eight terms in a series representing the cross section of the tire. In examining this calculation, one should recall that the surface against which the tire is pressed is assumed to be frictionless.

We hope that subsequent work on direct integration of the differential equations, sponsored by the National Aeronautics and Space Administration, will provide a much less expensive method of handling this problem.

XI. COMPARISON OF CALCULATION WITH EXPERIMENT

Calculation and experiment can be compared in a number of ways in this particular problem of the tire carrying a load against a flat surface. In reaching the final deformation analysis in the digital computer program, one must determine the length of the contact patch between the tire and the frictionless plane. One way of assessing the accuracy of computer results is to compare this contact patch length with a known contact patch length. This contact length is stored in the computer and can be printed out, as was done in the two cases described involving a 0.5 in. and a 1.0 in. deflection of a standard automotive tire.

Experiments can readily be performed on contact patch length since the tire tread can be inked and pressed against a flat cardboard to form a "foot-print." This was done for the two deflections indicated in the program; the contact patches are shown in Figs. 8 and 9. In these figures the predicted lengths of these patches are drawn out; agreement at a deflection level of 0.5 in. is quite good, and at the 1.0 in. level the predicted contact patch length is only slightly shorter than the measured one.

Examination of these contact patch lengths shows that the digital computer program is producing approximately the correct answer.

It is possible to integrate the pressure distribution over this contact patch, to determine the total load carried by the tire, and to compare this total load with the actual measured load during the experiment, but this was not done because the provision for such integration of pressure

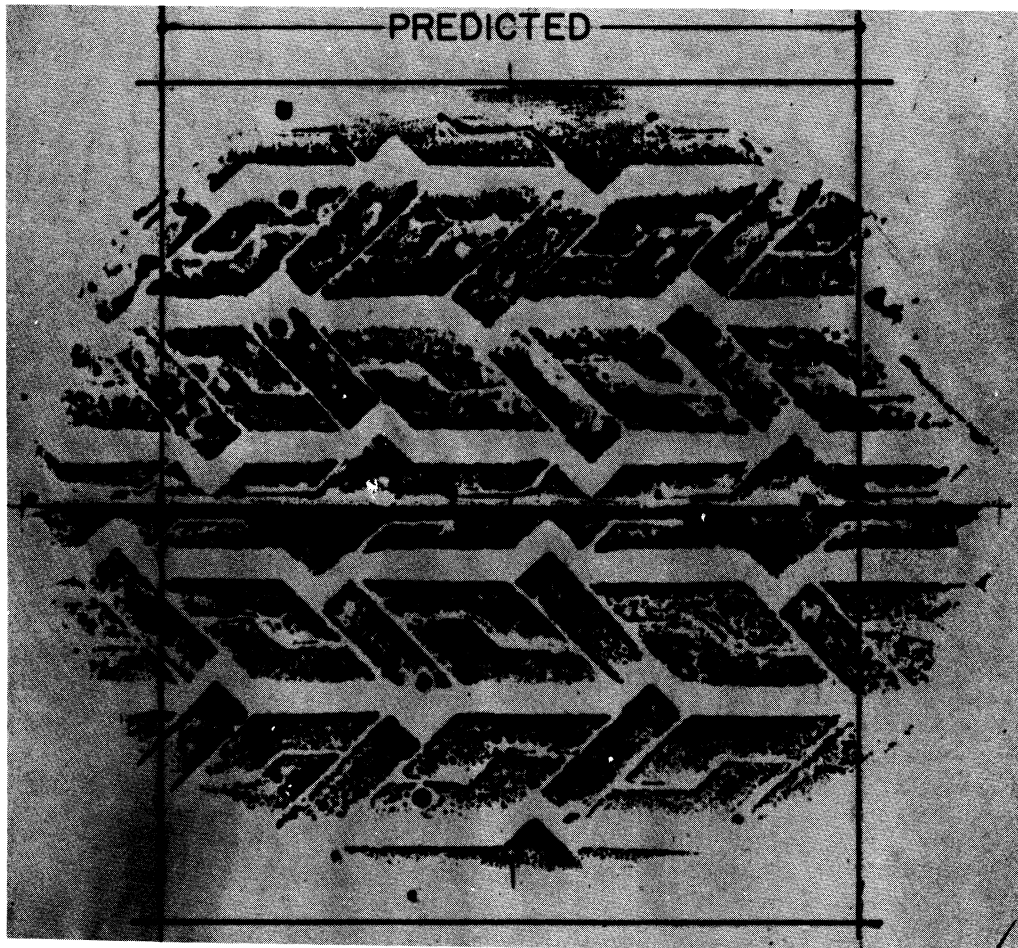


Fig. 8. Real and predicted contact patch at 0.5 in. deflection.

distributions was not readily available in the program.

Calculation and experiment can also be compared by considering the cross sections of the deformed tire. The most obvious place at which to do this is the centerline of the contact patch, denoted in this report by the angle $\theta = 0$. In Figs. 10 and 11, for both the 0.5 in. and 1.0 in. deflection con-

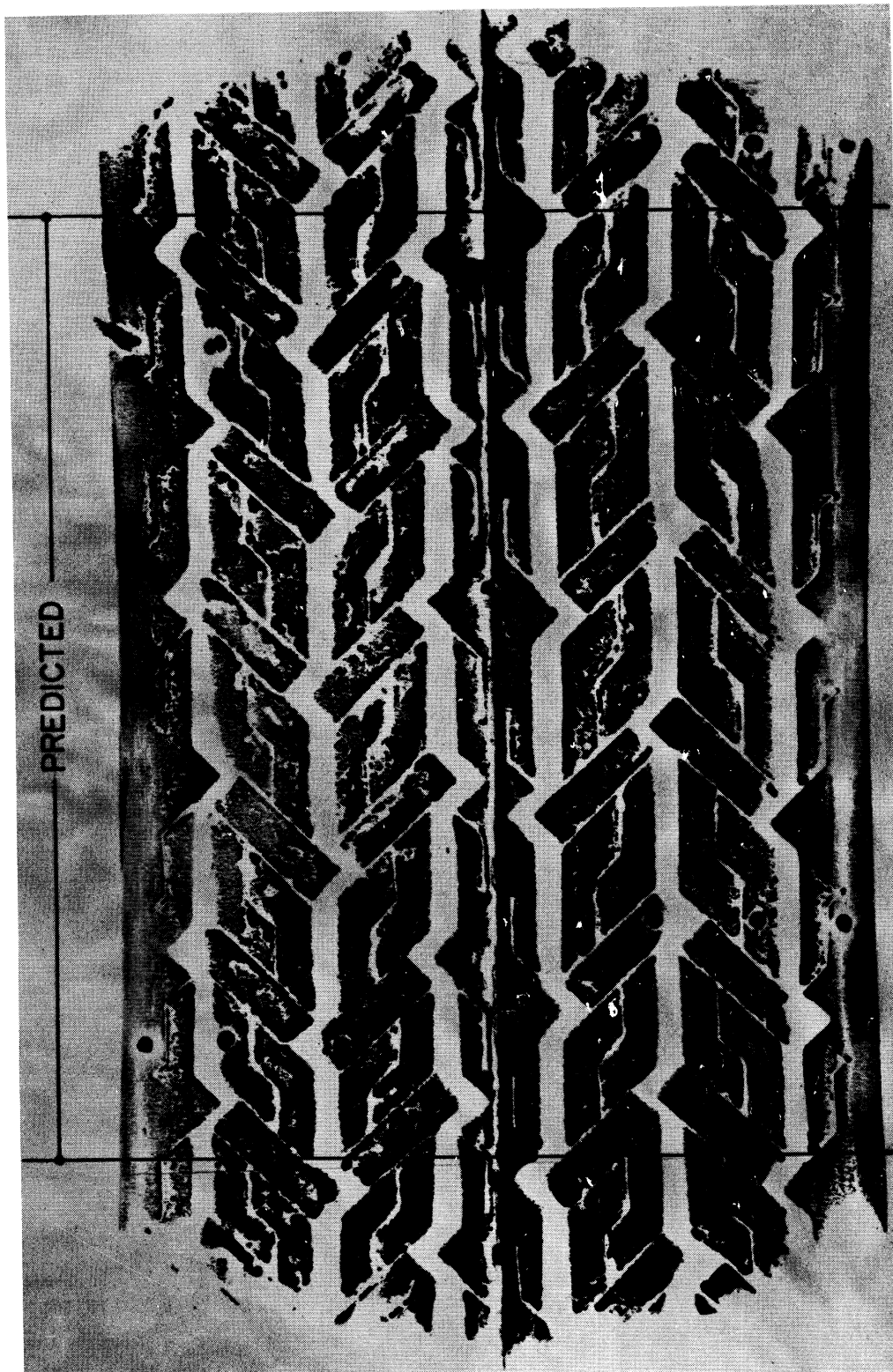


Fig. 9. Real and predicted contact patch at 1.0 in. deflection.

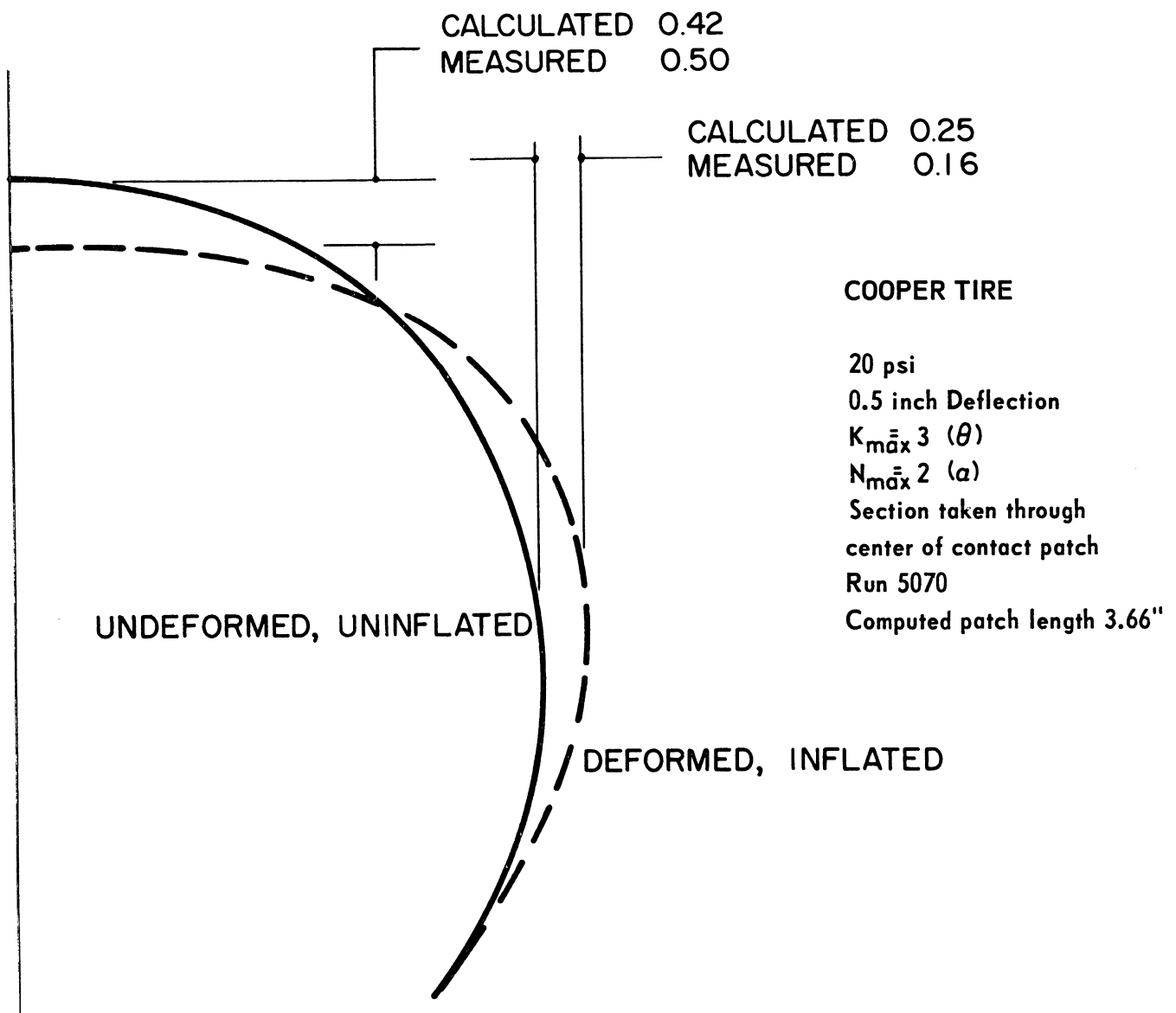


Fig. 10. Deflected and undeflected tire carcass midlines, from calculation, for 0.5 in. deflection.

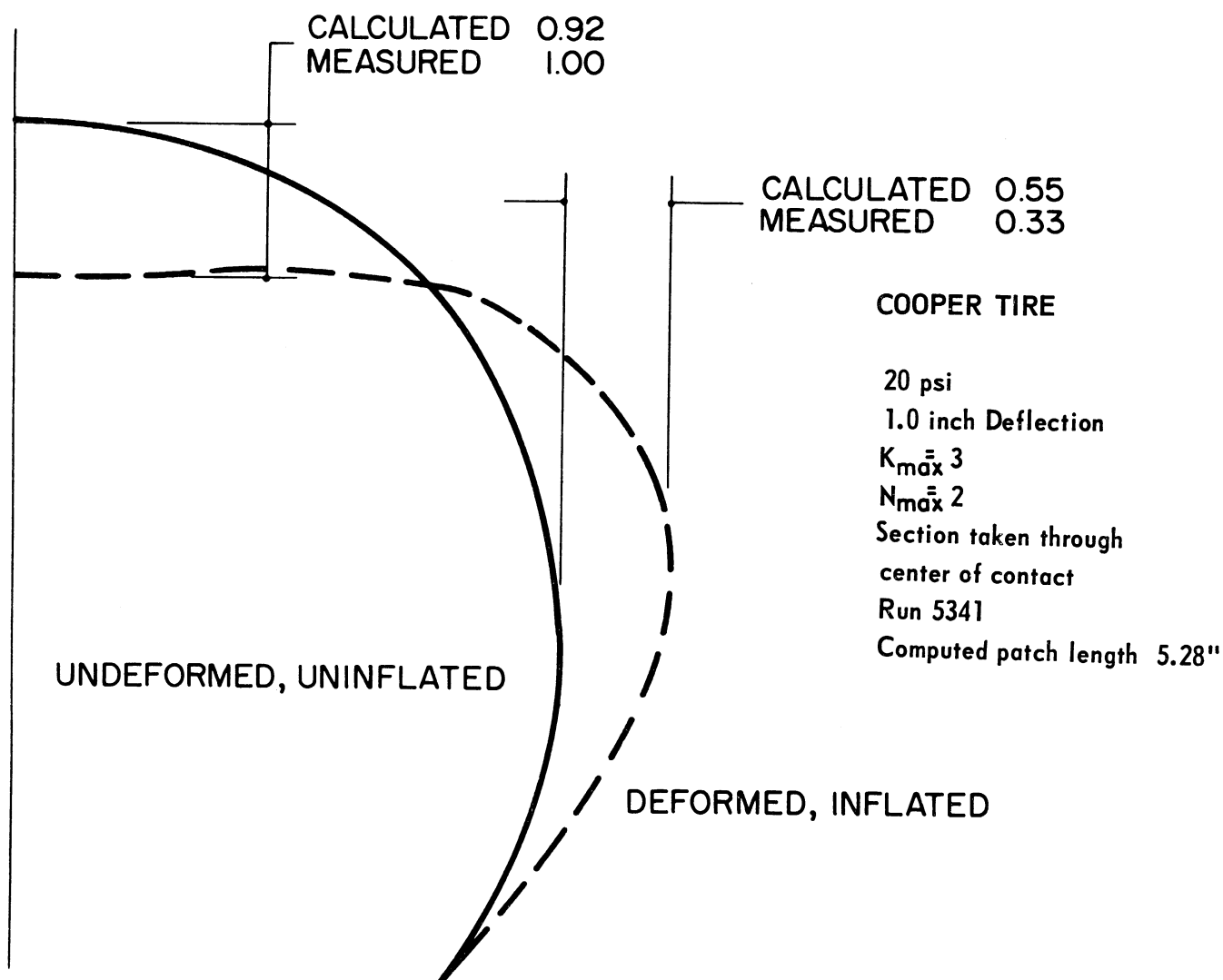


Fig. 11. Deflected and undeflected tire carcass midlines, from calculation, for 1.0 in. deflection.

ditions, the deflected carcass midlines are compared with the uninflated carcass midlines. The deflection at the crown is apparently very close to what one would expect, since the imposed deflections indicated on the figures are only slightly greater than the carcass midline deflections. This should be so because some deflection occurs in the tread itself. Rough calculations indicate that this tread deflection is accurate enough to account for the observed carcass midline deflection as shown in Figs. 10 and 11.

There is some discrepancy between the deflections of the side wall and the deflections measured with calipers at the widest point of the tire. In general, the side wall deflection is about 1.5 times as great as that observed on the actual tire, and, at the moment, there is no definite explanation for this discrepancy. However, the necessary truncation of these series at a very few terms is probably the cause, since representation of such a curve by a three term Fourier series will not give very accurate results.

In the final, completed version of the digital computer program, provision will be made for calculation of cord loads and interply stresses at various places throughout the contact patch of the tire, as well as in sections of the tire not in contact with the roadway. At the moment this set of calculations has not been performed, although it could easily be done by including in the present program techniques previously developed for the inflation program. No data on cord loads or interply stresses are presented in this report, but they will be made available in the succeeding technical report dealing with the actual use of this digital computer program.

XII. ACKNOWLEDGMENTS

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