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THE BANACH ALGEBRA OF CONTINUOUS MAPPINGS FROM A COMPACT HAUSDORFF SPACE TO A BANACH ALGEBRA

by
William K. Smith

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THE BANACH ALGEBRA OF CONTINUOUS MAPPINGS FROM A COMPACT HAUSDORFF SPACE TO A BANACH ALGEBRA

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A study is made of the Banach algebra B^X of the set of all continuous mappings from a compact Hausdorff space X to a Banach algebra B. Here B is assumed to have identity e and to be commutative. The scalars may be either real or complex; in the former case it is assumed that b^2+e has an inverse for each $b \in B$; in the latter it is assumed that $b \in B$ implies $b^* \in B$, where $b^*(M)=\overline{b(M)}$ for each maximal ideal M in B. It is shown that the maximal ideal space $\mathfrak{N}$ of B^X is homeomorphic to $X \times \mathfrak{M}$, where $\mathfrak{M}$ is the maximal ideal space of B. Thus each ideal $M \in \mathfrak{M}$ determines a subset $\mathfrak{N}(M)$ of $\mathfrak{N}$ homeomorphic to X. This fact is exploited to obtain a theorem stating that X and Y are homeomorphic if an isomorphism exists between B^X and B^Y which also provides an isomorphism between the constant mappings in B^X and those in B^Y; moreover, this condition is necessary.

Next, further properties of B^X are investigated in the case that B is semi-simple and a set of necessary and sufficient conditions is given describing when, for given algebras A and B, there is a compact space X such that A is equivalent to B^X. The conditions are based on the sets $\mathfrak{N}(M)$ mentioned above, although the points of X are bounded linear multiplicative transformations from A to a subalgebra B'

equivalent to B. Throughout the numerical-valued functions $b(M)$, $M \in \mathfrak{M}$ and $f(N)$, $f \in B^X$, $N \in \mathfrak{N}$, are used extensively.

If B is not semi-simple a characterization based on these methods does not seem possible. It is shown however that if B has radical P then the radical Q of B^X is exactly P^X. Also if B is a direct sum of B/P and P then B^X/Q is equivalent to $(B/P)^X$ and B^X is a direct sum of B^X/Q and Q.

Consideration is given to the relation between the maximal ideals of a Banach algebra and the T-sets Myers has defined in a Banach space. Roughly speaking, the former are related to "minimum value" whereas the latter are related to "maximum value". A method of enlarging each maximal ideal to a set more closely related to the T-set is given, although it is pointed out that since the ideals depend only upon the algebraic structure of the algebra whereas the definition of T-set involves the norm of the space the analogy probably cannot be pushed too far.

The final section specializes B to be a regular algebra as studied by Šilov, i.e. one such that Gelfand's topology for $\mathfrak{M}$ and the Stone intersection topology mentioned by Gelfand and Šilov lead to homeomorphic spaces. It is shown that B^X is regular if and only if B is regular.

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§1. Introduction

In studying the system of continuous functions defined on a topological space X and having values in some space one encounters two basic questions: Does the system of functions determine the topological space? and What characterizes the system of functions? The answers to one or both of these questions have been given by various writers when the image space is the field of real numbers and when the system of functions is treated as a lattice, an ordered group, a linear lattice, a Banach space and a Banach algebra, among others. And Jerison [4] has generalized some of these results to the case where the image space is a strictly convex Banach space, the functions being treated as a Banach space.

This work gives consideration to these questions for X a compact Hausdorff space and a Banach algebra B as image space, the set of functions forming a Banach algebra. It is first (§3) found that under certain conditions the space of maximal ideals in B^X is homeomorphic to $X \times \mathfrak{M}$, where $\mathfrak{M}$ is the space of maximal ideals in B. This makes it possible then (§4) to state conditions under which the isomorphism of B^X and B^Y implies the homeomorphism of X and Y.

The image algebra B is then (§5) specialized to be semi-simple (i.e. having no radical) and the properties of the algebra B^X are studied. From this study a characterization of B^X is obtained (§6). The case where B has non-trivial radical is next (§7) touched on lightly.

In § 8 a study is made of the analogy between the T-sets Myers [6] defined in a Banach space and the maximal ideals in a Banach algebra. It is found that, roughly speaking, the former are related to "maximum value" and the latter to "minimum value." A different approach to a homeomorphism theorem is given.

The "regular" algebras studied by Šilov [8] are considered briefly in §9 where it is shown that, for compact X, B^X is regular if and only if B is regular.

Many of the results are true for either a real or complex Banach algebra, although some basic restrictions on B are stated in §§ 2 and 3.

§2. Preliminaries; definitions, notation

Let B be a Banach algebra (normed ring) in the sense of Gelfand [2] , i.e.

i) B is a linear normed complete space.

ii) a commutative and associative multiplication is defined in B, satisfying:

$$b_1(b_2+b_3)=b_1b_2+b_1b_3$$

$$\lambda(b_1b_2)=(\lambda b_1)b_2, \quad \lambda \text{ a scalar.}$$

iii) B has an identity e with respect to multiplication and $\|e\|=1$ ($\|b\|$ denotes the norm of $b \in B$).

iv) $\|b_1b_2\| \leq \|b_1\| \cdot \|b_2\|$, $b_1, b_2 \in B$.

The field of scalars over which B is defined may be either the field of real numbers or the field of complex numbers;

however, in the former case it will be assumed that B satisfies

v) for each $b \in B$, b^2+e has an inverse.

The space of maximal ideals in B will be denoted by $\mathfrak{M}$. The topology of $\mathfrak{M}$ will be the first or preferred topology of Gelfand, which makes $\mathfrak{M}$ a compact Hausdorff space and which makes each $b \in B$ correspond to a <u>continuous</u> numerical-valued function on $\mathfrak{M}$. For $b \in B, M \in \mathfrak{M}$ these functions will be denoted by $b(M)$; their values will be complex numbers if the scalar field is complex, real numbers (thanks to v)) if the scalar field is the field of reals. This topology on $\mathfrak{M}$ is defined in terms of neighborhoods, where if $M_0 \in \mathfrak{M}$, a $V(M_0)$ is determined by $\varepsilon > 0$ and $b_1, \ldots, b_n \in B$ as

$$V(M_0) = \{M \in \mathfrak{M} \mid |b_i(M) - b_i(M_0)| < \varepsilon, \ i = 1, \ldots, n\}.$$

As Gelfand has shown, B is homomorphic to a subalgebra of the algebra of all continuous functions on $\mathfrak{M}$; the kernel of the homomorphism is the radical of B.

Let X be a compact Hausdorff topological space and let B^X be the set of all continuous (hence bounded) functions defined on X with values in B. If $f, g \in B^X$, define $f+g$ and fg as

$$(f+g)(x) = f(x)+g(x)$$

$$(fg)(x) = f(x)g(x).$$

Also, for $f \in B^X$ define the norm of f as

$$\|f\| = \sup_{x \in X} \|f(x)\|.$$

With these definitions B^X becomes a Banach algebra. The

identity in B^X is the function which is identically equal to e; denote it by e', i.e. $e'(x)=e$, all $x \in X$. More generally, the functions constant on X form a subalgebra B' of B^X equivalent to B: if $b \in B$, then $b'(x)=b$, all $x \in X$, and $\|b'\| = \sup_{x \in X} \|b'(x)\| = \|b\|$.

Denote the space of maximal ideals in B^X by $\mathfrak{N}$.

§3. The space $\mathfrak{N}$.

In this section it will be shown that under certain conditions every ideal of $\mathfrak{N}$ is determined by a point $x \in X$ and a maximal ideal $M \in \mathfrak{M}$, and in fact $\mathfrak{N}$ is homeomorphic to $X \times \mathfrak{M}$.

To this end I begin with a known [10] result.

Lemma 3.1. If $x_o \in X$ and $M_o \in \mathfrak{M}$ then the set $(x_o, M_o) = \{f \in B^X \mid f(x_o) \in M_o\}$ is a maximal ideal of B^X.

Proof. It is trivial to show that (x_o, M_o) is an ideal. Suppose (x_o, M_o) is properly contained in a (proper) maximal ideal N; then there exists $g \in N$ such that $g(x_o)=b_o \notin M_o$. The maximality of M_o in B implies that there exist $m_o \in M_o$ and $b_1 \in B$ such that $e=b_1 \cdot b_o + m_o$. Let $f \in B^X$ be defined by $f=e'-gb_1'$; then $f \in (x_o, M_o)$, since $f(x_o)=e'(x_o)-g(x_o)b_1'(x_o)=e-b_o b_1 = m_o \in M_o$. But from this it follows that $e' \in N$, for

$$e' = gb_1' + (e' - gb_1') = gb_1' + f$$

where $g \in N$, so $gb_1' \in N$ and $f \in (x_o, M_o) \subset N$. This means $N=B^X$, contradicting the assumption that N is proper. Thus, (x_o, M_o) is a maximal ideal.

The following lemma is due to Yood [10] .

Lemma 3.2. For any $f \in B^X$ the numerical-valued function $f(x,M)$ defined by $f(x,M)=f(x)(M)$ is continuous on $X \times \mathcal{M}$.

Proof. Let $(x_0,M_0) \in X \times \mathcal{M}$; let $\varepsilon>0$ be given.

Let

$$V(x_0)=\{x \in X \mid \|f(x)-f(x_0)\| < \varepsilon/2\};$$

let

$$V(M_0)=\{M \in \mathcal{M} \mid |f(x_0)(M)-f(x_0)(M_0)| < \varepsilon/2\}.$$

Let $V(x_0,M_0)=V(x_0)\times V(M_0)$. Then, if $(x,M) \in V(x_0,M_0)$,

$$\begin{aligned} |f(x,M)-f(x_0,M_0)| &\leq |f(x,M)-f(x_0,M)|+|f(x_0,M)-f(x_0,M_0)| \\ &= |f(x)(M)-f(x_0)(M)|+|f(x_0)(M)-f(x_0)(M_0)| \\ &= |(f(x)-f(x_0))(M)|+|f(x_0)(M)-f(x_0)(M_0)| \\ &\leq \|f(x)-f(x_0)\|+|f(x_0)(M)-f(x_0)(M_0)| \\ &< \varepsilon/2+\varepsilon/2 = \varepsilon, \end{aligned}$$

so $f(x,M)$ is continuous.

Theorem 3.1. Let B be a real (i.e. having real scalars) Banach algebra. Then for every maximal ideal N_0 in B^X there is a point $x_0 \in X$ and a maximal ideal $M_0 \in \mathcal{M}$ such that $N_0=\{f \in B^X \mid f(x_0) \in M_0\}$.

Proof. The method of proof is contrapositive. If N_0 is not of the stated type, then for every $x_0 \in X$ and every $M_0 \in \mathcal{M}$ there is $f_0 \in N_0$ such that $f_0(x_0) \notin M_0$. This means that for every $(x_0,M_0) \in X \times \mathcal{M}$ there is $f_0 \in N_0$ such that $f_0(x_0,M_0) \neq 0$. Since $f(x,M)$ is continuous on $X \times \mathcal{M}$, each (x_0,M_0) has a neighborhood $V(x_0,M_0)$ such that $(x,M) \in V(x_0,M_0)$ implies $f_0(x,M) \neq 0$. The compactness of $X \times \mathcal{M}$ assures the existence of

a finite number of neighborhoods $V_1, \ldots, V_n$ which cover $X \times \mathcal{M}$ and corresponding functions $f_1, \ldots, f_n \in N_o$ such that $(x,M) \in V_i$ implies $f_i(x,M) \neq 0$. Let

$$f = f_1^2 + \ldots + f_n^2; \tag{3.1}$$

then $f \in N_o$ and moreover $f(x,M)$ is nowhere zero. This last means that for each $x \in X$, $f(x)$ is in no maximal ideal in B, so that $(f(x))^{-1}$ exists. Now, for b in the set of regular elements in B the mapping $\varphi(b) = b^{-1}$ is continuous. Thus, if

$$g(x) = (f(x))^{-1},$$

then $g = \varphi f$, so g is continuous, i.e. $g \in B^X$. But $g = f^{-1}$ and hence f cannot be in the maximal ideal N_o, a contradiction. Therefore, for some $x_o \in X$, $M_o \in \mathcal{M}$ and for every $f \in N_o$, $f(x_o) \in M_o$. Since N_o is maximal it must consist of all such f.

Theorem 3.2. Let B be a complex Banach algebra satisfying

i) to each element $b \in B$ there corresponds an element b^* such that $b^*(M) = \overline{b(M)}$ for all $M \in \mathcal{M}$ ($\overline{\lambda}$ denotes the complex conjugate of the number λ);

ii) the mapping $\psi: B \longrightarrow B$ defined by $\psi(b) = b^*$ is continuous on B.

Then the conclusion of theorem 3.1 holds.

Proof. The proof is the same as that of theorem 3.1 except for slight modifications as follows. For each f_i, $i = 1, \ldots, n$, define a function g_i by

$$g_i(x) = (f_i(x))^*;$$

then ii) guarantees that g_i is continuous on X, so $g_i \in B^X$.

Next, replace the definition of f in (3.1) by

$$f=f_1 g_1+\cdots+f_n g_n \qquad (3.2).$$

Then $f\in N_0$, and the remainder of the proof is as before.

Lemma 3.3. If B is without radical, then the mapping $\gamma(b)=b^*$, where $b^*(M)=\overline{b(M)}$, all $M\in\mathfrak{M}$, is a continuous mapping.

Proof. Let B_1 be the algebra with the same set of elements and the same algebraic structure as B, but with norm defined as $\|b\|_1=\|b^*\|$. Then B_1 will be a Banach algebra if, under this norm, B_1 is complete. This is readily seen to be so. In fact, let $\{b_n\}$ be a sequence in B_1 such that $\|b_n-b_m\|_1\longrightarrow 0$. This means $\|b_n^*-b_m^*\|\longrightarrow 0$, so, by the completeness of B, there exists $b_0^*\in B$ such that $\|b_n^*-b_0^*\|\longrightarrow 0$. But this says exactly that $\|b_n-b_0\|_1\longrightarrow 0$, and hence B_1 is complete. Clearly B and B_1 are isomorphic under the identity mapping, so by theorem 17 of Gelfand [2] B and B_1 are homeomorphic. It now follows that the mapping $\gamma(b)=b^*$ is continuous, for if $\|b_n-b_0\|_1\longrightarrow 0$ then B_1 homeomorphic to B implies $\|b_n-b_0\|\longrightarrow 0$, i.e. $\|b_n^*-b_0^*\|\longrightarrow 0$.

Combining theorem 3.2 and lemma 3.3 leads immediately to the following theorem.

Theorem 3.3. Let B be a complex Banach algebra without radical such that to each element $b\in B$ there corresponds an element b^* such that $b^*(M)=\overline{b(M)}$ for all $M\in\mathfrak{M}$. Then the conclusion of theorem 3.1 holds.

Lemma 3.4. The two maximal ideals (x_1,M_1) and (x_2,M_2)

of B^X are equal if and only if $x_1=x_2$ and $M_1=M_2$.

Proof. The "if" part is obvious. Suppose $M_1 \not\subset M_2$. Then there exists $b \in M_1$, $b \notin M_2$. Let $b'(x)=b$, all $x \in X$, so $b' \in (x_1, M_1)$, $b' \notin (x_2, M_2)$.

Finally, suppose $x_1 \neq x_2$. Then there exists $\lambda : X \longrightarrow [0,1]$, $\lambda(x_1)=0$, $\lambda(x_2)=1$. Set $f(x)=\lambda(x).e$, so $f \in (x_1, M_1)$, $f \notin (x_2, M_2)$.

Lemma 3.5. If $(x,M) \in X \times \mathcal{M}$ corresponds to $N \in \mathcal{N}$ and if $f \in B^X$, then $f(x,M)=f(x)(M)$ is equal to $f(N)$.

Proof. i) Let $f \in N$, so $f(N)=0$. If N corresponds to (x,M), then $f(x) \in M$ and $f(x)(M)=0=f(N)$.

ii) Let $f=e'$, the identity in B^X. Then $e'(N)=1$. Also, $e'(x)=e$, all $x \in X$, so $e'(x)(M)=e(M)=1=e'(N)$.

iii) Let f be arbitrary in B^X. Let $f(N)=\lambda$; define $g=f-\lambda e'$. Then $g(N)=(f-\lambda e')(N)=f(N)-\lambda e'(N)=\lambda-\lambda=0$, so $g \in N$ and $g(N)=g(x,M)$, where (x,M) corresponds to N. Thus

$$\begin{aligned} 0=g(N)=g(x)(M) &= \left[(f-\lambda e')(x)\right](M) \\ &=(f(x)-\lambda e'(x))(M)=(f(x)-\lambda e)(M) \\ &=f(x)(M)-\lambda e\ (M)=f(x)(M)-\lambda \\ &=f(x,M)-f(N), \end{aligned}$$

so $f(x,M)=f(N)$.

Theorem 3.4. Under the hypotheses of either theorem 3.1 or theorem 3.2 or theorem 3.3 the space $\mathcal{N}$ is homeomorphic to $X \times \mathcal{M}$.

Proof. Let $\eta(x,M)=N$. By lemmas 3.1 and 3.4 and any one of theorems 3.1, 3.2, or 3.3, η is a 1-1 mapping of $X \times \mathcal{M}$

onto $\mathfrak{N}$. Since both $X\times\mathfrak{M}$ and $\mathfrak{N}$ are compact Hausdorff spaces, homeomorphism will follow if η is shown to be continuous. Let $\eta(x_\circ,M_\circ)=N_\circ$, let

$$V(N_\circ)=\{N\in\mathfrak{N} \mid |f_i(N)-f_i(N_\circ)|<\varepsilon,\ i=1,\dots,n\}.$$

Let

$$V_i(x_\circ)=\{x\in X \mid \|f_i(x)-f_i(x_\circ)\|<\varepsilon/2\},\ i=1,\dots,n,$$

and let

$$V(x_\circ)=\bigcap_{i=1}^{n} V_i(x_\circ).$$

Let $f_i(x_\circ)=b_i$, let

$$V(M_\circ)=\{M\in\mathfrak{M} \mid |b_i(M)-b_i(M_\circ)|<\varepsilon/2,\ i=1,\dots,n\}.$$

Finally, let $V(x_\circ,M_\circ)=V(x_\circ)\times V(M_\circ)$ and let $(x,M)\in V(x_\circ,M_\circ)$, $\eta(x,M)=N$. Then for $i=1,\dots,n$ (using lemma 3.5),

$$\begin{aligned}
|f_i(N)-f_i(N_\circ)| &= |f_i(N)-f_i(x_\circ)(M)+f_i(x_\circ)(M)-f_i(N_\circ)| \\
&\le |f_i(x,M)-f_i(x_\circ)(M)|+|f_i(x_\circ)(M)-f_i(x_\circ,M_\circ)| \\
&= |f_i(x)(M)-f_i(x_\circ)(M)|+|b_i(M)-b_i(M_\circ)| \\
&= |(f_i(x)-f_i(x_\circ))(M)|+|b_i(M)-b_i(M_\circ)| \\
&\le \|f_i(x)-f_i(x_\circ)\|+|b_i(M)-b_i(M_\circ)| \\
&< \varepsilon/2+\varepsilon/2=\varepsilon,
\end{aligned}$$

since $x\in V(x_\circ)$ and $M\in V(M_\circ)$.

Thus η is a continuous, hence a homeomorphic mapping.

§4. The homeomorphism theorem

In the sequel it will be assumed that a complex B with radical satisfies the hypotheses of theorem 3.2 and a complex B without radical satisfies the hypotheses of theorem 3.3.

The literature abounds with answers (largely in the affirmative) to the question whether or not a structure (lattice, Banach space, sub-space of a Banach space, Banach algebra, etc.) of real- or complex-valued functions defined on a compact Hausdorff space determines the space. In the case where the image space of the functions is a Banach space the result is not generally true, however. M.Jerison [4], drawing on an example of R.H. Fox [1] , has in fact given an example (see below) of a Banach space B and non-homeomorphic compact Hausdorff spaces X and Y such that the Banach spaces B^X and B^Y are equivalent. On the other hand, Jerison has shown for a wide class of Banach spaces that the Banach-space equivalence of B^X and B^Y, X and Y compact Hausdorff, implies the homeomorphism of X and Y.

Now Jerison's counter-example involves, and his theorem fails to hold in the case of, a B which is itself the set of numerical-valued functions defined on some compact Hausdorff space. And this fact points up a possible difficulty of obtaining an analogous theorem in the present case where B is a Banach algebra; for all Banach algebras are either equivalent to the algebra of all functions on the compact Hausdorff space of maximal ideals or are isomorphic or homomorphic to a dense subalgebra thereof.

Moreover, theorem 3.4 sheds further light on the matter. In the first place there is the following rather trivial consequence: whenever the Banach algebra B has space of

maximal ideals $\mathcal{M}$ which is such that $X\times\mathcal{M}$ homeomorphic to $Y\times\mathcal{M}$, X and Y compact Hausdorff, implies X homeomorphic to Y, then it follows that B^X isomorphic to B^Y implies X homeomorphic to Y. In particular, there is the following theorem.

Theorem 4.1. Let B be a Banach algebra with a finite number of maximal ideals. Let X and Y be compact Hausdorff spaces. Then if B^X is isomorphic to B^Y the space X is homeomorphic to Y.

This theorem, then, covers the case where B is any Euclidean space, multiplication being defined coordinate-wise.

Secondly, the Jerison-Fox counter-example applies as well to Banach algebras. In fact, let

$\mathcal{M}$ = the unit interval

$X=\{(\lambda,\mu)\mid \frac{1}{2}\leq\lambda^2+\mu^2\leq 1;\ \text{or } 0\leq\lambda\leq 2,\ \mu=0\}$

$Y=\{(\lambda,\mu)\mid \frac{1}{2}\leq\lambda^2+\mu^2\leq 1;\ \text{or } 1\leq\lambda\leq 2,\ \mu=0;\ \text{or } \lambda=0,\ 1\leq\mu\leq 2\}$;

then X and Y are not homeomorphic although $X\times\mathcal{M}$ and $Y\times\mathcal{M}$ are. Hence, if we let $B=\mathcal{C}(\mathcal{M})$ = the set of all continuous real-valued functions defined on $\mathcal{M}$, then B^X has space of maximal ideals $X\times\mathcal{M}$ and B^X is equivalent to the algebra of all continuous real-valued functions on $X\times\mathcal{M}$. Similarly B^Y has space of maximal ideals $Y\times\mathcal{M}$ and B^Y is equivalent to $\mathcal{C}(Y\times\mathcal{M})$. Therefore, as Banach algebras, B^X and B^Y are equivalent, whereas X and Y are not homeomorphic. Finally it should be remarked that the "cancellation" problem (using = to denote

"is homeomorphic to," does $X\times\mathfrak{M}=Y\times\mathfrak{M}$ imply $X=Y$?) is far from simple, and the Banach algebra approach to it is probably not sufficiently powerful to give much of a solution. Consequently it is not likely that it will be possible to obtain a homeomorphism theorem for Banach algebras analogous to that obtained by Jerison for Banach spaces.

The homeomorphism theorem which appears below makes use of the subalgebra B' of the constant functions in B^X. Theorem 8.2 gives a different approach.

In the subalgebra B', as was remarked in §2, the element b' is the function which for all $x\in X$ takes on the value b. If M is a maximal ideal in B, then M' is the corresponding maximal ideal in B'.

Lemma 4.1. If $N=(x,M)\in\mathfrak{N}$, then $N\cap B'=M'$.

Proof. Since

$$N=(x,M)=\left\{f\in B^X \,\middle|\, f(x)\in M\right\},$$

it follows that

$$N\cap B'=(x,M)\cap B'=\left\{b'\in B' \,\middle|\, b'(x)\in M\right\}.$$

But $b'\in B'$ implies $b'(x)=b$, all $x\in X$, so $b'\in(x,M)\cap B'$ implies $b\in M$ and $b'\in M'$. Conversely, let $b'\in M'$; then $b'(x)=b\in M$, all $x\in X$, so $b'\in(x,M)\cap B'$. Hence, $(x,M)\cap B'=M'$.

Lemma 4.2. If $N\in\mathfrak{N}$ and $N\cap B'=M_0'\in\mathfrak{M}'$, then $N=(x,M_0)$ for some $x\in X$.

Proof. We know that $N=(x,M_1)$ for some $M_1\in\mathfrak{M}$. By lemma 4.1, then, $N\cap B'=M_1'$, and by hypothesis $N\cap B'=M_0'$. Hence $M_0'=M_1'$ and by the equivalence $M_0=M_1$, which is to say $N=(x,M_0)$.

Definition. For $M \in \mathcal{M}$, let

$$\eta(M) = \{N \in \eta \mid N \cap B' = M'\}.$$

Theorem 4.2. For any $M_0 \in \mathcal{M}$, $\eta(M_0)$ is homeomorphic to X.

Proof. By theorem 3.4 η is homeomorphic to $X \times \mathcal{M}$, and $\eta(M_0)$, as defined above, is a subset of η. But it follows from lemmas 3.4, 4.1 and 4.2 that under this homeomorphism (between η and $X \times \mathcal{M}$) $\eta(M_0)$ corresponds to $X \times M_0$ which is homeomorphic to X, so $\eta(M_0)$ is homeomorphic to X.

Theorem 4.3. Let X and Y be compact Hausdorff spaces and let B^X be isomorphic to B^Y. Assume further, if the isomorphism between B^X and B^Y is provided by a mapping Ψ, that Ψ furnishes an isomorphism between the subalgebra B' of constant functions in B^X and B'*, the subalgebra of constant functions in B^Y. Then X and Y are homeomorphic.

Proof. For $M' \in \mathcal{M}'$, $N \in \eta$, let $\Psi(M') = M'^* \in \mathcal{M}'^*$, $\Psi(N) = N^* \in \eta^*$. Then

$$\begin{aligned} \Psi[\eta(M)] &= \Psi\{N \in \eta \mid N \cap B' = M'\} \\ &= \{N^* \in \eta^* \mid N^* \cap B'^* = M'^*\} \\ &= \eta^*(M). \end{aligned}$$

Now the topology of the space of maximal ideals depends only upon the algebraic structure of the Banach algebra. It follows, then, from the above equation and the hypothesis that B^X and B^Y are isomorphic that $\eta(M)$ is homeomorphic to $\eta^*(M)$. But $\eta(M)$ is homeomorphic to X and $\eta^*(M)$ is

homeomorphic to Y, so the conclusion of the theorem follows.

Conversely, suppose X and Y are homeomorphic compact Hausdorff spaces, the homeomorphism being given by a 1-1, bicontinuous mapping φ, $\varphi(y)=x$. Let B be a Banach algebra. For each $f \in B^X$ define $\Psi(f)=g \in B^Y$ by

$$g(y)=f(\varphi(y)).$$

Suppose $b'(x)=b$, all $x \in X$; let $\Psi(b')=b'^*$. Then

$$b'^*(y)=b'(\varphi(y))=b, \text{ all } y \in Y.$$

Thus Ψ takes constant functions into constant functions. Moreover, it is easily verified that Ψ is an isomorphic (and an isometric) mapping of B^X on B^Y. Thus the following theorem has been proven:

Theorem 4.4. Let X and Y be homeomorphic compact Hausdorff spaces. Let B be a Banach algebra. Then there exists an isomorphism between B^X and B^Y which is also an isomorphism between the subalgebra B' of constant functions in B^X and the subalgebra B'* of constant functions in B^Y.

§5. Properties of the algebra B^X

In this and the following section B will be a Banach algebra as described in §§2, 3 and 4, with the additional restriction that B is without radical; thus if B is complex it will be assumed that B contains the conjugate elements. Some properties of the algebra B^X will be discussed. First, however, three lemmas should be mentioned.

Lemma 5.1. If the Banach algebra A contains a subalgebra B' which has the same identity e' as A, if M' is a maximal ideal in B', N is a maximal ideal in A and $M' \subset N$; then, for each $b' \in B'$, $b'(M') = b'(N)$.

Proof. i) Let $b' \in M'$; then $b' \in N$ and $b'(M') = b'(N) = 0$. ii) Let $b' = e'$; then $b'(M') = b'(N) = 1$. iii) Let b' be an arbitrary element of B'; let $b'(M') = \lambda$. Then

$$(b' - \lambda e')(M') = b'(M') - \lambda e'(M')$$
$$= b'(M') - \lambda = \lambda - \lambda = 0,$$

so $b' - \lambda e' \in M'$ and hence $b' - \lambda e' \in N$. It follows from i) that

$$0 = (b' - \lambda e')(N) = b'(N) - \lambda e'(N) = b'(N) - \lambda, \text{ so}$$

$b'(N) = \lambda = b'(M')$.

Lemma 5.2 (Šilov [7]). If the algebra B' is contained in the algebra A, then each maximal ideal M' of B' is contained in some maximal ideal N of A.

Lemma 5.3. Let A and B' be algebras, $B' \subset A$, having the same identity e'. Then the intersection $N \cap B'$ of a maximal ideal N of A with B' is a maximal ideal of B'.

Proof. It is routine to show that $N \cap B'$ is an ideal in B'. Further, $N \cap B'$ is a proper ideal since the identity e' of B' and A is not an element of N. It remains to show that $N \cap B'$ is maximal in B'. Assume $N \cap B'$ not maximal. Then $N \cap B'$ is contained in a proper maximal ideal M' of B', and there is $m \in M'$, $m \notin N \cap B'$; in particular, $m \notin N$. This means there exist scalar λ and an $n \in N$ such that $e' = \lambda m + n$. Since $n = e' - \lambda m$,

it follows that $n \in B'$, so $n \in N \cap B' \subset M'$. But this means $e' = \lambda m + n \in M'$, which cannot be. Thus $N \cap B'$ is a maximal ideal in B'.

The properties of B^X that will be utilized are the following.

1. B^X is without radical.

Proof. It will be recalled that the radical of a Banach algebra is exactly the intersection of all maximal ideals. Suppose $f \in \bigcap_{N \in \mathcal{N}} N$; then for each $x \in X$ $f(x)$ lies in every maximal ideal $M \in \mathcal{M}$, i.e. $f(x)=0$, since B has only the trivial radical. Thus f is the zero of B^X.

2. The space $\mathcal{N}$ of maximal ideals in B^X is homeomorphic to $X \times \mathcal{M}$, and each maximal ideal in $\mathcal{N}$ is fixed, i.e. $N \in \mathcal{N}$ implies

$$N = \{f \mid f(x) \in M, \text{ some } x \in X, \text{ some } M \in \mathcal{M}\}.$$

This is the result of § 3.

3. If $f \in B^X$, $N = (x, M)$, then $f(N) = f(x)(M)$.

Proof. This was proven in § 3. It is stated again for emphasis.

4. The constant functions in B^X form a subalgebra B' equivalent to B; the identity of B' is that of B^X.

5. The intersection of a maximal ideal $N = (x, M)$ with B' is the maximal ideal M' that corresponds to M under the equivalence between B' and B: $N \cap B' = M'$, $N = (x, M)$.

This is the result of lemma 4.1.

6. For arbitrary $M \in \mathcal{M}$, the set $\mathcal{N}(M)$ is defined

$$\eta(M)=\{N\in\eta \mid N\cap B'=M'\}.$$

All sets $\eta(M)$, $M\in\mathfrak{M}$, are homeomorphic to X (theorem 4.1), hence to each other. Let M_o be an arbitrary but fixed element of $\mathfrak{M}$, let the homeomorphism between $\eta(M)$ and $\eta(M_o)$ be given by η_M; η_M: $\eta(M)\longrightarrow\eta(M_o)$, where $\eta_M(x,M)=(x,M_o)$.

7. It follows from 6 that, for $N_1\in\eta(M_1)$ and $N_2\in\eta(M_2)$, $\eta_{M_1}(N_1)=\eta_{M_2}(N_2)$ if and only if $N_1=(x_o,M_1)$ and $N_2=(x_o,M_2)$, for some $x_o\in X$. If in this case we define $N_1\sim N_2$ [i.e. $N_1\sim N_2$ if and only if $\eta_{M_1}(N_1)=\eta_{M_2}(N_2)$ where $N_1\in\eta(M_1)$, $N_2\in\eta(M_2)$, then it is clear that $\sim$ is an equivalence relation. Define $[N_o]=\{N\in\eta \mid N\sim N_o\}$. Then, if $N_o=(x_o,M_o)$

$$[N_o]=\{N\in\eta \mid N=(x_o,M),\ M\in\mathfrak{M}\}=x_o\cdot\mathfrak{M}.$$

8. For each $f\in B^X$ and each class $[N_i]$ there is a unique $b'\in B'$ such that $f(N)=b'(N)$, each $N\in[N_i]$.

Proof. Let $N_1=(x_1,M_o)$, let $f(x_1)=b$, let $b'\in B'$ correspond to b. Then for any $N\in[N_1]$, $N=(x_1,M)$,

$$\begin{aligned} f(N)&=f(x_1,M)=f(x_1)(M)\\ &=b(M)=b'(M')\\ &=b'(N), \end{aligned}$$

by lemma 5.1. The uniqueness of b' follows from the fact that B' is without radical, so that each element b' is uniquely determined by the set of values $b'(M')$, $M'\in\mathfrak{M}'$.

An alternative formulation of 8 is

8'. For each $f\in B^X$ and each class $[N_i]$ there is a

unique $b \in B$ such that $b(M)=f(\eta_M^{-1}(N_1))$, each $M \in \mathfrak{M}$, where $N_1 \in \eta(M_0)$.

Proof. Let $N_1=(x_1,M_0)$; then $\eta_M^{-1}(N_1)=(x_1,M)$ and $f(\eta_M^{-1}(N_1))=f(x_1,M)=b(M)$, where $b=f(x_1)$.

9. If we define $T_{[N_1]}(f)=b'$, where b' is the element of B' whose existence was stated in 8, then $T_{[N_1]}$ is a linear multiplicative transformation from B^X to B'. Also, for each class $[N]$, $\|T_{[N]}\|=1$.

Proof. Let $N_1=(x_1,M_0)$, $T_{[N_1]}(f_1)=b_1'$, $T_{[N_1]}(f_2)=b_2'$; thus $b_1=f_1(x_1)$ and $b_2=f_2(x_1)$. Then $T_{[N_1]}(f_1 f_2)=b_1'b_2'$, since $(f_1 f_2)(x_1)=f_1(x_1)f_2(x_1)=b_1 b_2$; also, $T_{[N_1]}(f_1+f_2)=b_1'+b_2'$, since $(f_1+f_2)(x_1)=f_1(x_1)+f_2(x_1)=b_1+b_2$. Finally, since $\|f\|=\sup_{x \in X}\|f(x)\|$, $\|T_{[N]}(f)\| \leq \|f\|$, so $\|T_{[N]}\| \leq 1$. But $T_{[N]}(b')=b'$, so $\|T_{[N]}\|=1$.

10. For each $T_{[N_0]}$ the kernel is the ideal $I_{[N_0]}=I(x_0)$, where, if $N_0=(x_0,M_0)$,

$$I(x_0)=\{f \mid f(x_0)=0\}=\bigcap_{N \in [N_0]} N.$$

Proof. The first equality follows from the definition of $T_{[N_0]}$ and the second from the facts that $[N_0]=\{(x_0,M) \mid M \in \mathfrak{M}\}$ and $\bigcap_{M \in \mathfrak{M}} M=0$, since B has zero radical.

11. The residue class algebra $B^X/I_{[N_0]}$ is equivalent to B (or B').

Proof. Denote the elements of $B^X/I_{[N_0]}$ by $F,G,\ldots$. If $f_1,f_2 \in F$, then $f_1-f_2 \in I_{[N_0]}$, or $(f_1-f_2)(x_0)=0$, so $f_1(x_0)=f_2(x_0)=b$, say. Clearly, if $f \in B^X$ and $f(x_0)=b$, then

$f_1 - f \in I_{[N_0]}$, so $f \in F$. Thus the residue class F consists of exactly all $f \in B^X$ such that $f(x_0)=b$ (note that $b' \in F$). A similar statement being true for each element of $B^X / I_{[N_0]}$, the mapping ψ defined by $\psi(F)=b$, where $F=\{f \mid f(x_0)=b\}$, is a well-defined mapping of $B^X / I_{[N_0]}$ into B. Since B^X contains B' and since $b' \in B'$ in the class F implies $\psi(F)=b$, it follows that ψ maps B^X onto B. Clearly, if $F \neq G$, $\psi(F) \neq \psi(G)$, so ψ is 1-1. (It is obvious from the foregoing that each $b' \in B'$ determines an element of $B^X / I_{[N_0]}$ and the elements so determined make up all of $B^X / I_{[N_0]}$.) Next, from the way the operations are defined in $B^X / I_{[N_0]}$ it follows immediately that ψ is an isomorphism.

Finally, ψ is an isometry, for if $\psi(F)=b$,

$$\|F\| = \inf_{f \in F} \|f\| = \inf_{f \in F} \left[\sup_{x \in X} \|f(x)\| \right] \geq \inf_{f \in F} \|f(x_0)\| = \|b\| ;$$

conversely, since $b' \in F$,

$$\|F\| = \inf_{f \in F} \|f\| \leq \|b'\| = \|b\| .$$

Hence, $\|\psi(F)\| = \|F\|$, and ψ is an equivalence between $B^X / I_{[N_0]}$ and B.

12. For each $f \in B^X$ there is a class $[N_f]$ such that $\|T_{[N_f]}(f)\| = \|f\|$.

Proof. Since $T_{[N_0]}(f)=b'$, where $f(x_0)=b$, and since there is $x_f \in X$ such that $\|f(x_f)\| = \|f\|$, the result follows.

12'. For each $f \in B^X$ there is an ideal $I_{[N_f]} = I(x_f)$ such that $\|f\| = \|F\|$, where F is the residue class of $B^X / I_{[N_f]}$ which contains f.

Proof. Let $x_f \in X$ be such that $\|f(x_f)\| = \|f\|$ — since X is compact such x_f exists. If $F \in B^X / I(x_f)$ contains f, then $\|F\| = \|f(x_f)\|$, since $\psi(F) = f(x_f)$ and ψ is an isometry. Thus $\|f\| = \|F\|$.

13. There is contained in B^X a closed subalgebra $\mathcal{C}$ which is equivalent to the algebra $\mathcal{C}(X)$ of all continuous numerical-valued functions on X, viz.

$$\mathcal{C} = \{ f \in B^X \mid f(x) = \lambda_x e, \text{ all } x \in X \}.$$

Since $N_1 \sim N_2$ implies $N_1 = (x, M_1)$ and $N_2 = (x, M_2)$ and since if $N = (x, M)$, $f(N) = f(x)(M)$, it follows that

$$\mathcal{C} = \{ f \in B^X \mid N_1 \sim N_2 \text{ implies } f(N_1) = f(N_2) \}.$$

To see this, suppose first $f \in \mathcal{C}$ and $N_1 \sim N_2$, i.e. $N_1 = (x, M_1)$ and $N_2 = (x, M_2)$. Then $f(N_1) = f(x)(M_1) = \lambda_x e(M_1) = \lambda_x$ and $f(N_2) = f(x)(M_2) = \lambda_x e(M_2) = \lambda_x$. Secondly, suppose that f is such that $N_1 \sim N_2$ always implies $f(N_1) = f(N_2)$. This means f is constant on the class $[N_1]$, i.e. $N \in [N_1]$ implies $f(N) = \lambda$, say. This means, if $N_1 = (x, M_1)$, that $f(x) = \lambda e$. Since this is true for all $N \in \eta$, $f(x) = \lambda_x e$, all $x \in X$, so $f \in \mathcal{C}$.

Suppose, now, that B is a Banach algebra of the type under consideration, i.e. B is without radical; if the scalars are real, then $b^2 + e$ has an inverse, any $b \in B$, and

if the scalars are complex B contains the conjugate elements. Let A be a Banach algebra without radical, let A contain a subalgebra B' equivalent to B, A and B' having the same identity e'. Let the space of maximal ideals be $\mathfrak{N}$ in A, $\mathfrak{M}$ in B, $\mathfrak{M}'$ in B'. By lemma 5.2 each M' of B' is contained in at least one N of A. Conversely, by lemma 5.3 each $N \in \mathfrak{N}$ intersects B' in a maximal ideal $M' \in \mathfrak{M}'$. Define

$$\mathfrak{N}(M) = \{N \in \mathfrak{N} \mid N \cap B' = M'\}.$$

Then $\mathfrak{N} = \bigcup_{M \in \mathfrak{M}} \mathfrak{N}(M)$ and $\mathfrak{N}(M_1) \cap \mathfrak{N}(M_2) = \emptyset$. If the $\mathfrak{N}(M)$ are all homeomorphic we can define, for arbitrary but fixed M_o, η_M to be the homeomorphism which takes $\mathfrak{N}(M)$ onto $\mathfrak{N}(M_o)$. Further, we can define, for $N_1 \in \mathfrak{N}(M_1)$ and $N_2 \in \mathfrak{N}(M_2)$, $N_1 \sim N_2$ if and only if $\eta_{M_1}(N_1) = \eta_{M_2}(N_2)$. Then $\sim$ is clearly an equivalence relation. Finally, define $[N_o] = \{N \mid N \sim N_o\}$. Thus, if $N_o \in \mathfrak{N}(M_o)$, then $[N_o] = \{N \mid N = \eta_M^{-1}(N_o),\ M \in \mathfrak{M}\}$.

Theorem 5.1. Let A and B be Banach algebras without radicals satisfying the following conditions:

1). A contains a subalgebra B' equivalent to B; A and B' having the same identity.

2). For arbitrary but fixed $M_o \in \mathfrak{M}$ the sets

$$\mathfrak{N}(M) = \{N \in \mathfrak{N} \mid N \cap B' = M'\}$$

are homeomorphic to $\mathfrak{N}(M_o)$ for all $M \in \mathfrak{M}$, the homeomorphic mappings being designated by η_M: $\mathfrak{N}(M) \longrightarrow \mathfrak{N}(M_o)$.

3). For each $a \in A$ and each class $[N_o] = \{N \mid N \sim N_o\}$ (where $N_1 \sim N_2$ if and only if $N_i \in \mathfrak{N}(M_i)$, $i = 1,2$, implies $\eta_{M_1}(N_1) = \eta_{M_2}(N_2)$) there is a $b' \in B'$ determined by $b'(N) = a(N)$ for each $N \in [N_o]$, and $\|b'\| \leq \|a\|$.

Then there is a completely regular space X such that A is isomorphic to a subalgebra of B^X (here B^X denotes the algebra of bounded continuous functions from X to B) which contains the constant functions and which separates points of X.

Proof. Denote by $T_{[N_0]}(a)$ the b' which is determined by each class $[N_0]$ and each $a \in A$. Then $N_1 \not\sim N_2$ implies $T_{[N_1]} \neq T_{[N_2]}$. For, $N_1 \not\sim N_2$ means, if $N_1 \in \mathfrak{N}(M_1)$ and $N_2 \in \mathfrak{N}(M_2)$, that $\eta_{M_1}(N_1) \neq \eta_{M_2}(N_2)$; therefore, since each η_M is a homeomorphism, $N_1 = \eta_{M_1}^{-1}(\eta_{M_1}(N_1)) \neq \eta_{M_1}^{-1}(\eta_{M_2}(N_2)) = N_2'$, say. Hence, there exists an $a \in A$, $a \in N_1$, $a \notin N_2'$. Let $T_{[N_1]}(a) = b_1'$, $T_{[N_2]}(a) = b_2'$. Then $b_1'(N_1) = a(N_1) = 0$ and $b_2'(N_2') = a(N_2') \neq 0$ (note that $N_2' \sim N_2$). From this it follows that $b_1' \neq b_2'$, for $N_1 \in \mathfrak{N}(M_1)$ and $N_2' \in \mathfrak{N}(M_1)$, and if $b_1' = b_2' = b'$, say, then $b'(N_1) = b'(N_2') = b'(M_1')$, by lemma 5.1. Thus, the assertion is proved, and there is a 1-1 correspondence between the classes $[N]$ and the transformations $T_{[N]}$, since it is trivial that $N_1 \sim N_2$ implies $T_{[N_1]} = T_{[N_2]}$.

Next, for each class $[N_0]$ the transformation $T_{[N_0]}$ is linear, multiplicative and bounded. To show this, let $T_{[N_0]}(a_1) = b_1'$, $T_{[N_0]}(a_2) = b_2'$, so that $a_1(N) = b_1'(N)$ and $a_2(N) = b_2'(N)$ for each $N \in [N_0]$. Then $T_{[N_0]}(a_1 + a_2) = b_3'$ where, for each $N \in [N_0]$, $b_3'(N) = (a_1 + a_2)(N) = a_1(N) + a_2(N) = b_1'(N) + b_2'(N)$; thus $b_3' = b_1' + b_2'$. Similarly, $T_{[N_0]}(a_1 a_2) = b_1' b_2' = T_{[N_0]}(a_1) T_{[N_0]}(a_2)$. That $T_{[N_0]}$ is bounded follows from condition 3).

Let $\mathfrak{T}$ be the set of all linear multiplicative transformations from A into B'. Topologize $\mathfrak{T}$ by defining as neighborhoods of $T_0 \in \mathfrak{T}$ the sets

$$V(T_o)=\{T\in\mathcal{T} \mid \|T(a_i)-T_o(a_i)\|<\varepsilon,\ i=1,\dots,n\},$$

where $\varepsilon>0$ and $a_1,\dots,a_n\in A$ are arbitrary. It is routine to show that with this topology $\mathcal{T}$ becomes a uniform topological space and thus a completely regular space.

Now let X be the set of transformations $T_{[N]}$. Then X, as a subset of $\mathcal{T}$, is a completely regular space. For each $a\in A$ it is now possible to define a function f_a from X to B, in the following way: let $f_a(T_{[N]})=b$, where $T_{[N]}(a)=b'$ [equivalently, if ψ provides the equivalence between B' and B, $\psi(b')=b$, then $f_a(T_{[N]})=\psi(T_{[N]}(a))=\psi(b')=b$]. It follows from the way the topology is defined in $\mathcal{T}$ ($T_{[N]}(a)$ is a continuous function of $T_{[N]}$ for each a) and from the fact that B' and B are equivalent that f_a is continuous. Also, since from condition 3) $\|T_{[N]}(a)\|\leq\|a\|$, $\|f_a(T_{[N]})\|=\|T_{[N]}(a)\|\leq\|a\|$ we see that f_a is bounded.

Next, the correspondence $a\longrightarrow f_a$ is an isomorphism. The proof is simple. Let $a_1\longrightarrow f_{a_1}$, $a_2\longrightarrow f_{a_2}$, let $a_1+a_2\longrightarrow g$. Then $g(T_{[N]})=\psi(T_{[N]}(a_1+a_2))=\psi(T_{[N]}(a_1)+T_{[N]}(a_2))=$ $=\psi(T_{[N]}(a_1))+\psi(T_{[N]}(a_2))=f_{a_1}(T_{[N]})+f_{a_2}(T_{[N]})$; i.e. $a_1+a_2\longrightarrow f_{a_1}+f_{a_2}$. Similarly $a_1a_2\longrightarrow f_{a_1}f_{a_2}$. And if $a_1\neq a_2$, then there is $N\in\mathcal{N}$ such that $a_1(N)\neq a_2(N)$. This means $T_{[N]}(a_1)\neq T_{[N]}(a_2)$ so that $f_{a_1}(T_{[N]})\neq f_{a_2}(T_{[N]})$, i.e. $f_{a_1}\neq f_{a_2}$.

It is a consequence of the way the transformation $T_{[N]}$ is defined that, for each class [N] and each $b'\in B'$, $T_{[N]}(b')=b'$. From this we have that $f_b'(T_{[N]})=b$, all $T_{[N]}\in X$, so the constant functions are contained in the subalgebra

of B^X. Finally, $T_{[N_1]} \neq T_{[N_2]}$ implies there is $a \in A$ such that $T_{[N_1]}(a) \neq T_{[N_2]}(a)$. Hence $f_a(T_{[N_1]}) \neq f_a(T_{[N_2]})$, so the subalgebra separates points of X. Thus the theorem is proved.

Theorem 5.2. Let A and B be Banach algebras without radicals satisfying conditions 1), 2), 3) of theorem 5.1 and

4) For each $a \in A$ there is a class $[N_a]$ such that if b' is determined by a and $[N_a]$ then $\|b'\| = \|a\|$.

Then there is a completely regular space X such that A is equivalent to a closed subalgebra of B^X which contains the constant functions and which separates the points of X.

Proof. Drawing on theorem 5.1, we need prove only that the correspondence $a \longrightarrow f_a$ is an isometry and that the subalgebra is closed.

By 3) $\|f_a(T_{[N]})\| \leq \|a\|$, so $\sup_{T_{[N]} \in X} \|f_a(T_{[N]})\| \leq \|a\|$,

and by 4) $\|f_a\| = \sup_{T_{[N]} \in X} \|f_a(T_{[N]})\| = \|a\|$,

so the correspondence $a \longrightarrow f_a$ is isometric.

Since A is complete the equivalent subalgebra of B^X is closed, and theorem 5.2 is proved.

The natural question whether or not the hypotheses of theorems 5.1 and 5.2 are necessary for their conclusions is not an easy one to answer. An attempt will now be made to provide an answer, although the results obtained are incomplete.

To begin with, it is clear that 1) is necessary for theorem 5.2 but not for theorem 5.1. For if A is equivalent to a subalgebra of B^X which contains the constant functions,

these form a subalgebra of A equivalent to B. On the other hand, if A is only isomorphic to such a subalgebra then the constant functions are in general isomorphic and not equivalent to B.

In order to discuss conditions 2), 3) and 4) it is necessary to obtain some information about the space $\mathcal{M}$ of maximal ideals in the algebra B^X of all continuous bounded mappings from a completely regular space X to B. In this case each set $(x,M)=\{f \in B^X \mid f(x) \in M\}$ is a maximal ideal of B^X but not all maximal ideals are of this type. The following lemma confirms what one would expect.

Lemma 5.4. If X is completely regular then $X \times \mathcal{M}$ can be (homeomorphically) embedded in the space $\mathcal{M}$ of maximal ideals of B^X.

Proof. As remarked above, each point of $X \times \mathcal{M}$ determines a point of $\mathcal{M}$. Use this to map $X \times \mathcal{M}$ into $\mathcal{M}$, $\psi(x,M)=N \in \mathcal{M}$. Since (lemma 3.4) the ideals (x_1,M_1) and (x_2,M_2) are equal if and only if $x_1=x_2$ and $M_1=M_2$, ψ is a 1-1 mapping. To see that ψ is continuous let $\psi(x_0,M_0)=N_0$. Let

$$V(N_0)=\{N \in \psi(X \times \mathcal{M}) \mid |f_i(N)-f_i(N_0)|<\varepsilon, i=1,\ldots,n\},$$

$$V(x_0)=\{x \in X \mid \|f_i(x)-f_i(x_0)\|<\varepsilon/2, i=1,\ldots,n\},$$

$$V(M_0)=\{M \in \mathcal{M} \mid |f_i(x_0)(M)-f_i(x_0)(M_0)|<\varepsilon/2, i=1,\ldots,n\}.$$

Let $V(x_0,M_0)=V(x_0) \times V(M_0)$. If $(x,M) \in V(x_0,M_0)$ and if $\psi(x,M)=N$, then for $i=1,\ldots,n$

$$\begin{aligned}|f_i(N)-f_i(N_\circ)| &= |f_i(x,M)-f_i(x_\circ,M_\circ)| \\ &\leq |f_i(x,M)-f_i(x_\circ,M)| + |f_i(x_\circ,M)-f_i(x_\circ,M_\circ)| \\ &\leq |f_i(x)-f_i(x_\circ)| + |f_i(x_\circ)(M)-f_i(x_\circ)(M_\circ)| \\ &< \varepsilon/2+\varepsilon/2 = \varepsilon ;\end{aligned}$$

i.e. $N \in V(N_\circ)$ and ψ is continuous.

I next show that ψ^{-1} is a continuous mapping from $\psi(X\times\mathcal{M})$ to $X\times\mathcal{M}$. Let $N_\circ \in \psi(X\times\mathcal{M})$; let $\psi^{-1}(N_\circ)=(x_\circ,M_\circ)$. Let a $V(x_\circ,M_\circ)=V(x_\circ)\times V(M_\circ)$ be given, where $V(M_\circ)=$ $=\{M\in\mathcal{M} \mid |b_i(M)-b_i(M_\circ)|<\varepsilon, i=1,\ldots,n\}$. Since X is completely regular there is a function $\mu : X \longrightarrow [0,\varepsilon]$, $\mu(x_\circ)=0$, $\mu(x)=\varepsilon$, $x \notin V(x_\circ)$. Now for $i=1,\ldots,n$ define $f_i(x)=b_i'(x)$ and define $f_{n+1}(x)=\mu(x)e$. Let

$$V(N_\circ)=\{N\in\psi(X\times\mathcal{M}) \mid |f_i(N)-f_i(N_\circ)|<\varepsilon, i=1,\ldots,n+1\}.$$

If $N\in V(N_\circ)$, let $\psi^{-1}(N)=(x,M)$. I assert $x\in V(x_\circ)$ and $M\in V(M_\circ)$, i.e. $(x,M)\in V(x_\circ,M_\circ)$. For, if $i=1,\ldots,n$,

$$\begin{aligned}|b_i(M)-b_i(M_\circ)| &= |f_i(x)(M)-f_i(x)(M_\circ)| \\ &= |f_i(x,M)-f_i(x_\circ,M_\circ)| \\ &= |f_i(N)-f_i(N_\circ)| < \varepsilon,\end{aligned}$$

so $M\in V(M_\circ)$. Also, suppose $x\notin V(x_\circ)$; then

$$\begin{aligned}|f_{n+1}(N)-f_{n+1}(N_\circ)| &= |f_{n+1}(x,M)-f_{n+1}(x_\circ,M_\circ)| \\ &= |f_{n+1}(x)(M)-f_{n+1}(x_\circ)(M_\circ)| \\ &= |\mu(x)-\mu(x_\circ)| \\ &= \mu(x) = \varepsilon,\end{aligned}$$

so $N\notin V(N_\circ)$. It follows that ψ^{-1} is continuous and $\psi(X\times\mathcal{M})$ is a subset of η which is homeomorphic to $X\times\mathcal{M}$.

Thus η is a compact extension of $X\times\mathcal{M}$, although our

interest remains centered on $X\times\mathfrak{M}$. In fact, if we define, for X completely regular,

$$\mathfrak{N}(M_\circ)=\{N\in X\times\mathfrak{M} \mid N\cap B'=M_\circ'\},$$

and define the class $[N_\circ]$ in terms of elements of $X\times\mathfrak{M}$, then it is clear that for B^X itself conditions 2) and 3) hold. For, $\mathfrak{N}(M)=X\times M$, so all sets $\mathfrak{N}(M)$ are homeomorphic to X. And the class $[N]=x\times\mathfrak{M}$; so each $f\in B^X$ and each class $[N]$ determines a $b\in B$, viz. $b=f(x)$, such that $b'(N)=f(N)$, each $N\in[N]$, and $\|b'\|=\|b\|\leq\|f\|$.

However, even for all of B^X condition 4) of theorem 5.2 need not hold. In general, of course, 4) holds only if X is compact.

Moreover, the conclusion of theorems 5.1 and 5.2 involve not B^X itself but only a subalgebra A' which contains the constant functions and which separates points of X. If for this A' we define

$$(x,M)'=\{f\in A' \mid f(x)\in M\}=(x,M)\cap A',$$

then (x,M)' is a maximal ideal of A', and the first part of the proof of lemma 5.4 remains valid. But the proof that ψ^{-1} is continuous requires that A' contain a subalgebra equivalent to C(X). For such an A' conditions 2) and 3) hold.

Remark. An equivalent formulation of conditions 3) and 4) can be obtained as follows. Define for each class $[N_\circ]$

$$I\,[N_\circ]=\bigcap_{N\in[N_\circ]}N.$$

Then $I\,[N_\circ]$ is a closed ideal and so $A/I\,[N_\circ]$ is a Banach algebra.

I assert $\frac{\mathbf{A}}{I\,[N_o]}$ contains a subalgebra isomorphic to B'. To show this consider $I\,[N_o] \cap B'$. If $N_o \in \mathfrak{N}(M_o)$ then

$$[N_o] = \{N \mid N = \eta_M^{-1}(N_o),\ M \in \mathfrak{M}\}.$$

Thus

$$I\,[N_o] = \bigcap_{N \sim N_o} N = \bigcap_{M \in \mathfrak{M}} \eta_M^{-1}(N_o),$$

and, since $\bigcap_{M \in \mathfrak{M}} M = 0$, it follows that $I\,[N_o] \cap B' = 0 \in B'$.

Consequently each $b' \in B'$ determines a different residue class $F_{b'} \in \frac{\mathbf{A}}{I\,[N_o]}$, and from the way the operations are defined in $\frac{\mathbf{A}}{I\,[N_o]}$ we can conclude that the mapping $b' \longrightarrow F_{b'}$ is an isomorphism.

The alternative formulation of 3) and 4) is then

3') For each class $[N_o]$ the algebra $\frac{\mathbf{A}}{I\,[N_o]}$ is equivalent to B'.

4') For each $a \in \mathbf{A}$ there is a class $[N_a]$ such that if $a \in F_a$ then $\|a\| = \|F_a\|$.

<u>Proof that 3) and 4) are equivalent to 3') and 4').</u> First assume 3') and 4').

Let $[N_o]$ be arbitrary, let $a_o \in \mathbf{A}$ be arbitrary. Then there exists $b' \in B'$ such that $a_o \equiv b' \pmod{I[N_o]}$. This means $a_o - b' \in I\,[N_o] = \bigcap_{N \sim N_o} N$, or $(a_o - b')(N) = 0$ for each $N \in [N_o]$, so $a_o(N) = b'(N)$, each $N \in [N_o]$. Moreover, since $\frac{\mathbf{A}}{I\,[N_o]}$ is equivalent to B',

$$\|b'\| = \|F_{b'}\| = \inf_{a \in F_{b'}} \|a\| \leq \|a_\circ\| ,$$

so condition 3) is satisfied. To show 4) holds use 4'). Let $F_a \longleftrightarrow b' \in B'$, so $\|F_a\| = \|b'\|$. Then $\|a\| = \|b'\|$.

Now assume 3) and 4). Let $a \in A$ and $[N_\circ]$ be arbitrary, let $I[N_\circ] = \bigcap_{N \sim N_\circ} N$. If $b' \in B'$ is determined by a and $[N_\circ]$, as in 3), then $a(N) = b'(N)$, each $N \in [N_\circ]$, so $(a-b')(N) = 0$, $N \in [N_\circ]$, or $a-b' \in I[N_\circ]$. In other words, each $a \in A$ lies in the same residue class as some $b' \in B'$. As shown above (p.28), each $b' \in B'$ determines a residue class of $\frac{A}{I[N_\circ]}$, so $\frac{A}{I[N_\circ]}$ is isomorphic to B'. Next, let $a \in F_a \in \frac{A}{I[N_\circ]}$, $F_a \longrightarrow b' \in B'$. By 3), $\|a\| \geq \|b'\|$, so $\|F_a\| = \inf_{a \in F_a} \|a\| \geq \|b'\|$. But $b' \in F_a$, so $\|F_a\| \leq \|b'\|$. Thus $\|F_a\| = \|b'\|$, and $\frac{A}{I[N_\circ]}$ is equivalent to B'.

Finally, it is obvious that 4') follows from 4).

§6. Characterization of B^X.

The preceding § provides a basis for a characterization of the algebra B^X.

Theorem 6.1. Let A and B be Banach algebras without radical. In order that there exist a compact Hausdorff space X such that A is equivalent to B^X it is necessary and sufficient that conditions 1), 2), 3), 4) of theorem 5.2 be satisfied, along with the condition

5). i) For each $a \in A$, $N_1 \in \mathcal{N}$, $\delta > 0$, let $b' \in B'$ be determined by a and $[N_1]$. Then there exist $q \in N_1$, $\delta' > 0$ such that $|q(N)| < \delta'$ implies $\|b'_N\| < \delta$, where $b'_N \in B'$ is determined by a-b' and $[N]$.

ii) Let $\mathcal{C} \subset A$ be the set

$$\mathcal{C} = \{c \in A \mid N_1 \sim N_2 \Longrightarrow c(N_1) = c(N_2)\}.$$

Then for any N_1, N_2, $N_1 \not\sim N_2$ implies there exists $c \in \mathcal{C}$ such that $c(N_1) \not\perp c(N_2)$.

Proof. Properties 4, 5, 6, 8, 12, and 13 of § 5 show that conditions 1), 2), 3), 4) and 5ii) are necessary. To show that 5i) is necessary let $f \in B^X$, $N_1 = (x_1, M_1)$, $\delta > 0$ be given. Let $f(x_1) = b$, so $T_{[N_1]}(f) = b'$. Let $f' = f - b'$ and $q(x) = \|f'(x)\| e$. Also, take $\delta' = \delta$. Then $q \in N_1$, for

$$q(x_1) = \|f'(x_1)\| e = \|f(x_1) - b'(x_1)\| e = \|b - b\| e = 0 \in M_1.$$

Also, if $N = (x, M)$, then

$$q(N) = q(x)(M) = \|f'(x)\| \cdot e(M) = \|f'(x)\|.$$

Consequently $|q(N)| < \delta$ implies $\|f'(x)\| < \delta$, or $\|T_{[N]}(f')\| < \delta$. Thus, if $T_{[N]}(f') = b'_N$, then $\|b'_N\| < \delta$. This is 5i).

The sufficiency. Suppose A satisfies conditions 1)-5). Then by theorem 5.2 A is equivalent to a closed subalgebra of B^X where X is completely regular. It remains to show that X is compact and A is equivalent to all of B^X.

That X is compact will be shown by proving, first, that $\mathcal{N}(M_\circ)$ is a closed, hence compact, subset of $\mathcal{N}$, and, secondly, that X is homeomorphic to $\mathcal{N}(M_\circ)$. To show $\mathcal{N}(M_\circ)$ is closed in $\mathcal{N}$, recall that $\mathcal{N}(M_\circ) = \{N \in \mathcal{N} \mid N \wedge B' = M'_\circ\}$. Hence, if

$N_1 \notin \mathcal{F}(M_0)$ then there exists $b_0' \in N_1 \cap B'$, $b_0' \in M_0'$. Then $b_0'(N_1)=0$, $b_0'(M_0')\neq 0$. Let ε be such that $0<\varepsilon<|b_0'(M_0')|$, let

$$V(N_1)=\{N\in\mathcal{F}_1 \mid |b_0'(N)-b_0'(N_1)|<\varepsilon\}$$
$$=\{N\in\mathcal{F}_1 \mid |b_0'(N)|<\varepsilon\}.$$

If $N\in\mathcal{F}_1(M_0)$ then $M_0'\subset N$, so for all $b'\in B'$, by lemma 5.1, $b'(N)=b'(M_0')$. Thus, $N\in\mathcal{F}_1(M_0)$ implies $|b_0'(N)|>\varepsilon$, so $\mathcal{F}_1(M_0)\cap V(N_1)=\emptyset$. It has been shown that an arbitrary point not in $\mathcal{F}_1(M_0)$ has a neighborhood not intersecting $\mathcal{F}_1(M_0)$. It follows that $\mathcal{F}_1(M_0)$ is closed in the compact space $\mathcal{F}_1$, so $\mathcal{F}_1(M_0)$ is compact.

To prove that X and $\mathcal{F}_1(M_0)$ are homeomorphic use will be made of a lemma which will now be interposed.

Lemma 6.1. Let λ be a continuous real-valued function defined on X, $\lambda(T_{N_1})=\lambda\in R_1$. For each $N\in\mathcal{F}_1$ let $\rho(N)=\lambda(T_{N_1})$. Then ρ is continuous on $\mathcal{F}_1$.

Proof of lemma 6.1. Let $N_1\in\mathcal{F}_1$, $\varepsilon>0$ be given. Since λ is continuous on X there is $V(T_{N_1})$ such that $T_{N_1}\in V(T_{N_1})$ implies $|\lambda(T_{N_1})-\lambda(T_{N_1})|<\varepsilon$. Suppose

$$V(T_{N_1})=\{T_{N_1} \mid \|T_{N_1}(a_i)-T_{N_1}(a_i)\|<\varepsilon,\ i=1,\dots,k\},$$

where $a_1,\dots,a_k\in A$. Let $T_{N_1}(a_i)=b_i'$, let $a_i'=a_i-b_i'$. By hypothesis 5i) there exist $q_1,\dots,q_k\in N_1$ and $\varepsilon_1',\dots,\varepsilon_k'>0$ such that $|q_i(N)|<\varepsilon_i'$ implies $\|T_N(a_i')\|<\varepsilon$. Let $\varepsilon'=\min(\varepsilon_1',\dots,\varepsilon_k')$. Let

$$V(N_1)=\{N\in\mathcal{F}_1 \mid |q_i(N)-q_i(N_1)|<\varepsilon',\ i=1,\dots,k\}$$
$$=\{N\in\mathcal{F}_1 \mid |q_i(N)|<\varepsilon',\ i=1,\dots,k\},$$

since $q_i\in N_1$ implies $q_i(N_1)=0$. Then, if $N\in V(N_1)$,

$|q_i(N)| < \delta'$, $i=1,\ldots,k$, so $\|T_{[N]}(a_i')\| < \delta$; i.e.

$$\delta > \|T_{[N]}(a_i - b_i')\| = \|T_{[N]}(a_i) - T_{[N]}(b_i')\|$$
$$= \|T_{[N]}(a_i) - b_i'\|$$
$$= \|T_N(a_i) - T_{[N_1]}(a_i)\|,$$

which is to say $T_{[N]} \in V(T_{[N_1]})$. Thus, $N \in V(N_1)$ implies $T_{[N]} \in V(T_{[N_1]})$, which in turn implies

$$|\rho(N) - \rho(N_1)| = |\lambda(T_{[N]}) - \lambda(T_{[N_1]})| < \varepsilon.$$

Hence ρ is continuous at N_1. Since N_1 is arbitrary, ρ is continuous on η. This concludes the proof of lemma 6.1.

Now, for each $N \in \eta(M_0)$ define $\varphi(N) = T_{[N]}$. I will show φ is a 1-1 continuous transformation from $\eta(M_0)$ onto X. Since $\eta(M_0)$ is compact Hausdorff it will follow that X is compact and homeomorphic to $\eta(M_0)$. Suppose $N_1, N_2 \in \eta(M_0)$, $N_1 \neq N_2$. Then $\eta_{|M_0}(N_1) = N_1 \not\sim N_2 = \eta_{|M_0}(N_2)$, so $N_1 \not\sim N_2$. From this we conclude that $T_{[N_1]} \neq T_{[N_2]}$, as in the beginning of the proof of theorem 5.1. Also, if $T_{[N]} \in X$ then there is $N_0 \in [N] \cap \eta(M_0)$, so $\varphi(N_0) = T_{[N]}$. Hence φ is 1-1 onto. To show that φ is continuous let $N_0 \in \eta(M_0)$, let $\varphi(N_0) = T_{[N_0]}$. Let

$$V(T_{[N_0]}) = \{T_{[N]} \mid \|T_{[N]}(a_i) - T_{[N_0]}(a_i)\| < \varepsilon,\ i=1,\ldots,n\},$$

where $\varepsilon > 0$ and $a_1, \ldots, a_n \in A$ are arbitrary. Let $T_{[N_0]}(a_i) = b_i'$ and let $a_i' = a_i - b_i'$, $i=1,\ldots,n$. Then

$$T_{[N]}(a_i') = T_{[N]}(a_i - b_i') = T_{[N]}(a_i) - T_{[N]}(b_i')$$
$$= T_{[N]}(a_i) - b_i' = T_{[N]}(a_i) - T_{[N_0]}(a_i).$$

Also,

$$f_{a_i}(T_{[N]}) = \psi(T_{[N]}(a_i')) = \psi(T_{[N]}(a_i) - T_{[N_0]}(a_i))$$
$$= \psi(T_{[N]}(a_i)) - \psi(T_{[N_0]}(a_i)) = f_{a_i}(T_{[N]}) - f_{a_i}(T_{[N_0]}).$$

Let $\lambda_i(T_{[N]}) = \|f_{a_i}(T_{[N]})\|$ (note that $\lambda_i(T_{[N_\circ]}) = 0$); then the λ_i are continuous real-valued functions on X, since the f_{a_i} are continuous from X to B and the norm is a continuous function on a Banach space. For each $N \in \mathcal{N}$ define $\rho_i(N) = \lambda_i(T_{[N]})$, $i = 1, \ldots, n$. Then $\rho_i(N) = 0$ for $N \sim N_\circ$, so, in particular, $\rho_i(N_\circ) = 0$. By lemma 6.1 the ρ_i are continuous real-valued functions on $\mathcal{N}$. By a theorem of Gelfand and Šilov [3] A is isomorphic to a subalgebra of $\mathcal{C}(\mathcal{N})$ which is dense in $\mathcal{C}(\mathcal{N})$. Hence, there exist $r_i \in A$, $i = 1, \ldots, n$, such that $|r_i(N) - \rho_i(N)| < \varepsilon/3$, any $N \in \mathcal{N}$.

Let

$$V(N_\circ) = \left\{ N \in \mathcal{N}(M_\circ) \;\middle|\; |r_i(N) - r_i(N_\circ)| < \varepsilon/3,\ i = 1, \ldots, n \right\}.$$

Let $N \in V(N_\circ)$, $\varphi(N) = T_{[N]}$; then, for $i = 1, \ldots, n$,

$$\begin{aligned} \|T_{[N]}(a_i) - T_{[N_\circ]}(a_i)\| &= \|f_{a_i}(T_{[N]}) - f_{a_i}(T_{[N_\circ]})\| = \|f_{a_i}(T_{[N]})\| \\ &= |\lambda_i(T_{[N]})| = |\rho_i(N)| = |\rho_i(N) - \rho_i(N_\circ)| \\ &\leq |\rho_i(N) - r_i(N)| + |r_i(N) - r_i(N_\circ)| + |r_i(N_\circ) - \rho_i(N_\circ)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

The conclusion is that $N \in V(N_\circ)$ implies $\varphi(N) \in V(T_{[N_\circ]})$, so φ is continuous and X is homeomorphic to $\mathcal{N}(M_\circ)$, hence X is compact.

Consider now the set $\mathcal{C} \subset A$:

$$\mathcal{C} = \left\{ c \in A \;\middle|\; N_1 \sim N_2 \Longrightarrow c(N_1) = c(N_2) \right\}.$$

First of all $\mathcal{C}$ is non-empty, since $e'(N) = 1$, all $N \in \mathcal{N}$, so $e' \in \mathcal{C}$, and, in fact, $\lambda e' \in \mathcal{C}$ for any λ. Secondly, $\mathcal{C}$ is a subalgebra. To see this let $c_1, c_2 \in \mathcal{C}$, $N_1, N_2 \in \mathcal{N}$, $N_1 \sim N_2$; then

$$(c_1+c_2)(N_1)=c_1(N_1)+c_2(N_1)=c_1(N_2)-c_2(N_2)=(c_1-c_2)(N_2),$$
$$(c_1c_2)(N_1)=c_1(N_1)c_2(N_1)=c_1(N_2)c_2(N_2)=(c_1c_2)(N_2).$$

Also, $\mathfrak{C}$ is a closed subalgebra. For, let $\{c_n\}\in\mathfrak{C}$, $c_n\longrightarrow a\in A$. Let $N_1\sim N_2$, let $\varepsilon>0$ be arbitrary. Then

$$\begin{aligned}|a(N_1)-a(N_2)| &\leq |a(N_1)-c_n(N_1)| + |c_n(N_2)-a(N_2)| \\ &= |(a-c_n)(N_1)| + |(c_n-a)(N_2)| \\ &\leq 2\,|a-c_n| \\ &< 2\varepsilon,\end{aligned}$$

for sufficiently large n. Thus $a(N_1)=a(N_2)$ and $a\in\mathfrak{C}$. Next, for each $N\in\mathfrak{N}$ and each $c\in\mathfrak{C}$, $T_N(c)=c(N)e'$. This follows from the definition of T_N and the definition of $\mathfrak{C}$. By hypothesis 4)

$$|c| = \sup_{T_N\in X} |f_c(T_N)|.$$

The foregoing properties of $\mathfrak{C}$ say exactly that $\mathfrak{C}$ is equivalent to a closed subalgebra of C(X), the algebra of all continuous numerical-valued functions on X. But hypothesis 5ii) and the Stone-Weierstrass theorem [9] then imply that $\mathfrak{C}$ is equivalent to the whole of C(X).

With this result it can be shown that A is equivalent to all of B^X. Since A is complete it is sufficient to show that A is equivalent to a dense subset of B^X; i.e. if $f\in B^X$ and $\varepsilon>0$ are given it must be shown there exists $a\in A$ such that $\|f_a-f\|<\varepsilon$. Since X is compact f(X) is a compact subset of B. Therefore (Leray and Schauder [5]) there is a continuous mapping $\varphi: f(X)\longrightarrow B$ such that $\|f(x)-\varphi(f(x))\|<\varepsilon$

for all $x \in X$ and $\varphi(f(X))$ is contained in a finite-dimensional linear subspace of B. Let $g(x) = \varphi(f(x))$; then $g \in B^X$ and $\|g-f\| < \varepsilon$. Let $b_1, \ldots, b_n$ be a basis for the subspace of B which contains g(X). For each $x \in X$, $g(x) = \alpha_1(x)b_1 + \ldots + \alpha_n(x)b_n$, where each $\alpha_i(x)$ is a uniquely determined continuous numerical-valued function on X. But, as just shown, $\mathcal{C}$ is equivalent to C(X); let $c_i \in \mathcal{C}$ correspond to α_i, $i=1,\ldots,n$, under this equivalence. Let $a_i = c_i b_i'$. Then $f_{a_i}(x) = \alpha_i(x)b_i'(x) = \alpha_i(x)b_i$, and if $a = a_1 + \ldots + a_n$, $f_a(x) = \alpha_1(x)b_1 + \ldots + \alpha_n(x)b_n = g(x)$, so the required $a \in \mathbf{A}$ has been found, and A is equivalent to B^X. This completes the proof of theorem 6.1.

Remark. It is readily seen that in the demonstration just concluded the following generalization of the Stone-Weierstrass theorem has been proved.

Theorem 6.2. Let X be a compact Hausdorff space and let B be a Banach algebra of the type under consideration. Let A be a closed subalgebra of the algebra B^X which contains all the constant functions, and let $\mathcal{C}$ be the subalgebra of B^X consisting of those functions f such that $f(x) = \lambda_x e$, all $x \in X$. The necessary and sufficient condition that $\mathbf{A} = B^X$ is that for every pair of distinct points $x_1, x_2 \in X$ there exist $f \in \mathcal{C} \cap \mathbf{A}$ such that $f(x_1) \neq f(x_2)$.

An immediate corallary of theorem 6.2 is the following.

Theorem 6.3. Let X and Y be compact Hausdorff spaces, let A be a closed subalgebra of $C(X \times Y)$, the algebra of continuous numerical-valued functions on $X \times Y$. Then either of

the following is a necessary and sufficient condition that $A=C(X\times Y)$:

1) $x_1\neq x_2$ implies there exists $f\in A$, f constant on Y, such that $f(x_1,Y)\neq f(x_2,Y)$; and A contains the functions constant on X.

2) $y_1\neq y_2$ implies there exists $g\in A$, g constant on X, such that $g(X,y_1)\neq g(X,y_2)$; and A contains the functions constant on Y.

§ 7. The case of a radical

The results of §§ 5 and 6 do not hold in case B has non-zero radical P, for it is then no longer true that the collection of numbers $b(M)$, $M\in\mathfrak{M}$, uniquely determines b. Indeed, not too much can be said, although the following facts are of some interest.

Lemma 7.1. If the Banach algebra B has radical P then B^X has a radical Q and $Q=P^X$, the set of all continuous functions from X to P, X compact Hausdorff.

Proof. It will be recalled that the radical is equal to the intersection of all maximal ideals. Let $f\in\bigcap_{N\in\mathfrak{N}} N=\bigcap_{(x,M)\in X\times\mathfrak{M}}(x,M)$. Then $f(X)\subset\bigcap_{M\in\mathfrak{M}} M$, i.e. $f(X)\subset P$, so, letting $Q=\bigcap_{N\in\mathfrak{N}} N$, $Q\subseteq P^X$. On the other hand, if $f\in P^X$, then $f(X)\subset P=\bigcap_{M\in\mathfrak{M}} M$, so $f\in Q$. Hence, $Q=P^X$.

Theorem 7.1. If B has radical P, then $B^X/_Q$ is isomorphic to a subalgebra of $(B/_P)^X$, where Q is the radical of B^X.

Proof. Suppose $f,g \in B^X$ and $f \equiv g \pmod{Q}$. Then $f-g \in Q$, so $f(x)-g(x) \in P$ for every $x \in X$; that is to say for every $x \in X$ $f(x)$ and $g(x)$ lie in the same residue class of B (mod P). Consequently, if $F \in B^X/_Q$ is the residue class containing f and g, then F provides a mapping from X into $B/_P$. That is, if $f(x)=b \in \bar{b} \in B/_P$, and if $f \in F \in B^X/Q$, then $F(x)=\bar{b}$.

This function F is continuous on X. For, let $x_0 \in X$, let $f \in F$, and choose $V(x_0)$ such that $x \in V(x_0)$ implies $\|f(x)-f(x_0)\| < \varepsilon$, $\varepsilon > 0$ arbitrary. Then for $x \in V(x_0)$

$$\|F(x)-F(x_0)\| \leq \|f(x)-f(x_0)\| < \varepsilon,$$

so F is continuous on X, i.e. $F \in (B/_P)^X$. Since the zero of $B/_P$ is P and the zero of $B^X/_Q$ is Q and since $Q=P^X$, it follows that $B^X/_Q$ is isomorphic to a subalgebra of $(B/_P)^X$.

In order to obtain a more satisfactory result it seems necessary to restrict B further.

Theorem 7.2. Suppose B has radical P. Suppose further that B contains a subalgebra S equivalent to $B/_P$, where if $\bar{b} \in B/_P$ corresponds to $s \in S$, then $s \in \bar{b}$. Then $B^X/_Q$ is equivalent to $(B/_P)^X$, i.e. $B^X/_{P^X}=(B/_P)^X$, symbolically.

Proof. Because of theorem 7.1 it is necessary to prove only that $B^X/_Q$ is isomorphic to all of $(B/_P)^X$ and that norms are preserved under the isomorphism.

Let $G \in (B/_P)^X$, $G(x)=\bar{b} \in B/_P$. Let $\bar{b}$ correspond to $s \in S \subset B$ and define $f(x)=s$. Since G is continuous so is f, i.e. $f \in B^X$.

The residue class F containing f then corresponds to the function $G \in ({}^{B}/_{P})^{X}$. Thus ${}^{B^X}/_{Q}$ is isomorphic to all of $({}^{B}/_{P})^{X}$. Moreover, norms are preserved. First, let $F \in {}^{B^X}/_{Q}$. Then

$$\|F\| = \inf_{f \in F} \|f\| = \inf_{f \in F} \left[\sup_{x \in X} \|f(x)\|\right] \geq \inf_{f \in F} \left[\sup_{x \in X} \|f(x) (\text{mod } P)\|\right]$$

$$= \sup_{x \in X} \|F(x)\| .$$

To show $\|F\| \leq \sup_{x \in X} \|F(x)\|$, let $\bar{f} \in F$ be the function defined in the first part of the proof; i.e. if $F(x) = \bar{b}$ and $\bar{b} \longleftrightarrow s$, then $\bar{f}(x) = s$. Thus

$$\sup_{x \in X} \|F(x)\| = \sup_{x \in X} \|\bar{f}(x)\| = \|\bar{f}\| \geq \inf_{f \in F} \|f\| = \|F\| .$$

Therefore $\|F\| = \sup_{x \in X} \|F(x)\|$ and ${}^{B^X}/_{Q} = ({}^{B}/_{P})^{X}$.

Remark. The hypotheses of theorem 7.2 are sufficient to guarantee that B is the direct sum of ${}^{B}/_{P}$ and P, since if $b \in B$, b in the residue class $\bar{b}$ (mod P), there is unique $s \in S \subset B$, $s \longleftrightarrow \bar{b}$, such that $s \in \bar{b}$. Consequently, $b - s = p \in P$, or $b = s + p$. It is clear that this representation is unique. But the proof of theorem 7.2 shows that B^X contains a subalgebra equivalent to ${}^{B^X}/_{Q}$, viz. the collection of $\bar{f}$. Thus we can write

$$\begin{aligned} B^X &= {}^{B^X}/_{Q} + Q, \text{ direct} \\ &= {}^{B^X}/_{P^X} + P^X, \text{ direct} \\ &= ({}^{B}/_{P})^{X} + P^X, \text{ direct,} \end{aligned}$$

or, since as just stated $B = {}^{B}/_{P} + P$, direct, there is a "binomial theorem"

$$({}^{B}/_{P} + P)^{X} = ({}^{B}/_{P})^{X} + P^X, \text{ direct.}$$

An example of a B satisfying the above conditions is a primary algebra, i.e. one with one maximal ideal M. In this case each $b \in B$ can be represented uniquely in the form $b = \lambda e + m$, $m \in M$, $\lambda = b(M)$, so

$$B = {}^{B}/_{M} + M, \text{ direct,}$$

where ${}^{B}/_{M} = R_1 e$ or $Z_1 e$, depending on whether B is a real algebra with scalars R_1 or complex with scalars Z_1.

§8. Relation between T-set in a Banach space and maximal ideal in a Banach algebra

Throughout this section it will be assumed that any Banach space or Banach algebra under consideration is real.

In the study of real-valued continuous functions defined on a compact Hausdorff space X and of continuous functions from such an X to a Banach space, Myers [6] and Jerison [4] have made profitable use of the T-sets and the corresponding functionals F_T originally introduced by Myers [6]. These are defined as follows.

Definition 8.1. A T-set of a Banach space is a subset T maximal with respect to the property that for every finite set of elements $b_1, \ldots, b_n \in T$, $\left\| \sum_{i=1}^{n} b_i \right\| = \sum_{i=1}^{n} \|b_i\|$.

Definition 8.2. Corresponding to each T-set the functional F_T is defined as

$$F_T(b) = \inf_{t \in T} \left[\|b + t\| - \|t\| \right].$$

These sets and functionals for a Banach space bear a

certain similarity to the maximal ideals and the corresponding functionals which have been exploited in the preceding paragraphs. For one thing both types of sets satisfy a maximality condition; and for another, if $B=C(X)$ for compact X then each point x_0 of X determines a pair of T-sets $\pm T_0$ and $F_{T_0}(f)=f(x_0)$, whereas if C(X) is considered as a Banach algebra then each point x_0 of X determines a maximal ideal M_0 and $M_0(f)=f(x_0)$. However, it would seem that it is not possible to push the analogy too far, since the maximal ideal depends upon the algebraic structure of the Banach algebra and the T-set depends upon the definition of the norm in the Banach space.

The following tables will illustrate the points of similarity and of difference between T-sets and maximal ideals. As usual X is compact Hausdorff. For the purpose only of these tables the functional corresponding to a maximal ideal M will be denoted as $F_M(b)$ instead of b(M), as earlier.

Table 1: T-sets and maximal ideals in C(X) treated as a Banach space and as a Banach algebra.

Banach Space [6]	Banach Algebra
$T=\left\{f \mid \lvert f(x_\circ)\rvert = \lVert f\rVert = \sup_{x \in X} \lvert f(x)\rvert\right\}$	$M=\left\{f \mid f(x_\circ)=0\right\}$
i.e. f takes on its absolute maximum at $x_\circ$	i.e. f takes on its absolute minimum at $x_\circ$
$f \in T \Rightarrow F_T(f)=\lVert f\rVert$	$f \in M \Rightarrow F_M(f)=0$
$\cup T=C(X)$	$\cup M=$all singular elements $\not\in C(X)$

Table 2: T-sets in the Banach space B^X where B is a Banach space and maximal ideals in the Banach algebra B^X where B is a Banach algebra.

Banach Space [4]	Banach Algebra
T is a T-set in B	M is a maximal ideal in B
T-set $(x_\circ, T) =$ $= \{f \mid f(x_\circ) \in T, \|f(x_\circ)\| = \|f\|\}$, i.e. $(x_\circ, T)$ does not consist of all f such that $f(x_\circ) \in T$, but only those such that also $\|f(x_\circ)\| = \|f\|$.	maximal ideal $(x_\circ, M) =$ $= \{f \mid f(x_\circ) \in M\}$, i.e. $(x_\circ, M)$ consists of <u>all</u> f such that $f(x_\circ) \in M$.
$F_{(x_\circ, T)}(f) = \|f(x_\circ)\|$, $f(x_\circ) \in T$	$F_{(x_\circ, M)}(f) = 0$, $f(x_\circ) \in M$
$F_{(x_\circ, T)}(f) < \|f(x_\circ)\|$, $f(x_\circ) \notin T$	$\|F_{(x_\circ, M)}(f)\| > 0$, $f(x_\circ) \notin M$
$\bigcup_{T \text{ of } B} (x_\circ, T) = \{f \mid \|f(x_\circ)\| = \|f\|\}$	$\bigcup_{M \text{ of } B} (x_\circ, M) = \{f \mid f(x_\circ) \text{ is singular}\}$
$\bigcup (x, T) = B^X$	$\bigcup (x, M) =$ set of all singular f $\neq B^X$

Two observations can be made from a study of tables 1 and 2, viz.

i) T-sets are related to "maximum value," whereas maximal ideals are related to "minimum value,"

ii) every element of a Banach space lies in some T-set, whereas for Banach algebras the corresponding statement is that every singular element lies in some maximal ideal.

A development based on these observations follows.

In the Banach algebra B define the functional μ as $\mu(b)=\inf_{M\in\mathfrak{M}} |b(M)|$. Since b(M) is continuous on the compact space $\mathfrak{M}$ there exists at least one point of $\mathfrak{M}$ at which this minimum is assumed. Note that b is a regular element if and only if $\mu(b)>0$. Adjoin to each $M\in\mathfrak{M}$ all regular $b\in B$ which have $|b(M)| = \mu(b)$. Let the resulting set be denoted by M*. Then, since each $b\in B$ lies in at least one M*, $\bigcup_{M\in\mathfrak{M}} M^* = B$. Since $e(m)\equiv 1$, it follows that $\mu(e)=1$ and $e\in\bigcap_{M\in\mathfrak{M}} M^*$. Further, $\mu(\lambda e)=|\lambda|$, and $\lambda e\in\bigcap_{M\in\mathfrak{M}} M^*$ for all real λ.

The analogy between the T-sets and the M*-sets is stronger than between the T-sets and the maximal ideals. This is indicated by the following (cf. theorem 4.1 of [4]).

Theorem 8.1. Let X be compact Hausdorff and let B be a (real) Banach algebra so that a maximal ideal in B^X is a set

$$(x_\circ, M_\circ)=\{f \mid f(x_\circ)\in M_\circ\}.$$

Then

$$(x_\circ, M_\circ)^* = \{f \mid f(x_\circ) \in M_\circ^*, \ \mu(f(x_\circ)) = \mu(f)\}.$$

(The same symbol μ is used for both B and B^X).

Proof. First,

$$(x_\circ, M_\circ)^* \subseteq \{f \mid f(x_\circ) \in M_\circ^*, \ \mu(f(x_\circ)) = \mu(f)\}.$$

If $f \in (x_\circ, M_\circ)$ the assertion is obvious, since then $f(x_\circ) \in M_\circ$ and $\mu(f(x_\circ)) = \mu(f) = 0$. Next, suppose f is regular and $f \in (x_\circ, M_\circ)^*$. This means $|f(x_\circ, M_\circ)| = \mu(f) > 0$. Let $f(x_\circ) = b_\circ$, a regular element of B, since f is regular in B^X. I will show $b_\circ \in M_\circ^*$ and $\mu(b_\circ) = \mu(f)$. Now $f(x_\circ, M_\circ) = b_\circ(M_\circ)$, so $\mu(b_\circ) \leq |f(x_\circ, M_\circ)| = \mu(f)$. Suppose $\mu(b_\circ) < \mu(f)$. Then there is $M_1 \in \mathfrak{M}$ such that $|b_\circ(M_1)| < |b_\circ(M_\circ)|$, which is to say $|f(x_\circ, M_1)| < |f(x_\circ, M_\circ)|$, contradicting $|f(x_\circ, M_\circ)| = \mu(f)$, and so $\mu(b_\circ) = \mu(f) = |b_\circ(M_\circ)|$, and $b_\circ \in M_\circ^*$.

Second,

$$(x_\circ, M_\circ)^* \supseteq \{f \mid f(x_\circ) \in M_\circ^*, \ \mu(f(x_\circ)) = \mu(f)\}.$$

Suppose $f(x_\circ) \in M_\circ^*$ and $\mu(f(x_\circ)) = \mu(f)$. If $\mu(f(x_\circ)) = \mu(f) = 0$, then $f(x_\circ) \in M_\circ$ and $f \in (x_\circ, M_\circ) \subset (x_\circ, M_\circ)^*$. Next, suppose $\mu(f(x_\circ)) = \mu(f) > 0$. This implies $f(x_\circ)$ is regular in B and f is regular in B^X. Further, since $f(x_\circ) = b_\circ \in M_\circ^*$, it states that $|b_\circ(M_\circ)| = \mu(b_\circ) = \mu(f)$. But $|b_\circ(M_\circ)| = |f(x_\circ, M_\circ)|$, so $|f(x_\circ, M_\circ)| = \mu(f)$, and $f \in (x_\circ, M_\circ)^*$. The theorem is proved.

There follows a list of properties of the function μ, some of which have already been stated.

1. $\mu(b)=0$ if and only if b is a singular element.
2. $\mu(b)>0$ if and only if b is a regular element.
3. $\mu(e)=1$.
4. $\mu(-b)=\mu(b)$.
5. $\mu(\lambda b)=|\lambda|\mu(b)$.
6. $\mu(b_1b_2)\geq\mu(b_1)\mu(b_2)$.

   Proof. We have

$$\mu(b_1b_2)=\inf|b_1b_2(M)|=\inf|b_1(M)\,b_2(M)|$$
$$=\inf\left[|b_1(M)|\cdot|b_2(M)|\right]\geq\inf|b_1(M)|\inf|b_2(M)|$$
$$=\mu(b_1)\mu(b_2).$$

7. For b_1,b_2 regular, $\mu(b_1b_2)=\mu(b_1)\mu(b_2)$ if and only if b_1 and b_2 lie in the same M_o^*.

   Proof. a) Suppose $b_1,b_2\in M_o^*$; then $|b_1(M_o)|=\mu(b_1)$, $|b_2(M_o)|=\mu(b_2)$, and $\mu(b_1b_2)=\inf\limits_{M\in\mathfrak{M}}|b_1(M)b_2(M)|\leq$

$$\leq|b_1(M_o)|\cdot|b_2(M_o)|=\mu(b_1)\mu(b_2).$$

   It follows from 6 that $\mu(b_1b_2)=\mu(b_1)\mu(b_2)$.

   b) Conversely, suppose $\mu(b_1b_2)=\mu(b_1)\mu(b_2)$. Let $b_1b_2\in M_o^*$, so $\mu(b_1b_2)=|b_1(M_o)b_2(M_o)|=$

$$=|b_1(M_o)|\cdot|b_2(M_o)|.$$

   But $|b_1(M_o)|\cdot|b_2(M_o)|>$

$$>\mu(b_1)\mu(b_2)$$

   unless $b_1,b_2\in M_o^*$.

8. For b_1,b_2 regular, $b_1,b_2\in M_o^*$, we have $b_1b_2\in M_o^*$. This is a corollary of the proof of 7.
9. For b regular, $\mu(b^{-1})=\dfrac{1}{\sup|b(M)|}$. In general, $\mu(b)\mu(b^{-1})\leq 1$, and equality holds if and only if $|b(M)|$ is a constant $\neq 0$.

Proof. First,

$$\mu(b^{-1}) = \inf |b^{-1}(M)| = \inf \left| \frac{1}{b(M)} \right| = \frac{1}{\sup\limits_{M \in \mathfrak{M}} |b(M)|} .$$

Also,

$$1 = u(e) = u(bb^{-1}) \geq u(b)\, u(b^{-1}).$$

By 7, $u(b)\, u(b^{-1}) = 1$ if and only if b, b^{-1} lie in the same M_0^*. This means $|b(M_0)| = \inf\limits_{M \in \mathfrak{M}} |b(M)|$ and also by the first part of 9,

$$u(b^{-1}) = |b^{-1}(M_0)| = \frac{1}{|b(M_0)|} = \frac{1}{\sup\limits_{M \in \mathfrak{M}} |b(M)|} ,$$

so $|b(M_0)| = \sup |b(M)| = \inf |b(M)|$. Hence $|b(M)|$ is the same for all $M \in \mathfrak{M}$.

10. If B is without radical and $\mathfrak{M}$ is connected, then $\mu(b)\, \mu(b^{-1}) = 1$ if and only if $b = \lambda e$. In this case $\bigcap\limits_{M \in \mathfrak{M}} M^* = R_1 e$.

Proof. If $b = \lambda e$, then $\mu(b) = |\lambda|$, $\mu(b^{-1}) = |\lambda|^{-1}$, so $\mu(b)\, \mu(b^{-1}) = 1$. If $u(b)\, u(b^{-1}) = 1$, then by 9 $|b(M)|$ is constant ($= \lambda$) on $\mathfrak{M}$. Since $\mathfrak{M}$ is connected, either $b(M) \equiv \lambda$ or $b(M) \equiv -\lambda$. But if B has no radical, then $b(M)$ uniquely determines b. Moreover, $(\lambda e)(M) \equiv \lambda$, so either $b = \lambda e$ or $b = -\lambda e$. If $b \neq 0$ is in every M^* then $b(M)$ is constant, so $b = \lambda e$. Also, if $b = \lambda e$, then b is in every M^*.

11. If $B = C(\mathfrak{M})$, then for all regular b, $\mu(b^{-1}) = \|b\|^{-1}$. This follows from 9.

12. If $B=C(\mathfrak{M})$, then μ is a continuous function on the set $\mathcal{G}$ of regular elements of B.

Proof. By 11, $\mu(b)=\|b^{-1}\|^{-1}$. Set $\tau(b)=b^{-1}$, $\nu(b)=\|b\|$, $\rho(\lambda)=\lambda^{-1}$, $\lambda>0$. Then τ and ν are continuous on $\mathcal{G}$ and ρ is continuous on the set of positive reals. Hence $\mu=\rho\nu\tau$ is continuous.

13. If B is without radical and $\mathfrak{M}$ is connected, then for b regular $\|b\|=\mu(b)$ if and only if $b=\lambda e$, and $\bigcap_{M\in\mathfrak{M}} M^*=R_1 e$.

The proof is the same as in 10.

14. Suppose B is without radical and $\mathfrak{M}$ is connected. Let $\mathcal{C}$ be the subalgebra of B^X consisting of all $f\in B^X$ such that $f(x)=\lambda_x e$, every $x\in X$. Then, by 10,

$$\mathcal{C}=\bigcap_{(x,M)\in X\times\mathfrak{M}}(x,M^*)=\bigcap_{(x,M)\in X\times\mathfrak{M}}\{f\mid f(x)\in M^*\}.$$

As a consequence of 14 and Gelfand's work the following theorem holds.

Theorem 8.2. Let X and Y be compact Hausdorff spaces, let B be a Banach algebra without radical and with connected space $\mathfrak{M}$ of maximal ideals. Let

$$\mathcal{C}_X=\bigcap_{(x,M)\in X\times\mathfrak{M}}(x,M^*)=\bigcap_{(x,M)\in X\times\mathfrak{M}}\{f\in B^X\mid f(x)\in M^*\}$$

$$\mathcal{C}_Y=\bigcap_{(y,M)\in Y\times\mathfrak{M}}(y,M^*)=\bigcap_{(y,M)\in Y\times\mathfrak{M}}\{f\in B^Y\mid f(y)\in M^*\}.$$

If the algebras $\mathcal{C}_X$ and $\mathcal{C}_Y$ are isomorphic then X and Y are homeomorphic.

§9. B a regular (Šilov) algebra

(In this section B is either real or complex, subject to the restrictions of §§ 2, 3 and 4.)

Šilov [8] has made an extensive study of a type of Banach algebra which he calls regular; a regular algebra has also been called a Šilov algebra.

Definition 9.1. The Banach algebra B is "regular" if and only if for every closed set $F \subset \mathfrak{M}$ and $M_0 \notin F$ there exists $b \in B$ such that $b(M)=0$, for $M \in F$ and $b(M_0) \neq 0$.

In this section it will be shown that for X compact B^X is regular if and only if B is regular. To this end it will be convenient to recall an alternative method, proposed by Gelfand and Šilov [3], of topologizing the space of maximal ideals in B. This is the Stone intersection topology which defines an ideal M_0 to be a limit point of a subset $S \subset \mathfrak{M}$ if $M_0 \supset \bigcap_{M \in S} M$. I will denote by $\mathfrak{M}_s$ the space so topologized, retaining $\mathfrak{M}$ as the symbol for the space with the original topology. In general $\mathfrak{M}_s$ is a compact T_1 space but not necessarily Hausdorff. Thus it is not generally true that $\mathfrak{M}$ and $\mathfrak{M}_s$ are homeomorphic. The following known result will be of use in the sequel.

Lemma 9.1. The spaces $\mathfrak{M}$ and $\mathfrak{M}_s$ are homeomorphic if and only if B is regular.

Proof. Assume first that B is regular. The mapping that assigns to each $M \in \mathfrak{M}_s$ the same $M \in \mathfrak{M}$ is clearly 1-1 onto.

Moreover it is continuous. For, let G be open in $\mathfrak{M}$, $F = \mathfrak{M} - G$, $M_0 \in G$; then by hypothesis there is $b \in B$ such that $b(M_0) \neq 0$, $b(M) = 0$, $M \in F$. As Gelfand and Šilov [3] point out, the set $\{M \mid b(M) \neq 0\}$ is a vicinity $V(M_0)$ of M_0 in $\mathfrak{M}_S$; and $V(M_0) \cap F = \emptyset$. Thus $G_S = \bigcup_{M \in G} V(M)$ is open in $\mathfrak{M}_S$. The mapping is continuous; both spaces being compact and $\mathfrak{M}$ being Hausdorff, the homeomorphism follows.

Next assume $\mathfrak{M}$ and $\mathfrak{M}_S$ homeomorphic. Let $F \subset \mathfrak{M}$ be closed, let $M_0 \notin F$. By hypothesis F is closed in $\mathfrak{M}_S$, so there is $V(M_0) = \{M \mid b(M) \neq 0\}$, some $b \in B$, such that $V(M_0) \cap F = \emptyset$. It follows that $b(M_0) \neq 0$, $b(M) = 0$, $M \in F$.

The principal result of this § can now be obtained.

Theorem 9.1. If X is compact, then B^X is regular if and only if B is regular.

Proof. First let B be regular so that, by lemma 9.1, $\mathfrak{M}$ and $\mathfrak{M}_S$ are homeomorphic. First I will show that $\mathfrak{M}_S$ and $X \times \mathfrak{M}_S$ are homeomorphic. If $N_0 = \{f \mid f(x_0) \in M_0\}$, define $\varphi(N_0) = (x_0, M_0) \in X \times \mathfrak{M}_S$. Then φ is a 1-1 mapping onto $X \times \mathfrak{M}_S$. Also, φ is continuous. Let $V(x_0, M_0)$ be given. Then there exist $V(x_0)$, $V_1(M_0)$ such that $V(x_0) \times V_1(M_0) \subset V(x_0, M_0)$. Also, there exists $V(M_0) = \{M \mid b(M) \neq 0\}$, some $b \in B$, such that $V(M_0) \subset V_1(M_0)$; then $V(x_0) \times V(M_0) \subset V(x_0, M_0)$. There exists $\alpha : X \longrightarrow [0,1]$, $\alpha(x_0) = 1$, $\alpha(x) = 0$, $x \notin V(x_0)$. Let $f(x) = \alpha(x)b$. Define $V(N_0) = \{N \mid f(N) \neq 0\}$. If $N \in V(N_0)$, $\varphi(N) = (x, M)$, then $x \in V(x_0)$, $M \in V(M_0)$, so $(x, M) \in V(x_0, M_0)$. For, if $x \notin V(x_0)$, then $f(N) = f(x)(M) = \alpha(x)b(M) = 0(M) = 0$; and if $M \notin V(M_0)$, then

$b(M)=0$, and $f(N)=\alpha(x)b(M)=\alpha(x).0=0$.

Thus φ is a 1-1 continuous mapping of $\mathfrak{N}_S$ on $X\times\mathfrak{M}_S$. Since $\mathfrak{N}_S$ is compact and $X\times\mathfrak{M}_S$ $(=X\times\mathfrak{M})$ is Hausdorff, it follows that φ^{-1} is continuous, i.e. $\mathfrak{N}_S$ and $X\times\mathfrak{M}_S$ are homeomorphic.

The "if" part of the theorem now comes readily. Using $=$ to mean "is homeomorphic to" we have:

$$\mathfrak{N}_S = X\times\mathfrak{M}_S = X\times\mathfrak{M} = \mathfrak{N}.$$

Drawing again on lemma 9.1 we conclude that B^X is regular.

Next, assume that B^X is a regular algebra. Let $F\subset\mathfrak{M}$ be closed, let $M_0\notin F$, let $x_0\in X$ be arbitrary. Then the set $x_0\times F$ is closed in $X\times\mathfrak{M}$ and $(x_0,M_0)\notin x_0\times F$. By hypothesis there is $f\in B^X$ such that $f(x_0,M_0)\neq 0$, $f(x_0,M)=0$, $M\in F$, i.e. $(x_0,M)\in x_0\times F$. Let $f(x_0)=b_0$; then $b_0(M_0)=f(x_0,M_0)\neq 0$, and, for $M\in F$, $b_0(M)=f(x_0,M)=0$. Thus B is regular, and the theorem is proved.

Remark. Šilov was chiefly concerned with regular algebras without radical. It should be noted that the above result holds whether or not B has radical.

The above theorem suggests a consideration of the relation between the properties of B and of B^X. A few comments on this subject follow; some of the facts have been mentioned and proven earlier, most of them are fairly obvious.

1) B^X has radical if and only if B has radical (§7).
2) B^X is equivalent to $C(X\times\mathfrak{M})$ if and only if B is equivalent to $C(\mathfrak{M})$. If B is without radical

(so that B^X is also semi-simple) and B is not equivalent to $C(\mathfrak{M})$ then, by a theorem of Gelfand and Šilov [3], B is isomorphic to a dense subalgebra of $C(\mathfrak{M})$; in this case B^X is isomorphic to a dense subalgebra of $C(X \times \mathfrak{M})$.

3) If B is complex and X is connected, then, by a theorem of Gelfand, B^X is a direct sum of subalgebras (each with identity) if and only if B is also the direct sum of subalgebras.

On the other hand the structure of B^X is generally more complicated than that of B. Thus if B is primary (i.e. has only one maximal ideal) and X has more than one point, then B^X is not primary. Also, B^X can be a field only in the trivial case that B is a field and X contains exactly one point. And clearly the structure of B^X depends upon X. For example, even if B cannot be represented as a direct sum, B^X can be provided X has, say, two components. And if B is finite-dimensional, B^X will be finite-dimensional only if X has a finite number of points.

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