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REGULAR CONVERGENCE IN A PARACOMPACT SPACE

by

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INTRODUCTION

The convergence concept pervades most of mathematics. For example, much of analysis is concerned with sequences of functions converging to a limit function. Under appropriately strong forms of convergence, the limit function inherits certain properties the members of the sequence may possess. Similar questions arise in topology concerning sequences of subsets of a topological space.

Let X be a compact Hausdorff topological space. For a sequence (A_i) where each A_i is a subset of X, let lim sup A_i denote those $x \in X$ with the property that each neighborhood of x contains points from infinitely many of the A_i. Let lim inf A_i denote those $x \in X$ with the property that each neighborhood of x contains points from all but a finite number of the A_i. If lim sup A_i = lim inf A_i, the common set A is the limit of the sequence (A_i). Since lim sup A_i is always a closed subset of X, the limit set A is always closed. Also, a sequence of subsets of X converges to the limit set A if and only if the sequence consisting of the closures of the members converges to A. For this reason we consider only sequences whose members are closed subsets. This definition of convergence was introduced by Zarankiewicz [16]. It follows easily that, if (A_i) converges to A where each A_i is a closed connected subset of X, then A is a closed connected subset of X. However, a stronger type of convergence is required

to insure the limit set A will be, for example, a simple closed curve if each A_i is a simple closed curve.

G. T. Whyburn [13] introduced the concept of a sequence (A_i) of closed subsets of a compact metric space X converging regularly to a subset A of X. He required that the sequence (A_i) converge in the above sense and that for each positive number e there exist a positive number d and an integer j such that two points of A_i at distance less than d are contained in a connected subset of A_i with diameter less than e for all i larger than j. Notice that in the special case where each A_i equals A, the above definition requires that A be uniformly locally connected. He showed, among other things, that the limit set A is always locally connected and that the simple closed curve property is inherited by the limit set A.

G. T. Whyburn [13], also, introduced the concept of a sequence (A_i) of closed subsets of a compact metric space X converging n-regularly to a subset A of X. He used the Vietoris homology theory in formulating his definition and showed that O-regular convergence corresponds to the above regular convergence. Over a period of years several authors investigated this situation. P. A. White [12] contains a summary of the results obtained. For example, E. G. Begle [1] obtained the following extension of the above result concerning simple closed curves: If a sequence (A_i) of closed orientable classical 2-manifolds

converges 1-regularly to a non-degenerate set A, then A is a closed orientable classical 2-manifold and A is homeomorphic with all but a finite number of the A_i. The lack of a topological characterization of classical n-manifolds with n larger than 2 has prevented further progress in this direction. However, using the generalized manifold introduced by R. L. Wilder [14], E. G. Begle [1] obtained the following result: If a sequence (A_i) of compact orientable generalized n-manifolds converges (n-1)-regularly to a n-dimensional set A, then A is a compact orientable generalized n-manifold.

G. S. Young, Jr. [15] investigated n-regular convergence of a sequence (A_i) of closed subsets of a locally compact complete metric space X. He extended the above 2-dimensional result of E. G. Begle to non-closed classical 2-manifolds with boundary.

P. A. White [10] considered the problem of extending the regular convergence theory to sequences (A_i) of closed subsets of a possibly non-metric, compact Hausdorff topological space. However, his definition of regular convergence differs from the classical definition. He fails to relate his definition to the classical one because of an error in the proof of [10, Theorem. 10]. In proving [10, Theorem. 10] he uses [10, Corollary 7.2] to show that the classical definition implies the definition introduced earlier in his paper. However, the

hypothesis of this corollary requires that the sequence converge in this earlier sense.

E. E. Floyd [5] defined n-regular convergence for a sequence (A_i) of closed subsets of a possibly non-metric, compact Hausdorff topological space X. He used the Čech homology theory with coefficients in a field or compact group and established the following result: If a sequence (A_i) of closed subsets of a compact Hausdorff topological space X converges n-regularly to A, then the n-dimensional Čech homology group of A_i is isomorphic with the n-dimensional Čech homology group of A for all but a finite number of the A_i.

It is our intention to systematize the main results concerning n-regular convergence using a uniform method and placing them in the framework of a general, possibly non-metrizable, paracompact topological space. In a certain sense, which we will make precise later, this is a natural setting for the n-regular convergence theory.

Let X be a paracompact topological space. Let C(X) denote the family of non-empty closed subsets of X. For a collection $\underline{u}$ of open subsets of X, let $N(\underline{u})$ denote those elements of C(X) that have a non-empty intersection with each $U \in \underline{u}$ and are contained in the union of all $U \in \underline{u}$. We give C(X) the topology generated by sets of the form $N(\underline{u})$ where $\underline{u}$ is a locally finite collection of open subsets of X. Since X may not be a

metric space, it is natural to consider nets in C(X) in the sense of J. L. Kelley [6] rather than sequences in C(X). We say a net (A_i) in C(X) converges to A if it converges in the above topology. By a closed covering of X we mean a locally finite collection $\underline{e}$ of closed subsets of X such that each point $x \in X$ is in the interior of some $E \in \underline{e}$. For a closed subset F of X, we let $H^q(F)$ denote the q-dimensional reduced Čech cohomology group of F with coefficients in a principal ideal domain L. If $\underline{e}$ and $\underline{d}$ are closed coverings of X, we write $\underline{d} \overset{n}{>} \underline{e}$ if there exists a function π from $\underline{d}$ into $\underline{e}$ with $D \subset \pi D$ for each $D \in \underline{d}$ and such that the natural homomorphism of $H^q(\pi D)$ into $H^q(D)$ is trivial for all $q \leq n$. If $\underline{e}$ is a closed covering of X and F is a closed subset of X, we let $\underline{e} \cap F$ denote the closed covering of F consisting of elements of the form $E \cap F$ where $E \in \underline{e}$. A net (A_i) in C(X) converges cohomologically n-regularly to A if (A_i) converges to A in C(X) and for each closed covering $\underline{e}$ of X there exists a closed covering $\underline{d}$ of X and j from the index set I for the net such that $\underline{d} \cap A_i \overset{n}{>} \underline{e} \cap A_i$ for all i larger than j. We use the abbreviation n-rc converges for this type of convergence.

We indicate a few of the results proven herein. If (A_i) is a net in C(X) n-rc converging to A, there exists a $j \in I$ such that $H^n(A_i)$ and $H^n(A)$ are isomorphic for all i larger than j. If (A_i) is a net in C(X) (n-1)-rc converging to a non-degenerate set A and each A_i is a connected compact orientable n-dimensional generalized manifold over a field, then A is a connected compact orientable n-dimensional generalized manifold over the field.

If (A_i) is a net in $C(X)$ (n-1)-rc converging to a non-degenerate set A, each A_i is a connected n-dimensional generalized manifold over a field, and each $x \in A$ has an open neighborhood W such that $W \cap A_i$ is orientable for all $i \in I$, then A is a connected n-dimensional generalized manifold over the field.

CHAPTER I

1.0 We consider convergence of nets in this chapter. Throughout this paper we let X denote a fixed topological space that is at least paracompact. If S is a subset of X, we denote the closure of S by S^- and the interior by S^o. By a nbd. of $x \in S$ we mean a subset N of X with $x \in N^o$.

1.1 Definition. A set I is a directed set if there is a reflexive, transitive, binary relation $>$ defined on I with the additional property that for $i, j \in I$ there exists $k \in I$ with $k > i$ and $k > j$. A net is a function whose domain is a directed set. A net is in a set Y provided its range is a subset of Y. If I is a directed set, we denote the net consisting of the pairs (i, y_i) by (y_i). A net (y_i) in a topological space Y converges to $y \in Y$ if for each nbd. N of y there exists $j \in I$ such that $y_i \in N$ for all $i > j$.

1.2 Definition. Let C(X) denote the collection of non-empty closed subsets of X. For a collection $\underline{u}$ of open subsets of X, let $N(\underline{u})$ denote those elements of C(X) that have a non-empty intersection with each $U \in \underline{u}$ and are contained in the union of all $U \in \underline{u}$. We give C(X) the topology generated by sets of the form $N(\underline{u})$ where $\underline{u}$ is a locally finite collection of open subsets of X. A net (A_i) in C(X) converges to A means the net converges in this topology.

1.3 Definition. If (A_i) is a net in C(X) let lim inf A_i denote the set of all $x \in X$ with the property that for each nbd. N of x there exists $j \in I$ such that $A_i \cap N \neq \emptyset$ if $i > j$. Let lim sup A_i denote the set of all $x \in X$ with the property that for

each nbd. N of x and for each $j \in I$ there exists $i \in I$ such that $i > j$ and $A_i \cap N \neq \emptyset$. If lim inf A_i equals lim sup A_i, we write lim $A_i = A$ where A is the common set. Since lim sup A_i is always closed, the limit set A is closed.

1.4 Proposition. Let (A_i) be a net in C(X). If (A_i) converges to A in C(X), then lim $A_i = A$. If X is compact and lim $A_i = A$, then (A_i) converges to A in C(X).

Proof. Suppose (A_i) converges to A in C(X). Let $x \in A$ and let U be an open nbd. of x. Choose an open subset V of X such that $A \in N(\underline{u})$ where $\underline{u} = (U, V)$. Choose $j \in I$ so that $A_i \in N(\underline{u})$ for all $i > j$. Then $A_i \cap U \neq \emptyset$ for all $i > j$ so that $A \subset$ lim inf A_i. Let $x \in$ lim sup A_i and suppose $x \notin A$. Choose an open nbd U of x with $U^- \cap A = \emptyset$. Since $X - U^-$ is a nbd. of A in C(X), there exists $j \in I$ such that $A_i \subset (X - U^-)$ for all $i > j$. Hence, $A_i \cap U = \emptyset$ for all $i > j$ which contradicts the choice of x. Hence, $x \in A$ so that lim sup $A_i \subset A$. Hence, lim $A_i = A$.

Suppose lim $A_i = A$ and that X is compact. Let $N(\underline{u})$ be a nbd. of A in C(X) and let W denote the union of all $U \in \underline{u}$. For each $x \in X - W$, choose an open nbd. U_x of x and $j_x \in I$ such that $A \cap U_x^- = \emptyset$ and $A_i \cap U_x = \emptyset$ for all $i > j_x$. Since X-W is compact, a finite number of the U_x's cover X-W. Choose $j' \in I$ such that it is larger than each of the j_x's in the corresponding finite set. Since X is compact, the number of elements in $\underline{u}$ will be finite. Hence, we may choose $j'' \in I$ such that $A_i \cap U \neq \emptyset$ for all $U \in \underline{u}$ and all $i > j''$.

Choose $j \in I$ such that $j > j'$ and $j > j''$. Then $A_i \in N(\underline{u})$ for all $i > j$. Hence, (A_i) converges to A in C(X).

Corollary. If X is compact, lim satisfies the axioms of a convergence class for C(X) as given in J. L. Kelley [6].

Example. The following example shows that $\lim A_i = A$ does not imply convergence in C(X) in general. Let X be the plane and let A_i be the graph of the equation $x^2 + i\,y^2 = i$ where i is a positive integer. Then $\lim A_i$ consists of the lines $y=1$, $y=-1$ while (A_i) does not converge to these lines in C(X).

L. Vietoris [9] showed, if X is compact, then C(X) is a compact Hausdorff topological space. E. Michael [7] observed the following proposition is valid

1.5 Proposition. Let (A_i) be a net in C(X) converging to A. If there exists $j \in I$ such that A_i is connected for all $i > j$, then A is connected.

Proof. If A is not connected, $A = B \cup C$ where B and C are non-empty open subsets of A with $B \cap C = \emptyset$. Choose open subsets U and V of X with $U \cap A = B$, $V \cap A = C$, and $U \cap V = \emptyset$. If $\underline{u} = (U, V)$, there exists $j \in I$ such that $A_i \in N(\underline{u})$ for all $i > j$. Hence, all but a finite number of the A_i are not connected which contradicts our assumptions. Hence, A is connected

1.6 Example. Let X be the plane and let A_i be the set of points (x, y) with $y=\sin(1/x)$ and $(1/i) \leq x \leq (1/\pi)$ for i a positive integer. Then (A_i) converges to the set A of points (x, y) with $y=\sin(1/x)$ and $0 < x \leq (1/\pi)$ together with the points on the line segment between $(0, 1)$ and $(0, -1)$. Observe that each A_i is locally connected, but A is not locally connected.

Remark. The above example shows a stronger form of convergence is required to insure, for example, that A will be a simple closed curve if each A_i is a simple closed curve.

CHAPTER II

2.0 Let L be a fixed ring that we assume to be at least a principal ideal domain. We denote the q-dimensional reduced Čech cohomology group of X with coefficients in L by $H^q(X)$. We denote the q-dimensional Čech cohomology of X with compact supports and coefficients in L by $h^q(X)$. We recall, if U is an open subset of X with $\bar{U}$ compact, then $h^q(U) = H^q(X, X-U)$. If E and D are closed subsets of X with $D \subset E$, there is a natural homomorphism of $H^q(E)$ into $H^q(D)$. We denote the image of this homomorphism by $IH^q(E \mid D)$ and the kernel by $KH^q(E \mid D)$. If U and V are open subsets of X with $V \subset U$, there is a natural homomorphism of $h^q(V)$ into $h^q(U)$. We denote the image of this homomorphism by $Ih^q(V \mid U)$ and the kernel by $Kh^q(V \mid U)$.

2.1 By a closed covering of X, we mean a locally finite collection $\underline{e}$ of closed subsets of X such that each $x \in X$ is in the interior of some $E \in \underline{e}$. If $\underline{e}$ and $\underline{d}$ are collections of closed subsets of X, $\underline{d}$ refines $\underline{e}$ and we write $\underline{d} > \underline{e}$ if there exists a function π from $\underline{d}$ into $\underline{e}$ such that $D \subset \pi D$ for each $D \in \underline{d}$. If $\underline{d} > \underline{e}$ and for each $D \in \underline{d}$ every element of $\underline{d}$ that meets D is contained in πD, we write $\underline{d} \overset{*}{>} \underline{e}$. If $\underline{d} > \underline{e}$ and for each $D \in \underline{d}$, $IH^q(\pi D \mid D) = 0$ for all $q \leq n$, we write $\underline{d}^n > \underline{e}$. If $\underline{d} > \underline{e}$ and for all $D^{i_0}, \cdots, D^{i_s} \in \underline{d}$, $IH^q(\pi D^{i_0} \cap \cdots \cap \pi D^{i_s} \mid D^{i_0} \cap \cdots \cap D^{i_s}) = 0$ for all $q \leq n$, we write $\underline{d}^n >> \underline{e}$. It follows easily that, if $\underline{e}$, $\underline{d}$, and $\underline{c}$ are collections of closed subsets of X with $\underline{d} \overset{*}{>} \underline{e}$ and $\underline{c}^n > \underline{d}$, then $\underline{d}^n >> \underline{e}$.

Remark. If $\underline{e}$ is a collection of closed subsets of X, we let $H^q(\underline{e})$ denote the q-dimensional reduced cohomology group of the nerve of $\underline{e}$. If $\underline{e}$ is a closed covering of X, it follows easily from the

paracompactness of X that there exists an open covering $\underline{u}$ of X and a one to one function g of $\underline{e}$ into $\underline{u}$ such that g(E) contains E for each $E \in \underline{e}$ and $g(E^{i_0}) \cap \cdots \cap g(E^{i_q}) \neq \emptyset$ if and only if $E^{i_0} \cap \cdots \cap E^{i_q} \neq \emptyset$. Since X is paracompact, open coverings of this type are cofinal in the family of open coverings of X. Hence, we may use closed coverings of X to compute the Čech cohomology groups of X. For a closed covering $\underline{e}$ of X, there is a natural homomorphism of $H^q(\underline{e})$ into $H^q(X)$. We denote the image by $IH^q(\underline{e} \mid X)$ and the kernel by $KH^q(\underline{e} \mid X)$. In short, this notation always refers to the natural homomorphism from the group indicated on the left to the group indicated on the right.

E. E. Floyd [5] proved a homology version of the following theorem with X compact and the coefficient group L a field or compact group. Using sheaf theory, E. Dyer [4] proved a cohomology version with X compact and L a principal ideal domain. Recently, in a course at the University of Michigan, C. N. Lee used sheaf theory to prove the following version.

2.2 Theorem. Let $E_{-1}, E_0, \cdots, E_n$ be closed subsets of X with $E_j \subset E_{j-1}$ for $j = 0, 1, \cdots, n$. Let $\underline{e}_j$ be a closed covering of E_j for $j = -1, 0, \cdots, n$ such that $\underline{e}_n \overset{n}{>>} \cdots \overset{n}{>>} \underline{e}_0 \overset{n}{>>} \underline{e}_{-1}$. Then

1. $IH^q(E_{-1} \mid E_n) \subset IH^q(\underline{e}_n \mid E_n)$ for all $q \leq n$,

2. $KH^q(\underline{e}_0 \mid E_0) \subset KH^q(\underline{e}_0 \mid \underline{e}_n)$ for all $q \leq n+1$.

The results of this paper follow for the most part from applications of this theorem.

2.3 Definition. X is q-clc if for each $x \in X$ and each closed nbd. E of x there exists a closed nbd. D of x with $D \subset E$ such that $IH^q(E \mid D) = 0$. If X is locally compact, X is 0-clc if and only if it is locally connected. If the coefficient domain L is a field, X is q-clc if and only if X is q-lc in the sense of R. L. Wilder [14]. X is clc^n if X is q-clc for all $q \leq n$.

2.4 Definition. If $\underline{e}$ is a closed covering of a space X and A is a closed subset of X, let $\underline{e} \cap A$ denote the closed covering of A consisting of elements of the form $E \cap A$ where $E \in \underline{e}$.

2.5 Definition. A net (A_i) in C(X) converges cohomologically n-regularly to A if (A_i) converges to A in C(X) and for each closed covering $\underline{e}$ of X there exists a closed covering $\underline{d}$ of X and $j \in I$ such that $\underline{d} \cap A_i \overset{n}{>} \underline{e} \cap A_i$ for all $i > j$. We use the abbreviation n-rc convergence for this type of convergence.

2.6 Proposition. Let X be compact and let (A_i) be a net in C(X) converging to A. Suppose for each $x \in A$ and each nbd. E of x there exists a nbd. D of x with $D \subset E$ and $j \in I$ such that $IH^q(E \cap A_i \mid D \cap A_i) = 0$ for all $q \leq n$ and all $i > j$. Then (A_i) n-rc converges to A.

Proof. Let $\underline{e}$ be a closed covering of X. For $x \in E^o$, $E \in \underline{e}$, choose D_x and j_x with the above properties as stated in the hypothesis. Since X is compact a finite number of the D_x's cover X. Let $\underline{d}$ be such a cover and choose j such that $j > j_x$ for each j_x corresponding to an element of $\underline{d}$. Defining the function π from $\underline{d}$ into $\underline{e}$ in the obvious manner, we see that $\underline{d} \cap A_i \overset{n}{>} \underline{e} \cap A_i$ for all $i > j$.

Remark. If X is a compact metric space and L is a field, n-rc convergence agrees with n-regular convergence defined in

terms of the Vietoris homology theory. If X is a compact space and L is a field, n-rc convergence agrees with the n-regular convergence defined in terms of the Čech homology theory.

2.7 Proposition. Let (A_i) be a net in C(X) n-rc converging to A. If $\underline{e}$ is a closed covering of X, there exists a closed covering $\underline{d}$ of X and $j \in I$ such that $\underline{d} \wedge A_i \overset{n}{\gg} \underline{e} \wedge A_i$ for all $i > j$.

Proof. Let $\underline{e}$ be a closed covering of X. Since X is paracompact, we may choose a closed covering $\underline{c}$ such that $\underline{c} \overset{*}{>} \underline{e}$. Since (A_i) n-rc converges to A, we may choose a closed covering $\underline{d}$ and $j \in I$ such that $\underline{d} \wedge A_i \overset{n}{>} \underline{c} \wedge A_i$ for all $i > j$. Hence, from 2.1 we conclude that $\underline{d} \wedge A_i \overset{n}{\gg} \underline{e} \wedge A_i$ for all $i > j$.

Remark. For a regular topological space every open covering has an open star refinement if and only if the space is paracompact. We used this in the above proof in choosing $\underline{c}$ so that $\underline{c} \overset{*}{>} \underline{e}$. In this sense, a paracompact topological space provides a natural setting for applying 2.2 to the theory of regular convergence.

2.8 Proposition. Let (A_i) be a net in C(X) n-rc converging to A. If $\underline{e}$ is a closed covering of X, there exists a closed covering $\underline{b}$ of X with $\underline{b} > \underline{e}$ such that, for each closed covering $\underline{c}$ of X with $\underline{c} > \underline{b}$, there exists a closed covering $\underline{d}$ with $\underline{d} > \underline{c}$ and $j \in I$ such that for all $i > j$

1. $IH^q(\underline{c} \wedge A_i \mid A_i) = H^q(A_i)$ for all $q \leq n$,

2. $KH^q(\underline{c} \wedge A_i \mid A_i) \subset KH^q(\underline{c} \wedge A_i \mid \underline{d} \wedge A_i)$ for all $q \leq n+1$.

Proof. Let $\underline{e}$ be a closed covering of X. Choose closed coverings $\underline{e}_0, \underline{e}_1, \ldots, \underline{e}_{n-1}, \underline{b}$ of X and $j' \in I$ such that

$$\underline{b} \cap A_i \overset{n}{>>} \underline{e}_{n-1} \cap A_i \overset{n}{>>} \quad \cdot \overset{n}{>>} \underline{e}_0 \cap A_i \overset{n}{>>} \underline{e} \cap A_i$$

for all $i > j'$ by 2.7. Then with $A_i = E_j$ for each j in 2.2, we have $H^q(A_i) \subset IH^q(\underline{b} \cap A_i \mid A_i)$ for all $q \leq n$ and all $i > j'$. Hence, $IH^q(\underline{c} \cap A_i \mid A_i) = H^q(A_i)$ for all $q \leq n$ and all $i > j'$ since $\underline{c} > \underline{b}$.

Choose closed coverings $\underline{d}_1, \ldots, \underline{d}_{n-1}, \underline{d}$ of X and $j'' \in I$ such that

$$\underline{d} \cap A_i \overset{n}{>>} \underline{d}_{n-1} \cap A_i \overset{n}{>>} \ldots \overset{n}{>>} \underline{d}_1 \cap A_i \overset{n}{>>} \underline{c} \cap A_i$$

for all $i > j''$. Again by 2.2, we have $KH^q(\underline{c} \cap A_i \mid A_i) \subset KH^q(\underline{c} \cap A_i \mid \underline{d} \cap A_i)$ for all $q \leq n+1$ and all $i > j''$. Choose $j \in I$ such that $j > j'$ and $j > j''$ to obtain the proposition.

2.9 Proposition. Let (A_i) be a net in C(X) n-rc converging to A. Let $x \in A$ and let E and D be closed nbds. of x with $D \subset E^o$. If $\underline{e}$ is a closed covering of X, there exists a closed covering $\underline{d}$ of X and $j \in I$ such that $\underline{d} \cap (D \cap A_i) \overset{n}{>>} \underline{e} \cap (E \cap A_i)$ for all $i > j$.

Proof. Let $\underline{e}$ be a closed covering of X. Choose $\underline{c}$ a closed covering of X such that $\underline{c} >^* \underline{e}$. Choose a closed nbd F of x such that $D \subset F^o$ and $F \subset E^o$. Form a new covering $\underline{c}'$ by replacing each element C of $\underline{c}$ that meets E^o and X-E by the elements $C \cap E$ and $C \cap (X - F^o)$. Since (A_i) n-rc converges to A, we may choose a closed covering $\underline{d}$ with $\underline{d} > \underline{c}'$ and $j \in I$ such that $\underline{d} \cap A_i \overset{n}{>} \underline{c}' \cap A_i$ for all $i > j$. Hence, $\underline{d} \cap (D \cap A_i) \overset{n}{>} \underline{c}' \cap (E \cap A_i)$

for all $i > j$. But $\underline{c}' >^* \underline{e}$ so that we have that $\underline{d} \wedge (D \wedge A_i)^n \gg \underline{e} \wedge (E \wedge A_i)$ for all $i > j$.

2.10 Proposition. Let (A_i) be a net in $C(X)$ n-rc converging to A. Let $x \in A$ and let E and D be closed nbds. of x with $D \subset E^o$. If $\underline{e}$ is a closed covering of X, there exists a closed covering $\underline{d}$ of X with $\underline{d} > \underline{e}$ and $j \in I$ such that for all $i > j$

1. $IH^q(E \wedge A_i \mid D \wedge A_i) \subset IH^q(\underline{d} \wedge (D \wedge A_i) \mid D \wedge A_i)$ for all $q \leq n$,
2. $KH^q(\underline{e} \wedge (E \wedge A_i) \mid E \wedge A_i) \subset KH^q(\underline{e} \wedge (E \wedge A_i) \mid \underline{d} \wedge (D \wedge A_i))$ for all $q \leq n+1$.

Proof. Let $x \in A$ and let E, D, and $\underline{e}$ be as stated in the proposition. Choose closed sets $E_0, E_1, \cdots, E_{n-1}$ such that $E_0 \subset E^o$, $E_j \subset E^o_{j-1}$ for $j = 1, 2, \cdots, n-2$, and $D \subset E^o_{n-1}$. Choose closed coverings $\underline{e}_0, \underline{e}_1, \cdots, \underline{e}_{n-1}, \underline{d}$ of X and $j \in I$ such that

$$\underline{d} \wedge (D \wedge A_i)^n \gg \underline{e}_{n-1} \wedge (E_{n-1} \wedge A_i)^n \gg \cdots \gg^n \underline{e}_0 \wedge (E \wedge A_i)^n \gg \underline{e} \wedge (E \wedge A_i)$$

for all $i > j$ by 2.9. Then from 2.2 we have that for all $i > j$

$$IH^q(E \wedge A_i \mid D \wedge A_i) \subset IH^q(\underline{d} \wedge D \wedge A_i) \mid D \wedge A_i)$$

for all $q \leq n$ and

$$KH^q(\underline{e} \wedge E \wedge A_i) \mid E \wedge A_i) \subset KH^q(\underline{e} \wedge (E \wedge A_i) \mid \underline{e}_{n-1} \wedge (E_{n-1} \wedge A_i))$$

for all $q \leq n+1$. Since $\underline{d} \wedge (D \wedge A_i) > \underline{e}_{n-1} \wedge (E_{n-1} \wedge A_i)$, it follows that

$$KH^q(\underline{e} \wedge (E \wedge A_i) \mid E \wedge A_i) \subset KH^q(\underline{e} \wedge (E \wedge A_i) \mid \underline{d} \wedge (D \wedge A_i))$$

for all $i > j$ and all $q \leq n+1$.

2.11 Definition A closed covering $\underline{e}$ of X is in general position relative to a closed subset F of X if $E^{i_0} \cap \cdots \cap E^{i_q} \cap F \neq \emptyset$ implies that $(E^{i_0})^0 \cap \cdots \cap (E^{i_q})^0 \cap F \neq \emptyset$ where $E^{i_0}, \cdots, E^{i_q}$ are elements of $\underline{e}$.

2.12 Proposition. Let $\underline{e}$ be a closed covering of a space X, and let $F_1, \cdots, F_s$ be closed subsets of X with $F_s \subset \cdots \subset F_1$. Then there exists a closed covering $\underline{d}$ of X with $\underline{d} > \underline{e}$ and such that $\underline{d}$ is in general position relative to F_r for $r = 1, \cdots, s$.

Proof. Let $\underline{e}$ be a closed covering of X. We may assume the elements of $\underline{e}$ are well ordered. Let E^1 denote the first element in this ordering. For each $i_0, \cdots, i_q$ select a point $x(i_0, \cdots, i_q)$ in $(E^{i_0})^0 \cap \cdots \cap (E^{i_q})^0 \cap (E^1)^0 \cap F_1$ if possible. We, also, assume the point is chosen so that it is contained in the smallest possible F_r. Since $\underline{e}$ is locally finite, there will be only a finite number of such points. Let G^1 denote the union of the finite number of elements of $\underline{e}$ that meet E^1. Then $E^1 - G^1$ is an open subset of X contained in $(E^1)^0$. Since X is normal, we may choose a closed set D^1 such that $(E^1 - G^1) \subset (D^1)^0$, $D^1 \subset (E^1)^0$, and such that $(D^1)^0$ contains the above mentioned finite set of points. Then $(D^1, E^2, E^3, \cdots)$ is a closed covering of X, and $D^1 \cap E^{i_0} \cap \cdots \cap E^{i_q} \cap F_r \neq \emptyset$ for some r implies $(D^1)^0 \cap (E^{i_0})^0 \cap \cdots \cap (E^{i_q})^0 \cap F_r \neq \emptyset$. A transfinite induction proof based on this construction shows there exists the desired covering.

2.13 Proposition. Let (A_i) be a net in C(X) converging to A. Let F be a closed subset of X with $F^0 \cap A \neq \emptyset$, and let

$\underline{e}$ be a closed covering of X in general position relative to $F \frown A$. Then there exists $j \in I$ such that $\underline{e} \frown (F \frown A_i)$ and $\underline{e} \frown (F \frown A)$ may be identified for all $i > j$.

Proof. Let $\underline{e}$ and F be as in the proposition. Let $\underline{u}$ be the collection of open subsets of X consisting of F^o, X-F, and elements of the form $(E^{i_0})^o \frown \cdots \frown (E^{i_q})^o$ where $E^{i_0} \frown \cdots \frown E^{i_q} \frown (F \frown A) \neq \emptyset$. Since (A_i) converges to A, there exists $j \in I$ such that $A_i \in N(\underline{u})$ for all $i > j$. But $A_i \in N(\underline{u})$ implies $E^{i_0} \frown \cdots \frown E^{i_q} \frown (F \frown A) \neq \emptyset$ if and only if $E^{i_0} \frown \cdots \frown E^{i_q} \frown (F \frown A_i) \neq \emptyset$. Hence, we may make the assertion.

Corollary. Let (A_i) be a net in C(X) converging to A. Let $\underline{e}$ be a closed covering of X in general position relative to A. Then there exists $j \in I$ such that $\underline{e} \frown A_i$ and $\underline{e} \frown A$ may be identified for all $i > j$.

2.14 Definition. If $\underline{e}$ is a collection of closed subsets of X and E is a closed subset of X, we let $h^q(\underline{e} \frown (X-E))$ denote the Čech relative group $H^q(\underline{e}, \underline{e} \frown E)$.

2.15 Lemma. Let E and D be closed subsets of X with $D \subset E^o$. Let $\underline{e}$ be a closed covering of X. Then there exists a closed covering $\underline{d}$ of X such that

$$Kh^q(\underline{e} \frown (X-E) \mid (X-E) \subset Kh^q(\underline{e} \frown (X-E) \mid \underline{d} \frown (X-D))$$

for all $q \leq n$ where n is a fixed non-negative integer.

Proof. It is clearly enough to show we can do this for any fixed q. Let $\underline{e}$ be a closed covering of X. $Kh^q(\underline{e} \frown (X-E) \mid X-E)$ is a submodule of a finitely generated L-module. Since L is a principal ideal domain, $Kh^q(\underline{e} \frown (X-E) \mid X-E)$ is finitely generated, say by $z_1, \cdots, z_s$. For each z_r choose a closed covering $\underline{c}_r$ of X with

$\underline{c}_r > \underline{e}$ such that $z_r \in Kh^q(\underline{e}\frown(X-E) \mid \underline{c}_r\frown(X-E))$. Choose a closed covering $\underline{d}$ of X such that $\underline{d} > \underline{c}_r$ for $r = 1, \cdots, s$. Then $z_r \in Kh^q(\underline{e}\frown(X-E) \mid \underline{d}\frown(X-E))$ for $r = 1, \cdots, s$. Since there is a natural homomorphism of $h^q(X-E)$ into $h^q(X-D)$, we have the assertion.

Proposition. Let (A_i) be a net in C(X) n-rc converging to A. Let $x \in A$ and let E and D be closed nbds. of x with $D \subset E^o$. If $\underline{e}$ is a closed covering of X, there exists a closed covering $\underline{d}$ of X and $j \in I$ such that

1. $Ih^q(A_i-E \mid A_i-D) \subset Ih^q(\underline{d}\frown(A_i-D) \mid A_i-D)$ for all $q \leq n$ and all $i > j$.

2. $Kh^q(\underline{e}\frown(A_i-E) \mid A_i-E) \subset Kh^q(\underline{e}\frown(A_i-E) \mid \underline{d}\frown(A_i-D))$ for all $q \leq n+1$ and all $i > j$.

Proof. Let $x \in A$ and let E, D, and $\underline{e}$ be as stated in the proposition. Choose F a closed nbd. of x with $D \subset F^o$ and $F \subset E^o$. Let $\underline{e}' = (E, X-F^o)$. Choose a closed covering $\underline{b}$ with the properties stated in 2.8 relative to $\underline{e}$. Choose $\underline{c}$ a closed covering of X with $\underline{c} > \underline{b}$, $\underline{c} >^* \underline{e}'$, and such that $\underline{c}$ is in general position relative to A, $E\frown A$, and $D\frown A$. Choose a closed covering $\underline{d}$ and $j \in I$ with the properties stated in 2.8 relative to $\underline{c}$. Since this holds for any refinement of $\underline{d}$, we may assume that $\underline{d}$ has the above general position properties and that $\underline{d}$ has the properties expressed in 2.10 if we substitute $\underline{c}$ for $\underline{e}$ in the proposition. By the lemma we may also assume that for a fixed i' with $i' > j$

$$Kh^q(\underline{c}\frown(A_{i'}-E) \mid A_{i'}-E) \subset Kh^q(\underline{c}\frown(A_{i'}-E) \mid \underline{d}\frown(A_{i'}-D))$$

for all $q \leq n+1$. If $i > j$ and $q \leq n+1$, we have in the following diagram:

$$\begin{array}{ccccc}
H^{q-1}(\underline{d}\cap(D\cap A_i)) & \xrightarrow{\psi_1} & h^q(\underline{d}\cap(A_i-D)) & \xrightarrow{\phi_1} & H^q(\underline{d}\cap A_i) \\
 & & \uparrow \pi_1' & & \uparrow \pi_2' \\
 & & h^q(\underline{c}\cap(A_i-E)) & \xrightarrow{\phi_2} & H^q(\underline{c}\cap A_i) \\
 & & \downarrow \pi_1 & & \downarrow \pi_2 \\
 & & h^q(A_i-E) & \xrightarrow{\phi_3} & H^q(A_i)
\end{array}$$

where the π's are natural homomorphisms and the rows are part of an exact cohomology sequence, that the kernel of π_2 is contained in the kernel of π_2'. We also have that the kernel of π_1 is contained in the kernel of π_1' for a fixed i' with $i' > j$ and all $q \leq n+1$.

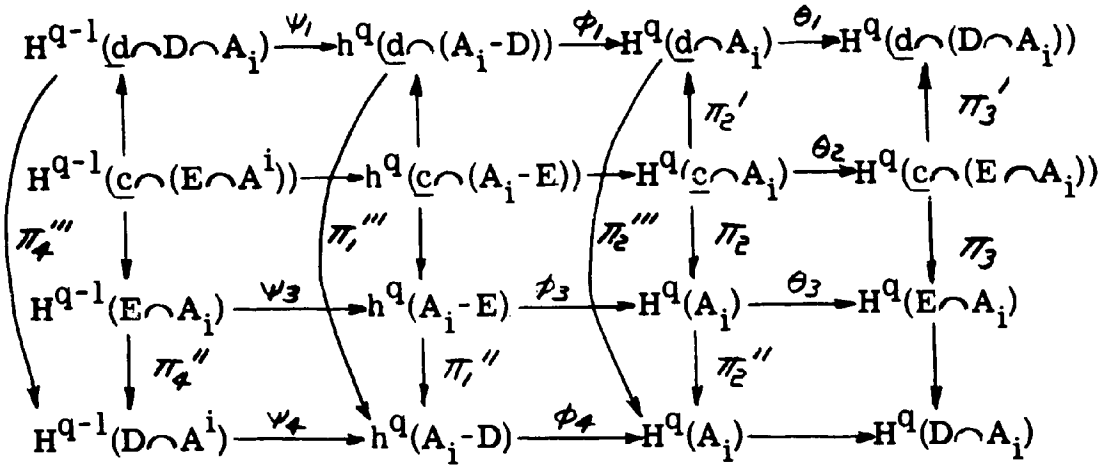
Let $\pi_1 z = 0$ for some $z \in h^q(\underline{c}\cap(A_i-E))$.

Then $\pi_2 \phi_2 z = 0$ so that $\pi_2' \phi_2 z = 0$. Hence, $\phi_1 \pi_1' z = 0$ and by exactness there exists w with $\psi_1 w = \pi_1' z$. For the fixed i' mentioned above, $\psi_1 w = 0$. But ψ_1 is independent of i by 2.13 for all $i > j$. Hence $\pi_1' z = 0$ for all $i > j$ and all $q \leq n+1$. Since $\underline{c} > \underline{e}$, it follows

$$Kh^q(\underline{e}\cap(A_i-E) \mid A_i-E) \subset Kh^q(\underline{e}\cap(A_i-E) \mid \underline{d}\cap(A_i-D))$$

for all $i > j$ and all $q \leq n+1$.

Consider the following diagram:



where the π's are natural homomorphisms and the rows are part of an exact cohomology sequence. We let $K\pi$ ($I\pi$) denote the kernel (image) of a homomorphism π. Then if $q \leq n$ and $i > j$, we have $I\pi_2 = I\pi_2''' = H^q(A_i)$, $K\pi_3 \subset K\pi_3'$, and $I\pi_4'' \subset I\pi_4'''$. We show that $I\pi_1'' \subset I\pi_1'''$. Let $x \in h^q(A_i - E)$.

If $\phi_3 x = 0$, choose y such that $\psi_3 y = x$. Choose z such that $\pi_4''' z = \pi_4'' y$. Then $\pi_1''' \psi_1 z = \psi_4 \pi_4''' z = \psi_4 \pi_4'' y = \pi_1'' \psi_3 y = \pi_1'' x$.

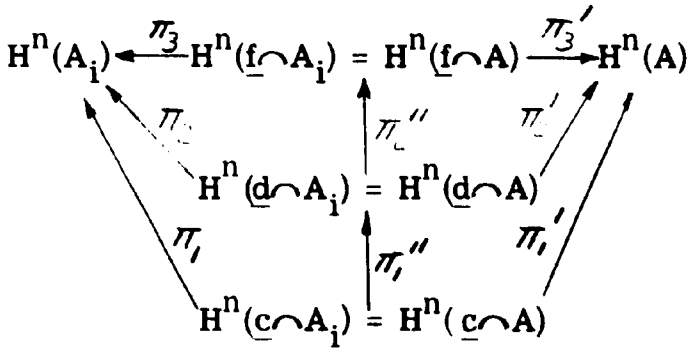
If $\phi_3 x \neq 0$; choose u such that $\pi_2 u = \phi_3 x$. Then $\pi_3 \theta_2 u = \theta_3 \pi_2 u = \theta_3 \phi_3 x = 0$. Hence $\pi_3' \theta_2' u = 0$ so that $\theta_1' \pi_2' u = 0$. Choose v such that $\phi_1 v = \pi_2' u$. Then $\phi_4 \pi_1''' v = \pi_2'' \pi_2' u = \pi_2'' \phi_3 x = \phi_4 \pi_1'' x$. Hence $\phi_4(\pi_1''' v - \pi_1'' x) = 0$. Now $\phi_3 x \neq 0$ implies that a cocycle determining x is not the coboundary of an element of $E \cap A_i$. But $\underline{d} > \underline{c}$ implies $\underline{d} >^* \underline{e}'$ so that $\pi_1''' v = \pi_1'' x$.

E. E. Floyd [5] proved a homology version of the following theorem with L a field or compact group. Our proof is a cohomology version of his argument.

2.16 Theorem. Let (A_i) be a net in C(X) n-rc converging to A. Then there exists $j \in I$ such that $H^n(A_i)$ and $H^n(A)$ are isomorphic for all $i > j$.

Proof. Choose a closed covering $\underline{b}$ of X with the properties indicated in 2.8. Let $\underline{c}$ be a closed covering of X in general position relative to A and such that $\underline{c} > \underline{b}$. Then by 2.8 and 2.12 we may choose a closed covering $\underline{d}$ of X in general position relative to A with $\underline{d} > \underline{c}$ and $j' \in I$ such that $IH^n(\underline{c} \cap A_i \mid A_i) = H^n(A_i)$ and $KH^n(\underline{c} \cap A_i \mid A_i) = KH^n(\underline{c} \cap A_i \mid \underline{d} \cap A_i)$ if $i > j'$. Similarly, we may choose a closed covering $\underline{f}$ of X in general position relative to A with $\underline{f} > \underline{d}$ and $j'' \in I$ such that $KH^n(\underline{d} \cap A_i \mid A_i) = KH^n(\underline{d} \cap A_i \mid \underline{f} \cap A_i)$

if $i > j''$. Choose $j \in I$ with $j > j'$ and $j > j''$ and such that the properties of 2.13 hold for c, d, and f. Consider the following diagram:



where the π's are natural homomorphisms and the indicated equalities hold by 2.13.

Let $z \in H^n(A)$. We may assume that d was chosen so that there exists $w \in H^n(d\frown A)$ with $\pi_2' w = z$. Hence, there exists $u \in H^n(c\frown A_i)$ with $\pi_1 u = \pi_2 w$. We may assume f was chosen so that $\pi_2'' \pi_1'' u = \pi_2'' w$. Hence $\pi_3' \pi_2'' w = \pi_2' w = \pi_1' u = z$. Hence $IH^n(c\frown A \mid A) = H^n(A)$.

Let $\pi_1' u = 0$ for some $u \in H^n(c\frown A)$. We may assume now that f is chosen so that $\pi_2'' \pi_1'' u = 0$. Then $\pi_2 \pi_1'' u = 0$ so that $\pi_1'' u = 0$. Hence $KH^n(c\frown A \mid A) = KH^n(c\frown A \mid d\frown A)$.

It follows that for $i > j$, $H^n(A_i)$ and $H^n(A)$ are each isomorphic with $IH^n(c\frown A \mid d\frown A)$.

The last half of this argument applies for dimension $(n+1)$, also.

Corollary. Let (A_i) be a net in $C(X)$ n-rc converging to A. If $H^{n+1}(A_i) = 0$ for all i, then $H^{n+1}(A) = 0$.

2.17 Theorem. Let (A_i) be a net in $C(X)$ n-rc converging to A. Then A is clc^n.

Proof. Let $x \in A$ and let E be a closed nbd. of x. Choose a closed nbd. F of x with $F \subset E^o$. Let $\underline{e} = (E, X - F^o)$ and choose a closed covering $\underline{d}$ of X and $j' \in I$ such that $\underline{d} \wedge A_i \overset{n}{>} \underline{e} \wedge A_i$ for all $i > j$. Choose $D \in \underline{d}$ such that $x \in D^o$. Then $IH^q(E \wedge A_i \mid D \wedge A_i) = 0$ for all $q \leq n$ and all $i > j'$. Choose a closed nbd. G of x with $G \subset D^o$. We show that $IH^q(E \wedge A \mid G \wedge A) = 0$ for all $q \leq n$.

Let $\underline{c}$ be a closed covering of X in general position relative to $E \wedge A$, $D \wedge A$, and $G \wedge A$. If $z \in H^q(E \wedge A_i)$ it may be represented by an element $z(c) \in H^q(\underline{c} \wedge (E \wedge A))$ for such a $\underline{c}$. By 2.10 choose a closed covering $\underline{d}$ of X and $j'' \in I$ such that $KH^q(\underline{c} \wedge (D \wedge A_i) \nmid D \wedge A_i) \subset KH^q(\underline{c} \wedge (D \wedge A_i) \mid \underline{d} \wedge (G \wedge A_i))$ for all $q \leq n$ and all $i > j''$. We may assume $\underline{d}$ has the above general position properties of $\underline{c}$. Choose $j \in I$ such that $j > j'$ and $j > j''$ and such that the property expressed in 2.13 holds for the relevant nerves. Let π_1 denote the homomorphism of $H^q(\underline{c} \wedge (E \wedge A))$ into $H^q(\underline{d} \wedge (G \wedge A))$, π_2 the homomorphism of $H^q(\underline{c} \wedge (E \wedge A_i))$ into $H^q(\underline{c} \wedge (D \wedge A_i))$ and π_3 the homomorphism of $H^q(\underline{c} \wedge (D \wedge A_i))$ into $H^q(\underline{d} \wedge (G \wedge A_i))$ where $i > j$. Observe that $\pi_1 = \pi_3 \pi_2$. Hence, $\pi_1 z(\underline{c}) = \pi_3 \pi_2 z(\underline{c}) = 0$ for all $q \leq n$. Hence, $z \in KH^q(E \wedge A \nmid G \wedge A)$ for all $q \leq n$.

Corollary. Let U be an open subset of X with U^- compact. Let (A_i) be a collection of compact subsets of U such that each point of U is in an element of some A_i and $i > i'$ if and only if $A_{i'} \subset A_i$. If for each $x \in U^-$ and each closed nbd. E of x, there exists a closed nbd. D of x with $D \subset E$ and $j \in I$ such that $IH^q(E \wedge A_i \mid D \wedge A_i) = 0$ for all $i > j$ and all $q \leq n$, then U^- is clc^n.

Remark. E. G. Begle [1] claimed the homology version of the above theorem is not true for the Vietoris theory

with coefficients in a ring with unit element.

2.18 Definition. For X locally compact and $x \in X$, we write $p^r(x; X) = k$, k a non-negative interger, if for each open nbd. U of x there exists open nbds. V and W of x with $W^- \subset V$, $V^- \subset U$, such that for any open nbd. W' of x with $W' \subset W$, $h^r(W' \mid V) = h^r(W \mid V)$ and is a free L-module of rank k.

2.19 Definition. Let U be an open subset of X. For X locally compact, we say X has the n-dimensional Poincare duality property inside U if for each pair of open subsets P and Q with $Q \subset P \subset U$, $IH^{n-q}(P \mid Q)$ is isomorphic with $Ih^q(Q \mid P)$ for all $q \leq n$. For this definition we assume ordinary cohomology groups are used in dimension 0.

2.20 Theorem. Let X be locally compact and let (A_i) be a net in C(X) n-rc converging to A. Suppose for each $x \in A$ there exists an open nbd. W of x and $j \in I$ such that A_i has the n-dimensional Poincare duality property inside $W \cap A_i$ for all $i > j$. Then $p^r(x; A) = 0$ for all $x \in A$ and all $r < n$, and $p^n(x; A) \leq 1$ for all $x \in A$.

Proof. Let $x \in A$ and choose W and $j' \in I$ such that A_i has the n-dimensional Poincare duality property inside $W \cap A_i$ for all $i > j'$. Let U be an open nbd. of x with U^- compact. Choose open nbds. P and Q of x with $Q^- \subset P$ and $P^- \subset (W \cap U)$. Choose an open nbd. V of x with $V^- \subset Q$ and $j'' \in I$ such that $IH^q(Q \cap A_i \mid V \cap A_i) = 0$ for all $q \leq n$ and all $i > j''$ by the n-rc convergence.

We show that $Ih^q(V \cap A \mid U \cap A) = 0$ for all $q < n$. Let $\underline{e}$ be a closed covering of X in general position relative to A, A-U, A-P, A-Q, and A-V. If $z \in h^q(V \cap A)$, it may be represented by an element $z(\underline{e}) \in h^q(\underline{e} \cap (V \cap A))$ for such an $\underline{e}$. By the proposition of 2.15, there exists a closed covering $\underline{d}$ of X and $j''' \in I$ such that

$$Kh^q(\underline{e}\frown(P\frown A_i) \mid P\frown A_i) \subset Kh^q(\underline{e}\frown(P\frown A_i) \mid \underline{d}\frown(U\frown A_i))$$

for all $q \leq n+1$ and all $i > j'''$. We may assume that $\underline{d}$ has the above general position properties of $\underline{e}$. Choose $j \in I$ such that $j > j'$, $j > j''$, $j > j'''$, and such that the property expressed in 2.13 holds for the relevant nerves. Since $I\!H^{n-q}(Q\frown A_i \mid V\frown A_i) = 0$ for all $q < n$ and all $i > j$, the same is the case for $Ih^q(V\frown A_i \mid Q\frown A_i)$ and hence for $Ih^q(V\frown A_i \mid P\frown A_i)$. Hence, $Ih^q(\underline{e}\frown(V\frown A_i) \mid \underline{d}\frown(U\frown A_i)) = 0$ for all $q < n$ and all $i > j$. Arguing as in 2.17, it follows that $Ih^q(V\frown A \mid U\frown A) = 0$ for all $q < n$. Hence, $p^r(x; A) = 0$ for all $x \in A$ and all $q < n$.

Since $I\!H^0(Q\frown A_i \mid V\frown A_i)$ is isomorphic with L when ordinary cohomology groups are used, the above argument yields $p^n(x; A) \leq 1$ for all $x \in A$.

CHAPTER III

3.0 We consider n-rc convergence of nets in this chapter where the elements of the net are generalized manifolds.

3.1 If X is locally compact, we write $\dim_L X \leq n$ if and only if $h^{n+1}(U) = 0$ for each open subset U of X. This gives the cohomology dimension introduced by Cohen [3].

If $\dim_L X \leq n$, then $h^q(U) = h^q(X-U) = 0$ for all $q \leq n+1$ and U an open subset of X. If $\dim_L X$ is finite, then $\dim_L X \leq n$ if and only if $p^r(x; X) = 0$ for all $x \in X$ and all $r \geq n+1$. A. Borel [2] contains a proof for this.

3.2 Definition. A generalized n-manifold over L is a locally compact Hausdorff topological space M such that

1. $\dim_L M$ is finite,
2. $p^r(x; M) = 0$ for all $x \in M$ and all $r \neq n$,
3. $p^n(x; M) = 1$ for all $x \in M$.

We use the standard abbreviation n-gm for such a topological space.

An orientable n-gm is an n-gm M such that for each component W of M and U a connected open subset of W with $\bar{U}$ compact, the natural homomorphism of $h^n(U)$ into $h^n(W)$ is an isomorphism. A locally orientable n-gm is an n-gm M such that each $x \in M$ has an open orientable n-gm for a neighborhood.

Remark. If F is a proper closed subset of a connected locally orientable n-gm, then $h^n(F) = 0$. If M is a connected n-gm, $h^n(F) = 0$ for all proper closed subsets of M, and $h^n(M)$ is isomorphic with L, then M is an orientable n-gm. A. Borel [2] contains proofs for these statements.

3.3 Proposition. Let X be locally compact with $\dim_L X$ finite. Let (A_i) be a net in C(X) n-rc converging to A where $\dim_L A_i \leq n$ for each $i \in I$. Then (A_i) q-rc converges to A for any q and $\dim_L A \leq n$.

Proof. Let $\underline{e}$ be a closed covering of X. Choose a closed covering $\underline{f}$ of X such that $\underline{f} > \underline{e}$ and each element of $\underline{f}$ is compact. Choose a closed covering $\underline{d}$ of X and $j \in I$ such that $\underline{d} \wedge A_i^n > \underline{f} \wedge A_i$ for all $i > j$. Let π denote the function from $\underline{d}$ into $\underline{f}$. Then, by 3.1 $H^q(\pi D \wedge A_i) = h^q(\pi D \wedge A_i) = 0$ for any $q \geq n+1$ and any $D \in \underline{d}$. Hence, $\underline{d} \wedge A_i^q > \underline{f} \wedge A_i$ for any q and all $i > j$. Hence, $\underline{d} \wedge A_i^q > \underline{e} \wedge A_i$ for any q and all $i > j$. Hence, (A_i) q-rc converges to A for any q.

Let $x \in A$ and let U be an open nbd. of x. Choose V and open nbd. of x with $\bar{V} \subset U$. Using the proposition of 2.15, part 2., we may argue as in 2.20 to show that $Ih^q(V \wedge A \mid U \wedge A) = 0$ for any fixed q with $q \geq n+1$. Hence, $p^r(x; A) = 0$ for all $r \geq n+1$. Hence, $\dim_L A \leq n$ by 3.1.

3.4 Theorem. Let X be locally compact and let (A_i) n-rc converge to A where each A_i is a connected n-gm over L and $\dim_L X$ is finite. Suppose for each $x \in A$ there exists an open nbd. W of x and $j \in I$ such that A_i has the n-dimensional Poincare duality property inside $W \wedge A_i$ for all $i > j$. Then A is a connected n-gm over L.

Proof. By 1.5, 2.20, 3.1, and 3.3, we need only show that $p^n(x; A) \neq 0$ for all $x \in A$.

Suppose $p^n(x; A) = 0$ for some $x \in A$. Choose an open nbd W of x and $j' \in I$ such that A_i has the n-dimensional Poincare duality property inside $W \wedge A_i$ for all $i > j'$. Let U be an open nbd. of x

with $U \subset W$ and U^- compact. Since $p^n(x; A) = 0$, there exists an open nbd P of x with $P^- \subset U$ such that $Ih^n(P \frown A \mid U \frown A) = 0$. By the n-rc convergence choose an open nbd. V of x with $V^- \subset P$ and $j'' \in I$ such that $H^o(U \frown A_i \mid V \frown A_i)$ is isomorphic with L for all $i > j''$ where ordinary cohomology groups are used. Since U and V are contained in W, $Ih^n(V \frown A_i \mid U \frown A_i)$ is isomorphic with L for all $i > j''$. By part 1. of the proposition in 2.15, we may choose a closed covering $\underline{e}$ of X in general position relative to A-U, A-P, and A-V so that $Ih^n(V \frown A_i \mid P \frown A_i) \subset Ih^n(\underline{e} \frown (P \frown A_i) \mid P \frown A_i)$ for large i. By the lemma in 2.15, we may choose a closed covering $\underline{d}$ of X with the above general position properties so that $Kh^n(\underline{e} \frown (U \frown A) \mid U \frown A) \subset Kh^n(\underline{e} \frown (U \frown A) \mid \underline{d} \frown (U \frown A))$. Hence, $Kh^n(\underline{e} \frown (P \frown A) \mid U \frown A) \subset Kh^n(\underline{e} \frown (P \frown A) \mid \underline{d} \frown (U \frown A))$. But $Ih^n(\underline{e} \frown (P \frown A) \mid U \frown A) = 0$ by the choice of P. Hence, for large i $Ih^n(\underline{e} \frown (P \frown A_i) \mid \underline{d} \frown (U \frown A_i)) = 0$. This together with the above assertion concerning $Ih^n(V \frown A_i \mid P \frown A_i)$ contradicts the fact that $Ih^n(V \frown A_i \mid U \frown A_i) \neq 0$ for large i.

3.5 Theorem. Let X be locally compact with $\dim_L X$ finite. Let (A_i) be a net in C(X) (n-1)-rc converging to A where each A_i is a connected n-gm over a field L and A is non-degenerate. Suppose for each $x \in A$ there exists an open nbd. W of x such that $W \frown A_i$ is an orientable n-gm over L for all $i \in I$. Then A is a connected n-gm over L.

Proof. We first show that (A_i) n-rc converges to A. Let $\underline{e}$ be a closed covering of X. Choose a closed covering $\underline{f}$ with each of its elements compact, $\underline{f} > \underline{e}$, and such that for each $F \in \underline{f}$ with $F^o \frown A = 0$ there exists a W as in the hypothesis with $F \subset W \frown E^o$ and

$(W \frown A) - F \neq \emptyset$ for some $E \in \underline{e}$. Choose a closed covering $\underline{d}$ of X with $\underline{d} > \underline{f}$ and $j' \in I$ such that $\underline{d} \frown A_i \overset{n-1}{>} \underline{f} \frown A_i$ for all $i > j'$. Since (A_i) converges to A in C(X), there exists $j \in I$ with $j > j'$ such that $(W \frown A_i) - D \neq \emptyset$ for some W where $D^o \frown A \neq \emptyset$ provided $i > j$. Hence, $H^n(D \frown A_i) = h^n(D \frown A_i) = 0$ for all $i > j$ and all $D \in \underline{d}$. Hence, $\underline{d} \frown A_i \overset{n}{>} \underline{e} \frown A_i$ for all $i > j$.

Since an orientable n-gm over a field has the n-dimensional Poincare duality property, it follows from 3.4 that A is a connected n-gm over L.

3.6 Theorem. Let X be compact with $\dim_L X$ finite. Let (A_i) be a net in C(X) (n-1)-rc converging to A where each A_i is a connected compact orientable n-gm over a field L and A is non-degenerate. Then A is a connected compact orientable n-gm over L.

Proof. By 3.5, A is a connected compact n-gm over L. We show that A is orientable.

Let D be a closed nbd. of some $x \in A$ with $A - D \neq \emptyset$. We show that $h^n(D \frown A) = H^n(D \frown A) = 0$. Since $\dim_L A = n$, this will follow provided $IH^n(A \mid D \frown A) = 0$. Choose E a closed nbd. of x with $D^o \subset E$ and $A - E \neq \emptyset$. An element of $H^n(A)$ would be determined by a closed covering $\underline{e}$ of X in general position relative to $E \frown A$ and $D \frown A$. Choose a closed covering $\underline{d}$ of X with these general position properties such that $KH^n(\underline{e} \frown (E \frown A_i) \mid E \frown A_i) \subset KH^n(\underline{e} \frown (E \frown A_i) \mid \underline{d} \frown (D \frown A_i))$ for large i be 2.10. Since A_i is orientable, $h^n(E \frown A_i) = H^n(E \frown A_i) = 0$ for large i. Hence, for large i $KH^n(\underline{e} \frown (E \frown A_i) \mid \underline{d} \frown (D \frown A_i)) = H^n(\underline{e} \frown (E \frown A_i))$. Hence, for large i $IH^n(\underline{e} \frown A_i \mid \underline{d} \frown (D \frown A_i)) = 0$. Hence,

$IH^n(e\frown A \mid \underline{d}\frown(D\frown A)) = 0$. Hence, $IH^n(A \mid D\frown A) = 0$.

By 2.16, $h^n(A) = H^n(A)$ is isomorphic with $H^n(A_i) = h^n(A_i)$ for large i. Hence, $h^n(A)$ is isomorphic with L. Hence, A is orientable by the remark in 3.2.

Remark. This theorem generalizes the result of E. G. Begle [1] mentioned in the introduction. Notice that the requirement that A be n-dimensional is not necessary.

P. A. White [10] gives a proof for this theorem with the assumption that $\dim_L A = n$. However, as mentioned in the introduction, his definition of regular convergence is not proven to be equivalent with the standard definition.

Example. Let X be the plane and let A_i denote the set points (x, y) such that $x^2 + y^2 = (1/i)$ where i ranges over the positive integers. Then (A_i) 0-rc converges to the point (0, 0). Hence, the above non-degeneracy assumption is necessary.

3.7 Definition. By an n-gm with boundary B we mean a locally compact Hausdorff topological space M and a closed subset B such that

1. B is an (n-1)-gm,
2. M-B is an n-gm,
3. $p^r(x; M) = 0$, for all $x \in B$ and all r.

For convenience, we say that M is a 1. o. n-gm with 1. o. boundary B, when M is an n-gm with boundary B and M-B and B are locally orientable.

This definition is given by F. A. Raymond [8] and the following two propositions proven by him.

3.8 Proposition. Let M_i be a l. o. (orientable) n-gm with l. o. (orientable) boundary B_i for i = 1, 2. If $M = M_1 \cup M_2$ and $M_1 \cap M_2 = B_1 = B_2$, then M is a locally orientable (orientable) n-gm.

3.9 Proposition. Let M be a connected locally orientable (orientable) n-gm, and let M' be a connected locally orientable (orientable) (n-1)-gm imbedded as a closed subset of M. If M-M' is separated, then M-M' is the union of exactly two disjoint connected open sets each of whose frontiers is M', and onto each of which M' fits as a l. o. (orientable) n-gm with l. o. (orientable) boundary.

3.10 Proposition. Let (A_i) and (B_i) be nets in C(X) n-rc converging to A and B respectively. If $(A_i \cap B_i)$ (n-1)-rc converges to $A \cap B$, then $(A_i \cup B_i)$ n-rc converges to $A \cup B$.

Proof. Let $\underline{e}$ be a closed covering of X. Choose a closed covering $\underline{f}_1$ of X and $j_1 \in I$ such tnat $\underline{f}_1 \cap A_i \overset{n}{>} \underline{e} \cap A_i$ for all $i > j_1$. Choose a closed covering $\underline{f}_2$ of X and $j_2 \in I$ such that $\underline{f}_2 \cap B_i \overset{n}{>} \underline{e} \cap B_i$ for all $i > j_2$. Choose a closed covering $\underline{f}$ such that $\underline{f} > \underline{f}_1$ and $\underline{f} > \underline{f}_2$. Choose a closed covering $\underline{d}$ of X and $j \in I$ with $j > j_1$ and $j > j_2$ such that $\underline{d} \cap (A_i \cap B_i) \overset{n-1}{>} \underline{f} \cap (A_i \cap B_i)$ for all $i > j$. Let π denote the map of $\underline{d}$ into $\underline{f}$, and π' the map of $\underline{f}$ into $\underline{e}$. For any $D \in \underline{d}$ and $i > j$ consider the following diagram:

$$
\begin{array}{ccccc}
 & & H^q(E\cap(A_i\cup B_i)) & \xrightarrow{\phi_1} & H^q(E\cap A_i)\oplus H^q(E\cap B_i) \\
 & & \downarrow \theta_1 & & \downarrow \psi_2 \\
H^{q-1}(F\cap(A_i\cap B_i)) & \xrightarrow{\Delta_2} & H^q(F\cap(A_i\cup B_i)) & \xrightarrow{\phi_2} & H^q(F\cap A_i)\oplus H^q(F\cap B_i) \\
\downarrow \psi_1 & & \downarrow \theta_2 & & \\
H^{q-1}(D\cap(A_i\cap B_i)) & \xrightarrow{\Delta_3} & H^q(D\cap(A_i\cup B_i)) & &
\end{array}
$$

where $f = \pi D$, $E = \pi' F$, the rows are a portion of an exact Mayer-Vietoris cohomology sequence, and the vertical homomorphisms are natural. If $q \leq n$ and $i > j$, the homomorphisms ψ_1 and ψ_2 have trivial images. Using this and the exactness of the rows, it follows that the image of $\theta_2 \theta_1$ is trivial for all $q \leq n$ and all $i > j$. Hence, the map $\pi' \pi$ of $\underline{d}$ into $\underline{e}$ gives that $\underline{d} \cap (A_i \cup B_i) \overset{n}{>} \underline{e} \cap (A_i \cup B_i)$ for all $i > j$. Hence, $(A_i \cup B_i)$ n-rc converges to $A \cup B$.

3.11 Theorem. Let X be compact with $\dim_L X$ finite and let (A_i) be a net in C(X) (n-1)-rc converging to a non-degenerate set A where each A_i is a connected compact orientable n-gm with l. o. boundary B_i over a field L. If (B_i) (n-2)-rc converges to a non-degenerate set B, A is a connected compact orientable n-gm with connected orientable boundary B over L.

Proof. For each i let C_i denote the double of A_i formed by attaching two distinct copies of A_i along B_i and let C denote the double of A. F. A. Raymond [8] shows that in the above situation each B_i is actually connected and orientable. Hence, by 3.8 C_i is a connected compact orientable n-gm. By 3.10, (C_i) n-rc converges to C. Hence, C is a connected compact orientable n-gm. Similarly, B is a connected compact orientable (n-1)-gm. Since B separates C, A is a connected compact orientable n-gm with connected orientable boundary B by 3.9.

Remark. P. A. White [11], using the paper referred to in 3.6, proved this result with the additional assumption that $\dim_L A = n$.

Remark. It is not known whether a n-gm is always

locally orientable. Should this be the case, 3.5 together with the above argument would give a similar result for the non-compact case.

Since a separable metric n-gm is a classical manifold for $n \leq 2$, it is locally orientable. Hence, we may state the following result.

3.12 Theorem. Let X be locally compact and metrizable with $\dim_L X$ finite. Let (A_i) be a net in C(X) 1-rc converging to a non-degenerate set A where each A_i is a classical connected 2-manifold with connected boundary B_i. Suppose for each $x \in A$ there exists an open nbd. W of x such that $W \cap A_i$ is a classical orientable 2-manifold with boundary $W \cap B_i$ for all $i \in I$. If (B_i) 0-rc converges to a non-degenerate set B, then A is a connected 2-manifold with connected boundary B provided A is separable.

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