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THE STABILITY OF LAMINAR FLOW IN AN ELASTIC CYLINDER

by

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TABLE OF CONTENTS

	Page
ACKNOWLEDGMENTS	ii
LIST OF FIGURES	iv
I. INTRODUCTION	1
II. THE PRIMARY FLOW	3
III. THE STABILITY PROBLEM	5
A. Differential Equations	5
B. Boundary Conditions	11
IV. SOLUTION OF THE STABILITY PROBLEM	13
A. General	13
B. Solution for Long Waves	13
C. Solution for Small Reynolds Number	18
D. Solution for Large Reynolds Number	24
V. RESULTS AND CONCLUSIONS	35
VI. SUGGESTIONS FOR FURTHER WORK	37
APPENDICES	38
A. Stability of Poiseuille Flow Relative to Disturbances in Planes Perpendicular to the Axis of the Cylinder	39
B. Asymptotic Expansions for the Whittaker Functions	41
C. Asymptotic Solution of a Nonhomogeneous Differential Equation with a Turning Point	44
BIBLIOGRAPHY	50

LIST OF FIGURES

Figure		Page
1	Definition Sketch	3

I. INTRODUCTION

The problem of the stability of laminar flows relative to infinitesimal disturbances represents an eigenvalue problem governed in part by the Orr-Sommerfeld equation. In order to completely specify this eigenvalue problem, boundary conditions are also needed. These boundary conditions will, of course, depend upon the nature of the particular boundaries present in the problem under study. Probably the most obvious type of boundary is a rigid, stationary wall, which requires the vanishing of the normal and tangential velocity components on it. In recent years more and more attention has been devoted to the study of flows involving more complicated boundaries. One such flow, which involves a "free" boundary, is the flow of a liquid down an inclined plane. This problem was first formulated correctly by Yih [19] and was subsequently studied by Benjamin [1]. Recently Yih has re-examined this problem [21]; the results of his investigation indicate that for vertical inclination the flow is unstable for all finite Reynolds number. The instability present in this problem is a surface instability, or gravity wave.

As another example, the problem of the stability of the boundary layer over a flexible surface has been treated, within the framework of the well-established Tollmien-Schlicting stability theory, by Benjamin [2]. Hains and Price [9] have studied, by a numerical method, the effect of wall flexibility on the stability of plane Poiseuille flow. Both plane Poiseuille flow and boundary layer flow exhibit instability with rigid boundaries; the effect of boundary flexibility on these flows is stabilizing, i.e., the critical Reynolds number at which instability first occurs is higher with the flexible boundary than with a rigid boundary. Can boundary flexibility ever be destabilizing? The present

work is motivated by a desire to answer this question. The flow whose stability is studied here is the flow of a fluid through an elastic cylinder of circular cross-section. The corresponding rigid-boundary case is the classical problem of flow in a pipe. This latter flow is generally considered stable* for all finite Reynolds number. The method of solution used involves iterative procedures for small Reynolds number and small wavenumber; for large Reynolds number, the exact form of the secular equation is treated by an asymptotic analysis.

The present problem has some bearing on phenomena which occur in nature and in industrial practice. The flow of blood in blood vessels involves a flexible boundary. (It must be noted that this problem involves other complications in addition to the flexible boundary [8].) In industry widespread use is made of flexible hoses. It is hoped that the present problem will provide some insight into the stability of these flows.

* Item 6 in the Bibliography examines this problem and contains references to earlier works.

II. THE PRIMARY FLOW

The present problem is an investigation of the stability of a steady, laminar flow of an incompressible, viscous fluid in a flexible, elastic cylinder of circular cross-section. The classical problem of the stability of flow in a rigid cylinder represents a special case of this work. The undeformed radius of the cylinder is a. The origin of the coordinates $\bar{R}$, θ, and Z is at the center of the cylinder with the Z axis parallel to the axis of the cylinder. (See Figure 1.) The flow is motivated by a constant piezometric-head gradient. The effect of gravity is ignored. It is negligible if the tube is vertical, or if the tension in the wall is very high.

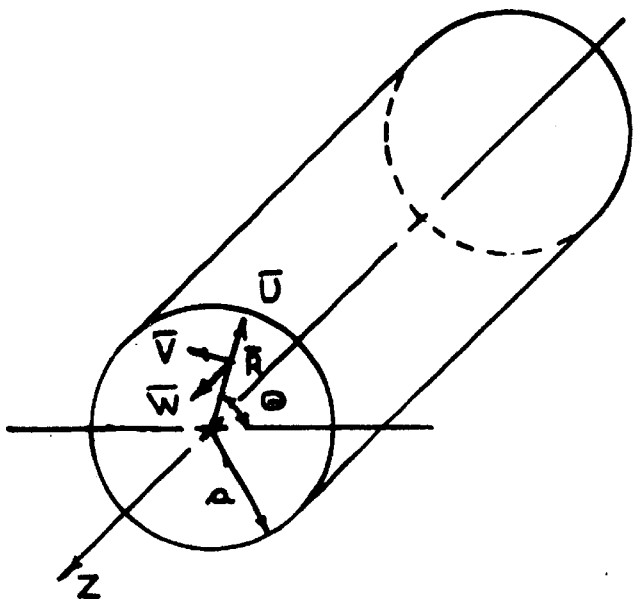

Figure 1. Definition Sketch.

The Navier-Stokes equations which govern the axisymmetric primary flow are

$$0 = -\bar{P}_{,z} + \mu\left\{\bar{W}_{,\bar{R}\bar{R}} + \frac{1}{\bar{R}}\bar{W}_{,\bar{R}}\right\}, \qquad (2\text{-}1)$$

$$0 = -\bar{P}_{,\bar{R}},$$

where $\bar{W}$ is the velocity component in the Z direction, $\bar{P}$ is the dynamic part of pressure, μ represents the viscosity, and comma followed by a subscript indicates partial differentiation with respect to that quantity. Since the velocity components in the $\bar{R}$ and θ directions ($\bar{U}$ and $\bar{V}$, respectively) are zero and $\bar{W}$ is a function of $\bar{R}$ alone, the equation of continuity is automatically satisfied.

Equations (2-1) are easily solved for the velocity and pressure distributions. The results are

$$\bar{W} = W_0\left\{1 - \frac{\bar{R}^2}{a^2}\right\}, \qquad (2\text{-}2)$$

where

$$W_0 = -\frac{\bar{P}_{,z}\, a^2}{4\mu} \qquad (2\text{-}3)$$

and

$$\bar{p} = \bar{P}_{,z}\, Z + \text{constant}, \qquad (2\text{-}4)$$

with $\bar{P}_{,z}$ the constant pressure gradient. The boundary conditions used in deriving (2-2) are no-slip at the boundary, $\bar{R} = a$, and $\bar{W}$ finite for $\bar{R} = 0$. The primary flow described by (2-2) is the well-known Hagen-Poiseuille flow.

In the primary flow, before any disturbances are introduced, the elastic cylinder is assumed to be undeformed and to behave as a rigid cylinder. In order to accomplish this, a pressure opposite to the fluid pressure, (2-4), is supposed to be imposed externally on the elastic cylinder.

III. THE STABILITY PROBLEM

A. Differential Equations

The equations to be satisfied are the Navier-Stokes equations and the continuity equation. We will consider disturbances that are rotationally symmetric. Since the primary flow is also rotationally symmetric, the appropriate forms of the Navier-Stokes equations are, with u^*, v^*, and w^* indicating the velocity components in the directions of increasing $\bar{R}$, θ, and Z, respectively,

$$\rho\left\{u^*_{,T} + u^* u^*_{,\bar{R}} + w^* u^*_{,Z} - \frac{v^{*2}}{\bar{R}}\right\} = -p^*_{,\bar{R}} + \mu\left\{\Delta^2 u^* - \frac{u^*}{\bar{R}^2}\right\}, \tag{3-1}$$

$$\rho\left\{v^*_{,T} + u^* v^*_{,\bar{R}} + w^* v^*_{,Z} + \frac{u^* v^*}{\bar{R}}\right\} = \mu\left\{\Delta^2 v^* - \frac{v^*}{\bar{R}^2}\right\}, \tag{3-2}$$

and

$$\rho\left\{w^*_{,T} + u^* w^*_{,\bar{R}} + w^* w^*_{,Z}\right\} = -p^*_{,Z} + \mu\,\Delta^2 w^*, \tag{3-3}$$

in which p^* is the pressure, T is the time, ρ is the density, and

$$\Delta^2 \equiv \frac{\partial^2}{\partial \bar{R}^2} + \frac{1}{\bar{R}}\frac{\partial}{\partial \bar{R}} + \frac{\partial^2}{\partial Z^2} .$$

The corresponding form of the continuity equation is

$$(u^* \bar{R})_{,\bar{R}} + (w^* \bar{R})_{,Z} = 0. \tag{3-4}$$

As in the usual procedure of stability analysis, the disturbances will be assumed small so that terms quadratic in them can be neglected. Let the velocity components and pressure in the disturbed state be described by

$$\left.\begin{aligned} w^* &= \bar{W}(\bar{R}) + w'(\bar{R}, Z, T), \\ u^* &= u'(\bar{R}, Z, T), \\ p^* &= \bar{P}(Z) + p'(\bar{R}, Z, T), \end{aligned}\right\} \tag{3-5}$$

and

$$v^* = v'(R, Z, T),$$

where the primed quantities represent the disturbances in the velocity and pressure. Substitution of (3-5) into (3-1), (3-2), and (3-3) and neglect of the second-order terms in the primed quantities yields

$$\rho\{u'_{,T} + \bar{W} u'_{,Z}\} = -p'_{,\bar{R}} + \mu\left\{\Delta^2 u' - \frac{u'}{\bar{R}^2}\right\}, \tag{3-6}$$

$$\rho\{w'_{,T} + u' \bar{W}_{,\bar{R}} + \bar{W} w'_{,Z}\} = -p'_{,Z} + \mu \Delta^2 w', \tag{3-7}$$

and

$$\rho\{v'_{,T} + \bar{W} v'_{,Z}\} = \mu\left\{\Delta^2 v' - \frac{v'}{\bar{R}^2}\right\}. \tag{3-8}$$

In obtaining (3-6) and (3-7), use has been made of the fact that $\bar{W}$ and $\bar{P}$ are solutions of (2-1). In terms of (3-5), the continuity equation is

$$(u'\bar{R})_{,\bar{R}} + (w'\bar{R})_{,Z} = 0. \tag{3-9}$$

In order to simplify the solution of (3-6) - (3-9), it is convenient to introduce the following non-dimensional quantities:

$$\left.\begin{aligned} r &= \frac{\bar{R}}{a}, & z &= \frac{Z}{a}, & t &= \frac{T W_0}{a}, \\ u &= \frac{u'}{W_0}, & w &= \frac{w'}{W_0}, & v &= \frac{v'}{W_0}, \\ W &= \frac{\bar{W}}{W_0}, & p &= \frac{p'}{\rho W_0^2}, & R &= \frac{\rho W_0 a}{\mu}, \end{aligned}\right\} \tag{3-10}$$

where R is the Reynolds number. Introducing (3-10) into (3-6) - (3-8) gives the following (dimensionless) equations:

$$u_{,t} + W u_{,z} = - p_{,r} + \frac{1}{R}\left\{\nabla^2 u - \frac{u}{r^2}\right\}, \tag{3-11}$$

$$v_{,t} + W v_{,z} = \frac{1}{R}\left\{\nabla^2 v - \frac{v}{r^2}\right\}, \tag{3-12}$$

and

$$w_{,t} + u W_{,r} + W w_{,z} = - p_{,z} + \frac{1}{R}\nabla^2 w, \tag{3-13}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The dimensionless continuity equation is

$$(ur)_{,r} + (wr)_{,z} = 0. \tag{3-14}$$

Equation (3-14) permits the use of a stream function ψ in terms of which the velocity components u and w can be expressed as follows:

$$u = - \frac{1}{r}\psi_{,z}, \qquad w = \frac{1}{r}\psi_{,r}. \tag{3-15}$$

If a sinusoidal disturbance with an exponential time-factor is assumed, the perturbation quantities can be written as

$$\psi = \phi(r)\, e^{i\alpha(z-ct)}, \tag{3-16}$$

$$p = q(r)\, e^{i\alpha(z-ct)}, \tag{3-17}$$

and

$$w = \tilde{w}(r)\, e^{i\alpha(z-ct)}, \tag{3-18}$$

where α is the wavenumber (wavelength = $2\pi a/\alpha$) and $c = c_r + i c_i$, c_r being the wave velocity and αc_i being the rate of amplification or damping. Since α is assumed positive, c_i must be less than zero for stability. With (3-16) in (3-15), u and w become

$$u = -i\alpha \frac{\phi(r)}{r} e^{i\alpha(z-ct)}, \quad w = \frac{\phi'(r)}{r} e^{i\alpha(z-ct)}, \tag{3-19}$$

where prime now signifies ordinary differentiation with respect to r. In (3-19) significance is attached only to the real parts of u and w. Now using (3-19) and (3-17) in (3-11) and (3-13), we have

$$\begin{aligned} -\frac{\alpha^2 c\phi}{r} + \frac{W\alpha^2\phi}{r} &= -q' \\ &+ \frac{1}{R}\left\{-\frac{i\alpha\phi''}{r} + \frac{i\alpha\phi'}{r^2} + \frac{i\alpha^3\phi}{r}\right\} \end{aligned} \tag{3-20}$$

and

$$\begin{aligned} -\frac{i\alpha c\phi'}{r} - \frac{i\alpha\phi W'}{r} + \frac{W i\alpha\phi'}{r} &= -i\alpha q \\ &+ \frac{1}{R}\left\{\frac{\phi'''}{r} - \frac{\phi''}{r^2} + \frac{\phi'}{r^3} - \frac{\alpha^2\phi'}{r}\right\}. \end{aligned} \tag{3-21}$$

Elimination of q from (3-20) and (3-21) gives

$$\phi^{III} - \frac{2\phi'''}{r} + \frac{3\phi''}{r^2} - \frac{3\phi'}{r^3} - \alpha^2\left(2\phi'' - \frac{2\phi'}{r} - \alpha^2\phi\right) \tag{3-22}$$

$$+ i\alpha R\left\{(c-W)\left(\phi'' - \frac{1}{r}\phi' - \alpha^2\phi\right) + \left(W'' - \frac{W'}{r}\right)\phi\right\} = 0.$$

From (2-2) and (3-10), W(r) is

$$W(r) = 1 - r^2 \tag{3-23}$$

so that (3-22) becomes

$$\phi^{III} - \frac{2\phi'''}{r} + \frac{3\phi''}{r^2} - \frac{3\phi'}{r^3} - \alpha^2\left(2\phi'' - \frac{2\phi'}{r} - \alpha^2\phi\right) \tag{3-24}$$

$$- i\alpha R(W-c)\left(\phi'' - \frac{1}{r}\phi' - \alpha^2\phi\right) = 0.$$

This is the axisymmetric counterpart of the Orr-Sommerfeld equation, and is available in the literature on the stability of Poiseuille flow.

Introduction of (3-18) into (3-12) yields

$$\tilde{w}'' + \frac{1}{r}\tilde{w}' - \left\{\alpha^2 + \frac{1}{r^2} + i\alpha R(W-c)\right\}\tilde{w} = 0. \tag{3-25}$$

Equation (3-24) governs the behavior of an axial symmetric disturbance. This equation is independent of (3-25) which represents a disturbance in planes perpendicular to the axis of the cylinder. This second type of disturbance is shown in the appendix to be stable so that it is sufficient to consider only (3-24), i.e., the original assumption of rotational symmetry leads to a consideration of axial symmetry.

In the disturbed state, the flexible cylinder is subjected to a radial loading due to the perturbation in the fluid pressure and velocity. The axisymmetric equilibrium condition (in the radial direction) for an elastic cylinder has been given by Flügge [7]. With some change of notation, the result is (The subsequent development is based on the assumption $\alpha << 1$. The error is of order α^2.)

$$-\frac{\bar{E}h}{1-\nu^2}\frac{\bar{\eta}}{a^2} + L_{\bar{R}} = 0,$$

with $\bar{\eta}$ the displacement of the cylinder in the radial direction (positive outward), $L_{\bar{R}}$ the loading per unit area in the radial direction, h the cylinder thickness, $\bar{E}$ Young's modulus, and ν Poisson's ratio. Using d'Alembert's principle, the loading can be expressed as that due to the normal stress fluctuation minus the effect of the inertia force,

$$L_{\bar{R}} = p'(a, Z, T) - 2\mu w'_{,\bar{R}}(a, Z, T) - \bar{m}\,\bar{\eta}_{,TT},$$

($\bar{m}$ is the mass per unit area of the cylinder), so that the dynamic equation of motion is

$$-\frac{\bar{E}h}{1-\nu^2}\frac{\bar{\eta}}{a^2} + p'(a,Z,T) - 2\mu w'_{,\bar{R}}(a,Z,T) = \bar{m}\,\bar{\eta}_{,TT}. \quad (3\text{-}26)$$

This equation is derived on the basis of the membrane theory of elasticity with the displacement of the cylinder in the axial direction neglected. The assumption is made that the effect of the axial displacement of the cylinder wall is negligible in comparison to the effect of the radial displacement.

In (3-26), p' and $w'_{,\bar{R}}$ are evaluated at $\bar{R} = a$ rather than $\bar{R} = a + \bar{\eta}$ because expansion of these quantities in a Taylor series about $\bar{R} = a$ yields terms of higher order, $\bar{\eta}$ being assumed the same order as the fluid disturbances. With $\bar{\eta} = \eta a$ and (3-10), (3-26) can be non-dimensionalized. The result is

$$-E\eta + p(1, \zeta, t) - \frac{a}{R} w_{,n}(1, \zeta, t) = m\,\eta_{,tt} \quad (3\text{-}27)$$

where

$$E = \frac{Eh}{(1-\nu^2)\rho W_0^2 a} \qquad \text{and} \qquad m = \frac{\bar{m}}{\rho a}.$$

B. Boundary Conditions

We require that no "gaps" be formed between the fluid and the cylinder walls, i.e., the fluid velocity at the cylinder in the radial direction equals the cylinder velocity in the radial direction. To the first order, this condition requires that

$$u = \eta_{,t} \qquad \text{at} \quad r = 1, \tag{3-28}$$

or, in terms of (3-19),

$$\eta_{,t} = -i\alpha\phi(1)\, e^{i\alpha(z-ct)}.$$

Solving for η gives

$$\eta = \frac{\phi(1)}{c}\, e^{i\alpha(z-ct)}. \tag{3-29}$$

Since the fluid is viscous and the axial displacement of the cylinder is neglected, the no-slip condition at the cylinder wall requires (to the first order) that

$$w + W'\eta = 0 \qquad \text{at} \quad r = 1.$$

Using (3-19), (3-23), and (3-29), the no-slip condition becomes

$$\phi'(1) - \frac{2}{c}\phi(1) = 0. \tag{3-30}$$

Using (3-29), (3-17), and (3-19), the equation of motion for the cylinder, (3-27), becomes

$$-\frac{E}{c}\phi(1) + q(1) - \frac{2}{R}\left\{-\phi'(1) + \phi''(1)\right\} = m\left\{-\alpha^2 c\,\phi(1)\right\},$$

or, after using (3-21) to eliminate the pressure and using (3-30),

$$\phi'''(1) - (1 + 2i\alpha)\phi''(1)$$

$$+ \left(1 + 2i\alpha - \frac{i\alpha R E}{2} - \alpha^2 + \frac{i\alpha^3 R \kappa^2 m}{2}\right)\phi'(1) = 0. \tag{3-31}$$

Two boundary conditions are thus provided by (3-30) and (3-31). If E is infinite (which corresponds to a rigid cylinder wall), (3-30) and (3-31) reduce to

$$\phi'(1) = \phi(1) = 0,$$

which signifies the vanishing of the normal and tangential velocity components at the wall.

The two remaining conditions are due to the lack of singularities on the axis of the cylinder. From (3-19), these are

$$\frac{\phi(r)}{r} = 0 \quad \text{at } r = 0 \tag{3-32}$$

and

$$\frac{\phi'(r)}{r} \text{ finite} \quad \text{at } r = 0. \tag{3-33}$$

In what follows, (3-32) and (3-33) will be referred to as regularity conditions. The use of them will allow the rejection of solutions to our equations that are singular at r = 0.

IV. SOLUTION OF THE STABILITY PROBLEM

A. General

Equation (3-24) and the associated boundary conditions constitute an eigenvalue problem. To insure the existence of a nontrivial solution, a relation

$$c = c(\alpha, \alpha R)$$

(E and m fixed) must exist. Since c is complex, we also have

$$c_r = c_r(\alpha, \alpha R), \quad c_i = c_i(\alpha, \alpha R).$$

The neutral stability curve obtained by setting $c_i = 0$ is a graph indicating the relation (for various values of c_r) between α and R in the α-R plane. Although it is possible, as we shall see, to obtain a general expression for the secular equation of the eigenvalue problem, this expression is not readily adaptable to computation in all cases, so that it is convenient to consider first the behavior of the system in certain sectors of the α-R plane using iterative procedures proposed by Yih [21]. The general expression for the secular equation will then be derived and studied for the important case of large Reynolds number.

B. Solution for Long Waves

Recalling that the wavelength = $2\pi a/\alpha$, we see that for long waves, α is small. To obtain a solution for this case, we let

$$\left.\begin{aligned} \phi &= \phi_0 + \alpha \phi_1 + \alpha^2 \phi_2 + \cdots \\ \text{and} \qquad c &= c_0 + \alpha c_1 + \alpha^2 c_2 + \cdots \end{aligned}\right\} \tag{4-1}$$

Using (4-1) in the differential equation (3-24) and in the boundary conditions (3-30) and (3-31) and collecting powers of α gives

$$\phi_0^{III} - \frac{2}{r}\phi_0''' + \frac{3}{r^2}\phi_0'' - \frac{3}{r^3}\phi_0'$$

$$+ \alpha\left\{\phi_1^{III} - \frac{2}{r}\phi_1''' + \frac{3}{r^2}\phi_1'' - \frac{3}{r^3}\phi_1' - iR(w-c_0)(\phi_0'' - \frac{1}{r}\phi_0')\right\}$$

$$+ \alpha^2\left\{\phi_2^{III} - \frac{2}{r}\phi_2''' + \frac{3}{r^2}\phi_2'' - \frac{3}{r^3}\phi_2' - 2(\phi_0'' - \frac{1}{r}\phi_0')\right.$$

$$\left. - iR(w-c_0)(\phi_1'' - \frac{1}{r}\phi_1') + iRc_1(\phi_0'' - \frac{1}{r}\phi_0')\right\}$$

$$+ \cdots = 0,$$

$$c_0\phi_0' - 2\phi_0 + \alpha(c_0\phi_1' + c_1\phi_0' - 2\phi_1)$$

$$+ \alpha^2(c_0\phi_2' + c_1\phi_1' + c_2\phi_0' - 2\phi_2) + \cdots = 0,$$

and

$$\phi_0''' - \phi_0'' + \phi_0' + \alpha\left\{\phi_1''' - \phi_1'' + \phi_1' - 2i(\phi_0'' - \phi_0') - \frac{iER}{2}\phi_0'\right\}$$

$$+ \alpha^2\left\{\phi_2''' - \phi_2'' + \phi_2' - 2i(\phi_1'' - \phi_1') - \frac{iER}{2}\phi_1' - \phi_0'\right\} + \cdots = 0,$$

with the last two expressions evaluated at r = 1. Now equating the coefficients of the powers of α to zero yields a series of differential equations with corresponding boundary conditions that may be solved successively. The regularity conditions must also be satisfied by each ϕ_i(i=0, 1, 2...).

a. Zeroth-Order Approximation

The differential equation is

$$\phi_o^{IV} - \frac{2}{r}\phi_o''' + \frac{3}{r^2}\phi_o'' - \frac{3}{r^3}\phi_o' = 0, \tag{4-2}$$

and the boundary conditions are

$$\epsilon_o \phi_o' - 2\phi_o = 0 \quad \text{at} \quad r = 1 \tag{4-3}$$

and

$$\phi_o''' - \phi_o'' + \phi_o' = 0 \quad \text{at} \quad r = 1. \tag{4-4}$$

Equation (4-2) is an equation of the Euler type in ϕ_o'. By making the substitution $r = \exp s$, (4-2) becomes an ordinary differential equation with constant coefficients and is easily solved by elementary techniques.[10] The result for ϕ_o is

$$\phi_o = A_o r^4 + B_o r^2 + C_o r^2 \ln r + D_o \tag{4-5}$$

where the term $r^2 \ln r$ is due to a repeated root in the complementary equation. The regularity conditions demand that

$$C_o = D_o = 0$$

while (4-3) gives

$$\epsilon_o (4A_o + 2B_o) - 2(A_o + B_o) = 0. \tag{4-6}$$

From (4-4), we have

$$A_o = 0. \tag{4-7}$$

Using (4-7) in (4-6), we find that in order to insure a nontrivial solution we must have

$$c_o = 1. \tag{4-8}$$

We conclude that in the limit of vanishing wave number the system is neutrally stable. The eigenfunction corresponding to (4-8) is

$$\phi_o = B_o r^2$$

where B_o is arbitrary. For convenience, we take B_o as unity, and

$$\phi_o = r^2. \tag{4-9}$$

b. First-Order Approximation

The differential equation is

$$\phi_1^{IV} - \frac{2}{r}\phi_1''' + \frac{3}{r^2}\phi_1'' - \frac{3}{r^3}\phi_1' = i R(W - c_o)(\phi_o'' - \frac{1}{r}\phi_o').$$

Using the expression for ϕ_o, we see that the equation for ϕ_1 is identical with that for ϕ_o so that the proper expression for ϕ_1 can be written immediately,

$$\phi_1 = A_1 r^4 + B_1 r^2. \tag{4-10}$$

The boundary conditions require

$$\phi_1' + 2c_1 - 2\phi_1 = 0 \quad \text{at } r = 1 \tag{4-11}$$

and

$$\phi_1''' - \phi_1'' + \phi_1' - iRE = 0 \text{ at } r = 1. \tag{4-12}$$

Now (4-11) requires

$$4A_1 + 2B_1 + 2c_1 - 2(A_1 + B_1) = 0$$

or

$$A_1 + c_1 = 0, \tag{4-13}$$

while from (4-12)

$$24 A_1 - (12 A_1 + 2B_1) + 4A_1 + 2B_1 - i R E = 0$$

or

$$A_1 = \frac{i R E}{16} . \tag{4-14}$$

We conclude that

$$c_1 = - \frac{i R E}{16} . \tag{4-15}$$

Note that B_1 is not determined by the above analysis but that B_1 would involve a term of the form $B_1 r^2$ in ϕ. Since we have already chosen the coefficient of r^2 as unity, for consistency we must take $B_1 = 0$. The first order approximation is then

$$\phi_1 = \frac{i R E}{16} r^4 . \tag{4-16}$$

Neglecting terms of order α^2 in the expansions, (4-1), the eigenfunction and eigenvalue are

$$\phi = r^2 + \frac{i \alpha R E}{16} r^4 , \tag{4-17}$$

$$c = 1 - \frac{i \alpha R E}{16} . \tag{4-18}$$

The expression for c (with negative imaginary part) clearly indicates that for small α an increase in Reynolds number and/or an increase in E (increasing E implies the cylinder wall is increasing in rigidity) represents a stabilizing factor.

C. Solution for Small Reynolds Number

To study the behavior of the system for small Reynolds number, it is convenient to follow Sexl [15] and rewrite (3-24) in terms of the vorticity, $u_{,z} - w_{,r}$, [11] which, with the exponential factor omitted, will be denoted by Φ. Hence

$$\Phi = \phi'' - \frac{1}{r}\phi' - \alpha^2\phi, \tag{4-19}$$

and (3-24) becomes

$$\Phi'' - \frac{1}{r}\Phi' - \{\alpha^2 - i\alpha R(c - w)\}\Phi = 0. \tag{4-20}$$

Now we expand* Φ and c in terms of $i\alpha R$,

$$\left.\begin{aligned} \Phi &= \Phi_0 + i\alpha R\,\Phi_1 + (i\alpha R)^2\Phi_2 + \cdots \\ \text{and} \\ c &= c_0 + i\alpha R\,c_1 + (i\alpha R)^2 c_2 + \cdots \end{aligned}\right\} \tag{4-21}$$

We can then proceed in the spirit of section IV-B, solving successively the linear systems resulting from equating each power of $i\alpha R$ to zero.

a. Zeroth Order Approximation

The differential equations are

$$\phi_0'' - \frac{1}{r}\phi_0' - \alpha^2\phi_0 = \Phi_0 \tag{4-22}$$

and

$$\Phi_0'' - \frac{1}{r}\Phi_0' - \alpha^2\Phi_0 = 0. \tag{4-23}$$

Letting

$$\Phi_i = r\,F_i \quad \text{and} \quad \phi_i = r\,f_i, \tag{4-24}$$

(i = 0, 1 . . .), and defining a linear operator L by

---

* ϕ will, of course, have a corresponding expansion.

$$L \equiv D^2 + \frac{1}{r} D - \frac{1}{r^2} + (i\alpha)^2$$

(D = d/dr), the equations become

$$L f_0 = F_0 \qquad \text{and} \qquad L F_0 = 0,$$

or, after combination,

$$L^2 f_0 = 0. \tag{4-25}$$

Yih [20] has given the solutions for $L^n f = 0$. With n = 2, the solutions are the solutions of (4-25). These are, however, well known independently. In the present case only the solutions that are regular at the origin are required. The result is

$$f_0 = A_0 r I_0(\alpha r) + B_0 I_1(\alpha r)$$

where $I_n(\alpha r)$ is the Bessel function of the nth order and of imaginary argument $i\alpha r$. From (4-24), the result for ϕ_0 is

$$\phi_0 = A_0 r^2 I_0(\alpha r) + B_0 r I_1(\alpha r). \tag{4-26}$$

The boundary conditions require (at r = 1)

$$\kappa_0 \phi_0' - 2\phi_0 = 0 \tag{4-27}$$

and

$$\phi_0''' - (1 + 2i\alpha)\phi_0'' + (1 + 2i\alpha - \alpha^2 - \frac{i\alpha E^*}{2})\phi_0' = 0, \tag{4-28}$$

where E^* denotes the quantity

$$RE = \frac{\bar{E} h}{(1-\nu^2)\mu W_0},$$

which appears in the boundary condition (3-31). Since W_o approaches zero as R approaches zero (μ fixed), the term involving E^* may appear troublesome. From (2-3), however, we see that, for fixed pressure gradient, μW_o is finite.

Using (4-26) in (4-27) and (4-28), after some simplification, we have

$$A_o\{c_o(2I_o+\alpha I_1)-2I_1\}+B_o\{c_o\alpha I_o-2I_1\}=0 \qquad (4\text{-}29)$$

and

$$A_o\left\{2\alpha^2 I_o-4i\alpha^2 I_1-2i\alpha^3 I_o-i\alpha E^* I_o-\frac{i\alpha^2 E^*}{2}I_1\right\}$$

$$+B_o\left\{-2i\alpha^3 I_1-\frac{i\alpha^2 E^*}{2}I_o\right\}=0 \qquad (4\text{-}30)$$

where $I_n = I_n(\alpha)$ $(n = 0, 1)$.

To insure a nontrivial solution the determinant of the coefficients of A_o and B_o must vanish; from which, the result for c_o is

$$c_o = 4\alpha I_o^3 I_1+\alpha^2(I_o^2-I_1^2)\{8\alpha I_1^2+E^* 2I_oI_1$$

$$-E^*\alpha(I_o^2-I_1^2)\}-i\{[4\alpha^2 I_oI_1+I_o^2E^*][2I_oI_1-\alpha(I_o^2-I_1^2)]\}$$

$$\Big/\ 2\alpha^2\{I_o^4+\alpha^4(I_o^2-I_1^2)^2\} \qquad (4\text{-}31)$$

To insure stability, the imaginary part of c_o must be negative. Since the denominator in the expression for c_o is always positive ($\alpha > 0$), for a stable configuration we must have

$$2I_oI_1 > \alpha(I_o^2-I_1^2).$$

Computation shows that this condition is satisfied and that therefore the system is stable in the limit for vanishing Reynolds number. It is interesting to note that the system is stable even for E^* equal zero which implies an extremely flexible cylinder wall. Increasing E^* is therefore a stabilizing factor at small Reynolds number.

Now for small α

$$I_0 \sim 1, \quad I_1 \sim \frac{\alpha}{2},$$

so that for α equal zero, we have

$$c_0 = 1$$

in agreement with (4-8). For large α

$$I_1 / I_0 = 1 - (2\alpha)^{-1}$$

so that, for this case, we may write

$$c_0 = \left\{ \frac{3}{\alpha} + \frac{E^*}{4\alpha^2} \right\} - i \left\{ \frac{1}{\alpha} + \frac{E^*}{4\alpha^2} \right\}.$$

Thus $c_0 = 0$ at $\alpha = \infty$.

The eigenfunction can be readily determined. From (4-30),

$$B_0 = - \frac{A_0 \left\{ 4\alpha I_1 + 2\alpha^2 I_0 + E^* \left(I_0 + \frac{\alpha}{2} I_1 \right) + i\, 2\alpha I_0 \right\}}{2\alpha^2 I_1 + \frac{\alpha E^*}{2} I_0}.$$

Since the eigenfunction is determined only up to an arbitrary multiplicative constant, A_0 can be chosen as unity; the complete eigenfunction then follows from (4-26).

b. First Order Approximation

In studying the behavior of the system as R departs from zero, we will consider the case $E^* = 0$, since the effect of non-zero E^* is definitely stabilizing.

The differential equations, in terms of (4-24), are

$$L f_1 = F_1 \quad \text{and} \quad L F_1 = (W - c_0) F_0 .$$

Combination gives

$$L^2 f_1 = (W - c_0) F_0 . \tag{4-32}$$

Now $F_0 = 2\alpha I_1(\alpha r)$, as is easily shown by using $Lf_0 = F_0$, so that

$$L^2 f_1 = 2\alpha (1 - r^2 - c_0) I_1(\alpha r). \tag{4-33}$$

The homogeneous solution is known from the result for f_0. To obtain a particular solution, the following identities are useful:

$$L^2 \left\{ r^{m+1} I_m(\alpha r) \right\} = 4\alpha^2 m(m+1) r^{m-1} I_{m-2}(\alpha r)$$

$$= 4\alpha^2 m(m+1) r^{m-1} I_m(\alpha r)$$

$$+ 8\alpha (m-1) m (m+1) r^{m-2} I_{m-1}(\alpha r).$$

The complete solution to (4-33) is then

$$f_1 = \frac{(1 - c_0)}{4\alpha} r^2 I_1(\alpha r) - \frac{1}{24\alpha} r^4 I_3(\alpha r)$$

$$+ A_1 r I_0(\alpha r) + B_1 I_1(\alpha r),$$

or, returning to ϕ_1,

$$\phi_1 = \frac{(1-c_0)}{4\alpha} r^3 I_1(\alpha r) - \frac{1}{24\alpha} r^5 I_3(\alpha r)$$

$$+ A_1 r^2 I_0(\alpha r) + B_1 r I_1(\alpha r). \qquad (4\text{-}34)$$

The boundary conditions for ϕ_1 are (with m = 0)

$$c_0 \phi_1' + c_1 \phi_0' - 2\phi_1 = 0$$

and

$$\phi_1''' - (1 + 2i\alpha)\phi_1'' + (1 + 2i\alpha - \alpha^2)\phi_1' = 0.$$

The expression for ϕ_1 and the corresponding boundary conditions contain as unknowns A_1, B_1, and c_1. For consistency, however, A_1 must be zero since the coefficient of the term $r^2 I_0(\alpha r)$ has already been chosen as unity. Thus two unknowns remain to be determined through use of the two boundary conditions.

Now writing ϕ_1 as

$$\phi_1 = \phi_{1p} + B_1 \phi_{1h}$$

and defining two linear operators by

$$L_1 \phi = c_0 \phi' - 2\phi$$

and

$$L_2 \phi = \phi''' - (1 + 2i\alpha)\phi'' + (1 + 2i\alpha - \alpha^2)\phi',$$

solving for c_1 yields

$$c_1 = \frac{1}{\phi_0'} \left\{ \frac{L_2 \phi_{1p} L_1 \phi_{1h}}{L_2 \phi_{1h}} - L_1 \phi_{1p} \right\}.$$

Letting $\Delta c = i\alpha R\, c_1$ denote the second term in the expansion for c, after some simplification, we have

$$\Delta c = \frac{i\alpha R\, I_1 \left\{ \alpha (I_0^2 - I_1^2) - i I_0^2 \right\} \left\{ L_1 \phi_{1\beta} - \frac{i (c_0 \alpha I_0 - 2 I_1)}{2\alpha^3 I_1} L_2 \phi_{1\beta} \right\}}{I_0^4 + \alpha (I_0^2 - I_1^2)^2} . \tag{4-35}$$

Considering α large it is easy to show that the imaginary part of Δc is negative; for intermediate values of α, it is necessary to resort to numerical computation to determine the sign of Δc. Computation fails to uncover a value of α for which $\mathrm{Im}(\Delta c)$ is zero or positive. We conclude that the system is stable for small Reynolds number.

D. Solution for Large Reynolds Number

For the case of large Reynolds number it is convenient to again consider the differential equation (3-24) in terms of the vorticity, i.e., to rewrite (3-24) in the equivalent form

$$\phi'' - \frac{1}{r}\phi' - \alpha^2 \phi = \Phi \tag{4-19}$$

and

$$\Phi'' - \frac{1}{r}\Phi' - \left\{ \alpha^2 - i\alpha R (c - w) \right\} \Phi = 0 . \tag{4-20}$$

If a solution to (4-20) is found, a solution for ϕ (in integral form) follows by applying the method of variation of parameters. To obtain a solution of (4-20), introduce a new independent variable defined by

$$\rho = (-i\alpha R)^{1/2} r^2 \equiv \omega r^2 \tag{4-36}$$

and a parameter k defined by

$$k = \frac{\omega(1-\kappa)}{4} - \frac{\alpha^2}{4\omega} \quad . \tag{4-37}$$

Equation (4-20) then becomes

$$\Phi_{,\rho\rho} + \left\{ -\frac{1}{4} + \frac{k}{\rho} \right\} \Phi = 0. \tag{4-38}$$

This is the Whittaker equation [18]. We require only the solution that is regular at the origin. In the notation used by Buchholz [3], we have

$$\Phi(\rho) = \mathcal{M}_{k,\frac{1}{2}}(\rho)$$

where $\mathcal{M}_{k,m}(s)$ is the Whittaker function defined by

$$\mathcal{M}_{k,m}(\rho) \equiv \frac{\rho^{m+\frac{1}{2}} e^{-\frac{\rho}{2}}}{\Gamma(1+2m)} \, {}_1F_1\left[\tfrac{1}{2} + m - k ; 1 + 2m ; \rho\right]$$

with $\Gamma(1 + 2m)$ the gamma function and ${}_1F_1\ [\frac{1}{2} + m + k;\ 1 + 2m;\ s]$ the confluent hypergeometric function, or Kummer function, defined by

$${}_1F_1\left[\tfrac{1}{2} + m - k ; 1 + 2m ; \rho\right] = \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2} + m - k + m)\,\Gamma(1+2m)}{\Gamma(1+2m+m)\,\Gamma(\frac{1}{2}+m-k)} \frac{\rho^m}{m!} .$$

In terms of the original variable, Φ is

$$\Phi(r) = \mathcal{M}_{k,\frac{1}{2}}(\omega r^2) . \tag{4-39}$$

Homogeneous solutions to (4-19) are $r\,I_1(\alpha r)$ and $r\,K_1(\alpha r)$ ($K_1(\alpha r)$ is the Bessel function of the second kind, of the first order, and of imaginary argument $i\alpha r$) so that the method of variation of parameters yields for ϕ

$$\phi(r) = A\int_0^r \{ r I_1(\alpha r) K_1(\alpha a) - r K_1(\alpha r) I_1(\alpha a) \} m_{k,1/2}(\omega a^2)\, da + B r I_1(\alpha r). \tag{4-40}$$

This expression provides two independent solutions for ϕ which satisfy the regularity conditions at the origin. The solution $r I_1(\alpha r)$ can be considered as the "inviscid solution" since it is a solution of (3-24) for infinite Reynolds number.

If the two boundary conditions (3-30) and (3-31) are denoted by $B_1\phi = 0$ and $B_2\phi = 0$ respectively and $\phi^{(1)}$ and $\phi^{(2)}$ denote the linearly independent solutions given by (4-40), the secular equation is

$$S(\alpha R, \alpha, c, E, m) = \begin{vmatrix} B_1\phi^{(1)} & B_1\phi^{(2)} \\ B_2\phi^{(1)} & B_2\phi^{(2)} \end{vmatrix} = 0$$

from which, after some reduction,

$$(c\alpha I_0 - 2 I_1)\{ m_{k,1/2}(\omega r^2)_{,r} - 2i\alpha\, m_{k,1/2}(\omega r^2) \}_{r=1} + \{ i\alpha R E - m i\alpha^3 R c^2 + 2 i\alpha^3 c \} \int_0^1 I_1(\alpha a)\, m_{k,1/2}(\omega a^2)\, da = 0, \tag{4-41}$$

with $I_n = I_n(\alpha)$ $(n = 0, 1)$.

For E infinite (4-41) reduces to

$$\int_0^1 I_1(\alpha a)\, m_{k,1/2}(\omega a^2)\, da = 0$$

a result (with different notation) used by Pekeris[14] in connection with the stability of Poiseuille flow in a rigid cylinder.

The expression (4-41) involves no restrictions on the magnitudes of the various parameters. Restricting our attention now to the case of large Reynolds number, in order to solve (4-41) for the eigenvalue c, we must employ expressions for $\mathcal{M}_{k,m}(s)$ valid for large complex k and s. Corcos and Sellars [6] have indicated that the work of Pekeris in connection with the stability of flow in a rigid cylinder is incomplete. Comparison of his work with the present development indicates that the reason for the incompleteness lies in the fact that the expansion used by Pekeris is valid only for large complex s. For large Reynolds number the parameter k can be expressed as (α is assumed of order one),

$$k \sim \frac{(-i\alpha R)^{1/2}(1-c)}{4},$$

and, if we further restrict our attention to the determination of the neutral stability curve, c can be replaced by its real part c_r, so that

$$k = \frac{(-i\alpha R)^{1/2}(1-c_r)}{4}. \tag{4-42}$$

The argument of the Whittaker functions in (4-41) is

$$\Omega = (-i\alpha R)^{1/2} r^2$$

with $0 \leq r \leq 1$. The ratio

$$\frac{\Omega}{4k} = \frac{r^2}{1-c_r} \tag{4-43}$$

is therefore real for large R and real c. An asymptotic expansion of $\mathcal{M}_{k,m}(s)$ is available for this condition [3]. (This expansion is listed in the Appendix.) It is interesting to note that the asymptotic expansion for $\mathcal{M}_{k,m}(s)$ takes on different forms depending on whether the ratio

in (4-43) is less than, equal to, or greater than unity. In other words the critical point, where $1 - r^2 - c_r = 0 (s/4k = 1)$, plays an important role in the solution of the problem even though this point is an ordinary point of the differential equation. This introduction of an artificial difficulty at the critical point by the approximate method of solution is typical of hydrodynamic stability problems at large Reynolds number.

Asymptotic expansions are thus available to evaluate the Whittaker functions in (4-41); however, the evaluation of the integral involving the Whittaker function in (4-41) requires more careful consideration. A numerical quadrature is a possibility but the rapid oscillations of the integrand coupled with the fact that the form of the integrand changes three times (as (4-43) changes) over the interval of integration make a numerical quadrature an extremely difficult and delicate task. The method that will be used here avoids the difficulties of numerical quadrature by reducing the evaluation of the integral to the solution of a nonhomogeneous differential equation. An asymptotic solution to this equation valid for large R is then found.

Consider the two differential equations,

$$\Phi_{1,rr} - \frac{1}{r}\Phi_{1,r} - \alpha^2 \Phi_1 + i\alpha R (c_r - 1 + r^2)\Phi_1 = r\, I_1(\alpha r) \qquad \text{(i)}$$

and

$$\Phi_{,rr} - \frac{1}{r}\Phi_{,r} - \alpha^2 \Phi + i\alpha R (c_r - 1 + r^2)\Phi = 0. \qquad \text{(ii)}$$

Multiplying (i) by Φ, (ii) by Φ_1, and subtracting, we get

$$\Phi_{1,rr}\Phi - \Phi_{,rr}\Phi_1 - \frac{1}{r}\Phi_{1,r}\Phi + \frac{1}{r}\Phi_{,r}\Phi_1 = r\, I_1(\alpha r)\Phi,$$

or

$$\left\{ \left(\frac{\Phi_1}{R} \right)_{,R} \Phi - \left(\frac{\Phi}{R} \right)_{,R} \Phi_1 \right\}_{,R} = I_1(\alpha R)\, \Phi$$

Integration yields

$$\int I_1(\alpha R)\, \Phi(R)\, dR = \left\{ \left(\frac{\Phi_1}{R} \right)_{,R} \Phi - \left(\frac{\Phi}{R} \right)_{,R} \Phi_1 \right\} + C \quad . \qquad (4\text{-}44)$$

If we now choose

$$\Phi(R) = \mathcal{M}_{k,1/2}(\omega R^2),$$

the evaluation of the integral,

$$\int_0^1 I_1(\alpha a)\, \mathcal{M}_{k,1/2}(\omega a^2)\, da \qquad (4\text{-}45)$$

is reduced to finding and evaluating a solution of the differential equation,

$$\Phi_{1,RR} - \frac{1}{R}\, \Phi_{1,R} - \alpha^2 \Phi_1 + i\alpha R (\kappa_R - 1 + R^2) \Phi_1 = R\, I_1(\alpha R). \qquad (4\text{-}46)$$

A solution in integral form can easily be found since the homogeneous solutions are known. A solution of this form is, however, of no practical use since it requires the evaluation of integrals similar to the integral under question. We will therefore use the results given by Clark[4] for the asymptotic solution of a nonhomogeneous differential equation with a turning point. In the present problem the turning point corresponds to what we have called the critical point, i.e., the point where $c_r - 1 + r^2 = 0$. The importance of this point in the solution of the nonhomogeneous differential equation is evident from an examination of the asymptotic forms of the Whittaker functions.

For large R then the asymptotic solution for Φ_1 is

$$\Phi_1(\rho) = \frac{\lambda^{-2}\, G^3(\rho)\, \rho\, I_1(\alpha\rho)}{X(\rho)} \tag{4-47}$$

where the following definitions are employed:

$$\rho_0 \equiv (1-\kappa_\rho)^{1/2},$$

$$g(\rho) \equiv (\rho^2 - \rho_0^2)/2\rho_0,$$

$$X(\rho) \equiv \left\{\frac{3}{2}\int_{\rho_0}^{\rho} g(\alpha)^{1/2}\, d\alpha\right\}^{2/3},$$

$$\lambda^2 \equiv 2\rho_0\, i\alpha R,$$

and $$G(\rho) \equiv \left\{X(\rho)/g(\rho)\right\}^{1/4}.$$

For convenience the result for Φ_1 is established in the Appendix.

Now that Φ_1 is known, we can turn our attention to the evaluation of the integral (4-45), through the use of (4-44). Before proceeding it must be noted that in the neighborhood of $r = 0$ the asymptotic solution for Φ_1 is not valid. Examination of the power series expansions of $I_1(\alpha r)$ and $\mathcal{M}_{k,\,1/2}(\omega r^2)$ indicates, however, that the contribution to the integral near $r = 0$ is negligible so that we may write, for $1 \gg \delta > 0$,

$$\int_0^1 I_1(\alpha\rho)\, \mathcal{M}_{k,1/2}(\omega\rho^2)\, d\rho = \int_\delta^1 I_1(\alpha\rho)\, \mathcal{M}_{k,1/2}(\omega\rho^2)\, d\rho.$$

As we will see, δ need not be defined precisely. In addition, the use of an asymptotic expansion in obtaining (4-47) requires that r cannot be equal to r_o. This restriction will be discussed more fully shortly.

Now using (4-44) and introducing the notation,

$$\Phi^*(r) = \frac{\Phi_{1,r}\Phi - \Phi_{,r}\Phi_1}{r},$$

we have

$$\int_\delta^1 I_1(\alpha a)\, \mathcal{M}_{k,1/2}(\omega a^2)\, da = \Phi^*(1) - \Phi^*(\delta).$$

If we now examine the expressions for Φ and Φ_1, it is clear that $\Phi^*(1)$ is much greater than $\Phi^*(\delta)$ ($\mathcal{M}_{k,m}(\omega r^2)$ grows exponentially with the Reynolds number as r increases) so that, in keeping with the spirit of the asymptotic analysis, we write

$$\int_\delta^1 I_1(\alpha a)\, \mathcal{M}_{k,1/2}(\omega a^2)\, da = \left\{\Phi_{1,r}\Phi - \Phi_{,r}\Phi_1\right\}_{r=1}. \quad (4\text{-}48)$$

Using (4-48) in the secular equation yields

$$(c_r \alpha I_0 - 2 I_1)\left\{\mathcal{M}_{k,1/2}(\omega r^2)_{,r} - 2i\alpha\, \mathcal{M}_{k,1/2}(\omega r^2)\right\}_{r=1}$$

$$+ \left\{ i\alpha R E - i m \alpha^3 R c_r^2 + 2 i \alpha^3 c_r \right\}\left\{ \Phi_{1,r}\, \mathcal{M}_{k,1/2}(\omega r^2) \right.$$

$$\left. - \mathcal{M}_{k,1/2}(\omega r^2)_{,r}\, \Phi_1 \right\}_{r=1} = 0. \quad (4\text{-}49)$$

Equation (4-49), upon separation into real and imaginary parts, yields two equations from which α and R can be found for specified values of c_r (E and m fixed). We will now state the restrictions on c_r inherent in the present analysis. First, the assumption was made that k is large for large R; if, however, $c_r = 1$, this assumption is

no longer valid, for, with $c_r = 1$, from (4-37)

$$k = \frac{-\alpha^2 (-i\alpha R)^{1/2}}{4}$$

In addition, the assumption that r is not equal to r_o requires that c_r cannot equal zero (r equal unity is the value of interest). With these restrictions in mind and in view of the importance of the critical point in the stability of parallel flows at large Reynolds number [6], [13], we consider values of c_r such that

$$0 < c_r < 1.$$

Since

$$\mathcal{M}_{k,1/2}(\alpha)_{,\alpha} = \left\{ \frac{1}{\alpha} - \frac{k}{2} \right\} \mathcal{M}_{k,1/2}(\alpha) - \left\{ \frac{k^2-1}{2} \right\} \mathcal{M}_{k,3/2}(\alpha),$$

equation (4-49) can be simplified; after neglecting smaller order terms (k and ω are of order $(\alpha R)^{1/2}$) and with some rearrangement, the simplified (in the sense of large Reynolds number) result for (4-49) is

$$(c_r \alpha I_0 - 2 I_1) \left\{ -k\omega \left[\mathcal{M}_{k,1/2}(\omega r^2) + k \mathcal{M}_{k,3/2}(\omega r^2) \right]_{r=1} \right\}$$

$$= -\left\{ i\alpha RE - m i \alpha^3 R c_r^2 \right\} \left\{ \Phi_{1,r} \mathcal{M}_{k,1/2}(\omega r^2) \right.$$

$$\left. + \Phi_1 k\omega \left[\mathcal{M}_{k,1/2}(\omega r^2) + k \mathcal{M}_{k,3/2}(\omega r^2) \right] \right\}_{r=1}. \qquad (4\text{-}50)$$

Examination of the result for Φ_1 indicates that the imaginary part of (4-50) is satisfied if the imaginary part of the factor

$$(k\omega)^{-1} \left\{ 1 + k \frac{\mathcal{M}_{k,3/2}(\omega)}{\mathcal{M}_{k,1/2}(\omega)} \right\}^{-1} \qquad (4\text{-}51)$$

is zero. To evaluate this expression it is necessary to use the proper asymptotic expansion for the Whittaker functions. From (4-43), with r equal unity,

$$\frac{\Omega}{4k} = \frac{1}{1-\kappa_R}$$

or, recalling the restrictions on c_r, we see that

$$1 < \frac{\Omega}{4k} < \infty .$$

We are now able to determine the appropriate asymptotic expansions needed to evaluate (4-51). Since we are dealing with the ratio of two Whittaker functions, there is a considerable cancellation of common factors. In fact, we have the simple result

$$k \frac{m_{k,3/2}(\omega)}{m_{k,1/2}(\omega)} = -\left\{1 + Q k^{-1}\right\}$$

where

$$Q = Q_1(\gamma)\Big|_{m=\frac{3}{2}} - Q_1(\gamma)\Big|_{m=\frac{1}{2}} .$$

The function $Q_1(\gamma)$ is defined in the Appendix. Here we need only be concerned with the fact that Q is positive and real and dependent only on c_r. The expression (4-51) now becomes

$$(k\omega)^{-1}\left\{1 + k \frac{m_{k,3/2}(\omega)}{m_{k,1/2}(\omega)}\right\}^{-1} = -(Q\omega)^{-1} .$$

From (4-36)

$$\omega = (\alpha R)^{1/2} e^{-i\frac{\pi}{4}}$$

so that the condition for the satisfaction of the imaginary part of (4-50) is

$$Im\left\{Q^{-1}(\alpha R)^{-1/2} e^{i\frac{\pi}{4}}\right\} = 0.$$

This condition is satisfied only for αR equal to infinity. Since α was assumed of order one, the neutral stability curve in the α - R plane, then, lies at $R = \infty$. It is important to note that this result is valid for all E, including the classical case of the rigid wall ($E = \infty$). The result thus provides an independent demonstration of the stability relative to infinitesimal disturbances of the laminar flow of a viscous fluid in a pipe.

V. RESULTS AND CONCLUSIONS

In the present work an analysis on the stability relative to infinitesimal disturbances of laminar flow in a flexible elastic cylinder was performed. Such an analysis is of interest because of its relation to the phenomenon of transition from laminar flow to turbulent flow. If the infinitesimal disturbances grow with time, the primary laminar flow will be modified by the interdependent disturbances until that flow loses all trace of steadiness. Since there appears to be no stable laminar flow between the primary laminar flow and turbulent flow, the presumption is made that the growth with time (instability) of infinitesimal disturbances signifies ultimately the onset of turbulence.

The results of the stability analysis indicate that laminar flow in an elastic cylinder is stable relative to infinitesimal axisymmetric disturbances for all finite Reynolds number. These results also hold for the classical problem of laminar flow in a pipe. Thus the boundary flexibility fails to introduce any instability. In view of the fact that a free surface is generally a destabilizing factor so that, for such flows, instability can occur for all finite Reynolds number [21], these results might, at first glance, appear surprising. However, it must be noted that there are distinct differences between the flexible boundary of the present problem and a free surface. The instability occurring in a flow with a free surface is a gravity wave while in the confined flow considered here, even though the boundary is flexible, gravity plays no role in the stability problem. In addition, at a flexible boundary, the primary flow velocity is zero, whereas with a free surface the opposite is true. The vanishing of the primary flow velocity at the flexible boundary probably tends to minimize the effect of the boundary.

Since plane Poiseuille flow exhibits instability at large Reynolds

number, the reason for the absence of instability in axisymmetric Poiseuille flow is a matter of some interest. Corcos [5] has stated that the reason for this difference lies in the fact that for axisymmetric Poiseuille flow the inviscid solution of the Orr-Sommerfeld equation (see (4-40)) is regular in the interval of interest while for plane Poiseuille flow the inviscid solutions have singularities at the critical points. In this connection, it is worth mentioning that for plane Couette flow, which is generally considered to be stable [17], the inviscid solutions are also regular in the interval of interest. More study is needed to completely resolve this question.

From the results presented, the following conclusions can be drawn:

a) Axisymmetric flow in a pipe is stable for axisymmetric infinitesimal disturbances.

b) Axisymmetric flow in an elastic cylinder is also stable for axisymmetric infinitesimal disturbances so that the boundary flexibility fails to introduce any instability.

VI. SUGGESTIONS FOR FURTHER WORK

Since only axisymmetric disturbances were considered in the present work, a logical extension would consider the effect of asymmetric disturbances. Since the experimental work of Leite [12] in connection with Poiseuille flow in a rigid cylinder has shown that the asymmetric part of a disturbance decays more rapidly than the axisymmetric part, it would be interesting to perform an analytical study of this problem to attempt to verify his results.

The present analysis could be carried out for other types of wall properties. This would not greatly change the formulation of the problem, but would complicate somewhat the algebra involved in the calculations.

It would be interesting to consider the stability of flow in an elastic cylinder relative to large disturbances. In this case the results from linear elasticity would no longer apply. Since this flow is stable relative to infinitesimal disturbances, the point at which disturbances become "large" enough to cause instability is a matter of some interest. A complete answer to this question would entail a detailed examination of the factors causing the disturbance and would therefore require a more complete analysis.

An interesting variation of this problem would consider the stability of a non-Newtonian fluid in a flexible cylinder. Such a problem would more closely approximate the flow of blood.

APPENDICES

APPENDIX A

Stability of Poiseuille Flow Relative to Disturbances in Planes Perpendicular to the Axis of the Cylinder

The differential equation governing the disturbances in planes perpendicular to the axis of the cylinder is (3-25):

$$\tilde{v}'' + \frac{1}{r}\tilde{v}' - \left\{\alpha^2 + \frac{1}{r^2} + i\alpha R(1 - r^2 - c)\right\}\tilde{v} = 0. \quad \text{(A-1)}$$

The boundary conditions on v require that it vanish at the cylinder wall and at the axis of the cylinder, i.e.,

$$\tilde{v}(1) = \tilde{v}(0) = 0.$$

Making the substitutions

$$\tilde{v} = \frac{\bar{v}}{r}, \qquad x = r^2,$$

(A-1) becomes

$$\bar{v}_{,xx} + \left\{\frac{i\alpha R}{4} - \frac{\alpha^2 + i\alpha R(1-c)}{4x}\right\}\bar{v} = 0 \qquad \text{(A-2)}$$

while the boundary conditions are

$$\bar{v}(1) = 0, \qquad \frac{\bar{v}(x)}{x^{1/2}} = 0 \quad \text{at} \quad x = 0.$$

To show that this particular type of disturbance is stable, we must show that c_i is negative. Multiplying (A-2) by the complex conjugate of $\bar{v}$, denoted by $\bar{v}^*$, and integrating over zero to one, after using

the boundary conditions, the result is

$$-J_1^2 + \frac{i\alpha R}{4} J^2 - \left\{ \frac{\alpha^2}{4} + \frac{i\alpha R(1-c)}{4} \right\} J_2^2 = 0, \qquad \text{(A-3)}$$

where

$$J_1^2 = \int_0^1 \bar{v}_{,x} \bar{v}_{,x}^* \, dx, \qquad J^2 = \int_0^1 \bar{v} \bar{v}^* \, dx$$

and

$$J_2^2 = \int_0^1 \frac{\bar{v} \bar{v}^*}{x} \, dx.$$

Solving (A-3) for c_i yields

$$c_i = -\frac{\alpha^2}{\alpha R} - \frac{4}{\alpha R} \frac{J_1^2}{J_2^2}$$

which indicates that c_i is negative and that the disturbance under consideration is stable. This result has previously been given by Synge [16] and Pekeris [14].

APPENDIX B

Asymptotic Expansions for the Whittaker Functions

The asymptotic expansion of the Whittaker functions takes on different forms depending on the value of the ratio

$$\frac{\Omega}{4k} = \frac{R^2}{1-\kappa_R}$$

where

$$\Omega = (-i\alpha R)^{1/2} R^2$$

and

$$k = \frac{(-i\alpha R)^{1/2}(1-\kappa_R)}{4}$$

Case I: $1 < r^2/(1-c_r) < \infty$

$$M_{k,m}(\Omega) \sim (2\pi \tanh\gamma)^{-1/2} \exp\left\{-m\log k + \frac{m(4m^2-1)}{24k^2}\right\}$$

$$\times \left[2\sin\pi\left(\tfrac{1}{2}+m-k\right)\exp\left\{k(\sinh 2\gamma - 2\gamma)\right\}\left\{1 + \frac{Q_1(\gamma)}{k}\right.\right.$$

$$\left.\left. + \frac{Q_2(\gamma)}{k^2}\right\} + \exp\left\{\pi i\left(k-\tfrac{1}{2}-m\right) - k(\sinh 2\gamma - 2\gamma)\right\}\left\{1\right.\right.$$

$$\left.\left. - \frac{Q_1(\gamma)}{k} + \frac{Q_2(\gamma)}{k^2}\right\}\right]$$

where

$$\frac{\Omega}{4k} = \cosh^2 \gamma .$$

Case II: $0 < r^2/(1 - c_r) < 1$

$$M_{k,m}(\Omega) \sim \sqrt{2}\,(\pi \tan \gamma)^{-1/2} k^{-m} \exp\left\{\frac{m(4m^2-1)}{24k^2}\right\}$$

$$\times \left[\sin\left\{k(\pi - 2\gamma + \sin 2\gamma) - \pi m + \frac{\pi}{4}\right\}\left\{1 + \frac{Q_2(i\gamma)}{k^2}\right\}\right.$$

$$\left. - \cos\left\{k(\pi - 2\gamma + \sin 2\gamma) - \pi m + \frac{\pi}{4}\right\}\left\{i\,\frac{Q_1(i\gamma)}{k}\right\}\right]$$

where

$$\frac{\Omega}{4k} = \cos^2 \gamma .$$

In Cases I and II, we have neglected terms of order k^{-3} and higher and the functions $Q_1(\gamma)$ and $Q_2(\gamma)$ are given by

$$Q_1(\gamma) = \frac{1}{96 \tanh^3 \gamma}\left\{3(16m^2-1)\tanh^4 \gamma - 6\tanh^2 \gamma + 5\right\}$$

and

$$Q_2(\gamma) = \frac{1}{18432 \tanh^6 \gamma}\left\{9(16m^2-9)(16m^2-1)\tanh^8 \gamma\right.$$

$$+ 36(48m^2-7)\tanh^6 \gamma + 6(121-112m^2)\tanh^4 \gamma$$

$$\left. - 924\tanh^4 \gamma - 924\tanh^2 \gamma + 385\right\}.$$

Case III: $r^2/(1-c_r) \approx 1$

$$\mathcal{M}_{k,m}(a) \sim \frac{\sqrt{2}}{\pi}\Gamma\left(\tfrac{1}{3}\right)\left(\frac{a}{6}\right)^{1/6} k^{-m}$$

$$\times \left\{ \sin \pi\left(m-k+\tfrac{2}{3}\right)\left[1+\frac{\varepsilon\left(\varepsilon^2+\frac{1}{6}\right)}{3!\,\frac{8}{3}}\right]\right.$$

$$-\sin\pi\left(m-k-\tfrac{2}{3}\right)\Gamma\left(\tfrac{5}{6}\right)\pi^{-1/2}\left[\frac{\varepsilon}{\left(\frac{1}{3}a\right)^{1/3}}+\frac{1}{3\cdot 3!\left(\frac{1}{3}a\right)^{4/3}}\right.$$

$$\left.\left.\times\left(\varepsilon^4+2\varepsilon^2+2m^2-\tfrac{33}{35}\right)\right]\right\}$$

where

$$a-4k=2\varepsilon.$$

APPENDIX C

Asymptotic Solution of a Nonhomogeneous Differential Equation with a Turning Point

Following the method used by Clark [4], we will establish the asymptotic solution of a nonhomogeneous differential equation with a turning point. The reader is referred to Clark's original paper for a proof that the formal solution obtained does, in fact, represent an asymptotic solution to the differential equation.

The standard form of the differential equation treated by Clark is

$$U_{,yy} + \{\lambda^2 g(y) + \lambda F_1(y,\lambda)\} U = \lambda^2 F_2(y,\lambda) \qquad \text{(C-1)}$$

where λ is a large complex parameter, y is a real variable, and the real function g has a first order zero (turning point) at y = 0. Changing the dependent and independent variables by letting

$$X(y) = \left\{ \tfrac{3}{2} \int_0^y g(a)^{1/2}\, da \right\}^{2/3}, \qquad \text{(C-2)}$$

$$G(y) = \{ X(y) / g(y) \}^{1/4},$$

and

$$U(y) = G(y)\, \tilde{U}(x),$$

the standard form becomes

$$\tilde{U}_{,xx} + \{\lambda^2 X + \lambda q(x,\lambda)\} \tilde{U} = \lambda^2 f(x,\lambda), \qquad \text{(C-3)}$$

with

$$q(X(y), \lambda) = G^4(y) F_1(y, \lambda) + \lambda^{-1} G^3(y) G_{,yyy} \quad \text{(C-4)}$$

and

$$f(X(y), \lambda) = G^3(y) F_2(y, \lambda). \quad \text{(C-5)}$$

Now assuming that q and f are represented by the series

$$q(X, \lambda) = \sum_{m=0}^{\infty} \lambda^{-m} q_m(X), \quad f(X, \lambda) = \sum_{m=0}^{\infty} \lambda^{-m} f_m(X), \quad \text{(C-6)}$$

we attempt to find a formal solution of (C-3) in the form

$$\tilde{U}(X, \lambda) = \lambda^{2/3} \sum_{m=0}^{\infty} \lambda^{-m} \{ a_m(X) S(Y)$$

$$+ \lambda^{-1/3} b_m(X) S(Y)_{,Y} + \lambda^{-2/3} c_m(X) \}, \quad \text{(C-7)}$$

where a_n, b_n, and c_n are to be determined and S(Y) ($Y = \lambda^{2/3} X$) is an appropriate solution of $S_{,YY} + YS = 1$. Substitution of (C-7) into (C-3) and collecting similar terms yields a system of differential equations for the functions a_n, b_n, and c_n:

$$2 a_m' + q_0 b_m = - b_{m-1}'' - \sum_{k=0}^{m-1} q_{m-k} b_k,$$

$$2 X b_m' + b_m - q_0 b_m = a_{m-1}'' + \sum_{k=0}^{m-1} q_{m-k} a_k,$$

$$X c_m = f_m - a_m - 2 b_{m-1}' - c_{m-2}'' - \sum_{k=0}^{m-1} q_{m-k-1} c_k$$

(prime signifies differentiation with respect to X). Thus to solve any given equation of the standard form, we need only solve the differential equation recurrence system (f_n and q_n are known) and reverse the transformations of (C-2).

For the present problem, we require a solution of (4-46)

$$\Phi_{1,rr} - \frac{1}{r}\Phi_{1,r} - \alpha^2\Phi_1 + i\alpha R(c_r - 1 + r^2)\Phi = r I_1(\alpha r). \quad \text{(C-8)}$$

Letting

$$\Phi_1 = U r^{1/2}, \quad y = r - r_0 \quad \text{(C-9)}$$

($r_o = (1 - c_r)^{1/2}$), (C-8) becomes

$$U_{,yy} + \left\{ i\alpha R 2r_0 \left[y + \frac{y^2}{2r_0} \right] - \left[\alpha^2 + \frac{3}{4(y+r_0)^2} \right] \right\} U$$

$$= (y + r_0)^{1/2} I_1(\alpha(y + r_0)). \quad \text{(C-10)}$$

Comparison with (C-1) yields

$$\lambda^2 = i\alpha R 2r_0,$$

$$g(y) = (y + y^2/2r_0),$$

$$\lambda F_1(y, \lambda) = -\left\{ \alpha^2 + \frac{3}{4(y+r_0)^2} \right\}, \quad \text{(C-11)}$$

and $$\lambda^2 F_2(y, \lambda) = (y + r_0)^{1/2} I_1(\alpha(y + r_0)).$$

By comparing (C-11) with (C-4) and (C-5), it is easy to see that the series (C-6) take on extremely simple forms

$$g(X, \lambda) = \lambda^{-1} g_1(X), \quad f(X, \lambda) = \lambda^{-2} f_2(X).$$

Solving for a_n, b_n, and c_n we find that for $n = 0, 1$, these functions are zero, while for $n = 2$, we have

$$a_2(X) = f_2(0), \quad b_2(X) = 0, \quad c_2(X) = \frac{f_2(X) - f_2(0)}{X}.$$

Now making use of (C-7) with (C-2), the solution to (C-10) is

$$U(y) = G(y) \lambda^{-4/3} \left\{ f_2(0)\, S(Y(y)) + \lambda^{-2/3} \left[\frac{f_2(X(y)) - f_2(0)}{X(y)} \right] \right\} + O(\lambda^{-7/3}), \tag{C-12}$$

where G(y) and X(y) are obtained from (C-11) and (C-2). From (C-11), (C-5), and (C-6), f_2 is given by

$$f_2(X(y)) = G^3(y)(y + r_0)^{1/2} I_1(\alpha(y + r_0))$$

so that (C-12) becomes

$$U(y) = \lambda^{-4/3} \left\{ r_0^{1/2} I_1(\alpha r_0)\, S(\lambda^{2/3} X(y)) + \lambda^{-2/3} \frac{\left[G^3(y)(y + r_0)^{1/2} I_1(\alpha(y + r_0)) - r_0^{1/2} I_1(\alpha r_0) \right]}{X(y)} \right\}$$

(C-13)

with

$$X(y) = \left\{ \frac{3}{2} \int_0^y g(a)^{1/2}\, da \right\}^{2/3},$$

$$g(y) = (y + y^2/2r_0),$$

and $$G(y) = \{ X(y) / g(y) \}^{1/4}.$$

For y not equal to zero, $\lambda^{2/3}X(y)$ is large; in this case, as Clark shows, $S(\lambda^{2/3}X(y))$ has the asymptotic expansion:

$$S(\lambda^{2/3} X(y)) \sim \frac{1}{\lambda^{2/3} X(y)} + O(\lambda^{-8/3}).$$

The final result for U(y) is, therefore,

$$U(y) = \lambda^{-2} \frac{G^3(y)(y + r_0)^{1/2}\, I_1(\alpha(y + r_0))}{X(y)}.$$

Using (C-9), the solution for $\Phi_1(r)$ is

$$\Phi_1(r) = \frac{\lambda^{-2} G^3(r)\, r\, I_1(\alpha r)}{X(r)}, \qquad \text{(C-14)}$$

with

$$g(r) = (r^2 - r_0^2)/2r_0,$$

$$X(r) = \left\{ \frac{3}{2} \int_{r_0}^r g(a)^{1/2}\, da \right\}^{2/3},$$

$$\lambda^2 = 2r_0 i\alpha R,$$

and $$G(r) = \{ X(r) / g(r) \}^{1/4}.$$

It is instructive to consider the accuracy of our result. Clark shows that if $\tilde{U}(X, \lambda)$ is a solution of (C-3) while $\tilde{U}_N(X, \lambda)$ represents the formal solution (C-7) truncated at $n = N$ (in the present case, $N = 2$) then there exists, for $|\lambda|$ large, a number M_N independent of X and λ such that

$$| \tilde{U}(X, \lambda) - \tilde{U}_N(X, \lambda)| \leq M_N |\lambda|^{-N-\frac{1}{3}}.$$

We conclude that the result (C-14) is accurate to order $(\alpha R)^{-7/6}$.

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