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ON THE CONNECTEDNESS OF DIRECTED GRAPHS UNDER BINARY OPERATIONS

by

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of the requirements for the degree of
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PREFACE

The author would like to take this opportunity to thank his parents for making possible much of his college education and his wife for her understanding patience.

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CHAPTER I -- SUMMARY

This study is primarily concerned with the investigation of two closely related problems in the theory of directed graphs. Both problems involve the mutually exclusive categories of connectedness; C_0 (disconnected), C_1 (strictly weak), C_2 (strictly unilateral), and C_3 (strong).

The first problem is the determination of the possible categories of connectedness to which a directed graph, D, may belong when D is represented as the image, under some binary operation, of two directed graphs D_1 and D_2, each of which belongs to a specific category of connectedness.

The second problem is, then, the determination of necessary and sufficient conditions on D_1 and D_2 which will insure that D is in a particular category of connectedness among those possible.

Such an investigation depends, of course, upon the operation involved; and this study is limited to the consideration of three specific operations: the tensor product, the cartesian product, and composition. These operations have the property that the point set of the image of (D_1,D_2) under each operation is always the cartesian product of those of D_1 and of D_2.

These problems are solved completely for two of the operations under consideration, namely the cartesian pro-

duct and composition.

It is found that the category of connectedness to which the cartesian product $D = D_1 \times D_2$ belongs is entirely determined by the categories of connectedness to which D_1 and D_2 belong. Indeed, with but one exception, if D_1 is in C_i and D_2 is in C_j, then D is in C_k, where k is the minimum of i and j. The exception occurs when $i = j = 2$ and $k = 1$.

Hence, the solution of the first problem is also a solution of the second, in the case of the cartesian product.

Excluding trivial (one-point) directed graphs, it is found that, with two exceptions, the composition $D = D_1[D_2]$ is in C_k if and only if D_1 is in C_k. The two exceptions occur when D_1 is in C_2 and D_2 is in C_0 and C_1, respectively. In both these cases, $k = 1$ or 2, and $k = 2$ if and only if every point of D_1 belongs to a cycle of positive length in D_1.

The third operation, tensor product, is considerably more involved. In this instance, again with two exceptions, the tensor product $D = D_1 \boxtimes D_2$ is in C_k where k may assume any value in the set $\{ k' : k' \leq \min(i,j) \}$ if D_1 is in C_i and D_2 is in C_j. The first exception occurs when $i = j = 2$. Here, k may take only the values 0 or 1. The second exception occurs when $i = j = 3$, and here, k is restricted to the values 0 or 3. Hence, the first problem is completely solved for the tensor product.

The second problem for this operation, however, pre-

sents several difficulties; and it is solved only for $k = 2$ and $k = 3$.

For example, if $k = 3$, then from the above, one has $i = j = 3$. Thus, D_1 and D_2 are strong and may be written as unions of cycles of lengths $a_1,\ldots,a_t$ in D_1 and of lengths $b_1,\ldots,b_s$ in D_2. Excluding the possibility that D_1 or D_2 is a trivial directed graph, it is found that $D = D_1 \times D_2$ has $a_0 = g.c.d.(a_1,\ldots,a_t,b_1,\ldots,b_s)$ strong components which are also the weak components of D. Therefore, D is in C_3 if and only if D_1 and D_2 are in C_3 and $a_0 = 1$.

The problem is solved for $k = 2$ by investigating the receiving and transmitting subgraphs of the tensor product and using the above result.

These questions have been investigated for graphs. However, since a graph may be looked upon as being a symmetric directed graph, one has, in this instance, only to deal with the categories C_0 and C_3, and hence one might expect certain simplifications.

Indeed, such simplifications occur, but even for graphs the problem is sometimes fairly substantial. An example of this is a recent article by P. M. Weichsel wherein the operation under consideration is the Kronecker, or tensor, product. The interesting nature of his results prompted the present study, and Chapters III and IV are devoted to this particular operation.

Chapter V is devoted to the cartesian product and Chapter VI to composition.

CHAPTER II -- INTRODUCTION

This section has a two-fold purpose. Initially, the basic notation and definitions used in the body of the manuscript are introduced, and secondly, the final paragraphs are devoted to a general discussion of the problems dealt with in Chapters III, IV, V and VI.

2.1 DEFINITIONS AND NOTATION

For the sake of completeness, the following basic definitions are given.

DEFINITION 2.1A.

(1) A net N is an ordered 4-tuple $N = \langle X, V, f, s \rangle$ where X and V are sets, V is non-empty, and (when $X \neq \emptyset$) f and s are functions from X into V. If $X = \emptyset$, then N is considered to be the set V.

(2) X is called the set of lines of N and is denoted $X(N)$.

(3) V is called the set of points of N and is denoted $V(N)$.

It should be remarked that when no confusion is apt to arise, the points and lines of a net N' are denoted V', X', the points and lines of a net N_1 are denoted V_1, X_1, and so forth.

(4) Given x in X, $f(x) = v$ is called the first point of x, and $s(x) = v'$ is called the second point of x.

(5) x in X is called a loop if $f(x) = s(x)$.

(6) If x and x' are in X, and $x \neq x'$, then x is said to be parallel to x' if $f(x) = f(x')$ and $s(x) = s(x')$. Clearly, this relation is symmetric.

(7) A subnet N' of $N = \langle X,V,f,s\rangle$ is any 4-tuple $\langle X',V',f',s'\rangle$ where $X' \subset X$, $V' \subset V$, $f' = f|X'$, and $s' = s|X'$ that is itself a net.

(8) A subnet N' of a net N is said to span N if $V' = V$. This is denoted $N' \prec N$.

DEFINITION 2.1B.

(1) A directed graph (digraph) D is a net which possesses no loops and no parallel lines.

Note that, in this case, if x is a line of a digraph D, then the ordered pair $\langle f(x),s(x)\rangle$ completely determines x. Thus, a digraph may be thought of as a collection of points, V, and a distinguished subcollection of the set $V \times V - \Delta$ where Δ is the diagonal of V, that is $\Delta = \{(v,v) : v \text{ is in } V\}$.

(2) A subnet D' of a digraph D is easily seen to be a digraph, and is called a subgraph of D.

DEFINITION 2.1C. A net is said to be symmetric if, whenever x is in X, with $f(x) = v$ and $s(x) = v'$, then there exists a line x' in X such that $f(x') = v'$ and $s(x') = v$.

DEFINITION 2.1D. A multigraph is a symmetric net, and a graph is a symmetric digraph.

It is perhaps desirable to note that a graph is

often defined as a set of points, V, and a subcollection X of $V \times V - \Delta$ called lines, where as above, Δ is the diagonal of V. Here again, a line x is uniquely determined by an unordered pair of points. The correspondence $(v,v') \text{ --- } (\langle v,v' \rangle , \langle v',v \rangle)$ establishes the interchangeability of the definition with Definition 2.1D.

DEFINITION 2.1E. A net N is said to be finite if both V and X are finite sets. For a finite digraph D, $p = p(D)$ is used to denote the number of points of D and $q = q(D)$ the number of lines of D.

In keeping with the above notation, p' and q' are used to denote the number of points and lines of D', p_1 and q_1 of D_1, and so forth, whenever no confusion is apt to arise.

Throughout, finiteness is assumed, and the word digraph is taken to be synonymous with finite digraph. A similar convention is adopted for graphs.

DEFINITION 2.1F. Let D be a digraph and let $v_1,v_2,\ldots,v_n$ be in V (not necessarily distinct). Then

(1) $v_1,v_2,\ldots,v_n$ is said to be a sequence of length n-1 of D from v_1 to v_n if $\langle v_i,v_{i+1} \rangle$ is in X for all $i = 1,2,\ldots,n-1$. It is said to be a semi-sequence of length n-1 of D between v_1 and v_n if $\langle v_i,v_{i+1} \rangle$ or $\langle v_{i+1},v_i \rangle$ is in X for $i = 1,2,\ldots,n-1$. Thus every sequence is a semi-sequence.

A semi-sequence is closed if $v_1 = v_n$. Otherwise

it is open.

(2) $v_1, v_2, \ldots, v_n$ is said to be a path of length n-1 of D from v_1 to v_n if it is a sequence of length n-1 from v_1 to v_n with the property that $i \neq j$ implies $v_i \neq v_j$. Similarly, it is said to be a semi-path of length n-1 of D between v_1 and v_n if it is a semi-sequence of length n-1 between v_1 and v_n with the property that $i \neq j$ implies $v_i \neq v_j$.

(3) $v_1, v_2, \ldots, v_n$ is said to be a cycle of length n-1 in D if $v_1, v_2, \ldots, v_{n-1}$ is a path from v_1 to v_{n-1}, $v_1 = v_n$ and $\langle v_{n-1}, v_n \rangle$ is in X. It is said to be a semi-cycle of length n-1 in D if, $v_1, v_2, \ldots, v_{n-1}$ is a semi-path between v_1 and v_{n-1}, $v_1 = v_n$ and $\langle v_{n-1}, v_n \rangle$ or $\langle v_n, v_{n-1} \rangle$ is in X.

DEFINITION 2.1G. Let D be a digraph and $S \subset V$. Then the subgraph $\langle S \rangle$ of D defined as follows

$V(\langle S \rangle) = S$, and

$X(\langle S \rangle) = \{\langle u, u' \rangle : u, u' \in S \text{ and } \langle u, u' \rangle \in X\}$,

is called the subgraph generated by S. Thus, $D = \langle V \rangle$.

DEFINITION 2.1H.

(1) A digraph D is said to be weakly connected (weak) if, given any two distinct points v and v' of D, there exists a semi-path of D between v and v'. Otherwise, D is called disconnected.

(2) Let a relation $\sim$ be defined on V, where $v \sim v'$ if and only if there exists a v --- v' semi-path (that is, a semi-path of D between v and v').

Then clearly $\sim$ is an equivalence relation. If T is a $\sim$-equivalence class, and $W = \langle T \rangle$, then W is called a weak component of D.

W is a weak component of D if and only if W is a maximal weak subgraph of D. Also, if W_1 and W_2 are two weak components of D such that $V(W_1) \neq V(W_2)$ (in which case they are disjoint), then $X(W_1) \cap X(W_2) = \emptyset$. Further, if $W_1, W_2, \ldots, W_r$ are all the weak components of D, then $V = \bigcup_i V(W_i)$ and $X = \bigcup_i X(W_i)$.

(3) A digraph D is said to be unilaterally connected (unilateral) if, given any two distinct points v and v' of D, then there exists a $v \dashrightarrow v'$ path (that is, a path of D from v to v') or there exists a $v' \dashrightarrow v$ path in D.

A subgraph U of D is called a unilateral component of D if it is a maximal unilateral subgraph of D. Contrary to above, however, unilateral components of a given digraph need not be pointwise or linewise disjoint.

(4) A digraph is said to be strongly connected (strong) if, given any two distinct points v and v' of D, there exists a $v \dashrightarrow v'$ path in D and a $v' \dashrightarrow v$ path in D.

(5) Let $\approx$ be a relation defined on V by $v \approx v'$ if and only if there exists a $v \dashrightarrow v'$ path and a $v' \dashrightarrow v$ path in D. Note that this includes the case $v = v'$. Then clearly $\approx$ is an equivalence relation. If

T is a $\approx$-equivalence class and $S = \langle T \rangle$, then S is called a strong component of D. Thus, S is a strong component of D if and only if S is a maximal strong subgraph of D.

Here, as in (2), $S_1 = \langle T_1 \rangle$, $S_2 = \langle T_2 \rangle$ and $T_1 \neq T_2$ implies $X(S_1) \cap X(S_2) = \emptyset$. Further, if $S_1, S_2, \ldots, S_r$ are all the strong components of D, then $V = \bigcup_i V(S_i)$, but $\bigcup_i X(S_i)$ is not necessarily all of X.

DEFINITION 2.1I. A digraph D is called strictly weak if it is weak but not unilateral, and it is called strictly unilateral if it is unilateral but not strong.

DEFINITION 2.1J. Let $\mathfrak{D}$ be the set of all digraphs, and let $C_0 = \{D : D \in \mathfrak{D}$ and D is disconnected$\}$,

$C_1 = \{D : D \in \mathfrak{D}$ and D is strictly weak$\}$,

$C_2 = \{D : D \in \mathfrak{D}$ and D is strictly unilateral$\}$,

$C_3 = \{D : D \in \mathfrak{D}$ and D is strong$\}$.

The C_i $(i = 0,1,2,3)$ are called the connectedness categories of $\mathfrak{D}$.

Note that $\bigcup_i C_i = \mathfrak{D}$ and that $i \neq j$ implies $C_i \cap C_j = \emptyset$.

DEFINITION 2.1K. Let g be a map of $\mathfrak{D} \times \mathfrak{D}$ into $\mathfrak{D}$. Then g is called a binary digraph operation (or, briefly, operation).

2.2 GENERAL STATEMENT OF THE PROBLEM

This study will concern itself almost exclusively with operations g wherein $V(g(D_1,D_2)) = V_1 \times V_2$.

However, the general problem may be formulated without such a restriction. Briefly, and quite simply, it is this.

Suppose that g is a fixed operation. Then

(I) For each pair (i,j), $0 \leqq i,j \leqq 3$, determine the possible values of k for which there exists (D_1,D_2) in $C_i \times C_j$ with $g(D_1,D_2)$ in C_k. For each such k, call the triple (i,j,k) a <u>possible triple</u>.

(II) For each possible triple (i,j,k), find necessary and sufficient conditions on D_1 and D_2 which will insure that $g(D_1,D_2)$ is in C_k.

In the case of graphs, the situation is considerably simpler. A graph, thought of as a symmetric digraph, is clearly either disconnected or strong. These two possibilities are then called disconnected and connected respectively.

Many specific operations on graphs have been investigated. This has been done by various persons, among whom are Harary, [2,3,4]; Izbicki, [5]; Berge, [1]; Sabidussi, [7,8]; Weichsel, [9]; König, [6].

However, digraph operations have been investigated very little. Berge defines both the tensor product (produit) and cartesian product (somme) of digraphs. He

relates several applications of these operations, and raises various questions pertaining to their structure. But, the major source of information on digraph operations is probably an appendix to the manuscript, [4], by Harary, Norman and Cartwright, wherein a large number of digraph operations are defined and discussed generally.

CHAPTER III -- TENSOR PRODUCTS OF DIGRAPHS, CONNECTEDNESS

3.1 INTRODUCTION TO THE TENSOR PRODUCT

3.1.1 Equivalence of two standard definitions

The following definition is the basis of this and the next chapter.

DEFINITION 3.1A. The tensor product of two digraphs D_1 and D_2, denoted $D_1 \boxtimes D_2$, is that digraph whose points and lines are specified by

$V = V_1 \times V_2$, and

$X = \{ \langle (u,v),(u',v') \rangle : (u,v),(u',v') \in V, \langle u,u' \rangle \in X_1$ and $\langle v,v' \rangle \in X_2 \}$.

Numerous examples of the tensor product will be found in Section 3.7.

As an immediate consequence of this definition, one sees that, for the operation of tensor product, $p = p_1 p_2$ and $q = q_1 q_2$.

Also, when a digraph D is looked upon as an irreflexive relation on V, the tensor product of digraphs corresponds to the direct product of relations, wherein $(u,v)\ R_1 \boxtimes R_2\ (u',v')$ if and only if $u\ R_1\ u'$ and $v\ R_2\ v'$.

On the other hand, the tensor product may be defined in a manner which is possibly more motivated in a historical sense. This point of view was taken, for example, by Weichsel, [9], in his investigation of graphs. Since certain labeling procedures associated with this

approach are of use later, we will examine this more fully. In order to do so, the following definitions are needed.

DEFINITION 3.1B. Let D be a digraph with points $V = \{v_1, v_2, \ldots, v_p\}$. Let $A(D) = (a_{ij})$ be the p-by-p matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \langle v_i, v_j \rangle \text{ is in } X \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is called the adjacency matrix of D.

DEFINITION 3.1C. Two digraphs D and D' are said to be isomorphic if there exists a one-to-one map, μ, such that $\mu(V) = V'$ and $\langle v, v' \rangle$ is in X if and only if $\langle \mu v, \mu v' \rangle$ is in X'.

It should be noted that A completely describes D in the sense that two digraphs D and D' are isomorphic if and only if their adjacency matrices are permutationally equivalent. Further, any p-by-p matrix with zero entries on the diagonal determines a digraph uniquely to within isomorphism. Such a matrix will be symmetric if and only if its associated digraph is a graph.

DEFINITION 3.1D. The tensor product, $A \boxtimes B$, of the m-by-n matrix $A = (a_{ij})$ and the r-by-s matrix $B = (b_{ij})$ is the rm-by-sn matrix

$$A \boxtimes B = \begin{pmatrix} Ab_{11} & Ab_{12} & \cdots & Ab_{1n} \\ Ab_{21} & Ab_{22} & \cdots & Ab_{2n} \\ \vdots & \vdots & & \\ Ab_{m1} & Ab_{m2} & \cdots & Ab_{mn} \end{pmatrix}.$$

Throughout the following two chapters, we will often

use the expression $i \pmod n$ or $i(n)$ to denote the remainder of i in $\{1,2,\ldots,n\}$ on division by n. Of course, $(\mathrm{mod}\ n)$ is used in congruence relations also, but the meaning of the expression should always be clear from context.

Furthermore, we will use the expression $(i/n)^*$ to denote the smallest integer greater than or equal to i/n.

If D' is the digraph with adjacency matrix $A_1 \boxtimes A_2$, the assertion is, then, that $D_1 \boxtimes D_2$ is isomorphic with D'. In order to prove this, we need the following lemma.

LEMMA 3.1A. Let D_1 and D_2 be digraphs with adjacency matrices $A_1 = (a_{ij})$ and $A_2 = (b_{ij})$ and points $V_1 = \{u_1,\ldots,u_{p_1}\}$ and $V_2 = \{v_1,\ldots,v_{p_2}\}$, respectively. Further, let $A_1 \boxtimes A_2 = (c_{ij})$ and $n = p_1$. Then, assuming $i,j \leq p_1p_2$, the following statements are equivalent:

(1) $c_{ij} = 1$,

(2) $a_{i(n)j(n)}b_{(i/n)^*(j/n)^*} = 1$,

(3) $\langle u_{i(n)},u_{j(n)}\rangle \in X_1$ and $\langle v_{(i/n)^*},v_{(j/n)^*}\rangle \in X_2$.

PROOF. To begin with, the equivalence of (1) and (2) will be established. Let $n = p_1$. Then, $i = q_1n + r_1$ and $j = q_2n + r_2$ with $0 \leq r_1,r_2 < n$. These expressions are, of course, unique. Four cases must be considered.

(i) If $r_1 > 0$ and $r_2 > 0$, then

$c_{ij} = a_{r_1 r_2} b^{(q_1+1)(q_2+1)}$,

(ii) If $r_1 = 0$ and $r_2 > 0$, then $i = q_1 n$ and

$c_{ij} = a_{n r_2} b^{q_1 (q_2+1)}$,

(iii) If $r_1 > 0$ and $r_2 = 0$, then $j = q_2 n$ and

$c_{ij} = a_{r_1 n} b^{(q_1+1) q_2}$,

(iv) If $r_1 = 0$ and $r_2 = 0$, then $c_{ij} = a_{nn} b^{q_1 q_2}$.

Now note that in the expression $s = qt + r$, with $0 \leq r < t$, one has $r = s(t)$. Further, q is the greatest integer less than or equal to s/t. Thus, $q + 1$ is the least integer greater than or equal to s/t. The equality holds in the former (and thence the latter) if and only if $r = 0$.

Therefore, treating the case $r = 0$ subject to our convention about $i(n)$, it is clear that each of the four cases above is of the form $c_{ij} = a_{i(n)j(n)} b^{(i/n)^*(j/n)^*}$.

But, the latter expression obviously proves the equivalence of (1) and (2).

On the other hand, the equivalence of (2) and (3) is clear from the definition of the adjacency matrix since $a_{i(n)j(n)} b^{(i/n)^*(j/n)^*} = 1$ if and only if both $a_{i(n)j(n)} = 1$ and $b^{(i/n)^*(j/n)^*} = 1$.

LEMMA 3.1B. Let D_1 and D_2 be two digraphs with adjacency matrices A_1 and A_2 and points $V_1 = \{u_1, u_2, \ldots, u_{p_1}\}$ and $V_2 = \{v_1, v_2, \ldots, v_{p_2}\}$ respect-

tively. Then $D_1 \boxtimes D_2$ is isomorphic with the digraph D whose adjacency matrix is $A_1 \boxtimes A_2$.

PROOF. Since the Kronecker product of A_1 with A_2 is p_1p_2-by-p_1p_2, it is clear that both $V(D_1 \boxtimes D_2)$ and $V(D)$ have the same cardinality. Let $w_1, \ldots, w_{p_1p_2}$ be the points of D with adjacency specified by $A(D) = A_1 \boxtimes A_2 = (c_{ij})$. That is, $c_{ij} = 1$ if and only if $\langle w_i, w_j \rangle$ is in $X(D)$. By Lemma 3.1A, $c_{ij} = 1$ if and only if $\langle u_{i(n)}, u_{j(n)} \rangle \in X_1$ and $\langle v_{(i/n)^*}, v_{(j/n)^*} \rangle \in X_2$, where $n = p_1$.

Thus, let β be the map of $V(D)$ into $V(D_1 \boxtimes D_2)$ defined by $\beta : w_i \dashrightarrow (u_{i(n)}, v_{(i/n)^*})$. β is clearly a map.

If $\beta(w_i) = \beta(w_j)$, then $(u_{i(n)}, v_{(i/n)^*}) = (u_{j(n)}, v_{(j/n)^*})$, so that $i(n) = j(n)$ and $(i/n)^* = (j/n)^*$. But, $i = q_1n + r_1$ and $j = q_2n + r_2$ with $0 \leq r_1, r_2 < n$, in a unique way, so that, $i(n) = j(n)$ implies $r_1 = r_2$. Thus, when $i(n) = j(n)$ and $(i/n)^* = (j/n)^*$, one has $r_1 = r_2$ and $q_1 = q_2$, implying $i = j$ and $w_i = w_j$.

Consequently, β is one-to-one (and therefore onto).

Now, suppose that $\langle w_i, w_j \rangle$ is in $X(D)$. Then, from above, $c_{ij} = 1$, and using Lemma 3.1A, $\langle u_{i(n)}, u_{j(n)} \rangle$ is in X_1 and $\langle v_{(i/n)^*}, v_{(j/n)^*} \rangle$ is in X_2. Since digraphs have no loops, we may use the definition of $D_1 \boxtimes D_2$ to conclude that $\langle w_i, w_j \rangle$ being in $X(D)$ implies that

$\langle \beta(w_i), \beta(w_j) \rangle = \langle (u_{i(n)}, v_{(i/n)^*}), (u_{j(n)}, v_{(j/n)^*}) \rangle$ is in $X(D_1 \boxtimes D_2)$.

The converse, namely that $\langle \beta(w_i), \beta(w_j) \rangle$ being in $X(D_1 \boxtimes D_2)$ implies $\langle w_i, w_j \rangle$ is in $X(D)$, is proved by reversing the steps since Lemma 3.1A asserts an equivalence.

Thus, β is an isomorphism of $D_1 \boxtimes D_2$ and D, proving the lemma.

3.1.2 The structure of $([\mathfrak{D}], \boxtimes)$

THEOREM 3.1A. Let D_1 and D_2 be two digraphs. Then $D_1 \boxtimes D_2$ is isomorphic with $D_2 \boxtimes D_1$.

PROOF. The proof follows immediately from the well known fact that $A_1 \boxtimes A_2$ is permutationally equivalent to $A_2 \boxtimes A_1$.

It can also be shown that the map $(u,v) \dashrightarrow (v,u)$ of $V(D_1 \boxtimes D_2)$ into $V(D_2 \boxtimes D_1)$ is such an isomorphism.

Hence, if the notation of the Introduction is used, let $[\mathfrak{D}]$ denote the collection of isomorphism classes of $\mathfrak{D}$. Then the operation $\boxtimes$ may be extended to $[\mathfrak{D}]$ in the usual manner, yielding,

COROLLARY 3.1A. $([\mathfrak{D}], \boxtimes)$ is a commutative semi-group which is not a cancellation semi-group and has no identity.

PROOF. The verification that $([\mathfrak{D}], \boxtimes)$ is a

semi-group is immediate, and Theorem 3.1A yields the commutativity.

Now, suppose that [K] is the identity of $([\mathfrak{D}], \boxtimes)$. Then K in [K] implies that the cardinality of V(K) is one, since $D \boxtimes K$ must have $p(D)p(K) = p(D)$ points. But since K can have no loops, it can have no lines, and therefore $q = q_1 q_2 = 0$. Thus, suppose D is a digraph with at least one line (such a digraph clearly exists). Then $D \boxtimes K$ has no lines, and is, by assumption, D. This is a clear contradiction, and thus, [K] is no identity.

Now let D and D' be two digraphs with $p = p'$, but such that D and D' are not isomorphic. Such digraphs certainly exist. Then, if K_1 is the one-point digraph described in the Introduction, both $D \boxtimes K_1$ and $D' \boxtimes K_1$ have $p = p'$ points and no lines and are therefore isomorphic. Thus, $([\mathfrak{D}], \boxtimes)$ cannot be a cancellation semi-group.

However, even more can be said. For, if $\mathfrak{D}'$ is the set of all digraphs D for which $X = \emptyset$, then $\mathfrak{D} - \mathfrak{D}'$ still does not possess the cancellation property. This is seen in the following example.

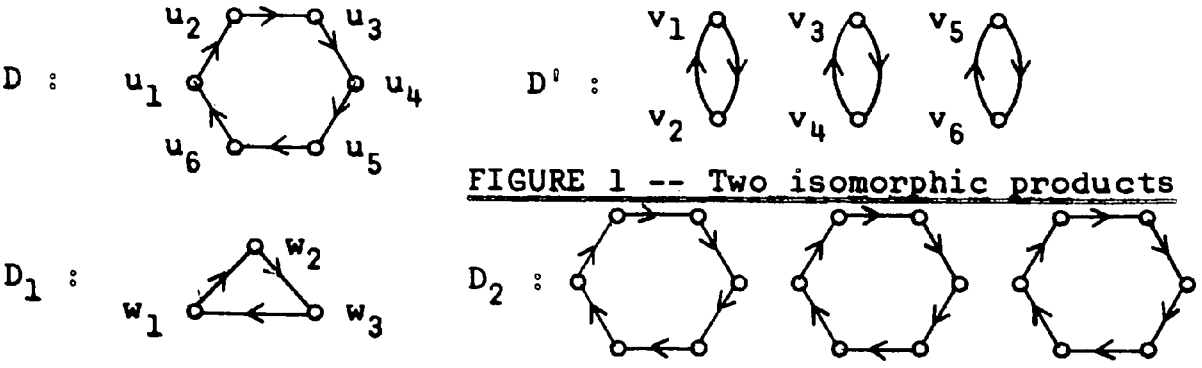


FIGURE 1 -- Two isomorphic products

Here, $D \boxtimes D_1 \approx D' \boxtimes D_1 \approx D_2$, but $D \not\approx D'$.

NOTE. The symbol K_1 will be reserved for the trivial digraph which consists of one point and no lines. Throughout this chapter and Chapter IV, it is assumed that the set of lines $X \neq \emptyset$ for each digraph $D \neq K_1$.

3.2 BASIC CONNECTEDNESS INEQUALITY AND CONSEQUENCES

Returning to the problem at hand, the following theorem gives some information regarding categories of connectedness.

THEOREM 3.2A. Let D_1 and D_2 be digraphs. If D_1 is in C_i and D_2 is in C_j, if $D_1 \boxtimes D_2$ is in C_k, then $k \leq \min(i,j)$.

PROOF. (a) Suppose that $\min(i,j) = 0$. Then using Theorem 3.1A, it may be assumed without loss of generality that $i = 0$. Then, D_1 has at least two distinct weak components. Let W_1 be a weak component of D_1, and let $B = V_1 - V(W_1)$. Clearly, by assumption, $B \neq \emptyset$. Then for all v_1 in $V(W_1)$ and v_2 in B, neither $\langle v_1,v_2\rangle$ nor $\langle v_2,v_1\rangle$ is in X_1 by definition of W_1.

Let $S = \{(v,u) : v \in B \text{ and } u \in V_2\}$ and

$T = \{(v,u) : v \in V(W_1) \text{ and } u \in V_2\}$.

From above, S and T are not empty. Further, since $V(W_1) \cup B = V_1$, one has $S \cup T = V(D_1 \boxtimes D_2)$.

The assertion is that there is no line in $X(D_1 \boxtimes D_2)$ with one end point in S and the other in T.

Suppose $\langle(v,u),(v',u')\rangle$ is in $X(D_1 \boxtimes D_2)$ with

(v,u) in S and (v',u') in T. By definition, then, $\langle v,v'\rangle$ is in X_1 with v in B and v' in $V(W_1)$. This is contrary to the construction of B.

It is clear that the same argument suffices if it is assumed that (v,u) is in T and (v',u') is in S. Thus, $D_1 \boxtimes D_2$ is in category C_0.

(b) Suppose that $\min(i,j) = 1$. Then, as above, one may assume without loss of generality that $i = 1$. So, by definition of C_1, there exist points v_1 and v_1' in V_1 with no directed $v_1 \dashrightarrow v_1'$ or $v_1' \dashrightarrow v_1$ path in D_1.

Now, let u_1 and u_1' be arbitrary points of D_2, and suppose $(v_1,u_1),(v_2,u_2),\ldots,(v_n,u_n) = (v_1',u_1')$ is a $(v_1,u_1) \dashrightarrow (v_1',u_1')$ path in $D_1 \boxtimes D_2$. Then from the definition of tensor product, all of the lines $\langle v_1,v_2\rangle,\langle v_2,v_3\rangle,\ldots,\langle v_{n-1},v_n\rangle = \langle v_{n-1},v_1'\rangle$ are in X_1. Whence, $v_1,v_2,\ldots,v_{n-1},v_1'$ is a $v_1 \dashrightarrow v_1'$ sequence in D_1. But, this implies the existence of a $v_1 \dashrightarrow v_1'$ path in D_1, contrary to supposition.

A similar argument suffices to show that for any u_1 and u_1' in V_2, there does not exist a $(v_1',u_1') \dashrightarrow (v_1,u_1)$ path in $D_1 \boxtimes D_2$. Thus, $D_1 \boxtimes D_2$ is in C_k, $k \leqq 1$.

(c) Suppose $\min(i,j) = 2$, and, without loss of generality, let $i = 2$. Then there exist points v_1 and v_1' in V_1 such that there is no $v_1 \dashrightarrow v_1'$ path in D_1.

Taking u_1 and u_1' in V_2, and supposing, as in (b), that $(v_1,u_1),(v_2,u_2),\ldots,(v_{n-1},u_{n-1}),(v_1',u_1')$ is a $(v_1,u_1) \dashrightarrow (v_1',u_1')$ path in $D_1 \boxtimes D_2$ yields (in the same manner) a conclusion that there exists a $v_1 \dashrightarrow v_1'$

path in D_1: a contradiction.

Thus, there is no $(v_1,u_1) \dashrightarrow (v_1',u_1')$ path in $D_1 \times D_2$, so that $D_1 \times D_2$ is in C_k with $k \leq 2$.

(d) When $\min(i,j) = 3$, the result is clear.

Thus, the theorem is proved.

This obviously implies that if D is not weak, then $D \times D'$ is not weak; if D is not unilateral, then $D \times D'$ is not unilateral; and if D is not strong, then $D \times D'$ is not strong, regardless of the nature of D'.

As a corollary to the proof of the preceding theorem, one has the following.

COROLLARY 3.2A. Let D_1 and D_2 be digraphs. Then there exists a $(v,u) \dashrightarrow (v',u')$ sequence in $D_1 \times D_2$ if and only if there exist a $v \dashrightarrow v'$ sequence in D_1 and a $u \dashrightarrow u'$ sequence in D_2 both having the same length.

PROOF. The necessity is clear since if $(v,u) = (s_0,t_0),(s_1,t_1),\cdots,(s_n,t_n) = (v',u')$ is a $(v,u) \dashrightarrow (v',u')$ sequence in $D_1 \times D_2$, then, as in the preceding proof, one finds that

$v = s_0,s_1,\cdots,s_n$ is a $v \dashrightarrow v'$ sequence of length n in D_1, and

$u = t_0,t_1,\cdots,t_n = u'$ is a $u \dashrightarrow u'$ sequence of length n in D_2.

The sufficiency follows at once from the definition of tensor product.

Since a path is a sequence, and since the existence of a $(u,v) \dashrightarrow (u',v')$ sequence in $D_1 \blacksquare D_2$ implies the existence of a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \blacksquare D_2$, one can also state

COROLLARY 3.2B. There exists a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \blacksquare D_2$ if and only if there exist $u \dashrightarrow u'$ and $v \dashrightarrow v'$ sequences of the same length in D_1 and D_2 respectively.

3.3 THE STRUCTURE OF PATHS AND CYCLES

3.3.1 Definitions

Let us note the following. If D is a digraph and $P: v_0,v_1,\ldots,v_n$ is a $v_0 \dashrightarrow v_n$ path of D, then one can associate a unique subgraph $\ll P \gg$ of D with P in the following way. P is a path if and only if the lines $x_i = \langle v_{i-1},v_i \rangle$ exist for $i = 1,\ldots,n$ and the points v_j are distinct. Thus, given any path P, we may define

$$V(\ll P \gg) = \{v_0,v_1,\ldots,v_n\} \quad \text{and}$$

$$X(\ll P \gg) = \{x_1,x_2,\ldots,x_n\}.$$

The association of P with $\ll P \gg$ is clearly unique.

In the same way, if $L: v_0,v_1,\ldots,v_{n-1},v_0$ is a cycle in D, then $P: v_0,v_1,\ldots,v_{n-1}$ is a $v_0 \dashrightarrow v_{n-1}$ path in D. A subgraph $\ll L \gg$ is associated with L in the following manner.

$$V(\ll L \gg) = V(\ll P \gg) \quad \text{and}$$

$$X(\ll L \gg) = X(\ll P \gg) \cup \{\langle v_{n-1}, v_0 \rangle\} .$$

Here, the cycle L and the cycles L^i: $v_i, \ldots, v_{n-1}, v_0, \ldots, v_i$ $i = 1, \ldots, n-1$, are in some sense the "same", and the association of $\ll L \gg$ with $\{L, L^1, \ldots, L^{n-1}\}$ is obviously unique.

DEFINITION 3.3A. When P is a path of D of length n and $\ll P \gg = D$ (as it surely may), then D is called the path of length n and is denoted P_n. When, on the other hand, L is a cycle of D of length n and $\ll L \gg = D$, then D is called the cycle of length n and is denoted L_n.

Note that P_n has $n+1$ distinct points and L_n has n distinct points while both have n distinct lines. Often, P and L will be used to denote both the subgraphs and the sequences, but no confusion should arise.

DEFINITION 3.3B. The union of digraphs D_1 and D_2, denoted $D_1 \cup D_2$, is the digraph D whose points and lines are given by $V = V_1 \cup V_2$ and $X = X_1 \cup X_2$. The union is called disjoint whenever $V_1 \cap V_2 = \emptyset$.

3.3.2 Paths

We now consider some special digraphs.

THEOREM 3.3A. Let $n \geq m$. Then,

$$P_n \boxplus P_m = (n-m+1)P_m + 2P_{m-1} + 2P_{m-2} + \cdots + 2P_1 + 2K_1$$

where $+$ denotes disjoint union, the coefficients of the P_j's denote the number of copies of P_j in the union, and

K_1 is the digraph with a single point and no lines.

PROOF. Before giving a precise proof, some plausibility will be demonstrated.

Consider P_n and P_m to be labeled as shown at the right. Then consider the points of $P_n \times P_m$ to be arranged as is shown below.

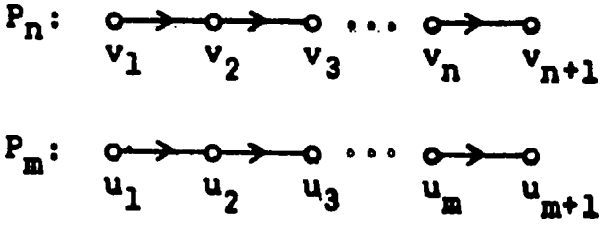


FIGURE 2 -- P_n and P_m

In the following illustration, it is understood that (i,j) represents the point (v_i,u_j).

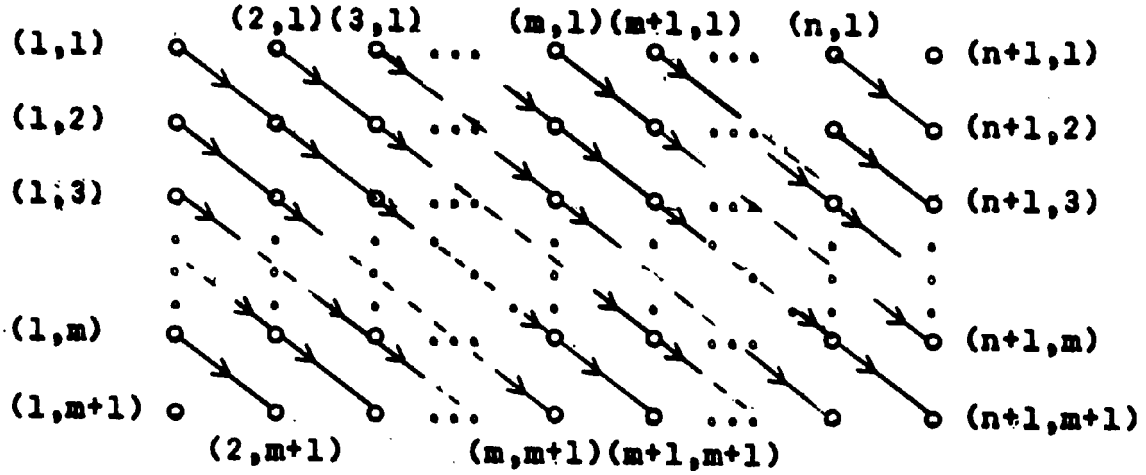


FIGURE 3 -- $P_n \times P_m$

Each path indicated which ends at points $(m+1,m+1)$, $\ldots,(n+1,m+1)$ has length m. Those "preceding" and "following" this group have decreasing lengths, and the points $(1,m+1)$ and $(n+1,1)$ are isolated. These paths must exist, by the definition of tensor product; but are there any others?

Since $n \geq m$ by assumption, $n = qm + r$ with

$q \geq 1$ and $0 \leq r < m$.

To simulate the above situation rigorously, let P_n and P_m be labeled as follows:

$V(P_n) = \{v_1, \ldots, v_{n+1}\}$, $X(P_n) = \{\langle v_1, v_2\rangle, \langle v_2, v_3\rangle, \ldots, \langle v_n, v_{n+1}\rangle\}$,

$V(P_m) = \{u_1, \ldots, u_{m+1}\}$, $X(P_m) = \{\langle u_1, u_2\rangle, \langle u_2, u_3\rangle, \ldots, \langle u_m, u_{m+1}\rangle\}$.

Then consider the following paths in $P_n \boxtimes P_m$.

Group I

(v_1, u_{m+1})

$(v_1, u_m), (v_2, u_{m+1})$

$\circ \quad \circ \quad \circ$

$(v_1, u_j), (v_2, u_{j+1}), \ldots, (v_{m-j+1}, u_m), (v_{m-j+2}, u_{m+1})$

$\circ \quad \circ \quad \circ$

$(v_1, u_2), (v_2, u_3), \ldots, (v_{m-1}, u_m), (v_m, u_{m+1})$

$(v_1, u_1), (v_2, u_2), \ldots, (v_m, u_m), (v_{m+1}, u_{m+1})$

Group II

$(v_2, u_1), (v_3, u_2), \ldots, (v_{m+1}, u_m), (v_{m+2}, u_{m+1})$

$\circ \quad \circ \quad \circ$

$(v_{km+r'}, u_1), (v_{km+r'+1}, u_2), \ldots, (v_{(k+1)m+r'+1}, u_{m+1})$

$\circ \quad \circ \quad \circ$

$(v_{n-m+1}, u_1), (v_{n-m+2}, u_2), \ldots, (v_n, u_m), (v_{n+1}, u_{m+1})$

$\circ \quad \circ \quad \circ$

(v_{n+1}, u_1)

It is evident that each of these is in $P_n \boxtimes P_m$ by the definition of tensor product, and that they are, indeed, paths since their projections are paths.

In Group I, if (v_i, u_j) is on one of these paths P, then $i \leq j$, and also P contains the subpath

$(v_i,u_j),(v_{i+1},u_{j+1}),\ldots,(v_{i+k+1},u_{m+1})$ where $j + k = m$. Thus, k is completely determined by j since m is given. Whence, $v_{i+k+1} = v_{i+m-j+1}$ is completely determined by i and j. But since each path in Group I has a distinct point of the form (v_s,u_{m+1}), it follows that the paths in Group I are pointwise disjoint. That is, if two such paths both contain the point (v_i,u_j), then they both contain the point $(v_{i+m-j+1},u_{m+1})$, implying the paths are the same.

Similarly, suppose that (v_i,u_j) is on a path in Group II. Then $i \geq j$. If $j = 1$, then there is clearly only one such path containing (v_i,u_j). If $j \geq 1$, then any path containing (v_i,u_j) must also contain the point (v_{i-1},u_{j-1}) preceding it by construction of these paths. Indeed, let $j - k = 1$. Then, by construction, a path containing (v_i,u_j) must also contain the subpath $(v_{i-k},u_1),(v_{i-k+1},u_{j-k+1}),\ldots,(v_i,u_j)$. Thus, $v_{i-k} = v_{i-(j-1)}$ is completely determined by i and j, and clearly the term (v_{i-k},u_1) completely determines the path in Group II.

Therefore, as for Group I, the paths in Group II are pointwise disjoint.

However, in Group I, $i \leq j$, and in Group II $i \geq j$. So, the paths in Group I are pointwise disjoint from those of Group II, and consequently, all the paths are pointwise disjoint.

On the other hand, it is clear that these paths' are

(in order, from top to bottom),

$K_1, P_1, P_2, \ldots, P_{m-1}, P_m, \ldots, P_m, P_{m-1}, P_{m-2}, \ldots, P_1, K_1$, where the number of P_m's is $n-m+1$; that is, from the path with "first point" (v_1, u_1) through the path with "first point" (v_{n-m+1}, u_1), since $km + r'$ assumes all the values between 1 and $n-m$.

Thus the number of distinct points in these paths is

$2[m + (m-1) + \ldots + 1] + (n-m+1)(m+1) = (n+1)(m+1)$, that is, all the points of $P_n \boxtimes P_m$.

Evidently, it remains only to show that if $\langle (v_i, u_j), (v', u') \rangle$ is a line in $P_n \boxtimes P_m$, then this line belongs to one of the above paths.

To see this, the definition of tensor product assures us that $\langle v_i, v' \rangle$ and $\langle u_j, u' \rangle$ are lines in P_n and P_m respectively. But, this implies that $v' = v_{i+1}$ and $u' = u_{j+1}$ by the labeling of P_n and P_m. Then, since (v_i, u_j) is on some path by the above, (v_{i+1}, u_{j+1}) must follow it by construction of the paths. But then, by definition of a path, the line $\langle (v_i, u_j), (v', u') \rangle = \langle (v_i, u_j), (v_{i+1}, u_{j+1}) \rangle$ is on this path, as was to be shown.

This concludes the proof of Theorem 3.3A.

3.3.3 Cycles and the structure of $L_n \boxtimes L_m$

Referring to Definition 3.3A, we state the following theorem.

THEOREM 3.3B. Let L_n and L_m be cycles. Then, $L_n \boxtimes L_m = (m,n)L_{[m,n]}$, where (m,n) is the g.c.d. and $[m,n]$ the l.c.m. of m and n.

PROOF. Here again, an intuitive description will be given before proceeding to a rigorous proof. Suppose L_n and L_m are as shown at the right. Below is the point set of $L_n \boxtimes L_m$ arranged conveniently, where (i,j) represents (v_i,u_j). It is supposed that $n \geq m$.

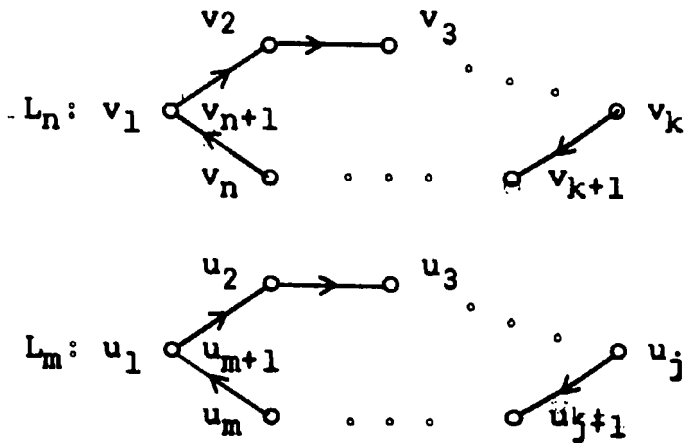


FIGURE 4 -- L_n and L_m

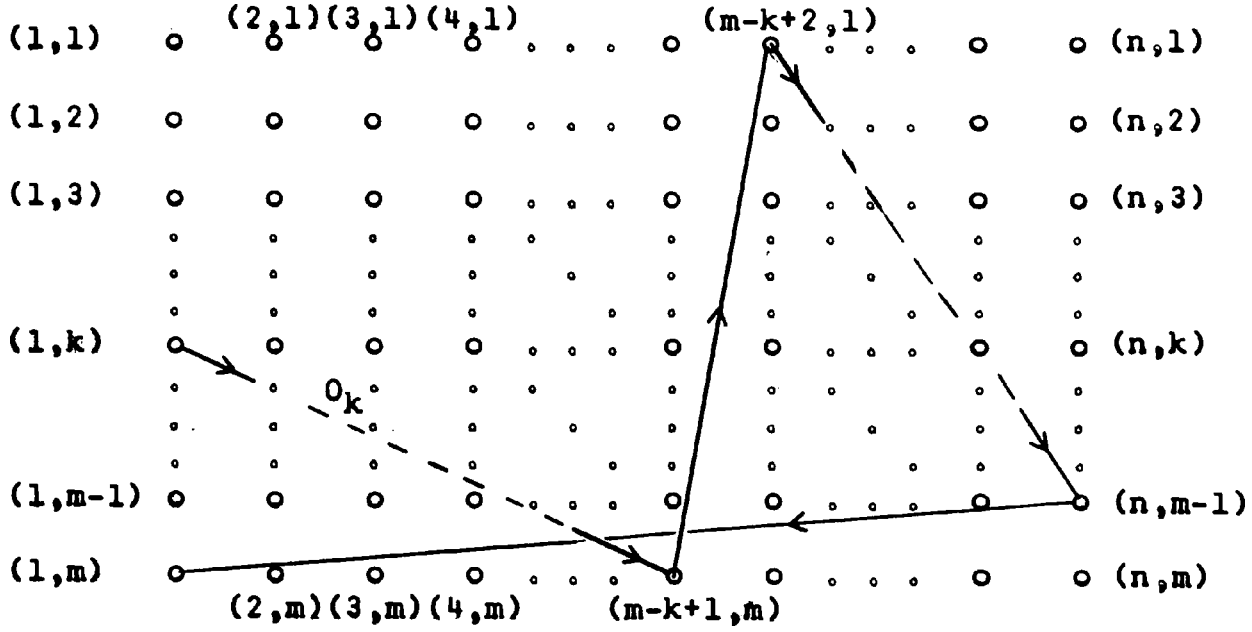


FIGURE 5 -- O_k in $V(L_n \boxtimes L_m)$

With this picture in mind, let us label L_n and L_m as follows:

L_n: $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ and

L_m: $u_1, u_2, \ldots, u_m, u_{m+1} = u_1$.

Consider paths in $L_n \boxtimes L_m$ of the following form:

O_k: $(v_1, u_k), (v_2, u_{k+1}), \ldots, (v_{m-k+1}, u_m), (v_{m-k+2}, u_1), \ldots$

$\ldots, (v_{am-k+j}, u_{j-1}), \ldots, (v_{q(k)m-k+r(k)}, u_{r(k)-1})$,

where the general term is (v_{am-k+j}, u_{j-1}) for judicious choice of a and j (that is, restricting j so that $2 \leq j \leq m+1$), $n = q(k)m-k+r(k)$ where $2 \leq r(k) \leq m+1$, for all k, $1 \leq k \leq m$. Note, as with the general j, $r(k)$ can always be chosen to be between 2 and $m+1$, inclusive.

Each O_k has length $n-1$ since the v's are all distinct.

Further, let $x_k = \langle (v_n, u_{r(k)-1}), (v_1, u_{r(k) \pmod m}) \rangle$, and X_k be the subgraph composed of the line x_k and its end points.

Then,

LEMMA 3.3A. $L_n \boxtimes L_m = [\bigcup_1^m O_k] \cup [\bigcup_1^m X_k]$.

PROOF. First, it is evident that O_k is completely specified when k is given, and that

$r(k) = r(t)$ if and only if $k = t$. . . . . (*)

The latter is so since $n+k = q(k)m+r(k)$, $r(k) \leq m+1$

and $n+t = q(t)m+r(t)$, $r(t) \leq m+1$.

Thus, $r(k) = r(t)$ implies

$n-r(k) = [q(k)-1]m + (m-k) = [q(t)-1]m + (m-t)$, where,

since $k,t \geq 1$, $m-k$ and $m-t$ are less than m. The uniqueness of this expression is guaranteed by the division algorithm, and therefore $m-k = m-t$, or $k = t$.

On the other hand, $k = t$ obviously implies that $r(k) = r(t)$.

Suppose now that the point (v_i,u_j) is on O_k and O_t. Then by construction, they both contain the subpath

$$(v_i,u_j),(v_{i+1},u_{j+1}),\ldots,(v_{i+s},u_m),(v_{i+s+1},u_1),\ldots,(v_{m+i+s},u_m),$$

$$(v_{m+i+s+1},u_1),\ldots,(v_{am+i+s+b},u_b)$$

where $am+i+s+b = n$. But then, $r(k)-1 = b = r(t)-1$, so that $r(k) = r(t)$ and, from (*), $k = t$.

Thus, the paths O_k are pairwise pointwise disjoint.

But, since each O_k has length $n-1$ (and therefore has n distinct points), and since there are m such paths, one must have $V(\bigcup_1^m O_k) = V(L_n \boxtimes L_m)$.

Now, let $x = \langle(v_i,u_j),(v',u')\rangle$ be a line in $X(L_n \boxtimes L_m)$. Then the following cases are possible.

(1) $v' = v_{i+1}$ and $u' = u_{j+1}$

(2) $v' = v_{i+1}$ and $u' = u_1$ (here, $u_j = u_m$)

(3) $v' = v_1$ and $u' = u_{j+1}$ (here, $v_i = v_n$)

(4) $v' = v_1$ and $u' = u_1$ (here, $v_i = v_n$ and $u_j = u_m$).

In case (1), x is on some O_k by the previous part. That is, (v_i,u_j) is on some O_k and, by construction of O_k, it must contain (v_{i+1},u_{j+1}) which is (v',u'). Hence O_k contains the line x.

In case (2), consider the path $(v_i,u_m),(v_{i+1},u_1),\ldots,(v_{m+i},u_m),(v_{m+i+1},u_1),\ldots,(v_{am+i+b},u_b)$, where $am+i+b = n$. Then $b = r(k)-1$ for some k since $r: \{1,2,\ldots,m\} \rightarrow \{2,3,\ldots,m+1\}$ is one-to-one and onto. But, by construction then, O_k contains the subpath above and therefore the line x.

In case (3), x has the form $\langle(v_n,u_i),(v_1,u_{i+1})\rangle$. But again, since r is one-to-one and onto, $i = r(k)-1$ for some k, and x is, by definition, x_k.

In case (4), x is of the form $\langle(v_n,u_m),(v_1,u_1)\rangle$. But this is clearly also some x_k.

Thus in each case, x is in $X\{[\bigcup_1^m O_k] \cup [\bigcup_1^m X_k]\}$. Whence, $L_n \cap L_m = [\bigcup_1^m O_k] \cup [\bigcup_1^m X_k]$, proving the lemma.

Now, consider $r(k)$. Let $M = \{1,2,\ldots,m\}$ and $M' = \{2,3,\ldots,m+1\}$. Then by definition of r and by (*), r is a one-to-one map of M onto M' as noted above. Thus, $r \pmod m$ is a one-to-one map of M onto M, considering $r \pmod m: k \rightarrow r(k) \pmod m$.

In this way, $r(r(k) \pmod m)$ is defined for all k in M. This procedure may be repeated, and the expression $r(r(r\cdots r(k))) \pmod m$ will be taken to mean $r(r(r\cdots r(k) \pmod m) \pmod m) \pmod m) \pmod m$.

DEFINITION 3.3C. Let $r^i(k)$ be defined to be $r(r(r\cdots r(k))) \pmod m$ where r appears i times, $0 \leq i \leq m$, and $r^0(k)$ is understood to be k.

LEMMA 3.3B. $r^i(k) = r^j(k)$ for $i \geq j$, if and only if $r^{i-j}(k) = k$.

PROOF. Consider $r^{i-1}(k)$ and $r^{j-1}(k)$. Since $r(r^{i-1}(k))$ and $r(r^{j-1}(k))$ are in M' (above) and assumed equal when reduced modulo m, they must be equal in M'. Thus, by (*), above, $r^{i-1}(k) = r^{j-1}(k)$ in M.

By a finite argument, then, $r^{i-j+1}(k) = r^1(k)$, where $r^1(k) = r(k) \pmod m$. Then, as above, $r(r^{i-j}(k)) \equiv r(k) \pmod m$, and hence they are equal in M'. Thus, applying (*), $r^{i-j}(k) = k$

On the other hand, the sufficiency is clear.

LEMMA 3.3C. Suppose $j \in M - \{m\}$. Then, $r^i(k) \neq k$ for all i, $1 \leq i \leq j \leq m$, if and only if

$$Y_j(k) = O_k \cup X_k \cup O_{r^1(k)} \cup X_{r^1(k)} \cup \cdots \cup O_{r^{j-1}(k)} \cup X_{r^{j-1}(k)}$$

is a path in $L_n \boxplus L_m$. Otherwise, Y_{i_o} is a cycle with $i_o n$ distinct lines and points, where $i_o \leq j$ is the least member of M such that $r^{i_o}(k) = k$.

PROOF. (a) From Lemma 3.3B, it follows that $r^i(k) \neq k$ for all i, $1 \leq i \leq j$, if and only if $r^s(k) \neq r^t(k)$ for all $s \neq t$, $0 \leq s,t \leq j$.

Thus, $r^i(k) \neq k$ for all i, $1 \leq i \leq j$, is equivalent to the statement that $k, r^1(k), \ldots, r^j(k)$ are all distinct. But, evidently, $O_k, O_{r^1(k)}, \ldots, O_{r^j(k)}$ and $X_k, X_{r^1(k)}, \ldots, X_{r^j(k)}$ are distinct if and only if $k, r^1(k), \ldots, r^j(k)$ are distinct.

Since $Y_j(k)$ is clearly a sequence by construction, $r^i(k) \neq k$ for all i, $1 \leq i \leq j$, if and only if $Y_j(k)$ is a path. In this case, it obviously has length jn.

(b) Let i_o be the least $i \leq j$ such that $r^{i_o}(k) = k$. Then $r^i(k) \neq k$ for all i, $1 \leq i \leq i_o-1$. Using part (a), then, $Y_{i_o-1}(k)$ is a path of length $(i_o-1)n$. Further, $k, r^1(k), \ldots, r^{i_o-1}(k)$ are distinct, and thus, $0_k, \ldots, 0_{r^{i_o-1}(k)}$ are distinct. Hence, $Y' = Y_{i_o-1}(k) \cup 0_{r^{i_o-1}(k)}$ is a path of length $(i_o-1)n + (n-1)$.

But, $r^{i_o}(k) = k$ implies that the line $x_{r^{i_o-1}(k)} = \langle (v_n, u_{r(r^{i_o-1}(k))-1}), (v_1, u_{r^{i_o}(k)}) \rangle =$ $= \langle (v_n, u_{r(r^{i_o-1}(k))-1}), (v_1, u_k) \rangle$, where the latter point, (v_1, u_k), is the first point of 0_k. Thus, $Y_{i_o}(k) = Y' \cup X_{r^{i_o-1}(k)}$ is a cycle of length $(i_o-1)n + (n-1) + 1 = i_o n$, as claimed.

This completes the proof of Lemma 3.3C.

It should be noted that, using the obvious induction, in the latter case above, $Y_j(k)$, for $j > i_o$, is a sequence obtained by "retracing" lines of $Y_{i_o}(k)$, that is, a sequence whose distinct lines are precisely those of $Y_{i_o}(k)$.

Also, any two such cycles in $L_n \times L_m$ are either the same, or they are pointwise disjoint, since a point in common implies some 0_k in common, which in turn implies that they are the same cycle by the remark of the preceding paragraph.

LEMMA 3.3D. $an = bm$ for $0 < a < m$, if and only if $r^a(k) = k$ for all k, $1 \leq k \leq m$.

PROOF. Suppose $an = bm$, $0 < a < m$. Then from a previous expression, $n = q(r^i(k))m - r^i(k) + r(r^i(k))$ for all i and k, so that,

$an = aq(r^i(k))m + a[r(r^i(k)) - r^i(k)] = bm$. Whence one has $a[r(r^i(k)) - r^i(k)] = m[b - aq(r^i(k))]$. Thus,

$$a \cdot \sum_{i=0}^{a-1} [r(r^i(k)) - r^i(k)] = m \cdot \sum_{i=0}^{a-1} [b - aq(r^i(k))] =$$

$$= mab - ma \cdot \sum_{i=0}^{a-1} q(r^i(k)). \qquad (**)$$

Now, $\sum_{i=0}^{a-1} [r(r^i(k)) - r^i(k)] =$

$$= \sum_{i=0}^{a-1} [r(r^i(k)) - r^{i+1}(k)] + [r(r^{a-1}(k)) - k].$$

But, either $r(r^i(k)) = r^{i+1}(k)$ or $r(r^i(k)) = r^{i+1}(k) + m$ (the latter only when $r(r^i(k)) = m+1$). Therefore,

$\sum_{i=0}^{a-2} [r(r^i(k)) - r^{i+1}(k)] = dm$ with $0 < d < a$.

Thus, cancelling $a \neq 0$ in (**) yields:

$r(r^{a-1}(k)) - k = bm - dm - m \cdot \sum_{i=0}^{a-1} q(r^i(k))$, so that,

$r(r^{a-1}(k)) - k \equiv 0 \pmod{m}$.

Clearly then, $r^a(k) - k \equiv 0 \pmod{m}$. But, $r^a(k)$ and k are in M, and therefore $r^a(k) - k$ must be zero. This proves the necessity, since k was arbitrary.

Conversely, if $a < m$ and $r^a(k) = k$ for all k, then $a \cdot n = m \cdot \sum_{i=0}^{a-1} q(r^i(k)) + \sum_{i=0}^{a-1} [r(r^i(k)) - r^i(k)]$

$$= m\cdot\sum_{i=0}^{a-1} q(r^i(k)) + dm + [r(r^{a-1}(k)) - k]$$

$$= m\cdot\sum_{i=0}^{a-1} q(r^i(k)) + dm + sm = bm, \text{ where } s \text{ is } 0 \text{ or } 1.$$

The last expressions were obtained by summing the above expression, $n = q(r^i(k))m - r^i(k) + r(r^i(k))$, from 0 to a-1, which one can do since $0 \prec a \prec m$. Thus, $an = bm$, proving the lemma.

From the last two lemmas, the following basic result is obviously derived.

LEMMA 3.3E. Suppose that $(m,n) \succ 1$ and a_o is m divided by the greatest common factor of m and n. Then $Y_{a_o}(k) = 0_k \cup X_k \cup 0_{r1(k)} \cup \cdots \cup 0_{ra_o-1(k)} \cup X_{ra_o-1(k)}$ is a cycle of length $a_o n$ in $L_n \boxtimes L_m$ for every k in M.

On the other hand, if $(m,n) = 1$, then $Y_a(k) = 0_k \cup X_k \cup 0_{r1(k)} \cup \cdots \cup 0_{ra-1(k)} \cup X_{ra-1(k)}$ is a path of length a·n in $L_n \boxtimes L_m$ for every k in M and all $a \prec m$.

PROOF. The proof is immediate.

Clearly in the latter case, $0_{rm-1(k)}$ will be distinct from all 0_j in $Y_{m-1}(k)$ since $k, r^1(k), \ldots, r^{m-1}(k)$ are all distinct (because $Y_{m-1}(k)$ is a path). But, since r is one-to-one and onto, and since $r^{m-1}(k) \neq r^j(k)$ for all j, $0 \leq j \leq m-2$, and $r^j(k) \neq k$, for all j, $1 \leq j \leq m-1$, one must have, using previous results, $r(r^{m-1}(k)) \neq r^j(k)$ for all j, $1 \leq j \leq m-1$, and therefore

$r(r^{m-1}(k))$ must be congruent to k modulo m. However, this clearly implies $r^m(k) = k$, so that the sequence $Y_{m-1}(k) \cup O_{r^{m-1}(k)} \cup X_{r^{m-1}(k)}$ is a cycle of length mn, and therefore, $L_n \boxtimes L_m = L_{mn} = (m,n)L_{[m,n]}$, where $(m,n) = 1$ and $[m,n] = mn$.

In the former case of Lemma 3.3E, since a_o is m divided by the greatest common factor of m and n, one has $(a_o,b_o) = 1$, where $a_o n = b_o m$. Thus, since a_o divides m, and $a_o n = [m,n]$ is the length of every cycle $Y_{a_o}(k)$, and since any two of these cycles are the same or pointwise disjoint, there must be precisely m/a_o such distinct cycles. But, $m/a_o = (m,n)$ by hypothesis, so that once again, $L_n \boxtimes L_m = (m,n)L_{[m,n]}$.

Now, although we assumed $m \leq n$, the symmetry of the result clearly implies the result stated in Theorem 3.3B. That is, for arbitrary n and m, $L_n \boxtimes L_m = (m,n)L_{[m,n]}$.

This completes the proof of Theorem 3.3B.

3.4 COMMUTATIVITY OF UNION AND TENSOR PRODUCT

It becomes necessary to derive some results about digraph unions.

THEOREM 3.4A. Let $D = \bigcup_1^n D_i$ and $D' = \bigcup_1^m D'_j$ where $D_1,\ldots,D_n$ are subgraphs of D and $D'_1,\ldots,D'_m$ are subgraphs of D'. Then $D \boxtimes D' = \bigcup_{i,j} [D_i \boxtimes D'_j]$.

PROOF. Let $\underline{D} = \bigcup_{i,j} [D_i \boxtimes D'_j]$. Then by definition

of union, $V(\underline{D}) = \bigcup_{i,j} \{ (v_m, v'_n) : v_m \in V(D_i), \ v'_n \in V(D'_j) \}$.

Thus, $V(\underline{D}) = \{ (v,v') : v \in \bigcup_i V(D_i) \text{ and } v' \in \bigcup_j V(D'_j) \}$.

But, $V = \bigcup_i V(D_i)$ and $V' = \bigcup_j V(D'_j)$, so that one obtains $V(\underline{D}) = \{ (v,v') : v \in V \text{ and } v' \in V' \} = V(D \boxtimes D')$.

Now, suppose $x = \langle (v_1,u_1),(v_2,u_2) \rangle$ is in $X(\underline{D})$. Then x is a line in, say, $D_i \boxtimes D'_j$, whence $\langle v_1,v_2 \rangle \in X_i$ and $\langle u_1,u_2 \rangle \in X'_j$. But since $X = \bigcup_i X_i$ and $X' = \bigcup_j X'_j$, $\langle v_1,v_2 \rangle \in X$ and $\langle u_1,u_2 \rangle \in X'$, so that x is in $X(D \boxtimes D')$. Hence $X(\underline{D}) \subset X(D \boxtimes D')$.

The converse is equally clear. For, suppose that x (as above) is in $X(D \boxtimes D')$. Then, $\langle v_1,v_2 \rangle \in X$ and $\langle u_1,u_2 \rangle \in X'$. But since $X = \bigcup_i X_i$, $\langle v_1,v_2 \rangle \in X_i$ for some i. Similarly, $\langle u_1,u_2 \rangle \in X'_j$ for some j. Thus, x is in $X(D_i \boxtimes D'_j)$ which is clearly a subset of $X(\underline{D})$. Hence $X(D \boxtimes D') \subset X(\underline{D})$.

Therefore, $X(\underline{D}) = X(D \boxtimes D')$, completing the proof of the theorem.

COROLLARY 3.4A. If $D = \bigcup_i^n L_{a_i}$ and $D' = \bigcup_j^m L'_{b_j}$, where $L_{a_1},\ldots,L_{a_n}$ are cycles of D and $L'_{b_1},\ldots,L'_{b_m}$ are cycles of D', then $D \boxtimes D' = \bigcup_{i,j} [L_{a_i} \boxtimes L'_{b_j}]$.

3.5 POSSIBLE CATEGORIES WHEN D AND D' ARE STRONG

The result above leads to the following theorem.

It is well known (see for example [3]) that a digraph D is in C_3 if and only if D is weak and can be written as a union of cycles (that is, every point and line of D is on a cycle).

THEOREM 3.5A. If $D \in C_3$ and $D' \in C_3$, then $D \boxtimes D' \in C_0 \cup C_3$.

PROOF. Since $D \in C_3$ implies that D is a union of cycles (and similarly for D'), one can apply the above corollary to get $D \boxtimes D' = \bigcup_{i,j} [L_{a_i} \boxtimes L'_{b_j}]$ where the notation should be clear.

But either $L_{a_i} \boxtimes L'_{b_j}$ is a cycle or a disjoint union of cycles from Theorem 3.3B. Thus, $D \boxtimes D'$ is a union of cycles.

If $D \boxtimes D'$ is weak, then it is strong, using the remark above. But, if $D \boxtimes D'$ is not weak, it is in C_0. Therefore, $D \boxtimes D'$ is in $C_0 \cup C_3$ as claimed.

3.6 POSSIBLE CATEGORIES WHEN D AND D' ARE STRICTLY UNILATERAL

DEFINITION 3.6A. Let D be a digraph and S a set of points $S \subsetneq V$. The coboundary of S, denoted $U(S)$, is the set of lines x, of D, having the property that if $x = \langle u,v \rangle$ then either (i) $u \in S$ and $v \in V - S$ or (ii) $v \in S$ and $u \in V - S$. Two lines x and x' of the coboundary $U(S)$ of S are called similarly directed when $f(x) \in S$ if and only if

$f(x') \in S$.

DEFINITION 3.6B. Let D be a digraph with $S \subsetneq V$, and let $U(S)$ be the coboundary of S. Then we let $U_0(S) = \{x : x \in U(S) \text{ and } f(x) \in S\}$ and $U_I(S) = \{x : x \in U(S) \text{ and } s(x) \in S\}$. Note that these sets are the equivalence classes of $U(S)$ under the equivalence relation "similarly directed".

DEFINITION 3.6C. Let D be a digraph with $S \subsetneq V$. Then, S (and $<S>$) is called isolated if $U(S) = \emptyset$, is called a receiver if $U_0(S) = \emptyset$ and $U_I(S) \neq \emptyset$, is called a transmitter if $U_0(S) \neq \emptyset$ and $U_I(S) = \emptyset$, and is called a pass if $U_0(S) \neq \emptyset$ and $U_I(S) \neq \emptyset$.

S is called an independent set (of V) if $X(<S>)$ is empty, and a dependent set otherwise.

It is well know that if D is in C_2, then D has a unique receiving strong component, S. Let $B = U(V(S))$ be the coboundary of $V(S)$ in D. Since S is a receiving subgraph, B is not empty and $U(V(S)) = U_I(V(S))$.

Let $K' = V - V(S)$, and $K = <K'>$. Then, $D = K \cup B \cup S$, where the subgraphs K, B, S are pairwise line disjoint and $V(K) \cap V(S) = \emptyset$. This decomposition of any digraph D in category C_2 leads to the following theorem.

THEOREM 3.6A. If $D_1 \in C_2$ and $D_2 \in C_2$, then $D_1 \boxplus D_2 \in C_0 \cup C_1$.

PROOF. To prove the theorem, it suffices to show that there exist points (u,v) and (u',v') in $V(D_1 \boxtimes D_2)$ such that no $(u,v) \dashrightarrow (u',v')$ path and no $(u',v') \dashrightarrow (u,v)$ path exists in $D_1 \boxtimes D_2$.

Let $D_i = K_i \cup B_i \cup S_i$ where, as above, S_i is the receiving strong component of D_i, $B_i = U(V(S_i))$, and $K_i = \langle V_i - V(S_i) \rangle$ for $i = 1,2$.

Also, take (u,v) in $K_1 \boxtimes S_2$ and (u',v') in $S_1 \boxtimes K_2$. Suppose $(u,v) = (u_1,v_1),\ldots,(u_n,v_n) = (u',v')$ is a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \boxtimes D_2$. Then, $v_1,\ldots,v_n$ is a $v \dashrightarrow v'$ sequence in D_2 where $v \in V(S_2)$ and $v' \in V(K_2)$. It is well known, however, that any path between K_2 and S_2 contains exactly one line of B_2.

Let $P'\colon v = v_1 = v'_1, v'_2, \ldots, v'_r = v_n = v'$ be a $v \dashrightarrow v'$ subpath of the sequence $v_1,\ldots,v_n$. Then $\langle v'_i, v'_{i+1} \rangle \in B_2$ for some i. But $\langle v'_i, v'_{i+1} \rangle$ being a line of P' implies v'_i is in $V(S_2)$, whereas v'_i, v'_{i+1} being in B_2 implies v'_i is in $V(K_2)$. Since $V(S_2) \cap V(K_2) = \emptyset$, this is clearly impossible, and hence there does not exist a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \boxtimes D_2$.

Conversely, if $(u',v') = (u_1,v_1),\ldots,(u_n,v_n) = (u,v)$ is a $(u',v') \dashrightarrow (u,v)$ path in $D_1 \boxtimes D_2$, then $u' = u_1,\ldots,u_n = u$ is a $u' \dashrightarrow u$ sequence in D_1. Let $P'\colon u' = u'_1, u'_2, \ldots, u'_s = u$ be a subpath of this sequence. Then u' is in $V(S_1)$ and u is in $V(K_1)$, and the line $\langle u'_i, u'_{i+1} \rangle$ is in both B_1 and P' for some i. This, in turn, leads to a contradiction as above.

Thus, there does not exist a $(u',v') \dashrightarrow (u,v)$ path in $D_1 \boxtimes D_2$.

Consequently, $D_1 \boxtimes D_2 \in C_0 \cup C_1$, proving the theorem.

3.7 DETERMINATION OF POSSIBLE TRIPLES FOR TENSOR PRODUCT

3.7.1 Table 1 -- Possible Triples

Now, recalling the notation of the Introduction, we are in a position to determine the possible triples, (i,j,k), for all i and j, $0 \leq i,j \leq 3$, and $g: \mathcal{D} \times \mathcal{D} \dashrightarrow \mathcal{D}$ defined by $g(D_1,D_2) = D_1 \boxtimes D_2$. This is done in the form of a table for convenience.

D_1 \ D_2	0	1	2	3
0	0	0	0	0
1	0	0,1	0,1	0,1
2	0	0,1	0,1	0,1,2
3	0	0,1	0,1,2	0,3

TABLE 1 -- Possible Triples for $\boxtimes$

To begin with, since $D_1 \boxtimes D_2 \approx D_2 \boxtimes D_1$, clearly Table 1 will be symmetric about the diagonal.

From Theorem 3.2A, Theorem 3.5A and Theorem 3.6A, it is easy to see that no other values are possible in Table 1.

Also, Theorem 3.3A guarantees that 0 is in some triple $(2,2,0)$, and Theorem 3.3B guarantees that the triples $(3,3,0)$ and $(3,3,3)$ are possible.

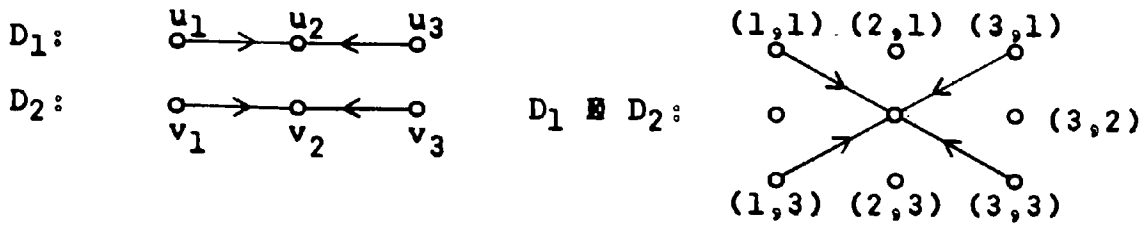
Since disconnected digraphs exist, Theorem 3.2A assures one that the triples $(i,0,0)$ are possible for $i = 0,1,2,3$.

3.7.2 Examples to show that certain triples are possible for tensor product

To see that the remaining values in Table 1 are assumed, consider the following simple examples.

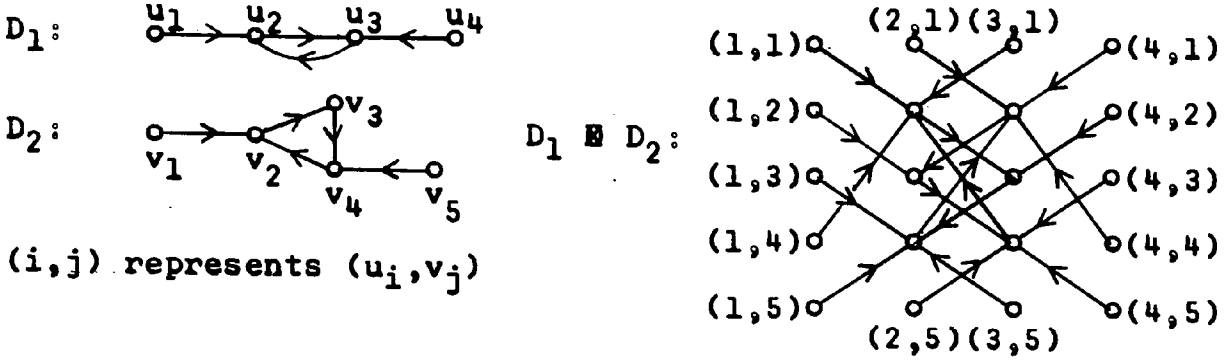
We first consider the triples $(1,1,0)$ and $(1,1,1)$.

FIGURE 6 -- The Triple $(1,1,0)$



In $D_1 \boxtimes D_2$, (i,j) represents the point (u_i,v_j). Here $D_1 \in C_1$, $D_2 \in C_1$ and $D_1 \boxtimes D_2 \in C_0$. Thus, the triple $(1,1,0)$ is possible.

FIGURE 7 -- The Triple $(1,1,1)$



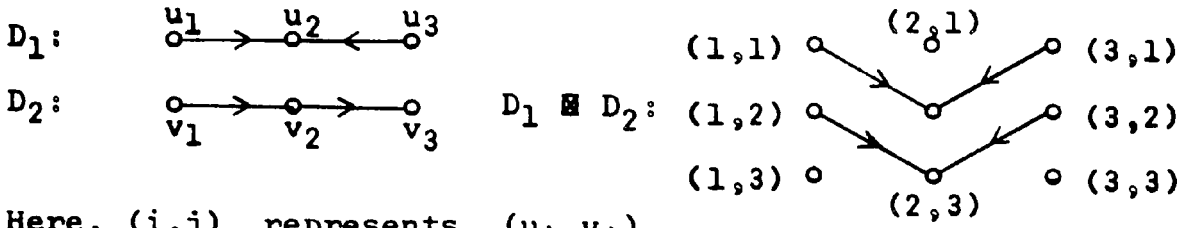
(i,j) represents (u_i,v_j)

Here, $D_1 \in C_1$, $D_2 \in C_1$ and $D_1 \boxtimes D_2 \in C_1$. Hence the triple $(1,1,1)$ is possible.

This verifies the $(1,1,i)$ entries in Table 1.

Now, let us consider the triples of the form $(1,2,i)$.

FIGURE 8 -- The Triple $(1,2,0)$



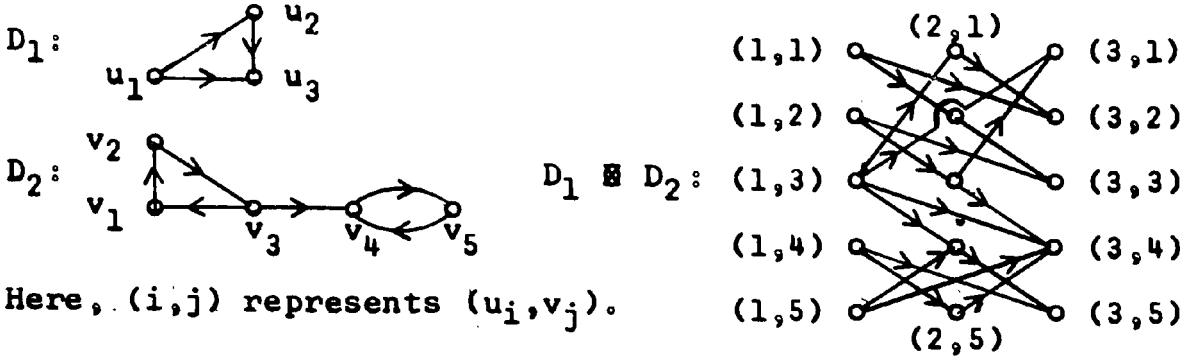
Here, (i,j) represents (u_i,v_j).

In this case, $D_1 \in C_1$, $D_2 \in C_2$, and $D_1 \boxtimes D_2 \in C_0$.

Consider the example of Figure 7. Take D_2 as given, and form D' from D_1 as follows: $V' = V_1$ and $X' = X_1 \cup \{\langle u_3,u_4\rangle\}$. Then, $D_1 \boxtimes D_2$ spans $D' \boxtimes D_2$, so that using Theorem 3.2A, $D' \boxtimes D_2 \in C_1$. Whence, in this case $D' \in C_2$, $D_2 \in C_1$ and $D' \boxtimes D_2 \in C_1$.

Therefore, from the above two cases, both triples $(1,2,0)$ and $(1,2,1)$ are possible in Table 1.

FIGURE 9 -- The Triple $(2,2,1)$



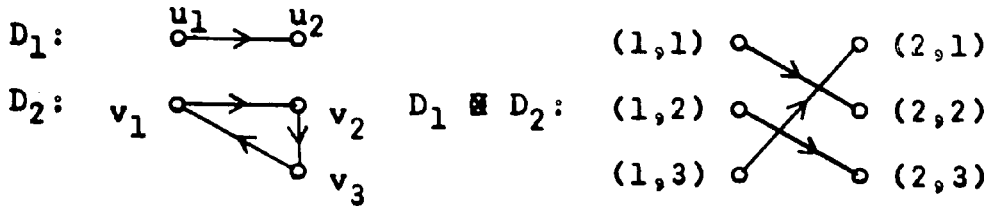
Here, (i,j) represents (u_i,v_j).

In this instance, $D_1 \in C_2$, $D_2 \in C_2$ and $D_1 \boxtimes D_2 \in C_1$. Since we have already seen that $(2,2,0)$ is possible,

this verifies the (2,2,i) entries in Table 1.

For triples of the form (2,3,i), one has the following examples.

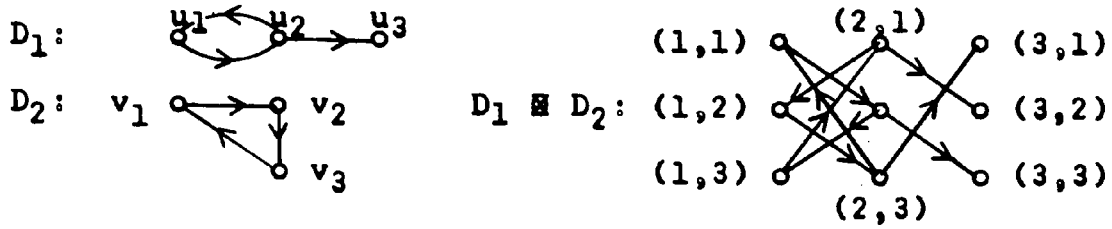
FIGURE 10 -- The Triple (2,3,0)



The pair (i,j) represents the point (u_i,v_j).

Here, $D_1 \in C_2$, $D_2 \in C_3$ and $D_1 \boxtimes D_2 \in C_0$. Thus, (2,3,0) is possible.

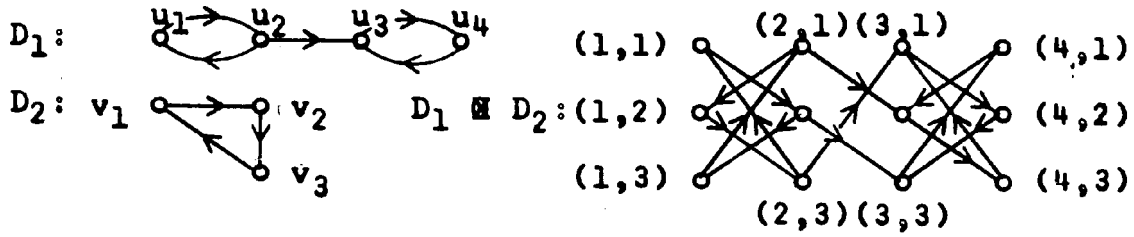
FIGURE 11 -- The Triple (2,3,1)



As usual, (i,j) represents (u_i,v_j).

In this case, $D_1 \in C_2$, $D_2 \in C_3$ and $D_1 \boxtimes D_2 \in C_1$.

FIGURE 12 -- The Triple (2,3,2)



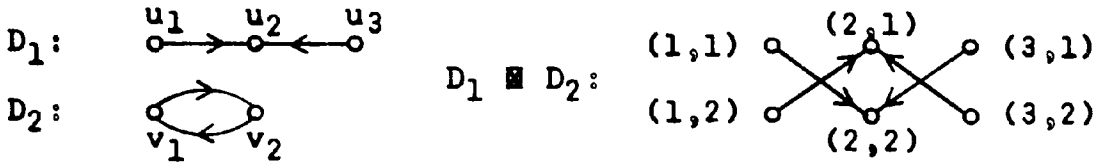
(i,j) represents the point (u_i,v_j).

In this instance, $D_1 \in C_2$, $D_2 \in C_3$ and $D_1 \boxtimes D_2 \in C_2$.

Therefore, these three cases verify the (2,3,i) entries.

It remains only to verify that (1,3,0) and (1,3,1) are possible.

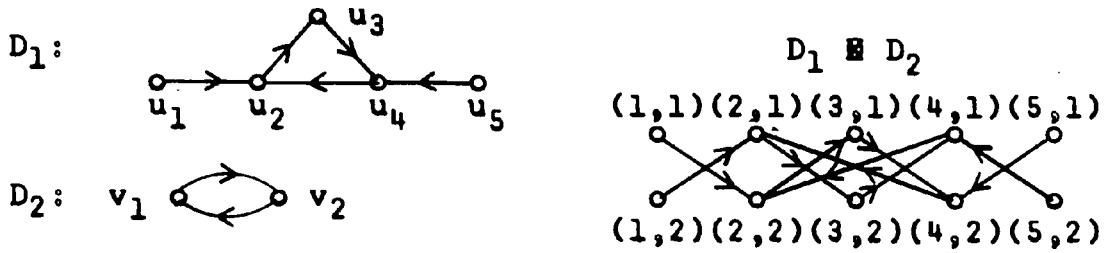
FIGURE 13 -- The Triple (1,3,0)



As usual, (i,j) represents the point (u_i, v_j).

Here, $D_1 \in C_1$, $D_2 \in C_3$ and $D_1 \boxtimes D_2 \in C_0$, so that (1,3,0) is possible.

FIGURE 14 -- The Triple (1,3,1)



The pair (i,j) represents the point (u_i, v_j).

In this instance, $D_1 \in C_1$, $D_2 \in C_3$ and $D_1 \boxtimes D_2 \in C_1$.

Thus, the above illustrations verify the triples (1,3,0) and (1,3,1).

This completes the verification that Table 1 represents the collection of all possible triples for tensor product.

CHAPTER IV -- TENSOR PRODUCTS OF DIGRAPHS: STRUCTURE

Referring to Table 1 of the previous chapter, one sees that if the triple (i,j,k) is possible and $k = 3$, then $i = j = 3$. However, since $(3,3,0)$ is also possible, if D_1 and D_2 are strong, it does not follow that $D_1 \boxtimes D_2$ is strong.

Clearly, a similar statement may be made about the triples $(2,3,2)$ and $(3,2,2)$ since these are the only possible triples, (i,j,k), in which k equals two.

In keeping with the problem outlined in the Introduction, necessary and sufficient conditions are derived in this chapter which will insure $D_1 \boxtimes D_2$ is in C_k for each of the possible triples, (i,j,k), above.

4.1 RECEIVING, TRANSMITTING AND ISOLATED SUBGRAPHS

In order to obtain some necessary and/or sufficient conditions on D_1 in C_i and D_2 in C_j that will insure that $D_1 \boxtimes D_2$ be in C_k for a specific k in a possible triple (i,j,k), the following is noted.

If $D_1 \boxtimes D_2$ has a weak (or strong) component $W \neq D_1 \boxtimes D_2$, then either there are subgraphs D_1' and D_2' of D_1 and D_2 respectively, such that $W = D_1' \boxtimes D_2'$ or there are not. Examples of the latter are obvious from Theorem 3.3B.

However, the following development yields information in case W does, indeed have a representation as $D_1' \boxtimes D_2'$.

4.1.1 Coboundaries in $D_1 \boxtimes D_2$

LEMMA 4.1A. Let D_1 and D_2 be two digraphs, $S_1 \subset V_1$, $S_2 \subset V_2$ with either $S_1 \neq V_1$ or $S_2 \neq V_2$. Then $x = \langle (u,v),(u',v') \rangle$ is in the coboundary $U[V(\langle S_1 \rangle \boxtimes \langle S_2 \rangle)]$ if and only if

(i) either (u,v) or (u',v') is in $S_1 \times S_2$, but not both, and

(ii) (1) $\langle u,u' \rangle \in U(S_1)$ and $\langle v,v' \rangle \in U(S_2) \cup X(\langle S_2 \rangle)$, or, (2) $\langle v,v' \rangle \in U(S_2)$ and $\langle u,u' \rangle \in U(S_1) \cup X(\langle S_1 \rangle)$.

PROOF. Note first that the fact that one of the S_i is a proper subset of V_i implies that $V(\langle S_1 \rangle \boxtimes \langle S_2 \rangle)$ is a proper subset of $V(D_1 \boxtimes D_2)$, so that, $U[V(\langle S_1 \rangle \boxtimes \langle S_2 \rangle)]$ is defined. Further, note that $V(\langle S_1 \rangle \boxtimes \langle S_2 \rangle) = S_1 \times S_2$.

For the necessity, suppose $x = \langle (u,v),(u',v') \rangle$ is in $U(S_1 \times S_2)$. Then by definition of coboundary, either (u,v) or (u',v') is in $S_1 \times S_2$, but not both. Further, since x is in $X(D_1 \boxtimes D_2)$, one has $\langle u,u' \rangle$ in X_1 and $\langle v,v' \rangle$ in X_2.

If (u,v) is in $S_1 \times S_2$, then clearly $\langle u,u' \rangle$ is in $U(S_1) \cup X(\langle S_1 \rangle)$ and $\langle v,v' \rangle$ is in $U(S_2) \cup X(\langle S_2 \rangle)$ since either u' is in S_1 or not, and v' is in S_2 or not.

Similarly, (u',v') being in $S_1 \times S_2$ implies $\langle u,u' \rangle$ is in $U(S_1) \cup X(\langle S_1 \rangle)$ and $\langle v,v' \rangle$ is in $U(S_2) \cup X(\langle S_2 \rangle)$.

Now, suppose that $\langle u,u' \rangle$ were not in $U(S_1)$. Then $\langle u,u' \rangle$ must be in $X(\langle S_1 \rangle)$ from above, so that

u and u' are in S_1. If, in addition, $\langle v,v' \rangle$ were not in $U(S_2)$, then $\langle v,v' \rangle$ would be in $X(\langle S_2 \rangle)$. So, if $\langle u,u' \rangle$ is not in $U(S_1)$, and $\langle v,v' \rangle$ is not in $U(S_2)$, then both f(x) and s(x) are in $S_1 \times S_2$. This is a clear contradiction.

Hence either $\langle u,u' \rangle$ is in $U(S_1)$ or $\langle v,v' \rangle$ is in $U(S_2)$. This, together with the above, establishes the necessity of conditions (i) and (ii).

To prove the sufficiency, note first that either (i) or (ii) implies that x is in $X(D_1 \blacksquare D_2)$. If (u,v) is in $S_1 \times S_2$, it is sufficient to show that (u',v') is not in $S_1 \times S_2$ in order that x be in $U(S_1 \times S_2)$. But, if (u',v') were in $S_1 \times S_2$, then $\langle u,u' \rangle$ would be in $X(\langle S_1 \rangle)$ and $\langle v,v' \rangle$ would be in $X(\langle S_2 \rangle)$, contrary to (i) and (ii). Thus, (u',v') is not in $S_1 \times S_2$, and x is in $U(S_1 \times S_2)$.

This completes the proof of Lemma 4.1A.

4.1.2 Receiving subgraphs in $D_1 \blacksquare D_2$

Having the above characterization of lines in a coboundary, one can prove the following.

THEOREM 4.1A. Let D_1 and D_2 be digraphs, $S_1 \subset V_1$, $S_2 \subset V_2$ with $S_1 \neq V_1$ or $S_2 \neq V_2$. Then $\langle S_1 \rangle \blacksquare \langle S_2 \rangle$ is a receiver if and only if the following holds for i = 1 and j = 2 or for j = 1 and i = 2.

S_i is a receiver, and

either (i) S_j is a receiver,

(ii) S_j is isolated and dependent or $S_j = V_j$.

(iii) S_j is a transmitter and dependent and S_i is independent,

or, (iv) S_j is a pass and S_i is independent.

The following lemma is needed for the proof of Theorem 4.1A.

LEMMA 4.1B. Let D_1 and D_2 be digraphs, $S_1 \subset V_1$, $S_2 \subset V_2$ with $S_1 \neq V_1$ or $S_2 \neq V_2$. Then, if $\langle S_1 \rangle \blacksquare \langle S_2 \rangle$ is a receiver, S_1 or S_2 must be a receiver.

PROOF OF LEMMA. Suppose S_1 and S_2 are not receivers. Then from Lemma 4.1A, at least one must not be isolated. Thus, there is a line $\langle u,u' \rangle$ in $U_0(S_1)$ since S_1 is not a receiver, where it is assumed without loss of generality that S_1 is not isolated.

Now, using Lemma 4.1A, there is a line $\langle v,v' \rangle$ in $U_0(S_2) \times X(\langle S_2 \rangle)$ since S_2 is not a receiver.

From this, one is able to conclude that (u,v) is in $S_1 \times S_2$ and (u',v') is not in $S_1 \times S_2$. Using this and the above, Lemma 4.1A then implies that the line $x = \langle (u,v),(u',v') \rangle$ is in $U_0(S_1 \times S_2)$, which is an obvious contradiction.

This concludes the proof of Lemma 4.1B.

PROOF OF THEOREM. To begin with, the following

statements are proved.

(I) If $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is a receiver and $U_I(S_2) = \emptyset$ or $S_2 = V_2$, then S_2 is dependent, and

(II) If $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is a receiver and $U_0(S_2) \neq \emptyset$, then S_1 must be independent.

First recall that throughout Chapters III and IV it is assumed that a point set of any digraph is a dependent set. Hence, in (I), if $S_2 = V_2$, then S_2 is dependent.

For the remainder of (I), since $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is a receiver, there exists $x = \langle (u,v),(u',v') \rangle$ in $U_I(S_1 \times S_2)$. So, (u,v) is not in $S_1 \times S_2$ and (u',v') is. Thus, one must have $\langle v,v' \rangle$ in X_2 with v' in S_2. If v is not in S_2, then $\langle v,v' \rangle$ is in $U_I(S_2)$ contrary to the assumption that $U_I(S_2) = \emptyset$. Therefore, v' is in S_2, and $\langle v,v' \rangle$ is in $X(\langle S_2 \rangle)$, implying that S_2 is dependent.

For (II), suppose that $\langle u,u' \rangle$ is in $X(\langle S_1 \rangle)$. Take $\langle v,v' \rangle$ in $U_0(S_2)$, which is, by assumption, not empty. Then v is in S_2 and v' is not in S_2. But, clearly u is in S_1, so that (u,v) is in $S_1 \times S_2$ and (u',v') is not in $S_1 \times S_2$. Hence, by Lemma 4.1A, one has $\langle (u,v),(u',v') \rangle$ in $U_0(S_1 \times S_2)$; a contradiction. Thus, $X(\langle S_1 \rangle) = \emptyset$, and S_1 is independent.

Let us now prove the necessity of the theorem.

From Lemma 4.1B, one sees that S_1 or S_2 must be a receiver. We shall assume that S_1 is a receiver. This may be done without loss of generality because of the obvious symmetry.

If we further suppose that S_2 is not a receiver, then there are four possibilities: namely,

(a) $U_I(S_2) = \emptyset$ and $U_O(S_2) = \emptyset$ (S_2 isolated)

(b) $S_2 = V_2$

(c) $U_I(S_2) = \emptyset$ and $U_O(S_2) \neq \emptyset$ (S_2 a transmitter), or

(d) $U_I(S_2) \neq \emptyset$ and $U_O(S_2) \neq \emptyset$ (S_2 a pass).

However, using (I) and (II), above, it is clear that (a) implies S_2 is isolated and dependent;

(b) implies $S_2 = V_2$ and S_2 is dependent;

(c) implies S_2 is a transmitter and dependent and S_1 is independent;

(d) implies S_2 is a pass and S_1 is independent.

But, then, (a) and (b) imply condition (ii), (c) implies condition (iii), and (d) implies condition (iv), for $i = 1$ and $j = 2$.

So, if $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is a receiver, S_1 is a receiver and S_2 is not a receiver, then either (ii), (iii) or (iv) must be true for $i = 1$ and $j = 2$. This proves the necessity of the conditions (i), (ii), (iii) and (iv) when it is assumed that S_1 is a receiver.

As previously remarked, however, the symmetry plus the fact that either S_1 or S_2 is a receiver proves the necessity of the conditions.

Now, in establishing the sufficiency of the conditions, one need only examine the case $i = 1$ and $j = 2$ because of the symmetry.

(A) Suppose S_1 and S_2 are receivers, and let $\langle u,u' \rangle$ be in $U_I(S_1)$ and $\langle v,v' \rangle$ be in $U_I(S_2)$. Then,

(u',v') is in $S_1 \times S_2$, and, using Lemma 4.1A, one has that $\langle(u,v),(u',v')\rangle$ is in $U_I(S_1 \times S_2)$.

We must now show that $U_0(S_1 \times S_2) = \emptyset$.

Let $x = \langle(u,v),(u',v')\rangle$ be in $U(S_1 \times S_2)$. Then either (u,v) or (u',v') is in $S_1 \times S_2$, but not both; and (I) $\langle u,u'\rangle$ is in $U(S_1)$ and $\langle v,v'\rangle$ is in

$$U(S_2) \cup X(\langle S_2\rangle)$$

or, (II) $\langle v,v'\rangle$ is in $U(S_2)$ and $\langle u,u'\rangle$ is in

$$U(S_1) \cup X(\langle S_1\rangle)$$

by Lemma 4.1A.

In case (I), u' is in S_1 and u is not in S_1 since S_1 is a receiver. Therefore, (u,v) is not in $S_1 \times S_2$, so that (u',v') is in $S_1 \times S_2$, and x is in $U_I(S_1 \times S_2)$. Since x was arbitrary, $U_I(S_1 \times S_2)$ must be $U(S_1 \times S_2)$, and $\langle S_1\rangle \boxtimes \langle S_2\rangle$ is a receiver.

(B) Suppose S_1 is a receiver, $X(\langle S_2\rangle) \neq \emptyset$, and S_2 is isolated or $S_2 = V_2$. Then there exist $\langle u,u'\rangle$ in $U_I(S_1)$ and $\langle v,v'\rangle$ in $X(\langle S_2\rangle)$. Here, u is not in S_1, u' is in S_1 and (u',v') is in $S_1 \times S_2$. Whence, by Lemma 4.1A, $\langle(u,v),(u',v')\rangle$ is in $U_I(S_1 \times S_2)$.

Thus, $U_I(S_1 \times S_2) \neq \emptyset$.

Now, suppose that $\langle(u,v),(u',v')\rangle$ is in $U(S_1 \times S_2)$. Then, by Lemma 4.1A, one must have $\langle u,u'\rangle$ in $U(S_1)$ and $\langle v,v'\rangle$ in $X(\langle S_2\rangle)$ since, by supposition, one cannot have $\langle v,v'\rangle$ in $U(S_2)$.

Therefore, u' is in S_1 since S_1 is a receiver, and v' is in S_2. That is, (u',v') is in $S_1 \times S_2$ so that, again, $U(S_1 \times S_2) = U_I(S_1 \times S_2)$; and consequently,

$\langle S_1 \rangle \blacksquare \langle S_2 \rangle$ is a receiver.

(C) For (iii), we suppose that S_1 is a receiver, S_2 a transmitter, and that both $X(\langle S_1 \rangle) = \emptyset$ and $X(\langle S_2 \rangle) \neq \emptyset$.

Then, the facts that S_1 is a receiver and $X(\langle S_2 \rangle) \neq \emptyset$ imply that $U_I(S_1 \times S_2) \neq \emptyset$. That is, if $\langle u,u' \rangle$ is in $U_I(S_1)$ and $\langle v,v' \rangle$ is in $X(\langle S_2 \rangle)$, then (u',v') is in $S_1 \times S_2$, (u,v) is not in $S_1 \times S_2$, and $\langle (u,v),(u',v') \rangle$ is in $U(S_1 \times S_2)$; the latter from Lemma 4.1A.

Now, if $x = \langle (u,v),(u',v') \rangle$ is in $U(S_1 \times S_2)$, then, using Lemma 4.1A, $\langle u,u' \rangle$ is in $U(S_1) \cup X(\langle S_1 \rangle) = U(S_1)$, the equality because $X(\langle S_1 \rangle) = \emptyset$. Thus, since S_1 is a receiver, u is not in S_1 and u' is in S_1. However, (u,v) or (u',v') is in $S_1 \times S_2$, but not both, by Lemma 4.1A, so that since u is not in S_1, it must be that (u',v') is in $S_1 \times S_2$. Once again, then, x is in $U_I(S_1 \times S_2)$, and $\langle S_1 \rangle \blacksquare \langle S_2 \rangle$ is a receiver.

(D) Here, we have that S_1 is a receiver, $U_0(S_2) \neq \emptyset$, $U_I(S_2) \neq \emptyset$ and $X(\langle S_1 \rangle) = \emptyset$.

In the same manner as in (C), $U_I(S_1 \times S_2) \neq \emptyset$, using the fact that S_1 is a receiver and $U_I(S_2) \neq \emptyset$.

Thus, using Lemma 4.1A, if $x = \langle (u,v),(u',v') \rangle$ is in $U(S_1 \times S_2)$, then $\langle u,u' \rangle$ must be in $U_I(S_1)$, since $X(\langle S_1 \rangle) = \emptyset$. But since (u,v) or (u',v') is in $S_1 \times S_2$ (but not both), and u is not in S_1, one has (u',v') in $S_1 \times S_2$, and again, $\langle S_1 \rangle \blacksquare \langle S_2 \rangle$ is a receiver.

Thus, any of the possibilities in Theorem 4.1A

imply that $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is a receiver when $i = 1$ and $j = 2$. As remarked, the symmetry assures that this is true for $j = 1$ and $i = 2$, concluding the proof of the theorem.

4.1.3 Transmitting subgraphs in $D_1 \boxtimes D_2$

There is a dual theorem giving necessary and sufficient conditions for $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ to be a transmitter, the proof of which is completely analagous to that of Theorem 4.1A. For this reason, the proof will be omitted, but the theorem and the lemma corresponding to Lemma 4.1B are as follows.

THEOREM 4.1B. Let D_1 and D_2 be digraphs, $S_1 \subset V_1$, $S_2 \subset V_2$ with $S_1 \neq V_1$ or $S_2 \neq V_2$. Then, $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is a transmitter if and only if the following holds for $i = 1$ and $j = 2$ or for $j = 1$ and $i = 2$.

S_i is a transmitter, and

either (i) S_j is a transmitter,

(ii) S_j is isolated and dependent or $S_j = V_j$,

(iii) S_j is a receiver and dependent and S_i is independent,

or, (iv) S_j is a pass and S_i is independent.

LEMMA 4.1C. Let D_1 and D_2 be digraphs, $S_1 \subset V_1$, $S_2 \subset V_2$, with $S_1 \neq V_1$ or $S_2 \neq V_2$. Then if $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is a transmitter, S_1 or S_2 must be a transmitter.

4.1.4 Isolated subgraphs in $D_1 \boxtimes D_2$

THEOREM 4.1C. Let D_1 and D_2 be digraphs, $S_1 \subset V_1$, $S_2 \subset V_2$ with $S_1 \neq V_1$ or $S_2 \neq V_2$. Then, $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is isolated if and only if

(1) S_1 is isolated and

(i) S_2 is isolated or $S_2 = V_2$,

or, (ii) S_1 is independent,

or (2) S_2 is isolated and

(i) S_1 is isolated or $S_1 = V_1$

or, (ii) S_2 is independent,

or (3) Neither S_1 nor S_2 is isolated, both S_1 and S_2 are independent, and $U_j(S_1) \neq \emptyset$ if and only if $U_j(S_2) = \emptyset$ for $j = 0, I$.

PROOF. It is clear that given S_1 and S_2, either S_1 and S_2 are isolated, S_1 is isolated and S_2 is not, S_2 is isolated and S_1 is not, or neither S_1 nor S_2 is isolated. Thus, (1) through (3) cover all possibilities in this regard. Recall that $V(\langle S_1 \rangle \boxtimes \langle S_2 \rangle)$ is the set $S_1 \times S_2$.

First, consider the necessity.

(1) Suppose S_1 is isolated. If there exist lines $\langle u,u' \rangle$ in $X(\langle S_1 \rangle)$ and $\langle v,v' \rangle$ in $U(S_2)$, then u and u' are in S_1 and either v or v' is in S_2. Hence, either (u,v) or (u',v') is in $S_1 \times S_2$. But, either v or v' is not in S_2, so that (u,v) or (u',v') is not in $S_1 \times S_2$. Then, Lemma 4.1A implies that $U(S_1 \times S_2)$

contains the line $\langle (u,v),(u',v') \rangle$.

Thus, if $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is isolated and S_1 is isolated, then either S_2 is isolated, S_1 is independent, or $S_2 = V_2$.

(2) Obviously, (2) is the symmetric counterpart of (1), so that, $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ isolated and S_2 isolated implies the remainder of (2).

(3) Now, suppose that $U(S_1) \neq \emptyset$ and $U(S_2) \neq \emptyset$. Then, if $\langle u,u' \rangle$ is in $X(\langle S_1 \rangle)$ and $\langle v,v' \rangle$ is in $U(S_2)$, one has (u',v') or (u,v) in $S_1 \times S_2$, but not both, so that Lemma 4.1A implies $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is not isolated. Thus, $X(\langle S_1 \rangle) = \emptyset$.

Similarly, $X(\langle S_2 \rangle) = \emptyset$.

Now, if $U_0(S_1) \neq \emptyset$, then $U_0(S_2) = \emptyset$ since $\langle u,u' \rangle$ in $U_0(S_1)$ and $\langle v,v' \rangle$ in $U_0(S_2)$ implies, using Lemma 4.1A, that $\langle (u,v),(u',v') \rangle$ is in $U_0(S_1 \times S_2)$.

On the other hand, if $U_0(S_2) = \emptyset$, then $U_I(S_2) \neq \emptyset$ since $U(S_2) \neq \emptyset$. But, if $\langle u,u' \rangle$ is in $U_I(S_1)$, take $\langle v,v' \rangle$ in $U_I(S_2)$. Then, as above, $\langle (u,v),(u',v') \rangle$ would be in $U(S_1 \times S_2)$. Hence, if $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is isolated, $U_I(S_1) = \emptyset$, and, since $U(S_1) \neq \emptyset$, one has $U_0(S_1) \neq \emptyset$ as desired.

Similarly, $U_I(S_1) \neq \emptyset$ if and only if $U_I(S_2) = \emptyset$.

Therefore, the necessity of (1), (2) or (3) is proved.

For the sufficiency, consider (1) first.

(1),(i) Here, S_1 and S_2 are isolated. Thus $U(S_1)$ and $U(S_2)$ are both empty, and from Lemma 4.1A, then,

$U(S_1 \times S_2) = \emptyset$, implying $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is isolated.

(1),(ii) In this case, $U(S_1) = \emptyset$ and $X(\langle S_1 \rangle) = \emptyset$. Then, $U(S_1) \cup X(\langle S_1 \rangle) = \emptyset$, and, referring to Lemma 4.1A, $U(S_1 \times S_2) = \emptyset$. Again one has that $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is isolated.

By symmetry, this also establishes the sufficiency of (2),(i) and (2),(ii).

(3) Note that in this instance, if $\langle (u,v),(u',v') \rangle$ is in $U(S_1 \times S_2)$, then $\langle u,u' \rangle$ is in $U(S_1)$ and $\langle v,v' \rangle$ is in $U(S_2)$, using Lemma 4.1A.

But, $\langle u,u' \rangle$ is in $U_0(S_1)$ if and only if $\langle v,v' \rangle$ is in $U_I(S_2)$, and $\langle u,u' \rangle$ is in $U_I(S_1)$ if and only if $\langle v,v' \rangle$ is in $U_0(S_2)$, by hypothesis.

Thus, either u is in S_1, u' is not in S_1, v is not in S_2, and v' is in S_2; or u' is in S_1, u is not in S_1, v' is not in S_2, and v is in S_2.

In either instance, neither (u,v) nor (u',v') is in $S_1 \times S_2$, contradicting the definition of $U(S_1 \times S_2)$.

Therefore, $U(S_1 \times S_2) = \emptyset$, and $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is isolated.

This completes the proof of Theorem 4.1C.

For the sake of completeness, one, of course, has that $\langle S_1 \rangle \boxtimes \langle S_2 \rangle$ is a pass if and only if it is neither isolated, a receiver, nor a transmitter.

COROLLARY 4.1A. Let D_1 and D_2 be digraphs. Then $D_1 \boxtimes D_2$ is in C_0 if there exist sets $S_1 \subset V_1$ and $S_2 \subset V_2$ satisfying (1), (2) or (3) of Theorem 4.1C.

The proof is immediate since the existence of an isolated subgraph in $D_1 \blacksquare D_2$ obviously implies $D_1 \blacksquare D_2$ is disconnected.

4.2 INDEPENDENT SETS AND R-T COMPATIBILITY

The case in which S_1 or S_2 is an independent set is important because quite often one is concerned with point receivers and transmitters in digraphs. When this is the situation, one has the following.

COROLLARY 4.2A. Suppose D_1 and D_2 are digraphs, $S_1 \subset V_1$, $S_2 \subset V_2$ with $S_1 \neq V_1$ or $S_2 \neq V_2$. Also suppose that both S_1 and S_2 are independent. Then $\langle S_1 \rangle \blacksquare \langle S_2 \rangle$ is a receiver (transmitter) if and only if either (1) S_1 and S_2 are receivers (transmitters),

(2) S_1 is a receiver (transmitter) and S_2 is a pass,

or, (3) S_2 is a receiver (transmitter) and S_1 is a pass.

COROLLARY 4.2B. Suppose D_1 and D_2 are digraphs, $S_1 \subset V_1$, $S_2 \subset V_2$ with $S_1 \neq V_1$ or $S_2 \neq V_2$. Also suppose that both S_1 and S_2 are independent. Then, $\langle S_1 \rangle \blacksquare \langle S_2 \rangle$ is isolated if and only if either

(1) S_1 or S_2 is isolated,

or, (2) Neither S_1 nor S_2 is isolated, and $U_j(S_1) \neq \emptyset$ if and only if $U_j(S_2) = \emptyset$ for $j = I, O$.

In particular,

COROLLARY 4.2C. If D_1 and D_2 are digraphs with u in V_1 and v in V_2, and if either

(i) u or v is isolated

or (ii) u is a transmitter (receiver) and v is a receiver (transmitter),

then (u,v) is isolated in $D_1 \boxtimes D_2$, and therefore $D_1 \boxtimes D_2$ is in C_0.

This result motivates the following definition.

DEFINITION 4.2A. Two digraphs D_1 and D_2 are said to be R-T compatible if the following conditions are satisfied.

(1) If D_1, (D_2 respectively), has a point receiver, then D_2, (D_1), has no point transmitter,

and (2) If D_1, (D_2), has a point transmitter, then D_2, (D_1), has no point receiver.

Then, using Corollary 4.2C, one clearly has:

COROLLARY 4.2D. If D_1 and D_2 are two digraphs, and if $D_1 \boxtimes D_2$ is weak, then D_1 and D_2 are R-T compatible.

It is obvious from Theorem 3.3B that R-T compatibility of digraphs D_1 and D_2 is not sufficient for $D_1 \boxtimes D_2$ to be weak since strong digraphs possess no transmitting and receiving subgraphs, and are therefore R-T compatible vacuously.

4.3 SPECIAL SUBGRAPHS OF $D_1 \times D_2$

4.3.1 Projections of strong components in the tensor product of strong digraphs

The following lemma will be found useful throughout the remainder of the chapter.

LEMMA 4.3A. If D_1 and D_2 are strong digraphs, and K is a strong component of $D_1 \times D_2$, then the projections of K are onto V_1 and V_2. That is, $P_1[V(K)] = V_1$ and $P_2[V(K)] = V_2$.

PROOF. Let

$S = \{ u : (u,v)$ is in $V(K)$ for some v in $V_2 \}$.

The assertion is, then, that $S = V_1$. Clearly, $S \subset V_1$.

First, let us note that, from Corollary 3.4A, K is a union of cycles. Thus, there is some cycle C in K. Let C be labeled $(u_0,v_0),\ldots,(u_n,v_n) = (u_0,v_0)$.

Then $S_1: u_0,u_1,\ldots,u_n = u_0$ is a closed sequence in S, and $S_2: v_0,v_1,\ldots,v_n = v_0$ is a closed sequence in D_2. It should be noted that K cannot be a point, and consequently, $n \geq 2$ throughout the above.

Let u be an element of V_1. We would like to show that there exists a point v of V_2 such that (u,v) is in $V(K)$, implying that u is in S.

Since D_1 is strong, there exists a $u \dashrightarrow u_0$ path, $W_1: u = u_0',u_1',\ldots,u_k' = u_0$ in D_1.

Taking the v_i's from S_2, let $k = an + b$ $(b < n)$ and let $v'_0 = v_{n-b}$, $v'_1 = v_{n-(b-1)}, \ldots$, $v'_b = v_0, \ldots$ $\ldots, v'_{n+b} = v_0, \ldots$, $v'_{n+b+t} = v_t, \ldots$, $v'_k = v_0$.

Since W_1 is a path, $(u, v_{n-b}) = (u'_0, v'_0), \ldots$, $(u'_k, v'_k) = (u_0, v_0)$ is a $(u, v_{n-b}) \dashrightarrow (u_0, v_0)$ path in $D_1 \boxtimes D_2$ with (u_0, v_0) on the cycle C.

We would now like to demonstrate the existence of a $C \dashrightarrow (u, v_{n-b})$ path, which would clearly imply that (u, v_{n-b}) is in K.

However, this need not be done directly. The existence of a $(u, v_{n-b}) \dashrightarrow C$ path implies that (u, v_{n-b}) is in the same weak component as C, and therefore it is in the same weak component as K. But, from Corollary 3.4A, the weak components of $D_1 \boxtimes D_2$ are also strong, and thus (u, v_{n-b}) and K are in the same strong component of $D_1 \boxtimes D_2$. However, this clearly implies that (u, v_{n-b}) is in $V(K)$, which in turn implies that u is in S.

Thus, $V_1 \subset S$, proving the lemma.

4.3.2 Particular receiving and transmitting subgraphs

LEMMA 4.3B. Let D_1 be in C_2 and D_2 be in C_3. Further, let S_r and S_t be the receiving and transmitting strong components, respectively, of D_1. Then,

(i) $S_r \boxtimes D_2$ is the maximal receiver in $D_1 \boxtimes D_2$ which is the union of receiving strong components of $D_1 \boxtimes D_2$,

(ii) $S_t \boxtimes D_2$ is the maximal transmitter in $D_1 \boxtimes D_2$ which is the union of transmitting strong components of $D_1 \boxtimes D_2$,

(iii) if $S_r \boxtimes D_2$ is strong, it is maximally so, and, (iv) if $S_t \boxtimes D_2$ is strong, it is maximally so.

PROOF. It is immediate that $S_r \boxtimes D_2$ is a receiver and that $S_t \boxtimes D_2$ is a transmitter in $D_1 \boxtimes D_2$, using Theorem 4.1A and Theorem 4.1B.

Further, $S_r \boxtimes D_2$ and $S_t \boxtimes D_2$ are unions of strong components of $D_1 \boxtimes D_2$. This may be seen as follows, for $S_r \boxtimes D_2$.

From Theorem 3.5A, $S_r \boxtimes D_2$ is a union of cycles since S_r is in C_3 and D_2 is in C_3, so that $S_r \boxtimes D_2$ is in $C_0 \cup C_3$. Thus, every weak component of $S_r \boxtimes D_2$ is strong. Let the components be $W_1, W_2, \ldots, W_s$. Then, $S_r \boxtimes D_2 = \bigcup_i W_i$. Clearly the W_i are receiving subgraphs.

Suppose W_j is not a strong component of $D_1 \boxtimes D_2$. Then W_j is a proper subgraph of W, where W is a strong component of $D_1 \boxtimes D_2$. It is clear that we may assume that $V(W_j)$ is a proper subset of $V(W)$ since the proper containment of W_j in W cannot be solely a matter of lines.

But, Lemma 4.3A implies that $P_1[V(W_j)] = V(S_r)$, so that $P_1[V(W_j)]$ is properly contained in $P_1[V(W)]$ since, obviously, $P_2[V(W_j)] = P_2[V(W)] = V(D_2)$.

But, $\langle P_1[V(W)] \rangle$ is strong since if u and u' are in $P_1[V(W)]$, then (u,v) and (u',v') are in

$V(W)$ for some v and v' in $V(D_2)$. However, W is strong by assumption, so there exist $(u,v) \dashrightarrow (u',v')$ and $(u',v') \dashrightarrow (u,v)$ paths in W, clearly implying the existence of $u \dashrightarrow u'$ and $u' \dashrightarrow u$ paths in $\langle P_1[V(W)]\rangle$.

However, from above, $P_1[V(W_j)] = V(S_r)$, so that S_r is properly contained in the strong digraph $\langle P_1[V(W)]\rangle$, a contradiction.

Thus, the W_i's are all strong components of $D_1 \otimes D_2$, and $S_r \otimes D_2 = \bigcup_i W_i$.

This, by the way proves (iii) by applying the above when $W_j = S_r \otimes D_2$.

To continue the proof of (i), suppose D' is a receiving subgraph of $D_1 \otimes D_2$ that can be written as a union of receiving strong components of $D_1 \otimes D_2$.

Let D'' be any one of these components of which D' is composed.

We saw above that $\langle P_1[V(D'')]\rangle$ is strong in D_1. Thus, there is a strong component D_1' of D_1 such that $D'' \subset D_1' \otimes D_2$.

Also we know that $P_1[V(D'')] = V(D_1')$ from Lemma 4.3A, so that if $\langle u,u'\rangle$ is in $U_0[V(D_1')]$, then (u,v) is in $V(D'')$ for some v in V_2. But u' is not in $V(D_1')$, so that for any v', (u',v') is not in $V(D_1' \otimes D_2)$, and thus (u',v') is not in $V(D'')$.

But, v is in V_2 and D_2 is strong, so that $od(v) > 0$. Hence $\langle v,v'\rangle$ is in X_2 for some v' in V_2.

Therefore, $x = \langle (u,v),(u',v') \rangle$ is in $X(D_1 \boxtimes D_2)$ with (u,v) in $V(D'')$ and (u',v') not in $V(D'')$. That is, x is in $U_0[V(D'')]$, which is a contradiction since D'' is a receiver by hypothesis.

Thus, D_1' is a strong component of D_1 with $U_0[V(D_1')] = \emptyset$. Since D_1 is in C_2, D_1' is not isolated, and therefore, $U_I[V(D_1')] = U[V(D_1')] \neq \emptyset$, implying that D_1' is a receiver.

However, a strictly unilateral digraph has a unique receiving strong component, and therefore $D_1' = S_r$. So, we have $D'' \subset S_r \boxtimes D_2$.

But D' was the union of receiving strong components of $D_1 \boxtimes D_2$, and D'' was an arbitrary one of these components. Thus, $D' \subset S_r \boxtimes D_2$, proving (i).

Because of the symmetry, a completely similar argument suffices to prove (ii) and (iv), and it will not be given here.

4.3.3 Sufficient conditions that $D_1 \boxtimes D_2$ be weak, and their consequences

LEMMA 4.3C. If D_1 is a unilateral digraph and D_2 is a strong digraph, and if there are subgraphs $D_1' \subset D_1$ and $D_2' \subset D_2$ such that $D_1' \boxtimes D_2'$ is strong but not a point, then $D_1 \boxtimes D_2$ is weak.

PROOF. By hypothesis, $D_1' \boxtimes D_2'$ contains a complete closed sequence

$$S: (u_0,v_0),(u_1,v_1),\ldots,(u_{n-1},v_{n-1}),(u_n,v_n) = (u_0,v_0)$$

with $n > 0$. Then, $u_0, u_1, \ldots, u_{n-1}, u_n = u_0$ is a complete closed sequence in D_1' and $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$ is a complete closed sequence in D_2'.

Let (u,v) be in $V(D_1 \boxtimes D_2)$.

Since D_1 is in $C_2 \cup C_3$, there is a $u \dashrightarrow u_0$ path or a $u_0 \dashrightarrow u$ path in D_1, and since D_2 is in C_3, there is a $v \dashrightarrow v_0$ and a $v_0 \dashrightarrow v$ path in D_2.

Suppose W_1: $u = w_0, w_1, \ldots, w_k = u_0$ is a $u \dashrightarrow u_0$ path in D_1 and that W_2: $v = w_0', w_1', \ldots, w_m' = v$ is a $v \dashrightarrow v_0$ path in D_2.

Suppose $k > m$. Then, since W_1 is a path, $(u,v) = (w_0, w_0'), \ldots, (w_m, w_m'), (w_{m+1}, v_1), \ldots, (w_k, v_{k-m(\text{mod } n)})$ is a $(u,v) \dashrightarrow D_1' \boxtimes D_2'$ path since $w_k = u_0$ and the point $(u_0, v_{k-m(\text{mod } n)})$ is in $V(D_1' \boxtimes D_2')$.

If, on the other hand, $k < m$, then $(u,v) = (w_0, w_0'), \ldots, (w_k, w_k'), (u_1, w_{k+1}'), \ldots, (u_{m-k(\text{mod } n)}, w_m')$ is a $(u,v) \dashrightarrow D_1' \boxtimes D_2'$ path since $w_m' = v_0$ and the point $(u_{m-k(\text{mod } n)}, w_m')$ is in $D_1' \boxtimes D_2'$.

If, instead, W_1': $u_0 = t_0, t_1, \ldots, t_s = u$ were a $u_0 \dashrightarrow u$ path in D_1, then a similar construction yields a $D_1' \boxtimes D_2' \dashrightarrow (u,v)$ path.

For example, if $s > t$, let $s - t = an + b$ $(b < n)$, and let $v_0' = v_{n-b}$, $v_1' = v_{n-b+1}, \ldots, v_b' = v_0 = v_n, \ldots, v_n' = v_{n-b}, \ldots, v_{n+b}' = v_0, \ldots, v_{an+b}' = v_0 = v_{s-t}'$. Here it is assumed that W_2': $v_0 = z_0, \ldots, z_t = v$ is a $v_0 \dashrightarrow v$ path.

Then,

$(u_0,v_0') = (t_0,v_0'),\ldots,(t_{s-t},v_{s-t}'),(t_{s-t+1},z_1),\ldots,(t_s,z_t)$

is a $D_1' \boxtimes D_2' \dashrightarrow (u,v)$ path since $(u,v) = (t_s,z_t)$.

The same type construction works if $s < t$, yielding a $D_1' \boxtimes D_2' \dashrightarrow (u,v)$ path.

Thus, every point (u,v) in $V(D_1 \boxtimes D_2)$ is connected via a path with $D_1' \boxtimes D_2'$. Since $D_1' \boxtimes D_2'$ was assumed to be strong, this obviously implies $D_1 \boxtimes D_2$ is weak, completing the proof of the lemma.

It might be well to remark that the assumption that D_2 be strong was necessary, in that if neither D_1 nor D_2 were strong, the existence of receivers and transmitters in both D_1 and D_2 implies the possibility of isolated subgraphs of $D_1 \boxtimes D_2$ in accordance with Theorem 4.1C.

As a corollary, one has the following.

COROLLARY 4.3A. If D_1 and D_2 are both strong digraphs, then $D_1 \boxtimes D_2$ is strong if and only if there are subgraphs $D_1' \subset D_1$ and $D_2' \subset D_2$ such that $D_1' \boxtimes D_2'$ is strong but not a point.

PROOF. Lemma 4.3C implies that $D_1 \boxtimes D_2$ is weak. But, from Theorem 3.5A, one then concludes that $D_1 \boxtimes D_2$ is strong.

Conversely, if $D_1 \boxtimes D_2$ is strong, then one may let $D_1' = D_1$ and $D_2' = D_2$.

Refering to Table 1, this can clearly be stated in

the form:

COROLLARY 4.3B. Given two digraphs D_1 and D_2, then $D_1 \boxtimes D_2$ is strong if and only if

(1) D_1 and D_2 are strong, and

(2) there exist $D_1' \subset D_1$ and $D_2' \subset D_2$ with $D_1' \boxtimes D_2' \neq K_1$ and $D_1' \boxtimes D_2'$ strong.

This is a characterization of strong connectedness under the tensor product. However, it is not as "useful" as one might desire. Such a characterization will be given later, in Theorem 4.6A and its attendant corollaries.

4.4 CHARACTERIZATION OF CATEGORY C_2

Now, as has been previously noted, a digraph D is in category C_2 if and only if D is weak and has a unique receiving strong component and a unique transmitting strong component.

Using this and the previous two lemmas, one obtains the following result.

THEOREM 4.4A. If D_1 is in C_2, D_2 is in C_3, S_r is the receiving strong component of D_1 and S_t is the transmitting strong component of D_1, then $D_1 \boxtimes D_2$ is in C_2 if and only if $S_r \boxtimes D_2$ and $S_t \boxtimes D_2$ are strong.

PROOF. For the necessity, we refer to the above remark and Lemma 4.3B. If $D_1 \boxtimes D_2$ is in C_2, with receiving strong component S_r' and transmitting strong component S_t', then, from Lemma 4.3B and above,

$S_r' = S_r \boxtimes D_2$ and $S_t' = S_t \boxtimes D_2$.

For example, $S_r' \subset S_r \boxtimes D_2$ by (i) of Lemma 4.3B since S_r' is a receiving strong component of $D_1 \boxtimes D_2$.

But, the above remark implies $S_r \boxtimes D_2 \subset S_r'$ since $S_r \boxtimes D_2$ is a union of strong components of $D_1 \boxtimes D_2$ both of which must be receivers in $D_1 \boxtimes D_2$. However, there is only one such strong component in $D_1 \boxtimes D_2$, namely S_r', so that we have the desired result.

In precisely the same way, $S_t \boxtimes D_2 = S_t'$.

Thus, clearly, $S_r \boxtimes D_2$ and $S_t \boxtimes D_2$ are strong; completing the proof of the necessity.

On the other hand, for the sufficiency, $V(S_r \boxtimes D_2)$ contains more than a single point since, as always in this chapter, we assume that V_2 has this property.

Hence, $S_r \boxtimes D_2 \neq K_1$ and $S_t \boxtimes D_2 \neq K_1$, and using Lemma 4.3C, we conclude that $D_1 \boxtimes D_2$ is weak.

But, from Lemma 4.3B, $S_r \boxtimes D_2$ is the maximal receiving subgraph in $D_1 \boxtimes D_2$ and $S_t \boxtimes D_2$ the maximal transmitting subgraph in $D_1 \boxtimes D_2$ that can be written as unions of receiving and transmitting components of $D_1 \boxtimes D_2$ respectively. Since both are strong, by Lemma 4.3B they are maximally strong, or strong components of $D_1 \boxtimes D_2$.

Thus, $S_r \boxtimes D_2$ is the only receiving strong component and $S_t \boxtimes D_2$ the only transmitting strong component of $D_1 \boxtimes D_2$. Hence, $D_1 \boxtimes D_2$ is weak and has a unique receiving strong component and a unique transmitting strong component, implying that $D_1 \boxtimes D_2$ is in C_2.

This concludes the proof of Theorem 4.4A.

COROLLARY 4.4A. Let D_1 and D_2 be digraphs. Then $D_1 \boxtimes D_2$ is in C_2 if and only if

(1) Either (D_1, D_2) or (D_2, D_1) is in $C_2 \times C_3$, and

(2) If $D_1 \in C_2$ with receiving strong component S_r and transmitting strong component S_t, then $S_r \boxtimes D_2$ and $S_t \boxtimes D_2$ are strong, or a similar criterion if $D_2 \in C_2$.

PROOF. To prove the necessity, it is only necessary to refer to Table 1 to conclude that if $D_1 \boxtimes D_2$ is in C_2, then (1) holds. But (1) and $D_1 \boxtimes D_2$ in C_2 imply (2) using Theorem 4.4A.

Obviously, the sufficiency follows immediately from Theorem 4.4A.

It should be noted, then, that if "useful" necessary and sufficient conditions for the tensor product of two strong digraphs to be strong were known, not only would this characterize the strong case, but it would finish the characterization of the strictly unilateral case using Corollary 4.4A.

So, for the present, we will turn our attention toward a characterization of the situation when $D_1 \boxtimes D_2$ is strong.

4.5 CHARACTERIZATION OF C_3

4.5.1 Certain sufficient conditions

We will begin by deriving some sufficient conditions

which will lend certain insight into the final result.

With sequences, as with paths, of a digraph, one can associate a subgraph. This is done as follows.

DEFINITION 4.5A. Let D be a digraph and S a sequence in D. Then S is associated with a subgraph $\ll S \gg$ of D as follows.
$V(\ll S \gg) = \{ v : v \text{ is a point in } S \}$, and
$\langle u,v \rangle$ is in $X(\ll S \gg)$ if and only if u and v are in $V(\ll S \gg)$, u is adjacent to v in S, and u precedes v in S. (Referring to Definition 2.1F, the latter clearly implies that $\langle u,v \rangle$ is in X).

For example, in the digraph D shown at the right, suppose S is given by $S: u_1, u_2, u_3, u_1, u_2$. Then, $\ll S \gg$ is the subgraph illustrated.

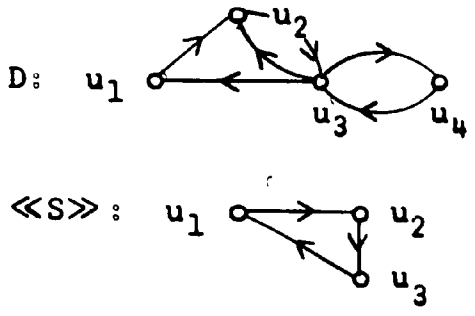


FIGURE 15 -- The subgraph $\ll S \gg$

It is obvious that many sequences may be associated with a single subgraph in this way.

Then one has:

LEMMA 4.5A. Let D_1 and D_2 be digraphs containing closed sequences S_1 and S_2 of lengths $n_1 > 0$ and $n_2 > 0$ respectively. If $(n_1, n_2) = 1$, then $\ll S_1 \gg \boxtimes \ll S_2 \gg$ is strong.

PROOF. Let S_1 and S_2 be labeled as follows.

S_1: $u_0, u_1, \cdots, u_{n_1} = u_0$ and S_2: $v_0, v_1, \cdots, v_{n_2} = v_0$.

We may assume without loss of generality that $n_2 > n_1$. Then, form the sequence S in $D_1 \boxtimes D_2$ whose points are (u_i', v_i'), $i = 1, 2, \cdots, n_1 n_2$, where $u_i' = u_{i(\text{mod } n_1)}$ and $v_i' = v_{i(\text{mod } n_2)}$.

Then $(i_1, i_1) = (j_1, j_1)$ if and only if $i_1 \equiv j_1 (\text{mod } n_1)$ and $i_1 \equiv j_1 (\text{mod } n_2)$, or equivalently, if and only if $i_1 \equiv j_1 (\text{mod } n_1 n_2)$ since $(n_1, n_2) = 1$.

This is so, even though as points, u_i may be the same as u_j for $i \neq j$ since S_1 is a sequence, and similarly for the v_k's.

Thus, S contains all possible (u_i, v_j) with $0 \leqq i \leqq n_1$ and $0 \leqq j \leqq n_2$, so surely the points of S are precisely those in $V(\ll S_1 \gg \boxtimes \ll S_2 \gg)$.

Furthermore, $(u_{n_1 n_2}', v_{n_1 n_2}') = (u_0, v_0)$ so that S is closed.

Hence, $\ll S_1 \gg \boxtimes \ll S_2 \gg$ contains a complete closed sequence and is, therefore, strong. This completes the proof of Lemma 4.5A.

COROLLARY 4.5A. Let D_1 and D_2 be strong digraphs with closed sequences S_1 and S_2 of lengths $n_1 > 0$ and $n_2 > 0$ respectively. If $(n_1, n_2) = 1$, then $D_1 \boxtimes D_2$ is strong.

This result follows immediately from Corollary 4.3B and the preceding lemma.

It should be remarked that, at present, no counter-

example or proof has been found to the conjecture that if $D_1 \boxtimes D_2$ is strong, then there will exist closed sequences S_1 in D_1 and S_2 in D_2 satisfying the conditions of Corollary 4.5A.

COROLLARY 4.5B: Let D_1 and D_2 be strong digraphs with cycles L_{n_1} and L_{n_2} of lengths $n_1 > 0$ and $n_2 > 0$ respectively. Then if $(n_1,n_2) = 1$, $D_1 \boxtimes D_2$ is strong.

This, of course, is an obvious special case of Corollary 4.5A. Referring to Theorem 3.3B, it also follows directly from Corollary 4.3B.

To see that this is not a necessary condition that $D_1 \boxtimes D_2$ be strong, consider the digraphs given at the right. Here $(6,15) = 3$ and $(10,15) = 5$. Note, however, that $(6,10,15) = 1$. Because $D_1 \boxtimes D_2$ has, in this case, 225 points, it will not be shown. However, it is strong.

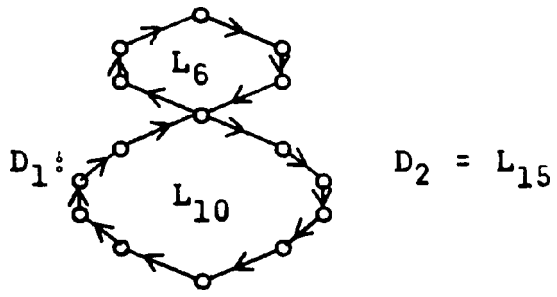


FIGURE 16 -- A counterexample

4.5.2 Additivity of r^b

Now recall that the expression $r^i(k)$ was defined in Definition 3.3C. Referring to this, we state the following lemma.

LEMMA 4.5B. r^b is additive modulo m for all b. That is, $r^b[(k+a) \pmod m] = \{r^b[k \pmod m] + a\} \pmod m$.

PROOF. (I) For $k < m$, we assert that $r^1(k+1) = [r^1(k) + 1] \pmod m$. Referring to the expression on page 29 for Q_K, r(k) is defined by the condition $n = q(k)m - k + r(k)$ where $2 \leq r(k) \leq m+1$. Thus, $n = q(k)m - k + r(k) = q(k+1)m - (k+1) + r(k+1)$, and hence $q(k)m + [r(k) - 1] = q(k+1)m + [r(k+1) - 2] = n + (k-1)$.

Hence, if $r(k) - 1 = m$, then one has from the above expression that $r^1(k+1) = [r^1(k) + 1] \pmod m$.

But, if $r(k) - 1 < m$, then the uniqueness of the division algorithm assures one that $r(k) - 1 = r(k+1) - 2$ using the above expression, so that, again one has $r^1(k+1) = [r^1(k) + 1] \pmod m$.

(II) If $k = m$, then $k+1 \equiv 1 \pmod m$, and one has $n = q(1)m - 1 + r(1) = q(m)m - m + r(m) = [q(m) - 1]m + r(m)$.

(a) If $r(m) < m$, then since $r(1) - 1 \leq m$, the division algorithm implies $r(1) - 1 \equiv r(m) \pmod m$. But, since $k = m$, this is $r[(k+1) \pmod m] \equiv [r(k) + 1] \pmod m$.

Thus, $r^1[(k+1) \pmod m] = [r^1(k) + 1] \pmod m$.

(b) If $r(m) \geq m$, then $0 \leq r(m) - m < m$ and $r(1) - 1 \leq m$, so that, by the division algorithm, one has $r(m) - m \equiv [r(1) - 1] \pmod m$, or $[r(m) + 1] - r(1) \equiv 0 \pmod m$.

Thus, $[r^1(k) + 1] \pmod m = r^1[(k+1) \pmod m]$.

(III) Let k be arbitrary. Then k (mod m) is

less than or equal to m, and therefore $k \pmod m$ falls into case (I) or (II). Hence

$$r^1[(k+1) \pmod m] = \{r^1[k \pmod m] + 1\} \pmod m.$$

(IV) Thus, for any k and a one has

$$\begin{aligned} r^1[(k+a) \pmod m] &= r^1[(k+a-1+1) \pmod m] \\ &= \{r^1[(k+a-1) \pmod m] + 1\} \pmod m \\ &= \{r^1[k \pmod m] + (a-1) + 1\} \pmod m \\ &= \{r^1[k \pmod m] + a\} \pmod m \end{aligned}$$

using the obvious induction.

(V) Now, assume that $r^{b-1}[(k+a) \pmod m] = \{r^{b-1}[k \pmod m] + a\} \pmod m$. Then,

$$\begin{aligned} r^b[(k+a) \pmod m] &= r\{r^{b-1}[(k+a) \pmod m]\} \pmod m \\ &= r^1\{[r^{b-1}(k \pmod m) + a] \pmod m\} \\ &= \{r^b[k \pmod m] + a\} \pmod m, \end{aligned}$$

proving Lemma 4.5B.

4.5.3 Further insight into the structure of $L_n \boxtimes L_m$

LEMMA 4.5C. Let L_a and L_b be cycles labeled L_a: $u_1, u_2, \ldots, u_a, u_1$ and L_b: $v_1, v_2, \ldots, v_b, v_1$. Then,

(i) $L_a \boxtimes L_b$ has (a,b) components,

(ii) For a given $u = u_k$, (u, v_i) and (u, v_j) are in the same strong component of $L_a \boxtimes L_b$ if and only if $i \equiv j \pmod{(a,b)}$, and

(iii) For a given $v = v_k$, (u_i, v) and (u_j, v) are in the same strong component of $L_a \boxtimes L_b$ if and only if $i \equiv j \pmod{(a,b)}$.

PROOF. Part (i) is immediate from Theorem 3.3B.

To prove part (ii), take (u,v_1) and let (u,v_k) be the point with the property that k' is the first integer greater than one such that $k = k' \pmod b$ and (u,v_k) is in the same component, say B_1, as (u,v_1). Lemmas 3.3C and 3.3D guarantee such a k' exists.

From Lemma 3.3C, it is easily seen that (u,v_k) and (u,v_1) are in the same strong component of $L_a \boxtimes L_b$ if and only if $r^d(1) = k$ for some d.

But, using Lemma 4.5B, and letting $t = k - 1$, one has $r^d(k) = r^d(1 + t) = [r^d(1) + t] \pmod b$

$$= (k + t) \pmod b = [k + (0 + 1)t] \pmod b.$$

Now suppose $r^d[(k + jt) \pmod b] = [k + (j + 1)t] \pmod b$. Then, $r^d\{[k + (j+1)t] \pmod b\} = r^d\{[(k+jt) + t] \pmod b\}$

$$= \{r^d[(k + jt) \pmod b] + t\} \pmod b$$

$$= \{[k + (j+1)t] + t\} \pmod b$$

$$= [k + (j + 2)t] \pmod b.$$

Thus, $r^d [k + (j+1)t] \pmod b = [k + (j+2)t] \pmod b$ for all j.

A simple induction argument then yields $r^d\{[k + (j+1)t] \pmod b\} = r^{(j+2)d}(k) = r^{(j+3)d}(1)$.

That is, $r^d(k) = (k + t) \pmod b$ from above, so $r^{(j+1)d}(k) = r^{jd}[r^d(k)] = r^{jd}[k + t \pmod b]$

$$= [r^{jd}(k) + t] \pmod b = [(k + jt) + t] \pmod b$$

$$= [k + (j + 1)t] \pmod b,$$

using the obvious induction step.

So, under the above assumptions on (u,v_1) and (u,v_k), one finds that,

$$B_1' = \{(u,v_1),(u,v_k),(u,v_{(k+t)(\bmod b)}),\ldots,(u,v_{(k+ct)(\bmod b)}),\ldots\}$$

is a subset of $V(B_1)$.

Now consider the point (u,v_i) for any i such that $2 \leq i \leq k'-1$. Since k' was assumed to be the smallest integer greater than one such that (u,v_k), with $k = k' \pmod b$, was in B_1, (u,v_i) is not in B_1.

But then, $(u,v_{[k+(j-1)t+(i-1)] \pmod b})$ is not in B_1 for any i with $2 \leq i \leq k'-1$ and any j since $r^{jd}(i) = [r^{jd}(1) + (i-1)] \pmod b = [k + (j-1)t + (i-1)] \pmod b$.

However,

$\{v_{[k+(j-1)t+(i-1)] \pmod b} : j \geq 0,\ 1 \leq i-1 \leq k'-2\} =$

$= V(L_b) - P_2[B_1']$ since, for any $s > k$, $s - k = mt + n$ with $m \geq 0$ and $0 < n < t$, and for any $s < k$, $s = t$ if $s = k-1$, and $s = n < k-1$ otherwise.

But when $i-1$ is zero or $k'-1$, the point $(u,v_{[k+(j-1)t+(i-1)] \pmod b})$ is in B_1'.

Thus, only the points of B_1' are points of B_1 of the form (u,v_s) since if (u,v_s) were in $V(B_1) - B_1'$, then $s = [k + (j_o - 1)t + (i_o - 1)] \pmod b = r^{j_o d}(i_o) =$ $= (j_o t + i_o) \pmod b$; implying that (u,v_{i_o}) is in B_1 with $i_o < k'$, a contradiction.

Further, for each (u,v_i) with $1 \leq i \leq k'-1$, $B_i' = \{(u,v_i),(u,v_{(i+t)(\text{mod } b)}),\ldots,(u,v_{(i+ct)(\text{mod } b)}),\ldots\}$ determines a unique strong component B_i of $L_a \boxtimes L_b$. This is so since, if $B_i' \cap B_j' = (u,v)$, then $(u,v) = (u,v_{(i+c't)(\text{mod } b)}) = (u,v_{(j+c''t)(\text{mod } b)})$ with, say, $i > j$. But, this implies that $r^{c'd}(i) = r^{c''d}(j)$, or if $c'' > c'$, $r^{(c''-c')d}(j) = i$, so that (u,v_i) is in

B_j'. This evidently implies $B_i' = B_j'$ by definition of B_i'. One reaches the same conclusion if $c' \succ c''$.

Now, by Lemma 4.3A, if K is a component of $L_a \boxtimes L_b$, then $P_1[V(K)] = V(L_a)$, so there is at least one (u,v_i) in K for any given u in $V(L_a)$. Hence, the B_i are the only components of $L_a \boxtimes L_b$. But, there are $k' - 1$ distinct B_i and (a,b) components (by part (i)).

Therefore, $k' - 1 = (a,b)$, which proves (ii) if $t \succ 0$ since, then, $k' - 1 = k - 1 = t$, and clearly, from above, (u,v_i) and (u,v_j) are in the same component if and only if $i \equiv j \pmod t$.

If $t = 0$, then $k' = b + 1$, and there are $b = (a,b)$ distinct components, each point (u,v_i) belonging to exactly one of these. Here, of course, for $i,j \leq b$ and $i \neq j$, one has $i \not\equiv j \pmod b$ and (u,v_i) and (u,v_j) are in different components, consistent with (ii).

For part (iii), consider (u_i,v) and (u_j,v) for fixed v in $V(L_b)$. We may assume that $i \leq j$. But, if $i = j$, we are clearly done. Thus assume $i \prec j$.

Then (u_j,v_k) (here, $v = v_k$) is on some path O_s, as are $(u_{j-1},v_{(k-1)(\mathrm{mod}\ b)})$, $(u_{j-2},v_{(k-2)(\mathrm{mod}\ b)})$, ..., and $(u_i,v_{[k-(j-i)](\mathrm{mod}\ b)})$.

That is,

$(u_i,v_{[k-(j-i)](\mathrm{mod}\ b)}),\ldots,(u_{j-1},v_{(k-1)(\mathrm{mod}\ b)}),(u_j,v_k)$

is a subpath of O_s.

So obviously, (u_i,v_k) and (u_j,v_k) are in the same strong component of $L_a \boxtimes L_b$ if and only if (u_i,v_k)

and $(u_i, v_{[k-(j-i)](\text{mod } b)})$ are in the same strong component. However, by part (ii), this will be the case if and only if $k \equiv \{[k-(j-i)] \pmod b\} \pmod{(a,b)}$. But this means $k \equiv [k-(j-i)+cb] \pmod{(a,b)}$, which is equivalent to $i \equiv j \pmod{(a,b)}$, as desired.

This completes the proof of Lemma 4.5C.

4.5.4 Extension of the results in Section 4.5.3

DEFINITION 4.5B. We say that the set of cycles $\{L_{a_i} : i = 1,2,\ldots,t\}$ possesses the path intersection property (P-I property) if the following conditions are satisfied.

(1) $L_{a_i} \cap L_{a_{i+1}} \neq \emptyset$ for $i = 1,2,\ldots,t-1$,

and (2) If the intersection $[\bigcup_{j=1}^{i} L_{a_j}] \cap L_{a_{i+1}}$ contains more that one point, and if u and u' are two distinct such points, then there is some path between u and u' common to both $L_{a_{i+1}}$ and $[\bigcup_{j=1}^{i} L_{a_j}]$ for $i = 1,2,\ldots,t-1$.

LEMMA 4.5D. Let $L_{a_1},\ldots,L_{a_t}$ be cycles possessing the P-I property, and let L_{b_1} be a cycle. Then

(1) If $D' = \bigcup_{i=1}^{t} L_{a_i}$, then $D = D' \boxtimes L_{b_1}$ has $a_o = (a_1,\ldots,a_t,b_1)$ strong components,

(2) If L_{b_1} is labeled $v_1,v_2,\ldots,v_{b_1},v_1$, then for u in V', (u,v_i) and (u,v_j) are in the same strong component of D if and only if $i \equiv j \pmod{a_o}$, and

(3) If L_{a_k} is labeled $u_1,u_2,\ldots,u_{a_k},u_1$ for any k, and

v is in $V(L_{b_1})$, then (u_i,v) and (u_j,v) are in the same strong component of D if and only if $i \equiv j \pmod{a_o}$.

PROOF. The proof is by induction on t, the number of cycles L_{a_i}.

From Lemma 4.5C, the statements (1), (2) and (3) are true if $t = 1$.

Suppose (1), (2) and (3) are true for $t-1$ cycles. Then, by choice, $L_{a_1},\ldots,L_{a_{t-1}}$ possess the P-I property, and we may assume (1), (2) and (3) for $D_1 = D_1' \boxtimes L_{b_1}$, where $D_1' = \bigcup_{i=1}^{t-1} L_{a_i}$.

Let u be in $V(L_{a_{t-1}}) \cap V(L_{a_t})$. Then we have:

(A) (u,v_i) and (u,v_j) are in the same strong component, C, of D_1 if and only if $i \equiv j \pmod{a'}$ where $a' = (a_1,\ldots,a_{t-1},b_1)$, and

(B) (u,v_i) and (u,v_j) are in the same strong component of $L_{a_t} \boxtimes L_{b_1}$ if and only if $i \equiv j \pmod{a''}$ where $a'' = (a_t,b_1)$.

Now, let $\{B_i\}$ be the set of strong components of D_1 with (u,v_i) in $V(B_i)$. Similarly, let $\{B_i'\}$ be the set of strong components of $L_{a_t} \boxtimes L_{b_1}$ with (u,v_i) in $V(B_i')$.

Then we assert that $B_i \cap B_j' \neq \emptyset$ if and only if there is a point (u,v_s) in both. The sufficiency is obvious.

The necessity of this may be seen as follows. If

$[\bigcup_{j=1}^{t-1} L_{a_j}] \cap L_{a_t}$ contains only one point (which must be u) then clearly we are done. If not, suppose (u',v_t) is in $B_i \cap B'_j$ with $u' \neq u$. Then label L_{a_t} as follows: L_{a_t}: $u = u_0, u_1, \ldots, u_r = u', u_{r+1}, \ldots, u_n = u$. The P-I property implies that either P: $u = u_0, u_1, \ldots, u_r = u'$ or P': $u' = u_r, u_{r+1}, \ldots, u_n = u$ is common to both L_{a_t} and $[\bigcup_{j=1}^{t-1} L_{a_j}]$.

Assume without loss of generality that it is P'. Then there is a $(u',v_t) \dashrightarrow (u,v_s)$ path in <u>both</u> B_i and B'_j for some s. But, this clearly implies (u,v_s) is in both B_i and B'_j, as desired.

Now, (u,v_s) is in both B_i and B'_j if and only if $i \equiv s \pmod{a'}$ and $j \equiv \pmod{a''}$ using (A) and (B) above, and this happens if and only if $i \equiv j \pmod{(a',a'')}$ or equivalently $i \equiv j \pmod{a_o}$.

Hence $B_i \cap B'_j \neq \emptyset$ if and only if $i \equiv j \pmod{a_o}$.

Now it is evident that (u,v_i) and (u,v_j) are in the same component of $D = D' \cup L_{b_1}$ if and only if there is a sequence $B_i = B_{i_1}, B'_{i_2}, B_{i_3}, \ldots, B_{i_{r-1}}, B'_{i_r} = B'_j$ such that $B_{i_k} \cap B'_{i_{k+1}} \neq \emptyset$ and $B'_{i_n} \cap B_{i_{n+1}} \neq \emptyset$ for $k = 1,2,\ldots,r-1$ and $n = 2,3,\ldots,r-2$.

But from above, this happens if and only if there exists a sequence $i = i_1, i_2, \ldots, i_r = j$ such that

$i \equiv i_2 \pmod{a_o}$

$i_2 \equiv i_3 \pmod{a_o}$

$\cdots$

$i_{r-1} \equiv j \pmod{a_o}$ which, of course, happens if and only

if $i \equiv j \pmod{a_o}$.

Thus, we have (u,v_i) and (u,v_j) is the same strong component of D if and only if $i \equiv j \pmod{a_o}$, where u is in $V(L_{a_{t-1}}) \cap V(L_{a_t})$.

Therefore, $(u,v_1),(u,v_2),\ldots,(u,v_{a_o})$ all belong to distinct components of D, and any other point of the form (u,v_k) belongs to one of these.

Since D' and L_{b_1} are strong, if K is any strong component of D, then it contains some point of the form (u,v_k) for any given u in $V(D')$ using Lemma 4.3A. Hence, since there are a_o distinct strong components of D containing points of the form (u,v_k), there must be $a_o = (a_1,a_2,\ldots,a_t,b_1)$ strong components of D, proving (1).

Now, if u is any point of D', then u is in $V(L_{a_k})$ for some k. Suppose L_{a_k} is labeled as follows, L_{a_k}: $u_1,u_2,\ldots,u_{a_k},u_1$, and let $u = u_n$. Further, suppose that u_m is in $V(L_{a_k}) \cap V(L_{a_{k+1}})$. Note that this is always possible except when $k = t$. In this case, take u_m in $V(L_{a_t}) \cap V(L_{a_{t-1}})$ as above. The following argument is still valid.

Then, (u_n,v_i) and (u_n,v_j) are in the same strong component of D if and only if the points $(u_m,v_{(i+m-n)(\mathrm{mod}\ b_1)})$ and $(u_m,v_{(j+m-n)(\mathrm{mod}\ b_1)})$ are in the same component of D.

We may apply the same argument as above to the two subgraphs $D_3 = [\bigcup_{i=1}^{k} L_{a_i}] \boxtimes L_{b_1}$ and $D_4 = [\bigcup_{i=k+1}^{t} L_{a_i}] \boxtimes L_{b_1}$.

Let B_i be the strong component of D_3 containing the point (u_m, v_i) and B_i' the component of D_4 containing the same point for $i = 1, 2, \ldots, b_1$. As above, we conclude that $B_i \cap B_j' \neq \emptyset$ if and only if $i \equiv j \pmod{a_o}$; and also as above, $(u_m, v_{(i+m-n)(\text{mod } b_1)})$ is in the same component as $(u_m, v_{(j+m-n)(\text{mod } b_1)})$ if and only if one has $[(i - j)(\text{mod } b_1)] \equiv 0 \pmod{a_o}$, or $i \equiv j \pmod{a_o}$.

This proves (2).

Part (3) is a consequence of (2). For, suppose u_i and u_j are points of L_{a_k}, which is assumed labeled L_{a_k}: $u_1, u_2, \ldots, u_{a_k}, u_1$. From Theorem 3.4A, the digraph $D = \bigcup_{i=1}^{t} [L_{a_i} \boxtimes L_{b_1}]$. Let $v = v_m$ be in $V(L_{b_1})$. Then, if $i < j$, (u_i, v_m) and (u_j, v_m) are in the same component of D if and only if $(u_j, v_{(m+j-i)(\text{mod } b_1)})$ and (u_j, v_m) are in the same component of D since, if we assume that L_{b_1} is labeled $v_1, v_2, \ldots, v_{b_1}, v_1$, then the path $(u_i, v_m), (u_{i+1}, v_{(m+1)(\text{mod } b_1)}), \ldots, (u_j, v_{(m+j-i)(\text{mod } b_1)})$ is in a subpath of some O_n in $L_{a_k} \boxtimes L_{b_1}$.

But then, using part (2), (u_i, v_m) and (u_j, v_m) are in the same component of D if and only if one has $m \equiv [(m + j - i)(\text{mod } b_1)] \pmod{a_o}$.

As before, this is equivalent to $i \equiv j \pmod{a_o}$, as desired.

Thus, Lemma 4.5D is proved.

4.5.5 Heuristic description of general aims

At this point, it would be well to state the general

aim of this development. We wish to show that if D_1 and D_2 are strong digraphs with cycles $L_{a_1},\ldots,L_{a_t}$ and $L_{b_1},\ldots,L_{b_s}$ respectively, then $D_1 \boxtimes D_2$ will have $a_o = (a_1,\ldots,a_t,b_1,\ldots,b_s)$ strong components.

Now, we assert that the cycles of, say, D_1 could be rearranged in the form $L_{a_i'},\ldots,L_{a_t'}$, wherein $L_{a_i'}$ is not necessarily different from $L_{a_j'}$ when i is different from j, in such a manner that the set $L_{a_i'}$ possesses the P-I property.

This is so since the graph D_C defined by
$V(D_C) = \{L_{a_i} : i = 1,2,\ldots,t\}$ and
$X(D_C) = \{(L_{a_i},L_{a_j}) : L_{a_i} \cap L_{a_j} \neq \emptyset \text{ and } i \neq j\}$
is connected because D is strong, and hence D_C contains a complete sequence.

Now, such a complete sequence may be generated, albeit inefficiently, as follows. Begin at an arbitrary point, $L_{a_1'}$, and continue along lines subject only to the restriction that lines already used be avoided if possible. When the latter is no longer possible, backtrack until it is once again possible to choose a "new" line. When, using this procedure, it is not possible to traverse a line not already included, then we are finished.

Hence, if we assume we have chosen $\mathcal{L} = \{L_{a_i'} : i=1,\ldots,s\}$ $(s < t)$ such that it possesses the P-I property, then take any cycle L_c such that $L_c \cap L_{a_s'} \neq \emptyset$. We may assume without loss of generality that L_c is not in $\mathcal{L}$.

If the set $\mathcal{L} \cup \{L_c\}$ possesses the P-I property,

we are done. So, assume that there are points u and u' in $L_c \cap [\bigcup_{j=1}^{s} L_{a_j'}]$ such that u and u' have no path between them common to L_c and $[\bigcup_{j=1}^{s} L_{a_j'}]$. Then by assumption, L_c is $u = u_1, \ldots, u_r = u', u_{r+1}, \ldots, u_n = u$, and we may assume without loss of generality that there is no u_i $(i = 1, \ldots, r)$ in $L_c \cap [\bigcup_{j=1}^{s} L_{a_j'}]$.

Further there is a $u' \dashrightarrow u$ path, P, in $\bigcup_{j=1}^{s} L_{a_j'}$ since the latter is clearly strong.

Let $P': u = u_1, u_2, \ldots, u_r = u'$. Then $P \cup P'$ is a union of cycles at least one of which does not belong to $[\bigcup_{j=1}^{s} L_{a_j'}]$, and any such cycle may be chosen as $L_{a_{s+1}'}$ since it will intersect $[\bigcup_{j=1}^{s} L_{a_j'}]$ only on the path P.

Then proceed from $L_{a_{s+1}'}$.

Since such a procedure does generate a complete sequence, as remarked above, we obtain a set of cycles $\{L_{a_i'}\}$ possessing the P-I property, as asserted.

Here, clearly $(a_1', a_2', \ldots, a_t') = (a_1, a_2, \ldots, a_t)$.

Similarly, $D_2 = \bigcup_{i=1}^{s'} L_{b_i'}$ with $\{L_{b_j'}\}$ possessing the P-I property.

So, given that D_1 and D_2 are strong digraphs with cycles $L_{a_1}, \ldots, L_{a_t}$ and $L_{b_1}, \ldots, L_{b_s}$ respectively, we may write

$D_1 = \bigcup_{i=1}^{t'} L_{a_i'}$ and $D_2 = \bigcup_{j=1}^{s'} L_{b_j'}$ with $\{L_{a_i'}\}$ and $\{L_{b_j'}\}$ possessing the P-I property, and

$(a_1', \ldots, a_{t'}', b_1', \ldots, b_{s'}') = (a_1, \ldots, a_t, b_1, \ldots, b_s)$.

The approach will then be as follows. The digraph

$D' = [\bigcup_{i=1}^{t'} L_{a_i'}] \boxtimes L_{b_1'} = \bigcup_{i=1}^{t'} [L_{a_i'} \boxtimes L_{b_1'}]$ has

$a' = (a_1', a_2', \ldots, a_{t'}', b_1')$ strong components according to Lemma 4.5D. In a similar fashion,

$D'' = L_{a_1'} \boxtimes [\bigcup_{i=1}^{s'} L_{b_i'}] = \bigcup_{i=1}^{s'} [L_{a_1'} \boxtimes L_{b_i'}] \approx \bigcup_{i=1}^{s'} [L_{b_i'} \boxtimes L_{a_1'}]$

has $a'' = (a_1', b_1', \ldots, b_{s'}')$ strong components.

Having done this, certain subgraphs D^i of $D_1 \boxtimes D_2$ will be defined for $i = 1, 2, \ldots, \min(s', t')$ having the properties (1) $D^1 = D' \cup D''$, and

(2) $D_1 \boxtimes D_2 = \bigcup_{i=1}^{N} D^i$ where $N = \min(s', t')$.

Then by induction on k, it will be shown that $\bigcup_{i=1}^{k} D^i$ has $a_o = (a', a'')$ strong components for all $k \leq \min(s', t')$. But, the latter and (2) will clearly imply that $D_1 \boxtimes D_2$ has a_o components also.

4.6 RIGOROUS DEVELOPMENT OF SECTION 4.5.5

In accordance with the outline of Section 4.5.5, we state the following lemma.

LEMMA 4.6A. Let D_1 and D_2 be strong digraphs with cycles $L_{a_1}, \ldots, L_{a_t}$ and $L_{b_1}, \ldots, L_{b_s}$ respectively. Let $D_1 = \bigcup_{i=1}^{t'} L_{a_i'}$ and $D_2 = \bigcup_{j=1}^{s'} L_{b_j'}$ with $\{L_{a_i'}\}$ and $\{L_{b_j'}\}$ possessing the P-I property. Further, let $D' = \bigcup_{i=1}^{t'} [L_{a_i'} \boxtimes L_{b_1'}]$ and $D'' = \bigcup_{j=1}^{s'} [L_{a_1'} \boxtimes L_{b_j'}]$, and let $a_o = (a_1, \ldots, a_t, b_1, \ldots, b_s)$. Then,

(1) $D' \cup D''$ has a_o strong components,

(2) If u is in $V(L_{a'_i})$ and v_m and v_n are in $V(L_{b'_j})$ for some j, then (u,v_m) and (u,v_n) are in the same strong component of $D' \cup D''$ if and only if $m \equiv n \pmod{a_o}$, assuming that $L_{b'_j}$ is labeled $L_{b'_j}$: $v_1,\ldots,v_{b'_j},v_1$, and

(3) If v is in $V(L_{b'_1})$ and u_m and u_n are in $V(L_{a'_j})$ for some j, then (u_m,v) and (u_n,v) are in the same strong component of $D' \cup D''$ if and only if $m \equiv n \pmod{a_o}$, assuming that $L_{a'_j}$ is labeled $L_{a'_j}$: $u_1,\ldots,u_{a'_j},u_1$.

PROOF. For (1), take u in $V(L_{a'_1})$, and let $B_1, B_2,\ldots, B_{b'_1}$ be the components of D' such that (u,v_j) is in B_j, for each v_j in $V(L_{b'_1})$.

Also let $B'_1, B'_2,\ldots, B'_{b'_1}$ be the components of D'' such that (u,v_j) is in B'_j.

Then $B_i \cap B'_j \neq \emptyset$ if and only if there is some (u,v_s) in both, which happens if and only if there is an s with $i \equiv s \pmod{a'}$, $a' = (a_1,\ldots,a_t,b'_1)$ and

$$j \equiv s \pmod{a''},\quad a'' = (a'_1,b_1,\ldots,b_s).$$

Thus, $B_i \cap B'_j \neq \emptyset$ if and only if $i \equiv j \pmod{a_o}$ where $a_o = (a',a'') = (a_1,\ldots,a_t,b_1,\ldots,b_s)$.

Then for (u,v_i) and (u,v_j) in $V(L_{a'_1} \boxplus L_{b'_1})$, one clearly has that (u,v_i) and (u,v_j) belong to the same component of $D' \cup D''$ if and only if there is a sequence $B_i = B_{i_1}, B'_{i_2},\ldots, B_{i_{n-1}}, B'_{i_n} = B'_j$ with

$B_{i_k} \cap B'_{i_{k+1}} \neq \emptyset$ and $B'_{i_k} \cap B_{i_{k+1}} \neq \emptyset$ for the obvious values of k in each instance.

But from the previous remark, this is true if and only if $i \equiv i_2 \pmod{a_o}$, $i_2 \equiv i_3 \pmod{a_o}, \ldots, i_{n-1} \equiv j \pmod{a_o}$, or equivalently, if and only if $i \equiv j \pmod{a_o}$.

Now, any component B of $D' \cup D''$ intersects either D' or D'', and therefore must contain some B_n or B'_m which, in turn intersect B'_n and B_m respectively. Hence B contains $B_n \cup B'_n$ for some n, and therefore contains a point (u,v_n) for the given u and some v_n in $V(L_{b'_1})$.

On the other hand, if (u,v_1) is in B, then from above, $(u,v_{(ja_o+1)(\mathrm{mod}\, b'_1)})$ is in B for all j, and if (u,v_m) is in B, then $m = [(ja_o + 1) \pmod{b'_1}]$ for some j.

Thus, the points $(u,v_1),(u,v_2),\ldots,(u,v_{a_o})$ are in different components of $D' \cup D''$, and if B is any component of $D' \cup D''$, it must contain one of these points. This obviously implies that $D' \cup D''$ has exactly a_o distinct strong components, completing the proof of part (1) of Lemma 4.6A.

We prove part (2) by induction on j. From the proof of part (1), the statement is true for $j = 1$. Assume that it is true for $j = k < s'$, and take v_m and v_n in $V(L_{b'_{k+1}})$ where we assume $L_{b'_{k+1}}$ to be labeled $L_{b'_{k+1}}$: $v_1, v_2, \ldots, v_{b'_{k+1}}, v_1$. Further assume that $n > m$.

Then, (u_s,v_m) and (u_s,v_n) are in the same

strong component of $D' \cup D''$ if and only if (u_s, v_m) and $(u_{[s-(n-m)](\text{mod } a_1')}, v_m)$ are in the same component, where $L_{a_1'}$ is assumed labeled $u_1, u_2, \ldots, u_{a_1'}, u_1$.

Suppose v_t is in $V(L_{b_k'}) \cap V(L_{b_{k+1}'})$. Then (u_s, v_m) and (u_s, v_n) are in the same component of $D' \cup D''$ if and only if $(u_{(s+b_{k+1}'-m+t)(\text{mod } a_1')}, v_t)$ and $(u_{(s+b_{k+1}'-n+t)(\text{mod } a_1')}, v_t)$ are in the same component using the preceding expressions.

However, these points are of the form (u_d, v_c') and (u_e, v_c') in $L_{a_1'} \cap L_{b_k'}$, with, say, $d \leqq e$. Here, it is assumed that $L_{b_k'}$ is labeled $v_1', v_2', \ldots, v_{b_k'}', v_1'$, and $v_t = v_c'$ for some c, since v_t is in both $V(L_{b_k'})$ and $V(L_{b_{k+1}'})$ by assumption. But, these are in the same strong component of $D' \cup D''$ if and only if (u_d, v_c') and $(u_d, v'_{[c-(e-d)](\text{mod } b_k')})$ are, which by hypothesis happens if and only if $c \equiv \{[c - (e - d)] \pmod{b_k'}\} \pmod{a_o}$, or equivalently, $hb_k' + rc + e \equiv d \pmod{a_o}$ where $r = h = 0$ if $e - d \prec c$ for some $c \prec b_k'$, and $h \succ 0$, $r = 2$ if $e - d \succeq c$ for all such c.

The former case is equivalent to $d \equiv e \pmod{a_o}$, and in the latter case, we may label the cycle $L_{b_k'}$ so that $c = b_k'$, and again get $d \equiv e \pmod{a_o}$.

Thus, (u_s, v_m) and (u_s, v_n) are in the same strong component of $D' \cup D''$ if and only if one has $[(s + b_{k+1}' - m + t)(\text{mod } a_1')] \equiv [(s + b_{k+1}' - n + t)(\text{mod } a_1')]$ modulo a_o, or, if and only if $[(n-m)(\text{mod } a_1')] \equiv 0 \pmod{a_o}$.

Therefore, (u_s,v_m) and (u_s,v_n) are in the same strong component of $D' \cup D''$ if and only if $n \equiv m \pmod{a_o}$. This proves part (2).

The symmetry of the situation then establishes part (3), concluding the proof of the lemma.

We will also need the following lemma.

LEMMA 4.6B. Suppose $D_1 = L_{a_1} \cup L_{a_2}$ where L_{a_i} is a cycle of length a_i, and $D_2 = L_{b_1} \cup L_{b_2}$ where L_{b_j} is a cycle of length b_j, are strong digraphs. Then, $D = D_1 \boxtimes D_2 = \bigcup_{i,j} [L_{a_i} \boxtimes L_{b_j}]$ has the same number of strong components as has $D' \cup D''$, where

$D' = [L_{a_1} \boxtimes L_{b_1}] \cup [L_{a_2} \boxtimes L_{b_1}]$ and

$D'' = [L_{a_1} \boxtimes L_{b_1}] \cup [L_{a_1} \boxtimes L_{b_2}]$.

PROOF. If u is a point of L_{a_1}, and L_{b_1} is assumed labeled $v_1,v_2,\ldots,v_{b_1},v_1$, then we know from Lemma 4.6A that (u,v_i) and (u,v_j) are in the same strong component of $D' \cup D''$ if and only if $i \equiv j \pmod{a_o}$ where $a_o = (a_1,a_2,b_1,b_2)$.

In particular this is true when u is a point of $L_{a_1} \cap L_{a_2}$, which cannot be empty since D_1 is strong.

So, take u in $V(L_{a_1}) \cap V(L_{a_2})$, and let $B_1,\ldots,B_{b_2}$ be the components of $D' \cup D''$ containing the points $(u,v_1'),\ldots,(u,v_{b_2}')$ respectively, for L_{b_2} labeled $v_1',\ldots,v_{b_2}',v_1'$. Also let $B_1',\ldots,B_{b_2}'$ be the components of $L_{a_2} \boxtimes L_{b_2}$ containing these same points respectively.

Then $B_i \cap B'_j \neq \emptyset$ if and only if there exists a point (u,v_s) in both.

Hence, $B_i \cap B'_j \neq \emptyset$ if and only if $i \equiv s \pmod{a_o}$ and $j \equiv s \pmod{(a_2,b_2)}$, or equivalently $i \equiv j \pmod{a_o}$ since $(a_o,(a_2,b_2)) = a_o$.

However (u,v'_i) and (u,v'_j) are in the same strong component of D if and only if there is a sequence $B_i = B_{i_1}, B'_{i_2}, B_{i_3}, B'_{i_4},\ldots,B_{i_{t-1}},B'_{i_t} = B'_j$ such that $B_{i_m} \cap B'_{i_{m+1}} \neq \emptyset$ and $B'_{i_n} \cap B_{i_{n+1}} \neq \emptyset$ for $m = 1,\ldots,t-1$ and $n = 2,\ldots,t-2$.

This, of course, happens if and only if $i \equiv i_2 \pmod{a_o}$, $i_2 \equiv i_3 \pmod{a_o},\ldots, i_{t-1} \equiv j \pmod{a_o}$, or if and only if $i \equiv j \pmod{a_o}$.

Thus, the points $(u,v'_1),(u,v'_2),\ldots,(u,v'_{a_o})$ belong to distinct strong components of D. On the other hand, given any strong component of D, it must contain a point of the form (u,v'_s) since it contains some component of $L_{a_1} \otimes L_{b_2}$ which in turn has this property, using Lemma 4.3A. But a component of D containing (u,v_s) must contain one of the above a_o points, and hence there must be a_o distinct strong components in D, proving Lemma 4.6B.

Continuing with the outline given previously, we state the following definition.

<u>DEFINITION 4.6A</u>. Let $D_1 = \bigcup_{i=1}^{t'} L_{a'_i}$ and $D_2 = \bigcup_{j=1}^{s'} L_{b'_j}$ with $\{L_{a'_i}\}$ and $\{L_{b'_j}\}$ possessing the P-I property. Let $D^k = [\bigcup_{j=k}^{s'} (L_{a'_k} \otimes L_{b'_j})] \cup [\bigcup_{i=k}^{t'} (L_{a'_i} \otimes L_{b'_k})]$ for $k \leq \min(s',t')$.

Here, clearly $D^1 = D' \cup D''$, where D' and D'' are as given in Lemma 4.6A.

LEMMA 4.6C. Let $D = \bigcup_{k=1}^{N} D^k$ with $N = \min(s', t')$. Then $D = D_1 \boxtimes D_2$.

PROOF. Clearly $D \subset D_1 \boxtimes D_2 = \bigcup_{i,j} [L_{a_i'} \boxtimes L_{b_j'}]$.

But, if $L_{a_i'} \boxtimes L_{b_j'}$ is arbitrary with $i \leq t'$ and $j \leq s'$, then either $i \leq j$ or $j < i$.

When $i \leq j$, $L_{a_i'} \boxtimes L_{b_j'} \subset \bigcup_{k=i}^{s'} [L_{a_i'} \boxtimes L_{b_k'}] \subset D^i$, and if $j < i$, $L_{a_i'} \boxtimes L_{b_j'} \subset \bigcup_{k=j}^{t'} [L_{a_k'} \boxtimes L_{b_j'}] \subset D^j$.

Thus, $L_{a_i'} \boxtimes L_{b_j'} \subset D^i \cup D^j \subset D$, so that $D = D_1 \boxtimes D_2$.

THEOREM 4.6A. If D_1 and D_2 are strong digraphs with cycles $L_{a_1}, \ldots, L_{a_t}$ and $L_{b_1}, \ldots, L_{b_s}$ respectively, where L_n has length n, then $D_1 \boxtimes D_2$ has $a_o = (a_1, \ldots, a_t, b_1, \ldots, b_s)$ strong components (which are, of course, also weak components).

PROOF. As before, we let $D_1 = \bigcup_{i=1}^{t'} L_{a_i'}$ and $D_2 = \bigcup_{j=1}^{s'} L_{b_j'}$ with $\{L_{a_i'}\}$ and $\{L_{b_j'}\}$ both possessing the P-I property.

Form the union $D_m' = \bigcup_{k=1}^{m} D^k$, where D^k is defined in Definition 4.6A, and $m \leq \min(s', t')$.

Then, we prove by induction that:

(1) D_m' has a_o strong components for all m such that $1 \leq m \leq \min(s', t')$,

(2) If $L_{a_i'} \boxplus L_{b_j'} \subset D_m'$, u is in $V(L_{a_i'})$ and v_c and v_d are in $V(L_{b_j'})$, then (u,v_c) and (u,v_d) are in the same strong component of D_m' if and only if $c \equiv d \pmod{a_o}$, where $L_{b_j'}$ is assumed labeled $v_1,v_2,\cdots,v_{b_1'},v_1$, and

(3) If $L_{a_i'} \boxplus L_{b_j'} \subset D_m'$, u_c and u_d are in $V(L_{a_i'})$ and v is in $V(L_{b_j'})$, then (u_c,v) and (u_d,v) are in the same strong component of D_m' if and only if $c \equiv d \pmod{a_o}$, where $L_{a_i'}$ is assumed labeled $u_1,u_2,\cdots,u_{a_i'},u_1$.

To begin with, we see that for $m = 1$, we have $D_m' = D_1' = D^1 = D' \cup D''$. But, if $L_{a_i'} \boxplus L_{b_j'}$ is a subgraph of D_1', then $a_i' = a_1'$ or $b_j' = b_1'$. Refering to Lemma 4.6A, then, we see that (1), (2) and (3) are true if $m = 1$.

Suppose (1), (2) and (3) are true for $m-1 < \min(s',t')$, and consider $D_{m'}'$. Here $D_m' = D_{m-1}' \cup D'$, where, $D' = D^m$.

Referring once again to Lemma 4.6A, one finds that

(A) D' has $a' = (a_m',\cdots,a_t',b_m',\cdots,b_s')$ strong components,

(B) If u is in $V(L_{a_m'})$, v_i and v_j are in $V(L_{b_k'})$ for some k, $m \leq k \leq s'$, then (u,v_i) and (u,v_j) are in the same strong component of D' if and only if $i \equiv j \pmod{a'}$, assuming $L_{b_k'}$ to be labeled $v_1,v_2,\cdots,v_{b_k'},v_1$, and

(C) If u_i and u_j are in $V(L_{a_k'})$ for some k, with $m \leq k \leq t'$, and v is in $V(L_{b_m'})$, then (u_i,v)

and (u_j,v) are in the same component of D' if and only if $i \equiv j \pmod{a'}$, assuming that $L_{a'_k}$ is labeled $u_1,u_2,\ldots,u_{a'_k},u_1$.

Now, heuristically, we have the situation shown below: recalling, however, that $L_{a'_i}$ is not necessarily distinct from $L_{a'_j}$ for $i \neq j$ (and similarly for the $L_{b'}$s). The pair (i,j) represents $L_{a'_i} \boxtimes L_{b'_j}$.

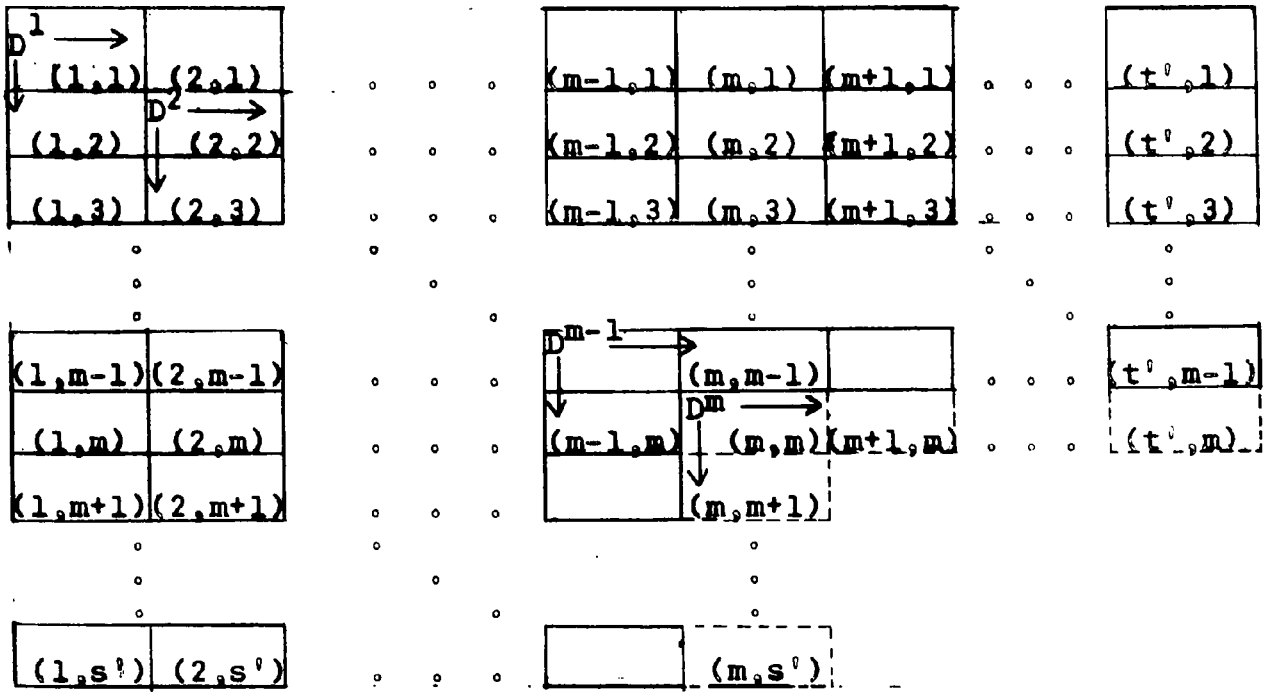


FIGURE 17 -- The union of the D^k

If the addition of D' to D'_{m-1} changes the number of strong components between D'_{m-1} and D'_m, then D'_m will have fewer than D'_{m-1}. This is so since, for every u in $V(L_{a'_m}) \cap V(L_{a'_{m-1}})$, one has (u,v) in D' for every v in $V(L_{b'_m})$, which is the case since $L_{a'_{m-1}} \boxtimes L_{b'_m} \subset D'_{m-1}$.

But, if B' is a component of D', then it contains

either some component, B_1', of $\bigcup_{j=m}^{s'} [L_{a_m'} \otimes L_{b_j'}]$ or some component, B_2', of $\bigcup_{i=m}^{t'} [L_{a_i'} \otimes L_{b_m'}]$, and therefore contains one of both since such components intersect in $L_{a_m'} \otimes L_{b_m'}$.

Suppose $B \supset B_1'$. Then $P_2[V(B_1')] = \bigcup_{j=m}^{s'} V(L_{b_j'})$ where, clearly, the latter contains $V(L_{b_m'})$. Hence, for every v_n in $V(L_{b_m'})$, there is a u_k in $V(L_{a_m'})$ such that (u_k, v_n) is in $V(B_1')$. If u_k is not in $V(L_{a_m'}) \cap V(L_{a_{m-1}'})$, and u_j is, with say, $k > j$, then $(u_j, v_{[n-(j-k)](\text{mod } b_m')})$ is in the same component as (u_k, v_n), namely B_1', and u_j is in $V(L_{a_m'}) \cap V(L_{a_{m-1}'})$.

Therefore, every component B' of D' contains a point (u,v) of $L_{a_m'} \otimes L_{b_m'}$ with u in $V(L_{a_m'})$ $V(L_{a_{m-1}'})$. This clearly implies that B' intersects <u>some</u> component, B, of D_{m-1}' since (u,v) is in $V(D_{m-1}')$. Hence the number of components decreases if it changes.

But the number of components can decrease only if some component of D' intersects two distinct components of D_{m-1}'. Thus, for the proof of (1), it suffices to show that any component of D' intersects at most one component of D_{m-1}'. This, too, will be proved by induction.

Let B' be a component of D'. Then from above, B' contains a point (u_i, v_j) of $L_{a_m'} \otimes L_{b_m'}$ with the point u_i in $V(L_{a_m'}) \cap V(L_{a_{m-1}'})$. This point also belongs to a component, B_1, of D_{m-1}', and hence $B' \cap B_1 \neq \emptyset$.

Suppose (u_c, v_d) is in $B' \cap B_2$ for some other component, B_2, of D_{m-1}', and suppose (u_c, v_d) is a

point in $L_{a'_m} \boxtimes L_{b'_m}$. Now, if $u_c \neq u_i$, then a translation of the previous types will assure that some (u_i, v_k) is in $B' \cap B_2$ if and only if (u_c, v_d) is, where $k = \{[d - (i-c)] \pmod{b'_m}\}$ or $k = d + c - i \pm eb'_m$ with $e > 0$.

Thus, (u_i, v_j) and (u_i, v_k) are in B', but not both in B_1, implying $j \equiv k \pmod{a'}$ and $j \not\equiv k \pmod{a_o}$ which is a clear contradiction since a_o divides a'.

Therefore, if $B' \cap B_2 \neq \emptyset$, the point of intersection is in $L_{a'_m} \boxtimes L_{b'_{m+i}}$ for some i, $0 < i \leq s'-m$, or in $L_{a'_{m+j}} \boxtimes L_{b'_m}$ for some j, $0 < j \leq t'-m$.

Suppose that $B' \cap B_2 = \emptyset$ in all $L_{a'_m} \boxtimes L_{b'_{m+i}}$ for $i \leq k < s'-m$, and suppose that $B' \cap B_2 \neq \emptyset$ in $L_{a'_m} \boxtimes L_{b'_{m+k+1}}$, with $B_2 \neq B_1$.

Then we have $B' \cap B_1 \neq \emptyset$ in $L_{a'_m} \boxtimes L_{b'_{m+k}}$ and $B' \cap B_2 \neq \emptyset$ in $L_{a'_m} \boxtimes L_{b'_{m+k+1}}$ with $B_1 \neq B_2$.

So, if we look at the subgraph

$$\underline{D} = (L_{a'_{m-1}} \boxtimes L_{b'_{m+k}}) \cup (L_{a'_m} \boxtimes L_{b'_{m+k}}) \cup (L_{a'_{m-1}} \boxtimes L_{b'_{m+k+1}}),$$

we see clearly that $\underline{D} \cup (L_{a'_m} \boxtimes L_{b'_{m+k+1}})$ has fewer components than does $\underline{D}$, contradicting Lemma 4.6B.

Therefore, for any component B_2 of D'_{m-1} not equal to B_1, $B' \cap B_2 = \emptyset$ in all $L_{a'_m} \boxtimes L_{b'_{m+i}}$ for $i \leq s' - m$, or equivalently, in all subgraphs of the form $L_{a'_m} \boxtimes L_{b'_j}$ with $m \leq j \leq s'$.

In precisely the same way, $B' \cap B_2 = \emptyset$ in all subgraphs $L_{a'_i} \boxtimes L_{b'_m}$ with $m \leq i \leq t'$, so that one has $B' \cap B_2 = \emptyset$ in all of D'. This, of course, implies that

B' intersects only B_1, so that, given any component B' of D', it intersects exactly one component, B, of D'_{m-1}.

But, then, we can clearly conclude that D'_m has the same number of components as does D'_{m-1} for all m. However, D'_1 has a_o strong components, and therefore, D'_m has a_o strong components for all $m \leq \min(s',t')$, proving part (1).

For the proof of (2), let $L_{a'_i} \boxtimes L_{b'_j} \subset D'_m$ with u in $V(L_{a'_i})$ and v_c and v_d in $V(L_{b'_j})$.

The assertion is that (u,v_c) and (u,v_d) are in the same component of D'_m if and only if $c \equiv d \pmod{a_o}$ with $L_{b'_j}$ assumed labeled cyclically.

This is true for $m = 1$ by Lemma 4.6A, as already remarked.

Suppose it is true for all $k \leq m-1 < \min(s',t')$. Then if $L_{a'_i} \boxtimes L_{b'_j}$ is in D'_{m-1}, we are obviously done. So, suppose $L_{a'_i} \boxtimes L_{b'_j} \subset D' = D^m$.

Let B'_1 and B'_2 be components of D', and suppose (u,v_c) is in $V(B'_1)$ and (u,v_d) is in $V(B'_2)$. Then clearly, (u,v_c) and (u,v_d) belong to the same component of D'_m if and only if there is some component B of D'_{m-1} such that $B \cap B'_1 \neq \emptyset$ and $B \cap B'_2 \neq \emptyset$.

But, if $u = u_k$, and u_n is in $V(L_{a'_i}) \cap V(L_{a'_{i-1}})$, then $(u_n, v_{(c+n-k)(\mathrm{mod}\ b'_j)})$ and $(u_n, v_{(d+n-k)(\mathrm{mod}\ b'_j)})$ belong to B'_1 and B'_2 respectively, and also to B (recalling that they must belong to B since, for example, B'_1 intersects exactly one component of D'_{m-1}).

But both these points can belong to B in $L_{a'_{i-1}} \boxtimes L_{b'_j}$ if and only if $\{[(c+n-k) - (d+n-k)] \pmod{b'_j}\} \equiv 0 \pmod{a_o}$ by the induction hypothesis, or equivalently, if and only if $c \equiv d \pmod{a_o}$.

This proves (2).

Now, (3) follows from (2) using the symmetry of the situation, completing the proof of (1), (2) and (3).

Thus, in particular, $D'_{min(s',t')} = D_1 \boxtimes D_2$ from Lemma 4.6C, and $D'_{min(s',t')}$ has a_o strong components from above, proving Theorem 4.6A.

Having also obtained the following facts, we state the corollary:

COROLLARY 4.6A. If D_1 and D_2 are strong digraphs with cycles $L_{a_1},\ldots,L_{a_t}$ and $L_{b_1},\ldots,L_{b_s}$ respectively, and $a_o = (a_1,\ldots,a_t,b_1,\ldots,b_s)$, then

(1) $D_1 \boxtimes D_2$ is strong if and only if $a_o = 1$,

(2) If u is in $V(L_{a_n})$ and v_i and v_j are in $V(L_{b_m})$ where it is assumed that L_{b_m} is labeled $v_1,\ldots,v_{b_m},v_1$, then (u,v_i) and (u,v_j) are in the same strong component of $D_1 \boxtimes D_2$ if and only if $i \equiv j \pmod{a_o}$, and

(3) If u_i and u_j are in $V(L_{a_n})$ and v is in $V(L_{b_m})$ where it is assumed that L_{a_n} is labeled $u_1,\ldots,u_{a_n},u_1$, then (u_i,v) and (u_j,v) are in the same strong component of $D_1 \boxtimes D_2$ if and only if $i \equiv j \pmod{a_o}$.

PROOF. Part (1) is immediately evident from the

theorem, and parts (2) and (3) are equally evident since these conditions are true for $D'_{\min(s',t')}$, and since $L_{a_n} \boxtimes L_{b_m}$ is some $L_{a'_i} \boxtimes L_{b'_j}$ and conversely.

4.7 APPLICATIONS OF THE CHARACTERIZATION OF C_3

4.7.1 2-cycles and graphs

COROLLARY 4.7A. Suppose D_1 and D_2 are strong digraphs such that D_1 or D_2 has a cycle of length two. Then $D_1 \boxtimes D_2$ will be strong if and only if D_1 or D_2 has a cycle of odd length. Otherwise, $D_1 \boxtimes D_2$ will have two strong components which are also the weak components of $D_1 \boxtimes D_2$.

PROOF. If D_1 has cycles $L_{a_1},\ldots,L_{a_t}$ and D_2 has cycles $L_{b_1},\ldots,L_{b_s}$, then $D_1 \boxtimes D_2$ is strong if and only if $a_o = (a_1,\ldots,a_t,b_1,\ldots,b_s)$ equals one, by Corollary 4.6A.

Suppose that some a_i or b_j is 2. Then if $a_o = 1$, some a_i or b_j must be an odd number, implying the existence of an odd cycle in D_1 or D_2.

Conversely, if D_1 or D_2 has an odd cycle, say of length $2m - 1$, then $(2,2m-1) = 1$, so that $a_o = 1$ and $D_1 \boxtimes D_2$ is strong.

Clearly, if $a_o \neq 1$, then $a_o = 2$. But then, $D_1 \boxtimes D_2$ has two strong components by Theorem 4.6A. These strong components are, of course, the weak components of $D_1 \boxtimes D_2$ also.

COROLLARY 4.7B. If D_1 and D_2 are connected graphs, then $D_1 \boxtimes D_2$ will be connected if and only if D_1 or D_2 has an odd cycle. Otherwise $D_1 \boxtimes D_2$ will have two components.

PROOF. Since D_1 and D_2 are connected graphs, they are strong symmetric digraphs. Thus, D_1 and D_2 are strong digraphs both of which contain a cycle of length two. Applying Corollary 4.7A yields the desired result.

The above result (Corollary 4.7B) is known, and is due to Weichsel, [9]. As remarked in the Introduction, it was this particular result that stimulated the present investigation.

4.7.2 Category C_2

Refering to Theorem 4.4A, we can state the following.

COROLLARY 4.7C. If D_1 is in C_2, D_2 is strong, S_r the receiving strong component of D_1, and S_t the transmitting strong component of D_1, and if a_o is the g.c.d. of the cycle lengths of cycles of S_r, b_o the g.c.d. of the cycle lengths of cycles of S_t, and d_o the g.c.d. of the cycle lengths of cycles of D_2, then $D_1 \boxtimes D_2$ is in C_2 if and only if $(a_o, d_o) = 1$ and $(b_o, d_o) = 1$.

PROOF. The result follows directly from Theorem 4.4A wherein it is stated that $D_1 \boxtimes D_2$ will be in C_2 if and only if $S_r \boxtimes D_2$ and $S_t \boxtimes D_2$ are strong, and from Corollary 4.6A.

Of course, the criteria of Corollary 4.7C are ammenable to simplification as per Corollary 4.7A whenever S_r, S_t, or D_2 has a cycle of length two. This is particularly true whenever D_2 has such a cycle.

4.8 DISCUSSION OF WEAK CONNECTEDNESS

Before proceeding to the investigation of other operations, it should be remarked that, to date, no necessary and sufficient conditions are known that will insure that $D_1 \boxtimes D_2$ will be weak.

If such a criterion were known, it would complete the investigation of connectedness of tensor products of digraphs, at least in theory, since not weak characterizes category C_0 and weak but not unilateral characterizes category C_1. The above results, when combined, yield a means of characterizing the unilateral case, of course.

After considerable investigation, the writer feels one can say almost certainly that a characterization of weak connectedness will involve more of what might well be called "fine structure" of the digraphs D_1 and D_2.

That is, at present it appears that any such result will depend more on the manner in which D_1 and D_2 are "put together" than do the above results, which depend only upon connectedness of various kinds and lengths of cycles.

This is clearly true for the necessary condition of R-T compatibility (cf., Definition 4.2A and Corollary 4.2D). However, certain results indicate an even more general

dependence upon lengths of certain types of paths and their relation to certain strong components.

The results alluded to are not presented here because of their prohibitive length balanced against the fact that they lend very limited insight into an extremely small class of weak digraphs.

CHAPTER V -- CARTESIAN PRODUCTS OF DIGRAPHS

5.1 DEFINITIONS AND STRUCTURE OF $([\mathfrak{D}], \times)$

DEFINITION 5.1A. The cartesian product of two digraphs D_1 and D_2, denoted $D_1 \times D_2$, is the digraph D whose points and lines are

$V(D) = V_1 \times V_2 = \{(u,v) : u \in V_1 \text{ and } v \in V_2\}$, and

$X(D) = \{ \langle (u,v),(u',v') \rangle : (u,v),(u',v') \in V(D)$, and

either $u = u'$ and $\langle v,v' \rangle \in X_2$

or $v = v'$ and $\langle u,u' \rangle \in X_1 \}$.

THEOREM 5.1A. Given two digraphs D_1 and D_2: $D_1 \times D_2$ is isomorphic to $D_2 \times D_1$.

PROOF. If we define a map $\mu : V(D_1 \times D_2) \longrightarrow V(D_2 \times D_1)$ by $\mu(u,v) = (v,u)$, then it is clear that μ is one-to-one and onto.

Now, let $x = \langle (u,v),(u',v') \rangle$ be in $X(D_1 \times D_2)$. Then either, (i) $u = u'$ and $\langle v,v' \rangle$ is in X_2

or, (ii) $v = v'$ and $\langle u,u' \rangle$ is in X_1.

In either case, it is clear that the pair $x' = \langle (v,u),(v',u') \rangle$ is in $X(D_2 \times D_1)$ by definition of the cartesian product. But, $x' = \mu x = \langle \mu(u,v), \mu(u',v') \rangle$, so that if x is in $X(D_1 \times D_2)$, then $\mu x = x'$ is in $X(D_2 \times D_1)$.

The converse is equally evident from (i) and (ii), and therefore $D_1 \times D_2 \simeq D_2 \times D_1$.

THEOREM 5.1B. $([\mathfrak{D}], \times)$ is a commutative semi-group with identity.

PROOF. That $([\mathfrak{D}], \times)$ is a semi-group is a straightforward verification, and the commutativity follows from Theorem 5.1A.

The one point digraph K_1 determines the identity class $[K_1]$. To see this, let D_1 be an arbitrary digraph. Then, if we denote K_1 by $\{v_1\}$, we have $V(D_1 \times K_1) = \{(u,v_1) : u \in V_1\}$ and $X(D_1 \times K_1) = \{\langle (u,v_1),(u',v_1)\rangle : u,u' \in V_1 \text{ and } \langle u,u'\rangle \in X_1\}$.

Thus, the map $\flat(u,v_1) = u$ is a one-to-one map of $V(D_1 \times K_1)$ onto V_1 having the property that, for all $x = \langle (u,v_1),(u',v_1)\rangle$ in $X(D_1 \times K_1)$, the image $\langle \flat(u,v_1), \flat(u',v_1)\rangle = \langle u,u'\rangle$ is in X_1 and conversely. That is, $\flat$ is an isomorphism between $D_1 \times K_1$ and D_1.

Since $([\mathfrak{D}], \times)$ is commutative, however, it follows that $[K_1]$ is the identity of this semi-group.

Because of this relationship, we will allow the possibilities that digraphs in this chapter may be K_1 and may have independent point sets.

5.2 DISCONNECTEDNESS

5.2.1 Characterization of disconnectedness

THEOREM 5.2A. Let D_1 and D_2 be digraphs. Then $D_1 \times D_2$ is disconnected if and only if D_1 or D_2 is disconnected.

PROOF. For the sufficiency, we may, using Theorem 5.1A, assume without loss of generality that D_1 is disconnected. Then there exist points u and u' in V_1 such that there is no semi-path between them in D_1.

Take arbitrary points v and v' in V_2, and suppose that $S: (u,v)=(u_0,v_0),(u_1,v_1),\ldots,(u_n,v_n)=(u',v')$ is a (u,v) --- (u',v') semi-path in $D_1 \times D_2$.

By our assumption, $u_0 \neq u_n$, and we are assured that there exists $u_j \neq u_0$ in S. Suppose that u_{i_1} is such that i_1 is the first j with u_j different from $u_{i_0} = u_0$ in S. Then, $u_0 = u_1 = \cdots = u_{i_1-1} \neq u_{i_1}$.

Using Definitions 2.1F and 4.5A, then $v_0 \neq v_1 \neq \cdots \neq v_{i_1-1} = v_{i_1}$, and either $\langle(u_{i_1},v_{i_1}),(u_{i_1-1},v_{i_1-1})\rangle$ or $\langle(u_{i_1-1},v_{i_1-1}),(u_{i_1},v_{i_1})\rangle$ is in $X(D_1 \times D_2)$. Hence, $\langle u_{i_1},u_{i_1-1}\rangle$ or $\langle u_{i_1-1},u_{i_1}\rangle$ is in X_1, so that $\langle u_{i_1},u_{i_0}\rangle$ or $\langle u_{i_0},u_{i_1}\rangle$ is in X_1 by choice of i_1.

Now, assume that sequence $u_0 = u_{i_0},u_{i_1},\ldots,u_{i_k}$ of points in $P_1[V(S)]$ has been found satisfying

(1) $i_j \succ i_t$ if and only if $j \succ t$, and

(2) $\langle u_{i_j},u_{i_{j+1}}\rangle$ or $\langle u_{i_{j+1}}, u_{i_j}\rangle$ is in X_1 for each j, $0 \leqq j \leqq k-1$.

Then choose $u_{i_{k+1}}$ in $P_1[V(S)]$ such that it is the first u_j with $j \succ i_k$ such that $u_j \neq u_{i_k}$.

If no such $u_{i_{k+1}}$ exists, then $u_{i_k} = u_n = u'$, and the above sequence of points defines a u_0 --- u_n semi-

sequence in D_1, implying the existence of a u --- u' semi-path in D_1, which is a clear contradiction.

But, for such a $u_{i_{k+1}}$, the sequence $u_{i_0}, u_{i_1}, \ldots, u_{i_{k+1}}$ clearly satisfies condition (1). For (2), note that $u_{i_{k+1}} \neq u_{i_{k+1}-1}$, and therefore, as above,

$\langle u_{i_{k+1}}, u_{i_{k+1}-1} \rangle = \langle u_{i_{k+1}}, u_{i_k} \rangle$ or

$\langle u_{i_{k+1}-1}, u_{i_{k+1}} \rangle = \langle u_{i_k}, u_{i_{k+1}} \rangle$ is in X_1.

Thus there exists a sequence of points of D_1, S_1: $u = u_{i_0}, u_{i_1}, \ldots, u_{i_s} = u'$ satisfying (1) and (2), and in particular (2). So, S_1 is a u --- u' semi-sequence in D_1, and therefore there exists a u --- u' semi-path in D_1; a contradiction.

Hence, S cannot exist, and therefore, for arbitrary v and v' in V_2, (u,v) and (u',v') belong to different weak components of $D_1 \times D_2$, implying that $D_1 \times D_2$ is disconnected.

To prove the necessity, suppose D_1 and D_2 are weak, and let (u,v) and (u',v') be arbitrary points of $D_1 \times D_2$.

Then there exist u --- u' and v --- v' semi-paths S_1: $u=u_0, u_1, \ldots, u_n=u'$ and S_2: $v=v_0, v_1, \ldots, v_m=v'$ in D_1 and D_2 respectively.

Thus, clearly $(u,v) = (u_0,v_0), (u_1,v_0), \ldots$ $\ldots, (u_n,v_0), (u_n,v_1), (u_n,v_2), \ldots, (u_n,v_m) = (u',v')$ is a (u,v) --- (u',v') semi-path in $D_1 \times D_2$, and hence $D_1 \times D_2$ is weak.

This is an obvious contradiction which completes the proof of Theorem 5.2A.

Since a digraph D is weak if and only if it is not disconnected, one evidently has:

COROLLARY 5.2A. Let D_1 and D_2 be digraphs. Then $D_1 \times D_2$ is weak if and only if D_1 and D_2 are weak.

5.2.2 Structure of weak components

LEMMA 5.2A. Let D_1 and D_2 be digraphs. Then W is a weak component of $D_1 \times D_2$ if and only if there are weak components D_1' of D_1 and D_2' of D_2 such that $W = D_1' \times D_2'$.

PROOF. Let W be a weak component of $D_1 \times D_2$. Consider $P_1[V(W)] = \{ u : (u,v) \in V(W) \text{ for some } v \in V_2 \}$.

Take u and u' in $P_1[V(W)]$. Then (u,v) and (u',v') are in $V(W)$ for some v and v' in V_2. But, since W is a weak component of $D_1 \times D_2$, there exists a (u,v) --- (u',v') semi-path

$S: (u,v) = (u_0,v_0),(u_1,v_1),\ldots,(u_n,v_n) = (u',v')$ in W.

From the proof of Theorem 5.2A, either $u = u'$ or there exists a u --- u' semi-path in $\langle P_1[V(W)] \rangle$. Thus, $\langle P_1[V(W)] \rangle$ is weak.

In the same way, $\langle P_2[V(W)] \rangle$ is also weak. But then, Corollary 5.2A implies that

$W' = \langle P_1[V(W)] \rangle \times \langle P_2[V(W)] \rangle$ is weak.

Clearly, however, $V(W) \subset V(W')$, and if $x = \langle(u,v),(u',v')\rangle$ is in $X(W)$, then either

(i) $u = u'$ in $P_1[V(W)]$ and $\langle v,v'\rangle$ $X(\langle P_2[V(W)]\rangle)$, or

(ii) $v = v'$ in $P_2[V(W)]$ and $\langle u,u'\rangle$ $X(\langle P_1[V(W)]\rangle)$, and therefore, x is in $X(W')$.

Thus, $W \subset W'$. However, W is a weak component by hypothesis, and W' is weak from above, so that $W = W'$, or $W = \langle P_1[V(W)]\rangle \times \langle P_2[V(W)]\rangle$.

To prove the necessity, then, it will suffice to show that $\langle P_1[V(W)]\rangle$ and $\langle P_2[V(W)]\rangle$ are weak components of D_1 and D_2 respectively.

We know that $\langle P_1[V(W)]\rangle$ is weak. So, suppose that $\langle P_1[V(W)]\rangle \subsetneq W_1$, where W_1 is a weak component of D_1. Then there is a point u in $V(W_1) - P_1[V(W)]$ such that $\langle u,u'\rangle$ or $\langle u',u\rangle$ is in the coboundary $U[P_1[V(W)]]$. So, (u',v') is in $V(W)$ for some v' in V_2, and (u,v') is in $V(D_1 \times D_2) - V(W)$ with $\langle u,u'\rangle$ or $\langle u',u\rangle$ in X_1. Thus, $\langle(u,v'),(u',v')\rangle$ or $\langle(u',v'),(u,v')\rangle$ is in $U(W)$, a clear contradiction.

Therefore, $\langle P_1[V(W)]\rangle$ is a weak component of D_1, and in precisely the same manner, $\langle P_2[V(W)]\rangle$ is a weak component of D_2.

From above, then, $W = D_1' \times D_2'$, where D_1' and D_2' are weak components of D_1 and D_2 respectively.

To prove the sufficiency, let D_1' and D_2' be weak components of D_1 and D_2 respectively. Then, from Corollary 5.2A, $D_1' \times D_2'$ is weak in $D_1 \times D_2$.

But, if $\langle(u,v),(u',v')\rangle$ is in $U[V(D_1' \times D_2')]$, then

either (i) $u = u'$ and $\langle v,v'\rangle$ is in X_2

or, (ii) $v = v'$ and $\langle u,u'\rangle$ is in X_1, and also, by definition of $U[V(D_1 \times D_2)]$, either

(iii) (u,v) is in $V(D_1 \times D_2)$ and (u',v') is not, or

(iv) (u',v') is in $V(D_1 \times D_2)$ and (u,v) is not.

If (i) is the case, then for (iii) or (iv), $u = u'$ is in V_1' since, if not, neither (u,v) nor (u',v') would be in $V(D_1' \times D_2') = V_1 \times V_2$. But then, (iii) and (iv) become (iii)* v is in V_2' and v' is not,

and (iv)* v' is in V_2' and v is not.

Thus, case (i) leads to the contradictory conclusion that $\langle v,v'\rangle$ is in $U(V_2')$.

Similarly, case (ii) leads to the conclusion that $\langle u,u'\rangle$ is in $U(V_1')$, which is, again, a contradiction.

Therefore, one must have $U[V(D_1' \times D_2')] = \emptyset$, implying that $D_1' \times D_2'$ is a weak component of $D_1 \times D_2$.

This completes the proof of Lemma 5.2A.

5.3 PATH STRUCTURE OF THE CARTESIAN PRODUCT

LEMMA 5.3A. Let D_1 and D_2 be digraphs. Then there exists a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \times D_2$ if and only if there exists a $u \dashrightarrow u'$ path in D_1 and a $v \dashrightarrow v'$ path in D_2.

PROOF. Let $S: (u,v)=(u_0,v_0),\ldots,(u_n,v_n)=(u',v')$ be a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \times D_2$. We can clearly assume that $(u,v) \neq (u',v')$, and using Theorem 5.1A, we can assume without loss of generality that $u \neq u'$.

Because of the latter assumption, we may let u_{i_1} be the first u_j different from $u_{i_0} = u_0$ in $P_1(S)$. Then $u_0 = u_1 = \cdots = u_{i_1-1} \neq u_{i_1}$. But, using Definitions 2.1F and 4.5A, this implies that $\langle u_{i_0}, u_{i_1} \rangle$ is in X_1.

Now, assume that a sequence of points $u_0 = u_{i_0}, u_{i_1}, \cdots, u_{i_k}$ in $P_1(S)$ has been found satisfying (1) $i_j \succ i_t$ if and only if $j \succ t$, and (2) $\langle u_{i_j}, u_{i_{j+1}} \rangle$ is in X_1 for each j, $0 \leq j \leq k-1$.

Then choose $u_{i_{k+1}}$ such that i_{k+1} is the first $j \succ i_k$ with $u_j \neq u_{i_k}$ in $P_1(S)$.

If no such $u_{i_{k+1}}$ exists, then $u_{i_k} = u_n$, and $u = u_{i_0}, u_{i_1}, \cdots, u_{i_k} = u_n = u'$ is a $u \dashrightarrow u'$ sequence in D_1, implying the existence of a $u \dashrightarrow u'$ path in D_1.

If $u_{i_{k+1}}$ exists, then $u_{i_{k+1}} \neq u_{i_{k+1}-1} = u_{i_k}$, and $v_{i_{k+1}} = v_{i_{k+1}-1}$, so that $\langle u_{i_{k+1}}, u_{i_{k+1}-1} \rangle = \langle u_{i_{k+1}}, u_{i_k} \rangle$ is in X_1, and by choice, $i_{k+1} \succ i_k$.

Thus, $u_{i_0}, u_{i_1}, \cdots, u_{i_{k+1}}$ satisfies (1) and (2), and hence, there is a sequence of points in $P_1(S)$, $u = u_{i_0}, u_{i_1}, \cdots, u_{i_s} = u'$ satisfying (2), which implies the existence of a $u \dashrightarrow u'$ path in D_1. So that clearly, using Theorem 5.1A, there is also a $v \dashrightarrow v'$ path in D_2.

Conversely, if $u = u_0, u_1, \cdots, u_n = u'$ and $v = v_0, v_1, \cdots, v_m = v'$ are $u \dashrightarrow u'$ and $v \dashrightarrow v'$ paths in D_1 and D_2 respectively, then $(u,v) = (u_0, v_0), (u_1, v_0), \cdots, (u_n, v_0), (u_n, v_1), \cdots, (u_n, v_m) = (u', v')$ is a $(u,v) \dashrightarrow (u', v')$ path in $D_1 \times D_2$, concluding the proof.

5.4 FORMS OF CONNECTEDNESS

5.4.1 Characterization of a unilateral product

THEOREM 5.4A. Let D_1 and D_2 be digraphs. Then $D_1 \times D_2$ is unilateral if and only if D_1 and D_2 are unilateral and D_1 or D_2 is strong.

PROOF. The proof depends heavily upon Lemma 5.3A.

For the necessity, let u and u' be in V_1, and v and v' be in V_2. Then (u,v) and (u',v') are in $V(D_1 \times D_2)$, and since $D_1 \times D_2$ is unilateral, there exists (i) a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \times D_2$

or, (ii) a $(u',v') \dashrightarrow (u,v)$ path in $D_1 \times D_2$.

Using Lemma 5.3A, (i) implies that there is a $u \dashrightarrow u'$ path in D_1 and a $v \dashrightarrow v'$ path in D_2, and (ii) implies there is a $u' \dashrightarrow u$ path in D_1 and a $v' \dashrightarrow v$ path in D_2. In either event, then, D_1 and D_2 are unilateral.

But, if both D_1 and D_2 are in C_2, then there exist points u and u' in D_1 and v and v' in D_2 such that no $u' \dashrightarrow u$ path exists in D_1 and no $v \dashrightarrow v'$ path exists in D_2. So, if there were a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \times D_2$, then there would be a $v \dashrightarrow v'$ path in D_2, a contradiction. And, if there were a $(u',v') \dashrightarrow (u,v)$ path in $D_1 \times D_2$, then there would be a $u' \dashrightarrow u$ path in D_1, again a contradiction. Thus, if $D_1 \times D_2$ is unilateral, then D_1 and D_2 must be unilateral, and D_1 or D_2 must be strong.

For the sufficiency, we may suppose without loss of generality that D_1 is strong and D_2 is unilateral. Take (u,v) and (u',v') arbitrarily in $V(D_1 \times D_2)$. Then one must have

(1) (i) a $u \dashrightarrow u'$ path in D_1, and
(ii) a $u' \dashrightarrow u$ path in D_1, and

(2) (iii) a $v \dashrightarrow v'$ path in D_2, or
(iv) a $v' \dashrightarrow v$ path in D_2.

If (iii) is the case, then using (i) and Lemma 5.3A, there is a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \times D_2$.

If (iv) is the case, then using (ii) and Lemma 5.3A, there is a $(u',v') \dashrightarrow (u,v)$ path in $D_1 \times D_2$.

Thus, there is a $(u,v) \dashrightarrow (u',v')$ path or a $(u',v') \dashrightarrow (u,v)$ path in $D_1 \times D_2$, which implies that $D_1 \times D_2$ is unilateral.

This, in turn, completes the proof of Theorem 5.4A.

5.4.2 Characterization of C_1 under cartesian product

Now, since $D_1 \times D_2$ is in C_1 if and only if $D_1 \times D_2$ is weak but not unilateral, we may state:

THEOREM 5.4B. Let D_1 and D_2 be digraphs. Then $D_1 \times D_2$ is in C_1 if and only if

(1) D_1 and D_2 are weak digraphs, and

(2) If D_1, (D_2 respectively), is strong, then D_2, (D_1 respectively), is strictly weak.

PROOF. From Corollary 5.2A, condition (1) is equivalent to $D_1 \times D_2$ being weak. But, from Theorem 5.4A,

$D_1 \times D_2$ is not unilateral if and only if

(3) either (i) D_1 or D_2 is not unilateral,
or (ii) D_1 and D_2 are strictly unilateral.

Otherwise, D_1 and D_2 are unilateral, and D_1 or D_2 is strong.

Thus, by the remark preceeding the statement of the theorem, $D_1 \times D_2$ is in C_1 if and only if (1) and (3) are true. But, if D_1 were strong, then (ii) is not true, so that (i) must be true, implying D_2 is strictly weak.

Similarly, if D_2 is strong, then D_1 must be strictly weak, and therefore, if $D_1 \times D_2$ is in C_1, then (1) and (2) must be true.

Conversely, (1) implies that $D_1 \times D_2$ is weak, and (2) implies (3) since, if neither D_1 nor D_2 is strong, then D_1 and D_2 are strictly unilateral or D_1 or D_2 is not unilateral. But if, say, D_1 is strong, then D_2 is strictly weak, and therefore not unilateral.

So, (1) and (2) imply that $D_1 \times D_2$ is weak but not unilateral, or equivalently that $D_1 \times D_2$ is strictly weak.

5.4.3 Characterization of C_3 under cartesian product

THEOREM 5.4C. Let D_1 and D_2 be digraphs. Then, $D_1 \times D_2$ is strong if and only if D_1 and D_2 are strong.

PROOF. For the sufficiency, suppose that (u,v) and (u',v') are arbitrary points of $D_1 \times D_2$. Then since

D_1 and D_2 are strong, there exist

(i) a $u \dashrightarrow u'$ path in D_1,

(ii) a $u' \dashrightarrow u$ path in D_1,

(iii) a $v \dashrightarrow v'$ path in D_2, and

(iv) a $v' \dashrightarrow v$ path in D_2.

Using Lemma 5.3A, then, (i) and (iii) imply that there is a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \quad D_2$, and (ii) and (iv) imply that there is a $(u',v') \dashrightarrow (u,v)$ path in $D_1 \times D_2$. Thus, $D_1 \times D_2$ is strong.

Conversely, if $D_1 \times D_2$ is strong, and if u and u' are in V_1 and v and v' are in V_2, then (u,v) and (u',v') are in $V(D_1 \times D_2)$, and there is

(i) a $(u,v) \dashrightarrow (u',v')$ path in $D_1 \times D_2$, and

(ii) a $(u',v') \dashrightarrow (u,v)$ path in $D_1 \times D_2$.

Using Lemma 5.3A, (i) implies that there is a $u \dashrightarrow u'$ path in D_1 and a $v \dashrightarrow v'$ path in D_2, and (ii) implies that there is a $u' \dashrightarrow u$ path in D_1 and a $v' \dashrightarrow v$ path in D_2. Hence, D_1 and D_2 are strong.

This concludes the proof of Theorem 5.4C.

5.4.4 Characterization of C_2 under cartesian product

Now, by definition, $D_1 \times D_2$ is in C_2 if and only if $D_1 \times D_2$ is unilateral but not strong. But, $D_1 \times D_2$ is not strong if and only if D_1 or D_2 is not strong by Theorem 5.4C. Thus, using Theorem 5.3A, one has:

THEOREM 5.4D. Let D_1 and D_2 be digraphs. Then, $D_1 \times D_2$ is in C_2 if and only if either

(1) D_1 is strong and D_2 is in C_2, or

(2) D_2 is strong and D_1 is in C_2.

PROOF. Suppose D_1 and D_2 are unilateral and D_1 or D_2 is strong, but not both. Then $D_1 \times D_2$ is unilateral using Theorem 5.4A, but $D_1 \times D_2$ is not strong by Theorem 5.4C. Hence $D_1 \times D_2$ is in C_2.

Conversely, suppose $D_1 \times D_2$ is in C_2. Then Theorem 5.4A implies that D_1 and D_2 are unilateral and that D_1 or D_2 is strong. But, D_1 and D_2 cannot both be strong since this would imply that $D_1 \times D_2$ was strong by Theorem 5.4C. Thus, if D_1 is strong, then D_2 must be in C_2, and similarly, if D_2 were strong, then D_1 would, of necessity, be in C_2.

This completes the proof of Theorem 5.4D.

5.5 TABLE 2 -- POSSIBLE TRIPLES UNDER CARTESIAN PRODUCT

Referring to Chapter II, we give the possible triples in table form. Here, Table 2, below, implicitly contains all necessary and sufficient conditions that the triple (i,j,k) be possible since each pair (i,j) determines a unique possible triple, (i,j,k).

D_1 \ D_2	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	2
3	0	1	2	3

TABLE 2 -- Possible triples for $\times$

Verification is immediate using Theorems 5.2A, 5.4B, 5.4C, 5.4D.

CHAPTER VI -- COMPOSITION OF DIGRAPHS

6.1 PRELIMINARIES

6.1.1 Definitions and line structure of $D_1[D_2]$

DEFINITION 6.1A. Let D_1 and D_2 be digraphs. Then the digraph $D_1 \dashv\!\rightarrow D_2$ is defined as follows:

$V(D_1 \dashv\!\rightarrow D_2) = \{u : u \in V_1 \text{ or } u \in V_2\} = V_1 \cup V_2$, and

$X(D_1 \dashv\!\rightarrow D_2) = \{\langle u,u'\rangle : u,u' \in V_1 \cup V_2$, and either $\langle u,u'\rangle \in X_1 \cup X_2$ or $u \in V_1$ and $u' \in V_2\}$.

DEFINITION 6.1B. Let D_1 and D_2 be digraphs, and let $V_1 = \{u_1,u_2,\ldots,u_{p_1}\}$. Then let D_{2i} be defined for $i = 1,2,\ldots,p_1$ as follows:

$V_{2i} = \{(u_i,v) : v \in V_2\}$,

$X_{2i} = \{\langle (u_i,v),(u_i,v')\rangle : \langle v,v'\rangle \in X_2\}$.

Clearly $D_{2i} \approx D_2$ for each $i = 1,\ldots,p_1$.

We then define the composition of D_2 into D_1, denoted $D_1[D_2]$ to be that digraph with points and lines:

$V(D_1[D_2]) = V_1 \times V_2$, and

$$X(D_1[D_2]) = [\bigcup_{i=1}^{p_1} X_{2i}] \cup [\bigcup_{i,j} \{X(D_{2i} \dashv\!\rightarrow D_{2j}) : \langle u_i,u_j\rangle \in X_1\}].$$

LEMMA 25. Let D_1 and D_2 be digraphs. Then, $x = \langle (u,v),(u',v')\rangle$ is in $X(D_1[D_2])$ if and only if either (1) $\langle u,u'\rangle$ is in X_1,

or (2) $u = u'$ and $\langle v,v'\rangle$ is in X_2.

PROOF. For the necessity, we first note the obvious, namely, that either $u = u'$ or $u \neq u'$.

Suppose $u \neq u'$. Then (u,v) is in V_{2i} for some i, and (u',v') is in V_{2j} for some j, with $i \neq j$.

So, since $x = \langle (u,v),(u',v') \rangle$ is in $X(D_1[D_2])$, and x is in no D_{2k}, x must be in $X(D_{2i} \dashv\!\!\rightarrow D_{2j})$, which implies $D_{2i} \dashv\!\!\rightarrow D_{2j}$ is a subgraph of $D_1[D_2]$. This, however, implies that $\langle u_i,u_j \rangle \equiv \langle u,u' \rangle$ is in X_1, so that, if $u \neq u'$, then (1) holds.

But, if $u = u'$, then (u,v) and (u',v') are in the same D_{2j} by definition of $D_1[D_2]$. However, by definition of D_{2j}, if x is in X_{2j}, then $\langle v,v' \rangle$ is in X_2, so that if $u = u'$, then (2) holds.

To prove the sufficiency, suppose (1) is true, and let $x = \langle (u,v),(u',v') \rangle$. Since $\langle u,u' \rangle$ is in X_1, $u \neq u'$ and u is in D_{2i}, u' is in D_{2j} with $i \neq j$. But, then, $D_{2i} \dashv\!\!\rightarrow D_{2j}$ must be a subgraph of $D_1[D_2]$, so that x is in $X(D_1[D_2])$ since x is in $D_{2i} \dashv\!\!\rightarrow D_{2j}$.

On the other hand, if (2) is true, then (u,v) and (u',v') are distinct points of some D_{2i}. But, using the definition of D_{2i}, if $\langle v,v' \rangle$ is in X_2, then x is in $X_{2i} \subset X(D_1[D_2])$.

Thus the sufficiency is proved, completing the proof of the lemma.

6.1.2 Relation of composition to previous operations

Recalling the definition of a spanning subgraph, we state the following corollary.

COROLLARY 6.1A. If D_1 and D_2 are digraphs, then

(i) $D_1 \boxtimes D_2 \prec D_1[D_2]$, and

(ii) $D_1 \times D_2 \prec D_1[D_2]$.

PROOF. First we remark that

$V(D_1 \boxtimes D_2) = V(D_1 \times D_2) = V(D_1[D_2]) = V_1 \times V_2$.

For (i), suppose that $x = \langle (u,v),(u',v') \rangle$ is in $X(D_1 \boxtimes D_2)$. Then, $\langle u,u' \rangle$ is in X_1 by definition, so that, using Lemma 6.1A, x is in $X(D_1[D_2])$. Thus, $D_1 \boxtimes D_2 \prec D_1[D_2]$.

For (ii), suppose $x = \langle (u,v),(u',v') \rangle$ is in $X(D_1 \times D_2)$. Then, by definition of $D_1 \times D_2$, either

(a) $u = u'$ and $\langle v,v' \rangle$ is in X_2, or

(b) $v = v'$ and $\langle u,u' \rangle$ is in X_1.

Case (a) is the same as condition (2) of Lemma 6.1A, and hence x is in $X(D_1[D_2])$.

But, case (b) clearly implies condition (1) of Lemma 6.1A, yielding the same conclusion. Thus, $D_1 \times D_2 \prec D_1[D_2]$.

6.1.3 Structure of $([\mathfrak{D}],[\ ])$

THEOREM 6.1A. $([\mathfrak{D}],[\ ])$ is a non-commutative semi-group with identity.

PROOF. The fact that $([\mathfrak{D}],[\ ])$ is a semi-group follows directly from the validity of the statement: $\langle (u,v,w),(u',v',w') \rangle$ is in both $X(D_1[D_2[D_3]])$ and $X((D_1[D_2])[D_3])$ if and only if either

(i) $\langle u,u' \rangle$ is in X_1,

(ii) $u = u'$ and $\langle v, v' \rangle$ is in X_2, or

(iii) $u = u'$, $v = v'$ and $\langle w, w' \rangle$ is in X_3,

the verification of which is immediate upon twice applying Lemma 6.1A.

We assert that $[K_1]$ is the identity of $([\mathfrak{D}], [\ ])$. For, suppose $K_1 = \{u_1\}$. Then, $V(K_1[D_2]) = \{(u_1, v) : v \in V_2\}$, and $X(K_1[D_2]) = X_{21} = \{\langle (u_1, v), (u_1, v') \rangle : \langle v, v' \rangle \in X_2\}$.

As in Theorem 5.1A, $K_1[D_2]$ is isomorphic to D_2 under the map $\phi(u_1, v) = v$.

Also, if $K_1 = \{v_1\}$, then

$V(D_1[K_1]) = \{(u, v_1) : u \in V_1\}$, and

$X(D_1[K_1]) = \{\langle (u_i, v_1), (u_j, v_1) \rangle : \langle u_i, u_j \rangle \in X_1\}$ since $D_{2i} = (u_i, v_1)$ is isomorphic to K_1 for all $i = 1, \ldots, p_1$ in this instance.

Again, then, $D_1[K_1] \approx D_1$, so that $[K_1]$ is, indeed, the identity of $([\mathfrak{D}], [\ ])$.

So, as with the cartesian product, we will allow one point digraphs and digraphs with independent point sets in this chapter.

To see that $([\mathfrak{D}], [\ ])$ is not commutative, let D_1 and D_2 be as shown at the right.

Since w_o has degree 5 in $D_2[D_1]$, and the degree of every point in $D_1[D_2]$ is less than 5, they cannot be isomorphic.

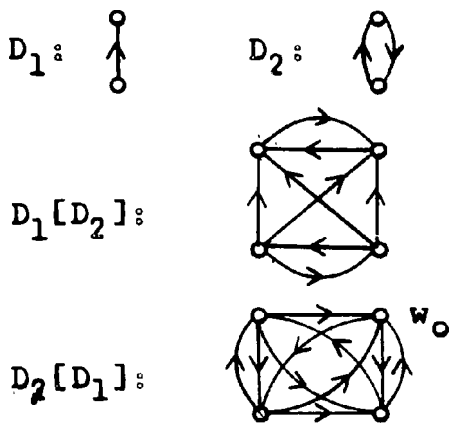


FIGURE 18 -- An example

6.2 DISCONNECTEDNESS UNDER COMPOSITION

THEOREM 6.2A. Let D_1 and D_2 be digraphs. Then $D_1[D_2]$ is disconnected if and only if either D_1 is disconnected, or $D_1 = K_1$ and D_2 is disconnected.

PROOF. For the necessity, it is clear that if $D_1 = K_1$, then D_2 must be disconnected in order that $K_1[D_2]$ be disconnected. So, suppose that $D_1 \neq K_1$.

Then if $D_1[D_2]$ is disconnected, there exist distinct points (u,v) and (u',v') in $V(D_1[D_2])$ such that there is no $(u,v) \dashrightarrow (u',v')$ semi-path in $D_1[D_2]$.

(I) Suppose $u = u'$, and suppose that D_1 were weak. Then there exists a line $\langle u,u_1\rangle$ or $\langle u_2,u\rangle$ in D_1 (that is, u is not isolated). If $\langle u,u_1\rangle$ is in X_1, then $D_{2i} \dashv\!\rightarrow D_{2j}$ is a subgraph of $D_1[D_2]$, where $u \equiv u_i$ and $u_1 \equiv u_j$.

Let (u_1,v_1) be a point of D_{2j}. Then by definition of $D_1[D_2]$, both $x = \langle(u,v),(u_1,v_1)\rangle$ and $x' = \langle(u',v'),(u_1,v_1)\rangle$ are in $X(D_{2i} \dashv\!\rightarrow D_{2j})$ and hence in $X(D_1[D_2])$ since $(u,v) \neq (u',v')$ and both are in D_{2i}.

But then, x,x' defines a $(u,v) \dashrightarrow (u',v')$ semi-path in $D_1[D_2]$: a contradiction. A similar conclusion is reached if it is assumed that $\langle u_2,u\rangle$ is in D_1. Thus, if $u = u'$, then D_1 must be in C_0.

(II) Now, suppose $u \neq u'$, and suppose D_1 is weak. Let $u = u_0,u_1,\ldots,u_n = u'$ be a $u \dashrightarrow u'$ semi-path $(n > 0)$. Then by Lemma 6.1A, one has, for any $i = 0,1,\ldots,n-1$,

$\langle (u_i,v_1),(u_{i+1},v_2)\rangle$ or $\langle (u_{i+1},v_1),(u_i,v_2)\rangle$ is in $X(D_1[D_2])$ for any v_1 and v_2 in V_2. Also, since $u_i \neq u_j$ for $i \neq j$ and $i,j = 0,1,\ldots,n$, one has $(u,v_1) \neq (u_i,v_2)$ for $i = 1,\ldots,n$ and any v_1 and v_2 in V_2.

Thus, $(u_0,v),(u_1,v'),(u_2,v'),\ldots,(u_n,v')$ is a (u,v) --- (u',v') semi-path in $D_1[D_2]$, once again a contradiction. Thus, D_1 must be disconnected.

To prove the sufficiency, suppose D_1 is disconnected. Then there exist distinct points u and u' in V_1 such that there is no u --- u' semi-path in D_1. Take v arbitrarily in V_2, and suppose there were a (u,v) --- (u',v) semi-path in $D_1[D_2]$, say $S\colon (u,v) = (u_0,v_0),(u_1,v_1),\ldots,(u_n,v_n) = (u',v)$. Then, for $i = 0,1,\ldots,n-1$, $\langle (u_i,v_i),(u_{i+1},v_{i+1})\rangle$ or $\langle (u_{i+1},v_{i+1}),(u_i,v_i)\rangle$ is in $X(D_1[D_2])$.

Let i_1 be the first i such that $u_i \neq u_{i_0} = u_0$. Such an i must exist since $u_n = u' \neq u = u_0$. Then, using Lemma 6.1A, the above implies that $\langle u_{i_0},u_{i_1}\rangle$ or $\langle u_{i_1},u_{i_0}\rangle$ is in X_1.

Suppose now, that a sequence $S_1\colon u_0 = u_{i_0},\ldots,u_{i_k}$ of points in $P_1(S)$ has been found satisfying

(i) $i_j \succ i_t$ if and only if $j \succ t$, and

(ii) $\langle u_{i_j},u_{i_{j-1}}\rangle$ or $\langle u_{i_{j-1}},u_{i_j}\rangle$ is in X_1 for $j = 1,\ldots,k$.

Then let i_{k+1} be the first $j \succ i_k$ such that $u_j \neq u_{i_k}$. If there is no such i_{k+1}, then $u_{i_k} = u_n = u'$, and S_1 defines a u --- u' semi-sequence in D_1, which

is a contradiction.

But, if i_{k+1} exists, then from above and Lemma 6.1A, $\langle u_{i_k}, u_{i_{k+1}} \rangle$ or $\langle u_{i_{k+1}}, u_{i_k} \rangle$ is in X_1, and hence S_2: $u_{i_0}, u_{i_1}, \ldots, u_{i_{k+1}}$ is a sequence of points of $P_1(S)$ satisfying (i) and (ii). Thus, there exists a sequence of points S': $u_0 = u_{i_0}, u_{i_1}, \ldots, u_{i_s} = u_n = u'$ in $P_1(S)$ satisfying (i) and (ii). This S', however, defines a u --- u' semi-sequence in D_1 contrary to the assumption that no u --- u' semi-path exists in D_1.

Therefore S must not exist, implying that $D_1[D_2]$ is disconnected.

If $D_1 = K_1$, then clearly $K_1[D_2]$ is disconnected when D_2 is disconnected. This concludes the proof of Theorem 6.2A.

The following corollary is immediate.

COROLLARY 6.2A. Let D_1 and D_2 be digraphs. Then $D_1[D_2]$ is weak if and only if D_1 is weak, and if $D_1 = K_1$, then D_2 is weak.

6.3. PATH STRUCTURE OF COMPOSITION

As with previous operations, let us determine the path structure of composition.

LEMMA 6.3A. Let D_1 and D_2 be digraphs. Then there exists a $(u,v) \dashrightarrow (u',v')$ path in $D_1[D_2]$ if and only if one of the following is true:

(1) $u = u'$ and there is a $v \dashrightarrow v'$ path in D_2,

(2) $u = u'$ and u belongs to a cycle of D_1 having positive length, or

(3) $u \neq u'$ and there is a $u \dashrightarrow u'$ path in D_1.

PROOF. To prove the necessity, suppose that (u,v) and (u',v') are distinct points of $D_1[D_2]$, and that $P: (u,v) = (u_0,v_0),\ldots,(u_n,v_n) = (u',v')$ is a $(u,v) \dashrightarrow (u',v')$ path in $D_1[D_2]$. Then $\langle(u_i,v_i),(u_{i+1},v_{i+1})\rangle$ is in $X(D_1[D_2])$ for $i = 0,\ldots,n-1$.

If $u = u'$, let $u_0 = u_{i_0}$ and let i_1 be the first j such that $j \succ i_0$ and $u_j \neq u_{i_0}$.

Then Lemma 6.1A implies $\langle u_{i_0},u_{i_1}\rangle$ is in X_1. As in the proof of the previous theorem, we may continue in this way and obtain a sequence (closed in this case) $u = u_{i_0},u_{i_1},\ldots,u_{i_s} = u' = u$. Thus, if $u = u'$, then either $u = u_0 = u_1 = \cdots = u_n$, or u belongs to a cycle of D_1 having positive length.

But, if $u = u_0 = u_1 = \cdots = u_n$, then $v_i \neq v_j$ for $i \neq j$ in P since P is a path. Further, since $\langle(u_i,v_i),(u_{i+1},v_{i+1})\rangle = \langle(u,v_i),(u,v_{i+1})\rangle$ is in $X(D_1[D_2])$, Lemma 6.1A implies that $\langle v_i,v_{i+1}\rangle$ is in X_2 for all i, $i = 0,\ldots,n-1$, and therefore, $v = v_0,v_1,\ldots,v_n = v'$ is a $v \dashrightarrow v'$ path in D_2.

Thus, if $u = u'$, then there is a $v \dashrightarrow v'$ path in D_2, or u belongs to a cycle of D_1 which has positive length.

If, on the other hand, $u \neq u'$, then there is, indeed, a first i_1 such that $i_1 \succ i_0$ and $u_{i_1} \neq u_{i_0}$,

and the above procedure generates a $u \dashrightarrow u'$ sequence, implying the existence of a $u \dashrightarrow u'$ path in D_1.

Therefore, if there is a $(u,v) \dashrightarrow (u',v')$ path in $D_1[D_2]$, then (1), (2) or (3) must hold.

For the sufficiency, suppose (1) is true. Let $v = v_0,v_1,\ldots,v_n = v'$ be a $v \dashrightarrow v'$ path in D_2 with $n > 0$ (the latter, since (u,v) and (u',v') are distinct and $u = u'$).

Then $(u,v) = (u,v_0),(u,v_1),\ldots,(u,v_n) = (u',v')$ is a $(u,v) \dashrightarrow (u',v')$ path in $D_1[D_2]$ since $v_i \neq v_j$ for $i \neq j$ implies that $(u,v_i) \neq (u,v_j)$ for $i \neq j$, and since $\langle v_i,v_{i+1}\rangle$ is in X_2 implies, using Lemma 6.1A, that the line $\langle (u,v_i),(u,v_{i+1})\rangle$ is in $X(D_1[D_2])$.

If (2) is true, let $u = u_0,u_1,\ldots,u_n = u$ be a cycle in D_1 containing u and with $n > 0$. Then, using Lemma 6.1A, $\langle (u_i,v),(u_{i+1},v')\rangle$ is in $X(D_1[D_2])$ for $i = 0,\ldots,n-1$ and for any v and v' in V_2.

Thus, if $v \neq v'$, then $(u,v) = (u_0,v),(u_1,v'),(u_2,v'),\ldots,(u_n,v') = (u',v')$ is a $(u,v) \dashrightarrow (u',v')$ path in $D_1[D_2]$ since only $u_0 = u_n$ and $v \neq v'$.

Finally, if (3) is true, then $u \neq u'$ and $u = u_0,u_1,\ldots,u_n = u'$ is a $u \dashrightarrow u'$ path in D_1. Thus, from Lemma 6.1A, $\langle (u_i,v),(u_{i+1},v')\rangle$ is in $X(D_1[D_2])$ for any v and v' in V_2. Hence, $(u,v) = (u_0,v),(u_1,v'),(u_2,v'),\ldots,(u_n,v') = (u',v')$ is a $(u,v) \dashrightarrow (u',v')$ path in $D_1[D_2]$, completing the proof of Lemma 6.3A.

6.4 FORMS OF CONNECTEDNESS

6.4.1 Characterization of unilateral composition

THEOREM 6.4A. Let D_1 and D_2 be digraphs. Then $D_1[D_2]$ is unilateral if and only if

(i) D_1 is unilateral and each point is on a positive cycle,

or, (ii) D_1 and D_2 are unilateral.

PROOF. Suppose that $D_1[D_2]$ is unilateral. If $D_1 = K_1$, then D_1 is unilateral and $K_1[D_2] \approx D_2$ is unilateral. So, suppose $D_1 \neq K_1$, and let u and u' be distinct elements of V_1, and let v be in V_2.

Then there is a $(u,v) \dashrightarrow (u',v)$ or a $(u',v) \dashrightarrow (u,v)$ path in $D_1[D_2]$. By Lemma 6.3A, then, there is a $u \dashrightarrow u'$ path or a $u' \dashrightarrow u$ path in D_1, so that D_1 is unilateral.

Now, if D_2 is not unilateral, then there are points v and v' in D_2 such that there does not exist a $v \dashrightarrow v'$ or a $v' \dashrightarrow v$ path in D_2. Take u in V_1. Then since $D_1[D_2]$ is unilateral, there is a $(u,v) \dashrightarrow (u,v')$ or a $(u,v') \dashrightarrow (u,v)$ path in $D_1[D_2]$. But, Lemma 6.3A then implies the existence of a cycle containing u since there is no $v \dashrightarrow v'$ or $v' \dashrightarrow v$ path in D_2.

Thus, either D_1 and D_2 are both unilateral, or D_1 is unilateral and every point belongs to a cycle of positive length.

Hence, the necessity of Theorem 6.4A is proved.

To prove the sufficiency of conditions (i) and (ii), suppose (u,v) and (u',v') are distinct points of $D_1[D_2]$.

(I) If $u = u'$ and D_1 is unilateral with each of its points on a cycle of positive length, then u is on a cycle $u = u_0,u_1,\ldots,u_n = u' = u$ of D_1 with $n \geq 0$. Thus, using Lemma 6.1A,
$(u,v) = (u_0,v),(u_1,v'),\ldots,(u_n,v') = (u,v') = (u',v')$ is a $(u,v) \dashrightarrow (u',v')$ path in $D_1[D_2]$.

(II) If $u = u'$ and D_2 is unilateral, then, $v \neq v'$ and there is a $v \dashrightarrow v'$ or a $v' \dashrightarrow v$ path in D_2. Suppose there is a $v \dashrightarrow v'$ path in D_2. Then Lemma 6.3A implies that there is a $(u,v) \dashrightarrow (u',v')$ path in $D_1[D_2]$. Similarly, if there is a $v' \dashrightarrow v$ path in D_2, then there is a $(u',v') \dashrightarrow (u,v)$ path in $D_1[D_2]$.

(III) If $u \neq u'$ and D_1 is unilateral, then there is a $u \dashrightarrow u'$ or a $u' \dashrightarrow u$ path in D_1, implying by Lemma 6.3A that there is a $(u,v) \dashrightarrow (u',v')$ or a $(u',v') \dashrightarrow (u,v)$ path in $D_1[D_2]$.

Hence, if (i) is assumed, then (I) and (III) guarantee that $D_1[D_2]$ is unilateral. If, on the other hand, (ii) is true, then (II) and (III) imply the same result.

This concludes the proof of Theorem 6.4A.

6.4.2 Characterization of C_1

COROLLARY 6.4A. Let D_1 and D_2 be digraphs. Then $D_1[D_2]$ is in C_1 if and only if

(1) D_1 is weak, and if $D_1 = K_1$, then $D_2 \in C_1$, and

(2) If D_1 is unilateral, $D_1 \neq K_1$, then

(i) $D_2 \in C_0 \cup C_1$, and

(ii) there exists a point u in $V(D_1)$ that does not lie on a cycle of D_1 having positive length.

PROOF. $D_1[D_2]$ is in C_1 if and only if $D_1[D_2]$ is weak and not unilateral.

Using Corollary 6.2A, this occurs if and only if (1) is true and $D_1[D_2]$ is not unilateral. But, using Theorem 6.4A, the latter happens if and only if (2) holds.

This completes the proof of Corollary 6.4A.

6.4.3 Characterization of C_3

THEOREM 6.4B. Let D_1 and D_2 be digraphs. Then $D_1[D_2]$ is strong if and only if either

(i) $D_1 = K_1$ and D_2 is strong, or

(ii) $D_1 \neq K_1$ and D_1 is strong.

PROOF. For the necessity, if $D_1 = K_1$, then clearly $D_2 \approx K_1[D_2]$ must be strong. If $D_1 \neq K_1$, let u and u' be distinct points of D_1, and let v be in V_2. Then since $D_1[D_2]$ is strong, there is a $(u,v) \dashrightarrow (u',v)$ path and a $(u',v) \dashrightarrow (u,v)$ path in $D_1[D_2]$, and using Lemma 6.1A, there is thus a $u \dashrightarrow u'$ and a $u' \dashrightarrow u$ path in D_1.

Therefore, either $D_1 = K_1$ and D_2 is strong, or

$D_1 \neq K_1$ and D_1 is strong, proving the necessity of conditions (i) and (ii).

For the sufficiency, we first note that if $D_1 = K_1$, then $K_1[D_2] \simeq D_2$ is strong if D_2 is strong.

Now, suppose that (u,v) and (u',v') are distinct points of $D_1[D_2]$.

(I) If $u = u'$, then $v \neq v'$. We may suppose that $D_1 \neq K_1$, so that u is on a cycle of D_1 of length greater than zero since D_1 is assumed strong. But then, Lemma 6.3A implies that there is a $(u,v) \dashrightarrow (u',v')$ and a $(u',v') \dashrightarrow (u,v)$ path in $D_1[D_2]$. Hence $D_1[D_2]$ is strong.

(II) If $u \neq u'$, then there is a $u \dashrightarrow u'$ and a $u' \dashrightarrow u$ path in D_1, implying that there is a $(u,v) \dashrightarrow (u',v')$ and a $(u',v') \dashrightarrow (u,v)$ path in $D_1[D_2]$. Once again, $D_1[D_2]$ is strong.

This concludes the proof of Theorem 6.4B.

6.4.4 Characterization of C_2

COROLLARY 6.4B. Let D_1 and D_2 be digraphs. Then $D_1[D_2]$ is in C_2 if and only if $D_1 \notin C_2$ and either

(i) D_2 is unilateral, or

(ii) every point of D_1 lies on a cycle of positive length.

PROOF. $D_1[D_2]$ is in C_2 if and only if $D_1[D_2]$ is unilateral but not strong.

Rephrasing Theorem 6.4A, we see that $D_1[D_2]$ is unilateral if and only if D_1 is unilateral and either

(a) D_2 is unilateral, or

(b) every point of D_1 lies on a cycle of positive length.

On the other hand, Theorem 6.4B implies that $D_1[D_2]$ is not strong if and only if D_1 is not strong: ignoring the case $D_1 = K_1$.

Thus, combining the above, one is led to the result of the corollary. It should be noted that we have chosen, here, to omit the case $D_1 = K_1$.

6.5 TABLE 3 -- POSSIBLE TRIPLES UNDER COMPOSITION

The above results lead to the following table of possible triples, where it is assumed that $p_1 > 1$, $p_2 > 1$.

D_2 \ D_1	0	1	2	3
0	0	1	1,2	3
1	0	1	1,2	3
2	0	1	2	3
3	0	1	2	3

TABLE 3 -- Possible triples for [ ]

Here, with two exceptions, the table implicitly contains the necessary and sufficient conditions that $D_1[D_2]$ be in C_n since each pair (i,j) determines a unique possible triple (i,j,k) except the pairs $(2,0)$ and $(2,1)$. In the case of the two exceptions, $D_1[D_2]$ is in C_2 if and only if every point of D_1 belongs to a cycle of D_1 having positive length.

APPENDIX I -- REACHABILITY

In this appendix, a brief investigation of a concept closely related to connectedness is carried out.

DEFINITION 1. Let D be a digraph with $V = \{v_1, v_2, \ldots, v_p\}$. Then v_j is said to be *reachable from* v_i if there exists a $v_i \dashrightarrow v_j$ path in D. The *reachability matrix*, $R(D)$, of D is defined by the conditions: $R \equiv (r_{ij})$ with

$$r_{ij} = \begin{cases} 1 & \text{if } v_j \text{ is reachable from } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Since v_i is always reachable from v_i, the entries r_{ii} are all equal to 1.

With this definition in mind, the purpose of the appendix is to develop expressions for $R(D_1 \circ D_2)$ in terms of A_1, A_2, R_1 and R_2 when the operation $\circ$ is $\boxplus$, $\times$, and $[\ ]$.

Obviously such results will depend upon the labeling of the digraphs D_1, D_2 and $D_1 \circ D_2$. Since the point set $V(D_1 \circ D_2) = V_1 \times V_2$ in all cases, we will assume, as in Lemma 3.1A, that $V_1 = \{u_1, \ldots, u_{p_1}\}$, $V_2 = \{v_1, \ldots, v_{p_2}\}$ and $V(D_1 \circ D_2) = \{w_1, \ldots, w_{p_1 p_2}\}$ where w_i is given by $w_i = (u_{i(n)}, v_{(i/n)^*})$ with $n = p_1$.

In the following definition, the various conventions of notation adhered to in this appendix are introduced.

DEFINITION 2. The symbol # is used to denote boolean operations. For example, if m and n are non-negative integers, the expression (m + n)# will represent the result of the addition of m and n in boolean terms.

If $A = (a_{ij})$ is a matrix, A # is understood to be the matrix $A\# = (a_{ij}\#)$.

Also, if $A = (a_{ij})$ is an n-by-n matrix, then the entries of A^m will be denoted $a_{ij}^{(m)}$.

Further, for any matrix $M = (a_{ij})$, the matrix $\text{diag}(M) = (a'_{ij})$ is defined by $a'_{ij} = \delta_{ij}a_{ij}$ where δ_{ij} is the Kronecker delta,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

We denote the n-by-n identity matrix by I_n, and the n-by-n universal matrix by J_n, where $J_n = (c_{ij})$ with $c_{ij} = 1$ for all i and j, $1 \leq i,j \leq n$.

The following result appears in the Harary, Norman and Cartwright manuscript, [4].

THEOREM 1. Let D be a digraph with $V = \{v_1, \ldots, v_p\}$ and adjacency matrix $A = (a_{ij})$. Then the i,j entry, $a_{ij}^{(n)}\#$ of $A^n\#$ equals 1 if and only if there exists in D at least one $v_i \dashrightarrow v_j$ sequence of length n.

Note that $a_{ij}^{(n)}\# = 1$ for some $n > 1$ if and only if v_i belongs to some cycle of length $m > 0$, where $km = n$.

THEOREM 2. $R(D_1 \times D_2) = R_1 \boxtimes R_2$.

PROOF. Let $R_1 = (a_{ij})$, $R_2 = (b_{ij})$ and $R(D_1 \times D_2) = (c_{ij})$. Further, let $R_1 \boxtimes R_2 = (c'_{ij})$, and $n = p_1$.

Then, from Lemma 3.1A, $c'_{ij} = a_{i(n)j(n)}b_{(i/n)*(j/n)*}$ where, by assumption, w_i in $D_1 \times D_2$ is the point $w_i = (u_{i(n)}, v_{(i/n)*})$.

Then $c_{ij} = 1$ if and only if there is a $(u_{i(n)}, v_{(i/n)*}) = w_i \dashrightarrow w_j = (u_{j(n)}, v_{(j/n)*})$ path in $D_1 \times D_2$, which, according to Lemma 5.3A, happens if and only if there exists a $u_{i(n)} \dashrightarrow u_{j(n)}$ path in D_1 and a $v_{(i/n)*} \dashrightarrow v_{(j/n)*}$ path in D_2.

However, the latter occurs if and only if $a_{i(n)j(n)} = 1$ and $b_{(i/n)*(j/n)*} = 1$, which, from above, occurs if and only if $c'_{ij} = 1$.

Thus, $c_{ij} = 1$ if and only if $c'_{ij} = 1$, proving the theorem.

THEOREM 3. $R(D_1 \boxtimes D_2) = [\sum_{k=1}^{p_1p_2} (A_1^k \boxtimes A_2^k) + I_{p_1p_2}]\ \#$.

PROOF. Let $R(D_1 \boxtimes D_2) = (c_{ij})$, $A_1^k \boxtimes A_2^k = (c_{ij}^k)$, and $\sum_{k=1}^{p_1p_2} (A_1^k \boxtimes A_2^k) + I_{p_1p_2} = (d_{ij})$. Also, let $A_1 = (a_{ij})$, $A_2 = (b_{ij})$.

The assertion is, then, that $c_{ij} = d_{ij}\ \#$.

But, obviously,

$$d_{ij} = \sum_{k=1}^{p_1p_2} c_{ij}^k + \delta_{ij},$$

where, δ_{ij} is the Kronecker delta.

CASE I. If $i = j$, then $c_{ij} = 1$ by definition of

$R(D_1 \boxtimes D_2)$. But, if $i \neq j$, $\delta_{ij} = 1$, so that $d_{ij}\# = (1 + \sum_k c_{ij}^k)\# = 1$ since $\sum_k c_{ij}^k \geq 0$.

Hence, $c_{ij} = d_{ij}\#$ if $i = j$.

Case II. When $i \neq j$, then $d_{ij} = \sum_{k=1}^{p_1p_2} c_{ij}^k$ since $\delta_{ij} = 0$.

By definition of $R(D_1 \boxtimes D_2)$, $c_{ij} = 1$ if and only if there is a $(u_{i(n)}, v_{(i/n)*}) = w_i \dashrightarrow w_j = (u_{j(n)}, v_{(j/n)*})$ path in $D_1 \boxtimes D_2$. By Corollary 4, this occurs if and only if there exist $u_{i(n)} \dashrightarrow u_{j(n)}$ and $v_{(i/n)*} \dashrightarrow v_{(j/n)*}$ sequences of the same length in D_1 and D_2 respectively.

But, this happens if and only if $a^{(m)}_{i(n)j(n)} \geq 1$ and $b^{(m)}_{(i/n)*(j/n)*} \geq 1$ for some m, using Theorem 22.

Thus, $c_{ij} = 1$ if and only if $c_{ij}^m = a^{(m)}_{i(n)j(n)}\, b^{(m)}_{(i/n)*(j/n)*} \geq 1$ for some m. Further, we may assume that $m \leq p_1p_2$ since if there is a $w_i \dashrightarrow w_j$ path P of length m in $D_1 \boxtimes D_2$, then certainly P contains at most p_1p_2 points, and hence P has length less than p_1p_2, so that $m \leq p_1p_2$.

But then,

$$d_{ij} = \sum_{k=1}^{p_1p_2} c_{ij}^k$$

so that $d_{ij}\# = 1$ if and only if there is some $m \leq p_1p_2$ such that $c_{ij}^m \geq 1$.

Therefore, $c_{ij} = 1$ if and only if $d_{ij}\# = 1$, or $c_{ij} = d_{ij}\#$, as desired.

This concludes the proof of Theorem 24.

THEOREM 25. For D_1 and D_2 digraphs, one has

$$R(D_1[D_2]) = \left\{\left[R_1 - I_{p_1} + \sum_{k=2}^{p_1p_2} \operatorname{diag}(A_1^k)\right] \boxtimes J_{p_2} + I_{p_1} \boxtimes R_2\right\}\#.$$

PROOF. Let $R(D_1[D_2]) = (c_{ij})$ and

$$[R_1 - I_{p_1} + \sum_{k=2}^{p_1 p_2} \mathrm{diag}(A_1^k)] \boxtimes J_{p_2} + I_{p_1} \boxtimes R_2 = (c'_{ij}).$$

The assertion is that $c'_{ij}\,\# = c_{ij}$.

Further, let $A_1 = (a_{ij})$, $R_1 = (r_{ij})$, $R_2 = (r'_{ij})$, and

$$R_1 - I_{p_1} + \sum_{k=2}^{p_1 p_2} \mathrm{diag}(A_1^k) = (b_{ij}).$$

Then, using Lemma 3.1A,

$c'_{ij} = b_{i(n)j(n)} \cdot 1 + \delta_{i(n)j(n)} \cdot r'_{(i/n)*(j/n)*}$. But clearly

$$b_{i(n)j(n)} = r_{i(n)j(n)} - \delta_{i(n)j(n)} + \sum_{k=2}^{p_1 p_2} \delta_{i(n)j(n)} \cdot a^{(k)}_{i(n)j(n)},$$

so that $c'_{ij} = r_{i(n)j(n)} - \delta_{i(n)j(n)} +$

$$+ \delta_{i(n)j(n)} r'_{(i/n)*(j/n)*} + \sum_{k=2}^{p_1 p_2} \delta_{i(n)j(n)} \cdot a^{(k)}_{i(n)j(n)}.$$

Using this expression, we examine two cases.

CASE I. $i(n) \neq j(n)$. In this case, $c'_{ij}\,\# = r_{i(n)j(n)}$.

CASE II. $i(n) = j(n)$. In this case,

$$c'_{ij}\,\# = \left\{ r'_{(i/n)*(j/n)*} + \sum_{k=2}^{p_1 p_2} a^{(k)}_{i(n)i(n)} \right\} \#.$$

Hence, we conclude from these that $c'_{ij}\,\# = 1$ if and only if

(1) $u_{i(n)} \neq u_{j(n)}$ and there is a $u_{i(n)} \dashrightarrow u_{j(n)}$ path in D_1 (from Case I),

(2) $u_{i(n)} = u_{j(n)}$ and there is a $v_{(i/n)*} \dashrightarrow v_{(j/n)*}$ path in D_2 (when $r'_{(i/n)*(j/n)*} = 1$),

or (3) $u_{i(n)} = u_{j(n)}$ and $u_{i(n)}$ belongs to a cycle of

D_1 of positive length (when $a^{(k)}_{i(n)i(n)} = 1$).

But referring to Lemma 6.3A, one sees immediately that $c_{ij} = 1$ if and only if (1), (2) or (3) is true.

Therefore, $c_{ij} = 1$ if and only if $c'_{ij} \neq 1$, concluding the proof of the theorem.

BIBLIOGRAPHY

[1] Berge, C., Théorie des graphes et ses applications, Paris, Dunod, 1958.

[2] Harary, F., On the group of the composition of two graphs, Duke Math. J., 26 (1959), 29-34.

[3] Harary, F., A graph theoretic method for the complete reduction of a matrix with a view toward finding its eigenvalues, J. Math. Physics, 38 (1959), 104-111.

[4] Harary, F., Norman, Z., Cartwright, D., Structural models: an introduction to the theory of directed graphs, New York, Wiley, (in press).

[5] Izbicki, H., Graphentransformationen, Monatshefte für Mathematik, 64 (1960), 135-175.

[6] König, D., Theorie der endlichen und unendlichen graphen, Leipzig, Akademie Verlag., 1939; reprinted New York, Chelsea, 1950.

[7] Sabidussi, G., The composition of graphs, Duke Math. J., 26 (1959), 693-696.

[8] Sabidussi, G., Graph multiplication, Math. Zeitschrift, 72 (1960), 446-457.

[9] Weichsel, P., The kronecker product of graphs, Proc. Amer. Math. Soc., 13 (1962), 47-52.