Construction Contracting Lotteries

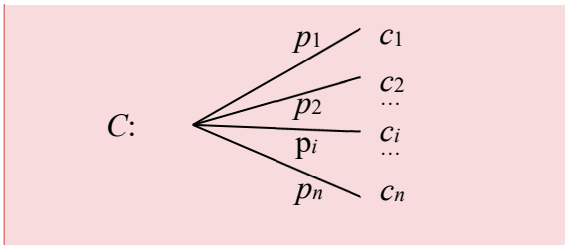
Consider a contractor who is about to sign a lump-sum contract to construct a project for a fixed amount of money L (a lump-sum). The actual cost of the project C is uncertain, and as a result, the contractor is about to acquire a gamble whose outcome is profit V given by:

$$V = L - C \quad \text{[IP1]}$$

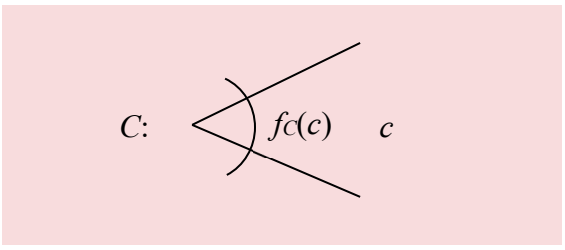
Before signing a lump-sum contract, the amount L is a decision variable whose value is determined either by competitive bidding analysis (a trade-off between the probability of winning and the profit to be made) or through negotiations with the owner. Thus, before signing a lump-sum contract, the contractor must analyze the project gamble as a profit lottery.

The contractor's uncertainty about project profit is entirely due to uncertainty in cost. There is no uncertainty about the lump-sum contract amount L that the owner will pay the contractor to perform the work.

A contractor estimates the probability distribution for project cost C based on the plans and specs and the contractor's overall experience and state of information. This probability distribution can be represented by a *cost* lottery C like the ones shown below.

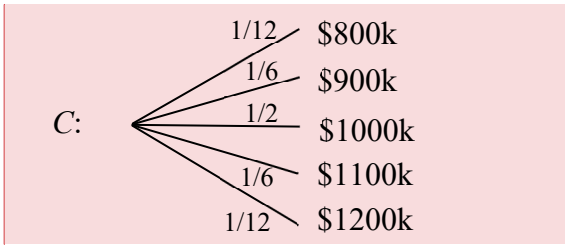


Project Cost, C, as a Discrete Cost Lottery

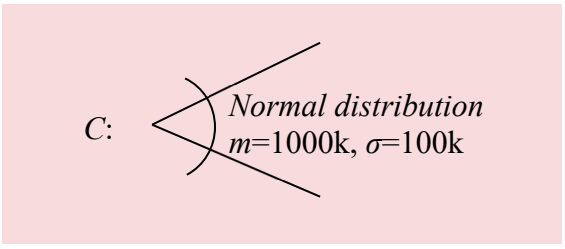


Project Cost, C, as a Continuous Cost Lottery [IP2]

The lottery on the left is a discrete cost lottery C with n branches, where each branch leads to cost value c_i with probability p_i. The lottery on the right is a continuous profit lottery C with an infinite number of branches, each leading to a cost value c with probability density $f_C(c)$. Typical examples of cost lotteries appear below:



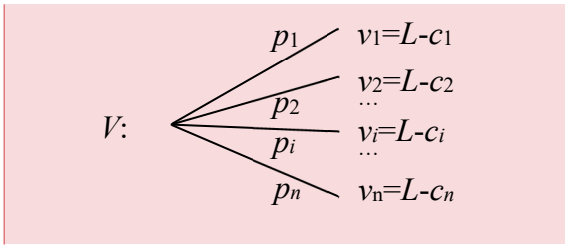
Example: Project Cost C as a Discrete Cost Lottery



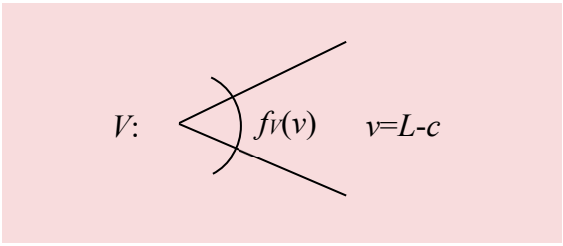
Example: Project Cost C as a Continuous Cost Lottery [IP3]

Notice that in the above lotteries, project costs are typically shown as positive numbers $c_i > 0$, even though they represent money that the contractor will have to pay. Thus, larger positive outcomes (high costs) are less desirable than smaller positive outcomes (low costs).

The total gamble in a lump-sum contract can be represented by a *profit* lottery, like the ones shown below, with outcomes $V=L-C$, where V is profit, L is the lump sum price specified in the contract (a fixed value), and C is the uncertain cost of the work.



Lump-Sum Contract Profit V as a Discrete Profit Lottery



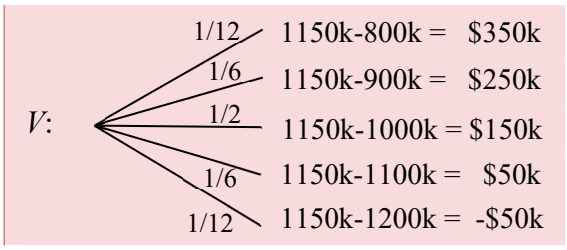
Lump-Sum Contract Profit V as a Continuous Profit Lottery[IP4]

The lottery on the left is a discrete profit lottery V with n branches, where each branch leads to profit v_i with probability p_i. On the right is a continuous profit lottery V with an infinite number of branches, where each branch leads to profit $v=L-c$ with probability density $f_V(v)$.

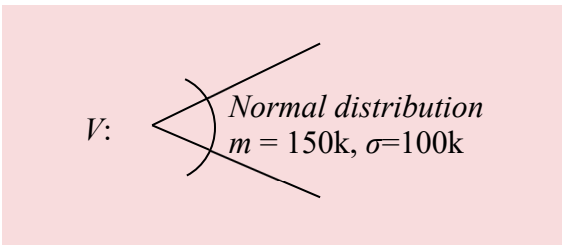
The probability distribution for each of the above profit lotteries V is the same as the probability distribution for the corresponding project cost lottery C when evaluated at $L-v$:

- $P[V = v_i] = P[L - C = v_i] = P[C = L - v_i = c_i]$
- $f_V(v) = f_C(L - v)$

Typical examples of profit lotteries for a lump-sum price L = \$1150k appear below:



Example: Project Cost C as a Discrete Cost Lottery



Example: Project Cost C as a Continuous Cost Lottery[IP5]

Notice that profit V and cost C increase in opposite directions. Thus, the probability $P[V=v_{high}]$ that profit V takes on a *high* value v_{high} is the same as the probability $P[C=c_{low}=L-v_{high}]$ that the cost C will take on the corresponding *low* value $c_{low}=L-v_{high}$.

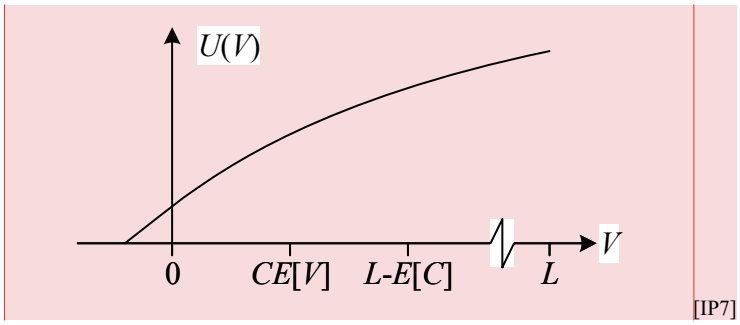
As a result, the 5% percentile for profit, v_5, corresponds to the 95% percentile for cost, c_{95}, and the 95% percentile for profit, v_{95}, corresponds to the 5% percentile for cost, c_5:

$$P[V \le v_5] = 5\% \quad \Rightarrow \quad P[L - C \le v_5] = P[C \ge L - v_5] = 5\% \quad \Rightarrow \quad P[C \le L - v_5 = c_{95}] = 95\%$$

$$P[V \le v_{95}] = 95\% \quad \Rightarrow \quad P[L - C \le v_{95}] = P[C \ge L - v_{95}] = 95\% \quad \Rightarrow \quad P[C \le L - v_{95} = c_5] = 5\%$$ [IP6]

Contractor's Current Net Asset Position (Wealth) and Utility Curve for Money

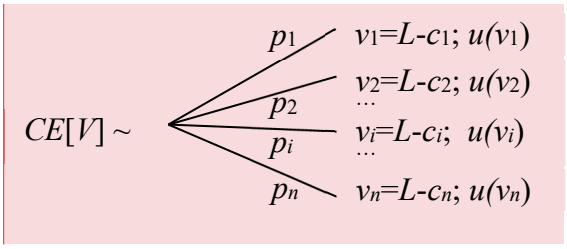
Shown below is an example of a contractor's utility curve for profit V. The utility curve increases as profit increases indicating that the contractor prefers more money to less money. This particular utility curve is concave indicating that the contractor is risk-averse, as is typically the case.



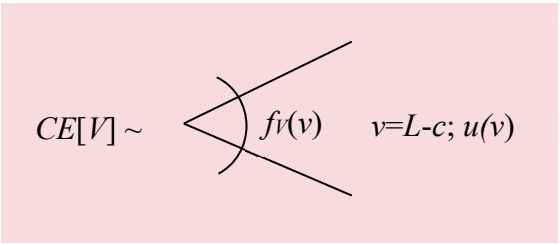
[IP7]

The horizontal axis shows profit V and measures changes to the contractor's current wealth, i.e., monetary additions (profit), or subtractions (loss), relative to a zero point on the axis that represents the contractor's current net asset position V_0. In the following discussion, V_0 is the contractor's net asset position right before signing a lump-sum contract for a new project and corresponds to the zero point on the V axis. Thus, positive values of V represent project profit and negative values represent project loss for the lump-sum contract that the contractor is about to sign.

By signing a lump-sum contract, a contractor buys a new profit gamble (a lottery that the contractor did not have before) whose possible monetary outcomes, $V=L–C$, are additions (profit), or subtractions (loss) relative to that original net asset position V_0, and thus have utility $u(V)$, where V (profit or loss) is measured from the zero point on the V axis.



Lump-sum Contract as a Discrete Profit Lottery V



Lump-sum Contract as a Continuous Profit Lottery V [IP8]

Shown on the left is a discrete profit lottery V for a new lump-sum contract where each branch leads with probability p_i to profit v_i whose utility is $u(v_i)$. On the right is a continuous profit lottery V for a new lump-sum contract where with probability density $f_V(v)$ each branch leads to profit v whose utility is $u(v)$.

Associated with each profit lottery is the expected utility of its profit outcomes $E[u(V)]$:

$$E[u(V)] = \sum p_i \cdot u(v_i) = \int u(v) f_V(v) dv$$

From utility theory we know that when faced with a choice a contractor prefers the lottery with the highest utility and that the utility of any lottery is the expected utility of its outcomes.

Also, the certain equivalent $CE[V]$ for a profit lottery V is that value of profit that has the same utility as the lottery, i.e., whose utility equals the expected utility of the lottery $E[u(V)]$:

$$u(CE[V]) = E[u(V)]$$

$$CE[V] = u^{-1}(E[u(V)])$$

For the profit lotteries shown in the above figures, the certain equivalent $CE[V]$ is shown to the left of each lottery. Because the utility of the certain equivalent $CE[V]$ and the expected utility of the lottery are the same, it follows that the certain equivalent $CE[V]$ is that special value of profit that would make the contractor indifferent (shown by the symbol ~) between the following two choices (a) and (b) that have equal utility:

(a) Receiving with certainty an amount of profit equal to $CE[V]$.

(b) Facing the profit lottery and receiving its uncertain (risky) profit (or loss) outcome V.

The certain equivalent for a project's profit lottery is central to a contractor's decision making. If a contractor can invest its resources in another venture with a higher certain equivalent (such as another project, a real estate development, the stock market, etc.) then the contractor would prefer that option than to sign a lump-sum contract for a fixed price L for the project under consideration.

In general, the certain equivalent $CE[V]$ for a profit lottery V is a function of the lump-sum contract amount L and the project cost lottery C.

$$CE[V] = CE[L - C]$$

However, immediately before signing a lump-sum contract for the project under consideration, the contractor's specific state of information (and uncertainty) about the project's cost remains constant (no new information is made available). Thus, given this particular state of information about the probability distribution of the project cost, the certain equivalent $CE[V]$ is only a function of the decision variable L.

It is important to point out that, in general, the certain equivalent $CE[V]$ is not a linear function of the lump-sum price L because it depends on the contractor's nonlinear utility curve. This means that if the contractor decides to increase by \$100 what he demands as a lump-sum amount L, the certain equivalent for profit $CE[V]$ will not increase by \$100.

Only the exponential utility curve (and its special case, the linear utility curve) have the following *delta*-property (that is discussed in detail in later sections):

$$CE[V] = CE[L - C] = L - CE[C]$$

Thus, when a contractor with an exponential (or linear) utility curve (sometimes called a "delta-contractor") increases the lump-sum amount L by Δ, i.e., to $L+\Delta$, then the certain equivalent for the project's profit, $CE[V]$, will also increase by Δ, to $CE[V]+\Delta$:

$$CE[V + \Delta] = CE[L + \Delta - C] = L + \Delta - CE[C] = CE[V] + \Delta$$

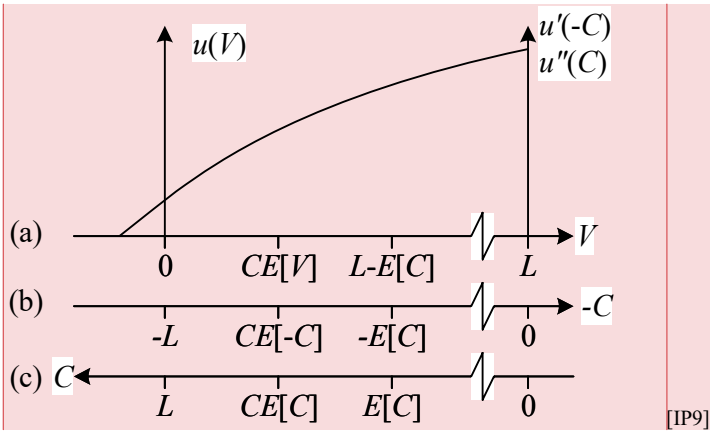
This means that only a "delta-contractor" can evaluate project-related lotteries irrespective of his initial net asset position V_0 and without having to know the lump-sum fixed price L.

Utility Functions for Profit and Cost Lotteries — Lump-Sum Contract for a fixed price *L*

After a lump-sum contract for a project is signed, the price *L* for doing the work is a fixed constant. Thus, gambles relating to this particular contract can now be analyzed using either a profit lottery or a cost lottery as described below. The choice depends on the contractor's preferred value measure for his utility curve: project profit or project cost (where costs can be shown using either negative or positive amounts). These value measures for the contractor's utility curve lead to the following three equivalent models (a) (b) (c) for project lotteries:

(a) A profit lottery where the value measure is profit $V=L–C$. In this case, the calculation of each possible profit lottery outcome v_i requires knowledge and the repeated use of the fixed contract price L along with a corresponding cost outcome c_i.
(b) A profit lottery where the value measure is the project's algebraic cost $-C$, which takes on negative values. In this case, the zero point on the monetary axis of the utility curve for the project is pre-shifted to the right by L, and the person calculating each possible profit lottery outcome $-c_i$ does not need to know the fixed contract price L.
(c) A cost lottery where the value measure is cost C that takes on positive values. In this case, the monetary axis of the project utility curve is pre-shifted to the right by L, and then its direction is flipped, so that cost C increases to the left. Thus, the person calculating the possible cost outcomes c_i does not need to know the fixed contract price L. Moreover, it is easier to use than (b) because all cost outcomes c_i are positive numbers.

Clearly, the contractor's attitude towards risk and his *underlying utility curve* do not change depending on which of the above three equivalent models for lotteries and value measures is used. Nevertheless, each of the above value measures must use its own utility function to describe the same utility curve as shown by the three monetary value axes (a), (b) and (c) below:



The value measure for each the above three alternative project lotteries corresponds to one of the three horizontal axes (a), (b) and (c) in the above figure. Notice that, as shown in the figure, there is only one utility curve. However, because each of the three value measures uses a different horizontal axis, this single utility curve must be represented by three utility functions, (a) $u(V)$, (b) $u'(–C)$ and (c) $u''(C)$, depending on the corresponding value measure.

For the three lottery alternatives (a), (b) and (c), to be equivalent, their results must be identical in every way: Corresponding lottery outcomes must have the same utility, lotteries must produce the same expected utilities (and thus must have *corresponding* certain equivalents), and choices between lotteries must lead to the same decisions.

Thus, for the three lottery models to be equivalent, the following utilities must be equal:

$$u(V) = u(L-C) = u'(-C) = u'(V-L) = u''(C) = u''(L-V)$$

For this to be true, it is necessary to move the "zero point" for the horizontal *x*-axes for cases (b) and (c) to the right by an amount L relative to the zero point for axis (a). Thus, the utility of zero cost on axes (b) and (c), i.e., $u'(0)$ and $u''(0)$, equals the utility of the lump-sum amount L on axis (a), i.e., $u(L)$.

$$u(L) = u'(0) = u''(0)$$

The following are specific examples of monetary values that have equal utilities:

$$u(L) = u'(0) = u''(0)$$
$$u(0) = u'(-L) = u''(L)$$
$$u(L-E[C]) = u'(-E[C]) = u''(E[C])$$
$$u(CE[V]) = u'(CE[V]-L) = u'(CE[-C]) = u''(L-CE[V]) = u''(CE[C])$$

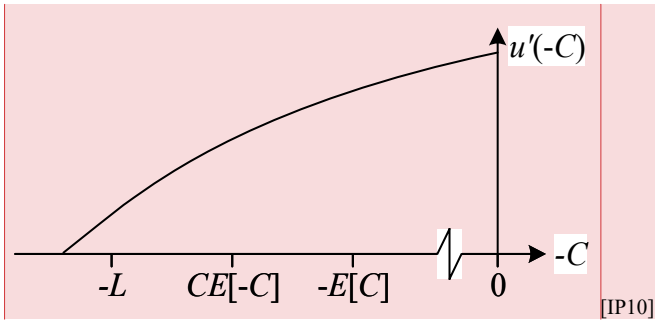
Why Would a Contractor Use Cost Lotteries?

A possible reason a contractor might prefer to model project-related lotteries as cost lotteries (using either alternative (b) or (c), above) is to have project-level management personnel focus on minimizing project cost (which they control) rather than on maximizing profit.

Another reason would be to keep the lump-sum amount L hidden and confidential. This is not possible when using profit lotteries as in (a) above, because the calculation of every profit outcome must reference the lump sum amount L. Alternatives (b) and (c) can easily hide the contract price L from view, simply by shifting the zero point for money on the utility curve.

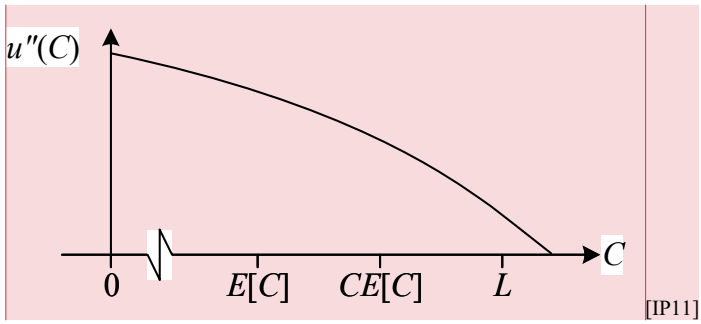
Utility Functions for Project Cost Lotteries — Lump-Sum Contract for a fixed price, *L*

A contractor, for example, could supply the following utility curve, $u'(-C)$, to a project manager (without including the monetary values shown on the cost axis) and ask that it be used for all decisions on a specific project. The value measure for this version of the utility curve is algebraic cost $-C$, which means that project costs (money paid out) are negative and thus appear to the left of the zero point. The utility curve $u'(-C)$ is identical to the utility curve $u(V)$ for profit, except that the zero point for the horizontal value axis has already been shifted to the right by the lump-sum amount L. Thus, the lump-sum contract price L does not have to be revealed and does not need to be used to construct project-related lottery outcomes $-c_i$.



[IP10]

Alternatively, the contractor could supply the project manager with the following utility curve $u''(C)$ where the value measure shows project costs C using positive numbers (again, without including the example labels shown on the cost axis). The utility curve $u''(C)$ is the same as $u'(-C)$ shown above, , except that it has been flipped horizontally so that utility decreases as project costs increase to the right. The fact that this utility curve u''(C) decreases as money takes on larger positive values, makes it obvious that positive monetary values represent project costs.



[IP11]

A convenient way to think about the above two utility curves for cost, u'(-C) and u''(C), is to consider that upon signing the lump-sum agreement for this project, the contractor's net asset position V_0 immediately increases by the lump-sum amount L to V_0+L. Thus, the zero point from which further changes to the contractor's new net asset position V_0+L will be measured (i.e., subtractions due to project costs), must be shifted to the right by L to reflect the contractor's new (higher) net asset position V_0+L. Lotteries concerning the project can now include only project costs (i.e., money paid out) because all revenue (money coming in) has already been accounted for by shifting the zero point for money to the right by L.

The following are important summary points from the above discussion:

- The utility of the outcomes of project-related lotteries must be measured based on net changes to the contractor's net asset position V_0. Such changes are measured by profit (or loss) and this requires that possible cost outcomes must be subtracted from the lump-sum amount L. Thus, the lump-sum amount L must be considered either explicitly as in model (a), or implicitly as in models (b) or (c).
- There are two ways in which the analysis of project-related lotteries (gambles) may include the value of the lump-sum, L:
    - Directly and explicitly, as in $u(V)$, i.e., model (a), or
    - Indirectly and implicitly, as in $u'(-C)$ and $u''(C)$, i.e., models (b) or (c), by shifting the zero point of the monetary value axis to the right by L.
- A contractor cannot produce the above utility functions for cost, $u'(-C)$ and u''(C), unless the lump-sum amount L is fixed and known.
- Thus, even though all project uncertainty may be due to project cost, it is not possible to analyze project-related lotteries (gambles) based on cost alone, when cost is measured relative to the original asset position V_0. To analyze project-related lotteries (gambles) based on cost, the utility of cost $u'(-C)$ and u''(C) must be measured relative to a new asset position V_0+L, which requires that the value for the lump-sum price L be known.

Relationships between Certain Equivalents for Profit and Cost Lotteries

As stated above, the three lottery alternatives (a), (b) and (c), that use utility functions and corresponding value measures (a) $u(V)$, (b) $u'(-C)$ and (c) $u''(C)$, are equivalent and their results are identical in every way: Corresponding lottery outcomes have the same utility, lotteries produce exactly the same expected utilities, lotteries have *corresponding* certain equivalents, and choices between lotteries lead to the same decisions.

Here we shall examine what it means for lotteries to have *corresponding* certain equivalents.

The utility of the certain equivalent of a lottery equals the expected utility of the lottery. Moreover, the expected utility of a lottery is the same, irrespective of whether the lottery is structured to use utility function (a) $u(V)$, or (b) $u'(-C)$ or (c) $u''(C)$. Thus, for the same underlying lottery, the following utilities in the left column are equal because they equal the expected the utility of the lottery. As a result, the following certain equivalent relationships in the right column must also be true:

$$\begin{array}{lll} \text{(a)} & u(CE[V]) = u(CE[L-C]) & \Rightarrow \quad CE[V] = CE[L-C] \\ \text{(b)} & u'(CE[-C]) = u'(CE[V]-L) & \Rightarrow \quad CE[-C] = CE[V]-L \\ \text{(c)} & u''(CE[C]) = u''(L-CE[V]) & \Rightarrow \quad CE[C] = L-CE[V] \end{array}$$ [IP12]

The following numerical example makes the above relationships easier to understand.

Suppose that the lump-sum contract amount is L= \$1,000k and that the certain equivalent for the profit lottery V is $CE[V]$= \$100k. Then, the corresponding certain equivalents when using utility functions (a) $u(V)$, (b) $u'(-C)$ or (c) $u''(C)$ would be as shown below:

$$\begin{array}{ll} \text{(a)} & CE[V] = CE[L-C] = \$100k \\ \text{(b)} & CE[-C] = CE[V]-L = \$100k - 1{,}000k = -\$900k \\ \text{(c)} & CE[C] = L - CE[V] = \$1{,}000k - 100k = \$900k \end{array}$$

The corresponding certain equivalents, (a) \$100k, (b) –\$900k, and (c) \$900k, do not have the same numerical values because they are measured relative to three different *value measure* scales (a), (b) and (c). Nevertheless, they have the same utility because when these three certain equivalents are added to the corresponding initial net asset position correctly, they all lead to the same *final net asset position,* $V_1 = V_0$ +\$100k, and thus to the same utility, $u(V_1) = u(V_0$ +\$100k):

$$\begin{array}{ll} \text{(a)} & V_1 = V_0 + CE[V] = V_0 + \$100k \\ \text{(b)} & V_1 = (V_0 + L) + CE[-C] = V_0 + L + CE[V] - L = V_0 + CE[V] = V_0 + \$100k \\ \text{(c)} & V_1 = (V_0 + L) - CE[C] = V_0 + L - (L - CE[V]) = V_0 + CE[V] = V_0 + \$100k \end{array}$$

Notice that the certain equivalent $CE[V]$ for profit lottery (a) represents money added to the initial asset position V_0. In contrast, the certain equivalents $CE[-C]$ and $CE[C]$ are added to or subtracted from a different initial asset position $(V_0 + L)$. The initial asset position, V_0, corresponds to the "zero" point for the profit scale (a), whereas the initial asset position $(V_0 + L)$ corresponds to the "zero" point of the cost scales (b) and (c). [IP13]

Interpretation of the Certain Equivalent for a Cost Lottery (given a Lump-Sum Price, *L*)

The certain equivalent, $CE[C]$, of a cost lottery is the maximum amount of money that a contractor who has signed a lump-sum contract for a price, L, would be willing to pay to have someone else (e.g., a subcontractor) perform all the work, so that the contractor would avoid facing the risk involved in the project's cost lottery.

Notice that for the contractor, the certain equivalent for cost $CE[C]$ is actually a *PISP*, a "personal indifferent selling price", for the project's cost lottery[IP14].

- $CE[C]$ is a selling price because after a contract has been signed to perform the work for a fixed price L, the contractor owns the project's cost lottery and is about to transfer ownership of that cost lottery (and the obligation to perform the work for the project) to someone else (e.g., a subcontractor). Clearly, the max selling price $CE[C]$ depends on L.
- For a contractor wishing to transfer (i.e., "sell") this cost lottery to someone else, the selling price is "negative" in the sense that the seller (the contractor) does not receive money but has to pay money to the buyer (the subcontractor) who will take over the cost lottery and assume responsibility for the work.
- A subcontractor who takes over all or part of the work for a project is buying a cost lottery for a negative buying price (he will be paid for doing the work). By signing a subcontract for a fixed amount S, the subcontractor buys a cost lottery he did not have before. Thus, the subcontract amount S is the subcontractor's *negative buying price* because it represents *money that he will receive*. Clearly, a subcontractor hopes that the actual cost outcome C for the cost lottery he will buy, will be less than the subcontract amount S that he will receive, and that he will make a profit $S–C$.

Expected Cost vs. Certain Equivalent of Cost Lotteries

When a contractor models project cost as $-C$ using alternative (b), i.e., as a *profit* lottery, the expected profit $E[-C]$ is *negative* (undesirable). A risk-averse contractor would be willing to pay more than the $E[-C]$ to avoid the risk of the lottery, Hence, for a *risk-averse* person, the certain equivalent $CE[-C]$ for *profit* lottery $-C$ is even "more negative" than $E[-C]$:

$$CE[-C] < E[-C] < 0 \quad \text{[IP15]}$$

Similarly, when a contractor models project cost C using alternative (c), i.e., as a *cost* lottery, the expected cost $E[C]$ is *positive* (undesirable). A risk averse contractor would be willing to pay more than the $E[C]$ to avoid the risk of the lottery. Thus, a *risk-averse* person would have a certain equivalent $CE[C]$ for cost lottery C that is even "more positive" than $E[C]$:

$$0 < E[C] < CE[C] \quad \text{[IP16]}$$

Clearly, the expected costs and the certain equivalents for the two lotteries (b) and (c) have the same absolute values and differ only in their algebraic sign: (b) uses negative numbers while (c) uses positive numbers.

$$E[C] = -E[-C]$$

$$CE[C] = -CE[-C] \quad \text{[IP17]}$$

Definition of Risk Premium for Cost Lotteries

The "risk premium" for a *profit* lottery and the "risk premium" for a *cost* lottery are defined differently so that both are *positive* for a risk-averse person. A *positive* "risk premium" represents the amount of money that a *risk-averse* contractor would be willing *to give up* to avoid the risk of a lottery that he owns by transferring that risk to someone else.

- For a profit lottery, a positive risk premium indicates that a risk-averse contractor would be willing to accept a selling price (the certain equivalent), that is less than the expected profit of the work, to avoid the risk of the lottery.
- For a cost lottery, a positive risk premium represents the amount of money that a risk-averse contractor would be willing to pay above the expected cost of the work to avoid the risk of the project's cost.

The standard definition for the risk premium of a *profit* lottery V is given by:

$$RP[V] = E[V] - CE[V] = L - E[C] - CE[V]$$

Alternative (b) models cost -C as a *profit lottery*, and thus its risk premium is:

$$RP[-C] = E[-C] - CE[-C] = -E[C] - (CE[V] - L) = L - E[C] - CE[V] = RP[V]$$

The risk premium for cost C modeled by *cost* lottery (c) which follows the opposite sign convention for *money received* and *money paid out,* follows the opposite sign convention:

$$RP[C] = CE[C] - E[C] = (L - CE[V]) - E[C] = RP[V]$$

The risk premiums (a) $RP[V]$, (b) $RP[-C]$, (c) $RP[C]$, for all three lottery models are equal:

$$RP[V] = RP[-C] = RP[C] \quad \text{[IP18]}$$

Positive risk premiums, $RP[V]$, $RP[-C]$ and $RP[C]$, indicate *risk-averse* behavior.

$$\text{Risk-averse behavior} \quad \Leftrightarrow \quad RP[V] = RP[C] = RP[-C] > 0$$

The risk premium $RP[-C]$ for *profit* lottery -C is subtracted from expected cost $E[-C]$ to obtain the certain equivalent $CE[-C]$. The risk premium $RP[C]$ for a *cost* lottery C, however, is *added* to the expected cost $E[C]$ to obtain the certain equivalent $CE[C]$. Thus, the certain equivalents $CE[.]$ shown below have the algebraic sign for $RP[.]$ on the right-hand-side reversed:

(b) For profit lottery, $-C$,	$CE[-C] = E[-C] - RP[-C]$	< 0
(c) For cost lottery, C,	$CE[C] = E[C] + RP[C]$	> 0 [IP19]

These expressions are consistent with each other because if we multiply both sides of the first equation by –1 we obtain the second (because $RP[-C] = RP[C]$):

$$CE[-C] = E[-C] - RP[-C]$$
$$-CE[-C] = -E[-C] + RP[-C]$$
$$CE[C] = E[C] + RP[C]$$

In the discussion that follows, it is important to remember that a *positive* risk premium for both *profit* and *cost* lotteries represents money that a risk averse contractor *gives up* to *avoid risk*.

PIBP – Buying Price for Project Cost Lottery – Risk-Adjusted Cost (RAC)

An important concept in decision analysis is that of PIBP, the *personal indifferent buying price*. A buying price arises when a person buys something that he does not already own. The term "indifferent" means that PIBP is that special buying price at which a person is truly indifferent between (a) buying that thing in exchange for an amount money equal to the PIBP, and (b) the status quo where there is zero change in the person's assets and wealth.

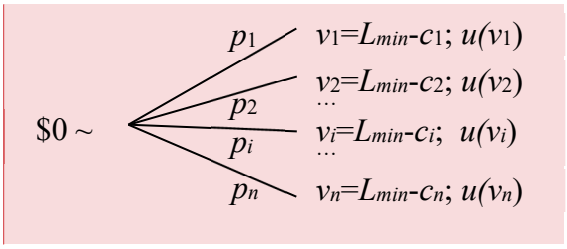
When a contractor signs a lump-sum contract to perform the work for a project for a fixed price L, he is essentially buying the cost lottery for that project, something that he did not have before. The buying price for the project cost gamble is the fixed price L. This is a "negative" buying price in the sense that the contractor does not pay L to acquire the cost lottery for the project, but instead must be paid L to take ownership of this undesirable gamble.

The PIBP is the maximum amount of money that a decision maker would be willing to pay to buy a desirable gamble that he does not already own. For a cost lottery, the PIBP is the minimum contract price L_{min} that the contractor would accept to take on the responsibility for the project cost lottery that he did not have to face before.

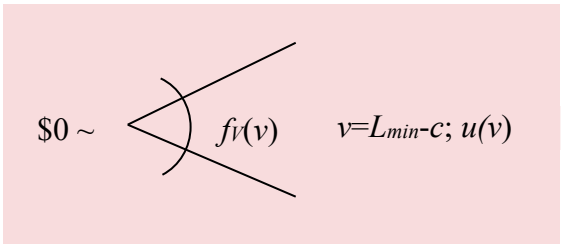
The indifference statement by which the PIBP = L_{min}, is computed is the following:

$$CE[V \mid L_{\min}] = CE[L_{\min} - C] = 0 \quad \text{[IP20]}$$

Hence, the PIBP = L_{min} is that special lump-sum price that makes the contractor indifferent between signing a lump-sum contract for a price L_{min} and \$0 (no contract, no change in wealth):



PIBP = L_{min} for a Discrete Profit Lottery V



PIBP = L_{min} for a Continuous Profit Lottery V [IP21]

Another term for PIBP = L_{min}, is "*Risk-Adjusted Cost*" or RAC. This term reflects the fact the contractor is indifferent between contracting to do the work at RAC or not at all. Doing the work for a price equal to RAC is the conceptual equivalent to "performing the work at cost"—there is no profit in it. Given the project's risk and the contractor's risk aversion, the contractor does not perceive any profit from doing the work at a lump-sum price L equal to the RAC.

As described earlier, the certain equivalent for cost $CE[C]$, is a function of the lump-sum price L and is back-calculated from the certain equivalent for profit $CE[V]$ as follows:

$$CE[C \mid L_{\min}] = L_{\min} - CE[V \mid L_{\min}] = L_{\min} \quad \text{[IP22]}$$

This means that the lump-sum price, L_{min} = PIBP = RAC, also equals the corresponding certain equivalent for cost $CE[C \mid L_{min}] = L_{min}$. Thus, a contractor with a contract for L_{min}, would be willing to make no profit and pay the entire price $CE[C \mid L_{min}] = L_{min}$ to a subcontractor to perform all the work.

Numerical Examples — Lump-Sum Contract for a fixed price *L*

This section uses numerical examples to illustrate some of the important points made above:

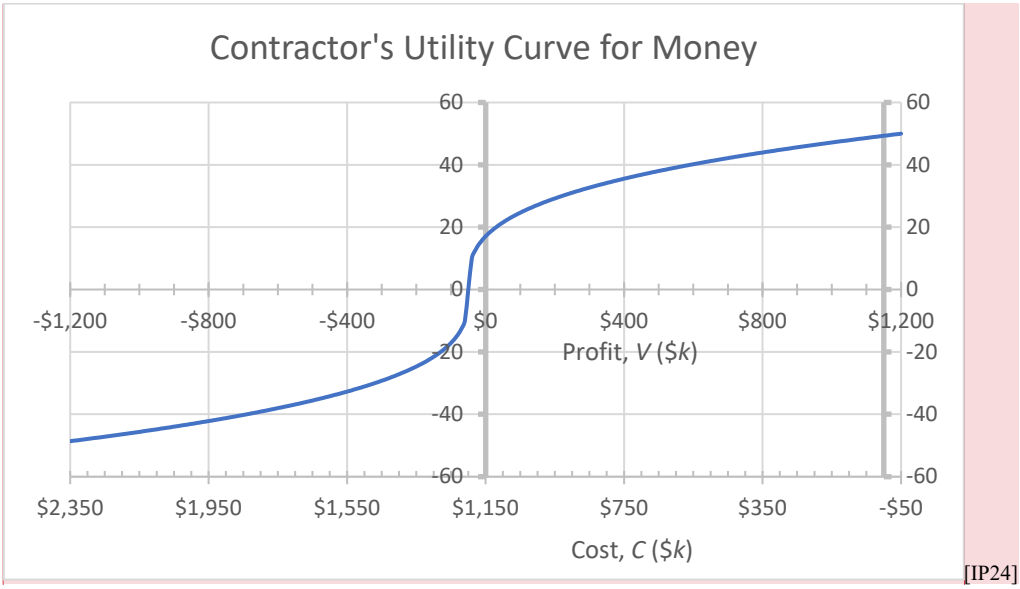
- Decisions about a project must be based on profit lotteries that measure profit added to, or loss subtracted from, the contractor's current net asset position (or wealth) V_0.
- For a lump-sum contract, it is not possible to construct an appropriate project profit lottery (or cost lottery) without knowing and taking into account the lump-sum fixed price L either directly or indirectly.
- A cost lottery that treats costs as money subtracted from the contractor's current asset position (or wealth) V_0 is not appropriate for making decisions because the resulting utilities for cost outcomes come from the wrong region of the utility curve. Such a cost lottery is equivalent to a profit lottery for a lump-sum contract with a fixed price L = \$0 — clearly an inappropriate model for any situation.

Consider a contractor with the following cubic-root utility curve for profit V (in \$*k*):

$$u(V) = \sqrt[3]{a + bV} = \sqrt[3]{5000 + 100V} \qquad \text{(profit } V \text{ is in \$}k\text{)} \quad \text{[IP23]}$$

The inverse of the utility function returns the profit $V(u)$ (in \$*k*) given its utility u:

$$V(u) = u^{-1}(u) = (u^3 - a)/b = (u^3 - 5000)/100 \quad \text{(profit } V \text{ is in \$}k\text{)}$$



[IP24]

This cubic-root utility curve was selected deliberately for the examples that follow because its properties make it easier to illustrate certain points:

(a) The cubic-root function is simple and can easily be inverted to find certain equivalents.

(b) It gives the utility for both positive and negative values of money (i.e., additions and subtractions relative to the current asset position V_0.

(c) It exhibits different attitudes towards risk for positive and negative values of money:

- For positive values of money, the utility curve is *concave* and thus exhibits *risk-averse* behavior.

- For negative values of money, the utility curve is *convex* and thus exhibits *gambling* (risk-seeking) behavior.
- This change in risk attitude makes the points below easier to appreciate.

Example Lump-Sum Contract — Correct Analysis based on Profit

Suppose that the contractor is about to sign a lump-sum contract for a fixed price L. Project cost C follows a distribution with the five outcomes c_i shown below (note that $E[C] = \$1{,}000k$ and $SD[C] = \$100k$). The profit lottery V also has five outcomes v_i as shown in the table.

Profit Lottery (Correct analysis based on profit for L =\$1,150k)			
Prob. $P[C=c_i] = P[V=v_i]$	Cost, C c_i (\$$k$)	Profit, $V=L-C$ $v_i = L - c_i$ (\$$k$)	Utility $u(v_i) = u(L-c_i) = u''(c_i)$
1/12	800	350	34.1995
1/6	900	250	31.0723
1/2	1,000	150	27.1442
1/6	1,100	50	21.5443
1/12	1,200	(50)	0.0000
$E[.]=$	$E[C] = \$1{,}000k$	$E[V] = \$150.000k$	$E[u(V)] = 25.1915$
$CE[.]=$	$CE[C] = \$1{,}040.132k$	$CE[V] = \$109.868k$	
$RP[.]=$	$RP[C] = \$40.132k$	$RP[V] = \$40.132k$	[IP25]

It is important to notice that for the lump-sum fixed price L =\$1,150k used in this example, all profit monetary outcomes are greater than -\$50k, and thus have positive utility values, which come from the region of the cubic-root utility curve that exhibits risk-averse behavior.

The expected utility and the certain equivalent of the profit lottery (shown above) are:

$$E[u(V)] = \sum_{i=1}^{5} P[V = v_i]u(v_i) = \sum_{i=1}^{5} P[C = L - v_i] \cdot \sqrt[3]{5000 + 100v_i} = 25.1915$$

$$CE[V] = V(E[u(V)]) = u^{-1}(E[u(V)]) = \{E[u(V)]^3 - 5000\}/100 = \$109.868k$$

- The certain equivalent for profit, $CE[V] = \$109.868k$, is the minimum amount to be received with certainty for which the contractor would be willing to sell the entire (risky) contract, including the lump-sum amount L, to another contractor.
- The expected profit of this risky contract, $E[V]$=\$150k, is greater that $CE[V]$=\$109.868k. The difference is by definition the risk premium, $RP[V] = \$40.132k$, i.e., the amount of expected profit that the contractor would be willing to give up to avoid risk:

$$RP[V] = E[V] - CE[V] = \$150k - \$109.868k = \$40.132k$$

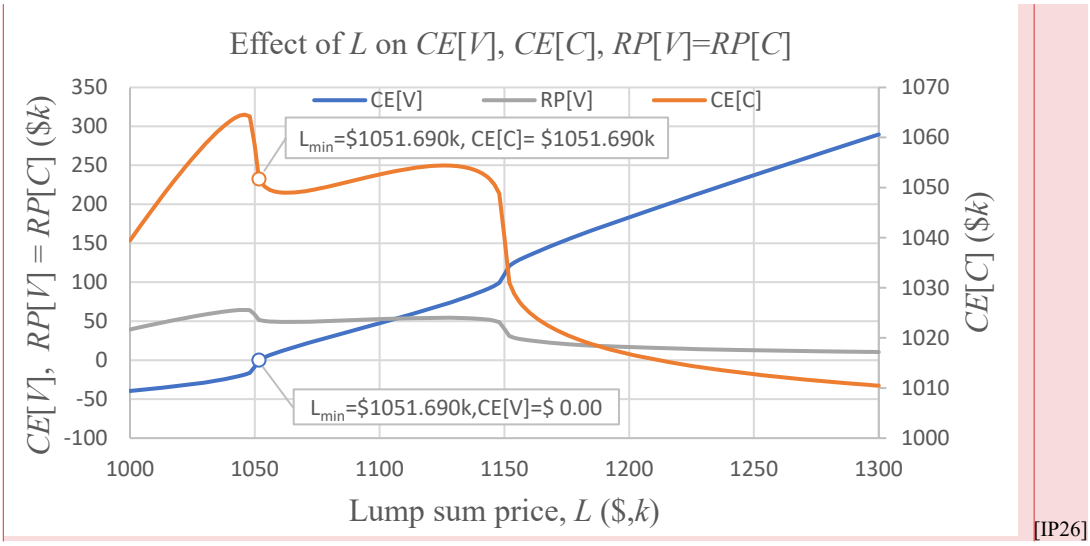
- The certain equivalent for cost $CE[C]$, and the risk premium for cost $RP[C]$, for the example lump-sum price of L =\$1,150k are back-calculated from $CE[V]$:

$$CE[C] = L - CE[V] = \$1{,}150k - \$109.868k = \$1{,}040.132k$$
$$RP[C] = CE[C] - E[C] = \$1{,}040.132k - \$1{,}000k = \$40.132k = RP[V]$$

- Both the certain equivalent for cost $CE[C]$, and the risk premium for cost $RP[C]$, depend on and cannot be computed without knowing and taking into account the lump-sum price L =\$1,150k and the certain equivalent for profit $CE[V]$.
- The risk premium for cost $RP[C]$ and the risk premium for profit $RP[V]$ for a lump-sum contract are equal and both depend on the lump-sum price L =\$1,150k.

Impact of the Lump-Sum Price *L*

The following graph illustrates the impact of the lump-sum price L on the certain equivalents for profit $CE[V]$, the certain equivalent for cost $CE[C] = L - CE[V]$, and the risk premiums for profit and cost $RP[V] = RP[C]$.



[IP26]

The RAC is L_{min}=\$1051.690, i.e., the contract price for which $CE[V \mid L_{min}] = \$0$. As expected, at L_{min}, the certain equivalent for cost has the same value, $CE[C \mid L_{min}] = L_{min} = \$1051.690$.

The certain equivalent for profit $CE[V]$ is a monotonically increasing function of L, whereas the certain equivalent for cost $CE[C] = L - CE[V]$ increases and decreases unexpectedly.

All three curves are quite sensitive to the lump-sum price L in the neighborhoods of L=\$1,050k and L=\$1,150k. This sensitivity is caused by the significant curvature of the example cubic-root utility curve in the neighborhoods of V = –\$100k (very risk-seeking behavior) and V=\$0k (very risk-averse behavior). When the lump-sum price is close to L=\$1,050k and L=\$1,150k, the corresponding profit lottery outcomes are in the neighborhoods of V=-\$100k and V=\$0k and take their utility values from these two regions of the utility curve that have very large curvatures (the region on the left is very risk-seeking and the region on the right is very risk-averse). This is the cause for the observed curve sensitivity in the above graphs.

The graphs for the risk premiums $RP[V]$=$RP[C]$ are identical. They are also identical in shape to the graph for the certain equivalent for cost $CE[C]$ but offset by \$1,000.

$$RP[V] = RP[C] = CE[C] - E[C] = CE[C] - \$1{,}000k$$

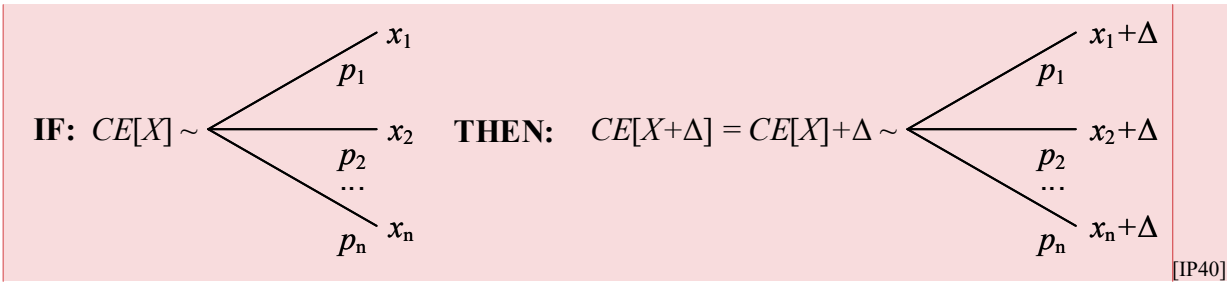
Lump-Sum Contracts and the Delta Property of the Exponential Utility Curve

The *delta*-property of the exponential utility curve can be defined as follows:

- If all *prizes* x_i of a lottery X are increased by the same amount Δ and become $x_i + \Delta$, then the *certain equivalent* of the lottery $CE[X]$ is also increased by the same amount Δ.

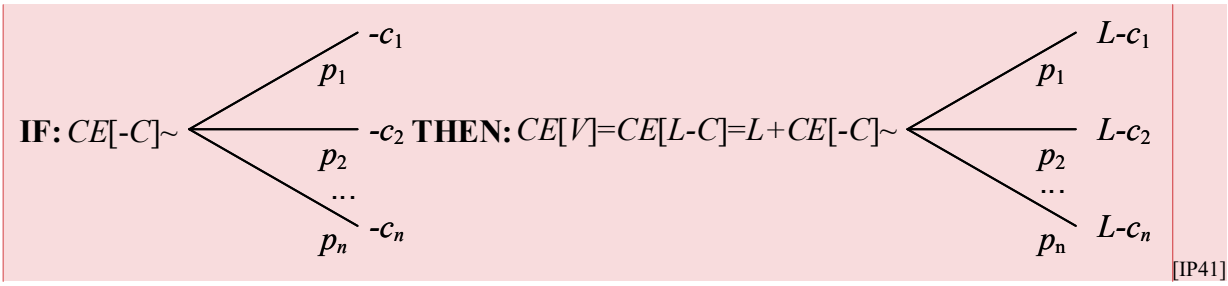
$$CE[X+\Delta] = CE[X] + \Delta$$

- The *delta-property* is illustrated graphically by the following *indifference* statements:



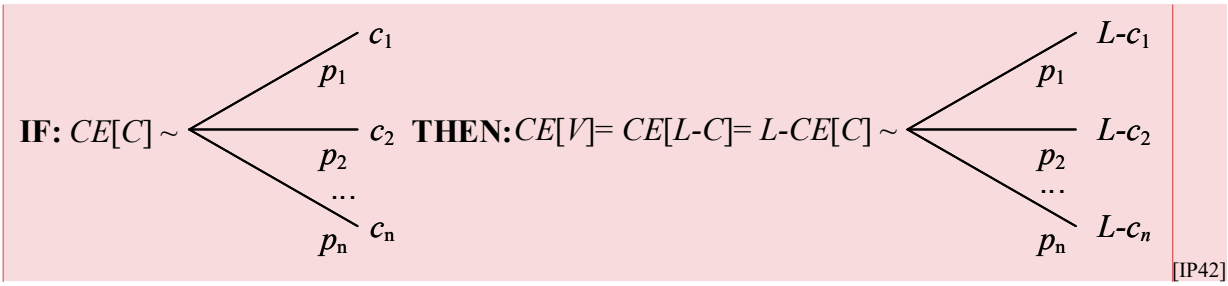
[IP40]

The delta-property can be applied directly to a lump-sum construction contract by starting with the project's cost lottery $-C$ that has negative outcomes $-c_i$, i.e., lottery (b), and then adding the constant L (i.e., L is the constant Δ) to derive the profit lottery outcomes $L-c_i$ as follows.



[IP41]

Based on lottery (b), the delta-property for a lump-sum contract can now be written in terms of the project's cost lottery C that has positive cost outcomes c_i, i.e., lottery (c), as follows.



[IP42]

Notice that the certain equivalent of lottery (b) for algebraic cost $-C$ is negative $CE[-C] < 0$, whereas the certain equivalent for lottery (c) where costs C are considered positive, is positive $CE[C] > 0$. The two certain equivalents differ only in algebraic sign:

$$CE[C] = -CE[-C]$$

In either case, the certain equivalent $CE[V]$ for profit lottery V that is derived by adding the lump-sum amount L to the above two cost lotteries (b) $-C$ and (c) C should be the same:

$$CE[V] = L - CE[C] = L + CE[-C]$$

Delta-Property and the Contractor's Current Net Asset Position V_0

The alternatives for profit and cost lotteries (a) (b) (c) presented earlier are equivalent because their corresponding monetary outcomes have identical utilities, and the lotteries have identical expected utilities. The reason that these profit and cost lotteries produce identical utility results, however, is that they all require and make use of the lump-sum contract price L either explicitly, as in profit lottery (a) V, or implicitly, as in profit lottery (b) $-C$, or cost lottery (c) C.

This observation led to the very important conclusion that a contractor with current net asset position V_0 cannot analyze decisions based only on project cost C without considering the lump-sum contract price L. The contractor's utility curve $u(V)$ gives the utility for net profit $V=L-C$ which is measured from a zero point that reflects this particular current net asset position V_0. Thus, the lump-sum price L is always required (either explicitly or implicitly) to read the correct utilities from the utility curve for any project-related lotteries.

The important consequence of the delta property is that a contractor with an exponential utility curve can analyze decisions based only on project cost C that are equally valid for any L.

For a "delta-contractor", project *profit* V is the difference between the lump sum amount L (a constant) and the cost C a random variable.

$$V = L - C$$

Thus, based on the *delta*-property, the certain equivalent for profit $CE[V]$ is obtained by adding the constant L to the certain equivalent $CE_0[-C]$:

$$CE[V] = CE[L - C] = L + CE_0[-C] = L - CE_0[C] \quad \text{[IP43]}$$

This result is identical to the general result derived earlier (that is valid for any utility curve):

$$CE[C] = L - CE[V] \quad \text{[IP44]}$$

The important difference is that the earlier evaluation of $CE[C]$ was based on the utility of costs measured from a "zero" monetary point associated with asset position V_0+L (i.e., by first increasing the contractor's current wealth by L), which required knowledge of the lump-sum price L. When using the delta property, the certain equivalent $CE_0[C]$ is calculated using the utility of costs measured from a "zero" monetary point associated with the initial net asset position V_0. Thus, the evaluation of $CE_0[C]$ does not require the lump-sum price L.

When the delta-property applies, the two certain equivalents $CE[C]$ and $CE_0[C]$ are equal for any lump-sum price L. As a result, the subscript "0" is not necessary (except when needed for clarity). I.e., the contractor's current net asset position or wealth does not matter:

$$CE[C] = CE_0[C] \quad \text{[IP45]}$$

Also, when the delta-property applies, the risk premium for profit $RP[V] = RP[C] = RP_0[C]$ does not depend on and is the same for any value of the lump-sum price L:

$$RP[V] = RP[C] = RP_0[C] = CE_0[C] - E[C] \quad \text{[IP46]}$$

Moreover, the delta-property makes the calculation of the risk adjusted cost RAC = PIBP = L_{min}, for which $CE[V]$=\$0, very simple:

$$RAC = L_{\min} = CE[C] = CE_0[C] \quad \text{[IP47]}$$

Numerical Examples — Delta Property and Lump-Sum Contracts

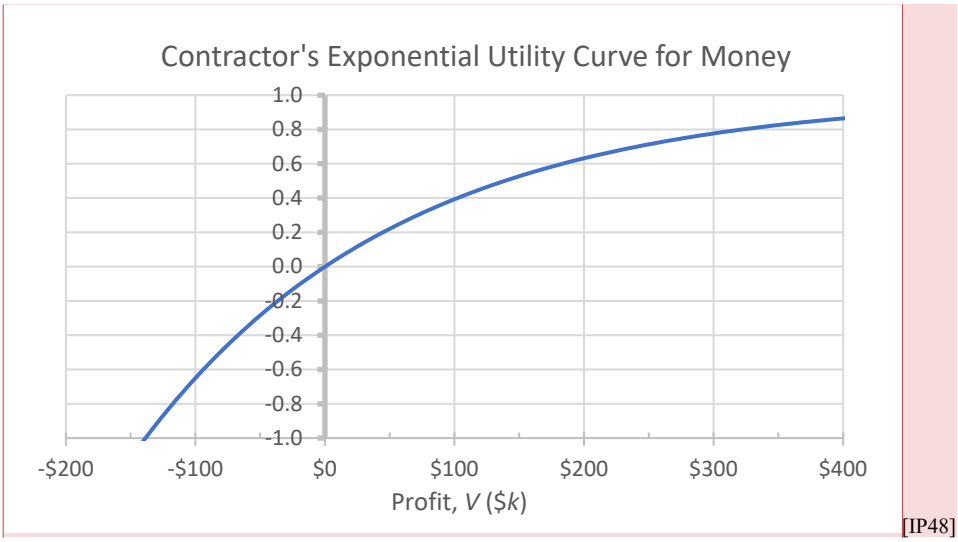
Consider a risk-averse contractor with an exponential utility curve for profit V (in \$$k$) with risk tolerance ρ =\$200k:

$$u(V) = a - be^{-V/\rho} = 1 - e^{-V/200} \quad \text{(profit } V \text{ is in \$}k\text{)}$$

The inverse of the utility function gives the profit $V(u)$ (in \$$k$) for a given utility value u:

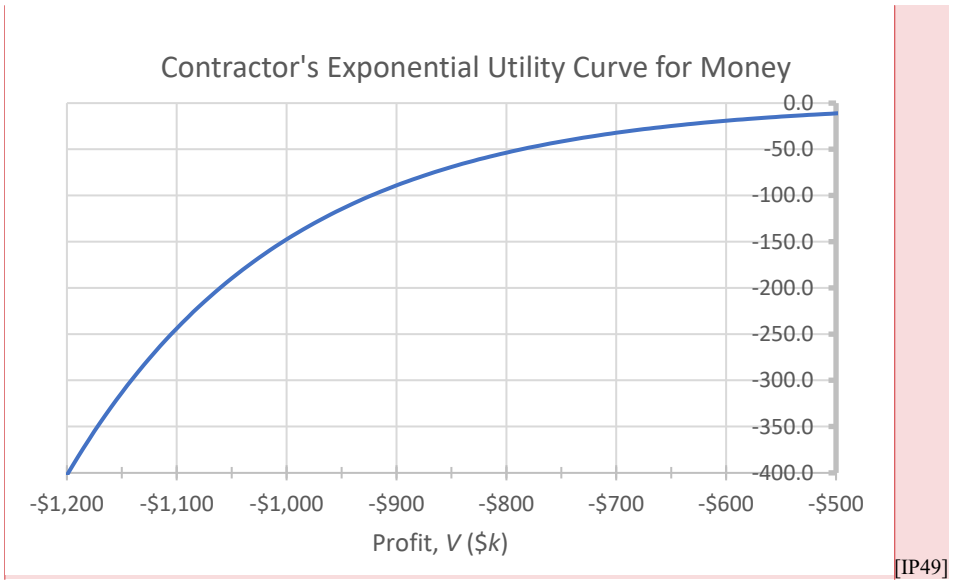
$$V(u) = u^{-1}(u) = -\rho \ln[(a-u)/b] = -200 \cdot \ln(1-u) \quad \text{(profit } V \text{ is in \$}k\text{)}$$

The graph below shows this exponential utility curve in the profit range -\$200k < V < \$400k.



[IP48]

Shown below is the same exponential utility curve for negative dollar values in the range -\$1,200k < V < \$-500k. It should be noted that even though the range of values for the profit and utility axes are different, the curvature of the exponential utility curve is the same:



[IP49]

Example Lump-Sum Contract — Analysis based on Profit

Let us consider again the previous example project whose cost C follows a discrete distribution with five outcomes c_i, that give $E[C] = \$1{,}000k$ and $SD[C] = \$100k$. The contractor has signed a lump-sum contract for a fixed price $L = \$1{,}150k$ and the resulting profit lottery V has the outcomes v_i and utilities $u(v_i)$ shown in the table.

Profit Lottery for $L = \$1{,}150k$			
Prob. $P[C=c_i] = P[V=v_i]$	Cost, C c_i ($k)	Profit, $V=L-C$ $v_i = L - c_i$ ($k)	Utility $u(v_i) = u(L-c_i)$
1/12	800	350	0.8262
1/6	900	250	0.7135
1/2	1,000	150	0.5276
1/6	1,100	50	0.2212
1/12	1,200	(50)	-0.2840
$E[.]=$	$E[C] = \$1{,}000k$	$E[V] = \$150.000k$	$E[u(V)] = 0.4648$
$CE[.]=$	$CE[C] = \$1{,}024.984k$	$CE[V] = \$125.016k$	
$RP[.]=$	$RP[C] = \$24.984k$	$RP[V] = \$24.984k$	[IP50]

The expected utility and the certain equivalent of the profit lottery (shown above) are:

$$E[u(V)] = \sum_{i=1}^{5} P[V = v_i]u(v_i) = \sum_{i=1}^{5} P[C = L - v_i](1 - e^{-v_i/200}) = 0.4648$$

$$CE[V] = V(E[u(V)]) = u^{-1}(E[u(V)]) = -200 \cdot \ln(1 - E[u(V)]) = \$125.016k$$

The fact that the contractor's behavior is risk-averse is indicated by the shape of the contractor's exponential utility curve as well as the above project profit lottery results:

- An exponential utility with positive risk tolerance $\rho = \$200k$ *and* positive scale parameter $b=1$ is concave and exhibits risk-averse behavior everywhere.
- The certain equivalent for profit $CE[V] = \$125.016k$ is smaller than the expected profit $E[V] = \$150.000k$ resulting in a positive risk premium for profit $RP[V] = \$24.984k$. A positive risk premium indicates risk-averse behavior.
- The certain equivalent for cost $CE[C] = \$1{,}024.984k$ is greater than the expected cost $E[C] = \$1000k$ resulting in a positive risk premium for cost $RP[C] = \$24.984k$. Again, a positive risk premium indicates risk-averse behavior.

In the example above, the certain equivalent for cost $CE[C]$, as well as the risk premium for cost $RP[C]$, have been back-calculated indirectly based on the specific example value $L=\$1{,}150k$, and the calculated certain equivalent for profit $CE[V]$, as follows:

$$CE[C] = L - CE[V] = \$1{,}150k - \$125.016k = \$1{,}024.984k$$

$$RP[C] = CE[C] - E[C] = \$1{,}024.984k - \$1{,}000k = \$24.984k = RP[V]$$ [IP51]

Example Lump-Sum Contract — Analysis based on Cost *C* for any Lump-Sum Price *L*

Because of the delta property of the exponential utility curve, it is possible to analyze the cost lottery *C* for the above project without first selecting and considering the lump-sum fixed price *L*. A contractor with an exponential utility curve may use the cost of the project *C* which follows a discrete distribution with five outcomes c_i, and still find the correct certain equivalent for cost *CE*[*C*] which would be the same for any value of *L*.

Cost Lottery — Ignores the value of the lump-sum price L (Correct analysis is possible because of the Delta Property)		
Prob. $P[C=c_i]$	Cost, *C* c_i (\$*k*)	Utility $u(-c_i) = 1 - e^{-(-c_i)/200}$
1/12	800	-53.5982
1/6	900	-89.0171
1/2	1,000	-147.4132
1/6	1,100	-243.6919
1/12	1,200	-402.4288
	$E[C]$ = \$1,000*k*	$E[u(C)]$ = -167.1603
	$CE_0[C]$ = \$1,024.984*k*	
	$RP_0[C]$ = \$24.984*k*	[IP52]

The expected utility and the certain equivalent of the above cost lottery are:

$$E[u(C)] = \sum_{i=1}^{5} P[C = c_i]u(-c_i) = \sum_{i=1}^{5} P[C = c_i](1 - e^{c_i/200}) = -167.1603$$

$$CE_0[C] = u^{-1}(E[u(C)]) = 200 \cdot \ln(1 - E[u(C)]) = \$1,024.984k$$

Notice that when the lump-sum price *L* is ignored, <u>all</u> cost monetary outcomes c_i become negative profit outcomes $-c_i$. As a result, they have the <u>negative</u> utility values shown in the table that are very different from the <u>positive</u> utility values for the profit outcomes shown in the previous table. Consequently, the expected utility for the profit lottery shown earlier and the cost lottery shown above are very different:

$$E[u(V)] = 0.4648 \quad \neq \quad E[u(C)] = -167.1603$$

Nevertheless, the <u>certain equivalents for cost</u> *CE*[*C*] and $CE_0[C]$ given by these two lotteries are <u>identical</u>:

Certain equivalents for cost, *CE*[*C*]	
Given by Profit Lottery (for *L*=\$1,150*k*)	Given by Cost Lottery (for any *L*)
CE[*C*] = \$1,024.984*k*	$CE_0[C]$ = \$1,024.984*k* [IP53]

Moreover, the relationships that connect the lump-sum price L, the certain equivalent for profit $CE[V]$, and the certain equivalents for cost $CE[C]$ and $CE_0[C]$, are exactly the same:

$$CE[C] = L - CE[V] = \$1{,}150k - \$125.016k = \$1{,}024.984k$$

$$CE[V] = L - CE_0[C] = \$1{,}150k - \$1{,}024.984k = \$125.016k$$

The difference is that for profit lottery V, we first calculate the certain equivalent for profit $CE[V]$, for a specific lump-sum price L, and based on these two values, we calculate the certain equivalent for cost, $CE[C]=L-CE[V]$. For the cost lottery C, we first calculate the certain equivalent for cost $CE_0[C]$, irrespective of L. Then, we can select any value for the lump-sum price L and calculate the certain equivalent for the associated profit, $CE[V]=L-CE_0[C]$. Thus, there exists a linear relationship between the lump-sum price L and the certain equivalent for profit $CE[V]$ as shown in the graph below.

The important conclusion is that the certain equivalent for cost $CE[C]$ equals the certain equivalent for cost $CE_0[C]$ (which does not depend on L). Thus, the certain equivalent for cost $CE[C]$ does not depend on L and has the same value \$1,024.984k for any lump-sum price L:

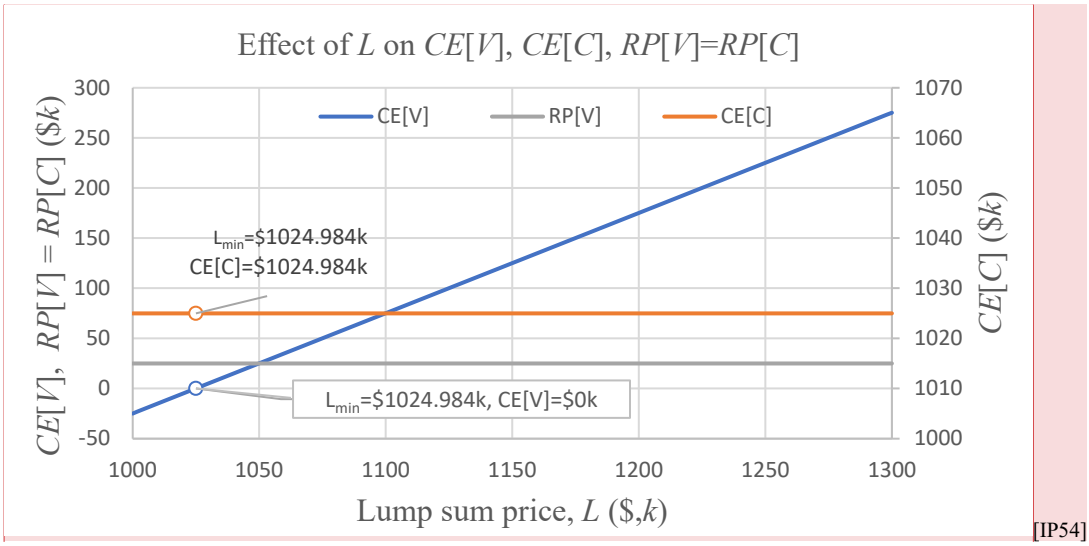
$$CE[C] = CE_0[C] = \$1{,}024.984k \text{ for any lump-sum price } L$$

Delta Property: Lump-Sum Price *L* changes *CE*[*V*], but not *CE*[*C*] or *RP*[*V*]=*RP*[*C*]

Because of the delta property of the exponential utility curve, the certain equivalent for profit $CE[V]$ is a simple linear function of the lump-sum price, L:

$$CE[V] = L - CE_0[C] = L - \$1{,}024.984k$$

Clearly, because the certain equivalent for cost $CE[C] = CE_0[C] = \$1{,}024.984k$, and the risk premiums for profit and cost $RP_0[C] = RP[V] = \$24.984k$, are determined using only the cost lottery C, they do not depend on the lump-sum price L. Also, RAC = $L_{min} = CE[C] = CE_0[C] = \$1{,}024.984k$. These relationships are illustrated in the graph below.



[IP54]

The linear relationship between the lump-sum price L and the certain equivalent for profit $E[V]$ makes it very easy to answer the following example questions A, B, and C.

A. The contractor would like the certain equivalent for profit $CE[V]$ to have a specific value, for example, to be equal to 10% of the expected project cost $E[C]$. Which lump-sum price L gives a certain equivalent for profit $CE[V]$ equal to $10\%E[C]$?[IP55]

For any risk-sensitive utility curve (other than the exponential) the answer to this question is found by a trial-and-error procedure where we select and test successive trial values L until $CE[V] = 10\%E[C]$.

The delta property, however, makes this problem remarkably simple:

$$\begin{aligned} CE[V] &= L - CE_0[C] = L - \$1,024.984k \\ &= 10\%E[C] = 10\%\$1,000k = \$100k \\ L - \$1,024.984k &= \$100k \\ L &= \$1,124.984k \end{aligned}$$

B. When modeling the same project for contractor "R", with a cubic-root utility curve for money (the name "R" is a mnemonic for "cubic Root"), we found a certain equivalent for profit, $CE_R[V] = \$109.868k$. This means that "R" would be willing to sell the entire contract to another contractor, including the lump sum amount $L = \$1{,}150k$ for no less than $Q = \$109.868k$ to be received with certainty. Would contractor "X", with an exponential utility curve for money (the name "X" is a mnemonic for "eXponential"), be willing to buy the contract from contractor "R" at this price $Q = \$109.868k$?[IP56]

If contractor "X" buys the entire contract from contractor "R" for the price $Q = \$109.868k$, then, because of the delta property, his certain equivalent for profit $CE_X[V]$ can be calculated directly from the certain equivalent for cost $CE_0[V]$=$1,024.984$k$ as follows:

$$CE_X[V] = (L - Q) - CE_0[C] = (\$1{,}150k - \$109.868k) - \$1,024.984k = \$15.148k$$

Notice that the constant L-$Q = \$1040.132k$ is the net price (similar to L) that contractor "X" would receive as payment to perform all the work for which his certain equivalent for cost is $CE_0[C]$=1024.984k$. Thus, contractor "X" would be willing to buy the entire contract for the minimum that "R" would accept $CE_R[V] = \$109.868k$, only if "X" finds the resulting certain equivalent for profit $CE_X[V] = \$15.148k$ acceptable, given other market opportunities available.

C. Instead of selling the entire contract, contractor "R" may choose instead to receive L and then to subcontract all the work to "X". In this case, the certain equivalent for cost for contractor "R" $CE_R[C] = \$1{,}040.132k$ is the most that "R" would be willing to pay to "X" to perform all the work. Would contractor "X" be willing to serve as subcontractor to contractor "R" and do all the work for a fixed price $S = \$1{,}040.132k$?[IP57]

Notice that this problem is identical to the one above. Again, because of the delta property, the certain equivalent for profit for contractor "X" $CE_X[V]$ can be calculated directly from his certain equivalent for cost $CE_0[C]$ as follows:

$$CE_X[V] = S - CE_0[C] = \$1,040.132k - \$1,024.984k = \$15.148k$$

The answer is exactly the same as before. Contractor "X" would be willing to serve as subcontractor to contractor "R" and perform all the work for a price of $S = CE_R[C] = \$1{,}040.132k$, only if he finds the resulting certain equivalent for profit $CE_X[V] = \$15.148k$ acceptable given other available opportunities.

Summary of Results for a Contractor and a Lump-Sum Contract[IP63]

The above results for a contractor and a lump-sum contract can be summarized as follows:

Utility Curve	Subcontract Price	Certain Equivalent, Risk Premium	Requires
Any	None	$CE[C] = L - CE[V]$ $RP[V] = RP[C] = E[V] - CE[V]$	L
Any	Any S	$CE[W] = L - S - CE[V']$ $RP[V'] = RP[W] = E[V'] - CE[V']$ $RP[V] - RP[V'] = RP[C] - RP[W]$	L, S
Any	$S^* = \max S$	$RP[V] - RP[V'] = RP^*[Z]$ $RP[V] - RP[V'] = S^* - E[Z]$	L, S^* (S^* found by trial and error)
Exponential	None	$CE[V] = L - CE_0[C]$ $RP[V] = RP[C] = RP_0[C] = CE_0[C] - E[C]$	
Exponential	Any S	$CE[V'] = L - S - CE_0[W]$ $RP[V'] = RP_0[W] = CE_0[W] - E[W]$ $RP[V] - RP[V'] = RP_0[C] - RP_0[W]$	
Exponential	$S^* = \max S$	$S^* = CE_0[C] - CE_0[W]$ $S^* = CE_0[Z \mid W]$	(S^* can be calculated directly)
Exponential	$S^* = \max S$	$RP[V] - RP[V'] = S^* - E[Z]$ $RP[V] - RP[V'] = CE_0[C] - CE_0[W] - E[Z]$	
Exponential	Any S	$CE_0[C] = CE_0[W] + CE_0[Z]$ $RP_0[C] = RP_0[W] + RP_0[Z]$ $RP[V] - RP[V'] = RP_0[Z]$	W and Z must be independent
Exponential	$S^* = \max S$	$S^* = CE_0[C] - CE_0[W] = CE_0[Z]$ $RP[V] - RP[V'] = CE_0[Z] - E[Z] = S^* - E[Z]$	W and Z must be independent

Normally distributed Lotteries — Certain Equivalent and Risk Premium

Consider a contractor with an exponential utility curve and risk tolerance ρ who has signed a lump–sum contract for a fixed price L.

The cost of the work C is described by a continuous *cost* lottery that follows a Normal distribution. The project profit $V = L–C$ is a linear function of project cost and is described by a continuous *profit* lottery that also follows a Normal distribution with the following parameters:

Cost Lottery, C	*Profit Lottery, V*
$C \sim N(E[C], Var[C])$ $E[C] > 0$ $Var[C] > 0$	$V \sim N(E[V], Var[V])$ $E[V] = E[L-C] = L - E[C]$ $Var[V] = Var[L-C] = Var[C]$ [IP70]

For an exponential utility curve for *profit* $u(V)$, the certain equivalent and risk premium for a Normally distributed *profit* lottery V are given by the closed-form formulas below. For completeness, the following table summarizes the closed-form results for the three profit and lottery models (a), (b), (c), described above.

Closed-form Formulas for the Certain Equivalent and Risk Premium of Lotteries when: • *The contractor's utility curve is exponential with risk tolerance ρ.* • *Profit lottery V and cost lottery C follow Normal distributions.*	
(a) Profit Lottery V	$CE[V] = E[V] - \frac{1}{2\rho} Var[V] = L - E[C] - \frac{1}{2\rho} Var[C]$ $RP[V] = E[V] - CE[V] = \frac{1}{2\rho} Var[V] = \frac{1}{2\rho} Var[C] = RP[C]$
(b) Profit Lottery $-C$	$CE[-C] = E[-C] - \frac{1}{2\rho} Var[-C] = -E[C] - \frac{1}{2\rho} Var[C]$ $RP[-C] = E[-C] - CE[-C] = \frac{1}{2\rho} Var[C] = RP[C] = RP[V]$
(c) Cost Lottery C	$CE[C] = -CE[-C] = E[C] + \frac{1}{2\rho} Var[C]$ $RP[C] = RP[-C] = CE[C] - E[C] = \frac{1}{2\rho} Var[C]$ [IP71]

Notice that the above results for *profit* lottery V can be derived from those for *cost* lottery C, and vice-versa, by applying the *delta* property:

$$CE[V] = CE[L-C] = L + CE[-C] = L - E[C] - \frac{1}{2\rho} Var[C]$$

$$RP[V] = RP[L-C] = RP[-C] = RP[C] = \frac{1}{2\rho} Var[C]$$

Numerical Examples — Delta Property and Normal PDFs for Lump-Sum Contract

Consider the risk-averse contractor described earlier with an exponential utility curve for profit V (in \$$k$) and risk tolerance ρ =\$200k:

$$u(V) = a - be^{-V/\rho} = 1 - e^{-V/200} \quad \text{(profit } V \text{ is in \$}k\text{)}$$

The inverse of the utility function returns the profit $V(u)$ (in \$$k$) given its utility u:

$$V(u) = u^{-1}(u) = -\rho \ln[(a-u)/b] = -200 \cdot \ln(1-u) \quad \text{(profit } V \text{ is in \$}k\text{)}$$

Consider also the previous example project whose cost C had a discrete distribution with $E[C]$ = \$1,000k, $SD[C]$ = \$100k. Here we assume that the project cost C follows a Normal distribution with the same $E[C]$ = \$1,000k, and $SD[C]$ = \$100k.

As before, suppose that the contractor has signed a lump-sum contract for a fixed price L = \$1,150k. The resulting Normally distributed cost and profit lotteries for a lump-sum contract are as follows:

Cost Lottery C	*Profit Lottery V for L*=\$1,150k
$C \sim N(E[C], Var[C])$ $E[C] = \$1,000k$ $Var[C] = (\$100k)^2$	$V \sim N(E[V], Var[V])$ $E[V] = L - E[C] = \$1,150k - 1,000k = \$150k$ [IP72] $Var[V] = Var[C] = (\$100k)^2$

The corresponding certain equivalents for profit and cost $CE[V]$ and $CE[C]$ and the associated risk premiums $RP[V]$ and $RP[C]$ given by the closed-form expressions described above are as follows:

Risk-averse contractor with exponential utility curve and risk tolerance ρ =\$200k: [IP73]

(a) Profit Lottery V	$CE[V] = E[V] - \frac{1}{2\rho} Var[V] = \$150k - \frac{(\$100k)^2}{2 \cdot \$200k} = \$125k$ $RP[V] = E[V] - CE[V] = \$150k - \$125k = \frac{(\$100k)^2}{2 \cdot \$200k} = \$25k$
(c) Cost Lottery C	$CE[C] = E[C] + \frac{1}{2\rho} Var[\mathrm{C}] = \$1,000k + \frac{(\$100k)^2}{2 \cdot \$200k} = \$1,025k$ $RP[C] = CE[C] - E[C] = \$1,025k - \$1,000k = \frac{(\$100k)^2}{2 \cdot \$200k} = \$25k$

These results clearly illustrate the delta property:

$$CE[V] = CE[L - C] = L + CE[-C] = \$1,150k - 1,025k = \$125k$$

Effect of Cost Variability on Contractor's Lump-Sum Price

Consider a contractor with an exponential utility curve and risk tolerance ρ and a project whose *cost* lottery C follows a Normal distribution. We shall investigate the effect of cost variability $Var[C]$ on the contractor's lump-sum price L.

Typically, a contractor determines the fixed amount L for a lump-sum contract by selecting an appropriate markup percentage x (such as 15%) and applying it to the expected cost $E[C]$:

$$L = (1+x)E[C] \quad \text{[IP89]}$$

Suppose that the contractor expects a gross profit percentage of "x_{rf} over costs" for *risk-free* work (indicated by the subscript *rf*). For example, x_{rf} = 10%. For such a risk-free project, the contractor's lump-sum price L_{rf} would be:

$$L_{rf} = (1+x_{rf})E[C] = E[C] + E[V_{rf}] \quad \text{[IP90]}$$

The certain equivalent for a project with no risk, i.e., when $Var[C]=0$, would be:

$$CE[V_{rf}] = E[V_{rf}] = L_{rf} - E[C] = x_{rf}E[C] \quad \text{[IP91]}$$

The certain equivalent for a project with risk, i.e., when $Var[C] > 0$, is:

$$CE[V] = L - E[C] - \frac{1}{2\rho}Var[C] \quad \text{[IP92]}$$

For the contractor to be indifferent between a *project with no risk* and a *project with risk* the two certain equivalents, $CE[V]$ and $CE[V_{rf}]$, must be equal:

$$CE[V] = CE[V_{rf}] \quad \text{[IP93]}$$

$$L - L_{rf} = L - E[C] - (L_{rf} - E[C]) = E[V] - E[V_{rf}] = CE[V] + RP[V] - CE[V_{rf}] = RP[V] = \frac{1}{2\rho}Var[C]$$

$$L = L_{rf} + RP[C] = L_{rf} + \frac{1}{2\rho}Var[C] = (1+x_{rf})E[C] + \frac{1}{2\rho}Var[C]$$

$$x = \frac{L - E[C]}{E[C]} = x_{rf} + \frac{Var[C]}{2\rho E[C]} \quad \text{[IP94]}$$

Thus, for the contractor to remain indifferent as the variability of project cost increases:

- The lump-sum price L must exceed the lump sum price for work with no risk L_{rf} by the risk premium for the corresponding project cost lottery $RP[C] = RP[V]$.
- As the variance of the project cost lottery $Var[C]$ increases, the lump-sum amount L increases linearly. Equivalently the lump-sum L increases with the square of the standard deviation of the project cost lottery $SD[C]$.
- The lump-sum L also increases linearly with the contractor's risk aversion coefficient $\gamma = 1/\rho$. The more risk-averse the contractor, the higher the lump sum price L.
- The markup percentage x starts at the markup for work with no risk x_{rf}, and also increases linearly with project cost variance $Var[C]$ and risk aversion, $\gamma = 1/\rho$.

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Cost-Plus-Percent-Fee Contracts — Certain Equivalent and Risk Premium

Consider a contractor *with an exponential utility curve* who has signed a *cost-plus-fee* contract for a project. The contractor's fee is specified in the contract as a percentage m of the actual project cost C. For example, m = 10% gives a *cost-plus-10%-fee* contract.[IP95]

The uncertain cost of the work is represented by a discrete or continuous *cost* lottery C (with positive outcomes c_i), or equivalently, by a *profit* lottery $-C$ (with negative outcomes $-c_i$).

The contractor's *profit* for the project V_m is a random variable equal to the fraction mC of the project cost and thus cannot be negative (in contrast to the profit V_L for a lump-sum contract):

$$V_m = mC \quad (>0) \text{[IP96]}$$

The *profit* lottery V_m is obtained by *multiplying* all outcomes of the *cost* lottery C by the fee percentage m. In this case, the *delta*-property does not apply and the certain equivalent for profit $CE[V_m]$ does not equal the certain equivalent for cost $CE_0[C]$ multiplied by m:

$$CE[V_m] = CE[mC] \neq mCE_0[C] \text{[IP97]}$$

Fortunately, the certain equivalent for profit $CE[V_m]$ can be derived easily for the case where the cost of the work C follows a Normal distribution with $E[C] > 0$ and $Var[C] > 0$:

$$C \sim N(E[C],\ Var[C]) \text{[IP98]}$$

In this case, the profit V_m also follows a Normal distribution $V_m \sim N(E[V_m],\ Var[V_m])$ where:

$$\begin{aligned} E[V_m] &= E[mC] = mE[C] \\ Var[V_m] &= Var[mC] = m^2 Var[C] \\ SD[V_m] &= mSD[C] \end{aligned} \text{[IP99]}$$

For a contractor with an exponential utility curve and a profit lottery V_m that follows a Normal distribution, the certain equivalent $CE[V_m]$ and the risk premium $RP[V_m]$ are given by:

$$CE[V_m] = E[V_m] - \frac{1}{2\rho} Var[V_m] = mE[C] - \frac{1}{2\rho} m^2 Var[C] \approx mE[C]$$

$$RP[V_m] = E[V_m] - CE[V_m] = \frac{1}{2\rho} m^2 Var[C] = m^2 RP_0[C] \approx 0 \text{[IP100]}$$

Notice that for a cost-plus-fee contract, the variance of profit $Var[V_m]$ is a very small fraction $m^2Var[C]=m^2Var[V_L]$ of the variance of the profit under a lump-sum contract. Thus,

The contractor's risk premium for a cost-plus-fee contract $RP[V_m]$ is significantly smaller than the risk premium for a lump sum contract $RP[V_L]$.[IP101]

For example, for a fee percentage m=10%,

$$m = 10\% \quad \Rightarrow \quad RP[V_m] = 1\%\, RP[V_L] \text{[IP102]}$$

Also, the standard deviation $SD[V_m] = m\ SD[V_L]$, is a small fraction of $SD[V_L]$. Thus, the *range* of possible profit outcomes for a cost-plus-fee contract is much narrower.

Contractor Indifference Conditions — Lump-Sum Contract vs. Cost-Plus-%Fee Contract

Consider a contractor with an exponential utility curve and risk tolerance ρ and a project whose cost is represented by a *cost* lottery C that follows a Normal distribution. We shall investigate the conditions for which the contractor would be indifferent between the following two contracts:

(a) A *lump-sum contract* for a fixed amount of money L.

(b) A *cost-plus-percent-fee contract*, where the contractor's fee mC is a percentage m of the actual project cost C.

In the following discussion it will be useful to relate the percentage m and the markup percentage x that corresponds to the fixed price L for the lump-sum contract:

$$x = \frac{L - E[C]}{E[C]} \quad \text{[IP103]}$$

The certain equivalents for the profit lotteries of the two contract types are:

Lump-Sum Contract:	$CE[V_L] = L - E[C] - \frac{1}{2\rho} Var[C]$ $CE[V_L] = xE[C] - \frac{1}{2\rho} Var[C]$
Cost-Plus-Percent-Fee Contract:	$CE[V_m] = mE[C] - \frac{1}{2\rho} m^2 Var[C]$ [IP104]

For a given project and contractor, the quantities ρ, $E[C]$, and $Var[C]$ have specific values, and as a result, the certain equivalent for the profit of a lump sum contract $CE[V_L]$ is an increasing function of the lump sum price L.

Similarly, for given values of ρ, $E[C]$, and $Var[C]$, the certain equivalent for the profit of a cost-plus-fee contract $CE[V_m]$ is an increasing function of the fee percentage m.

For the contractor to be indifferent between the two contract types for this project, the certain equivalents for the corresponding profit lotteries must be equal:

$$CE[V_L] = CE[V_m]$$

Substituting the above expressions for the certain equivalents gives:

$$L - E[C] - \frac{1}{2\rho} Var[C] = mE[C] - \frac{1}{2\rho} m^2 Var[C]$$

Thus, for a given project, the lump-sum amount L for which the contractor would be indifferent between a lump-sum contract and a cost-plus-fee contract with a fee percentage m is:

$$L = (1+m)E[C] + (1-m^2)\frac{1}{2\rho} Var[C]$$
$$L = (1+m)E[C] + (1-m^2)RP[C] \quad \text{[IP105]}$$

The above indifference condition that gives the lump-sum price L as a function of the percentage m is easier to understand when expressed in terms of the markup percentage x.

By substitution we obtain that the markup percentage x for which the contractor would be indifferent between a lump-sum contract, for amount L and a cost-plus-fee contract with a fee percentage m is given by:

$$x = m + (1 - m^2)\frac{Var[C]}{2\rho E[C]}$$
$$x = m + (1 - m^2)\frac{RP[C]}{E[C]} \quad \text{[IP106]}$$

These expressions indicate that for a contractor to be indifferent between a lump-sum contract for amount L and a cost-plus-percent-fee contract with a fee percentage m the following must be true:

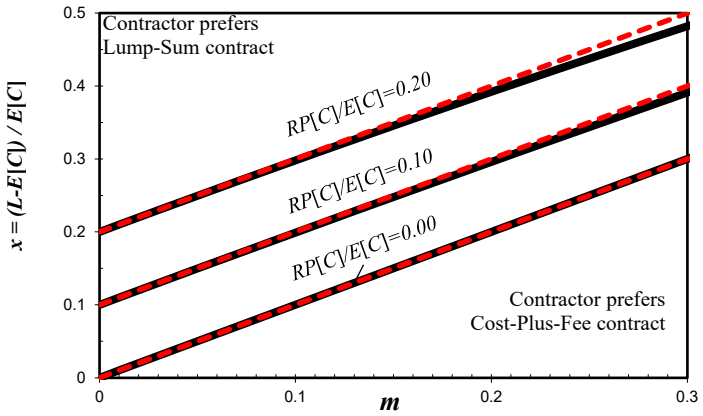
- The markup percentage x for the lump-sum contract must be greater than the fee percentage m. This is expected because a contractor assumes greater risk — may even lose money — under a lump-sum contract.
- The amount by which the markup percentage x for the lump-sum contract must exceed the fee percentage m is proportional to the "relative risk premium" of the lump sum contract $RP[C]/E[C]$.
- Equivalently, the amount by which the markup percentage x for the lump-sum contract exceeds the fee percentage m is proportional to both the relative variability of the project cost $Var[C]/E[C]$ and the contractor's risk aversion $1/\rho$. The riskier the project cost and the more risk-averse the contractor, the greater the amount by which the lump-sum markup percentage x must exceed the cost-plus-fee percentage m.

Contractor Indifference Curves for Lump-Sum Contract vs. Cost-Plus-%Fee Contract

Consider a contractor with an exponential utility curve and risk tolerance ρ and a project whose cost is represented by a *cost* lottery C that follows a Normal distribution with expected cost $E[C]$ and cost variance $Var[C]$. As shown above, the contractor would be indifferent between a lump-sum contract with markup percentage x and a cost-plus-percent-fee contract with fee percentage m when x and m satisfy the following condition:

$$x = m + (1-m^2)\frac{Var[C]}{2\rho E[C]} = m + (1-m^2)\frac{RP[C]}{E[C]} \quad \text{[IP107]}$$

The black curves in the figure below show graphs for the above condition for three example values of $Var[C]/(2\rho E[C]) = RP[C]/E[C]$: 0, 0.10, and 0.20.



Contractor is indifferent between

Lump-sum and Cost-plus-percent-fee contracts when:

$$x = \frac{L - E[C]}{E[C]} \approx m + \frac{RP[C]}{E[C]}$$

$$m \approx \frac{CE[V_L]}{E[C]} = \frac{CE[V_m]}{E[C]}$$

$$L \approx (1+m)E[C] + RP[C]$$

[IP108]

Conceptually, the three curves in the figure correspond to three projects, each having a different value of $RP[C]/E[C]$, or equivalently of $Var[C]/(2\rho E[C])$. Thus, the risk of each project increases as we move up in the graph, from a curve lower in the figure to a curve higher up.

The red dashed lines indicate that for the small cost-plus-fee percentages m used in practice, for example up to 30%, the relationship between m and x is almost linear with a slope equal to 1. The reason is that m^2 is very small, making the term $(1–m^2)$ very close to 1.

The bottom black curve in the figure above is the straight line, $x = m$, and represents a project with no risk, i.e., with $Var[C]=0$ and $RP[C]=0$. The slope of this straight line is 1, indicating that the contractor would be indifferent between a lump-sum contract and a cost-plus-fee contract at the same markup percentage. The *vertical offset* between the straight line, $x = m$, and one of upper curves is very close to $RP[C]/E[C]$ and represents the "percent markup premium", $RP[C]/E[C]$, that the contractor would require to undertake the additional risk of a lump-sum contract for that project.

Contractor Preferences — Lump-Sum Contract vs. Cost-Plus-Percent-Fee Contract

Additional insight is provided by considering just one of the curves in the above figure. The middle curve, for example, represents the indifference conditions for a contractor with risk tolerance ρ and a particular project with cost parameters $E[C]$ and $Var[C]$, such that $Var[C]/(2\rho E[C]) = 0.10$.

For the project that corresponds to the middle curve, each point (m, x) anywhere in the above graph, represents a possible decision where the contractor must choose between:

(a) A lump-sum contract with percentage markup x.

(b) A cost-plus-fee contract with a fee percentage m.

The contractor's choice depends on which of the two contract types has the highest certain equivalent for project profit at point (m, x).

Thus, all points (m, x) in the figure are divided by the contractor's indifference curve into two regions as follows:

- For points (m, x) above the middle curve, the contractor prefers a lump-sum contract. The corresponding fee percentage m is too low relative to the markup percentage x.
- For points (m, x) below the middle curve, the contractor prefers a cost-plus-fee contract. The corresponding markup percentage x is too low relative to the fee percentage m.
- For points (m, x) on the middle curve, the contractor would be indifferent between the two contract types because their certain equivalents have equal values $CE[V_L] = CE[V_m]$.[IP109]

Numerical Examples — Contractor Lump-Sum and Cost-Plus-Fee Lotteries

Consider the risk-averse contractor described earlier with an exponential utility curve for profit V (in \$$k$), and risk tolerance ρ =\$200k:

$$u(V) = a - be^{-V/\rho} = 1 - e^{-V/200} \quad \text{(profit } V \text{ is in \$}k\text{)}$$

As in the previous example, the total project cost C, follows a Normal distribution with $E[C]$ = \$1,000k, $SD[C]$ = \$100k.

The contractor will face the following cost C and profit lotteries V_L and V_m, for a lump-sum contract at a price L, and for a cost-plus-percent-fee contract at a fee percentage m:

Cost Lottery C	*Lump-Sum Profit Lottery* V_L, *for price* L	*Cost-Plus-%Fee Profit Lottery* V_m, *for fee percentage* m
$C \sim N(E[C], Var[C])$ $E[C] = \$1,000k$ $Var[C] = (\$100k)^2$	$V_L \sim N(E[V_L], Var[V_L])$ $E[V_L] = L - E[C] = L - 1,000k$ $Var[V_L] = Var[C] = (\$100k)^2$	$V_m \sim N(E[V_m], Var[V_m])$ $E[V_m] = mE[C] = m \cdot 1,000k$ $Var[V_m] = Var[mC] = m^2(\$100k)^2$

The cost lottery C, the profit lottery for a lump-sum contract V_L, and the profit lottery for a cost-plus-percent-fee contract V_m, have the following certain equivalents and risk premiums:

Cost Lottery: Contractor has exponential utility curve with risk tolerance ρ =\$200k	
Cost Lottery C	$CE[C] = E[C] + \frac{1}{2\rho}Var[C] = \$1,000k + \frac{(\$100k)^2}{2 \cdot \$200k} = \$1,025k$ $RP[C] = CE[C] - E[C] = \$1,025k - \$1,000k = \frac{(\$100k)^2}{2 \cdot \$200k} = \$25k$

Contractor's Profit Lottery for Lump-Sum Contract	
For price L, $V_L = L$-C	$CE[V_L] = E[V_L] - \frac{1}{2\rho}Var[V_L] = L - \$1,000k - \frac{(\$100k)^2}{2 \cdot \$200k} = L - \$1,025k$ $RP[V_L] = E[V_L] - CE[V_L] = \$150k - \$125k = \frac{(\$100k)^2}{2 \cdot \$200k} = \$25k$
For markup x, $V_L = (1+x)E[C]$-C	$CE[V_L] = xE[C] - \frac{1}{2\rho}Var[C] = \$1,000 \cdot x - \$25k$

Contractor's Profit Lottery for Cost-Plus-Percent-Fee Contract	
For fee percentage m, $V_m = mC$	$CE[V_m] = mE[C] - \frac{1}{2\rho}m^2Var[C] = \$1,000k \cdot m - \$25k \cdot m^2$ $RP[V_m] = m^2\frac{Var[C]}{2\rho} = \$25k \cdot m^2$

[IP110]

Numerical Examples — Contractor Indifference Curve: Lump-Sum vs Cost-Plus-%Fee

Earlier, we obtained the general condition between the lump-sum amount L and the fee percentage m that makes the contractor indifferent between a lump-sum contract and a cost-plus-fee contract by requiring that the respective certain equivalents be equal $CE[V_L] = CE[V_m]$:

$$L = (1+m)E[C] + (1-m^2)\frac{1}{2\rho}Var[C]$$

[IP111] $$L = (1+m)\cdot 1{,}000k + (1-m^2)\cdot 25k = \$25k(41+40m-m^2)$$

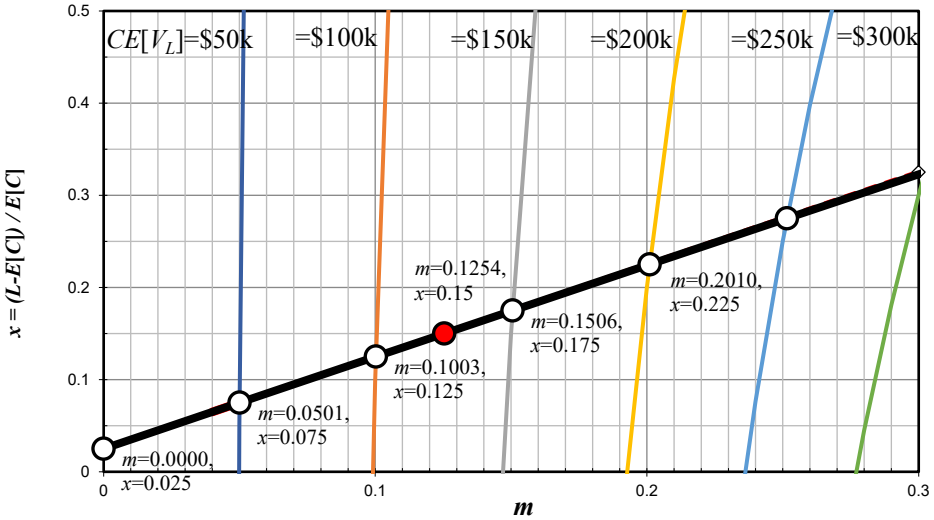
Similarly, we obtained the general condition for the markup percentage x that makes the contractor indifferent between a lump-sum contract and a cost-plus-percent-fee contract with a fee percentage m:

$$x = \frac{L-E[C]}{E[C]} = m + (1-m^2)\frac{RP[C]}{E[C]} \quad \text{[IP112]}$$

Using the data for the example project, the contractor would be indifferent between a lump-sum contract and a cost-plus-percent-fee contract, i.e., $CE[V_L] = CE[V_m]$, when:

$$x = m + 0.025(1-m^2) \quad \text{[IP113]}$$

This quadratic indifference relationship between the markup percentage x and the cost-plus-fee percentage m that would make a contractor with risk tolerance ρ=\$200k indifferent between a lump-sum contract and a cost-plus-percent-fee contract for $Var[C]/(2\rho E[C]) = 0.025$, is shown by the heavy black curve in the figure below. As explained earlier, because m^2 is very small, this black curve is very close to a straight line.



The black curve shows the white points (x,m) that produce the same CE[V] value for a given (constant) RP[C]/E[C]=0.025. Note, however, that the actual value of the CE[V] at each white point increases as we move on the black curve from a white point on the left to another white point on the right.
The previous figure shows that as RP[C]/E[C] increases, the black curve moves up.
As the black curve moves up because the RP[C]/E[C] increases, so do the white points.
For example, as RP[C]/E[C] increases, the white point for $CE[V_L]$=\$150k moves up and traces the gray iso-certain-equivalent curve where all points (x,m) give the same $CE[V_L]$=\$150k.
So points (x,m) on the black curve have equal CE[V] (with different values) for the same RP[C]/E[C]=0.025.
Points (x,m) on the gray curve produce the same $CE[V_L]$=\$150k but for different values of RP[C]/E[C].

Numerical Examples — Contractor Indifference Values: Lump-Sum vs Cost-Plus-%Fee

For the example project, it was assumed earlier that the contractor had signed a lump-sum contract for a fixed price $L = \$1{,}150k$ [IP114] or equivalently that he has used a markup percentage:

$$x = \frac{L - E[C]}{E[C]} = \frac{1{,}150k - 1{,}000k}{1{,}000k} = 15\% \quad \text{[IP115]}$$

As described above, the contractor would be indifferent between a lump-sum contract with a fixed price $L = \$1{,}150k$, or equivalently, a markup percentage x=15%, and a cost-plus-fee contract with fee percentage m that satisfies either of the following equations:

$L = (1+m)E[C] + (1-m^2)\frac{1}{2\rho}Var[C]$	$x = m + (1-m^2)\frac{RP[C]}{E[C]}$
$L = (1+m)\cdot 1{,}000k + (1-m^2)\cdot 25k = 1{,}150k$	$x = m + 0.025(1-m^2)$

When simplified, either of the above conditions gives the same quadratic equation as follows:

$$\begin{aligned} &m^2 - 40m - 1 + 40x = 0 \\ &m = 20 - \sqrt{401 - 40x} \\ &m = 12.54\% \end{aligned} \quad \text{[IP116]}$$

For x=15% we obtain the result m = 12.54%, which can be verified by comparing the resulting certain equivalent for the cost-plus-fee contract $CE[V_m]$ for m= 12.54% to the certain equivalent for the lump-sum contract $CE[V_L]$=\$125k found earlier:

$$\begin{aligned} CE[V_m] &= mE[C] - \frac{1}{2\rho}m^2 Var[C] \\ &= \$1{,}000k \cdot 0.1254 - \$25k \cdot 0.1254^2 = \$125k \end{aligned}$$

From here we see that the certain equivalent for a lump-sum contract with price $L = \$1{,}150k$ and the certain equivalent for a cost-plus-percent-fee contract at a percentage m= 12.54% are equal and thus the contractor would be indifferent between the two contract types:

$$CE[V_L] = CE[V_m] = \$125k$$

The indifference point (m=0.1254, x=0.15) is shown in the graph above by a red circle. The iso-certain-equivalent curve that passes through this point has value, $CE[V_L] = CE[V_m] = \$125k$.

We can also verify that because m^2 is indeed very small, m is given approximately by the following expression:

$$\begin{aligned} &x = m + 0.025(1-m^2) \\ &m \approx x - 0.025 \\ &m \approx 15\% - 2.5\% = 12.5\% \end{aligned} \quad \text{[IP117]}$$

Owner Indifference Conditions — Lump-Sum vs. Cost-Plus-Percent-Fee Contract[IP118]

Consider an owner with an exponential utility curve, with risk tolerance, R,[IP119] who is about to hire a contractor to do the work for a project. The cost of the project is uncertain and is represented by the *cost* lottery C that follows a Normal distribution. We shall investigate the conditions for which the owner would be indifferent between offering the contractor the following two contractual alternatives:

(a) A *lump-sum contract* for a fixed amount of money L.

(b) A *cost-plus-percent-fee contract*, where the contractor's fee mC, is a percentage m of the project actual cost C.[IP120]

For a lump-sum contract, the owner's cost will be the fixed amount L, with no risk[IP121]. For a cost-plus-percent-fee contract, the owner will pay the outcome of the Normal *cost* lottery C_m:

$$\begin{aligned} C_m &= (1+m)C \\ C_m &\sim N(E[C_m], Var[C_m]) \\ E[C_m] &= E[(1+m)C] = (1+m)E[C] \\ Var[C_m] &= Var[(1+m)C] = (1+m)^2 Var[C] \end{aligned} \quad \text{[IP122]}$$

The owner's certain equivalent for the cost-plus-percent-fee *cost* lottery, C_m, is given by:

$$CE[C_m] = E[C_m] + \frac{1}{2R} Var[C_m] = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R} \quad \text{[IP123]}$$

The owner's risk premium for a cost-plus-percent-fee *cost* lottery is:

$$RP[C_m] = CE[C_m] - E[C_m] = (1+m)^2 \frac{Var[C]}{2R}$$

Thus, the owner's certain equivalents for the cost lotteries of the two types of contracts are:

CE for Lump-Sum Contract, C_L:	$CE[C_L] = L$
CE for Cost-Plus-Percent-Fee Contract, C_m:	$CE[C_m] = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$

For the owner to be indifferent between the two contract types, the certain equivalents for the corresponding cost lotteries must be equal:

$$CE[C_L] = CE[C_m]$$

$$L = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R} \quad \text{[IP124]}$$

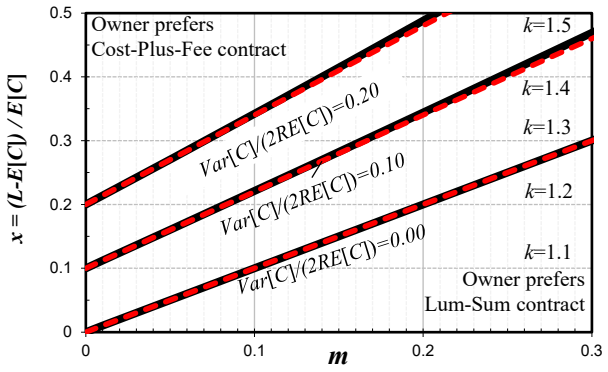
This indifference condition is easier to understand when expressed in terms of the contractor's markup, x. The markup percentage, x, for which the owner would be indifferent between a lump-sum contract, for amount, L, and a cost-plus-fee contract with a fee percentage, m, is:

$$x = \frac{L - E[C]}{E[C]} = m + (1+m)^2 \frac{Var[C]}{2RE[C]} = m + (1+m)^2 \frac{RP[C_m]}{E[C]} \quad \text{[IP125]}$$

The above indifference conditions for L and x, as functions of the fee percentage, m, indicate that for a risk-averse owner the following must be true:

- The owner would be indifferent between (a) paying an uncertain amount, $(1+m)C$, i.e., the uncertain actual cost of the work plus a fee percentage, and (b) paying (with certainty) a lump-sum amount, L, given by the above expression.
- The corresponding markup percentage, x, for the lump-sum price, L, is higher than the cost-plus-fee percentage, m. This was expected because the owner would be willing to pay a risk premium (over the expected cost of a cost-plus-fee contract) to sign a lump-sum contract and eliminate the uncertainty of project cost.
- The *percent spread* (i.e., the *relative risk premium* $RP[C_m]/E[C]$), by which the markup percentage x for the lump-sum contract exceeds the fee percentage m, is proportional to both the relative variability of the project cost $Var[C]/E[C]$, and the owner's risk aversion $1/R$. The riskier the project cost and the more risk-averse the owner, the greater the spread (i.e., the relative risk premium) by which the markup percentage x can exceed the fee percentage m. This is again as expected.

The black curves in the graph below show the relationship between the fee percentage, m, and the markup, x, that would make an owner with risk tolerance, R, indifferent between a cost-plus-fee contract and a lump-sum contract, for three values of $Var[C]/(2RE[C])$: 0, 0.10, and 0.20.



Owner is indifferent between

Lump-sum and Cost-plus-fee contracts

when:

$$x = \frac{L - E[C]}{E[C]} \approx (1 + 2\frac{Var[C]}{2RE[C]})m + \frac{Var[C]}{2RE[C]}$$

$$L \approx (1+m)E[C] + (1+2m)\frac{Var[C]}{2R}$$

[IP126]

The red straight lines indicate that for the small cost-plus-fee percentage values m seen in practice, and up to m=30%, the relationship between m and x is almost linear because m^2 is very small, making the term $(1+m)^2$ very close to $(1+2m)$. Thus, the slopes of the black curves are close to $\{1+2Var[C]/(2RE[C])\}$ and match the slopes of the corresponding red straight lines.

The bottom black curve is indeed the straight line $x = m$, with a slope equal to 1, and represents a project with no cost risk, $Var[C]=0$. In this case, the owner would be indifferent between a lump-sum contract and a cost-plus-fee contract at the same markup or fee percentage $x = m$. Each of two upper curves has a vertical offset, $(1+2m)Var[C])/(2RE[C])$, above this straight line, which represents the additional relative risk premium that the owner would be willing to pay with certainty to sign a lump-sum contract and thus avoid the uncertainty of a cost-plus-fee contract.

Owner Preferences — Lump-Sum Contract vs. Cost-Plus-Percent-Fee Contract

Focusing on one of the curves in the above figure provides additional insights. The middle curve, for example, represents the indifference conditions for a particular owner with risk tolerance R, and a particular project with cost parameters $E[C]$ and $Var[C]$, such that $Var[C]/(2RE[C])$ equals 0.10.

For this particular project, a point (m,x) anywhere in the space of the above figure represents a possible decision where the owner must choose between:

(a) A lump-sum contract with percentage markup x, and

(b) A cost-plus-percent-fee contract with a fee percentage m.

The points (m,x) in this space are divided by the owner's indifference curve into two regions as follows:

- For points (m, x) above the middle curve, the owner would prefer a cost-plus-fee contract. The lump-sum markup x is too high relative to the markup percentage m.
- For points (m, x) below the middle curve, the owner would prefer a lump-sum contract. The fee percentage m is too high relative to the markup percentage x.
- For points (m, x) on the middle curve, the owner would be indifferent between the two contract types.[IP127]

Numerical Examples — Owner Indifferent: Lump-Sum vs. Cost-Plus-Percent-Fee

Consider the same example project presented earlier, whose cost C follows a Normal distribution with the expected cost $E[C]$ and cost variance $Var[C]$ shown below. Suppose, also, that the owner of the project has an exponential utility curve, with risk tolerance R=\$250k.

The owner will face the following cost lotteries for a lump-sum contract and for a cost-plus-percent-fee contract:

Project Cost Lottery, C	*Lump-Sum Cost Lottery, $C_L = L$*	*Cost-Plus-Percent-Fee Cost Lottery, for fee percentage, m, $C_m = (1+m)C$*
$C \sim N(E[C], Var[C])$ $E[C] = \$1,000k$ $Var[C] = (\$100k)^2$	$C_L = L$ $= (1+x)E[C]$ $= (1+x)\$1,000k$	$C_m \sim N(E[C_m], Var[C_m])$ $E[C_m] = (1+m)E[C] = (1+m)1,000k$ $Var[C_m] = Var[(1+m)C] = (1+m)^2(\$100k)^2$

The owner's certain equivalent and risk premium for the lump-sum contract, shown below, are straightforward because by assumption a lump-sum contract has no cost risk:

Owner's Lump-Sum Contract Cost Lottery	
For price, L, $C_L = L$	$CE[C_L] = L = (1+x)E[C] = (1+15\%)\$1,000k = \$1,150k$ $RP[C_L] = \$0$

The owner's certain equivalent and risk premium for the cost-plus-fee cost lottery are:

Owner's Cost-Plus-Fee Contract Cost Lottery (Owner's risk tolerance R =\$250k)	
Owner's Cost-Plus-Percent-Fee Cost Lottery, $C_m = (1+m)C$	$CE[C_m] = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$ $= (1+m)1,000k + (1+m)^2 \frac{(\$100k)^2}{2 \cdot \$250k} = \$20k(51+52m+m^2)$ $RP[C_m] = (1+m)^2 \frac{Var[C]}{2R} = \$20k(1+m)^2$

[IP128]

Numerical Examples — Owner's Indifference Curve: Lump-Sum vs Cost-Plus-%Fee

Shown below are two versions of the general condition that makes the owner indifferent between a lump-sum contract and a cost-plus-percent-fee contract. These conditions were derived by setting the certain equivalents for the owner's project cost lotteries for the two contract types to be equal, $CE[C_L] = CE[C_m]$. The first version gives the markup percentage, x, whereas the second version is solved for the fee percentage, m.

$$x = \frac{L - E[C]}{E[C]} = m + (1+m)^2 \frac{Var[C]}{2RE[C]} = m + \frac{RP[C_m]}{E[C]}$$

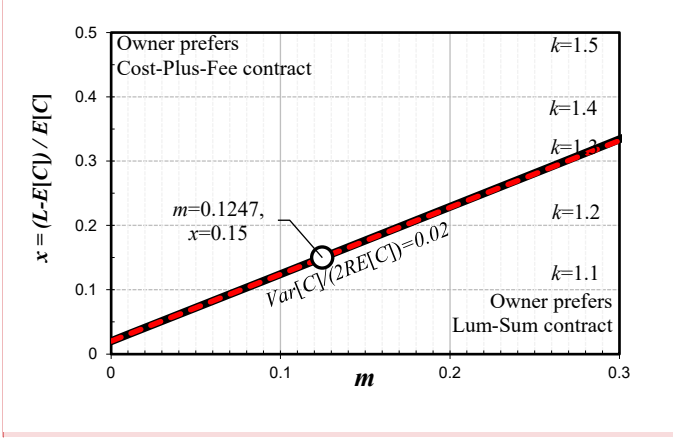
$$m = \frac{-q - 1 + \sqrt{+2q(1+x)+1}}{q}, \text{ where } q = \frac{Var[C]}{RE[C]} \quad \text{[IP129]}$$

For the example problem, the above owner indifference conditions can be simplified to:

$$x = m + 0.02(1+m)^2 = 0.02(1 + 52m + m^2)$$

$$m = -26 + \sqrt{675 + 50x} \quad \text{[IP130]}$$

These quadratic indifference conditions relate the markup percentage x and the cost-plus-fee percentage fee m that make the owner with risk tolerance R=\$250k indifferent between a lump-sum contract and a cost-plus-percent fee contract, for $Var[C]/(2RE[C]) = 0.02$. Both conditions are shown by the heavy black curve in the figure below.



[IP131]

Substituting the markup percentage x=0.15, that corresponds to a fixed price L=\$1,150k, into the above indifference relationship we obtain the fee percentage m=12.47%. Hence, the owner would be indifferent between a lump-sum contract for a fixed price, L=\$1,150k, or equivalently a markup percentage x=15%, and a cost-plus-fee contract with a fee percentage m=12.47%. Both have the same certain equivalent $CE[C_L] = CE[C_m]$ = \$1,150k.

The indifference point (m=0.1247, x=0.15) is shown in the graph above. The iso-certain-equivalent curve that passes through this point is the horizontal line x=0.15 that corresponds to k=1.15. The corresponding certain equivalent is:

$$CE[C_L] = CE[C_m] = k\,E[C] = 1.15*\$1{,}000\text{k} = \$1{,}150k$$

Both Owner and Contractor Indifferent Between Lump-Sum & Cost-Plus-Fee Contracts

Consider an owner with an exponential utility curve and risk tolerance R, who is about to hire a contractor with an exponential utility curve and risk tolerance ρ, to do the work for a particular project. The cost of the project C follows a Normal distribution with parameters $E[C]$ and $Var[C]$. We shall investigate the conditions that would make both the owner and the contractor indifferent between the following contractual alternatives:

Project Contract Choices for Owner and Contractor

(a) *Lump-sum contract* for a fixed amount of money $L = (1+x)E[C]$

(b) *Cost-plus-percent fee contract*, where the contractor's fee mC, is a percentage m of the actual project cost C.

Owner Indifference Conditions	Contractor Indifference Conditions
$L = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$	$L = (1+m)E[C] + (1-m^2)\frac{Var[C]}{2\rho}$
$x = \frac{L-E[C]}{E[C]} = m + (1+m)^2 \frac{Var[C]}{2RE[C]}$	$x = \frac{L-E[C]}{E[C]} = m + (1-m^2)\frac{Var[C]}{2\rho E[C]}$ [IP132]

Individually, the owner and the contractor would be indifferent between the two contract types when the lump-sum amount L (or markup percentage x), and the fee percentage m, satisfy the conditions on the left (for the owner) or the conditions on the right (for the contractor).

For both the owner and the contractor to be indifferent between the two contract types, at the same time, the lump-sum amount L (or markup percentage x), and the fee percentage m, must satisfy both the left and the right conditions simultaneously.

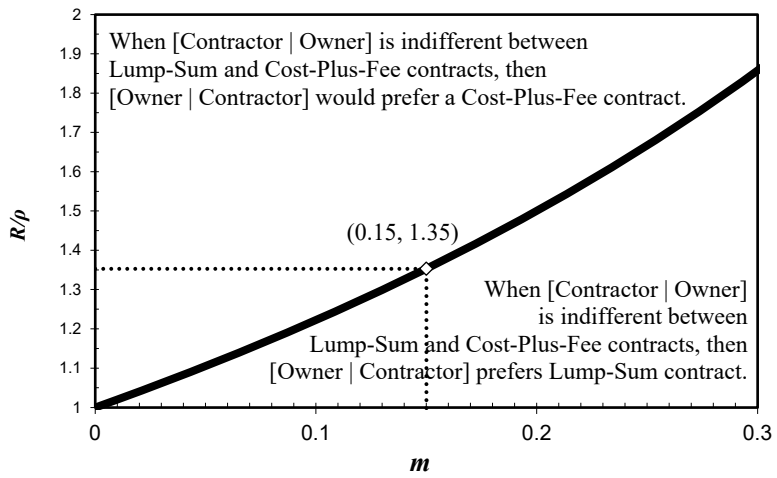
This equilibrium condition is found by setting the expression for L (or x) on the left column equal to the corresponding expression for L (or x) in the right column. After the algebraic cancellation of identical terms, we obtain the following equilibrium condition:

Fee percentage m^*, for which both the owner and the contractor are indifferent between a lump-sum contract and a cost-plus-fee contract.

$$\frac{R}{\rho} = \frac{1+m}{1-m} \quad \Leftrightarrow \quad m^* = \frac{\frac{R}{\rho}-1}{\frac{R}{\rho}+1}$$

For a specific owner with risk tolerance R, and a specific contractor with risk tolerance ρ, there is only one fee percentage m, (and consequently, only one lump-sum amount L, or markup percentage x), that makes both the owner and the contractor indifferent between a lump-sum contract for a certain fixed amount of money L, and a cost-plus-fee contract for the fee percentage m. [IP133]

A graph of this indifference condition between the ratio of the owner and contractor risk tolerances, R/ρ, and the cost-plus-fee percentage, m, is shown in the figure below.



Graph of

$$\frac{R}{\rho}=\frac{1+m}{1-m}$$

The Owner's and Contractor's preferences are expressed by the inequalities:

$$\frac{R}{\rho}\begin{matrix}>\\<\end{matrix}\frac{1+m}{1-m}$$ [IP134]

As a numerical example, let us assume that[IP135] the owner's risk tolerance R is 1.35 times greater than the contractor's risk tolerance, ρ. In this case, we see from the above figure that for $R/\rho=1.35$[IP136], both the owner and the contractor would be indifferent between:

- A cost-plus-fee contract with fee percentage, m=15%, and[IP137]
- A lump-sum contract, for a certain amount, L. The specific value of the lump-sum amount, L, can be determined by either the owner's or the contractor's indifference condition for L (or x) shown above, because, in this case, they are required to be equal and thus give the same value.

As an example, the specific values of L and x when $R/\rho=1.35$, shown below are calculated first using the owner's indifference conditions and thus L and x are functions of R. Using the substitution, $R=1.35\rho$, L and x are then converted to the equivalent expressions given by the contractor's indifference conditions and thus L and x are functions of ρ:

$R/\rho=1.35$, m=15%	Based on lump-sum amount, L	Based on markup percentage, x
Owner's indifference condition (uses R)	$L=(1+m)E[C]+(1+m)^2\frac{Var[C]}{2R}$ $=1.15E[C]+\frac{0.66125}{R}Var[C]$	$x=\frac{L-E[C]}{E[C]}=m+(1+m)^2\frac{Var[C]}{2RE[C]}$ $=0.15+\frac{0.66125}{R}\frac{Var[C]}{E[C]}$
Contractor's indifference condition (uses ρ)	$L=(1+m)E[C]+(1-m^2)\frac{Var[C]}{2\rho}$ $=1.15E[C]+\frac{0.48875}{\rho}Var[C]$	$x=\frac{L-E[C]}{E[C]}=m+(1-m^2)\frac{Var[C]}{2\rho E[C]}$ $=0.15+\frac{0.48875}{\rho}\frac{Var[C]}{E[C]}$ [IP138]

Characteristics of the Owner-Contractor Indifference Condition

The indifference condition for the fee percentage m^*, for which both the owner and the contractor are indifferent between a lump-sum contract and a cost-plus-percent-fee contract, has the following remarkable characteristics:

$$m^* = \frac{R/\rho - 1}{R/\rho + 1}$$

- The expression for the cost-plus-percent-fee percentage m^* is surprisingly simple.
- The cost-plus-percent-fee percentage m^* depends only on the ratio of the owner's and the contractor's risk tolerances R/ρ, but not their absolute values.
- The cost-plus-percent-fee percentage m^* does not depend on any characteristics of the project (for which the contract will be written) such as (a) the project expected cost $E[C]$, or (b) the project cost variance $Var[C]$.
- For a given owner and contractor (i.e., for a given ratio of owner and contractor risk tolerances R/ρ), the equilibrium fee percentage m^* is the same for any project, irrespective of project size and risk.
- The cost-plus-fee percentage m^* exists only when the owner is less risk averse than the contractor $R > \rho$. Otherwise, m^* is negative and thus meaningless.[IP139]

It should be pointed out that the above observations apply only to the cost-plus-percent-fee percentage m^*. The corresponding lump-sum contract amount L, for which both the owner and the contractor are indifferent between a lump-sum contract and a cost-plus-percent-fee contract depends on the particular values of the owner's and the contractor's risk tolerances R and ρ, as well as the project expected cost $E[C]$, and the project cost variance $Var[C]$.

These important differences in dependence between m^* and L are illustrated by the following example for the ratio of risk tolerances, R/ρ=1.35:[IP140]

- The indifference cost-plus-fee percentage, m=15%, is the same for any project that owner and contractor may contract between them and is given by:

$$m = \frac{R/\rho - 1}{R/\rho + 1} = \frac{1.35 - 1}{1.35 + 1} = 15\%$$ [IP141]

- In contrast, the corresponding indifference lump-sum contract amount, L,

$$L = 1.15E[C] + \frac{0.66125}{R} Var[C] = 1.15E[C] + \frac{0.48875}{\rho} Var[C]$$ [IP142]

depends on

a. The project to be contracted, and in particular, on the expected value and the variance of the project cost $E[C]$ and $Var[C]$.

b. The specific values of the risk tolerances R and ρ, and not just on their ratio R/ρ=1.35.

What type of Contract should an Owner offer a Contractor?[IP143]

The graph of the indifference condition between the cost-plus-fee percentage, m^*, and the ratio of the owner's and the contractor's risk tolerances, R/ρ, provides significant insight into risk pricing in construction and the selection of the appropriate contraction contract.

In particular, the above graph shows that:

(a) For the values of the fee percentage m encountered in practice, e.g., the entire range of the horizontal axis in the figure, $0 < m < 0.3$, the indifference condition is satisfied as an equality only for values of the risk-tolerance ratio R/ρ in the range $1 < R/\rho < 2$ (the entire range of the vertical axis in the figure).

(b) The indifference value for the fee percentage m^* is negative when $R/\rho < 1$.[IP144]

These observations lead to the following important conclusions about the ratio of the owner's and the contractor's risk tolerances R/ρ, and the owner's best contract choices.

A. $R/\rho > 2$ — The owner is significantly less risk averse than the contractor.
When the risk tolerance of the owner R is double (or more) than the risk tolerance of the contractor ρ, i.e., when $R > 2\rho$, then the owner is always better off with a cost-plus-percent-fee contract instead of a lump-sum contract.

B. $R/\rho < 1$ — The owner is more risk averse than the contractor.
When the risk tolerance of the owner R is less than or equal to the risk tolerance of the contractor ρ, i.e., when $R <= \rho$, then the owner is always better off with a lump-sum contract instead of a cost-plus-percent-fee contract.

C. $1 < R/\rho < 2$ — The owner is slightly less risk averse than the contractor.
When the owner's and contractor's risk tolerances have a ratio in the range, $1 < R/\rho < 2$, then the owner's choice between a lump-cost contract and a cost-plus-fee contract depends on the fee percentage m for the contractor's cost-plus-fee contract, as shown in the graph above.[IP145]

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Contract Preferred by Owner or Contractor when the Other Party is Indifferent[IP146]

Let us now consider an owner with risk tolerance R, and a contractor with risk tolerance ρ, such that the ratio of the risk tolerances has a specific value R/ρ, and investigate the conditions for the following preferences:

- When the contractor is indifferent between a lump-sum and a cost-plus-fee contract:

$$L = (1+m)E[C] + (1-m^2)\frac{Var[C]}{2\rho}$$

- The owner prefers a cost-plus-percent-fee contract:

$$L > (1+m)E[C] + (1+m)^2\frac{Var[C]}{2R}$$

Substituting the above equation into the inequality gives the following conditions:

$\frac{R}{\rho} > \frac{1+m}{1-m}$, or $m < m^* = \frac{\frac{R}{\rho}-1}{\frac{R}{\rho}+1}$ $\Leftrightarrow$	• When the Contractor is indifferent between a lump-sum and a cost-plus-percent-fee contract, • The Owner would prefer a cost-plus-percent-fee contract.[IP147]

Similarly, we shall now find the condition for which:

- When the owner is indifferent between a lump-sum and a cost-plus-percent-fee contract.

$$L = (1+m)E[C] + (1+m)^2\frac{Var[C]}{2R}$$

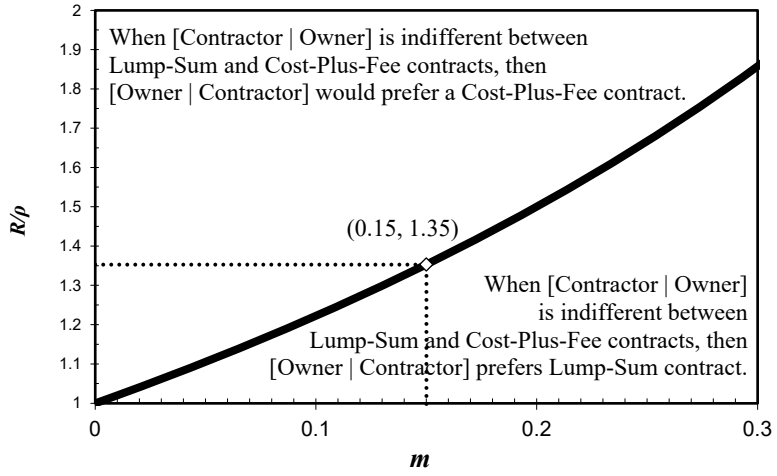
- The contractor prefers a cost-plus-percent-fee contract:

$$L < (1+m)E[C] + (1-m^2)\frac{Var[C]}{2\rho}$$

Surprisingly, a similar substitution shows that the resulting conditions are the same as above:

$\frac{R}{\rho} > \frac{1+m}{1-m}$, or $m < m^* = \frac{\frac{R}{\rho}-1}{\frac{R}{\rho}+1}$ $\Leftrightarrow$	• When the Owner is indifferent between a lump-sum and a cost-plus-percent-fee contract • The Contractor would prefer a cost-plus-percent-fee contract[IP148]

The figure below shows the two regions for points (m, R/ρ) defined by the above inequalities. Clearly, these regions are separated by the indifference condition between the ratio of the owner and contractor risk tolerances, R/ρ, and the cost-plus-fee percentage, m:



Graph of

$$\frac{R}{\rho} = \frac{1+m}{1-m}$$

The Owner's and Contractor's preferences are expressed by the inequalities:

$$\frac{R}{\rho} \lessgtr \frac{1+m}{1-m}$$

$$m \lessgtr m^* = \frac{R/\rho - 1}{R/\rho + 1}$$

The preferences shown by the above figure, however, are not immediately obvious. To further understand the regions above and below the equilibrium curve in the figure, we shall consider as an example, an owner and a contractor that have a ratio of risk tolerances $R/\rho = 1.35$. For this owner and this contractor, any cost-plus-fee contract with fee percentage m, is represented by a point (m,1.35) that lies on a horizontal line with ordinate $R/\rho = 1.35$ (see figure).

We shall first focus on the points (m,1.35) above and to the left of the equilibrium curve for which $m < m^* = 0.15$. Let us use as a representative example the point (0.05, 1.35) for which $R/\rho = 1.35$, and the cost-plus-fee percentage $m = 0.05$, is less than $m^* = 0.15$.

There are two ways to investigate the owner's and contractor's relative preferences for this point:

(a) Make the contractor indifferent between a cost-plus-5%-fee contract and a lump-sum contract, and then find which contract the owner would prefer.

(b) Make the owner indifferent between a cost-plus-5%-fee contract and a lump-sum contract, and then find which contract the contractor would prefer.

A. For the contractor to be indifferent between a cost-plus-5%-fee contract and a lump-sum contract, the lump-sum amount, L, is given by:

$$L = (1+m)E[C] + (1-m^2)\frac{Var[C]}{2\rho}$$

$$= 1.05E[C] + \frac{0.49875}{\rho}Var[C]$$

$$= 1.05E[C] + \frac{0.67478}{R}Var[C]$$

The analysis above indicates that the owner would prefer a cost-plus-5%-fee contract when the lump-sum amount, L, exceeds:

$$L > (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$$

$$L > 1.05E[C] + \frac{0.55125}{R} Var[C]$$

In this case, the above condition is true because:

$$L = 1.05E[C] + \frac{0.67478}{R} Var[C] \quad > \quad 1.05E[C] + \frac{0.55125}{R} Var[C]$$

From here we conclude that,

> The lump-sum amount L that makes the contractor indifferent to a cost-plus-5%-fee contract is too high for the owner. Thus, the owner prefers the cost-plus-5%-fee contract.

B. For the owner to be indifferent between a cost-plus-5%-fee contract and a lump-sum contract, the lump-sum amount L must be:

$$L = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$$

$$= 1.05E[C] + \frac{0.55125}{R} Var[C]$$

$$= 1.05E[C] + \frac{0.40745}{\rho} Var[C]$$

The contractor prefers a cost-plus-5%-fee contract when the lump-sum amount, L, is less than:

$$L < (1+m)E[C] + (1-m^2)\frac{Var[C]}{2\rho}$$

$$< 1.05E[C] + \frac{0.49875}{\rho} Var[C]$$

This is true in this case because.

$$L = 1.05E[C] + \frac{0.40745}{\rho} Var[C] \quad < \quad 1.05E[C] + \frac{0.49875}{\rho} Var[C]$$

From here we conclude that,

> The lump-sum amount L that makes the owner indifferent to a cost-plus-5%-fee contract is too low for the contractor. Thus, the contractor prefers a cost-plus-5%-fee contract.

Comparing cases (A) and (B) leads to the final conclusion:

> For the point $m = 0.05$, and $R/\rho = 1.35$, both the owner and the contractor would prefer a cost-plus-5%-fee contract rather than the lump-sum amount that would make the other party indifferent between a cost-plus-5%-fee contract and a lump-sum contract.

Exactly the opposite conclusions hold true for points (m,1.35) on the same horizontal line (i.e., for $R/\rho = 1.35$) for which $m > m^*=0.15$, i.e., to the right of the equilibrium curve.

As a specific example for $R/\rho = 1.35$, let us consider the point (0.25,1.35) for which the cost-plus-fee percentage $m = 25\%$ is greater than $m^*=0.15$.

Following an analysis similar to that above it is easy to show that for $R/\rho=1.35$ and $m = 25\%$:

- The lump-sum amount L that makes the contractor indifferent between a lump-sum contract and a cost-plus-25%-fee contract is low enough to make the owner prefer this lump-sum amount L to a cost-plus-25%-fee contract.
- Similarly, the lump-sum amount L that makes the owner indifferent between a lump-sum contract and a cost-plus-25%-fee contract is high enough to make the contractor prefer this lump-sum amount L to a cost-plus-25%-fee contract.

Thus, for the point $m = 0.25$, and $R/\rho = 1.35$, both the owner and the contractor would prefer "the lump-sum amount L that makes the other party indifferent between a cost-plus-25%-fee contract and a lump-sum contract", rather than a cost-plus-25%-fee contract.