Construction Contracting Lotteries

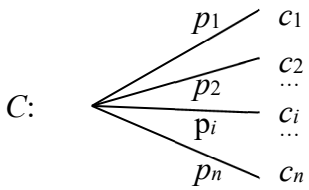
Consider a contractor who is about to sign a lump-sum contract to construct a project for a fixed amount of money L (a lump-sum). The actual cost of the project C is uncertain, and as a result, the contractor is about to acquire a gamble whose outcome is profit V given by:

$$V = L - C$$

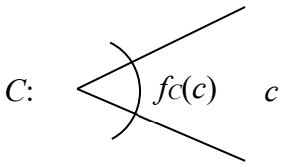
Before signing a lump-sum contract, the amount L is a decision variable whose value is determined either by competitive bidding analysis (a trade-off between the probability of winning and the profit to be made) or through negotiations with the owner. Thus, before signing a lump-sum contract, the contractor must analyze the project gamble as a profit lottery.

The contractor's uncertainty about project profit is entirely due to uncertainty in cost. There is no uncertainty about the lump-sum contract amount L that the owner will pay the contractor to perform the work.

A contractor estimates the probability distribution for project cost C based on the plans and specs and the contractor's overall experience and state of information. This probability distribution can be represented by a *cost* lottery C like the ones shown below.

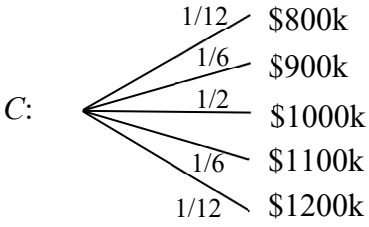


Project Cost, C, as a
Discrete Cost Lottery

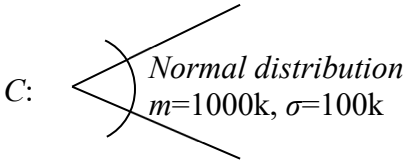


Project Cost, C, as a
Continuous Cost Lottery

The lottery on the left is a discrete cost lottery C with n branches, where each branch leads to cost value c_i with probability p_i. The lottery on the right is a continuous profit lottery C with an infinite number of branches, each leading to a cost value c with probability density $f_C(c)$. Typical examples of cost lotteries appear below:



Example: Project Cost C as a
Discrete Cost Lottery

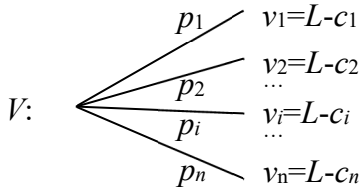


Example: Project Cost C as a
Continuous Cost Lottery

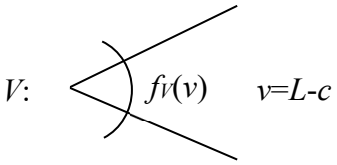
Notice that in the above lotteries, project costs are typically shown as positive numbers $c_i > 0$, even though they represent money that the contractor will have to pay. Thus, larger positive outcomes (high costs) are less desirable than smaller positive outcomes (low costs).

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The total gamble in a lump-sum contract can be represented by a *profit* lottery, like the ones shown below, with outcomes $V=L–C$, where V is profit, L is the lump sum price specified in the contract (a fixed value), and C is the uncertain cost of the work.



Lump-Sum Contract Profit V as a Discrete Profit Lottery



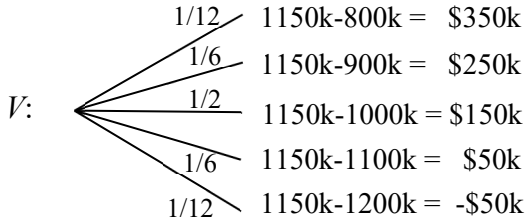
Lump-Sum Contract Profit V as a Continuous Profit Lottery

The lottery on the left is a discrete profit lottery V with n branches, where each branch leads to profit v_i with probability p_i. On the right is a continuous profit lottery V with an infinite number of branches, where each branch leads to profit $v=L–c$ with probability density $f_V(v)$.

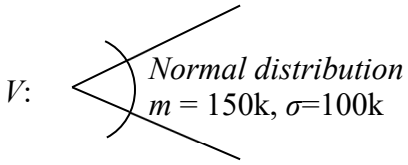
The probability distribution for each of the above profit lotteries V is the same as the probability distribution for the corresponding project cost lottery C when evaluated at $L–v$:

- $P[V = v_i] = P[L - C = v_i] = P[C = L - v_i = c_i]$
- $f_V(v) = f_C(L - v)$

Typical examples of profit lotteries for a lump-sum price L = \$1150k appear below:



Example: Project Cost C as a Discrete Cost Lottery



Example: Project Cost C as a Continuous Cost Lottery

Notice that profit V and cost C increase in opposite directions. Thus, the probability $P[V=v_{high}]$ that profit V takes on a *high* value v_{high} is the same as the probability $P[C=c_{low}=L–v_{high}]$ that the cost C will take on the corresponding *low* value $c_{low}=L–v_{high}$.

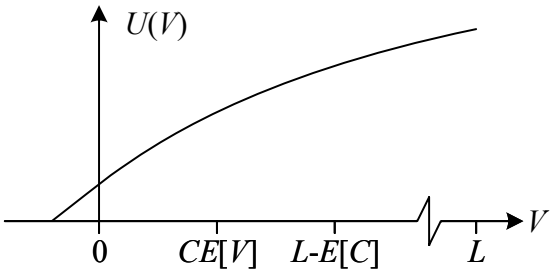
As a result, the 5% percentile for profit, v_5, corresponds to the 95% percentile for cost, c_{95}, and the 95% percentile for profit, v_{95}, corresponds to the 5% percentile for cost, c_5:

$$P[V \le v_5] = 5\% \quad \Rightarrow \quad P[L - C \le v_5] = P[C \ge L - v_5] = 5\% \quad \Rightarrow \quad P[C \le L - v_5 = c_{95}] = 95\%$$

$$P[V \le v_{95}] = 95\% \quad \Rightarrow \quad P[L - C \le v_{95}] = P[C \ge L - v_{95}] = 95\% \quad \Rightarrow \quad P[C \le L - v_{95} = c_5] = 5\%$$

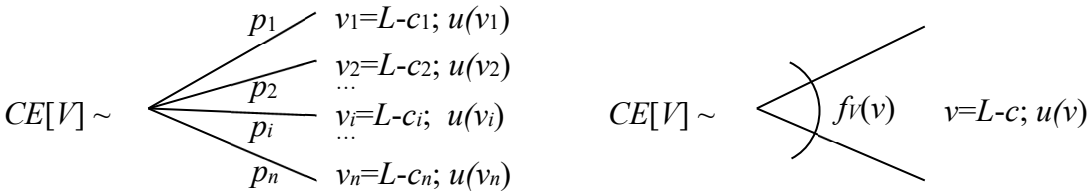
Contractor's Current Net Asset Position (Wealth) and Utility Curve for Money

Shown below is an example of a contractor's utility curve for profit V. The utility curve increases as profit increases indicating that the contractor prefers more money to less money. This particular utility curve is concave indicating that the contractor is risk-averse, as is typically the case.



The horizontal axis shows profit V and measures changes to the contractor's current wealth, i.e., monetary additions (profit), or subtractions (loss), relative to a zero point on the axis that represents the contractor's current net asset position V_0. In the following discussion, V_0 is the contractor's net asset position right before signing a lump-sum contract for a new project and corresponds to the zero point on the V axis. Thus, positive values of V represent project profit and negative values represent project loss for the lump-sum contract that the contractor is about to sign.

By signing a lump-sum contract, a contractor buys a new profit gamble (a lottery that the contractor did not have before) whose possible monetary outcomes, $V=L–C$, are additions (profit), or subtractions (loss) relative to that original net asset position V_0, and thus have utility $u(V)$, where V (profit or loss) is measured from the zero point on the V axis.



Lump-sum Contract as a Discrete Profit Lottery V

Lump-sum Contract as a Continuous Profit Lottery V

Shown on the left is a discrete profit lottery V for a new lump-sum contract where each branch leads with probability p_i to profit v_i whose utility is $u(v_i)$. On the right is a continuous profit lottery V for a new lump-sum contract where with probability density $f_V(v)$ each branch leads to profit v whose utility is $u(v)$.

Associated with each profit lottery is the expected utility of its profit outcomes $E[u(V)]$:

$$E[u(V)] = \sum p_i \cdot u(v_i) = \int u(v) f_V(v) dv$$

From utility theory we know that when faced with a choice a contractor prefers the lottery with the highest utility and that the utility of any lottery is the expected utility of its outcomes.

Also, the certain equivalent $CE[V]$ for a profit lottery V is that value of profit that has the same utility as the lottery, i.e., whose utility equals the expected utility of the lottery $E[u(V)]$:

$$u(CE[V]) = E[u(V)]$$

$$CE[V] = u^{-1}(E[u(V)])$$

For the profit lotteries shown in the above figures, the certain equivalent $CE[V]$ is shown to the left of each lottery. Because the utility of the certain equivalent $CE[V]$ and the expected utility of the lottery are the same, it follows that the certain equivalent $CE[V]$ is that special value of profit that would make the contractor indifferent (shown by the symbol ~) between the following two choices (a) and (b) that have equal utility:

(a) Receiving with certainty an amount of profit equal to $CE[V]$.

(b) Facing the profit lottery and receiving its uncertain (risky) profit (or loss) outcome V.

The certain equivalent for a project's profit lottery is central to a contractor's decision making. If a contractor can invest its resources in another venture with a higher certain equivalent (such as another project, a real estate development, the stock market, etc.) then the contractor would prefer that option than to sign a lump-sum contract for a fixed price L for the project under consideration.

In general, the certain equivalent $CE[V]$ for a profit lottery V is a function of the lump-sum contract amount L and the project cost lottery C.

$$CE[V] = CE[L - C]$$

However, immediately before signing a lump-sum contract for the project under consideration, the contractor's specific state of information (and uncertainty) about the project's cost remains constant (no new information is made available). Thus, given this particular state of information about the probability distribution of the project cost, the certain equivalent $CE[V]$ is only a function of the decision variable L.

It is important to point out that, in general, the certain equivalent $CE[V]$ is not a linear function of the lump-sum price L because it depends on the contractor's nonlinear utility curve. This means that if the contractor decides to increase by \$100 what he demands as a lump-sum amount L, the certain equivalent for profit $CE[V]$ will not increase by \$100.

Only the exponential utility curve (and its special case, the linear utility curve) have the following *delta*-property (that is discussed in detail in later sections):

$$CE[V] = CE[L - C] = L - CE[C]$$

Thus, when a contractor with an exponential (or linear) utility curve (sometimes called a "delta-contractor") increases the lump-sum amount L by Δ, i.e., to $L+\Delta$, then the certain equivalent for the project's profit, $CE[V]$, will also increase by Δ, to $CE[V]+\Delta$:

$$CE[V + \Delta] = CE[L + \Delta - C] = L + \Delta - CE[C] = CE[V] + \Delta$$

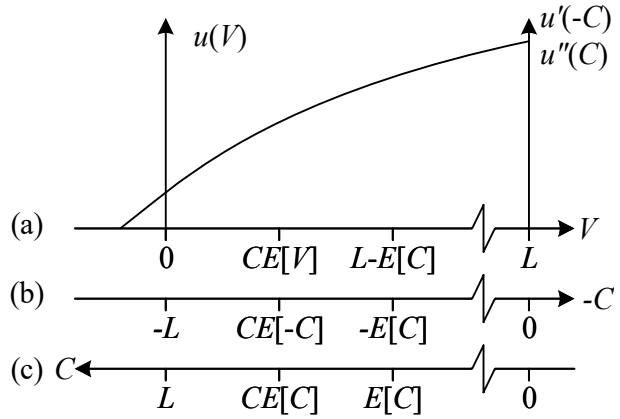
This means that only a "delta-contractor" can evaluate project-related lotteries irrespective of his initial net asset position V_0 and without having to know the lump-sum fixed price L.

Utility Functions for Profit and Cost Lotteries — Lump-Sum Contract for a fixed price *L*

After a lump-sum contract for a project is signed, the price *L* for doing the work is a fixed constant. Thus, gambles relating to this particular contract can now be analyzed using either a profit lottery or a cost lottery as described below. The choice depends on the contractor's preferred value measure for his utility curve: project profit or project cost (where costs can be shown using either negative or positive amounts). These value measures for the contractor's utility curve lead to the following three equivalent models (a) (b) (c) for project lotteries:

(a) A profit lottery where the value measure is profit $V=L-C$. In this case, the calculation of each possible profit lottery outcome v_i requires knowledge and the repeated use of the fixed contract price *L* along with a corresponding cost outcome c_i.
(b) A profit lottery where the value measure is the project's algebraic cost $-C$, which takes on negative values. In this case, the zero point on the monetary axis of the utility curve for the project is pre-shifted to the right by *L*, and the person calculating each possible profit lottery outcome $-c_i$ does not need to know the fixed contract price *L*.
(c) A cost lottery where the value measure is cost *C* that takes on positive values. In this case, the monetary axis of the project utility curve is pre-shifted to the right by *L*, and then its direction is flipped, so that cost *C* increases to the left. Thus, the person calculating the possible cost outcomes c_i does not need to know the fixed contract price *L*. Moreover, it is easier to use than (b) because all cost outcomes c_i are positive numbers.

Clearly, the contractor's attitude towards risk and his *underlying utility curve* do not change depending on which of the above three equivalent models for lotteries and value measures is used. Nevertheless, each of the above value measures must use its own utility function to describe the same utility curve as shown by the three monetary value axes (a), (b) and (c) below:



The value measure for each the above three alternative project lotteries corresponds to one of the three horizontal axes (a), (b) and (c) in the above figure. Notice that, as shown in the figure, there is only one utility curve. However, because each of the three value measures uses a different horizontal axis, this single utility curve must be represented by three utility functions, (a) $u(V)$, (b) $u'(-C)$ and (c) $u''(C)$, depending on the corresponding value measure.

For the three lottery alternatives (a), (b) and (c), to be equivalent, their results must be identical in every way: Corresponding lottery outcomes must have the same utility, lotteries must produce the same expected utilities (and thus must have *corresponding* certain equivalents), and choices between lotteries must lead to the same decisions.

Thus, for the three lottery models to be equivalent, the following utilities must be equal:

$$u(V) = u(L-C) = u'(-C) = u'(V-L) = u''(C) = u''(L-V)$$

For this to be true, it is necessary to move the "zero point" for the horizontal *x*-axes for cases (b) and (c) to the right by an amount *L* relative to the zero point for axis (a). Thus, the utility of zero cost on axes (b) and (c), i.e., $u'(0)$ and $u''(0)$, equals the utility of the lump-sum amount *L* on axis (a), i.e., $u(L)$.

$$u(L) = u'(0) = u''(0)$$

The following are specific examples of monetary values that have equal utilities:

$$u(L) = u'(0) = u''(0)$$
$$u(0) = u'(-L) = u''(L)$$
$$u(L-E[C]) = u'(-E[C]) = u''(E[C])$$
$$u(CE[V]) = u'(CE[V]-L) = u'(CE[-C]) = u''(L-CE[V]) = u''(CE[C])$$

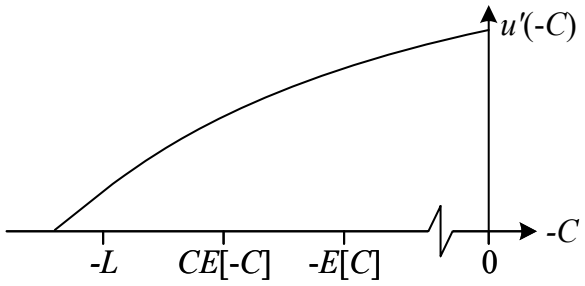
Why Would a Contractor Use Cost Lotteries?

A possible reason a contractor might prefer to model project-related lotteries as cost lotteries (using either alternative (b) or (c), above) is to have project-level management personnel focus on minimizing project cost (which they control) rather than on maximizing profit.

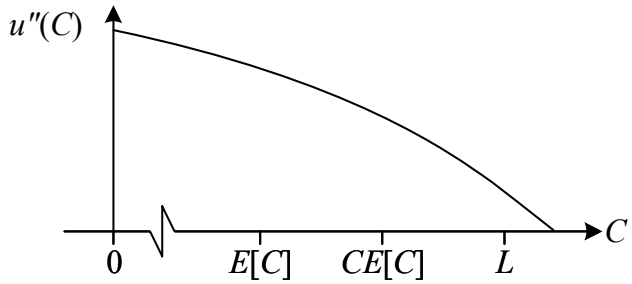
Another reason would be to keep the lump-sum amount *L* hidden and confidential. This is not possible when using profit lotteries as in (a) above, because the calculation of every profit outcome must reference the lump sum amount *L*. Alternatives (b) and (c) can easily hide the contract price *L* from view, simply by shifting the zero point for money on the utility curve.

Utility Functions for Project Cost Lotteries — Lump-Sum Contract for a fixed price, *L*

A contractor, for example, could supply the following utility curve, $u'(-C)$, to a project manager (without including the monetary values shown on the cost axis) and ask that it be used for all decisions on a specific project. The value measure for this version of the utility curve is algebraic cost $-C$, which means that project costs (money paid out) are negative and thus appear to the left of the zero point. The utility curve $u'(-C)$ is identical to the utility curve $u(V)$ for profit, except that the zero point for the horizontal value axis has already been shifted to the right by the lump-sum amount *L*. Thus, the lump-sum contract price *L* does not have to be revealed and does not need to be used to construct project-related lottery outcomes $-c_i$.



Alternatively, the contractor could supply the project manager with the following utility curve $u''(C)$ where the value measure shows project costs C using positive numbers (again, without including the example labels shown on the cost axis). The utility curve $u''(C)$ is the same as $u'(-C)$ shown above, , except that it has been flipped horizontally so that utility decreases as project costs increase to the right. The fact that this utility curve u''(C) decreases as money takes on larger positive values, makes it obvious that positive monetary values represent project costs.



A convenient way to think about the above two utility curves for cost, u'(-C) and u''(C), is to consider that upon signing the lump-sum agreement for this project, the contractor's net asset position V_0 immediately increases by the lump-sum amount L to V_0+L. Thus, the zero point from which <u>further changes</u> to the contractor's <u>new</u> net asset position V_0+L will be measured (i.e., subtractions due to project costs), must be shifted to the right by L to reflect the contractor's new (higher) net asset position V_0+L. Lotteries concerning the project can now include only project costs (i.e., money paid out) because all revenue (money coming in) has already been accounted for by shifting the zero point for money to the right by L.

The following are important summary points from the above discussion:

- The utility of the outcomes of project-related lotteries <u>must</u> be measured based on <u>net changes to the contractor's net asset position</u> V_0. Such changes are measured by <u>profit</u> (or loss) and this requires that possible <u>cost</u> outcomes must be subtracted from the lump-sum amount L. Thus, the lump-sum amount L must be considered either explicitly as in model (a), or implicitly as in models (b) or (c).
- There are two ways in which the analysis of project-related lotteries (gambles) may include the value of the lump-sum, L:
  - Directly and explicitly, as in $u(V)$, i.e., model (a), or
  - Indirectly and implicitly, as in $u'(-C)$ and $u''(C)$, i.e., models (b) or (c), by shifting the zero point of the monetary value axis to the right by L.
- A contractor cannot produce the above utility functions for cost, $u'(-C)$ and u''(C), unless the lump-sum amount L is fixed and known.
- Thus, even though all project uncertainty may be due to project cost, it is not possible to analyze project-related lotteries (gambles) based on cost <u>alone</u>, when cost is measured relative to the <u>original asset position</u> V_0. To analyze project-related lotteries (gambles) based on cost, the utility of cost $u'(-C)$ and u''(C) must be measured relative to a new asset position V_0+L, which requires that the value for the lump-sum price L be known.

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Relationships between Certain Equivalents for Profit and Cost Lotteries

As stated above, the three lottery alternatives (a), (b) and (c), that use utility functions and corresponding value measures (a) $u(V)$, (b) $u'(-C)$ and (c) $u''(C)$, are equivalent and their results are identical in every way: Corresponding lottery outcomes have the same utility, lotteries produce exactly the same expected utilities, lotteries have *corresponding* certain equivalents, and choices between lotteries lead to the same decisions.

Here we shall examine what it means for lotteries to have *corresponding* certain equivalents.

The utility of the certain equivalent of a lottery equals the expected utility of the lottery. Moreover, the expected utility of a lottery is the same, irrespective of whether the lottery is structured to use utility function (a) $u(V)$, or (b) $u'(-C)$ or (c) $u''(C)$. Thus, for the same underlying lottery, the following utilities in the left column are equal because they equal the expected the utility of the lottery. As a result, the following certain equivalent relationships in the right column must also be true:

$$\text{(a)} \quad u(CE[V]) = u(CE[L-C]) \quad \Rightarrow \quad CE[V] = CE[L-C]$$

$$\text{(b)} \quad u'(CE[-C]) = u'(CE[V]-L) \quad \Rightarrow \quad CE[-C] = CE[V]-L$$

$$\text{(c)} \quad u''(CE[C]) = u''(L-CE[V]) \quad \Rightarrow \quad CE[C] = L-CE[V]$$

The following numerical example makes the above relationships easier to understand.

Suppose that the lump-sum contract amount is L= \$1,000k and that the certain equivalent for the profit lottery V is $CE[V]$= \$100k. Then, the corresponding certain equivalents when using utility functions (a) $u(V)$, (b) $u'(-C)$ or (c) $u''(C)$ would be as shown below:

$$\text{(a)} \quad CE[V] = CE[L-C] = \$100k$$

$$\text{(b)} \quad CE[-C] = CE[V]-L = \$100k - 1{,}000k = -\$900k$$

$$\text{(c)} \quad CE[C] = L-CE[V] = \$1{,}000k - 100k = \$900k$$

The corresponding certain equivalents, (a) \$100k, (b) –\$900k, and (c) \$900k, do not have the same numerical values because they are measured relative to three different *value measure* scales (a), (b) and (c). Nevertheless, they have the same utility because when these three certain equivalents are added to the corresponding initial net asset position correctly, they all lead to the same *final net asset position,* $V_1 = V_0$ +\$100k, and thus to the same utility, $u(V_1) = u(V_0$ +\$100k):

$$\text{(a)} \quad V_1 = V_0 + CE[V] = V_0 + \$100k$$

$$\text{(b)} \quad V_1 = (V_0 + L) + CE[-C] = V_0 + L + CE[V] - L = V_0 + CE[V] = V_0 + \$100k$$

$$\text{(c)} \quad V_1 = (V_0 + L) - CE[C] = V_0 + L - (L - CE[V]) = V_0 + CE[V] = V_0 + \$100k$$

Notice that the certain equivalent $CE[V]$ for profit lottery (a) represents money added to the initial asset position V_0. In contrast, the certain equivalents $CE[-C]$ and $CE[C]$ are added to or subtracted from a different initial asset position ($V_0 + L$). The initial asset position, V_0, corresponds to the "zero" point for the profit scale (a), whereas the initial asset position ($V_0 + L$) corresponds to the "zero" point of the cost scales (b) and (c).

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Interpretation of the Certain Equivalent for a Cost Lottery (given a Lump-Sum Price, *L*)

The certain equivalent, $CE[C]$, of a cost lottery is the maximum amount of money that a contractor who has signed a lump-sum contract for a price, L, would be willing to pay to have someone else (e.g., a subcontractor) perform all the work, so that the contractor would avoid facing the risk involved in the project's cost lottery.

Notice that for the contractor, the certain equivalent for cost $CE[C]$ is actually a *PISP*, a "personal indifferent selling price", for the project's cost lottery.

- $CE[C]$ is a selling price because after a contract has been signed to perform the work for a fixed price L, the contractor owns the project's cost lottery and is about to transfer ownership of that cost lottery (and the obligation to perform the work for the project) to someone else (e.g., a subcontractor). Clearly, the max selling price $CE[C]$ depends on L.
- For a contractor wishing to transfer (i.e., "sell") this cost lottery to someone else, the selling price is "negative" in the sense that the seller (the contractor) does not receive money but has to pay money to the buyer (the subcontractor) who will take over the cost lottery and assume responsibility for the work.
- A subcontractor who takes over all or part of the work for a project is buying a cost lottery for a negative buying price (he will be paid for doing the work). By signing a subcontract for a fixed amount S, the subcontractor buys a cost lottery he did not have before. Thus, the subcontract amount S is the subcontractor's *negative buying price* because it represents *money that he will receive*. Clearly, a subcontractor hopes that the actual cost outcome C for the cost lottery he will buy, will be less than the subcontract amount S that he will receive, and that he will make a profit $S–C$.

Expected Cost vs. Certain Equivalent of Cost Lotteries

When a contractor models project cost as $-C$ using alternative (b), i.e., as a *profit* lottery, the expected profit $E[-C]$ is *negative* (undesirable). A risk-averse contractor would be willing to pay more than the $E[-C]$ to avoid the risk of the lottery, Hence, for a *risk-averse* person, the certain equivalent $CE[-C]$ for *profit* lottery $-C$ is even "more negative" than $E[-C]$:

$$CE[-C] < E[-C] < 0$$

Similarly, when a contractor models project cost C using alternative (c), i.e., as a *cost* lottery, the expected cost $E[C]$ is *positive* (undesirable). A risk averse contractor would be willing to pay more than the $E[C]$ to avoid the risk of the lottery. Thus, a *risk-averse* person would have a certain equivalent $CE[C]$ for cost lottery C that is even "more positive" than $E[C]$:

$$0 < E[C] < CE[C]$$

Clearly, the expected costs and the certain equivalents for the two lotteries (b) and (c) have the same absolute values and differ only in their algebraic sign: (b) uses negative numbers while (c) uses positive numbers.

$$E[C] = -E[-C]$$

$$CE[C] = -CE[-C]$$

Definition of Risk Premium for Cost Lotteries

The "risk premium" for a *profit* lottery and the "risk premium" for a *cost* lottery are defined differently so that both are *positive* for a risk-averse person. A *positive* "risk premium" represents the amount of money that a *risk-averse* contractor would be willing *to give up* to avoid the risk of a lottery that he owns by transferring that risk to someone else.

- For a profit lottery, a positive risk premium indicates that a risk-averse contractor would be willing to accept a selling price (the certain equivalent), that is less than the expected profit of the work, to avoid the risk of the lottery.
- For a cost lottery, a positive risk premium represents the amount of money that a risk-averse contractor would be willing to pay above the expected cost of the work to avoid the risk of the project's cost.

The standard definition for the risk premium of a *profit* lottery V is given by:

$$RP[V] = E[V] - CE[V] = L - E[C] - CE[V]$$

Alternative (b) models cost -C as a *profit lottery*, and thus its risk premium is:

$$RP[-C] = E[-C] - CE[-C] = -E[C] - (CE[V] - L) = L - E[C] - CE[V] = RP[V]$$

The risk premium for cost C modeled by *cost* lottery (c) which follows the opposite sign convention for *money received* and *money paid out,* follows the opposite sign convention:

$$RP[C] = CE[C] - E[C] = (L - CE[V]) - E[C] = RP[V]$$

The risk premiums (a) $RP[V]$, (b) $RP[-C]$, (c) $RP[C]$, for all three lottery models are equal:

$$RP[V] = RP[-C] = RP[C]$$

Positive risk premiums, $RP[V]$, $RP[-C]$ and $RP[C]$, indicate *risk-averse* behavior.

$$\text{Risk-averse behavior} \quad \Leftrightarrow \quad RP[V] = RP[C] = RP[-C] > 0$$

The risk premium RP[-C] for *profit* lottery -C is subtracted from expected cost E[-C] to obtain the certain equivalent CE[-C]. The risk premium $RP[C]$ for a *cost* lottery C, however, is *added* to the expected cost $E[C]$ to obtain the certain equivalent $CE[C]$. Thus, the certain equivalents CE[.] shown below have the algebraic sign for RP[.] on the right-hand-side reversed:

(b) For profit lottery, $-C$, $\quad CE[-C] = E[-C] - RP[-C] \quad < 0$

(c) For cost lottery, C, $\quad CE[C] = E[C] + RP[C] \quad > 0$

These expressions are consistent with each other because if we multiply both sides of the first equation by –1 we obtain the second (because $RP[-C] = RP[C]$):

$$CE[-C] = E[-C] - RP[-C]$$
$$-CE[-C] = -E[-C] + RP[-C]$$
$$CE[C] = E[C] + RP[C]$$

In the discussion that follows, it is important to remember that a *positive* risk premium for both *profit* and *cost* lotteries represents money that a risk averse contractor *gives up* to *avoid risk*.

PIBP – Buying Price for Project Cost Lottery – Risk-Adjusted Cost (RAC)

An important concept in decision analysis is that of PIBP, the *personal indifferent buying price*. A buying price arises when a person buys something that he does not already own. The term "indifferent" means that PIBP is that special buying price at which a person is truly indifferent between (a) buying that thing in exchange for an amount money equal to the PIBP, and (b) the status quo where there is zero change in the person's assets and wealth.

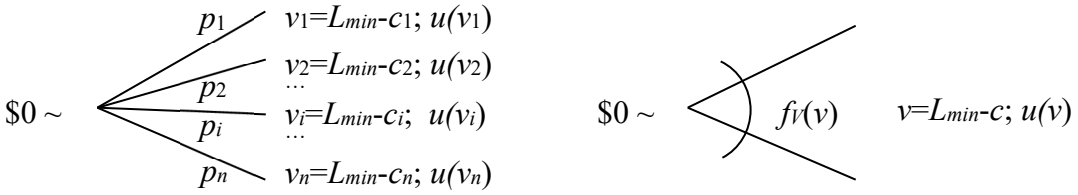
When a contractor signs a lump-sum contract to perform the work for a project for a fixed price L, he is essentially buying the cost lottery for that project, something that he did not have before. The buying price for the project cost gamble is the fixed price L. This is a "negative" buying price in the sense that the contractor does not pay L to acquire the cost lottery for the project, but instead must be paid L to take ownership of this undesirable gamble.

The PIBP is the maximum amount of money that a decision maker would be willing to pay to buy a desirable gamble that he does not already own. For a cost lottery, the PIBP is the minimum contract price L_{min} that the contractor would accept to take on the responsibility for the project cost lottery that he did not have to face before.

The indifference statement by which the PIBP = L_{min}, is computed is the following:

$$CE[V \mid L_{\min}] = CE[L_{\min} - C] = 0$$

Hence, the PIBP = L_{min} is that special lump-sum price that makes the contractor indifferent between signing a lump-sum contract for a price L_{min} and \$0 (no contract, no change in wealth):



PIBP = L_{min} for a Discrete Profit Lottery V

PIBP = L_{min} for a Continuous Profit Lottery V

Another term for PIBP = L_{min}, is "*Risk-Adjusted Cost*" or RAC. This term reflects the fact the contractor is indifferent between contracting to do the work at RAC or not at all. Doing the work for a price equal to RAC is the conceptual equivalent to "performing the work at cost"—there is no profit in it. Given the project's risk and the contractor's risk aversion, the contractor does not perceive any profit from doing the work at a lump-sum price L equal to the RAC.

As described earlier, the certain equivalent for cost $CE[C]$, is a function of the lump-sum price L and is back-calculated from the certain equivalent for profit $CE[V]$ as follows:

$$CE[C \mid L_{\min}] = L_{\min} - CE[V \mid L_{\min}] = L_{\min}$$

This means that the lump-sum price, L_{min} = PIBP = RAC, also equals the corresponding certain equivalent for cost $CE[C \mid L_{min}] = L_{min}$. Thus, a contractor with a contract for L_{min}, would be willing to make no profit and pay the entire price $CE[C \mid L_{min}] = L_{min}$ to a subcontractor to perform all the work.

Numerical Examples — Lump-Sum Contract for a fixed price *L*

This section uses numerical examples to illustrate some of the important points made above:

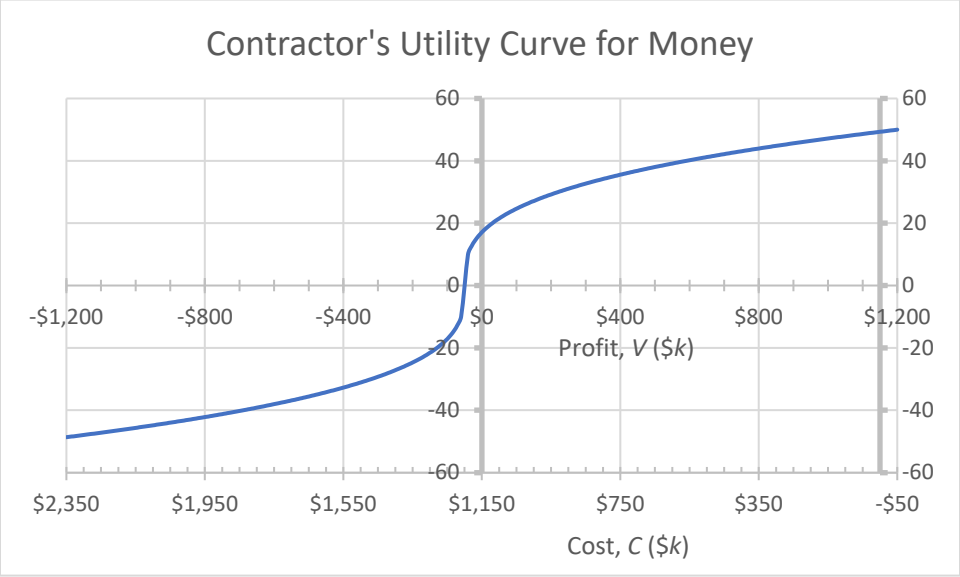
- Decisions about a project must be based on profit lotteries that measure profit added to, or loss subtracted from, the contractor's current net asset position (or wealth) V_0.
- For a lump-sum contract, it is not possible to construct an appropriate project profit lottery (or cost lottery) without knowing and taking into account the lump-sum fixed price L either directly or indirectly.
- A cost lottery that treats costs as money subtracted from the contractor's current asset position (or wealth) V_0 is not appropriate for making decisions because the resulting utilities for cost outcomes come from the wrong region of the utility curve. Such a cost lottery is equivalent to a profit lottery for a lump-sum contract with a fixed price L = \$0 — clearly an inappropriate model for any situation.

Consider a contractor with the following cubic-root utility curve for profit V (in \$*k*):

$$u(V) = \sqrt[3]{a + bV} = \sqrt[3]{5000 + 100V} \qquad \text{(profit } V \text{ is in \$}k\text{)}$$

The inverse of the utility function returns the profit $V(u)$ (in \$*k*) given its utility u:

$$V(u) = u^{-1}(u) = (u^3 - a)/b = (u^3 - 5000)/100 \quad \text{(profit } V \text{ is in \$}k\text{)}$$



This cubic-root utility curve was selected deliberately for the examples that follow because its properties make it easier to illustrate certain points:

(a) The cubic-root function is simple and can easily be inverted to find certain equivalents.

(b) It gives the utility for both positive and negative values of money (i.e., additions and subtractions relative to the current asset position V_0.

(c) It exhibits different attitudes towards risk for positive and negative values of money:

- For positive values of money, the utility curve is *concave* and thus exhibits *risk-averse* behavior.

- For negative values of money, the utility curve is *convex* and thus exhibits *gambling* (risk-seeking) behavior.
- This change in risk attitude makes the points below easier to appreciate.

Example Lump-Sum Contract — Correct Analysis based on Profit

Suppose that the contractor is about to sign a lump-sum contract for a fixed price L. Project cost C follows a distribution with the five outcomes c_i shown below (note that $E[C] = \$1{,}000k$ and $SD[C] = \$100k$). The profit lottery V also has five outcomes v_i as shown in the table.

Profit Lottery (Correct analysis based on profit for L =\$1,150k)			
Prob. $P[C=c_i] = P[V=v_i]$	Cost, C c_i (\$$k$)	Profit, $V=L-C$ $v_i = L - c_i$ (\$$k$)	Utility $u(v_i) = u(L-c_i) = u''(c_i)$
1/12	800	350	34.1995
1/6	900	250	31.0723
1/2	1,000	150	27.1442
1/6	1,100	50	21.5443
1/12	1,200	(50)	0.0000
$E[.]$=	$E[C] = \$1{,}000k$	$E[V] = \$150.000k$	$E[u(V)] = 25.1915$
$CE[.]$=	$CE[C] = \$1{,}040.132k$	$CE[V] = \$109.868k$	
$RP[.]$=	$RP[C] = \$40.132k$	$RP[V] = \$40.132k$	

It is important to notice that for the lump-sum fixed price L =\$1,150k used in this example, all profit monetary outcomes are greater than -\$50k, and thus have positive utility values, which come from the region of the cubic-root utility curve that exhibits risk-averse behavior.

The expected utility and the certain equivalent of the profit lottery (shown above) are:

$$E[u(V)] = \sum_{i=1}^{5} P[V = v_i]u(v_i) = \sum_{i=1}^{5} P[C = L - v_i] \cdot \sqrt[3]{5000 + 100v_i} = 25.1915$$

$$CE[V] = V(E[u(V)]) = u^{-1}(E[u(V)]) = \{E[u(V)]^3 - 5000\}/100 = \$109.868k$$

- The certain equivalent for profit, $CE[V] = \$109.868k$, is the minimum amount to be received with certainty for which the contractor would be willing to sell the entire (risky) contract, including the lump-sum amount L, to another contractor.
- The expected profit of this risky contract, $E[V]$=\$150k, is greater that $CE[V]$=\$109.868k. The difference is by definition the risk premium, $RP[V] = \$40.132k$, i.e., the amount of expected profit that the contractor would be willing to give up to avoid risk:

$$RP[V] = E[V] - CE[V] = \$150k - \$109.868k = \$40.132k$$

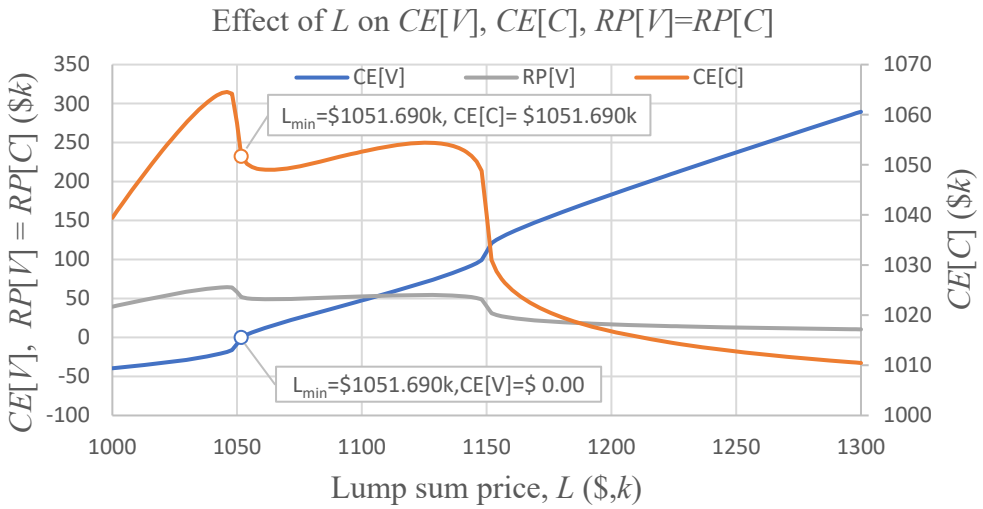
- The certain equivalent for cost $CE[C]$, and the risk premium for cost $RP[C]$, for the example lump-sum price of L =\$1,150k are back-calculated from $CE[V]$:

$$CE[C] = L - CE[V] = \$1{,}150k - \$109.868k = \$1{,}040.132k$$
$$RP[C] = CE[C] - E[C] = \$1{,}040.132k - \$1{,}000k = \$40.132k = RP[V]$$

- Both the certain equivalent for cost $CE[C]$, and the risk premium for cost $RP[C]$, depend on and cannot be computed without knowing and taking into account the lump-sum price L =\$1,150k and the certain equivalent for profit $CE[V]$.
- The risk premium for cost $RP[C]$ and the risk premium for profit $RP[V]$ for a lump-sum contract are equal and both depend on the lump-sum price L =\$1,150k.

Impact of the Lump-Sum Price *L*

The following graph illustrates the impact of the lump-sum price L on the certain equivalents for profit $CE[V]$, the certain equivalent for cost $CE[C] = L - CE[V]$, and the risk premiums for profit and cost $RP[V] = RP[C]$.



The RAC is L_{min}=\$1051.690, i.e., the contract price for which $CE[V \mid L_{min}] = \$0$. As expected, at L_{min}, the certain equivalent for cost has the same value, $CE[C \mid L_{min}] = L_{min} = \$1051.690$.

The certain equivalent for profit $CE[V]$ is a monotonically increasing function of L, whereas the certain equivalent for cost $CE[C] = L - CE[V]$ increases and decreases unexpectedly.

All three curves are quite sensitive to the lump-sum price L in the neighborhoods of L=\$1,050k and L=\$1,150k. This sensitivity is caused by the significant curvature of the example cubic-root utility curve in the neighborhoods of V = –\$100k (very risk-seeking behavior) and V=\$0k (very risk-averse behavior). When the lump-sum price is close to L=\$1,050k and L=\$1,150k, the corresponding profit lottery outcomes are in the neighborhoods of V=-\$100k and V=\$0k and take their utility values from these two regions of the utility curve that have very large curvatures (the region on the left is very risk-seeking and the region on the right is very risk-averse). This is the cause for the observed curve sensitivity in the above graphs.

The graphs for the risk premiums $RP[V]$=$RP[C]$ are identical. They are also identical in shape to the graph for the certain equivalent for cost $CE[C]$ but offset by \$1,000.

$$RP[V] = RP[C] = CE[C] - E[C] = CE[C] - \$1{,}000k$$

Incorrect Analysis — Cost Lottery Ignores Lump-Sum Price *L*

As mentioned earlier, a contractor must evaluate the monetary outcomes of a lump-sum contract as a profit lottery that requires the explicit or implicit consideration of the lump-sum contract price *L*. The only exception is when the "delta property" applies, i.e., when the contractor's utility curve is exponential (or linear). This special delta-property of the exponential utility curve will be illustrated using several examples in later sections.

In this section we shall illustrate the incorrect results obtained when a contractor (with a utility curve other that the linear or exponential) evaluates the monetary outcomes of a project as a cost lottery, without considering the lump-sum contract price *L*.

Suppose that the above contractor with a cubic-root utility curve evaluates the monetary outcomes of the example project by considering only the cost of the project *C* which follows a discrete distribution with the same five outcomes c_i.

Cost Lottery
(Incorrect analysis — Does not consider the lump-sum price L)

Prob. $P[C=c_i]$	Cost, C c_i (\$*k*)	Utility $u(c_i)=\sqrt[3]{5000+100(-c_i)}$
1/12	800	-42.1716
1/6	900	-43.9683
1/2	1,000	-45.6290
1/6	1,100	-47.1769
1/12	1,200	-48.6294
	$E[C]$ = \$1,000*k*	$E[u(C)]$ = -45.5721
	$CE[C]$ = \$996.451*k*	
	$RP[C]$ = -\$3.546*k*	

Notice that when the lump-sum price *L* is ignored, all cost monetary outcomes c_i are negative profit outcomes $-c_i$ that reduce the contractor's current net asset position V_0 and thus lie to the left of the zero point of the monetary axis of the cubic-root utility curve. As a result, these negative monetary values have the negative utility values shown the above table and correspond to the region of the utility curve that exhibits risk-seeking (gambling) behavior.

The expected utility and the certain equivalent of the improper cost lottery shown above are:

$$E[u(C)]=\sum_{i=1}^{5}P[C=c_i]u(c_i)=\sum_{i=1}^{5}P[C=c_i]\sqrt[3]{5000+100(-c_i)}=-45.5721$$

$$CE[C]=u^{-1}(E[u(C)])=-((-45.5721)^3-5000)/100=\$996.451k$$

The figure in the previous section shows a graph of the certain equivalent for cost $CE[C]$ as a function of the lump-sum price *L* that illustrates clearly that when the analysis is done correctly (i.e., by taking into account the lump-sum price, *L*) the certain equivalent for cost $CE[C]$ is always greater than the expected cost $E[C]$=\$1,000 for lump-sum prices, \$1000 < *L* < \$1300.

The certain equivalent $CE[C]$ = \$996.451 given by the cost lottery, is less than the expected cost $E[C]$=\$1,000 and would occur only when L=\$0, i.e., when the contractor would do the work for free (which is not realistic).

Moreover, the incorrect analysis above implies that the contractor would be not be willing to pay more than $CE[C] = \$996.451k$ to have another contractor perform all the work for the project. The fact that the certain equivalent for cost $CE[C]$ = \$996.451 is less than the expected cost of the entire contract $E[C]$ = \$1,000k, indicates gambling (risk-seeking) behavior. This gambling behavior is also indicated by the fact that the risk premium for the cost lottery is negative. A gambling (risk-seeking) contractor likes risk and thus would want to pay with certainty more than the expected cost to be willing to part from the risky cost lottery:

$$RP[C] = -(E[C] - CE[C]) = -\$1000k + \$996.451k = -\$3.546k$$

Another important point is that adding the lump-sum price L=\$1,150k to the incorrect certain equivalent for cost $CE[C]$= \$996.451, after the analysis is done incorrectly using the above cost lottery, does not give the correct certain equivalent for profit $CE[V]$= \$109.868:

$$CE[V] = \$109.868k \neq L - CE[C] = \$1{,}150k - \$996.451k = \$153.546k$$

Notice that because the cost outcomes of the incorrect cost lottery take their utility values from the risk-seeking region of the cubic-root utility curve, the resulting incorrect certain equivalent for profit calculated above, \$153.546k, exhibits risk-seeking behavior, too, because it exceeds the expected profit of \$150k. This would mean that the contractor would demand \$3.546k above the expected profit to sell a risky lump-sum contract to another contractor. Clearly, this behavior is very different from the correct risk-averse behavior shown earlier when using the correct profit lottery for this project.

Compound (Sequential) Cost Lotteries

Consider a risk-averse contractor who is under contract to perform the work for a project whose total cost C is described by a discrete or continuous *cost* lottery.

Suppose, further, that the project can be subdivided into two parts whose costs are represented by the two cost lotteries (random variables) W and Z such that:

$$C = W + Z$$

In this case, the random variable C can be viewed as a *compound* cost lottery that consists of two *sequential* lotteries where the cost lottery W is followed by the cost lottery Z (or *vice-versa*). Notice that, in general, the probabilities for the branches of lottery Z may depend on the outcomes of lottery W and *vice-versa*:

$$P[Z = z \mid W = w] \neq P[Z = z]$$

$$P[W = w \mid Z = z] \neq P[W = w]$$

Compound Cost Lotteries Cannot be Decoupled Unless Special Conditions Exist

In general, when the decision maker is risk-averse, the certain equivalent $CE[C]$ for the total project cost C does not equal the sum of the individual certain equivalents $CE[W]$ and $CE[Z]$:

$$CE[C] \neq CE[W] + CE[Z]$$

Decoupling two sequential lotteries so that each lottery may be evaluated independently of the other is convenient, but can only be done under the following special conditions:

- The decision maker has an exponential utility curve (he is a *delta*-person), and
- The two lotteries (random variables) W and Z are independent.

- Only under the above two special conditions do the following hold true:

$$CE[C] = CE[W] + CE[Z]$$

$$RP[C] = RP[W] + RP[Z]$$

Notice that both conditions are required. The above equalities are *not* true, even for a *delta*-person, when the component costs W and Z are *not* independent. Clearly, they are *not* true when the decision maker is *not* a delta-person, even if the component costs W and Z are independent. I.e., they are not true for any *risk-sensitive* utility curve other than the exponential.

These observations have direct implications for the pricing of risk in construction contracting. For example, a general contractor may wish to subcontract the work whose cost is represented by Z. In this case, it may appear that the contractor would be indifferent between (a) paying with certainty an amount $S=CE[Z]$ to the subcontractor and (b) facing the uncertain cost of the lottery Z. However, this is not true when the contractor is not a *delta*-person because the contractor's attitude towards risk would depend on his asset position and hence on the magnitude of the overall project, i.e., on L, C, S, and W. Moreover, even when the contractor is a *delta*-person, there may exist correlation between the cost Z of the work to be subcontracted and the cost o the remaining work W. As shown below, positive correlation increases risk, which increases the maximum amount S^* that a contractor would be willing to pay a subcontractor.

Subcontracting Part of the Work for a Lump-Sum Contract

Consider a *risk-sensitive* contractor who has signed a lump-sum contract for price L for a project whose total cost is described by cost lottery C.

The contractor is thus faced with profit lottery $V = L–C$ as shown below:

Contractor performs all work $C = W+Z$ (does not subcontract)	$V = L - C$ $E[V] = L - E[C]$ $CE[V] = u^{-1}(E[u(V)]) \quad \Rightarrow \quad CE[C] = L - CE[V]$ $RP[V] = E[V] - CE[V] = L - E[C] - CE[V] = CE[C] - E[C] = RP[C]$

Suppose that the project cost C can be broken down into the two cost lotteries W and Z:

$$C = W + Z$$

The contractor will perform the work with cost W and pay an amount S to a subcontractor to perform the work with cost Z. The contractor's resulting profit lottery V' is as follows:

Contractor performs work W and subcontracts work Z for price S.	$V' = L - S - W$ $E[V'] = L - S - E[W]$ $CE[V'] = u^{-1}(E[u(V')]) \quad \Rightarrow \quad CE[W] = L - S - CE[V']$ $RP[V'] = E[V'] - CE[V'] = CE[W] - E[W] = RP[W]$

Note that the certain equivalent $CE[V']$ is different from $CE[V]$. Its value is calculated as the inverse of the expected utility of the profit lottery $V' = L–S–W$. Clearly, to calculate the utilities of the profit outcomes for V', and thus to calculate the certain equivalent $CE[V']$, it is necessary to know the lump-sum price L and the subcontract price S.

Conceptually, the profit lottery $V' = L–S–W$ is more general and includes the original profit lottery $V = L–C$ as a special case (when Z=0 and S=0). It is also helpful to think of the amount, $L–S$, as the lump-sum price for doing the work with cost W. Thus, profit lottery $V' = L–S–W$ is similar to a lump-sum contract with price $L–S$ to perform the work with uncertain cost W.

Changing the subcontract price S changes the certain equivalent $CE[V']$. For example, as S increases, the remaining money $L–S$ for performing the remaining work W decreases, and thus the certain equivalent for the resulting profit $CE[V']$ also decreases.

The certain equivalent for the cost of the remaining work $CE[W]$ is that value that makes the contractor indifferent between receiving with certainty the certain equivalent $CE[V']$ and then paying with certainty the certain equivalent for cost $CE[W]$ to another contractor to do the associated work. (Note that the equality below is based on the fact that all monetary values, on both the left and right sides, are to be received or paid with certainty, i.e., without any risk.)

$$CE[V'] = L - S - CE[W]$$

$$RP[V'] = E[V'] - CE[V'] = \{L - S - E[W]\} - \{L - S - CE[W]\} = CE[W] - E[W] = RP[W]$$

The above formulas for $CE[W]$ and $RP[W]$ are true for any S and for any utility function.

It is also important to notice that the certain equivalent for the cost of the remaining work $CE[W]$ is calculated from the certain equivalent $CE[V']$ and not the other way around:

$$CE[W] = L - S - CE[V']$$

As a result $CE[W]$ depends on the value of the subcontract price S and on the lump-sum price L. The risk premiums $RP[V'] = RP[W]$ also depend on both the subcontract price S and on the lump-sum price L. This is illustrated clearly by the numerical example below.

Maximum Subcontract Price S^*

The maximum subcontract amount S^* for work Z that the contractor would be willing to pay is that specific value that makes the two certain equivalents for profit $CE[V]$ and $CE[V']$ equal:

$$E[u(V)] = E[u(V')] \qquad \text{or} \qquad CE[V] = CE[V'] \quad \Rightarrow \quad S^*$$

It is important to appreciate that the maximum acceptable subcontract amount S^* is defined to be that special certain equivalent $CE^*[Z]$ (i.e., the selling price, or *PISP*) for cost lottery Z, for a contractor that fulfills the following special conditions:

(a) The contractor's current asset position is V_0+L, and

(b) The contractor currently owns (i.e., must face) both cost lotteries W and Z, and

(c) The contractor will sell cost lottery Z and keep (face) cost lottery W.

Under these conditions, S^* and $CE^*[Z]$ are equal by definition:

$$S^* = CE[Z \mid W, V_0 + L] = CE^*[Z]$$

A risk-averse contractor would be willing to pay an amount S^* that exceeds the expected cost of the subtracted work $E[Z]$ and thus the associated risk premium $RP^*[Z]$ would be positive:

$$RP^*[Z] = RP[Z \mid W, V_0 + L] = CE[Z \mid W, V_0 + L] - E[Z] = S^* - E[Z]$$

The relationship between the three risk premiums $RP[V]$, $RP[V']$ and $RP^*[Z]$ is given by:

$$RP[V] = L - E[C] - CE[V] = L - E[Z] - E[W] - CE[V]$$

$$= \{S^* - E[Z]\} + \{L - S^* - E[W] - CE[V']\}$$

$$RP[V] = RP^*[Z] + RP[V'] = RP^*[Z] + RP[W]$$

$$RP[V] - RP[V'] = S^* - E[Z]$$

These relationships do not require the delta property. They are valid for any utility curve but only when the subcontract price is the maximum acceptable amount S^*. When $S = S^*$, the risk premium without subcontracting $RP[V]$ equals the maximum risk premium $RP^*[Z]$ that the contractor is willing to pay to the subcontractor, plus the risk premium $RP[V'] = RP[W]$ that the contractor demands for accepting the risk for the remaining work W.

Clearly, these relationships indicate that for a risk-averse contractor, $RP[V] > RP[V']$. Thus, subcontracting reduces the risk premium for profit (from $RP[V]$ to $RP[V']$), as well as the available risk premium for cost (from $RP[C]$ to $RP[W]$), by the difference between the maximum subcontract price S^* and the *expected cost* of the subcontracted work $E[Z]$.

In other words, the maximum amount that a risk-averse contractor is willing to pay a subcontractor above the expected cost of the work being subcontracted is $S^*–E[Z]$, and equals the reduction in risk premium brought about by subcontracting, $RP[V]–RP[V']$, or $RP[C]–RP[W]$.

Numerical Examples — Lump-Sum Contract for a fixed price, *L*, With Subcontracting

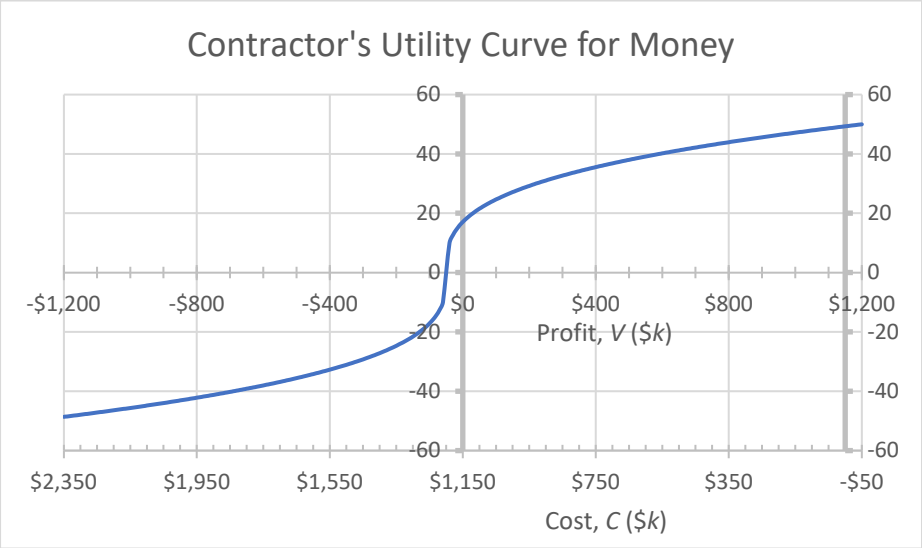
This section uses numerical examples to illustrate some of the important points made above.

Consider again the contractor described in the earlier example with the following cubic-root utility curve for profit V (in \$$k$):

$$u(V) = \sqrt[3]{a + bV} = \sqrt[3]{5000 + 100V} \qquad \text{(profit } V \text{ is in \$}k\text{)}$$

The inverse of the utility function gives the profit $V(u)$ (in \$$k$) given its utility u:

$$V(u) = u^{-1}(u) = (u^3 - a)/b = (u^3 - 5000)/100 \quad \text{(profit } V \text{ is in \$}k\text{)}$$



Example Lump-Sum Contract — Analysis based on Profit

Suppose that the contractor has signed a lump-sum contract for a fixed price L =\$1,150k for the previous example project whose cost C follows the discrete distribution with five outcomes c_i shown below. The resulting profit lottery V also has five outcomes v_i as shown in the table.

Profit Lottery for L =\$1,150k			
Prob. $P[C=c_i] = P[V=v_i]$	Cost, C c_i (\$$k$)	Profit, V=L-C $v_i = L - c_i$ (\$$k$)	Utility $u(v_i) = u(L-c_i) = u''(c_i)$
1/12	800	350	34.1995
1/6	900	250	31.0723
1/2	1,000	150	27.1442
1/6	1,100	50	21.5443
1/12	1,200	(50)	0.0000
$E[.]$=	$E[C]$ = \$1,000k	$E[V]$ = \$150.000k	$E[u(V)]$ = 25.1915
$CE[.]$=	$CE[C]$ = \$1,040.132k	$CE[V]$ = \$109.868k	
$RP[.]$=	$RP[C]$ = \$40.132k	$RP[V]$ = \$40.132k	

The project can be subdivided into two parts whose costs are represented by the two cost lotteries (random variables) W and Z such that:

$$C = W + Z$$

The contractor will perform the work with cost W whose cost outcomes w_i have the probabilities shown in the table below. The contractor will pay an amount S = \$420k to a subcontractor to perform the work with cost Z. (Note: As illustrated later, S = \$420k is less than the maximum subcontract price S^* = \$434.520k. that would be acceptable to the contractor.)

The analysis for the contractor's resulting profit lottery V', for S = \$420k, is as follows:

Profit Lottery for L =\$1,150k and S = \$420k $<$ S^* = \$434.520k			
Prob. $P[W=w_i] = P[V'=v_i']$	Cost, W w_i (\$$k$)	Profit, $V'=L-S-W$ $v_i' = L-S-w_i$ (\$$k$)	Utility $u(v_i') = u(L-S-w_i)$
1/12	500	230	30.3659
1/6	550	180	28.4387
1/2	600	130	26.2074
1/6	650	80	23.5133
1/12	700	30	20.0000
$E[.]$=	$E[W]$ = \$600k	$E[V']$ = \$130k	$E[u(V')]$ = 25. 9595
$CE[.]$=	$CE[W]$ = \$605.059k	$CE[V']$ = \$124.941k	
$RP[.]$=	$RP[W]$ = \$5.059k	$RP[V']$ = \$5.059k	

Notice that for the chosen lump-sum price L =\$1,150k and subcontract price S = \$420k, all profit outcomes v_i' are positive. Their utilities are also positive and clearly come from the region of the cubic-root utility curve that exhibits risk-averse behavior. The contractor's risk-averse attitude in this region is reflected in the results above and is discussed further below.

The expected utility and the certain equivalent of the profit lottery V' (shown above) are:

$$E[u(V')] = \sum_{i=1}^{5} P[V' = v_i'] \cdot u(v_i') = \sum_{i=1}^{5} P[W = L - S - v_i'] \cdot \sqrt[3]{5000 + 100 v_i'} = 25.9595$$

$$CE[V'] = V'(E[u(V')]) = u^{-1}(E[u(V')]) = \{E[u(V')]^3 - 5000\} / 100 = \$124.941k$$

- The certain equivalent for profit $CE[V']$ = \$124.941k for the example subcontract price S = \$420k, is greater than the certain equivalent for profit without subcontracting $CE[V]$ = \$109.868k. This means that the contractor would prefer the current subcontracting arrangement and would be willing to pay even more than S = \$420k to subcontract the work whose cost is Z. In fact, as shown below, the contractor would be willing to pay as much as S^* = \$434.520k to subcontract the work with cost Z.
- The risk premium for profit $RP[V']$ is positive indicating risk-averse behavior:

$$RP[V'] = E[V'] - CE[V'] = \$130k - \$124.941k = \$5.059k$$

- The certain equivalent for the cost of the remaining work $CE[W]$ and the associated risk premium $RP[W]$ are calculated from $CE[V'] = \$124.941k$ and depend also on the example lump-sum price $L = \$1{,}150k$ and the example subcontract price $S = \$420k$ as follows:

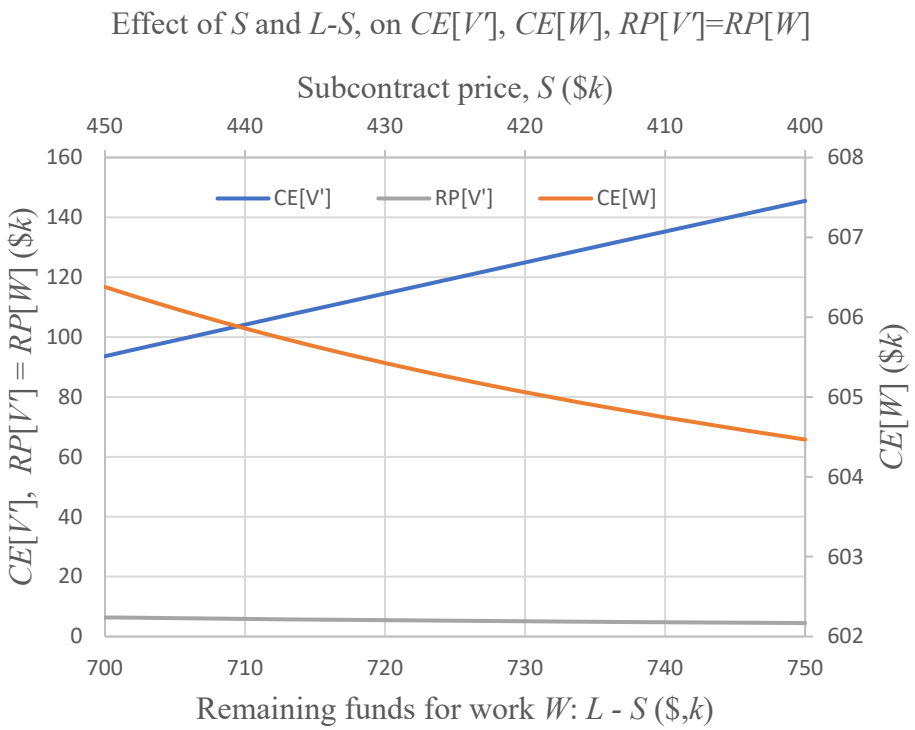
$$CE[W] = L - S - CE[V'] = \$1{,}150k - \$420k - \$124.941k = \$605.059k$$

$$RP[W] = CE[W] - E[W] = \$605.059k - \$600k = \$5.059k = RP[V']$$

- To avoid the risk of the remaining work with cost W, the contractor would be willing to pay up to $CE[W] = \$605.059k$, i.e., more than the expected cost $E[W] = \$600k$, to transfer (sell) the obligation to perform this work to another contractor.
- The risk premium $RP[W] = \$5.059k$ given by the difference $CE[W] - E[W]$ is positive indicating risk-averse behavior. Moreover, this confirms numerically that the two risk premiums $RP[W] = RP[V']$ are equal, which is always true for any L or S.
- The values of both the certain equivalent for cost $CE[W]$ and the risk premium for cost $RP[W]$ depend on the lump-sum price $L = \$1{,}150k$ and the subcontract price $S = \$420k$ and will change if either L or S change. This is illustrated by the graph below.

Impact of the Subcontract Price, *S*, and the Remaining Funds, *L-S*, for Work, *W*

Any subcontract price S leaves a fixed amount of money L-S to cover the cost of the work W that will not be subcontracted. The effects of the subcontract price S and the associated remaining funds L-S on the certain equivalents for profit $CE[V']$, the certain equivalent for cost $CE[W] = L - S - CE[V']$, and the risk premiums for profit and cost $RP[V'] = RP[W]$, are illustrated in the graph below.



It should be noted that there are two horizontal monetary axes in the above graph. The lower axis is for *L*-*S* and increases towards the right. The upper axis is for *S* and increases towards the left. The graph also has two *y*-axes: the left axis is for $CE[V']$ and $RP[V'] = RP[W]$; the right axis is for $CE[W] = L - S - CE[V']$.

As expected, the certain equivalent for profit $CE[V']$ increases as *L*-*S* increases, i.e., as either the lump-sum price *L* increases (more money is available to cover the cost of all the work, both *S* and *W*), or the subcontract price *S* decreases (more money out of the lump-sum price *L* is left after *S* is paid to cover the cost of the remaining work *W*).

For the specific range of values *L*–*S* in the graph $\$700k < L–S < \$750k$, the certain equivalent for cost $CE[W] = L–S–CE[V']$ decreases (indicating less risk-averse behavior) as *L*–*S* and $CE[V']$ increase. This is because as *L*–*S* increases the resulting values of possible profit outcomes v_i' increase, and thus their utilities come from less risk-averse regions of the utility curve.

As described earlier, the risk premiums $RP[V'] = RP[W]$ are always equal and thus their graphs coincide. The risk premiums $RP[V'] = RP[W]$ also decrease as *L*–*S* increases, for the same reasons that the certain equivalent for cost $CE[W]$ decreases.

The graphs for the risk premiums $RP[V'] = RP[W]$ are identical in shape to the graph for the certain equivalent for cost $CE[W]$ but their actual ordinate values (*y*-values) are smaller by $\$600k$. All three graphs decrease at the same rate (slope) as *L*–*S* increases.

$$RP[V'] = RP[W] = CE[W] - E[W] = CE[W] - \$600k$$

The reason that the curves for the risk premiums $RP[V'] = RP[W]$ do not appear to have as much sensitivity to *L*–*S* as the curve for $CE[W]$ is because $RP[V']=RP[W]$ are plotted relative to the left vertical axis, whereas $CE[W]$ is plotted relative to the right vertical axis and the left and right axes employ different scales.

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Maximum Subcontract Price S^*

The maximum price S^* that is acceptable to the contractor is found by a trial-and-error procedure that changes the value S until the certain equivalent for profit with subcontracting $CE[V']$ becomes equal to the certain equivalent for profit without subcontracting $CE[V]$:

$$CE[V'] = CE[V] \quad \Rightarrow \quad S^*$$

For this example, and for a contract price L =\$1,150k the maximum subcontract price (found by trial-and-error) is S^* = \$434.520k.

The final profit lottery is shown below for which the expected utility and the certain equivalent are equal to those without subcontracting.

$$E[u(V')] = E[u(V)] = 25.1915$$
$$CE[V'] = CE[V] = \$109.868k$$

Profit Lottery for L =\$1,150k and S^* = \$434.520k			
Prob. $P[W=w_i] = P[V'=v_i']$	Cost, W w_i (\$$k$)	Profit, $V'=L-S-W$ $v_i' = L-S-w_i$ (\$$k$)	Utility $u(v_i') = u(L-S-w_i)$
1/12	500	215.480	29.8316
1/6	550	165.480	27.8272
1/2	600	115.480	25.4829
1/6	650	65.480	22.6031
1/12	700	15.480	18.7084
$E[.]$=	$E[W]$ = \$600k	$E[V']$ = \$115.480k	$E[u(V')] = 25.1915$
$CE[.]$=	$CE[W]$ = \$605.612k	$CE[V']$ = \$109.868k	
$RP[.]$=	$RP[W]$ = \$5.612k	$RP[V']$ = \$5.612k	

The corresponding metrics for the work with cost Z when it is subcontracted for the maximum acceptable subcontract price S^* = \$434.520k are as follows:

Profit Lottery for L =\$1,150k and S^* = \$434.520k	
$CE^*[Z] = S^*$ =	\$434.520k
$E[Z] = E[C] - E[W]$ =	\$400.000k = \$1,000k - \$600k
$RP^*[Z] = S^* - E[Z]$ =	\$ 34.520k
$RP[V']$ =	\$ 5.612k
$RP[V] = RP^*[Z] + RP[V']$ =	\$ 40.132k = \$34.520k + \$5.612k

Notice that the reduction in risk premium $RP[V] - RP[V']$ = \$40.132k - \$5.612k = \$34.520k equals the risk premium $RP^*[Z] = S^* - E[Z]$ = \$34.520k, i.e., the amount by which the maximum subcontract price S^* = \$434.520k exceeds the expected cost $E[Z]$ = \$400k.

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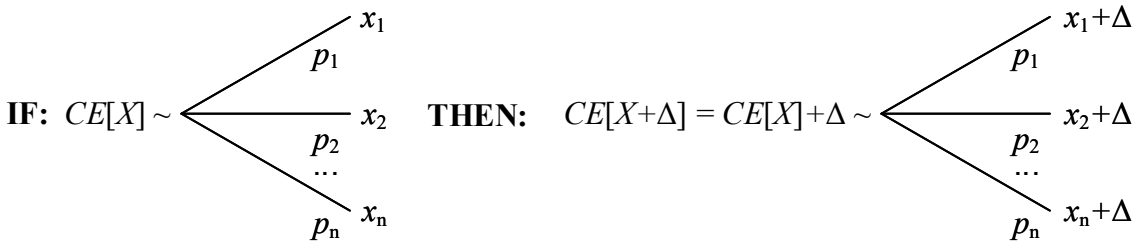
Lump-Sum Contracts and the Delta Property of the Exponential Utility Curve

The *delta*-property of the exponential utility curve can be defined as follows:

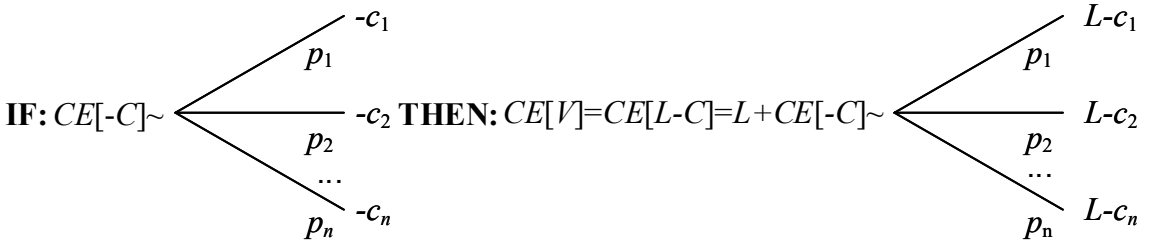
- If all *prizes* x_i of a lottery X are increased by the same amount Δ and become $x_i + \Delta$, then the *certain equivalent* of the lottery $CE[X]$ is also increased by the same amount Δ.

$$CE[X+\Delta] = CE[X] + \Delta$$

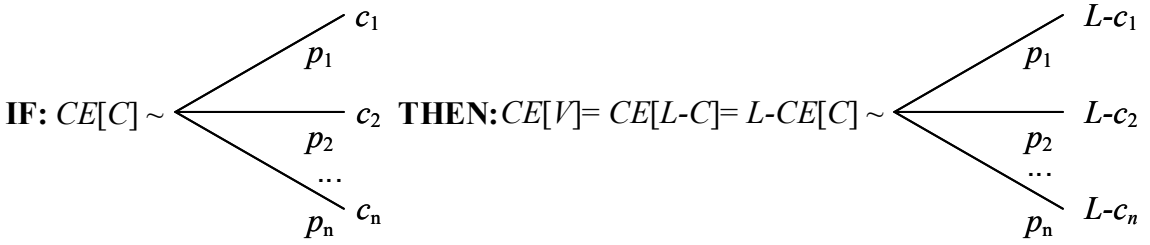
- The *delta-property* is illustrated graphically by the following *indifference* statements:



The delta-property can be applied directly to a lump-sum construction contract by starting with the project's cost lottery -C that has negative outcomes -c_i, i.e., lottery (b), and then adding the constant L (i.e., L is the constant Δ) to derive the profit lottery outcomes L-c_i as follows.



Based on lottery (b), the delta-property for a lump-sum contract can now be written in terms of the project's cost lottery C that has positive cost outcomes c_i, i.e., lottery (c), as follows.



Notice that the certain equivalent of lottery (b) for algebraic cost -C is negative $CE[-C] < 0$, whereas the certain equivalent for lottery (c) where costs C are considered positive, is positive $CE[C] > 0$. The two certain equivalents differ only in algebraic sign:

$$CE[C] = -CE[-C]$$

In either case, the certain equivalent $CE[V]$ for profit lottery V that is derived by adding the lump-sum amount L to the above two cost lotteries (b) -C and (c) C should be the same:

$$CE[V] = L - CE[C] = L + CE[-C]$$

Delta-Property and the Contractor's Current Net Asset Position V_0

The alternatives for profit and cost lotteries (a) (b) (c) presented earlier are equivalent because their corresponding monetary outcomes have identical utilities, and the lotteries have identical expected utilities. The reason that these profit and cost lotteries produce identical utility results, however, is that they all require and make use of the lump-sum contract price L either explicitly, as in profit lottery (a) V, or implicitly, as in profit lottery (b) $-C$, or cost lottery (c) C.

This observation led to the very important conclusion that a contractor with current net asset position V_0 cannot analyze decisions based only on project cost C without considering the lump-sum contract price L. The contractor's utility curve $u(V)$ gives the utility for net profit $V=L-C$ which is measured from a zero point that reflects this particular current net asset position V_0. Thus, the lump-sum price L is always required (either explicitly or implicitly) to read the correct utilities from the utility curve for any project-related lotteries.

The important consequence of the delta property is that a contractor with an exponential utility curve can analyze decisions based only on project cost C that are equally valid for any L.

For a "delta-contractor", project *profit* V is the difference between the lump sum amount L (a constant) and the cost C a random variable.

$$V = L - C$$

Thus, based on the *delta*-property, the certain equivalent for profit $CE[V]$ is obtained by adding the constant L to the certain equivalent $CE_0[-C]$:

$$CE[V] = CE[L - C] = L + CE_0[-C] = L - CE_0[C]$$

This result is identical to the general result derived earlier (that is valid for any utility curve):

$$CE[C] = L - CE[V]$$

The important difference is that the earlier evaluation of $CE[C]$ was based on the utility of costs measured from a "zero" monetary point associated with asset position V_0+L (i.e., by first increasing the contractor's current wealth by L), which required knowledge of the lump-sum price L. When using the delta property, the certain equivalent $CE_0[C]$ is calculated using the utility of costs measured from a "zero" monetary point associated with the initial net asset position V_0. Thus, the evaluation of $CE_0[C]$ does not require the lump-sum price L.

When the delta-property applies, the two certain equivalents $CE[C]$ and $CE_0[C]$ are equal for any lump-sum price L. As a result, the subscript "0" is not necessary (except when needed for clarity). I.e., the contractor's current net asset position or wealth does not matter:

$$CE[C] = CE_0[C]$$

Also, when the delta-property applies, the risk premium for profit $RP[V] = RP[C] = RP_0[C]$ does not depend on and is the same for any value of the lump-sum price L:

$$RP[V] = RP[C] = RP_0[C] = CE_0[C] - E[C]$$

Moreover, the delta-property makes the calculation of the risk adjusted cost RAC = PIBP = L_{min}, for which $CE[V]$=\$0, very simple:

$$RAC = L_{\min} = CE[C] = CE_0[C]$$

Numerical Examples — Delta Property and Lump-Sum Contracts

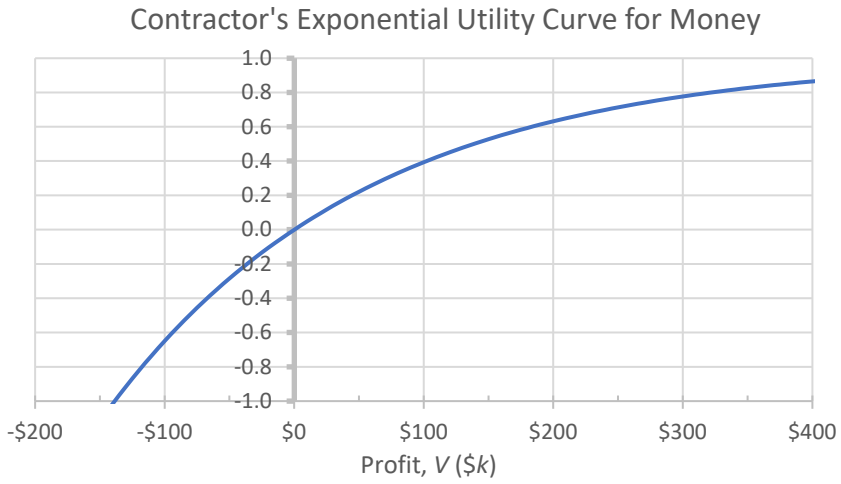
Consider a risk-averse contractor with an exponential utility curve for profit V (in \$$k$) with risk tolerance ρ =\$200k:

$$u(V) = a - be^{-V/\rho} = 1 - e^{-V/200} \quad \text{(profit } V \text{ is in \$}k\text{)}$$

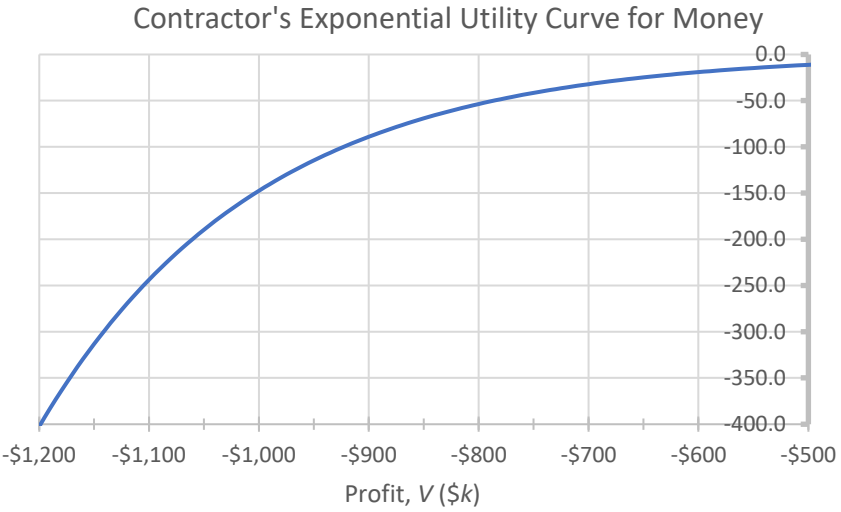
The inverse of the utility function gives the profit $V(u)$ (in \$$k$) for a given utility value u:

$$V(u) = u^{-1}(u) = -\rho \ln[(a-u)/b] = -200 \cdot \ln(1-u) \quad \text{(profit } V \text{ is in \$}k\text{)}$$

The graph below shows this exponential utility curve in the profit range -\$200k < V < \$400k.



Shown below is the same exponential utility curve for negative dollar values in the range -\$1,200k < V < \$-500k. It should be noted that even though the range of values for the profit and utility axes are different, the curvature of the exponential utility curve is the same:



Example Lump-Sum Contract — Analysis based on Profit

Let us consider again the previous example project whose cost C follows a discrete distribution with five outcomes c_i, that give $E[C] = \$1{,}000k$ and $SD[C] = \$100k$. The contractor has signed a lump-sum contract for a fixed price $L = \$1{,}150k$ and the resulting profit lottery V has the outcomes v_i and utilities $u(v_i)$ shown in the table.

Profit Lottery for $L = \$1{,}150k$			
Prob. $P[C=c_i] = P[V=v_i]$	Cost, C c_i ($k)	Profit, $V=L-C$ $v_i = L - c_i$ ($k)	Utility $u(v_i) = u(L-c_i)$
1/12	800	350	0.8262
1/6	900	250	0.7135
1/2	1,000	150	0.5276
1/6	1,100	50	0.2212
1/12	1,200	(50)	-0.2840
$E[.]=$	$E[C] = \$1{,}000k$	$E[V] = \$150.000k$	$E[u(V)] = 0.4648$
$CE[.]=$	$CE[C] = \$1{,}024.984k$	$CE[V] = \$125.016k$	
$RP[.]=$	$RP[C] = \$24.984k$	$RP[V] = \$24.984k$	

The expected utility and the certain equivalent of the profit lottery (shown above) are:

$$E[u(V)] = \sum_{i=1}^{5} P[V = v_i]u(v_i) = \sum_{i=1}^{5} P[C = L - v_i](1 - e^{-v_i/200}) = 0.4648$$

$$CE[V] = V(E[u(V)]) = u^{-1}(E[u(V)]) = -200 \cdot \ln(1 - E[u(V)]) = \$125.016k$$

The fact that the contractor's behavior is risk-averse is indicated by the shape of the contractor's exponential utility curve as well as the above project profit lottery results:

- An exponential utility with positive risk tolerance $\rho = \$200k$ *and* positive scale parameter $b=1$ is concave and exhibits risk-averse behavior everywhere.
- The certain equivalent for profit $CE[V] = \$125.016k$ is smaller than the expected profit $E[V] = \$150.000k$ resulting in a positive risk premium for profit $RP[V] = \$24.984k$. A positive risk premium indicates risk-averse behavior.
- The certain equivalent for cost $CE[C] = \$1{,}024.984k$ is greater than the expected cost $E[C] = \$1000k$ resulting in a positive risk premium for cost $RP[C] = \$24.984k$. Again, a positive risk premium indicates risk-averse behavior.

In the example above, the certain equivalent for cost $CE[C]$, as well as the risk premium for cost $RP[C]$, have been back-calculated indirectly based on the specific example value $L=\$1{,}150k$, and the calculated certain equivalent for profit $CE[V]$, as follows:

$$CE[C] = L - CE[V] = \$1{,}150k - \$125.016k = \$1{,}024.984k$$

$$RP[C] = CE[C] - E[C] = \$1{,}024.984k - \$1{,}000k = \$24.984k = RP[V]$$

Example Lump-Sum Contract — Analysis based on Cost *C* for any Lump-Sum Price *L*

Because of the delta property of the exponential utility curve, it is possible to analyze the cost lottery *C* for the above project without first selecting and considering the lump-sum fixed price *L*. A contractor with an exponential utility curve may use the cost of the project *C* which follows a discrete distribution with five outcomes c_i, and still find the correct certain equivalent for cost $CE[C]$ which would be the same for any value of *L*.

Cost Lottery — Ignores the value of the lump-sum price L
(*Correct analysis is possible because of the Delta Property*)

Prob. $P[C=c_i]$	Cost, *C* c_i (\$*k*)	Utility $u(-c_i) = 1 - e^{-(-c_i)/200}$
1/12	800	-53.5982
1/6	900	-89.0171
1/2	1,000	-147.4132
1/6	1,100	-243.6919
1/12	1,200	-402.4288
	$E[C]$ = \$1,000*k*	$E[u(C)]$ = -167.1603
	$CE_0[C]$ = \$1,024.984*k*	
	$RP_0[C]$ = \$24.984*k*	

The expected utility and the certain equivalent of the above cost lottery are:

$$E[u(C)] = \sum_{i=1}^{5} P[C = c_i]u(-c_i) = \sum_{i=1}^{5} P[C = c_i](1 - e^{c_i/200}) = -167.1603$$

$$CE_0[C] = u^{-1}(E[u(C)]) = 200 \cdot \ln(1 - E[u(C)]) = \$1,024.984k$$

Notice that when the lump-sum price *L* is ignored, <u>all</u> cost monetary outcomes c_i become negative profit outcomes $-c_i$. As a result, they have the <u>negative</u> utility values shown in the table that are very different from the <u>positive</u> utility values for the profit outcomes shown in the previous table. Consequently, the expected utility for the profit lottery shown earlier and the cost lottery shown above are very different:

$$E[u(V)] = 0.4648 \quad \neq \quad E[u(C)] = -167.1603$$

Nevertheless, the <u>certain equivalents for cost</u> $CE[C]$ and $CE_0[C]$ given by these two lotteries are <u>identical</u>:

Certain equivalents for cost, $CE[C]$	
Given by Profit Lottery (for *L*=\$1,150*k*)	Given by Cost Lottery (for any *L*)
$CE[C]$ = \$1,024.984*k*	$CE_0[C]$ = \$1,024.984*k*

Moreover, the relationships that connect the lump-sum price L, the certain equivalent for profit $CE[V]$, and the certain equivalents for cost $CE[C]$ and $CE_0[C]$, are exactly the same:

$$CE[C] = L - CE[V] = \$1{,}150k - \$125.016k = \$1{,}024.984k$$

$$CE[V] = L - CE_0[C] = \$1{,}150k - \$1{,}024.984k = \$125.016k$$

The difference is that for profit lottery V, we first calculate the certain equivalent for profit $CE[V]$, for a specific lump-sum price L, and based on these two values, we calculate the certain equivalent for cost, $CE[C]=L-CE[V]$. For the cost lottery C, we first calculate the certain equivalent for cost $CE_0[C]$, irrespective of L. Then, we can select any value for the lump-sum price L and calculate the certain equivalent for the associated profit, $CE[V]=L-CE_0[C]$. Thus, there exists a linear relationship between the lump-sum price L and the certain equivalent for profit $CE[V]$ as shown in the graph below.

The important conclusion is that the certain equivalent for cost $CE[C]$ equals the certain equivalent for cost $CE_0[C]$ (which does not depend on L). Thus, the certain equivalent for cost $CE[C]$ does not depend on L and has the same value \$1,024.984k for any lump-sum price L:

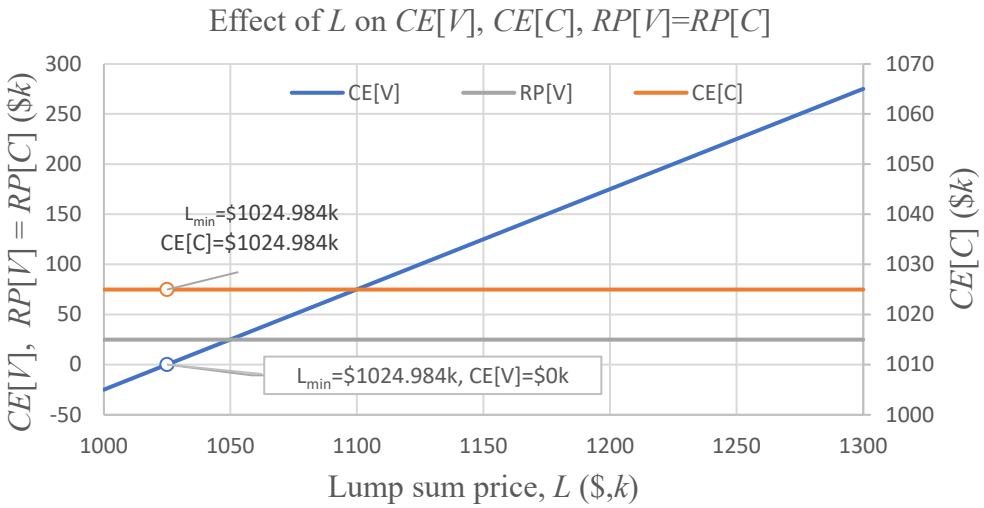
$$CE[C] = CE_0[C] = \$1{,}024.984k \quad \text{for any lump-sum price } L$$

Delta Property: Lump-Sum Price *L* changes *CE*[*V*], but not *CE*[*C*] or *RP*[*V*]=*RP*[*C*]

Because of the delta property of the exponential utility curve, the certain equivalent for profit $CE[V]$ is a simple linear function of the lump-sum price, L:

$$CE[V] = L - CE_0[C] = L - \$1{,}024.984k$$

Clearly, because the certain equivalent for cost $CE[C] = CE_0[C] = \$1{,}024.984k$, and the risk premiums for profit and cost $RP_0[C] = RP[V] = \$24.984k$, are determined using only the cost lottery C, they do not depend on the lump-sum price L. Also, RAC = $L_{min} = CE[C] = CE_0[C] = \$1{,}024.984k$. These relationships are illustrated in the graph below.



The linear relationship between the lump-sum price L and the certain equivalent for profit $E[V]$ makes it very easy to answer the following example questions A, B, and C.

A. The contractor would like the certain equivalent for profit $CE[V]$ to have a specific value, for example, to be equal to 10% of the expected project cost $E[C]$. Which lump-sum price L gives a certain equivalent for profit $CE[V]$ equal to $10\%E[C]$?

For any risk-sensitive utility curve (other than the exponential) the answer to this question is found by a trial-and-error procedure where we select and test successive trial values L until $CE[V] = 10\%E[C]$.

The delta property, however, makes this problem remarkably simple:

$$\begin{aligned} CE[V] &= L - CE_0[C] = L - \$1,024.984k \\ &= 10\%E[C] = 10\%\$1,000k = \$100k \\ L - \$1,024.984k &= \$100k \\ L &= \$1,124.984k \end{aligned}$$

B. When modeling the same project for contractor "R", with a cubic-root utility curve for money (the name "R" is a mnemonic for "cubic Root"), we found a certain equivalent for profit, $CE_R[V] = \$109.868k$. This means that "R" would be willing to sell the entire contract to another contractor, including the lump sum amount $L = \$1,150k$ for no less than $Q = \$109.868k$ to be received with certainty. Would contractor "X", with an exponential utility curve for money (the name "X" is a mnemonic for "eXponential"), be willing to buy the contract from contractor "R" at this price $Q = \$109.868k$?

If contractor "X" buys the entire contract from contractor "R" for the price $Q = \$109.868k$, then, because of the delta property, his certain equivalent for profit $CE_X[V]$ can be calculated directly from the certain equivalent for cost $CE_0[V]$=$1,024.984$k$ as follows:

$$CE_X[V] = (L - Q) - CE_0[C] = (\$1,150k - \$109.868k) - \$1,024.984k = \$15.148k$$

Notice that the constant L-$Q = \$1040.132k$ is the net price (similar to L) that contractor "X" would receive as payment to perform all the work for which his certain equivalent for cost is $CE_0[C]$=1024.984k$. Thus, contractor "X" would be willing to buy the entire contract for the minimum that "R" would accept $CE_R[V] = \$109.868k$, only if "X" finds the resulting certain equivalent for profit $CE_X[V] = \$15.148k$ acceptable, given other market opportunities available.

C. Instead of selling the entire contract, contractor "R" may choose instead to receive L and then to subcontract all the work to "X". In this case, the certain equivalent for cost for contractor "R" $CE_R[C] = \$1,040.132k$ is the most that "R" would be willing to pay to "X" to perform all the work. Would contractor "X" be willing to serve as subcontractor to contractor "R" and do all the work for a fixed price $S = \$1,040.132k$?

Notice that this problem is identical to the one above. Again, because of the delta property, the certain equivalent for profit for contractor "X" $CE_X[V]$ can be calculated directly from his certain equivalent for cost $CE_0[C]$ as follows:

$$CE_X[V] = S - CE_0[C] = \$1,040.132k - \$1,024.984k = \$15.148k$$

The answer is exactly the same as before. Contractor "X" would be willing to serve as subcontractor to contractor "R" and perform all the work for a price of $S = CE_R[C] = \$1,040.132k$, only if he finds the resulting certain equivalent for profit $CE_X[V] = \$15.148k$ acceptable given other available opportunities.

Exponential Utility — Independent Compound Cost Lotteries Can be Decoupled

In general, for a *risk-sensitive decision maker*, the certain equivalent $CE_0[C]$ for a compound lottery $C=W+Z$ does not equal the sum of the individual certain equivalents $CE_0[W]$ and $CE_0[Z]$ found by evaluating each lottery separately, starting from the same net asset position V_0:

$$C = W + Z$$

$$CE_0[C] \neq CE_0[W] + CE_0[Z]$$

The only exception is when both the following two conditions are true:

- The decision maker has an exponential utility curve (he is a *delta*-person), and
- The two lotteries (random variables) W and Z are independent.

Under these two special conditions, the certain equivalent for each cost lottery W and Z can be evaluated separately, starting from the same net asset position V_0, and is valid for any lump-sum price L. Moreover, the certain equivalent $CE_0[C]$ for the compound cost lottery $C=W+Z$ equals the sum of the individual certain equivalents $CE_0[W]$ and $CE_0[Z]$:

$$CE_0[C] = CE_0[W] + CE_0[Z]$$

A direct consequence of this equality is that the risk premium $RP_0[C]$ for the compound cost lottery $C=W+Z$, equals the sum of the individual risk premiums $RP_0[W]$ and $RP_0[Z]$:

$$\{CE_0[C] - E[C]\} = \{CE_0[W] - E[W]\} + \{CE_0[Z] - E[Z]\}$$

$$RP_0[C] = RP_0[W] + RP_0[Z]$$

$$RP[V] - RP[V'] = RP_0[Z]$$

Notice that because the above equalities require both conditions, they are *not* true, even for a *delta*-person, when the component costs W and Z are *not* independent. Clearly, they are *not* true when the decision maker is *not* a delta-person, even if the component costs W and Z are independent. I.e., they are not true for any *risk-sensitive* utility curve other than the exponential.

The case of a risk-sensitive general contractor, whose utility curve is not exponential, and who would like to subcontract the work with cost Z has already been described earlier. In that case, the maximum amount S^* that the contractor would be willing to pay the subcontractor does not equal the certain equivalent for cost $CE_0[Z]$ because the contractor's attitude towards risk depends on L, C, S, and W that describe the profit lottery with subcontracting:

$$S^* = CE[Z \mid W, V_0 + L] = CE^*[Z] \neq CE_0[Z]$$

Even when the contractor has an exponential utility curve (i.e., he is a *delta*-person), there may exist correlation between the cost of the work to be subcontracted Z and the remaining work W. As shown below, positive correlation increases risk, which increases the maximum amount S^* that the contractor would be willing to pay a subcontractor.

$$S^* \geq CE_0[Z]$$

Delta Property: Lump-Sum Contract with Subcontracting

Consider a contractor with an exponential utility curve (a *delta-contractor*) with a lump-sum contract for a fixed price L. Suppose that the project cost C can be separated into two parts whose costs are represented by the two cost lotteries (random variables) W and Z such that:

$$C = W + Z$$

The contractor will use its own forces to perform the work with cost W and will subcontract the work with cost Z for a price S. The resulting profit lottery V' is:

$$V' = L - S - W$$

In general, the certain equivalent for cost $CE[W]$ is computed based on (a) the certain equivalent for profit $CE[V']$, (b) the lump-sum amount L, and (c) the subcontract price S:

$$CE[W] = L - S - CE[V']$$

When the delta property applies, however, the certain equivalent $CE[W] = CE_0[W]$ can be calculated without considering the lump sum L or the subcontract amount S.

$$CE_0[W] = u^{-1}(E[u(W)])$$

As mentioned for $CE_0[C]$ earlier, the subscript "0" in $CE_0[W]$ indicates that the utilities $u(-w_i)$ for cost outcomes $-w_i$ are taken directly from the original exponential utility curve whose zero point on the money axis reflects the contractor's initial net asset position V_0.

Similarly, when the delta property applies, the risk premium $RP[W] = RP_0[W]$ can also be calculated without considering the lump sum L or the subcontract amount S.

$$RP[W] = CE[W] - E[W] = CE_0[W] - E[W] = RP_0[W]$$

The profit lottery V' is obtained by adding the constant $(L\text{-}S)$ to all outcomes $-w_i$ of the cost lottery W to derive the profit outcomes $v_i = L\text{-}S\text{-}w_i$. Thus, because of the delta-property we have:

$$CE[V'] = CE[L - S - W] = L - S - CE_0[W]$$

For any utility curve, the risk premium for the profit lottery V' equals the risk premium for the cost lottery W:

$$RP[V'] = RP[W], \text{ for any utility curve}$$

Because of the delta-property we also have that $RP[W] = RP_0[W]$, which makes the risk premium $RP[V']$ independent of both L and S:

$$RP[V'] = RP[W] = RP_0[W]$$

Delta Property: Maximum Subcontract Price S^* for Lump-Sum Contract

The maximum subcontract price S^* that the contractor would be willing to pay for the work with cost Z can be found easily by setting the certain equivalent for profit with subcontracting $CE[V']$ equal to the certain equivalent for profit without subcontracting $CE[V]$:

$$CE[V'] = L - S^* - CE[W] = CE[V] = L - CE[C]$$

$$S^* = CE_0[C] - CE_0[W]$$

Notice that because of the *delta*-property, S^* does *not* depend on the contract amount L.

By definition, S^* is the certain equivalent for lottery Z when the contractor owns both cost lotteries W and Z and will sell (i.e., subcontract) cost lottery Z while keeping cost lottery W:

$$S^* = CE[Z \mid W] = CE^*[Z] \neq CE_0[Z]$$

Only for a delta-contractor, and only when *W* and *Z* are *independent*, we have $S^* = CE_0[Z]$:

$$CE_0[C] = CE_0[W] + CE_0[Z]$$

$$S^* = CE_0[C] - CE_0[W] = CE_0[Z]$$

When *W* and *Z* are *not* independent, the relationship between S^* and $CE_0[Z]$ for the special case when costs are Normally distributed is derived in a separate section below and is shown to be:

$$S^* = E[Z] + \frac{1}{2\rho}\{Var[Z] + 2Cov[W,Z]\} = CE_0[Z] + \frac{1}{\rho}Cov[W,Z]$$

This means that the contractor would be willing *pay more* than $CE_0[Z]$ to subcontract work that not only eliminates its own risk (i.e., $Var[Z]$) but also reduces project variability (i.e., $Var[C]$) by eliminating *positive cost correlations* between W and Z (i.e., by eliminating $Cov[W,Z]$).

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Summary of Results for a Contractor and a Lump-Sum Contract

The above results for a contractor and a lump-sum contract can be summarized as follows:

Utility Curve	Subcontract Price	Certain Equivalent, Risk Premium	Requires
Any	None	$CE[C] = L - CE[V]$ $RP[V] = RP[C] = E[V] - CE[V]$	L
Any	Any S	$CE[W] = L - S - CE[V']$ $RP[V'] = RP[W] = E[V'] - CE[V']$ $RP[V] - RP[V'] = RP[C] - RP[W]$	L, S
Any	$S^* = \max S$	$RP[V] - RP[V'] = RP^*[Z]$ $RP[V] - RP[V'] = S^* - E[Z]$	L, S^* (S^* found by trial and error)
Exponential	None	$CE[V] = L - CE_0[C]$ $RP[V] = RP[C] = RP_0[C] = CE_0[C] - E[C]$	
Exponential	Any S	$CE[V'] = L - S - CE_0[W]$ $RP[V'] = RP_0[W] = CE_0[W] - E[W]$ $RP[V] - RP[V'] = RP_0[C] - RP_0[W]$	
Exponential	$S^* = \max S$	$S^* = CE_0[C] - CE_0[W]$ $S^* = CE_0[Z \mid W]$	(S^* can be calculated directly)
Exponential	$S^* = \max S$	$RP[V] - RP[V'] = S^* - E[Z]$ $RP[V] - RP[V'] = CE_0[C] - CE_0[W] - E[Z]$	
Exponential	Any S	$CE_0[C] = CE_0[W] + CE_0[Z]$ $RP_0[C] = RP_0[W] + RP_0[Z]$ $RP[V] - RP[V'] = RP_0[Z]$	W and Z must be independent
Exponential	$S^* = \max S$	$S^* = CE_0[C] - CE_0[W] = CE_0[Z]$ $RP[V] - RP[V'] = CE_0[Z] - E[Z] = S^* - E[Z]$	W and Z must be independent

Numerical Examples — Delta Property and Lump-Sum Contract with Subcontracting

Consider the risk-averse contractor in the previous example, whose exponential utility curve has risk tolerance ρ =\$200*k*. Suppose that the project can be divided into two parts whose costs are represented by the two cost lotteries *W* and *Z* such that the total project cost *C* is:

$$C = W + Z$$

The contractor will pay an amount *S* to subcontract the work with cost *Z* and perform the work with cost *W* whose cost outcomes w_i have the probabilities shown in the table below. Because of the delta-property, the contractor can compute the certain equivalent for cost $CE[W]=CE_0[W]$ for cost lottery *W* relative to his current net asset position V_0 as shown below, without the need to consider the lump sum price *L* or the subcontract amount *S*.

Cost Lottery — Does not require the value of the lump-sum price L, or the value of the subcontract price S
(*Correct analysis is possible because of the Delta Property*)

Prob. $P[W=w_i]$	Cost, *W* w_i (\$*k*)	Utility $u(-w_i) = 1 - e^{-(-w_i)/200}$
1/12	500	-11.1825
1/6	550	-14.6426
1/2	600	-19.0855
1/6	650	-24.7903
1/12	700	-32.1155
E[.]=	*E*[*W*] = \$600*k*	*E*[*u*(*W*)] = -19.7231
CE[.]=	*CE*[*W*] = \$606.250*k*	
RP[.]=	*RP*[*W*] = \$6.250*k*	

The expected utility and the certain equivalent shown above were computed as follows:

$$E[u(W)] = \sum_{i=1}^{5} P[W = w_i]u(-w_i) = \sum_{i=1}^{5} P[W = w_i](1 - e^{w_i/200}) = -19.7231$$

$$CE_0[W] = u^{-1}(E[u(W)]) = 200 \cdot \ln(1 - E[u(W)]) = \$606.250k$$

The certain equivalent for profit with subcontracting *CE*[*V'*] can be determined directly from the certain equivalent for cost by using the delta property (*L*, *S*, are constants) as follows:

$$V' = L - S - W$$

$$CE[V'] = CE[L - S - W] = L - S - CE[W]$$

$$CE[V'] = L - S - \$606.250k = \$1{,}150k - S - \$606.250k = \$543.75k - S$$

Thus, the certain equivalent for profit with subcontracting *CE*[*V'*] is a linear function of the lump-sum price *L* the subcontract price *S* and the certain equivalent for cost $CE[W]=CE_0[W]$.

Maximum Subcontract Price S^* for the Work with Cost, Z

The maximum subcontract price S^* that the contractor would be willing to pay for the work with cost Z can be found easily by setting the certain equivalent for profit with subcontracting $CE[V']$ equal to the certain equivalent for profit without subcontracting $CE[V]$:

$$CE[V'] = L - S^* - CE_0[W] = CE[V] = L - CE_0[C]$$

$$S^* = CE_0[C] - CE_0[W] = L - CE_0[W] - CE[V]$$

$$S^* = \$1{,}024.984k - \$606.250k = \$543.750k - \$125.016k = \$418.734k$$

Notice that because of the delta-property, $S^* = \$418.734k$ does not depend on the example lump-sum price $L = \$1{,}150k$ and thus would be the same for any other lump-sum price L.

The resulting profit lottery for $L = \$1{,}150k$ and $S^* = \$418.734k$ is shown below. $E[u(V')] = E[u(V)] = 0.4648$, and $CE[V'] = CE[V] = \$125.016k$, are equal to those without subcontracting.

Profit Lottery for $L = \$1{,}150k$ and $S^* = \$418.734k$

Prob. $P[W=w_i] = P[V'=v_i']$	Cost, W w_i ($k)	Profit, $V'=L-S-W$ $v_i' = L-S-w_i$ ($k)	Utility $u(v_i') = u(L-S-w_i)$
1/12	500	231.266	0.6854
1/6	550	181.266	0.5960
½	600	131.266	0.4812
1/6	650	81.266	0.3339
1/12	700	31.266	0.1447
$E[.]=$	$E[W] = \$600k$	$E[V'] = \$131.266k$	$E[u(V')] = 0.4648$
$CE[.]=$	$CE[W] = \$606.250k$	$CE[V'] = \$125.016k$	
$RP[.]=$	$RP[W] = \$6.250k$	$RP[V'] = \$6.250k$	

The expected utility and the certain equivalent shown above were determined as follows:

- $E[u(V')] = \sum_{i=1}^{5} P[V' = v_i'] \cdot u(v_i') = \sum_{i=1}^{5} P[W = L - S - v_i'] \cdot (1 - e^{-v_i'/200}) = 0.4648$
- $CE[V'] = V'(E[u(V')]) = u^{-1}(E[u(V')]) = -200 \cdot \ln(1 - E[u(V')]) = \$125.016k$

When the work with cost Z is subcontracted for the maximum acceptable subcontract price $S^* = \$418.734k$, the corresponding certain equivalents and risk premiums are as follows:
(Notice that because the cost lottery Z is not given, $CE_0[Z]$ cannot be determined directly based only on cost Z.)

Profit Lottery for L = any value, and $S^* = \$418.734k$

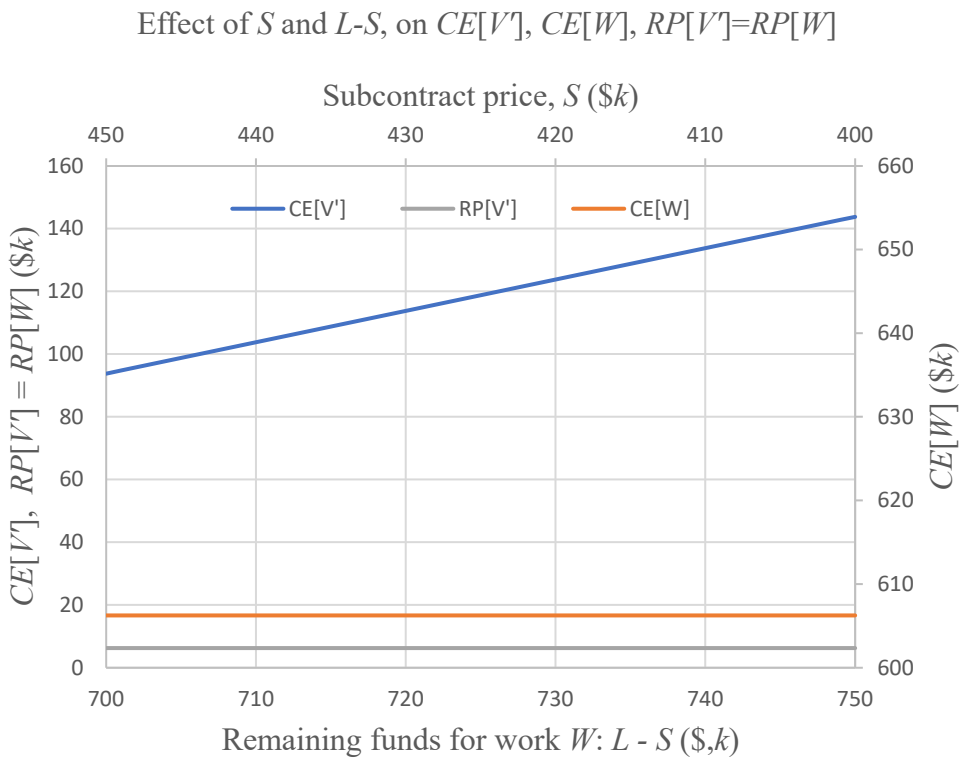
$CE^*[Z] = S^* =$	$\$418.734k$
$E[Z] = E[C] - E[W] =$	$\$400.000k = \$1{,}000k - \$600k$
$RP^*[Z] = S^* - E[Z] =$	$\$ 18.734k$
$RP[V'] =$	$\$ 6.250k$
$RP[V] = RP^*[Z] + RP[V'] =$	$\$ 24.984k = \$18.734k + \$6.250k$

Subcontract Price S and L-S affect $CE[V']$ but do not impact $RP[V'] = RP[W]$ or $CE[W]$

The certain equivalent for profit $CE[V'] = L–S–CE_0[W] = \$543.75k – S$ is a linear function of the subcontract price S. Also, $CE[V'] = L–S–CE_0[W] = (L – S) – \$606.250k$ is also a linear function of $(L–S)$, which is the money that will cover the cost of the remaining work W.

However, because of the delta-property, S and L-S do not affect the certain equivalent for cost $CE[W] = CE_0[W]$, or the risk premiums for profit and cost $RP[V'] = RP_0[W]$. These depend only on the cost lottery for the work W and thus their values are the same for any S and L.

These results are shown in the graph below, where only $CE[V']$ is a function of S and $(L – S)$.



It should be noted that there are two horizontal monetary axes in the above graph. The lower axis is for L-S and increases towards the right. The upper axis is for S and increases towards the left. The graph also has two vertical monetary axes:
The left axis is for $CE[V']$ and $RP[V'] = RP[W]$; the right axis is for $CE[W] = L – S – CE[V']$.

As expected, the certain equivalent for profit $CE[V']$ is a linear function of L-S and increases as L-S increases, i.e., as either the lump-sum price L increases (more money is available to cover the cost of all the work, both S and W), or the subcontract price S decreases (more money out of the lump-sum price L is left over after S is paid to cover the cost of the work W).

In contrast, the certain equivalent for cost $CE[W] = CE_0[W]$ and the risk premiums for profit and cost $RP[V'] = RP_0[W]$ remain constant because they are calculated from the cost lottery W based on the original net asset position V_0 and do not depend on either L or S.

Normally distributed Lotteries — Certain Equivalent and Risk Premium

Consider a contractor with an exponential utility curve and risk tolerance ρ who has signed a lump–sum contract for a fixed price L.

The cost of the work C is described by a continuous *cost* lottery that follows a Normal distribution. The project profit $V = L–C$ is a linear function of project cost and is described by a continuous *profit* lottery that also follows a Normal distribution with the following parameters:

Cost Lottery, C	*Profit Lottery, V*
$C \sim N(E[C], Var[C])$ $E[C] > 0$ $Var[C] > 0$	$V \sim N(E[V], Var[V])$ $E[V] = E[L-C] = L - E[C]$ $Var[V] = Var[L-C] = Var[C]$

For an exponential utility curve for *profit* $u(V)$, the certain equivalent and risk premium for a Normally distributed *profit* lottery V are given by the closed-form formulas below. For completeness, the following table summarizes the closed-form results for the three profit and lottery models (a), (b), (c), described above.

Closed-form Formulas for the Certain Equivalent and Risk Premium of Lotteries when: • *The contractor's utility curve is exponential with risk tolerance ρ.* • *Profit lottery V and cost lottery C follow Normal distributions.*	
(a) Profit Lottery V	$CE[V] = E[V] - \frac{1}{2\rho} Var[V] = L - E[C] - \frac{1}{2\rho} Var[C]$ $RP[V] = E[V] - CE[V] = \frac{1}{2\rho} Var[V] = \frac{1}{2\rho} Var[C] = RP[C]$
(b) Profit Lottery -C	$CE[-C] = E[-C] - \frac{1}{2\rho} Var[-C] = -E[C] - \frac{1}{2\rho} Var[C]$ $RP[-C] = E[-C] - CE[-C] = \frac{1}{2\rho} Var[C] = RP[C] = RP[V]$
(c) Cost Lottery C	$CE[C] = -CE[-C] = E[C] + \frac{1}{2\rho} Var[\mathrm{C}]$ $RP[C] = RP[-C] = CE[C] - E[C] = \frac{1}{2\rho} Var[C]$

Notice that the above results for *profit* lottery V can be derived from those for *cost* lottery C, and vice-versa, by applying the *delta* property:

$$CE[V] = CE[L-C] = L + CE[-C] = L - E[C] - \frac{1}{2\rho} Var[C]$$

$$RP[V] = RP[L-C] = RP[-C] = RP[C] = \frac{1}{2\rho} Var[C]$$

Numerical Examples — Delta Property and Normal PDFs for Lump-Sum Contract

Consider the risk-averse contractor described earlier with an exponential utility curve for profit V (in \$$k$) and risk tolerance ρ =\$200k:

$$u(V) = a - be^{-V/\rho} = 1 - e^{-V/200} \quad \text{(profit } V \text{ is in \$}k\text{)}$$

The inverse of the utility function returns the profit $V(u)$ (in \$$k$) given its utility u:

$$V(u) = u^{-1}(u) = -\rho \ln[(a-u)/b] = -200 \cdot \ln(1-u) \quad \text{(profit } V \text{ is in \$}k\text{)}$$

Consider also the previous example project whose cost C had a discrete distribution with $E[C]$ = \$1,000k, $SD[C]$ = \$100k. Here we assume that the project cost C follows a Normal distribution with the same $E[C]$ = \$1,000k, and $SD[C]$ = \$100k.

As before, suppose that the contractor has signed a lump-sum contract for a fixed price L = \$1,150k. The resulting Normally distributed cost and profit lotteries for a lump-sum contract are as follows:

Cost Lottery C	*Profit Lottery V for L*=\$1,150k
$C \sim N(E[C], Var[C])$	$V \sim N(E[V], Var[V])$
$E[C] = \$1,000k$	$E[V] = L - E[C] = \$1,150k - 1,000k = \$150k$
$Var[C] = (\$100k)^2$	$Var[V] = Var[C] = (\$100k)^2$

The corresponding certain equivalents for profit and cost $CE[V]$ and $CE[C]$ and the associated risk premiums $RP[V]$ and $RP[C]$ given by the closed-form expressions described above are as follows:

Risk-averse contractor with exponential utility curve and risk tolerance ρ =\$200k:	
(a) Profit Lottery V	$CE[V] = E[V] - \frac{1}{2\rho} Var[V] = \$150k - \frac{(\$100k)^2}{2 \cdot \$200k} = \$125k$ $RP[V] = E[V] - CE[V] = \$150k - \$125k = \frac{(\$100k)^2}{2 \cdot \$200k} = \$25k$
(c) Cost Lottery C	$CE[C] = E[C] + \frac{1}{2\rho} Var[\mathrm{C}] = \$1,000k + \frac{(\$100k)^2}{2 \cdot \$200k} = \$1,025k$ $RP[C] = CE[C] - E[C] = \$1,025k - \$1,000k = \frac{(\$100k)^2}{2 \cdot \$200k} = \$25k$

These results clearly illustrate the delta property:

$$CE[V] = CE[L - C] = L + CE[-C] = \$1,150k - 1,025k = \$125k$$

Normally Distributed Compound Cost Lotteries and Subcontracting

Consider a project where the total cost lottery C can divided into component cost lotteries W and Z that follow Normal distributions:

$$W \sim N(E[W], Var[W])$$

$$Z \sim N(E[Z], Var[Z])$$

Because the project total cost C is the sum of costs W and Z, it also follows a Normal distribution, even when W and Z are *not* independent. The expected value and the variance for the total cost C are given by the formulas for a sum of random variables:

$$C = W + Z$$

$$C \sim N(E[C], Var[C])$$

$$E[C] = E[W] + E[Z]$$

$$Var[C] = Var[W] + Var[Z] + 2Cov[W, Z]$$

For a contractor with an *exponential utility curve*, the certain equivalent for a *cost* lottery C that follows a Normal distribution is given by the formula:

$$CE_0[C] = E[C] + \frac{1}{2\rho} Var[C]$$

$$= E[W] + E[Z] + \frac{1}{2\rho}(Var[W] + Var[Z] + 2Cov[W, Z])$$

$$CE_0[C] = CE_0[W] + CE_0[Z] + \frac{1}{\rho} Cov[W, Z]$$

The certain equivalents $CE_0[W]$ and $CE_0[Z]$ for lotteries W and Z are found by analyzing each lottery W and Z separately, starting from the original net asset position V_0.

The above formula shows that the certain equivalent $CE_0[C]$ for the total cost C equals the sum of the certain equivalents $CE_0[W]$ and $CE_0[Z]$ of the component cost lotteries W and Z (when each is considered separately), plus an extra term caused by correlation between costs W and Z. In construction, correlations between costs tend to be *positive* because of *shared common causes*, such as location, geologic conditions, the weather, management, etc. Thus, for a risk-averse *delta*-contractor, the certain equivalent for the total cost is often *greater* than the sum of the certain equivalents for the component cost lotteries.

The risk premium $RP_0[C]$ exceeds the sum of the risk premiums $RP_0[W]$ and $RP_0[Z]$ by the same extra risk premium caused by correlation between the component costs W and Z:

$$RP_0[C] = CE_0[C] - E[C] = CE_0[W] + CE_0[Z] + \frac{1}{\rho} Cov[W, Z] - E[W] - E[Z]$$

$$RP_0[C] = RP_0[W] + RP_0[Z] + \frac{1}{\rho} Cov[W, Z]$$

$$RP_0[C] = \frac{1}{2\rho} Var[C] \qquad RP_0[W] = \frac{1}{2\rho} Var[W] \qquad RP_0[Z] = \frac{1}{2\rho} Var[Z]$$

Lump-Sum Contract—Subcontracting Fee *S* for Normally Distributed Project Profit

Consider a contractor with an exponential utility curve and risk tolerance ρ who has signed a lump–sum contract for a fixed price L.

The total cost for the project C, as well as its component costs W and Z, follow Normal distributions. Their expected value and variance are related as follows:

$$C = W + Z$$

$$E[C] = E[W] + E[Z]$$

$$Var[C] = Var[W] + Var[Z] + 2Cov[W, Z]$$

Suppose that the contractor will use its own forces to perform the work whose cost is represented by W and will subcontract the work whose cost is represented by Z.

The project profit V when the contractor does all work with its own forces, and the project profit V' when the work whose cost is represented by Z is subcontracted, follow Normal distributions with the following expected values and variances:

$$V = L - C \quad \Rightarrow \quad E[V] = L - E[C] \qquad Var[V] = Var[C]$$
$$V' = L - S - W \quad \Rightarrow \quad E[V'] = L - S - E[W] \qquad Var[V'] = Var[W]$$

The *maximum* fee S^* that a *delta*-contractor would be willing to pay a subcontractor is found by setting the certain equivalents for project profit before and after subcontracting to be equal:

$$CE[V] = E[V] - \frac{1}{2\rho} Var[V] = L - E[C] - \frac{1}{2\rho} Var[C]$$

$$CE[V'] = E[V'] - \frac{1}{2\rho} Var[V'] = L - S - E[W] - \frac{1}{2\rho} Var[W]$$

$$CE[V] = CE[V'] \quad \Rightarrow$$

C, W, Z, follow any distributions	*C, W, Z*, follow Normal distributions
$S^* = CE_0[C] - CE_0[W]$ $= CE_0[Z \mid W]$	$S^* = E[C] - E[W] + \frac{1}{2\rho}(Var[C] - Var[W])$ $= E[Z] + \frac{1}{2\rho}(Var[Z] + 2Cov[W, Z])$ $S^* = CE_0[Z] + \frac{1}{\rho} Cov[W, Z]$

In general, the maximum subcontracting fee S^* that a risk-averse *delta*-contractor would be willing to pay equals the *reduction in the certain equivalent* for the cost of the work brought about by subcontracting $CE_0[C]$–$CE_0[W]$. (This is true when C, W, Z, follow any distributions.)

When the component costs W and Z follow Normal distributions that are *positively* correlated, the maximum subcontracting fee S^* exceeds the certain equivalent of the work to be subcontracted $CE_0[Z]$ by $Cov[W,Z]/\rho$, because in addition to removing the work risk $Var[Z]$, subcontracting reduces total variability $Var[C]$ by eliminating the covariance $Cov[W,Z]$.

Subcontracting and Risk Premiums for Normally Distributed Project Profit

The risk premiums of the *profit* lotteries V and V' and cost lotteries C and W where costs and profits follow Normal distributions are equal and are given by the following expressions:

$$RP[V] = E[V] - CE[V] = \frac{1}{2\rho} Var[V] = \frac{1}{2\rho} Var[C] = RP_0[C]$$

$$RP[V'] = E[V'] - CE[V'] = \frac{1}{2\rho} Var[V'] = \frac{1}{2\rho} Var[W] = RP_0[W]$$

Notice that when the cost lotteries W, Z, C, follow Normal distributions, the risk premiums $RP[V] = RP_0[C]$ and $RP[V'] = RP_0[W]$, depend only on the contractor's risk tolerance ρ and the variance of costs $Var[C]$ and $Var[W]$.

Also, because of the delta-property, the risk premiums $RP[V] = RP_0[C]$ and $RP[V'] = RP_0[W]$ do not depend on the lump sum price L or the subcontract price S. This is true for any distribution of costs and profits (not just the Normal distribution).

Subcontracting the work with cost Z reduces the risk premium for profit (or cost) by:

$$\begin{aligned} RP[V] - RP[V'] &= RP_0[C] - RP_0[W] \\ &= \frac{1}{2\rho}(Var[C] - Var[W]) \\ &= \frac{1}{2\rho}(Var[Z] + 2Cov[W,Z]) \\ &= RP_0[Z] + \frac{1}{\rho} Cov[W,Z] \end{aligned}$$

Notice that subcontracting the work with cost Z for any amount $S < S^*$ reduces the risk premium for project profit from $RP[V]$ to $RP[V']$ by the same amount $(Var[C]\text{-}Var[W])/(2\rho)$, which is proportional to the reduction in the variance of the cost lottery that contractor will face.

The last relationship also shows that when costs W and Z are positively correlated (which is often the case), subcontracting the work with cost Z reduces the risk premium for project profit not just by the risk premium of the work to be subcontracted $RP_0[Z]$ but also by a factor $Cov[W,Z]/\rho$ that reflects the elimination of the extra risk caused by positive correlation.

The last relationship between risk premiums $RP[V]$ and $RP[V']$ can also be written in terms of the risk premiums for cost lotteries $RP_0[C]$ and $RP_0[W]$ as follows:

$$RP_0[C] = RP_0[W] + RP_0[Z] + \frac{1}{\rho} Cov[W,Z]$$

Thus, the risk premium $RP_0[C]$ for the total project cost C equals the sum of the individual risk premiums $RP_0[W]$ and $RP_0[Z]$ plus the same extra risk premium caused by correlation between the component costs W and Z.

Value of Risk-Reduction in Project Profit—New Technology

Subcontracting is just one method for reducing the overall risk facing a contractor. Other ways include the use of different construction methods or the introduction of new technology.

In general, any risk-reduction strategy may reduce the expected value of project cost or its variance or both. As a result, its impact can be modeled in a manner similar to subcontracting.

Suppose that the introduction of new technology changes the cost of the project from C to C_T which also follows a Normal distribution:

$$C_T \sim N(E[C_T], Var[C_T])$$

In general, new technology can be beneficial if it reduces the expected value of the project cost, or if it reduces the variance of the project cost or both.

$$E[C_T] < E[C]$$
$$Var[C_T] < Var[C]$$

Value of Variance Reduction for Normally Distributed Project Profit

To make the point clear, let us consider the special case where the introduction of new technology will reduce the variance of the total cost but will not change its expected value.

$$E[C_T] = E[C]$$
$$Var[C_T] < Var[C]$$

To find T^*, the *maximum* that a *delta*-contractor would be willing to spend for this new technology, we set the certain equivalents for the project profit $CE[V]$ and $CE[V']$, before and after the introduction of the new technology, to be equal:

$$CE[V] = E[V] - \frac{1}{2\rho} Var[V] = L - E[C] - \frac{1}{2\rho} Var[C]$$
$$CE[V'] = E[V'] - \frac{1}{2\rho} Var[V'] = L - T - E[C_T] - \frac{1}{2\rho} Var[C_T]$$
$$CE[V] = CE[V'] = CE[L - C] = CE[L - T^* - C_T] \quad \Rightarrow$$
$$T^* = \frac{1}{2\rho}(Var[C] - Var[C_T])$$
$$= RP[C] - RP[C_T] = RP[V] - RP[V']$$

The maximum amount T^* to spent on the new technology equals the reduction in the risk premium for profit (or cost), which is a function of *risk tolerance* ρ and the reduction in the cost variance from $Var[C]$ to $Var[C_T]$, or equivalently, the reduction in the profit variance from $Var[V]$ to $Var[V']$. Notice that, as was the case for subcontracting, T^* does not depend on the lump-sum contract amount L.

Thus, technology that can reduce variability (risk) has value, i.e., up to T^*.

Value of Variance Reduction & Subcontracting for Normally Distributed Project Profit

Consider the introduction of technology, as above, when the contractor will also subcontract for a price S the portion of the work whose cost is represented by Z. Suppose, further, that the new technology with cost T will reduce the variance of the *remaining* work from $Var[W]$ to $Var[W_T]$, but will not change its expected cost $E[W]$:

$$E[W_T] = E[W]$$
$$Var[W_T] < Var[W]$$

The initial project profit V, the profit after subcontracting V', and the profit after subcontracting and after applying the new technology V'', are given by:

$$V = L - C \quad \Rightarrow \quad E[V] = L - E[C] \qquad Var[V] = Var[C]$$
$$V' = L - S - W \quad \Rightarrow \quad E[V'] = L - S - E[W] \qquad Var[V'] = Var[W]$$
$$V'' = L - S - T - W_T \quad \Rightarrow \quad E[V''] = L - S - T - E[W] \qquad Var[V''] = Var[W_T]$$

$$E[C] = E[W] + E[Z]$$
$$Var[C] = Var[W] + Var[Z] + 2Cov[W, Z]$$

As above, the maximum subcontracting fee S^* is found by setting $CE[V]$ equal to $CE[V']$:

$$S^* = CE[Z] + \frac{1}{\rho} Cov[W, Z]$$

To find T^*, the *maximum* that a *delta*-contractor would be willing to spend on this new technology, we set $CE[V']$ equal to $CE[V'']$:

$$CE[V'] = E[V'] - \frac{1}{2\rho} Var[V'] = L - S - E[W] - \frac{1}{2\rho} Var[W]$$
$$CE[V''] = E[V''] - \frac{1}{2\rho} Var[V''] = L - S - T - E[W] - \frac{1}{2\rho} Var[W_T]$$

Set: $CE[V'] = CE[V'']$ $\Rightarrow$

$$T^* = \frac{1}{2\rho}(Var[W] - Var[W_T])$$
$$= RP[W] - RP[W_T] = RP[V'] - RP[V'']$$

Notice that the maximum amount to spent on the new technology T^* depends only on

- The *reduction* in variance $Var[W]$-$Var[W_T]$, and
- The risk tolerance ρ.

This means that the maximum amount to spent on the new technology T^* would have the same value for *any* lump-sum contact amount L and for *any* subcontract amount S.

Notice also that the same result can be found by applying only the *delta*-property, without requiring initially that the cost lotteries W and Z follow Normal distributions:

The general application of the *delta*-property gives:

$$CE[V'] = CE[L - S - W] = L - S - CE[W]$$
$$CE[V''] = CE[L - S - T^* - W_T] = L - S - T^* - CE[W_T]$$

To find the maximum value of the new technology T^* we set the certain equivalents before after the application of the new technology to be equal:

$$CE[V'] = CE[V'']$$

$$L - S - CE[W] = L - S - T^* - CE[W_T]$$

This gives the following general result for any cost distributions:

$$T^* = CE[W] - CE[W_T]$$

When, as in the example above, we also assume that $E[W] = E[W_T]$, then:

$$T^* = E[W] + RP[W] - E[W_T] - RP[W_T] = RP[W] - RP[W_T]$$

Notice that the above relationships do not require that costs follow Normal distributions. If $E[W] = E[W_T]$, and in addition costs do follow Normal distributions, then:

$$T^* = \frac{1}{2\rho}(Var[W] - Var[W_T])$$

It is important to point out that the above analysis depends on the special assumptions made for this specific example, and in particular, that the value of new technology is determined after the work with cost Z is subcontracted and that the new technology impacts only the variance of the remaining work $Var[W]$.

A different example could assume that new technology can reduce the variance of the total project cost $Var[C]$ and not just $Var[W]$. In that case, it might also be possible that the same new technology could also reduce the variance of the cost of the work to be subcontracted $Var[Z]$. Clearly, a reduction in $Var[Z]$ would reduce its certain equivalent $CE_0[Z]$ and this would reduce the maximum subcontract amount S^* that the contractor would be willing to pay to a subcontractor to perform the work with cost Z.

At the limit, if new technology could remove all uncertainty about the total project cost by eliminating its variance, i.e., $Var[C]=0$, then the maximum subcontract amount S^* that the contractor would be willing to pay to a subcontractor would equal the expected cost of the subcontracted work $E[Z]$, because that would be the cost of the work with certainty. I.e., the cost of the work Z would be a constant and not a random variable. In this scenario, the value of the new technology would be even greater because it would include the value of subcontracting. In contrast, the value of subcontracting would diminish.

Numerical Examples — Delta Property, Normal PDF, Subcontracting

Consider the risk-averse contractor described earlier with an exponential utility curve for profit V (in \$$k$) and risk tolerance ρ =\$200k:

$$u(V) = a - be^{-V/\rho} = 1 - e^{-V/200} \quad \text{(profit } V \text{ is in \$}k\text{)}$$

As in the previous example, the total project cost C follows a Normal distribution with $E[C]$ = \$1,000k, $SD[C]$ = \$100k.

The contractor has signed a lump-sum contract for a fixed price L =\$1,150k that gives the following cost and profit lotteries for a lump-sum contract:

Cost Lottery C	*Profit Lottery V for L*=\$1,150k
$C \sim N(E[C], Var[C])$ $E[C] = \$1,000k$ $Var[C] = (\$100k)^2$	$V \sim N(E[V], Var[V])$ $E[V] = L - E[C] = \$1,150k - 1,000k = \$150k$ $Var[V] = Var[C] = (\$100k)^2$

The above data result in the following certain equivalents and risk premiums:

Risk-averse contractor with exponential utility curve and risk tolerance ρ =\$200k:	
Profit Lottery V	$CE[V] = E[V] - \frac{1}{2\rho} Var[V] = \$150k - \frac{(\$100k)^2}{2 \cdot \$200k} = \$125k$ $RP[V] = E[V] - CE[V] = \$150k - \$125k = \frac{(\$100k)^2}{2 \cdot \$200k} = \$25k$
Cost Lottery C	$CE[C] = E[C] + \frac{1}{2\rho} Var[\mathrm{C}] = \$1,000k + \frac{(\$100k)^2}{2 \cdot \$200k} = \$1,025k$ $RP[C] = CE[C] - E[C] = \$1,025k - \$1,000k = \frac{(\$100k)^2}{2 \cdot \$200k} = \$25k$

The contractor will subcontract the work with cost Z for an amount S and perform the work with cost W. Both W and the resulting profit V' follow Normal distributions as shown below:

Cost Lottery W	*Profit Lottery* $V' = L - S^* - W$, *for* L=\$1,150k, *and* S^*=\$418.75k
$W \sim N(E[W], Var[W])$ $E[W] = \$600k$ $Var[W] = (\$50k)^2$	$V' \sim N(E[V'], Var[V'])$ $E[V] = L - S - E[W] = \$1,150k - 418.75k - 600k = \$131.25k$ $Var[V'] = Var[W] = (\$50k)^2$

Based on the above data, the cost lottery W for the work to be performed by the contractor, has the following certain equivalent and risk premium:

Cost Lottery W	$CE[W] = E[W] + \frac{1}{2\rho} Var[W] = \$600k + \frac{(\$50k)^2}{2 \cdot \$200k} = \$606.25k$ $RP[W] = CE[W] - E[W] = \$606.25k - \$600k = \frac{(\$50k)^2}{2 \cdot \$200k} = \$6.25k$

The difference between the certain equivalents for cost $CE_0[C]$- $CE_0[W]$, before and after the work Z is subcontracted, gives the maximum amount S^* that the contractor would be willing to pay to avoid the risk of the cost lottery Z when costs C and W follow any distributions:

$$\begin{aligned} S^* &= CE_0[C] - CE_0[W] \\ &= \$1,025k - 606.25k = \$418.75k \end{aligned}$$

For the example above, cost lotteries C, W, and Z, follow Normal distributions and as a result it is possible the use the specific formula below that gives the identical result:

$$E[Z] = E[C] - E[W] = \$1,000k - 600k = \$400k$$

$$Var[Z] + 2Cov[W, Z] = Var[C] - Var[W] = 7,500(\$k)^2 = (\$86.603k)^2$$

$$\begin{aligned} S^* &= CE_0[Z] + \frac{1}{\rho} Cov[W, Z] \\ &= E[Z] + \frac{1}{2\rho} \{Var[Z] + 2Cov[W, Z]\} \\ &= \$400k + \frac{1}{2 \cdot \$200k} \{7,500(\$k)^2\} = \$418.75k \end{aligned}$$

The certain equivalent and risk premium for the profit lottery $V' = L – S^* – W$, when the work Z will be subcontracted for the maximum acceptable subcontracting price S^* are shown below:

Profit Lottery V' *for* L=\$1,150k, *and* S^*=\$418.75k	$CE[V'] = E[V'] - \frac{1}{2\rho} Var[V'] = \$131.25k - \frac{(\$50k)^2}{2 \cdot \$200k} = \$125k$ $RP[V'] = E[V'] - CE[V'] = \$131.25k - \$125k = \frac{(\$50k)^2}{2 \cdot \$200k} = \$6.25k$

Value of Technology: New technology can reduce the variance of the remaining work W by

$$Var[Y] = Var[W] - Var[W_T] = (\$50k)^2 - (\$40k)^2 = (\$30k)^2$$

The maximum amount T^* that the contractor can spend on this new technology is:

$$T^* = \frac{1}{2\rho}(Var[W] - Var[W_T]) = \frac{1}{2\rho} Var[Y] = \frac{(\$30k)^2}{2 \cdot \$200k} = \$2.25k$$

Effect of Cost Variability on Contractor's Lump-Sum Price

Consider a contractor with an exponential utility curve and risk tolerance ρ and a project whose *cost* lottery C follows a Normal distribution. We shall investigate the effect of cost variability $Var[C]$ on the contractor's lump-sum price L.

Typically, a contractor determines the fixed amount L for a lump-sum contract by selecting an appropriate markup percentage x (such as 15%) and applying it to the expected cost $E[C]$:

$$L = (1+x)E[C]$$

Suppose that the contractor expects a gross profit percentage of "x_{rf} over costs" for *risk-free* work (indicated by the subscript *rf*). For example, x_{rf} = 10%. For such a risk-free project, the contractor's lump-sum price L_{rf} would be:

$$L_{rf} = (1+x_{rf})E[C] = E[C] + E[V_{rf}]$$

The certain equivalent for a project with no risk, i.e., when $Var[C]=0$, would be:

$$CE[V_{rf}] = E[V_{rf}] = L_{rf} - E[C] = x_{rf}E[C]$$

The certain equivalent for a project with risk, i.e., when $Var[C] > 0$, is:

$$CE[V] = L - E[C] - \frac{1}{2\rho}Var[C]$$

For the contractor to be indifferent between a *project with no risk* and a *project with risk* the two certain equivalents, $CE[V]$ and $CE[V_{rf}]$, must be equal:

$$CE[V] = CE[V_{rf}]$$

$$L - L_{rf} = L - E[C] - (L_{rf} - E[C]) = E[V] - E[V_{rf}] = CE[V] + RP[V] - CE[V_{rf}] = RP[V] = \frac{1}{2\rho}Var[C]$$

$$L = L_{rf} + RP[C] = L_{rf} + \frac{1}{2\rho}Var[C] = (1+x_{rf})E[C] + \frac{1}{2\rho}Var[C]$$

$$x = \frac{L - E[C]}{E[C]} = x_{rf} + \frac{Var[C]}{2\rho E[C]}$$

Thus, for the contractor to remain indifferent as the variability of project cost increases:

- The lump-sum price L must exceed the lump sum price for work with no risk L_{rf} by the risk premium for the corresponding project cost lottery $RP[C] = RP[V]$.
- As the variance of the project cost lottery $Var[C]$ increases, the lump-sum amount L increases linearly. Equivalently the lump-sum L increases with the square of the standard deviation of the project cost lottery $SD[C]$.
- The lump-sum L also increases linearly with the contractor's risk aversion coefficient $\gamma = 1/\rho$. The more risk-averse the contractor, the higher the lump sum price L.
- The markup percentage x starts at the markup for work with no risk x_{rf}, and also increases linearly with project cost variance $Var[C]$ and risk aversion, $\gamma = 1/\rho$.

Cost-Plus-Percent-Fee Contracts — Certain Equivalent and Risk Premium

Consider a contractor *with an exponential utility curve* who has signed a *cost-plus-fee* contract for a project. The contractor's fee is specified in the contract as a percentage m of the actual project cost C. For example, m = 10% gives a *cost-plus-10%-fee* contract.

The uncertain cost of the work is represented by a discrete or continuous *cost* lottery C (with positive outcomes c_i), or equivalently, by a *profit* lottery $-C$ (with negative outcomes $-c_i$).

The contractor's *profit* for the project V_m is a random variable equal to the fraction mC of the project cost and thus cannot be negative (in contrast to the profit V_L for a lump-sum contract):

$$V_m = mC \quad (>0)$$

The *profit* lottery V_m is obtained by *multiplying* all outcomes of the *cost* lottery C by the fee percentage m. In this case, the *delta*-property does not apply and the certain equivalent for profit $CE[V_m]$ does not equal the certain equivalent for cost $CE_0[C]$ multiplied by m:

$$CE[V_m] = CE[mC] \neq mCE_0[C]$$

Fortunately, the certain equivalent for profit $CE[V_m]$ can be derived easily for the case where the cost of the work C follows a Normal distribution with $E[C] > 0$ and $Var[C] > 0$:

$$C \sim N(E[C], Var[C])$$

In this case, the profit V_m also follows a Normal distribution $V_m \sim N(E[V_m], Var[V_m])$ where:

$$E[V_m] = E[mC] = mE[C]$$
$$Var[V_m] = Var[mC] = m^2 Var[C]$$
$$SD[V_m] = mSD[C]$$

For a contractor with an exponential utility curve and a profit lottery V_m that follows a Normal distribution, the certain equivalent $CE[V_m]$ and the risk premium $RP[V_m]$ are given by:

$$CE[V_m] = E[V_m] - \frac{1}{2\rho} Var[V_m] = mE[C] - \frac{1}{2\rho} m^2 Var[C] \approx mE[C]$$

$$RP[V_m] = E[V_m] - CE[V_m] = \frac{1}{2\rho} m^2 Var[C] = m^2 RP_0[C] \approx 0$$

Notice that for a cost-plus-fee contract, the variance of profit $Var[V_m]$ is a very small fraction $m^2 Var[C] = m^2 Var[V_L]$ of the variance of the profit under a lump-sum contract. Thus,

> *The contractor's risk premium for a cost-plus-fee contract $RP[V_m]$ is significantly smaller than the risk premium for a lump sum contract $RP[V_L]$.*

For example, for a fee percentage m=10%,

$$m = 10\% \quad \Rightarrow \quad RP[V_m] = 1\% RP[V_L]$$

Also, the standard deviation $SD[V_m] = m\ SD[V_L]$, is a small fraction of $SD[V_L]$. Thus, the *range* of possible profit outcomes for a cost-plus-fee contract is much narrower.

Contractor Indifference Conditions — Lump-Sum Contract vs. Cost-Plus-%Fee Contract

Consider a contractor with an exponential utility curve and risk tolerance ρ and a project whose cost is represented by a *cost* lottery C that follows a Normal distribution. We shall investigate the conditions for which the contractor would be indifferent between the following two contracts:

(a) A *lump-sum contract* for a fixed amount of money L.

(b) A *cost-plus-percent-fee contract*, where the contractor's fee mC is a percentage m of the actual project cost C.

In the following discussion it will be useful to relate the percentage m and the markup percentage x that corresponds to the fixed price L for the lump-sum contract:

$$x = \frac{L - E[C]}{E[C]}$$

The certain equivalents for the profit lotteries of the two contract types are:

Lump-Sum Contract:	$CE[V_L] = L - E[C] - \frac{1}{2\rho} Var[C]$ $CE[V_L] = xE[C] - \frac{1}{2\rho} Var[C]$
Cost-Plus-Percent-Fee Contract:	$CE[V_m] = mE[C] - \frac{1}{2\rho} m^2 Var[C]$

For a given project and contractor, the quantities ρ, $E[C]$, and $Var[C]$ have specific values, and as a result, the certain equivalent for the profit of a lump sum contract $CE[V_L]$ is an increasing function of the lump sum price L.

Similarly, for given values of ρ, $E[C]$, and $Var[C]$, the certain equivalent for the profit of a cost-plus-fee contract $CE[V_m]$ is an increasing function of the fee percentage m.

For the contractor to be indifferent between the two contract types for this project, the certain equivalents for the corresponding profit lotteries must be equal:

$$CE[V_L] = CE[V_m]$$

Substituting the above expressions for the certain equivalents gives:

$$L - E[C] - \frac{1}{2\rho} Var[C] = mE[C] - \frac{1}{2\rho} m^2 Var[C]$$

Thus, for a given project, the lump-sum amount L for which the contractor would be indifferent between a lump-sum contract and a cost-plus-fee contract with a fee percentage m is:

$$L = (1+m)E[C] + (1-m^2)\frac{1}{2\rho} Var[C]$$

$$L = (1+m)E[C] + (1-m^2)RP[C]$$

The above indifference condition that gives the lump-sum price L as a function of the percentage m is easier to understand when expressed in terms of the markup percentage x.

By substitution we obtain that the markup percentage x for which the contractor would be indifferent between a lump-sum contract, for amount L and a cost-plus-fee contract with a fee percentage m is given by:

$$x = m + (1 - m^2)\frac{Var[C]}{2\rho E[C]}$$

$$x = m + (1 - m^2)\frac{RP[C]}{E[C]}$$

These expressions indicate that for a contractor to be indifferent between a lump-sum contract for amount L and a cost-plus-percent-fee contract with a fee percentage m the following must be true:

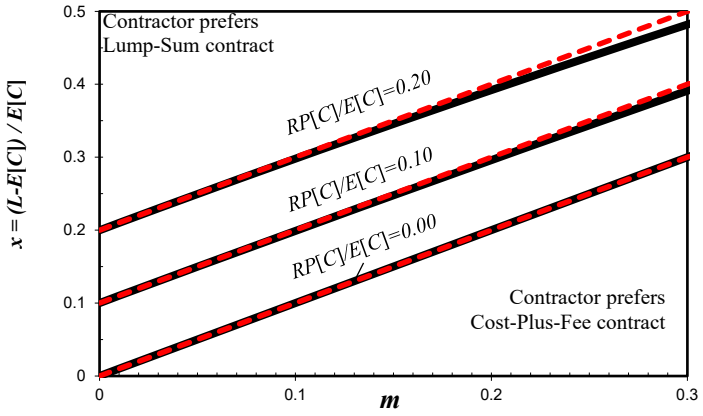
- The markup percentage x for the lump-sum contract must be greater than the fee percentage m. This is expected because a contractor assumes greater risk — may even lose money — under a lump-sum contract.
- The amount by which the markup percentage x for the lump-sum contract must exceed the fee percentage m is proportional to the "relative risk premium" of the lump sum contract $RP[C]/E[C]$.
- Equivalently, the amount by which the markup percentage x for the lump-sum contract exceeds the fee percentage m is proportional to both the relative variability of the project cost $Var[C]/E[C]$ and the contractor's risk aversion $1/\rho$. The riskier the project cost and the more risk-averse the contractor, the greater the amount by which the lump-sum markup percentage x must exceed the cost-plus-fee percentage m.

Contractor Indifference Curves for Lump-Sum Contract vs. Cost-Plus-%Fee Contract

Consider a contractor with an exponential utility curve and risk tolerance ρ and a project whose cost is represented by a *cost* lottery C that follows a Normal distribution with expected cost $E[C]$ and cost variance $Var[C]$. As shown above, the contractor would be indifferent between a lump-sum contract with markup percentage x and a cost-plus-percent-fee contract with fee percentage m when x and m satisfy the following condition:

$$x = m + (1-m^2)\frac{Var[C]}{2\rho E[C]} = m + (1-m^2)\frac{RP[C]}{E[C]}$$

The black curves in the figure below show graphs for the above condition for three example values of $Var[C]/(2\rho E[C]) = RP[C]/E[C]$: 0, 0.10, and 0.20.



Contractor is indifferent between

Lump-sum and Cost-plus-percent-fee contracts when:

$$x = \frac{L - E[C]}{E[C]} \approx m + \frac{RP[C]}{E[C]}$$

$$m \approx \frac{CE[V_L]}{E[C]} = \frac{CE[V_m]}{E[C]}$$

$$L \approx (1+m)E[C] + RP[C]$$

Conceptually, the three curves in the figure correspond to three projects, each having a different value of $RP[C]/E[C]$, or equivalently of $Var[C]/(2\rho E[C])$. Thus, the risk of each project increases as we move up in the graph, from a curve lower in the figure to a curve higher up.

The red dashed lines indicate that for the small cost-plus-fee percentages m used in practice, for example up to 30%, the relationship between m and x is almost linear with a slope equal to 1. The reason is that m^2 is very small, making the term $(1–m^2)$ very close to 1.

The bottom black curve in the figure above is the straight line, $x = m$, and represents a project with no risk, i.e., with $Var[C]=0$ and $RP[C]=0$. The slope of this straight line is 1, indicating that the contractor would be indifferent between a lump-sum contract and a cost-plus-fee contract at the same markup percentage. The *vertical offset* between the straight line, $x = m$, and one of upper curves is very close to $RP[C]/E[C]$ and represents the "percent markup premium", $RP[C]/E[C]$, that the contractor would require to undertake the additional risk of a lump-sum contract for that project.

Contractor Preferences — Lump-Sum Contract vs. Cost-Plus-Percent-Fee Contract

Additional insight is provided by considering just one of the curves in the above figure. The middle curve, for example, represents the indifference conditions for a contractor with risk tolerance ρ and a particular project with cost parameters $E[C]$ and $Var[C]$, such that $Var[C]/(2\rho E[C]) = 0.10$.

For the project that corresponds to the middle curve, each point (m, x) anywhere in the above graph, represents a possible decision where the contractor must choose between:

(a) A lump-sum contract with percentage markup x.

(b) A cost-plus-fee contract with a fee percentage m.

The contractor's choice depends on which of the two contract types has the highest certain equivalent for project profit at point (m, x).

Thus, all points (m, x) in the figure are divided by the contractor's indifference curve into two regions as follows:

- For points (m, x) above the middle curve, the contractor prefers a lump-sum contract. The corresponding fee percentage m is too low relative to the markup percentage x.
- For points (m, x) below the middle curve, the contractor prefers a cost-plus-fee contract. The corresponding markup percentage x is too low relative to the fee percentage m.
- For points (m, x) on the middle curve, the contractor would be indifferent between the two contract types because their certain equivalents have equal values $CE[V_L] = CE[V_m]$.

Contractor's Iso-Certain-Equivalent Curves

As the point (m, x) moves from left to right along the indifference curve for a particular project, the fee percentage m increases, and this increases the certain equivalent for the cost-plus-percent-fee contract $CE[V_m]$. Thus, the corresponding markup percentage x must also increase to keep the two certain equivalents for that point (m, x) equal to each other $CE[V_L] = CE[V_m]$.

The set of points, (m, x), on the indifference curves for different projects, whose certain equivalents are equal to each other and to a specific value $kE[C]$, i.e., $CE[V_L] = CE[V_m] = kE[C]$, define iso-certain-equivalent curves that are characterized by the parameter k. To construct these iso-certain-equivalent curves, for the above figure, we first observe that:

- At each point (m, x) on the same indifference curve (i.e., for the same project), the two certain equivalents are equal to each other $CE[V_L] = CE[V_m]$
- As the point (m, x) moves along the indifference curve from left to right, the corresponding certain equivalents increase in value and equal an increasing fraction k of the expected project cost $CE[V_L] = CE[V_m] = k\,E[C]$.
- The value of the certain equivalents $CE[V_L] = CE[V_m]$ at each point (m, x) on each of the above curves—as a fraction k of the expected project cost—can be found by using the above expressions for the certain equivalents of the two contract types.

We can start, for example, from the expression for the certain equivalent $CE[V_L]$:

$$CE[V_L] = xE[C] - \frac{1}{2\rho} Var[C]$$

Each point (m, x) must satisfy the following condition so that $CE[V_L] = CE[V_m]$.

$$x = m + (1 - m^2)\frac{Var[C]}{2\rho E[C]}$$

Rearranging terms to isolate the risk-related terms gives:

$$\frac{Var[C]}{2\rho} = \frac{x - m}{(1 - m^2)} E[C]$$

Thus, the value of the certain equivalent $CE[V_L]$ and also $CE[V_m]$ at each point (m, x) is:

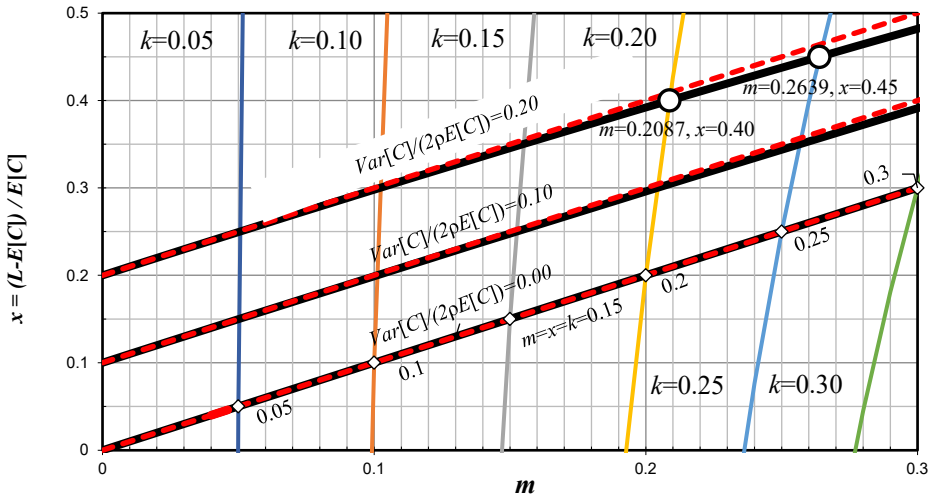
$$CE[V_L] = CE[V_m] = xE[C] - \frac{x - m}{1 - m^2} E[C] = (x - \frac{x - m}{1 - m^2})E[C]$$

Setting $k = CE[V_L]/E[C]$, and rearranging terms gives the iso-certain equivalent curve:

$$\frac{CE[V_L]}{E[C]} = x - \frac{x - m}{1 - m^2} = k$$

$$x = \frac{1}{m} + k(1 - \frac{1}{m^2})$$

The *iso-certain equivalent curves* in the (m, x) plane that correspond to $k = CE[V_L] / E[C]$ for values $k = 0.05, 0.10, 0.15, \ldots, 0.30$ are shown in the figure below.



The intersection of a black "project indifference curve" and one of the multicolored "*iso-certain equivalent curves*" is the point (m, x) that satisfies both the above conditions:

$$x = m + (1 - m^2)\frac{Var[C]}{2\rho E[C]} \qquad \text{and} \qquad x = \frac{1}{m} + k\left(1 - \frac{1}{m^2}\right)$$

Eliminating the markup percentage x gives the following quadratic equation:

$$\frac{Var[C]}{2\rho E[C]} m^2 - m + k = 0$$

Notice that this quadratic equation is exactly the same as the certain equivalent of a cost-plus-percent-fee contract $CE[V_m]$ when it is set equal to $kE[C]$:

$$CE[V_m] = mE[C] - \frac{1}{2\rho} m^2 Var[C] = kE[C]$$

The solution for a project with no risk, i.e., when $Var[C] / 2\rho E[C] = 0$, is $m = x = k$. For projects with risk, the solution always gives, $m > k$, which illustrates risk-averse behavior.

As an example, consider the upper indifference curve for which, $Var[C] / 2\rho E[C] = 0.20$, and the *iso*-curve for $k = 0.20$. The fee percentage at their intersection is $m = 0.2087$, as shown in the figure. At this intersection point, m=0.2087 > k=0.20, as required by risk aversion, but the "risk premium" is very small: m exceeds k by only 0.0087. In contrast, the corresponding markup percentage for this risky project is about 40%, which is quite high, indicating (a) that the project is risky and (b) that the cost-plus-fee percentage m = 20.87% is also quite high.

Notice that the "*iso-certain equivalent curves*" for smaller values of k are even closer to vertical lines and that the corresponding percentage "risk premiums" are even smaller. This indicates that the risk premiums for cost-plus-percent-fee contracts are quite small in practice.

Important Note: For a given project, the iso-certain-equivalent curves apply only to points on the contractor indifference curve for which $CE[V_L] = CE[V_m] = k\,E[C]$. They do not apply to other points (m, x) that are not on the contractor indifference curve for that project.

The table below shows the certain equivalents $CE[V_L] = CE[V_m]$ for the values of $Var[C]/2\rho E[C]$ that correspond to the three curves in the figure above.

$\frac{Var[C]}{2\rho E[C]}$	CE for a Lump-Sum Contract: $CE[V_L]=L-E[C]-\frac{1}{2\rho}Var[C]$	=	CE for a Cost-Plus-Fee Contract: $CE[V_m]=mE[C]-\frac{1}{2\rho}m^2Var[C]$
0.00	$CE[V_L] = L - E[C]$	=	$CE[V_m] = mE[C]$
0.10	$CE[V_L] = L - 1.10E[C]$	=	$CE[V_m] = mE[C] - 0.10m^2E[C]$
0.20	$CE[V_L] = L - 1.20E[C]$	=	$CE[V_m] = mE[C] - 0.20m^2E[C]$

Notice that in each row the certain equivalents for a lump-sum contract and a cost-plus-fee contract are equal to each other $CE[V_L] = CE[V_m]$ because of the indifference condition:

$$L=(1+m)E[C]+(1-m^2)\frac{1}{2\rho}Var[C]$$

As an example, we shall show that the certain equivalents are equal $CE[V_L] = CE[V_m]$ for the last row in the above table that corresponds to $Var[C]/2\rho E[C] = 0.20$:

$$\begin{aligned}CE[V_L] &= L-1.20E[C]\\ &= \{(1+m)E[C]+(1-m^2)0.20E[C]\}-1.20E[C]\\ &= \{1+m+0.20-0.20m^2-1.20\}E[C]\\ &= \{m-0.20m^2\}E[C]\\ &= CE[V_m]\end{aligned}$$

Thus, for points on the top black curve that correspond to $Var[C]/2\rho E[C] = 0.20$, the relative certain equivalents for profit as a percentage of expected cost $k=CE[V_L]/E[C] = CE[V_m]/E[C]$, are given directly by the expression for $CE[V_m]$:

$$m(1-0.20m) = k$$

For example, the intersection between the black curve for $Var[C]/2\rho E[C] = 0.20$ and the "iso-certain equivalent curve" for k =0.25 occurs at:

$$m(1-0.20m) = 0.25$$

$$m = 0.2639$$

The corresponding markup percentage, *x*, is given by the "iso-certain equivalent curve":

$$x = \frac{1}{m} + k\left(1-\frac{1}{m^2}\right)$$

$$x = 0.45$$

The same result is obtained by using the "project indifference curve":

$$x = m + (1-m^2)\frac{Var[C]}{2\rho E[C]}$$

$$x = 0.264 + (1-0.264^2)\times 0.2 = 0.45$$

Numerical Examples — Contractor Lump-Sum and Cost-Plus-Fee Lotteries

Consider the risk-averse contractor described earlier with an exponential utility curve for profit V (in \$$k$), and risk tolerance ρ =\$200k:

$$u(V) = a - be^{-V/\rho} = 1 - e^{-V/200} \quad \text{(profit } V \text{ is in \$}k\text{)}$$

As in the previous example, the total project cost C, follows a Normal distribution with $E[C]$ = \$1,000k, $SD[C]$ = \$100k.

The contractor will face the following cost C and profit lotteries V_L and V_m, for a lump-sum contract at a price L, and for a cost-plus-percent-fee contract at a fee percentage m:

Cost Lottery C	*Lump-Sum Profit Lottery* V_L, *for price* L	*Cost-Plus-%Fee Profit Lottery* V_m, *for fee percentage* m
$C \sim N(E[C], Var[C])$	$V_L \sim N(E[V_L], Var[V_L])$	$V_m \sim N(E[V_m], Var[V_m])$
$E[C] = \$1,000k$	$E[V_L] = L - E[C] = L - 1,000k$	$E[V_m] = mE[C] = m \cdot 1,000k$
$Var[C] = (\$100k)^2$	$Var[V_L] = Var[C] = (\$100k)^2$	$Var[V_m] = Var[mC] = m^2(\$100k)^2$

The cost lottery C, the profit lottery for a lump-sum contract V_L, and the profit lottery for a cost-plus-percent-fee contract V_m, have the following certain equivalents and risk premiums:

Cost Lottery: Contractor has exponential utility curve with risk tolerance ρ =\$200k	
Cost Lottery C	$CE[C] = E[C] + \frac{1}{2\rho}Var[\mathrm{C}] = \$1,000k + \frac{(\$100k)^2}{2 \cdot \$200k} = \$1,025k$ $RP[C] = CE[C] - E[C] = \$1,025k - \$1,000k = \frac{(\$100k)^2}{2 \cdot \$200k} = \$25k$

Contractor's Profit Lottery for Lump-Sum Contract	
For price L, $V_L = L$-C	$CE[V_L] = E[V_L] - \frac{1}{2\rho}Var[V_L] = L - \$1,000k - \frac{(\$100k)^2}{2 \cdot \$200k} = L - \$1,025k$ $RP[V_L] = E[V_L] - CE[V_L] = \$150k - \$125k = \frac{(\$100k)^2}{2 \cdot \$200k} = \$25k$
For markup x, $V_L = (1+x)E[C]$-C	$CE[V_L] = xE[C] - \frac{1}{2\rho}Var[C] = \$1,000 \cdot x - \$25k$

Contractor's Profit Lottery for Cost-Plus-Percent-Fee Contract	
For fee percentage m, $V_m = mC$	$CE[V_m] = mE[C] - \frac{1}{2\rho}m^2Var[C] = \$1,000k \cdot m - \$25k \cdot m^2$ $RP[V_m] = m^2\frac{Var[C]}{2\rho} = \$25k \cdot m^2$

Numerical Examples — Contractor Indifference Curve: Lump-Sum vs Cost-Plus-%Fee

Earlier, we obtained the general condition between the lump-sum amount L and the fee percentage m that makes the contractor indifferent between a lump-sum contract and a cost-plus-fee contract by requiring that the respective certain equivalents be equal $CE[V_L] = CE[V_m]$:

$$L = (1+m)E[C] + (1-m^2)\frac{1}{2\rho}Var[C]$$

$$L = (1+m)\cdot 1{,}000k + (1-m^2)\cdot 25k = \$25k(41+40m-m^2)$$

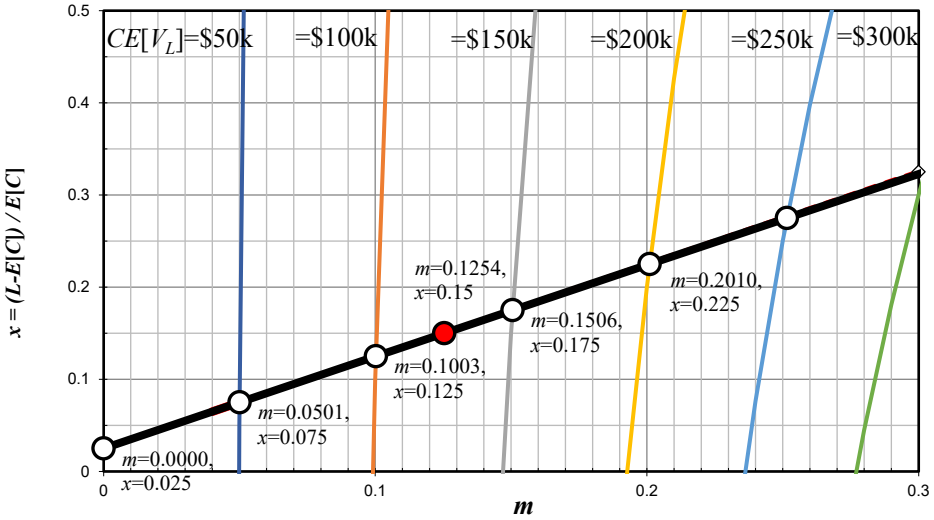
Similarly, we obtained the general condition for the markup percentage x that makes the contractor indifferent between a lump-sum contract and a cost-plus-percent-fee contract with a fee percentage m:

$$x = \frac{L-E[C]}{E[C]} = m + (1-m^2)\frac{RP[C]}{E[C]}$$

Using the data for the example project, the contractor would be indifferent between a lump-sum contract and a cost-plus-percent-fee contract, i.e., $CE[V_L] = CE[V_m]$, when:

$$x = m + 0.025(1-m^2)$$

This quadratic indifference relationship between the markup percentage x and the cost-plus-fee percentage m that would make a contractor with risk tolerance ρ=\$200k indifferent between a lump-sum contract and a cost-plus-percent-fee contract for $Var[C]/(2\rho E[C]) = 0.025$, is shown by the heavy black curve in the figure below. As explained earlier, because m^2 is very small, this black curve is very close to a straight line.



Numerical Examples — Contractor Indifference Values: Lump-Sum vs Cost-Plus-%Fee

For the example project, it was assumed earlier that the contractor had signed a lump-sum contract for a fixed price L =\$1,150k or equivalently that he has used a markup percentage:

$$x = \frac{L - E[C]}{E[C]} = \frac{1{,}150k - 1{,}000k}{1{,}000k} = 15\%$$

As described above, the contractor would be indifferent between a lump-sum contract with a fixed price L =\$1,150k, or equivalently, a markup percentage x=15%, and a cost-plus-fee contract with fee percentage m that satisfies either of the following equations:

$L = (1+m)E[C] + (1-m^2)\frac{1}{2\rho}Var[C]$	$x = m + (1-m^2)\frac{RP[C]}{E[C]}$
$L = (1+m)\cdot 1{,}000k + (1-m^2)\cdot 25k = 1{,}150k$	$x = m + 0.025(1-m^2)$

When simplified, either of the above conditions gives the same quadratic equation as follows:

$$m^2 - 40m - 1 + 40x = 0$$
$$m = 20 - \sqrt{401 - 40x}$$
$$m = 12.54\%$$

For x=15% we obtain the result m = 12.54%, which can be verified by comparing the resulting certain equivalent for the cost-plus-fee contract $CE[V_m]$ for m= 12.54% to the certain equivalent for the lump-sum contract $CE[V_L]$=\$125k found earlier:

$$CE[V_m] = mE[C] - \frac{1}{2\rho}m^2 Var[C]$$
$$= \$1{,}000k \cdot 0.1254 - \$25k \cdot 0.1254^2 = \$125k$$

From here we see that the certain equivalent for a lump-sum contract with price L =\$1,150k and the certain equivalent for a cost-plus-percent-fee contract at a percentage m= 12.54% are equal and thus the contractor would be indifferent between the two contract types:

$$CE[V_L] = CE[V_m] = \$125k$$

The indifference point (m=0.1254, x=0.15) is shown in the graph above by a red circle. The iso-certain-equivalent curve that passes through this point has value, $CE[V_L] = CE[V_m]$ = \$125k.

We can also verify that because m^2 is indeed very small, m is given approximately by the following expression:

$$x = m + 0.025(1-m^2)$$
$$m \approx x - 0.025$$
$$m \approx 15\% - 2.5\% = 12.5\%$$

Owner Indifference Conditions — Lump-Sum vs. Cost-Plus-Percent-Fee Contract

Consider an owner with an exponential utility curve, with risk tolerance, R, who is about to hire a contractor to do the work for a project. The cost of the project is uncertain and is represented by the *cost* lottery C that follows a Normal distribution. We shall investigate the conditions for which the owner would be indifferent between offering the contractor the following two contractual alternatives:

(a) A *lump-sum contract* for a fixed amount of money L.

(b) A *cost-plus-percent-fee contract*, where the contractor's fee mC, is a percentage m of the project actual cost C.

For a lump-sum contract, the owner's cost will be the fixed amount L, with no risk. For a cost-plus-percent-fee contract, the owner will pay the outcome of the Normal *cost* lottery C_m:

$$C_m = (1+m)C$$

$$C_m \sim N(E[C_m], Var[C_m])$$

$$E[C_m] = E[(1+m)C] = (1+m)E[C]$$

$$Var[C_m] = Var[(1+m)C] = (1+m)^2 Var[C]$$

The owner's certain equivalent for the cost-plus-percent-fee *cost* lottery, C_m, is given by:

$$CE[C_m] = E[C_m] + \frac{1}{2R} Var[C_m] = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$$

The owner's risk premium for a cost-plus-percent-fee *cost* lottery is:

$$RP[C_m] = CE[C_m] - E[C_m] = (1+m)^2 \frac{Var[C]}{2R}$$

Thus, the owner's certain equivalents for the cost lotteries of the two types of contracts are:

CE for Lump-Sum Contract, C_L: $CE[C_L] = L$

CE for Cost-Plus-Percent-Fee Contract, C_m: $CE[C_m] = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$

For the owner to be indifferent between the two contract types, the certain equivalents for the corresponding cost lotteries must be equal:

$$CE[C_L] = CE[C_m]$$

$$L = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$$

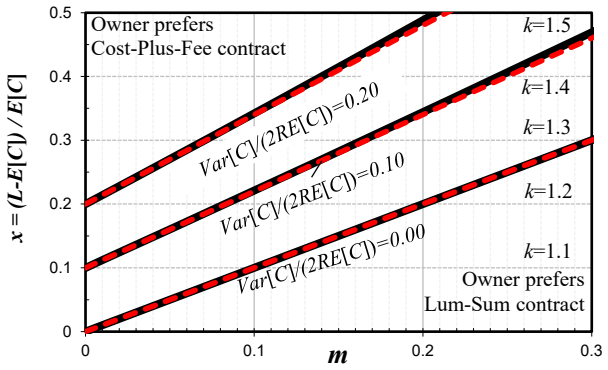
This indifference condition is easier to understand when expressed in terms of the contractor's markup, x. The markup percentage, x, for which the owner would be indifferent between a lump-sum contract, for amount, L, and a cost-plus-fee contract with a fee percentage, m, is:

$$x = \frac{L - E[C]}{E[C]} = m + (1+m)^2 \frac{Var[C]}{2RE[C]} = m + (1+m)^2 \frac{RP[C_m]}{E[C]}$$

The above indifference conditions for L and x, as functions of the fee percentage, m, indicate that for a risk-averse owner the following must be true:

- The owner would be indifferent between (a) paying an uncertain amount, $(1+m)C$, i.e., the uncertain actual cost of the work plus a fee percentage, and (b) paying (with certainty) a lump-sum amount, L, given by the above expression.
- The corresponding markup percentage, x, for the lump-sum price, L, is higher than the cost-plus-fee percentage, m. This was expected because the owner would be willing to pay a risk premium (over the expected cost of a cost-plus-fee contract) to sign a lump-sum contract and eliminate the uncertainty of project cost.
- The *percent spread* (i.e., the *relative risk premium* $RP[C_m]/E[C]$), by which the markup percentage x for the lump-sum contract exceeds the fee percentage m, is proportional to both the relative variability of the project cost $Var[C]/E[C]$, and the owner's risk aversion $1/R$. The riskier the project cost and the more risk-averse the owner, the greater the spread (i.e., the relative risk premium) by which the markup percentage x can exceed the fee percentage m. This is again as expected.

The black curves in the graph below show the relationship between the fee percentage, m, and the markup, x, that would make an owner with risk tolerance, R, indifferent between a cost-plus-fee contract and a lump-sum contract, for three values of $Var[C]/(2RE[C])$: 0, 0.10, and 0.20.



Owner is indifferent between

Lump-sum and Cost-plus-fee contracts when:

$$x = \frac{L - E[C]}{E[C]} \approx (1 + 2\frac{Var[C]}{2RE[C]})m + \frac{Var[C]}{2RE[C]}$$

$$L \approx (1+m)E[C] + (1+2m)\frac{Var[C]}{2R}$$

The red straight lines indicate that for the small cost-plus-fee percentage values m seen in practice, and up to m=30%, the relationship between m and x is almost linear because m^2 is very small, making the term $(1+m)^2$ very close to $(1+2m)$. Thus, the slopes of the black curves are close to $\{1+2Var[C]/(2RE[C])\}$ and match the slopes of the corresponding red straight lines.

The bottom black curve is indeed the straight line $x = m$, with a slope equal to 1, and represents a project with no cost risk, $Var[C]$=0. In this case, the owner would be indifferent between a lump-sum contract and a cost-plus-fee contract at the same markup or fee percentage $x = m$. Each of two upper curves has a vertical offset, $(1+2m)Var[C])/(2RE[C])$, above this straight line, which represents the additional relative risk premium that the owner would be willing to pay with certainty to sign a lump-sum contract and thus avoid the uncertainty of a cost-plus-fee contract.

Owner Preferences — Lump-Sum Contract vs. Cost-Plus-Percent-Fee Contract

Focusing on one of the curves in the above figure provides additional insights. The middle curve, for example, represents the indifference conditions for a particular owner with risk tolerance R, and a particular project with cost parameters $E[C]$ and $Var[C]$, such that $Var[C]/(2RE[C])$ equals 0.10.

For this particular project, a point (m,x) anywhere in the space of the above figure represents a possible decision where the owner must choose between:

(a) A lump-sum contract with percentage markup x, and

(b) A cost-plus-percent-fee contract with a fee percentage m.

The points (m,x) in this space are divided by the owner's indifference curve into two regions as follows:

- For points (m, x) above the middle curve, the owner would prefer a cost-plus-fee contract. The lump-sum markup x is too high relative to the markup percentage m.
- For points (m, x) below the middle curve, the owner would prefer a lump-sum contract. The fee percentage m is too high relative to the markup percentage x.
- For points (m, x) on the middle curve, the owner would be indifferent between the two contract types.

Owner's Iso-Certain-Equivalent Curves

When a point (m,x) moves from left to right along one of the three indifference curves, the fee percentage, m, increases, and this increases the certain equivalent for the *cost* lottery of a cost-plus-percent-fee contract, $CE[C_m]$, for that project. This causes the corresponding markup percentage, x, to also increase to keep the two certain equivalents for cost for that point (m,x) equal to each other, $CE[C_L] = CE[C_m]$.

It is important to notice that the owner prefers smaller (positive) values for the certain equivalents for cost. Thus, the owner's preferences increase as point (m,x) moves towards the southwest corner of the graph. Thus, the owner's preferences increase in the opposite direction from those of the contractor.

The points on each of the three indifference curves in the graph above have the following characteristics:

- The coordinates (m, x) for each individual point on each curve are such that m gives $CE[C_m]$.and x gives $CE[C_L]$, so that $CE[C_L] = CE[C_m]$ for the corresponding project.
- As a point (m, x) moves from left to right along the curve for a particular project, the corresponding fee percentage, m, increases, which increases the cost-plus-fee (and the lump-sum) certain equivalent, $CE[C_m]$=$CE[C_L]$, making the project less desirable.
- Also, as we move up from a lower curve to a higher curve (for the same fee percentage, m), the variance of the project cost lottery increases, which increases the certain equivalent for cost, thus making higher project curves less desirable.

The actual magnitude of the certain equivalents, $CE[C_L] = CE[C_m]$, at each point (m,x) along the black indifference curves in the figure, can be computed using either of the above expressions for the certain equivalents of the two types of contracts.

In this case, it is simplest to use the expression for the owner's certain equivalent for the cost of a lump-sum contract, $CE[C_L]$:

$$CE[C_L] = (1 + x)E[C]$$

It is important to notice that the value of the certain equivalent, $CE[C_L]$, and consequently of $CE[C_m] = CE[C_L]$, at each point (m,x) given by the above expression, is only a function of the markup percentage x, and not a function of the fee percentage, m.

Setting $k = CE[C_L]/E[C]$, and rearranging terms gives:

$$\frac{CE[C_L]}{E[C]} = 1 + x = k$$

$$x = k - 1$$

For the owner, all iso-certain-equivalent curves are straight lines parallel to the m axis. Thus, the certain equivalent, $CE[C_L] = kE[C]$, is the same for all points (m,x) on a horizontal line that crosses the x-axis at k–1. Moreover, the owner prefers smaller k values.

The coordinates (m, x) for each individual point on one of the three indifference curves are such that m gives $CE[C_m]$.and x gives $CE[C_L]$ so that $CE[C_L] = CE[C_m] = kE[C] = (1+x)E[C]$.

As an example, consider the iso-certain equivalent line for k=1.3, which is also the horizontal line x=0.3. The points (m,x) where this line intersects the three indifference curves have the same certain equivalents, $CE[C_L] = CE[C_m] = 1.3E[C]$, as follows:

Points (m,x) on project indifference curves and iso-certain equivalent line for k=1.3		
Lower Project Curve $\frac{Var[C]}{2RE[C]} = 0.00$	Middle Project Curve $\frac{Var[C]}{2RE[C]} = 0.10$	Upper Project Curve $\frac{Var[C]}{2RE[C]} = 0.20$
$1+m+0.00(1+m)^2 = 1.3$ $m-0.3=0$ $(m,x)=(0.3,0.3)$	$1+m+0.10(1+m)^2 = 1.3$ $m^2+12m-2=0$ $(m,x)=(0.1644,0.3)$	$1+m+0.20(1+m)^2 = 1.3$ $2m^2+14m-1=0$ $(m,x)=(0.0707,0.3)$

Each one of the three points, (m,x), {(0.3,0.3), (01644,0.3), (0.0707,0.3)} represents a different project where the owner is indifferent between (a) a lump-sum contract with the same markup percentage, x=30%, and (b) a cost-plus-fee contract for the corresponding percentage, m. Thus, each point represents a choice between two contractual alternatives for the same project. The owner is indifferent between all six alternatives because they all have the same certain equivalent, $1.3E[C]$.

The three lump-sum alternatives present no cost risk to the owner. They have the same markup percentage, x=30%, and hence the same fixed price, $L=(1+x)E[C]$, which is also the certain equivalent for all six alternatives.

In contract, the risk of the cost-plus-fee cost lottery facing the owner increases as the relative cost variability, $Var[C]/2RE[C]$, increases, from the lower project indifference curve to the upper project indifference curve. For the certain equivalent to stay the same, $1.3E[C]$, it is necessary for the fee percentage, m, to decrease as risk increases to counterbalance the increase in cost variability. That is the reason m decreases from 30% to 16.44% to 7.07%, as cost variability, $Var[C]/2RE[C]$, increases from 0.00, to 0.10, to 0.20.

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Owner's Certain-Equivalents for Project Indifference Curves

The table below shows the certain equivalents, $CE[C_L]$ and $CE[C_m]$, for the three values of $Var[C]/2RE[C]$ that correspond to the three project indifference curves in the above figure.

$\frac{Var[C]}{2RE[C]}$	*CE* for a Lump-Sum Contract: $CE[C_L] = L$	=	*CE* for a Cost-Plus-Fee Contract: $CE[C_m] = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$
0.00	$CE[C_L] = L$	=	$CE[V_m] = (1+m)E[C]$
0.10	$CE[C_L] = L$	=	$CE[C_m] = \{(1+m) + 0.10(1+m)^2\}E[C]$
0.20	$CE[C_L] = L$	=	$CE[C_m] = \{(1+m) + 0.20(1+m)^2\}E[C]$

In each row of the above table, the certain equivalent for the cost of a lump-sum contract, $CE[C_L]$, is shown on the left, and the certain equivalent for a cost-plus-fee contract, $CE[C_m]$, is shown on the right. Even though they appear different, the two certain equivalents, $CE[C_L]$ and $CE[C_m]$, equal each other because the points on each of the three project indifference curves satisfy the following (indifference) condition:

$$L = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$$

Thus, in each row of the above table we have,

$$CE[C_L] = L = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R} = CE[C_m]$$

Clearly, the certain equivalent for the cost lottery of a cost-plus-fee contract, $CE[C_m]$, increases as the fee percentage, m, increases, making the contractual arrangement less desirable to the owner.

Similarly, the certain equivalent for the cost of a lump-sum contract, $CE[C_L] = L$, for the project indifference curve in each row, also increases as the cost-plus-fee percentage, m, increases, to keep the two certain equivalents equal, $CE[C_L] = CE[C_m]$.

The above observations apply to contractual choices, (m,x), for the same project, for which the owner is indifferent (i.e., within the same table row).

We can also compare a project of lower risk to a project of higher risk. For the same fee percentage, m, the certain equivalent for cost for a cost-plus-fee contract, $CE[C_m]$, increases when moving down the table because the relative cost variability, $Var[C]/2RE[C]$ increases, making the corresponding project less desirable. This means that the certain equivalent for the cost of a lump-sum contract, $CE[C_L] = L$, also increases from row to row, as $Var[C]/2RE[C]$ increases, so that the two certain equivalents remain equal, $CE[C_L] = CE[C_m]$. This is made clear by the fact that each of the intersection points between one of the three curves and a vertical line (for m = constant) lies on a higher *iso-certain equivalent* horizontal line, making the project for the higher intersection point less desirable.

Numerical Examples — Owner Indifferent: Lump-Sum vs. Cost-Plus-Percent-Fee

Consider the same example project presented earlier, whose cost C follows a Normal distribution with the expected cost $E[C]$ and cost variance $Var[C]$ shown below. Suppose, also, that the owner of the project has an exponential utility curve, with risk tolerance R=\$250k.

The owner will face the following cost lotteries for a lump-sum contract and for a cost-plus-percent-fee contract:

Project Cost Lottery, C	*Lump-Sum Cost Lottery,* $C_L = L$	*Cost-Plus-Percent-Fee Cost Lottery, for fee percentage, m,* $C_m = (1+m)C$
$C \sim N(E[C], Var[C])$ $E[C] = \$1,000k$ $Var[C] = (\$100k)^2$	$C_L = L$ $= (1+x)E[C]$ $= (1+x)\$1,000k$	$C_m \sim N(E[C_m], Var[C_m])$ $E[C_m] = (1+m)E[C] = (1+m)1,000k$ $Var[C_m] = Var[(1+m)C] = (1+m)^2(\$100k)^2$

The owner's certain equivalent and risk premium for the lump-sum contract, shown below, are straightforward because by assumption a lump-sum contract has no cost risk:

Owner's Lump-Sum Contract Cost Lottery	
For price, L, $C_L = L$	$CE[C_L] = L = (1+x)E[C] = (1+15\%)\$1,000k = \$1,150k$ $RP[C_L] = \$0$

The owner's certain equivalent and risk premium for the cost-plus-fee cost lottery are:

Owner's Cost-Plus-Fee Contract Cost Lottery (*Owner's risk tolerance R* =\$250k)	
Owner's Cost-Plus-Percent-Fee Cost Lottery, $C_m = (1+m)C$	$CE[C_m] = (1+m)E[C] + (1+m)^2 \dfrac{Var[C]}{2R}$ $= (1+m)1,000k + (1+m)^2 \dfrac{(\$100k)^2}{2 \cdot \$250k} = \$20k(51+52m+m^2)$ $RP[C_m] = (1+m)^2 \dfrac{Var[C]}{2R} = \$20k(1+m)^2$

Numerical Examples — Owner's Indifference Curve: Lump-Sum vs Cost-Plus-%Fee

Shown below are two versions of the general condition that makes the owner indifferent between a lump-sum contract and a cost-plus-percent-fee contract. These conditions were derived by setting the certain equivalents for the owner's project cost lotteries for the two contract types to be equal, $CE[C_L] = CE[C_m]$. The first version gives the markup percentage, x, whereas the second version is solved for the fee percentage, m.

$$x = \frac{L - E[C]}{E[C]} = m + (1+m)^2 \frac{Var[C]}{2RE[C]} = m + \frac{RP[C_m]}{E[C]}$$

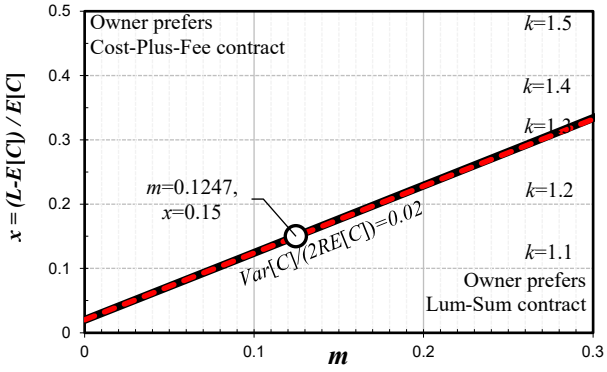
$$m = \frac{-q - 1 + \sqrt{+2q(1+x)+1}}{q}, \text{ where } q = \frac{Var[C]}{RE[C]}$$

For the example problem, the above owner indifference conditions can be simplified to:

$$x = m + 0.02(1+m)^2 = 0.02(1 + 52m + m^2)$$

$$m = -26 + \sqrt{675 + 50x}$$

These quadratic indifference conditions relate the markup percentage x and the cost-plus-fee percentage fee m that make the owner with risk tolerance R=\$250k indifferent between a lump-sum contract and a cost-plus-percent fee contract, for $Var[C]/(2RE[C]) = 0.02$. Both conditions are shown by the heavy black curve in the figure below.



Substituting the markup percentage x=0.15, that corresponds to a fixed price L=\$1,150k, into the above indifference relationship we obtain the fee percentage m=12.47%. Hence, the owner would be indifferent between a lump-sum contract for a fixed price, L=\$1,150k, or equivalently a markup percentage x=15%, and a cost-plus-fee contract with a fee percentage m=12.47%. Both have the same certain equivalent $CE[C_L] = CE[C_m]$ = \$1,150k.

The indifference point (m=0.1247, x=0.15) is shown in the graph above. The iso-certain-equivalent curve that passes through this point is the horizontal line x=0.15 that corresponds to k=1.15. The corresponding certain equivalent is:

$$CE[C_L] = CE[C_m] = k\,E[C] = 1.15 * \$1{,}000\text{k} = \$1{,}150k$$

Both Owner and Contractor Indifferent Between Lump-Sum & Cost-Plus-Fee Contracts

Consider an owner with an exponential utility curve and risk tolerance R, who is about to hire a contractor with an exponential utility curve and risk tolerance ρ, to do the work for a particular project. The cost of the project C follows a Normal distribution with parameters $E[C]$ and $Var[C]$. We shall investigate the conditions that would make both the owner and the contractor indifferent between the following contractual alternatives:

Project Contract Choices for Owner and Contractor

(a) *Lump-sum contract* for a fixed amount of money $L = (1+x)E[C]$

(b) *Cost-plus-percent fee contract*, where the contractor's fee mC, is a percentage m of the actual project cost C.

Owner Indifference Conditions	Contractor Indifference Conditions
$L = (1+m)E[C] + (1+m)^2 \dfrac{Var[C]}{2R}$	$L = (1+m)E[C] + (1-m^2)\dfrac{Var[C]}{2\rho}$
$x = \dfrac{L-E[C]}{E[C]} = m + (1+m)^2 \dfrac{Var[C]}{2RE[C]}$	$x = \dfrac{L-E[C]}{E[C]} = m + (1-m^2)\dfrac{Var[C]}{2\rho E[C]}$

Individually, the owner and the contractor would be indifferent between the two contract types when the lump-sum amount L (or markup percentage x), and the fee percentage m, satisfy the conditions on the left (for the owner) or the conditions on the right (for the contractor).

For both the owner and the contractor to be indifferent between the two contract types, at the same time, the lump-sum amount L (or markup percentage x), and the fee percentage m, must satisfy both the left and the right conditions simultaneously.

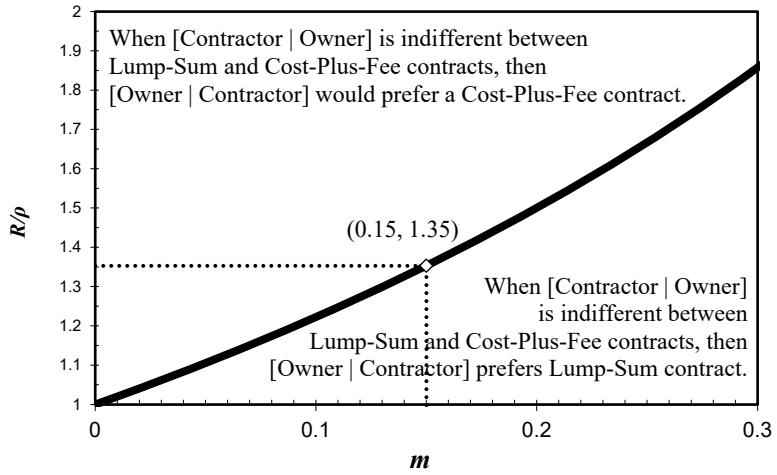
This equilibrium condition is found by setting the expression for L (or x) on the left column equal to the corresponding expression for L (or x) in the right column. After the algebraic cancellation of identical terms, we obtain the following equilibrium condition:

Fee percentage m^*, for which both the owner and the contractor are indifferent between a lump-sum contract and a cost-plus-fee contract.

$$\frac{R}{\rho} = \frac{1+m}{1-m} \quad \Leftrightarrow \quad m^* = \frac{\dfrac{R}{\rho}-1}{\dfrac{R}{\rho}+1}$$

For a specific owner with risk tolerance R, and a specific contractor with risk tolerance ρ, there is only one fee percentage m, (and consequently, only one lump-sum amount L, or markup percentage x), that makes both the owner and the contractor indifferent between a lump-sum contract for a certain fixed amount of money L, and a cost-plus-fee contract for the fee percentage m.

A graph of this indifference condition between the ratio of the owner and contractor risk tolerances, R/ρ, and the cost-plus-fee percentage, m, is shown in the figure below.



Graph of

$$\frac{R}{\rho}=\frac{1+m}{1-m}$$

The Owner's and Contractor's preferences are expressed by the inequalities:

$$\frac{R}{\rho}\begin{matrix}>\\<\end{matrix}\frac{1+m}{1-m}$$

As a numerical example, let us assume that the owner's risk tolerance R is 1.35 times greater than the contractor's risk tolerance, ρ. In this case, we see from the above figure that for R/ρ=1.35, both the owner and the contractor would be indifferent between:

- A cost-plus-fee contract with fee percentage, m=15%, and
- A lump-sum contract, for a certain amount, L. The specific value of the lump-sum amount, L, can be determined by either the owner's or the contractor's indifference condition for L (or x) shown above, because, in this case, they are required to be equal and thus give the same value.

As an example, the specific values of L and x when R/ρ=1.35, shown below are calculated first using the owner's indifference conditions and thus L and x are functions of R. Using the substitution, R=1.35ρ, L and x are then converted to the equivalent expressions given by the contractor's indifference conditions and thus L and x are functions of ρ:

R/ρ=1.35, m=15%	Based on lump-sum amount, L	Based on markup percentage, x
Owner's indifference condition (uses R)	$L=(1+m)E[C]+(1+m)^2\frac{Var[C]}{2R}$ $=1.15E[C]+\frac{0.66125}{R}Var[C]$	$x=\frac{L-E[C]}{E[C]}=m+(1+m)^2\frac{Var[C]}{2RE[C]}$ $=0.15+\frac{0.66125}{R}\frac{Var[C]}{E[C]}$
Contractor's indifference condition (uses ρ)	$L=(1+m)E[C]+(1-m^2)\frac{Var[C]}{2\rho}$ $=1.15E[C]+\frac{0.48875}{\rho}Var[C]$	$x=\frac{L-E[C]}{E[C]}=m+(1-m^2)\frac{Var[C]}{2\rho E[C]}$ $=0.15+\frac{0.48875}{\rho}\frac{Var[C]}{E[C]}$

Characteristics of the Owner-Contractor Indifference Condition

The indifference condition for the fee percentage m^*, for which both the owner and the contractor are indifferent between a lump-sum contract and a cost-plus-percent-fee contract, has the following remarkable characteristics:

$$m^* = \frac{R/\rho - 1}{R/\rho + 1}$$

- The expression for the cost-plus-percent-fee percentage m^* is surprisingly simple.
- The cost-plus-percent-fee percentage m^* depends only on the ratio of the owner's and the contractor's risk tolerances R/ρ, but not their absolute values.
- The cost-plus-percent-fee percentage m^* does not depend on any characteristics of the project (for which the contract will be written) such as (a) the project expected cost $E[C]$, or (b) the project cost variance $Var[C]$.
- For a given owner and contractor (i.e., for a given ratio of owner and contractor risk tolerances R/ρ), the equilibrium fee percentage m^* is the same for any project, irrespective of project size and risk.
- The cost-plus-fee percentage m^* exists only when the owner is less risk averse than the contractor $R > \rho$. Otherwise, m^* is negative and thus meaningless.

It should be pointed out that the above observations apply only to the cost-plus-percent-fee percentage m^*. The corresponding lump-sum contract amount L, for which both the owner and the contractor are indifferent between a lump-sum contract and a cost-plus-percent-fee contract depends on the particular values of the owner's and the contractor's risk tolerances R and ρ, as well as the project expected cost $E[C]$, and the project cost variance $Var[C]$.

These important differences in dependence between m^* and L are illustrated by the following example for the ratio of risk tolerances, R/ρ=1.35:

- The indifference cost-plus-fee percentage, m=15%, is the same for any project that owner and contractor may contract between them and is given by:

$$m = \frac{R/\rho - 1}{R/\rho + 1} = \frac{1.35 - 1}{1.35 + 1} = 15\%$$

- In contrast, the corresponding indifference lump-sum contract amount, L,

$$L = 1.15E[C] + \frac{0.66125}{R}Var[C] = 1.15E[C] + \frac{0.48875}{\rho}Var[C]$$

depends on

a. The project to be contracted, and in particular, on the expected value and the variance of the project cost $E[C]$ and $Var[C]$.

b. The specific values of the risk tolerances R and ρ, and not just on their ratio R/ρ=1.35.

What type of Contract should an Owner offer a Contractor?

The graph of the indifference condition between the cost-plus-fee percentage, m^*, and the ratio of the owner's and the contractor's risk tolerances, R/ρ, provides significant insight into risk pricing in construction and the selection of the appropriate contraction contract.

In particular, the above graph shows that:

(a) For the values of the fee percentage m encountered in practice, e.g., the entire range of the horizontal axis in the figure, $0 < m < 0.3$, the indifference condition is satisfied as an equality only for values of the risk-tolerance ratio R/ρ in the range $1< R/\rho < 2$ (the entire range of the vertical axis in the figure).

(b) The indifference value for the fee percentage m^* is negative when $R/\rho < 1$.

These observations lead to the following important conclusions about the ratio of the owner's and the contractor's risk tolerances R/ρ, and the owner's best contract choices.

A. $R/\rho > 2$ — The owner is significantly less risk averse than the contractor.
When the risk tolerance of the owner R is double (or more) than the risk tolerance of the contractor ρ, i.e., when $R > 2\rho$, then the owner is always better off with a cost-plus-percent-fee contract instead of a lump-sum contract.

B. $R/\rho < 1$ — The owner is more risk averse than the contractor.
When the risk tolerance of the owner R is less than or equal to the risk tolerance of the contractor ρ, i.e., when $R <= \rho$, then the owner is always better off with a lump-sum contract instead of a cost-plus-percent-fee contract.

C. $1 < R/\rho < 2$ — The owner is slightly less risk averse than the contractor.
When the owner's and contractor's risk tolerances have a ratio in the range, $1 < R/\rho < 2$, then the owner's choice between a lump-cost contract and a cost-plus-fee contract depends on the fee percentage m for the contractor's cost-plus-fee contract, as shown in the graph above.

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Contract Preferred by Owner or Contractor when the Other Party is Indifferent

Let us now consider an owner with risk tolerance R, and a contractor with risk tolerance ρ, such that the ratio of the risk tolerances has a specific value R/ρ, and investigate the conditions for the following preferences:

- When the contractor is indifferent between a lump-sum and a cost-plus-fee contract:

$$L = (1+m)E[C] + (1-m^2)\frac{Var[C]}{2\rho}$$

- The owner prefers a cost-plus-percent-fee contract:

$$L > (1+m)E[C] + (1+m)^2\frac{Var[C]}{2R}$$

Substituting the above equation into the inequality gives the following conditions:

$\frac{R}{\rho} > \frac{1+m}{1-m}$, or $m < m^* = \frac{\frac{R}{\rho}-1}{\frac{R}{\rho}+1}$ $\Leftrightarrow$	• When the Contractor is indifferent between a lump-sum and a cost-plus-percent-fee contract, • The Owner would prefer a cost-plus-percent-fee contract.

Similarly, we shall now find the condition for which:

- When the owner is indifferent between a lump-sum and a cost-plus-percent-fee contract.

$$L = (1+m)E[C] + (1+m)^2\frac{Var[C]}{2R}$$

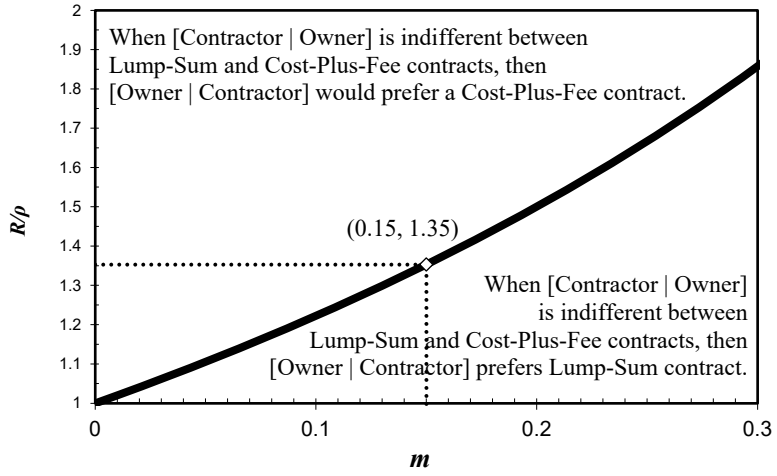
- The contractor prefers a cost-plus-percent-fee contract:

$$L < (1+m)E[C] + (1-m^2)\frac{Var[C]}{2\rho}$$

Surprisingly, a similar substitution shows that the resulting conditions are the same as above:

$\frac{R}{\rho} > \frac{1+m}{1-m}$, or $m < m^* = \frac{\frac{R}{\rho}-1}{\frac{R}{\rho}+1}$ $\Leftrightarrow$	• When the Owner is indifferent between a lump-sum and a cost-plus-percent-fee contract • The Contractor would prefer a cost-plus-percent-fee contract

The figure below shows the two regions for points (m, R/ρ) defined by the above inequalities. Clearly, these regions are separated by the indifference condition between the ratio of the owner and contractor risk tolerances, R/ρ, and the cost-plus-fee percentage, m:



Graph of

$$\frac{R}{\rho} = \frac{1+m}{1-m}$$

The Owner's and Contractor's preferences are expressed by the inequalities:

$$\frac{R}{\rho} \lessgtr \frac{1+m}{1-m}$$

$$m \lessgtr m^* = \frac{R/\rho - 1}{R/\rho + 1}$$

The preferences shown by the above figure, however, are not immediately obvious. To further understand the regions above and below the equilibrium curve in the figure, we shall consider as an example, an owner and a contractor that have a ratio of risk tolerances $R/\rho = 1.35$. For this owner and this contractor, <u>any</u> cost-plus-fee contract with fee percentage m, is represented by a point (m,1.35) that lies on a horizontal line with ordinate $R/\rho = 1.35$ (see figure).

We shall first focus on the points (m,1.35) <u>above</u> and to the <u>left</u> of the equilibrium curve for which $m < m^*=0.15$. Let us use as a representative example the point (0.05, 1.35) for which $R/\rho = 1.35$, and the cost-plus-fee percentage $m = 0.05$, is less than $m^*=0.15$.

There are two ways to investigate the owner's and contractor's relative preferences for this point:

(a) Make the <u>contractor indifferent</u> between a cost-plus-5%-fee contract and a lump-sum contract, and then find which contract the owner would prefer.

(b) Make the <u>owner indifferent</u> between a cost-plus-5%-fee contract and a lump-sum contract, and then find which contract the contractor would prefer.

A. For the contractor to be indifferent between a cost-plus-5%-fee contract and a lump-sum contract, the lump-sum amount, L, is given by:

$$L = (1+m)E[C] + (1-m^2)\frac{Var[C]}{2\rho}$$

$$= 1.05E[C] + \frac{0.49875}{\rho}Var[C]$$

$$= 1.05E[C] + \frac{0.67478}{R}Var[C]$$

The analysis above indicates that the owner would prefer a cost-plus-5%-fee contract when the lump-sum amount, L, exceeds:

$$L > (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$$

$$L > 1.05E[C] + \frac{0.55125}{R} Var[C]$$

In this case, the above condition is true because:

$$L = 1.05E[C] + \frac{0.67478}{R} Var[C] \quad > \quad 1.05E[C] + \frac{0.55125}{R} Var[C]$$

From here we conclude that,

> The lump-sum amount L that makes the contractor indifferent to a cost-plus-5%-fee contract is too high for the owner. Thus, the owner prefers the cost-plus-5%-fee contract.

B. For the owner to be indifferent between a cost-plus-5%-fee contract and a lump-sum contract, the lump-sum amount L must be:

$$L = (1+m)E[C] + (1+m)^2 \frac{Var[C]}{2R}$$

$$= 1.05E[C] + \frac{0.55125}{R} Var[C]$$

$$= 1.05E[C] + \frac{0.40745}{\rho} Var[C]$$

The contractor prefers a cost-plus-5%-fee contract when the lump-sum amount, L, is less than:

$$L < (1+m)E[C] + (1-m^2)\frac{Var[C]}{2\rho}$$

$$< 1.05E[C] + \frac{0.49875}{\rho} Var[C]$$

This is true in this case because.

$$L = 1.05E[C] + \frac{0.40745}{\rho} Var[C] \quad < \quad 1.05E[C] + \frac{0.49875}{\rho} Var[C]$$

From here we conclude that,

> The lump-sum amount L that makes the owner indifferent to a cost-plus-5%-fee contract is too low for the contractor. Thus, the contractor prefers a cost-plus-5%-fee contract.

Comparing cases (A) and (B) leads to the final conclusion:

> For the point $m = 0.05$, and $R/\rho = 1.35$, both the owner and the contractor would prefer a cost-plus-5%-fee contract rather than the lump-sum amount that would make the other party indifferent between a cost-plus-5%-fee contract and a lump-sum contract.

Exactly the opposite conclusions hold true for points (m,1.35) on the same horizontal line (i.e., for $R/\rho = 1.35$) for which $m > m^*=0.15$, i.e., to the right of the equilibrium curve.

As a specific example for $R/\rho = 1.35$, let us consider the point (0.25,1.35) for which the cost-plus-fee percentage $m = 25\%$ is greater than $m^*=0.15$.

Following an analysis similar to that above it is easy to show that for $R/\rho=1.35$ and $m = 25\%$:

- The lump-sum amount L that makes the contractor indifferent between a lump-sum contract and a cost-plus-25%-fee contract is low enough to make the owner prefer this lump-sum amount L to a cost-plus-25%-fee contract.
- Similarly, the lump-sum amount L that makes the owner indifferent between a lump-sum contract and a cost-plus-25%-fee contract is high enough to make the contractor prefer this lump-sum amount L to a cost-plus-25%-fee contract.

Thus, for the point $m = 0.25$, and $R/\rho = 1.35$, both the owner and the contractor would prefer "the lump-sum amount L that makes the other party indifferent between a cost-plus-25%-fee contract and a lump-sum contract", rather than a cost-plus-25%-fee contract.

Combined Indifferent Curves for Specific Ratio of Owner & Contractor Risk Tolerances

Consider an owner with risk tolerance, R, and a contractor with risk tolerance, ρ, such that the ratio of the risk tolerances has the specific value, $R/\rho = 1.35$.

We shall analyze the choice between a lump-sum contract and a cost-plus-fee contract for the same project by plotting in the same graph two indifference conditions, one for the owner and one for contractor, that relate the fee percentage, m, to the markup percentage, x.

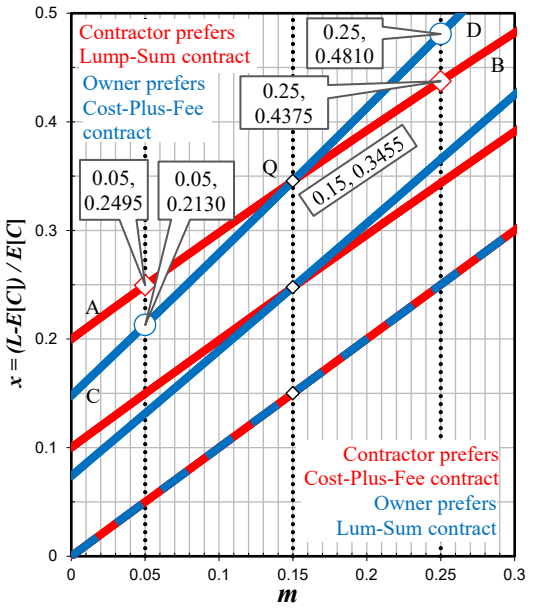
Project Contract Choices for Owner and Contractor

(a) *Lump-sum contract* for a fixed amount of money, $L = (1+x)E[C]$

(b) *Cost-plus-percent-fee contract*, where the contractor's fee, mC, is a percentage m of the actual project cost, C.

Owner Indifference Condition	Contractor Indifference Condition
$x = \dfrac{L-E[C]}{E[C]} = m + (1+m)^2 \dfrac{Var[C]}{2RE[C]}$	$x = \dfrac{L-E[C]}{E[C]} = m + (1-m^2) \dfrac{Var[C]}{2\rho E[C]}$

Clearly, the above two equations define two families of indifference curves that are characterized by the parameters, $E[C]$, $Var[C]$, R, and ρ.



$$\frac{R}{\rho} = \frac{1+m}{1-m} = \frac{1+0.15}{1-0.15} = 1.35$$

Contractor parameter values $\dfrac{Var[C]}{2\rho E[C]}$	Owner parameter values $\dfrac{Var[C]}{2RE[C]}$
0.00	0.00/1.35 = 0.00
0.10	0.10/1.35 = 0.073913
0.20	0.20/1.35 = 0.147826

The graph above shows three pairs of indifference curves, the blue family for the owner and the red family for the contractor, where each blue and red pair correspond to the same project. As shown in the table above, the owner's ratio of the project parameters for each of the three pairs of curves, $Var[C]/(2RE[C])$ has been scaled by a factor of 1.35 to reflect the ratio of the owner and contractor risk tolerances, $R/\rho = 1.35$.

As an example, the red curve, AQB, represents a project with $Var[C]/(2\rho E[C]) = 0.20$. The points (m,x) on this curve have coordinates such that the contractor would be indifferent between a lump-sum contract with markup percentage, x, and a cost-plus-percent-fee contract with fee percentage, m, for this project.

The corresponding blue curve, CQD, represents the same project with $Var[C]/(2RE[C]) = 0.20/1.35 = 0.147826$. The points (m,x) on this curve have coordinates such that the owner would be indifferent between a lump-sum contract with markup percentage, x, and a cost-plus-percent-fee contract with fee percentage, m, for the same project.

It is important to notice that the blue and red pairs of curves, for each of the three projects, intersect at the same fee percentage value, $m^*=0.15$. For $m^*=0.15$, the corresponding value for the markup percentage, x, is the same for both the owner and contractor, and this make them both indifferent between the two contract types. As shown earlier, this value, $m^*=0.15$, is the same for any project and depends only on the ratio of the risk tolerances, $R/\rho = 1.35$.

In the following discussion, we shall use as an example the contractor's red indifference curve AQB and the owner's blue indifference curve CQD, which for $R/\rho = 1.35$, and $Var[C]/(2\rho E[C]) = 0.20$, intersect at $m=0.15$, $x = 0.3455$, as shown in the figure.

For the same owner and contractor, $R/\rho = 1.35$, and the same project, $Var[C]/(2\rho E[C]) = 0.20$, let us now consider the fee percentage, $m = 0.05$, a value that is less than $m^*=0.15$. It should be pointed out that when the fee percentage, $m = 0.05$, was considered earlier, we concluded that when the owner, or the contractor, is indifferent between a lump-sum contract and a cost-plus-5%-fee contract, the other party would prefer the cost-plus-5%-fee contract.

The same conclusion can be reached by examining the above graph. As shown in the figure, the owner would be indifferent when faced with a choice between $x=0.2130$ and $m=0.05$ (this choice is indicated by a blue circle). Since the point for this contractual choice lies below the red contractor indifference curve AQB for the same project, the contractor would prefer the cost-plus-5%-fee contract as indicated by the red text for the lower region of the graph.

For the same value, $m=0.05$, the contractor would be indifferent between $x=0.2495$ and $m=0.05$ (this choice is indicated by a red diamond). Since this contractual choice lies above the blue owner indifference line CQD, the owner would prefer the cost-plus-5%-fee contract as indicated by the blue text for the upper region of the graph.

The opposite can be observed for the value $m=0.25 > m^*=0.15$. As shown in the figure, the owner would be indifferent when faced with a choice between $x=0.4810$ and $m=0.25$ (this choice is indicated by a blue circle). Since this point lies above the red contractor indifference line AQB, the contractor would prefer the lump-sum contract with markup percentage, $x=0.4810$, as indicated by the red text for the upper region of the graph.

For the same value, $m=0.25$, the contractor would be indifferent between $x=0.4375$ and $m=0.25$ (this choice is indicated by the red diamond). Since this point lies below the blue owner indifference line CQD for the same project, the owner would prefer the lump-sum contract with markup percentage, $x=0.4375$, as indicated by the blue text for the lower region of the graph.

Another important observation concerns the relative slopes of the blue and red indifference curves for the same project, indicated in the figure. An algebraic comparison of the corresponding slopes shows that the slope of the blue owner curve is always greater than the slope of the red contractor curve for the same project (which is approximately 1):

$$1+2\frac{Var[C]}{2RE[C]}>1$$

Thus, for any given value m that is less than m^*, the corresponding red diamond is always above the blue circle which means that when the owner, or the contractor, is indifferent between a lump-sum contract and a cost-plus-m-percent-fee contract, the other party would always prefer the cost-plus-m-percent-fee contract.

Similarly, for any given value m greater than m^*, the corresponding red diamond is always below the blue circle which means that when the owner, or the contractor, is indifferent between a lump-sum contract and a cost-plus-m-percent-fee contract, the other party would always prefer the lump-sum contract.

These are exactly the same conclusions we reached earlier.

It should also be noted that the vertical distance between the red diamond (contractor is indifferent) and the blue circle (owner is indifferent) is the difference between the corresponding markup percentages, x, for the same percentage fee, m. Clearly, the vertical distance between the corresponding blue and red points is small and gets even smaller as the percentage fee, m, approaches the equilibrium value, m^*, where both points coincide and the corresponding markup percentages, x, become equal.

Contractual Agreement Spaces

Let us consider the decision represented by some point (m,x) above the red and blue curves AQD. Given the choice represented by this point between a lump-sum contract with markup percentage, x, and a cost-plus-m-percent-fee contract, the owner would prefer the cost-plus-m-percent-fee contract, m, whereas the contractor would prefer a lump-sum contract with markup percentage, x. Thus, the owner and contractor would disagree as to the preferred contract type.

Similarly, a contract choice (m,x) below the curve segments CQB would also lead to disagreement as to the preferred contract type, because for any point (m,x) below CQB, the owner would prefer a lump-sum contract with markup percentage, x, whereas the contractor would prefer a cost-plus-m-percent-fee contract.

Thus, the larger regions above AQD and below CQB lead to disagreement as to the preferred contract type. Agreement exists only in the smaller regions AQC and DQB.

In particular, any point (m,x) in the left narrow triangular region AQC represents a contract choice where both the owner and the contractor prefer the cost-plus-m-percent-fee contract.

Similarly, any point (m,x) in the right narrow triangular region DQB represents a contract choice where both the owner and the contractor prefer the lump-sum contract with the specified markup percentage, x.

The clear conclusion from these observations is that the majority of points in the (m,x) space, (above AQD and below CQB) represent contract choices for which the owner and contractor would disagree about the preferred contract type. Successful contract negotiations between the owner and contractor can only be reached at a point (m,x) in the much smaller regions AQC and DQB.

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Contractual Agreement Spaces and Iso-Certain Equivalent Lines

Consider the project with $Var[C]/(2\rho E[C]) = 0.20$ in the above figure, and a contract choice represented by point (m,x) in the left triangular region AQC. In this region, given a choice between a lump-sum contract with markup percentage, x, and a cost-plus-m-percent-fee contract, both the owner and the contractor would prefer the cost-plus-m-percent-fee contract.

Now let us consider all points in region AQC that lie on the same vertical line that passes through this particular point (m,x). The certain equivalent associated with each of these points is the certain equivalent of the preferred contract alternative, which in this case is the cost-plus-m-percent-fee contract. Thus, all points on this vertical linear segment have the same certain equivalent, because they all have the same, m.

This means that the iso-certain equivalent lines for both the owner and the contractor in the region AQC, are vertical lines characterized by the corresponding value, m. However, the owner's and the contractor's preferences increase in opposite directions. The contractor prefers iso-certain equivalent lines to the right, i.e., for higher values, m, whereas the owner prefers iso-certain equivalent lines to the left, i.e., for lower values, m.

Similarly, given a contract choice represented by point (m,x) in the upper triangular region DQB, both the owner and the contractor would prefer a lump-sum contract with markup percentage, x. All points in region DQB that lie on the same horizontal line as point (m,x) have the same value x, and thus have the same certain equivalent, which is the certain equivalent for a lump-sum contract.

This means that the iso-certain equivalent lines for both the owner and the contractor in the region DQB, are horizontal lines characterized by the corresponding value, x. Clearly, the contractor prefers iso-certain-equivalent lines with higher values, x, whereas the owner prefers iso-certain-equivalent lines with lower values, x.

In summary, for the project with $Var[C]/(2\rho E[C]) = 0.20$, the owner's blue indifference curve and the contractor's red indifference curve divide the (m,x) space into four regions. The owner and contractor prefer the same contract type in only the two smaller regions, AQC and DQB. In AQC both prefer the cost-plus-m-percent-fee contract and the iso-certain equivalent lines are vertical lines where the owner prefers smaller m values and the contractor prefers larger m values. In the right triangular region DQB, both prefer a lump-sum contract with markup percentage, x. The iso-certain equivalent lines are horizontal lines where the owner prefers smaller x values and the contractor prefers larger x values.

Clearly, the identification of these four regions and the associated observations are important for guiding the selection of the appropriate contract type and the direction of negotiations between the owner and contractor.

Numerical Examples — Owner & Contractor Contract Preferences for Any Project

Consider the same owner and contractor as in the above numerical examples:

Owner has exponential utility curve with risk tolerance, $R = \$250k$.

Contractor has exponential utility curve with risk tolerance, $\rho = \$200k$.

At first, using only the ratio of the owner's and contractor's risk tolerances, R and ρ, we can find the percentage fee, m^*, that, for any project, makes both the owner and contractor indifferent between a cost-plus-percent-fee contract and a lump-sum contract:

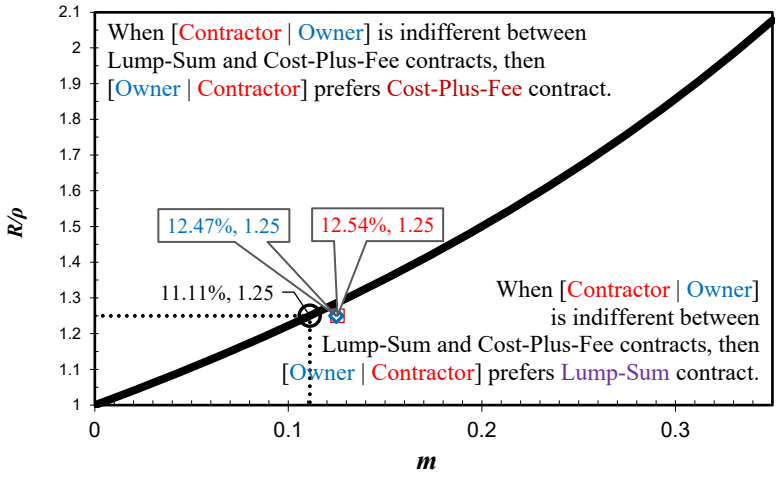
$$m^* = \frac{\frac{R}{\rho}-1}{\frac{R}{\rho}+1} = \frac{\frac{250}{200}-1}{\frac{250}{200}+1} = \frac{1.25-1}{1.25+1} = \frac{1}{9} = 0.111...$$

Thus, for the ratio of risk tolerances, $R/\rho = 1.25$, the percentage fee, m^*=11.11%, is the only value that makes both the owner and contractor indifferent between a cost-plus-m-percent-fee contract and a lump-sum contract, for any project.

Moreover, for any project, when the owner (or the contractor) is indifferent between a lump-sum contract and a cost-plus-fee contract with fee percentage, $m < 11.11\%$, then the other party (the contractor or the owner) would always prefer the cost-plus-m-percent-fee contract with that particular fee percentage, $m < 11.11\%$.

Conversely, for any project, when the owner (or the contractor) is indifferent between a lump-sum contract and a cost-plus-fee contract with fee percentage, $m > 11.11\%$, then the other party (the contractor or the owner) would always prefer the lump-sum contract.

These results will be illustrated in detail using the same example project described earlier.



Numerical Examples — Owner & Contractor Contract Preferences for a Specific Project

Consider the example project used earlier that has the following data:

- Project cost follows a Normal distribution with $E[C]$ = \$1,000k and $SD[C]$ = \$100k.
- Example lump sum amount, L = \$1,150k, markup percentage, x=15%.

We shall analyze several choices between a lump-sum contract and a cost-plus-fee contract for this project, using the following indifference conditions for the owner and contractor:

Project Contract Choices for Owner and Contractor

(a) *Lump-sum contract* for a fixed amount of money, $L = (1+x)E[C]$

(b) *Cost-plus-percent-fee contract*, where the contractor's fee, mC, is a percentage m of the actual project cost, C.

Owner Indifference Condition	**Contractor Indifference Condition**
$x = \dfrac{L - E[C]}{E[C]} = m + (1+m)^2 \dfrac{Var[C]}{2RE[C]}$	$x = \dfrac{L - E[C]}{E[C]} = m + (1-m^2) \dfrac{Var[C]}{2\rho E[C]}$
$x = m + 0.02(1+m)^2$	$x = m + 0.025(1-m^2)$
$m = -26 + \sqrt{675 + 50x}$	$m = 20 - \sqrt{401 - 40x}$

The table below shows the above indifference conditions and preferences for the example project data. Each row shows the lump-sum amount, L, and the markup percentage, x, along with the corresponding percentage fee, m, that make either the owner, or the contractor, (or both), indifferent between a lump-sum contract and a cost-plus-m-percent-fee contract.

Indifferent Party	*Indifference Parameters*			***Owner CE*** (\$k) R=\$250k			***Contractor CE*** (\$k) ρ=\$200k		
	L (\$k)	x	m	$CE[C_L]$		$CE[C_m]$	$CE[V_L]$		$CE[V_m]$
Owner	1,150.723	15.07%	12.54%	1,150.723	~	1,150.723	125.723	≻	125
Owner	1150	15%	12.47%	1,150	~	1,150	125	≻	124.312
Contractor	1150	15%	12.54%	1,150	≻	1,150.723	125	~	125
Contractor	1,149.312	14.931%	12.47%	1,149.312	≻	1,150	124.312	~	124.312
Both	1,135.802	13.580%	11.11%	1,135.802	~	1,135.802	110.802	~	110.802

To allow comparisons between rows, each row uses as input either the same markup percentage, x, or the same fee percentage, m, as another row and finds the corresponding value, x or m, that makes the party for that row indifferent. These equal values in the columns, x and m, are highlighted with the same color to make them easy to see. Also, to make comparisons easier, the rows in the table are sorted by the lump-sum amount, L, in decreasing order.

In the first two rows, the indifferent party is the owner. The owner's certain equivalents for cost for the two contract types are equal, $CE[C_L] = CE[C_m]$, and are shown in blue. In the next two rows, the indifferent party is the contractor. The contractor's certain equivalents for profit for the

two contract types are equal, $CE[V_L] = CE[V_m]$, and are shown in red. In each row, the certain equivalents for the other party are shown in green and black, where green indicates the preferred choice. Notice that for this example, when the owner or the contractor is indifferent between the two contract types, the other party (the contractor or the owner) always prefers the lump-sum contract choice.

For the specific ratio of owner and contractor risk tolerances, R/ρ=1.25, the fee percentage that makes both the owner and the contractor indifferent between a lump-sum and a cost-plus-fee contract is $m^* = 11.11\%$. Given that in the first four rows of the above table we have, $m > 11.11\%$, it is not surprising that both the owner and the contractor prefer a lump-sum contract when the other party is indifferent. This was already predicted in the previous section based on the curve that relates R/ρ and the fee percentage m^*=11.11% for which both the owner and contractor are indifferent.

The first row, for example, corresponds to a choice between (L1) a lump-sum contract for a fixed price, L = \$1,150.723*k*, and (F1) a cost-plus-fee contract with a fee percentage, m = 12.54%. The owner is indifferent between offering the contractor either contract (L1) or (F1) because $CE[C_L] = CE[C_m]$ = \$1,150.723*k*. However, the contractor would prefer (L1) with the higher $CE[V_L]$ = \$125.723*k*, to (F1) with the lower $CE[V_m]$ = \$125*k*.

The second row is similar and shows a choice between (L2) a lump-sum contract for a fixed price, L = \$1,150*k*, and (F2) a cost-plus-fee contract with a fee percentage, m = 12.47%. The owner is indifferent between offering the contractor contracts (L2) or (F2) because $CE[C_L] = CE[C_m]$ = \$1,150*k*. However, the contractor would prefer (L2) with the higher $CE[V_L]$ = \$125*k*, to (F1) with the lower $CE[V_m]$ = \$124.312*k*.

The third row shows a choice between (L3) a lump-sum contract for a fixed price, L = \$1,150*k*, and (F3) a cost-plus-fee contract with a fee percentage, m = 12.54%. In this case, it is the contractor who is indifferent between being offered (L3) or (F3) because $CE[V_L] = CE[V_m]$ = \$125*k*. However, the owner would prefer to offer (L3) with the lower certain equivalent for cost $CE[C_L]$ = \$1,150*k*, to (F3) with the higher $CE[C_m]$ = \$1,150.723*k*.

The fourth row shows a choice between (L4) a lump-sum contract for a fixed price, L = \$1,149.312*k*, and (F3) a cost-plus-fee contract with a fee percentage, m = 12.47%. The contractor is indifferent between being offered (L4) or (F4) because $CE[V_L] = CE[V_m]$ = \$124.312*k*. However, the owner prefers (L4) with the lower certain equivalent for cost $CE[C_L]$ = \$1,149.312*k*, to (F4) with the higher $CE[C_m]$ = \$1,150*k*.

The fifth row shows a choice between (L5) a lump-sum contract for a fixed price, L = \$1,135.802*k*, and (F5) a cost-plus-fee contract with a fee percentage, m = 11.11%. In this case, both the owner and the contractor are indifferent between these two contract choices (L5) and (F5) because $CE[C_L] = CE[C_m]$ = \$1,135.802*k* *and* $CE[V_L] = CE[V_m]$ = \$110.802*k*.

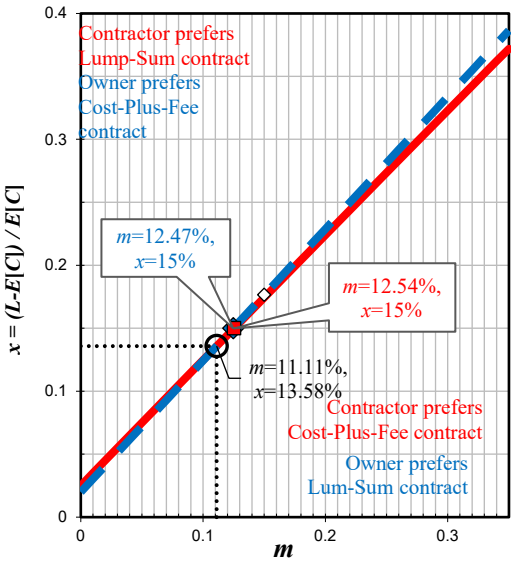
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Numerical Examples — Owner & Contractor Indifference Curves for a Specific Project

Consider the same example project used above:

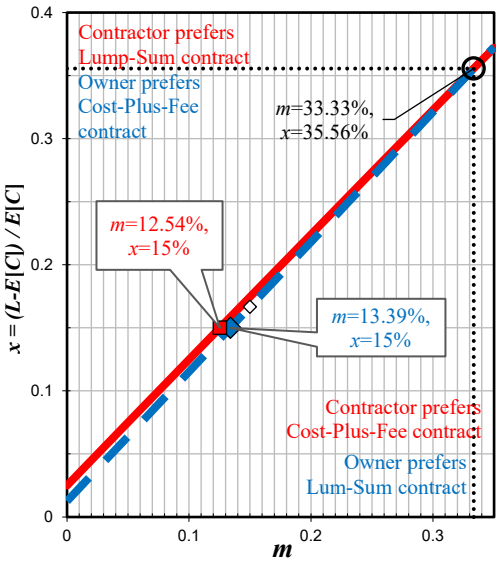
- Project cost follows a Normal distribution with $E[C]$ = \$1,000k and $SD[C]$ = \$100k.
- Example lump sum amount, L = \$1,150k, markup percentage, x=15%.
- Owner has exponential utility curve with risk tolerance, R = \$250k.
- Contractor has exponential utility curve with risk tolerance, ρ = \$200k.

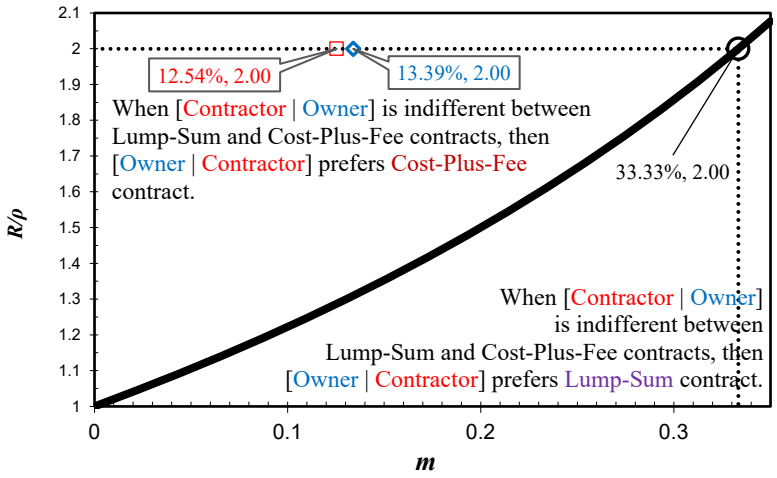
We shall analyze several choices between a lump-sum contract and a cost-plus-fee contract for this project, using the following indifference conditions for the owner and contractor:



Numerical Examples — Owner Significantly Less Risk-Averse than Contractor

Indifferent	*Indifference Parameters*			***Owner: CE*** *($k)*, R=$400k			***Contractor: CE*** *($k)*, ρ=$200k		
Party	*L ($k)*	*x*	*m*	$CE[C_L]$		$CE[C_m]$	$CE[V_L]$		$CE[V_m]$
Owner	1141.224	14.12%	12.54%	1141.224	~	1141.224	116.224	≺	125
Owner	1150	15%	13.39%	1150	~	1150	125	≺	133.479
Contractor	1150	15%	12.54%	1150	≺	1141.224	125	~	125
Contractor	1158.479	15.85%	13.39%	1158.479	≺	1150	133.479	~	133.479
Both	1355.556	35.56%	33.33%	1355.555	~	1355.556	330.556	~	330.556




2.1
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
R/p
0
0.1
0.2
0.3
m
12.54%, 2.00
13.39%, 2.00
33.33%, 2.00
When [Contractor | Owner] is indifferent between Lump-Sum and Cost-Plus-Fee contracts, then [Owner | Contractor] prefers Cost-Plus-Fee contract.
When [Contractor | Owner] is indifferent between Lump-Sum and Cost-Plus-Fee contracts, then [Owner | Contractor] prefers Lump-Sum contract.